

Stat 513
Assignment 1
Topic 1: Probability and Set Theory
Fall 2025

Total: 50 points

Due Sunday, September 7 at 23:59

1 Set Theory

Problem 1. Countability..

The *algebraic numbers* are defined as the set of roots of polynomials with integer coefficients. Formally,

$$A = \left\{ x : \exists N, a_0, a_1, \dots, a_N \in \mathbb{Z} \text{ s.t. } \sum_{i=0}^N a_i x^i = 0 \right\}.$$

Is A countable or uncountable? Show your answer by either demonstrating the existence of a bijection, or showing that no such bijection could exist.

Problem 2. Countability and Density of Sets

As in class, let $b_i(x)$ denote the i -th binary digit of $x \in (0, 1)$. The set of normal numbers between zero and one is defined as the following set:

$$A = \left\{ x \in (0, 1) \mid \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n b_i(x) = \frac{1}{2} \right\}.$$

- i) Show that A is dense in $(0, 1)$. *Hint:* For a given ϵ , look at the first n digits of the binary expansion for an appropriate value of n .
- ii) Show that the complement of A is also dense in $(0, 1)$.

Problem 3. Lim-sup and lim-inf of sets.

Consider a countable sequence of sets A_1, A_2, \dots . The lim-sup and lim-inf of this sequence are defined as follows:

$$\liminf A_i = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$$

and

$$\limsup A_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

- i) Let $A_i = [0, \frac{1}{i}]$ if i is odd and $A_i = [0, 1]$ if i is even. What are $\limsup A_i$ and $\liminf A_i$?

ii) Show that if $A_i \subset A_{i+1}$ for all i , then the \liminf and \limsup of $\{A_i\}_{i=1}^{\infty}$ are equal to each other and to the (infinite) union. Show the analogous result if $A_i \supset A_{i+1}$ with respect to the infinite intersection.

iii) Show that

$$(\liminf A_i)^c = \limsup A_i^c$$

and

$$(\limsup A_i)^c = \liminf A_i^c.$$

Problem 4. The Cantor Set

Consider the set of sequences of elements of $\{0, 1, 2\}$:

$$\mathcal{T} = \left\{ \{x_i\}_{i=1}^{\infty} \mid x \in \{0, 1, 2\} \right\}.$$

Similar to the set of binary sequences, we can define $t_i(x)$ as the i -th *ternary* digit of $x \in (0, 1)$, and establish a (psuedo) bijection with the unit interval:

$$f(\{x_i\}) = \sum_{i=1}^{\infty} \frac{t_i(x)}{3^i}.$$

(Note that this is not a true bijection as written because we would need to establish a condition for equivalent expansions similar to what we did for the binary digits, since $.022\dots$ and $.100\dots$ are both equal to $1/3$; however we will ignore this complication as justified by Problem 5(i)).

Given this definition, the *Cantor set* can be defined as

$$\mathcal{C} = \{x \in [0, 1] \mid t_i(x) \neq 1\}.$$

i) Show that the Cantor set as defined above is equivalent to defining collections of sets C_n for all $n \in \mathbb{N}$ through the following iterative process:

a) Initialization: Let $C_0 = \{[0, 1]\}$.

b) Construct C_{n+1} from C_n by removing the middle third of each of the intervals of C_n , i.e.

$$C_{n+1} = \left\{ \left[a, a + \frac{b-a}{3} \right], \left[a + \frac{2(b-a)}{3}, b \right] \mid \forall [a, b] \in C_n \right\}.$$

and taking the infinite intersection of the union of each of the C_n :

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \bigcup_{A \in C_n} A.$$

ii) Show that the Cantor set is closed (i.e., contains all of its limit points).

- iii) A set S is *nowhere dense* in \mathcal{X} if for all open subsets $E \subset \mathcal{X}$, S is not dense in E . Show that the Cantor set is nowhere dense in $(0, 1)$.
- iv) Using the uniform probability space, show that $P(\mathcal{C}) = 0$ by showing that $P(\mathcal{C}) < \epsilon$ for all $\epsilon > 0$.

2 Basic Probability

Problem 5. Binary Sequences

- i) Show that, when represented in base-2, $.1000\dots = .0111\dots = \frac{1}{2}$.
- ii) Taking (Ω, \mathcal{F}, P) to be $\Omega = (0, 1)$, \mathcal{F} as the Borel sets, and P as the uniform probability measure, show that

$$P(\{x \mid \exists N \text{ s.t. } b_j(x) = 0 \forall j \geq N\}) = 0.$$

(In other words, the probability that a given binary sequence ends in all zeros is zero). This allows us to use our bijection between the “non-terminating” binary sequences and the interval $(0, 1)$ without any loss of generality.

Problem 6. Infinite Sequences of Coin Flips

For the following parts, consider the event space of infinite sequences of zero-one coin flips:

$$\Omega = \left\{ \{x_i\}_{i=1}^{\infty} \mid x_i \in \{0, 1\} \right\}.$$

- i) Using the σ -algebra generated by evenly sized intervals of width $1/8$ (i.e., \mathcal{F} composed of the sets $A_i = (i/8, (i+1)/8)$ along with union and complements) derive the probability of the second and third coin flips being heads.
- ii) What is the smallest σ -algebra (i.e., composed of the fewest number of sets) that will allow you to evaluate the probability of the second and third coin flips being heads?
- iii) What is the smallest σ -algebra (i.e., composed of the fewest number of sets) that will allow you to evaluate the probabilities that the second and third coin flips take *any* value? (i.e, (H,H), (T,T), (H,T), (T,T)).

Problem 7. Probability of Union

Let P denote a probability function on sample space Ω and σ -algebra \mathcal{F} . Show that

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

Problem 8. Some Results on σ -algebras

- i) Let \mathcal{F} denote the collection of all countable subsets of $\Omega = \mathbb{R}$, and their complements. Show that
- a) \mathcal{F} is a σ -algebra.
 - b) If $P : \mathcal{F} \rightarrow [0, 1]$ is such that $P(A) = 0$ if A is countable, then (Ω, \mathcal{F}, P) forms a valid probability space.
- ii) Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ denote an increasing series of σ -algebras. Show by example that $\bigcup_i \mathcal{F}_i$ is not necessarily a σ -algebra.

Bonus Problem. Infinite Monkeys.

Let $\Omega = \left\{ \{x_i\}_{i=1}^{\infty} \mid x_i \in \{a, b, \dots, z\} \right\}$ denote the collection of infinite sequences of latin letters. Note that we can define a “uniform” probability space on Ω through a bijection between Ω and the interval $(0, 1)$ (using numbers represented in base 26).

Let $S = (x_1, \dots, x_n)$ denote a fixed sequence of n letters. Show that for any such sequence,

$$P \left(\left\{ \{x_i\} \in \Omega \mid S \text{ is a sub-sequence of } \{x_i\} \right\} \right) = 1.$$