Chapter 8

Inexact knowledge

8.1 THE EXPLANATORY TASK

Ignorance is a natural human state. The limits on what creatures of other species can know are not the limits of the universe; our species is unlikely to be very different. Our knowledge stands in more need of explanation than does our ignorance. Self-knowledge is no exception. We do not expect creatures of other species to be capable of knowing everything about themselves; we should not expect to be capable of knowing everything about ourselves. The same applies to our knowledge of our own creations. In some sense we create our language, but it does not follow that it is in every respect open to our gaze. Why should the boundaries of our terms not be invisible to us?

Such general considerations do not lift the whole burden of proof from the epistemic view of ignorance. For most vague terms, there is knowledge to be explained as well as ignorance. Although we cannot know whether the term applies in a borderline case, we can know whether it applies in many cases that are not borderline. The epistemic view may reasonably be expected to explain why the methods successfully used to acquire knowledge in the latter cases fail in the former. This chapter provides such an explanation.

Ignorance in borderline cases will be assimilated to a much wider phenomenon, a kind of ignorance that occurs wherever our knowledge is *inexact*. The notion of inexact knowledge, like that of vagueness, is best introduced by examples. Vision gives knowledge about the height of a tree, hearing about the loudness of a noise, touch about the temperature of a surface, smell about the age of an egg, taste about the constituents of a drink. Memory gives knowledge about the length of a walk, testimony about the physical characteristics of a criminal. The list could of course be continued indefinitely. In each case, the knowledge is inexact. One sees roughly but not exactly how many books a room contains, for example: it is certainly more than two hundred and less than twenty thousand, but one does not know the exact number. Yet there need be no relevant vagueness in the number. The inexactness was in the knowledge, not in the object about which it was acquired.

Section 8.2 examines a case of inexact knowledge in detail, to reveal its underlying structure. Section 8.3 draws the moral that each case of inexact knowledge is governed by a principle requiring knowledge to leave a *margin for error.* Section 8.4 applies the account of inexact knowledge to the problem of vagueness, using margin for error principles to explain our ignorance in borderline cases. It discusses the features that make knowledge involving vague concepts a distinct species of the genus, inexact knowledge. Vague terms are sharp but unstable in meaning; this instability is a distinctive source of inexactness. Section 8.5 characterizes vagueness in a concept as its indiscriminability from other possible concepts, and reconciles this with our knowledge of the meaning of vague terms. Section 8.6 relates inexact knowledge to the non-transitivity of indiscriminability more generally. Section 8.7 extends the notion of inexactness to cognitive attitudes weaker than knowledge, such as reasonable belief.

8.2 THE CROWD

I see a vast crowd in a stadium. I wonder how many people there are. Naturally, I cannot know exactly just by looking. My eyesight and ability to judge numbers are nothing like that good, and a few people may not even be visible from where I stand. Since I have no other source of relevant information at present, I do not know exactly how many people there are. For no number m do I know that there are exactly m people. Nevertheless, by looking I have gained some knowledge. I know that there are not exactly two hundred or two hundred thousand people; I do not know whether there are exactly twenty thousand people. For many numbers m, I do not know that there are not exactly m people. More precisely: for many numbers m, I do not have knowledge whose content is expressed by the result of replacing 'm' in 'There are not exactly m people' by a numeral designating m (e.g. '20,000'). Not under discussion is knowledge whose content is expressed by a sentence in which m is designated by a definite description, such as 'The number of people in the stadium minus one', for I may not know which number fits the description.

The least number principle says that every non-empty set of natural numbers has a least member. It is equivalent to the principle of mathematical induction, that if 0 has a property F, and whenever m has F so has $m + 1$, then every natural number has F. The set of natural numbers m such that I do not know that there are not exactly m people is non-empty, for it contains the number 20,000. By the least number principle, the set contains a least member n. Thus:

(1) I know that there are not exactly $n - 1$ people.

(2) I do not know that there are not exactly n people.

'n' abbreviates the definite description 'the least number m such that I do not know that there are not exactly m people'. '$n - 1$' abbreviates a corresponding description. However, by remarks at the end of the last paragraph, (1) and (2) do not concern knowledge in which numbers are designated by such descriptions; what makes them true or false is knowledge or the lack of it in which n or $n - 1$ is designated by an ordinary numeral such as '20,000' or '19,999'. The descriptions determine which numerals should replace them (in technical terms, they have wide scope).

