Stat 513
Fall 2025
Problem Set 6
Topic 6: Moment Generating Functions, Characteristic Functions, and Cumulants

Due Wednesday, November 19 at 23:59

Problem 1. Positivity of the Laplace Transform.

Show that

$$(-1)^n \frac{\partial^n}{\partial t^n} L(t) \geq 0$$

for all $n \in \mathbb{N}$.

Problem 2. Generating Moments.

Complete the proof that if for a random variable X the moment generating function $M_X(t)$ exists for $t \in (-\delta, \delta)$ where $\delta > 0$, then

$$\mathbb{E}[X^j] = M_X^{(j)}(0)$$

for all $j = 1, 2, \ldots$.

Problem 3. Deriving Characteristic Functions Derive characteristic functions for the following distributions:

a) X is a discrete random variable with

$$P(X = k) = \theta(1-\theta)^k$$

where $\theta \in (0, 1)$ and $k = 0, 1, \ldots$.

b) X is a continuous random variable with the following pdf:

$$f_X(x) = \frac{1}{2}(-|x|)$$

for $x \in \mathbb{R}$. This is the *Laplace distribution*.

Problem 4. Continuity of characteristic functions.

Show that characteristic functions $\phi(X)$ continuous with respect to t.

Problem 5. Poisson Distribution.

Recall that a random variable X is Poisson distributed with parameter $\lambda > 0$ if it is discrete and

$$P(X = k) = \frac{\lambda^k \exp(-\lambda)}{k!}$$

for all $k = 0, 1, \ldots$.

a) Derive the moment generating function of the Poisson distribution.

b) Show that each of the cumulants of the Poisson distribution are equal to λ.

Problem 6. Log-normal moment generating function.

A random variable X has a *log-normal* distribution if it is absolutely continuous with p.d.f.

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right).$$

Show that the moment generating function of the log-normal distribution does not exist, even $\mathbb{E}[X^n]$ exists for all $n = 0, 1,$.

Problem 7. Kurtosis.

Recall that Kurtosis of a random variable X is defined as

$$\frac{K_4(X)}{\mathrm{Var}[X]^2}$$

where $K_4(X)$ denotes the fourth cumulant of X, which has the following expression:

$$K_4(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^4\right] - 3\mathrm{Var}[X]^2.$$

a) Show that for any random variable X,

$$\mathrm{Kurtosis}(X) \geq -2.$$

b) Let $X \sim \mathrm{Bernoulli}(p)$ for $p \in (0, 1)$. Derive an expression for the Kurtosis of X, and show that it achieves its minimum value of $\mathrm{Kurtosis}(X) = -2$ at $p = \frac{1}{2}$.

Problem 8. Random Sum.

Let $J \sim \mathrm{Poisson}(\lambda)$, let $X_1, X_2, \ldots$ be independent and identically distributed random variables with mean μ and variance σ^2, and let

$$S = \sum_{n=1}^{J} X_j.$$

Assume that X_j have a moment generating function $M_X(t)$ which exists within a neighborhood of zero. Derive an expression for the moment generating function of S (hint: break up the expectation into an inner expectation conditioned on the value of J, and use indicator functions).

Bonus Problem.

Let X and Y denote random variables with characteristic functions $\phi_X(t)$ and $\phi_Y(t)$ respectively.

1. Is it true that $\phi_X(t) = \phi_Y(t)$ for all $t \in (-\delta, \delta)$ (but they could differ outside of this neighrbohood) implies $X \sim Y$? Why or why not?

2. Is it true that if $\mathbb{E}[X^j] = \mathbb{E}[Y^j]$ for all $j = 1, 2, \ldots$ then $X \sim Y$? Why or why not?