

Stat 513

Fall 2025

Problem Set 2

Topic 2: Random Variables, Distribution Functions, and Continuity

Total: 50 points

Due Sunday, September 21 at 23:59

1 Continuity

Problem 1. Continuity

- a) Show that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous if and only if $f^{-1}((a, b))$ is open for any open interval $(a, b) \subset \mathbb{R}$.
- b) Show that for a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$, for any fixed u the set

$$A = \{x \in \mathbb{R} : f(x) = u\}$$

is closed.

Problem 2. Limits of continuous functions

Consider continuous functions f_1, f_2, \dots on \mathbb{R} . Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be function defined as $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for all x . Show by example that f is not necessarily continuous. Conclude that a sequences of continuous random variables may converge in distribution to a discrete random variable (i.e., their distribution functions converge to a distribution function of a discrete random variable).

2 Distribution Functions

Problem 3. Properties of Distribution Functions

Let X denote a random variable on (Ω, \mathcal{F}, P) , with distribution function F . Define the notation

$$F(x-) = \lim_{h \downarrow 0} F(x - h)$$

and

$$F(x+) = \lim_{h \downarrow 0} F(x + h)$$

to denote the “limit from below/left” and the “limit from above/right”.

- a) Using a variation of the proof shown in class that $\lim_{x \rightarrow \infty} F(x) = 1$, show that

$$\lim_{x \rightarrow -\infty} F(x) = 0.$$

- b) Show that

$$F(x) = F(x-) + P(X = x).$$

- c) Conclude that F is continuous if and only if $P(X = x) = 0$ for all $x \in \mathbb{R}$.

Problem 4. Quantile Functions

Let X denote a random variable on probability space (Ω, \mathcal{F}, P) with distribution function F , define the *quantile function* $Q : (0, 1) \rightarrow \mathbb{R}$ as

$$Q(u) = \inf\{x : F(x) \geq u\}.$$

- a) Show that

$$P(X \leq Q(u)) \geq u.$$

This explains the naming of Q as the “quantile function”.

- b) What is the quantile function for $X \sim \text{Bernoulli}(1/2)$?
c) Show that the quantile function is monotonic non-decreasing.
d) Show that Q is a *psuedo-inverse* of F , in the sense that

$$Q(F(Q(u))) = Q(u) \text{ for all } u \in (0, 1)$$

and

$$F(Q(F(x))) = F(x) \text{ for all } x \in \mathbb{R}.$$

It may help to show that $Q(F(x)) \leq x$ for all $x \in \mathbb{R}$.

- e) Show that if F^{-1} is a function (in the sense that $F^{-1}(u)$ is a singleton element for all u), then $F^{-1} = Q$.

3 Random Variables

Problem 5. Equivalent Distributions

- a) Let X_1 and X_2 be random variables on probability space (Ω, \mathcal{F}, P) such that $X_1 \sim X_2$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ denote a function such that for borel sets B , $g^{-1}(B)$ is borel. Show that $g(X_1) \sim g(X_2)$.
- b) Let X_1 and X_2 be random variables defined on (Ω, \mathcal{F}, P) with distribution functions F_1 and F_2 . Show if that for any interval $(a, b) \subset \mathbb{R}$,

$$F_1(b) - F_1(a) = F_2(b) - F_2(a),$$

then $X_1 \sim X_2$. You can use the unique characterization result asserted in class.

Problem 6. Sequences and Sums of Random Variables

Consider the probability space (Ω, \mathcal{F}, P) for $\Omega = (0, 1)$, Borel sets \mathcal{F} and uniform probability measure P . As before, define $X_i = b_i(\omega)$ where b_i refers to the i -th digit in the binary representation of ω .

- a) Show that if X_i is a random variable for all i , then

$$S(\omega) = \sum_{i=1}^n X_i(\omega)$$

is a random variable for any value of n .

- b) **Changed to be an extra bonus problem.**

Problem 7. Inequalities of Random Variables

Let X_i be defined as in Problem 6.

- a) Let $Z_1 = X_1 + X_2$ and $Z_2 = X_2 + X_3$. Determine the value of $P(Z_1 > Z_2)$ by explicitly defining the subset of $(0, 1)$ corresponding to this event, and then taking its length.
- b) Calculate the same probability as in the previous part by defining the random variable $Y = Z_1 - Z_2$, and using the fact that the X_i are independent.

Problem 8. Independence of Random Variables Let X and Y denote two random variables defined on (Ω, \mathcal{F}, P) .

- a) Find non-constant functions g and h such that $g(X)$ and $h(Y)$ are independent, even when X and Y are not independent.
- b) Show that if X and Y are independent, then $g(X)$ and $h(Y)$ are independent for any pair of functions g and h .

Bonus Problem. Limits of Quantile Functions

- 1) Let $F_1, F_2, \dots, F_n, \dots$, and F be distribution functions with corresponding quantile distribution functions Q_1, Q_2, \dots , and Q . Suppose that

$$\lim_{n \rightarrow \infty} F_n(x) = F(x) \quad \text{for all continuity points } x \text{ of } F.$$

Show that

$$Q(u) \leq \liminf_n Q_n(u) \leq \limsup_n Q_n(u) \leq \lim_{h \rightarrow 0} Q(u+h)$$

and conclude that

$$\lim_{n \rightarrow \infty} Q(u) = Q(u) \quad \text{for all continuity points } u \text{ of } Q.$$

- 2) (From problem 6, note change from the previous version): Show that if $X_i(\omega) = b_i(\omega)$ are random variables in probability space (Ω, \mathcal{F}, P) , then

$$\left\{ \omega : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_n(\omega) = a \right\} \in \mathcal{F}$$

for any $a \in (0, 1)$.