

**Stat 513**  
**Fall 2025**  
**Problem Set 7**  
**Topic 7: Parametric Families of Distributions**

Due Sunday, November 14 at 23:59

**Problem 1. Identifiability**

Let  $X \sim \text{Bernoulli}(p)$ ,  $Y_1 \sim N(\mu_1, \sigma^2)$ ,  $Y_2 \sim N(\mu_2, \sigma^2)$ , and

$$Y = XY_1 + (1 - X)Y_2$$

where  $[p, \mu_1, \mu_2, \sigma] \in (0, 1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ . Consider the parametric family describing the distribution of  $Y$  with this parameter space. Is this parametric family identifiable?

**Problem 2. Exponential Families**

Show that the following distributions form exponential families with the specified parameters. For the first two, additionally use the cumulant function obtained from the exponential family form to derive the mean and variance of the distribution.

- a) Binomial( $n, p$ ) with  $p \in (0, 1)$  and *fixed*  $n$ .
- b) Poisson( $\lambda$ ) with  $\lambda > 0$ .
- c) Beta( $\alpha, \beta$ ) Distribution with  $\alpha > 0$  and  $\beta > 0$ , which is defined as an absolutely continuous distribution with the following pdf:

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}.$$

where  $\Gamma(\cdot)$  is the Gamma function.

**Problem 3. Characterization in Exponential Families**

Let  $X_1$  and  $X_2$  denote absolutely continuous one-dimensional random variables with density functions

$$f_1(x; \eta) = \exp(\eta x - k_1(\eta)) h_1(x)$$

$$f_2(x; \eta) = \exp(\eta x - k_2(\eta)) h_2(x)$$

respectively.

- a) Show that if  $k_1 = k_2$  for all  $\eta$ , then  $X_1 \sim X_2$ .
- b) Show that if  $\mathbb{E}[X_1; \eta] = \mathbb{E}[X_2; \eta]$  for all  $\eta$ , then  $X_1 \sim X_2$ .
- c) Show that if  $\mathbb{V}[X_1; \eta] = \mathbb{V}[X_2; \eta]$  for all  $\eta$ , then  $X_1$  does not necessarily have the same distribution as  $X_2$ .

#### Problem 4. Transforming Distributions

Show that

- a) If  $X \sim \text{Exponential}(\lambda)$ , then  $aX \sim \text{Exponential}(\lambda/a)$  for  $a > 0$ .
- b) If  $X \sim N(0, 1)$ , then  $X^2 \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$ .
- c) If  $X \sim N(0, 1)$  and  $Y \sim N(0, 1)$ , then  $X^2 + Y^2 \sim \text{Exponential}(\frac{1}{2})$ .
- d) If  $X$  denotes a Cauchy distributed random variable, then  $X^2 + \frac{1}{X^2} \sim 4X^2 + 2$ .

#### Problem 5. Order Statistics

Let  $X_1, \dots, X_n$  be i.i.d. from some distribution. The  $k$ -th *order statistic* is defined as the  $k$ -th random variable when arranged in a sorted order, i.e.  $X_{(1)} \leq X_{(2)} \leq \dots, X_{(n)}$ , where  $X_{(1)}$  is the minimum,  $X_{(n)}$  is the maximum, and the  $k$ -th order statistic is  $X_{(k)}$ .

- a) Say that  $X_1, \dots, X_n \sim \text{Uniform}(0, 1)$ . Show that  $X_{(k)} \sim \text{Beta}(k, n + 1 - k)$ .
- b) In the same set-up as (a), show that as  $n \rightarrow \infty$ , the first order statistic (i.e., a minimum) converges to an  $\text{Exponential}(n)$  distribution (formally,  $nX_{(1)} \xrightarrow{\mathcal{D}} \text{Exponential}(1)$ ).

#### Problem 6. Multivariate Normal

Let  $X \in \mathbb{R}^n$  denote a random vector with an  $n$ -dimensional multivariate normal distribution with mean zero and covariance matrix  $\Sigma$ . Let  $A \in \mathbb{R}^{n \times n}$  denote a real-valued matrix and consider the random vector  $Y = AX$ . What is the distribution of  $Y$ ?

#### Bonus Problem. Pareto Distribution and Conditional Expectation

Let  $X$  denote an absolutely continuous random variable with density function

$$f(x) = \frac{\alpha}{x^{\alpha+1}} \text{ for } x > 1.$$

This is referred to as the Pareto distribution with parameter  $\alpha > 0$ . Let  $Z = XY$  and  $W = X/Y$ . What are the values of  $E[X|Z]$  and  $E[W|Z]$ ?