Stat 513
Fall 2025
Problem Set 2
Topic 2: Random Variables, Distribution Functions, and Continuity

Total: 50 points

Due Sunday, September 21 at 23:59

1 Continuity

Problem 1. Continuity

a) Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}((a, b))$ is open for any open interval $(a, b) \subset \mathbb{R}$.

b) Show that for a continuous function $f : \mathbb{R} \to \mathbb{R}$, for any fixed u the set

$$A = \{x \in \mathbb{R} : f(x) = u\}$$

is closed.

Problem 1 Solution

a) *Direction 1:* Pre-image condition $\implies$ continuous.

Let $\epsilon > 0$. Then

$$(f(x) - \epsilon, f(x) + \epsilon)$$

is an open interval, so $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open.

By the definition of open, there exists a neighborhood of some width δ such that

$$N_\delta(x) \subset f^{-1}((f(x) - \epsilon, f(x) + \epsilon)).$$

This means that if $y \in N_\delta(x)$ (or, equivalently, $|x - y| < \delta$), then $y \in f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$, which implies that $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$ so $|f(x) - f(y)| < \epsilon$, as required.

Direction 2: Continuity $\implies$ pre-image condition.

Let $(a, b) \subset \mathbb{R}$ denote any open interval.

Let $x \in f^{-1}((a, b))$, which means that $f(x) \in (a, b)$. By the fact that this is an open interval, there exists a $\epsilon > 0$ such that $N_\epsilon(f(x)) \subset (a, b)$. By the definition of continuity, there exists a $\delta > 0$ such that $|x-y|<\delta$ implies $|f(y)-f(x)|<\epsilon$. This means that

$$N_\delta(x) \subset f^{-1}(N_\epsilon(f(x))) \subset f^{-1}((a,b))$$

Since x was arbitrary, this implies that $f^{-1}((a,b))$ is open, as required.

b) If $x \notin A$, then $f(x) \neq u$ by construction, so let $\epsilon = |f(x)-u| > 0$.

Then by the definition of continuity, there exists a δ such that $|y-x|<\delta$ implies $|f(x)-f(y)|<\epsilon$. Since for any $y \in N_\delta(x)$,

$$0 < |f(x)-u| - |f(x)-f(y)| \leq |f(y)-u|\,,$$

$f(y) \neq u$, so $N_\delta(x)$ cannot contain any points in A. As such, x cannot be a limit point of A, implying that A contains all of its limit points, and is therefore closed.

Problem 2. Limits of continuous functions

Consider continuous functions $f_1, f_2, \ldots$ on $\mathbb{R}$. Let $f:\mathbb{R}\to\mathbb{R}$ be function defined as $f(x) = \lim_{n\to\infty} f_n(x)$ for all x.

Show by example that f is not necessarily continuous. Conclude that a sequences of continuous random variables may converge in distribution to a discrete random variable (i.e., their distribution functions converge to a distribution function of a discrete random variable).

Problem 2 Solution Let

$$F_n(x) = \begin{cases} 0 & x \leq -\frac{1}{n} \\ 1+nx & -\frac{1}{n} < x < 0 \\ 1 & x \geq 0 \end{cases}$$

While each F_n is continuous, $F(x) = \lim_{n\to\infty} F_n(x) = 1$ if $x \geq 0$ and $F(x) = 0$ otherwise, which has a jump discontinuity at 0.

The function F is monotonic non-decreasing, goes from zero to one, and is right-continuous; as such, it is a distribution function corresponding to the discrete random variable X such that $P(X=0)=1$ and $P(X=x)=0$ otherwise. This is despite the fact that each F_n is a valid continuous distribution function, corresponding to a Uniform$(-1/n, 0)$ random variable.

2 Distribution Functions

Problem 3. Properties of Distribution Functions

Let X denote a random variable on $(\Omega, \mathcal{F}, P)$, with distribution function F. Define the notation

$$F(x-) = \lim_{h \downarrow 0} F(x-h)$$

and

$$F(x+) = \lim_{h \downarrow 0} F(x+h)$$

to denote the "limit from below/left" and the "limit from above/right".

a) Using a variation of the proof shown in class that $\lim_{x\to\infty} F(x) = 1$, show that

$$\lim_{x\to-\infty} F(x) = 0.$$

b) Show that

$$F(x) = F(x-) + P(X = x).$$

c) Conclude that F is continuous if and only if $P(X = x) = 0$ for all $x \in \mathbb{R}$.

