# Stat 513
## Fall 2025
## Problem Set 2
## Topic 2: Random Variables, Distribution Functions, and Continuity

Total: 50 points

Due Sunday, September 21 at 23:59

# 1 Continuity

**Problem 1. Continuity**

a) Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}((a, b))$ is open for any open interval $(a, b) \subset \mathbb{R}$.

b) Show that for a continuous function $f : \mathbb{R} \to \mathbb{R}$, for any fixed $u$ the set

$$A = \{x \in \mathbb{R} : f(x) = u\}$$

is closed.

**Problem 2. Limits of continuous functions**

Consider continuous functions $f_1, f_2, \ldots$ on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be function defined as $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x$. Show by example that $f$ is not necessarily continuous. Conclude that a sequences of continuous random variables may converge in distribution to a discrete random variable (i.e., their distribution functions converge to a distribution function of a discrete random variable).

# 2 Distribution Functions

**Problem 3. Properties of Distribution Functions**

Let $X$ denote a random variable on $(\Omega, \mathcal{F}, P)$, with distribution function $F$. Define the notation

$$F(x-) = \lim_{h\downarrow 0} F(x - h)$$

and

$$F(x+) = \lim_{h\downarrow 0} F(x + h)$$

to denote the "limit from below/left" and the "limit from above/right".

a) Using a variation of the proof shown in class that $\lim_{x\to\infty} F(x) = 1$, show that

$$\lim_{x\to-\infty} F(x) = 0.$$

b) Show that

$$F(x) = F(x-) + P(X = x).$$

c) Conclude that $F$ is continuous if and only if $P(X = x) = 0$ for all $x \in \mathbb{R}$.

**Problem 4. Quantile Functions**

Let $X$ denote a random variable on probability space $(\Omega, \mathcal{F}, P)$ with distribution function $F$, define the *quantile function* $Q : (0,1) \to \mathbb{R}$ as

$$Q(u) = \inf\{x : F(x) \geq u\}.$$

a) Show that

$$P(X \leq Q(u)) \geq u.$$

This explains the naming of $Q$ as the "quantile function".

b) What is the quantile function for $X \sim \text{Bernoulli}(1/2)$?

c) Show that the quantile function is monotonic non-decreasing.

d) Show that $Q$ is a *psuedo-inverse* of $F$, in the sense that

$$Q(F(Q(u))) = Q(u) \text{ for all } u \in (0,1)$$

and

$$F(Q(F(x))) = F(x) \text{ for all } x \in \mathbb{R}.$$

It may help to show that $Q(F(x)) \leq x$ for all $x \in \mathbb{R}$.

e) Show that if $F^{-1}$ is a function (in the sense that $F^{-1}(u)$ is a singleton element for all $u$), then $F^{-1} = Q$.

## 3 Random Variables

**Problem 5. Equivalent Distributions**

a) Let $X_1$ and $X_2$ be random variables on probability space $(\Omega, \mathcal{F}, P)$ such that $X_1 \sim X_2$. Let $g : \mathbb{R} \to \mathbb{R}$ denote a function such that for borel sets $B$, $g^{-1}(B)$ is borel. Show that $g(X_1) \sim g(X_2)$.

b) Let $X_1$ and $X_2$ be random variables defined on $(\Omega, \mathcal{F}, P)$ with distribution functions $F_1$ and $F_2$. Show that if

$$F_1(b) - F_1(a) = F_2(b) - F_2(a)$$

then $X_1 \sim X_2$.

**Problem 6. Sequences and Sums of Random Variables**
Consider the probability space $(\Omega, \mathcal{F}, P)$ for $\Omega = (0, 1)$, Borel sets $\mathcal{F}$ and uniform probability measure $P$. As before, define $X_i = b_i(\omega)$ where $b_i$ refers to the $i$-th digit in the binary representation of $\omega$.

a) Show that if $X_i$ is a random variable for all $i$, then

$$S(\omega) = \sum_{i=1}^{n} X_i(\omega)$$

is a random variable for any value of $n$.

b) Show that

$$Y(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_n(\omega)$$

is a valid random variable as well.

**Problem 7. Inequalities of Random Variables**
Let $X_i$ be defined as in Problem 6.

a) Let $Z_1 = X_1 + X_2$ and $Z_2 = X_2 + X_3$. Determine the value of $P(Z_1 > Z_2)$ by explicitly defining the sub-interval of $(0, 1)$ corresponding to this event, and then taking its length.

b) Calculate the same probability as in the previous part by defining the random variable $Y = Z_1 - Z_2$, and using the fact that the $X_i$ are independent.

**Problem 8. Independence of Random Variables** Let $X$ and $Y$ denote two random variables defined on $(\Omega, \mathcal{F}, P)$.

a) Find non-constant functions $g$ and $h$ such that $g(X)$ and $h(Y)$ are independent, even when $X$ and $Y$ are not independent.

b) Show that if $X$ and $Y$ are independent, then $g(X)$ and $h(Y)$ are independent for any pair of functions $g$ and $h$.

**Bonus Problem. Limits of Quantile Functions**

Let $F_1, F_2, \ldots, F_n, \ldots$, and $F$ be distribution functions with corresponding quantile distribution functions $Q_1, Q_2, \ldots$, and $Q$. Suppose that

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \text{for all continuity points } x \text{ of } F.$$

Show that

$$Q(u) \leq \liminf_n Q_n(u) \leq \limsup_n Q_n(u) \leq \lim_{h\to 0} Q(u+h)$$

and conclude that

$$\lim_{n\to\infty} Q(u) = Q(u) \quad \text{for all continuity points } u \text{ of } Q.$$