Stat 513

Fall 2025

Problem Set 3

Topic 3: Integration and Expectation

Total: 50 points

Due Sunday, October 5 at 23:59

Problem 1. Cantor Function.

Let $\mathcal{C}$ denote the Cantor set. Recall the definition of the Cantor function:

$$F(x) = \begin{cases} \sum_{i=1}^{\infty} \frac{t_i(x)/2}{2^i} & x \in \mathcal{C} \\ \sup_{y \leq x : y \in \mathcal{C}} c(y) & x \in [0,1] \cap \mathcal{C}^c \end{cases}$$

where t_i is the i-ith ternary digit of x (see PS1). Formally fill in some of the following details from class:

a) Show that the Cantor function is continuous at all $x \in (0, 1)$.

b) Show that the Cantor function is not absolutely continuous.

c) Show that $F'(x) = 0$ almost everywhere.

Problem 2. Almost everywhere.

a) Show that if $0 \leq f, g$ and $f = g$ almost everywhere (with respect to a measure μ), then

$$\int g d\mu = \int f d\mu.$$

b) Show that if $0 \leq f, g$ and $f \leq g$ almost everywhere, then

$$\int f d\mu \leq \int g d\mu$$

Problem 3. General definition of the integral.

Let $f : \Omega \to \mathbb{R}$ denote a positive function, i.e. $f \geq 0$. Let

$$SF(f) = \left\{ s(x) = \sum_{i=1}^{N} x_i \mathbb{1}\{x \in A_i\} \middle| s \leq f \right\}$$

denote the set of simple functions that are less than f (where $\{A_i\}_{i=1}^{N}$ denotes a finite partition of Ω). Show that the following two definitions of the integral are equivalent:

a)

$$\int f d\mu = \sup_{s \in SF(f)} \sum_{i=1}^{N} x_i \mu(A_i)$$

b)

$$\int f d\mu = \sup \sum_{i=1}^{N} \left[\inf_{x \in A_i} f(x) \right] \mu(A_i).$$

where the suprenum is over all finite partitions $\{A_i\}_{i=1}^N$ of Ω.

Problem 4. Integral of simple functions.

Prove that if f is a (non-negative) simple function, i.e.

$$f = \sum_{i=1}^{N} x_i \mu(A_i)$$

for a finite partition $\{A_i\}_{i=1}^N$ of Ω, then $\int f d\mu = \sum_{i=1}^N x_i \mu(A_i)$ when using the second definition of Problem 2.

Bonus Problem.

Come up with a distribution function that is not absolutely continuous (with proof) that is not the Cantor function or based on the Cantor set.