Problem 3 Solution

a) We will first show the lemma that if $A_1 \subset A_2 \dots$ is a decreasing collection of sets, then

$$P(\lim A_i) = P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} P(A_i).$$

From the previous homework, we can see that

$$1 - P(\lim A_i) = P(\lim A_i^c)$$

where A_i^c are an *increasing* collection of sets, since $A_i \subset A_{i+1} \dots$

As such, from the lemma in class we have

$$P(\lim A_i^c) = \lim_{i\to\infty} P(A_i^c) = \lim_{i\to\infty} 1 - P(A_i) = 1 - \lim_{i\to\infty} P(A_i)$$

and the lemma follows by subtracting one from each side/multiplying by negative one.

Now, similar to the proof in class, let $\{x_i\}_i$ denote a decreasing sequence such that $x_i \to -\infty$, and define

$$A_i = \{\omega : X(\omega) \leq x_i\}.$$

We can see that $P(A_i) = F(x_i)$, that this is a decreasing collection of sets, and that $\bigcap_{n=1}^{\infty} A_i = \emptyset$. So,

$$0 = P(\emptyset) = P\left(\bigcap_{n=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_i) = \lim_{x\to-\infty} F(x)$$

as required.

b) It suffices to show that

$$F(x-) = P(X < x)$$

because

$$P(X < x) = P(X \leq x) - P(X = x) = F(x) - P(X = x).$$

Let $\{h_i\}_{i=1}^{\infty}$ be a decreasing sequence such that $h_i \downarrow 0$, and define

$$A_i = \{\omega : X(\omega) \leq x - h_i\}.$$

Then A_i form an increasing collection of sets, $P(A_i) = F(x - h_i)$, and $\bigcup_{i=1}^{\infty} A_i = \{\omega : X(\omega) < x\}$, so

$$P(X < x) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = P(\lim A_i) = \lim_{i\to\infty} P(A_i) = \lim_{i\to\infty} F(x - h_i) = \lim_{h\downarrow 0} F(x - h) = F(x-).$$

c) This follows directly from the above and the fact that f is continuous if and only if it is right-continuous (implying $F(x+) = F(x)$) and left-continuous, $F(x-) = F(x)$ which is true when $P(X = x) = 0$ for all x.

Note that if a function is left-continuous and right-continuous at x it is continuous at x; consider any sequence $\{x_i\}$, if x_i does not contain infinite elements greater than/less than x, then $f(x_i)$ will converge to $f(x)$. Otherwise, consider the two sub-sequences of $\{x_i\}$ consisting of values greater than x and that consisting of values less than x. For both of these sub-sequences, $f(x_i)$ will converge to $f(x)$, so $f(x_i)$ must converge to $f(x)$ for the entire sequence.

Problem 4. Quantile Functions

Let X denote a random variable on probability space $(\Omega, \mathcal{F}, P)$ with distribution function F, define the *quantile function* $Q : (0, 1) \to \mathbb{R}$ as

$$Q(u) = \inf\{x : F(x) \geq u\}.$$

a) Show that

$$P(X \leq Q(u)) \geq u.$$

This explains the naming of Q as the "quantile function".

b) What is the quantile function for $X \sim \text{Bernoulli}(1/2)$?

c) Show that the quantile function is monotonic non-decreasing.

d) Show that Q is a *psuedo-inverse* of F, in the sense that

$$Q(F(Q(u))) = Q(u) \text{ for all } u \in (0, 1)$$

and

$$F(Q(F(x))) = F(x) \text{ for all } x \in \mathbb{R}.$$

It may help to show that $Q(F(x)) \leq x$ for all $x \in \mathbb{R}$.

e) Show that if F^{-1} is a function (in the sense that $F^{-1}(u)$ is a singleton element for all u), then $F^{-1} = Q$.

Problem 4 Solution

a) Consider the set $A = \{x \in \mathbb{R} : F(x) \geq u\}$. For any n, we can find a $x_n \in A$ such that $x_n - Q(u) < \frac{1}{n}$. Since $F(x_n) \geq u$ by definition, $\lim_{n\to\infty} F(x_n) \geq u$, so because $F(x) = \lim_{n\to\infty} F(x_n)$, it follows that $F(x) \geq u$.

b) The quantile function for the random variable $X \sim \text{Bernoulli}(1/2)$ is

$$Q(u) = \begin{cases} 0 & u \in (0, 1/2] \\ 1 & u \in [1/2, 1). \end{cases}$$

c) Let $u < v$. Then consider the sets

$$A = \{x \in \mathbb{R} : F(x) \geq u\}$$
$$B = \{x \in \mathbb{R} : F(x) \geq v\}.$$

Since $A \supset B$, it follows that $\inf A \leq \inf B$, so $Q(u) \leq Q(v)$.

d) To first show that $Q(F(x)) \leq x$, consider

$$A = \{y \in \mathbb{R} : F(y) \leq F(x)\}.$$

Since $x \in A$, it must be that $Q(F(x)) = \inf A \leq x$.

For the theorem, let's begin with showing the direction

$$Q(F(Q(u))) = Q(u) \text{ for all } x \in \mathbb{R}.$$

We have shown that $F(Q(u)) \geq u$, which implies that $Q(F(Q(u))) \geq Q(u)$ by monotonicity of Q. Denoting $x^* = Q(u)$, we can see that $Q(F(x^*)) \leq x^*$, which means that $Q(F(Q(u))) \leq Q(u)$, and as such $Q(F(Q(u))) = Q(u)$.

Now let's show

$$F(Q(F(x))) = F(x).$$

We have shown that $Q(F(x)) \leq x$, which by monotonicity of F implies that $F(Q(F(x))) \leq F(x)$. Denoting $u^* = F(x)$, we can see that $F(Q(u^*)) \geq u^*$, so $F(Q(F(x))) \geq F(x)$ and as such $F(Q(F(x))) = F(x)$.

e) From the previous part, $F(Q(F(x))) = F(x)$ which implies that $Q(F(x)) = x$ for all $x \in \mathbb{R}$, which implies that $Q = F^{-1}$.

3 Random Variables

Problem 5. Equivalent Distributions

a) Let X_1 and X_2 be random variables on probability space $(\Omega, \mathcal{F}, P)$ such that $X_1 \sim X_2$. Let $g : \mathbb{R} \to \mathbb{R}$ denote a function such that for borel sets B, $g^{-1}(B)$ is borel. Show that $g(X_1) \sim g(X_2)$.

b) Let X_1 and X_2 be random variables defined on $(\Omega, \mathcal{F}, P)$ with distribution functions F_1 and F_2. Show if that for any interval $(a, b) \subset \mathbb{R}$,

$$F_1(b) - F_1(a) = F_2(b) - F_2(a),$$

then $X_1 \sim X_2$.

Problem 5 Solution

a)

$$\begin{aligned}
&P(g(X) \in B) && (1)\\
&P(\{\omega : g(X(\omega)) \in B\}) && (2)\\
&= P(\{\omega : X(\omega) \in g^{-1}(B)\}) && (3)\\
&= P(\{\omega : Y(\omega) \in g^{-1}(B)\}) \qquad \text{because } X \sim Y && (4)\\
&= P(\{\omega : g(Y(\omega)) \in B\}) && (5)\\
&= P(g(Y) \in B) && (6)
\end{aligned}$$

b) Consider the intervals $(-n, x)$ where x is arbitrary. We can see that

$$F_1(x) - F_1(-n) = F_2(x) - F_2(-n).$$

Taking the limit of both sides as $n \to \infty$, and the fact that $\lim_{x \to -\infty} F(x) = 0$ we get that $F_1(x) = F_2(x)$ for all x, which from what we've shown in class implies that X_1 and X_2 have the same distribution.

Problem 6. Sequences and Sums of Random Variables

Consider the probability space $(\Omega, \mathcal{F}, P)$ for $\Omega = (0, 1)$, Borel sets $\mathcal{F}$ and uniform probability measure P. As before, define $X_i = b_i(\omega)$ where b_i refers to the i-th digit in the binary representation of ω.

a) Show that if X_i is a random variable for all i, then

$$S(\omega) = \sum_{i=1}^{n} X_i(\omega)$$

is a random variable for any value of n.

b) **Addendum: Moved to Bonus Problem.**

Problem 6 Solution

a) We will proceed by induction. $S_1(\omega) = X_1(\omega)$ satisfies the conditions for a random variable because $X_1(\omega)$ does. Assume that $S_n(\omega)$ satisfies the conditions. Let $A \in \mathcal{F}$ and consider the pre-image

$$S_{n+1}^{-1}(A) = \{\omega \in (0,1) : S_{n+1}(\omega) \in A\}.$$

This is equivalent to

$$\{\omega \in (0,1) : S_n(\omega) + X_{n+1}(\omega) \in A\}.$$

Consider the set $A^- = \{x \in \mathbb{R} : x + 1 \in A\}$. Then the overall event can be written as

$$\{\omega \in (0,1) : S_n(\omega) \in A, X_{n+1}(\omega) = 0\} \cup \{\omega \in (0,1) : S_n(\omega) \in A^-, X_{n+1}(\omega) = 1\}.$$

We can see that the first set of this union can be written as

$$\{\omega \in (0,1) : S_n(\omega) \in A\} \cap \{\omega \in (0,1) : X_{n+1}(\omega) = 1\}.$$

Both of these sets are in $\mathcal{F}$, the first by the induction assumption and the second by the fact that X_{n+1} is a random variable. As such their intersection is in $\mathcal{F}$. Since this logic holds analogously for the other set in the union above, $S_{n+1}^{-1}(A) \in \mathcal{F}$.

b) Moved.

Problem 7. Inequalities of Random Variables

Let X_i be defined as in Problem 6.

a) Let $Z_1 = X_1 + X_2$ and $Z_2 = X_2 + X_3$. Determine the value of $P(Z_1 > Z_2)$ by explicitly defining the subset of $(0,1)$ corresponding to this event, and then taking its length.

b) Calculate the same probability as in the previous part by defining the random variable $Y = Z_1 - Z_2$, and using the fact that the X_i are independent.

Problem 7 Solution

a) This corresponds to the set

$$(4/8, 5/8) \cup (6/8, 7/8),$$

so

$$P((4/8, 5/8) \cup (6/8, 7/8)) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

b) We can re-write the probability as $P(Y > 0) = P(X_1 - X_3 > 0)$. Since X_1 and X_3 are independent, this will be equal to the probability $P(X_1 = 1)P(X_3 = 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Problem 8. Independence of Random Variables Let X and Y denote two random variables defined on $(\Omega, \mathcal{F}, P)$.

a) Find non-constant functions g and h such that $g(X)$ and $h(Y)$ are independent, even when X and Y are not independent.

b) Show that if X and Y are independent, then $g(X)$ and $h(Y)$ are independent for any pair of functions g and h.

Problem 8 Solution

a) There are many possible examples; let, for example, $\Omega = (0, 1)$ and $X(\omega) = Y(\omega) = \omega$. X and Y are clearly dependent (since they are the same), however we have shown in class that
$g(X(\omega)) = g(\omega) = b_1(\omega)$ and $h(X(\omega)) = h(\omega) = b_2(\omega)$ are independent.

b) Let A and B denote Borel sets. Then

$$\begin{aligned}
&P(g(X) \in A, h(Y) \in B) \\
=&P(\{\omega : g(X(\omega)) \in A, h(Y(\omega)) \in B\}) \\
=&P(\{\omega : X(\omega) \in g^{-1}(A), Y(\omega) \in h^{-1}(B)\}) \\
=&P(\{\omega : X(\omega) \in g^{-1}(A)\})P(\{\omega : Y(\omega) \in h^{-1}(B)\}) \qquad \text{by the independence of X and Y} \\
=&P(\{\omega : g(X(\omega)) \in A\})P(\{\omega : h(Y(\omega)) \in B\}).
\end{aligned}$$