I do not know exactly how many people there are. I might guess but, even if I guessed right, that would not be *knowing* exactly how many people there are. If I judge that there are exactly m people, and in fact there are, then for all I know there are really $m - 1$ or $m + 1$; I do not know that there are not. *A fortiori*, if I do not judge that there are exactly m people, when in fact there are, I do not know that there are exactly m. Either way, if there are exactly m people, then I do not know that there are not exactly $m - 1$ or $m + 1$. These reflections are quite independent of the value of 'm'. There can be *no* number m such that there are exactly m people and I know that there not, say, exactly $m - 1$, for if there are exactly m and I judge that there are not exactly $m - 1$, I am merely guessing. Anyone who can know by looking from where I am that there are not exactly $m - 1$, when in fact there are exactly m, has much better eyesight than I have (including X-ray eyes to see through various obstacles) and a much greater ability to judge numbers. For every number m, if there are exactly m people, then I do not know that there are not exactly $m - 1$.[1]

The foregoing reflections are mine. *I* know that my eyesight and ability to judge numbers are not good enough for me to know by looking that there are not $m - 1$ people, when in fact there are m. I know that, for every number m, if there are exactly m people, then I do not know that there are not exactly $m - 1$. Indeed, I may be assumed to have pedantically instantiated this knowledge for each relevant number. For each such m, I know that if there are exactly m people, then I do not know that there are not exactly $m - 1$. For example, I know that if there are exactly 20,000 people, then I do not know that there are not exactly 19,999. Since the number n in (1) and (2) is one of the many for which I have instantiated my knowledge, it follows too that

(3) I know that if there are exactly n people, then I do not know that there are not exactly $n - 1$ people.

But now a contradiction threatens. By (1), I *do* know that there are not exactly $n - 1$ people. If I combine this knowledge with my knowledge in (3), I can apparently deduce, and thereby come to know, that there are not exactly n people, which contradicts (2). What has gone wrong?

The argument can be made more explicit. There is a limit to how much I can add to my knowledge of the number of people by reflecting on the limitations of my eyesight and ability to judge numbers, and making deductions from what I thereby know. It is indeed plausible, although this point will not be pressed, that such processes add *nothing* to the knowledge of the number that I have already gained. Be that as it may, the supposition can legitimately be built into the example that I go to the trouble of deducing in a logically competent way all propositions of the form 'There are not exactly m people' that can be deduced from what I know, for every relevant value of 'm'. After some time I have completed the task (the number of people in the stadium may be assumed not to change over this period). 'n' in (1)–(3) can be defined with respect to a time t at which this reflective equilibrium has been attained. Since the defining description for 'n' does not figure in the content of the knowledge under discussion, this procedure does not involve circularity, or a vicious regress. t can also be taken as the time of the knowing at issue in (1)–(3). Thus (3) says that I know at t that if there are exactly n people, then I do not know at t that there are not exactly $n - 1$ people. There is nothing viciously self-referential or ungrounded about this present knowledge of present (rather than past) ignorance, for it is based on general considerations about my eyesight and ability to judge numbers, not on a futile attempt to survey all the propositions I do not presently know. Now if it follows from what I know at t that there are not exactly n people, then by hypothesis I have already deduced that by t, and so come to know it. Thus at t:

(4) If I know some propositions, and from those propositions it logically follows that there are not exactly n people, then I know that there are not exactly n people.

Note that (4) is not a general principle about knowledge; it is merely a description of my state in a particular situation (the same goes for (1)–(3), of course). The paradox is that each of (1)–(4) appears to be true with respect to the envisaged mundanely possible situation, yet they appear to be mutually inconsistent.

One thought is that I might know my premises and competently deduce my conclusion without thereby coming to know it because, although the premises are probable enough to count as known, the conclusion is not. For a logical consequence of two propositions may be less probable than each of them. Thus (4) might fail. For several reasons, this is an unpromising diagnosis.

(a) It is counter-intuitive to suppose that making competent deductions from what one knows is not in general a way of extending one's knowledge.
(b) The connection assumed between knowledge and probability has not been made out. No degree of probability less than 1 by itself makes knowledge out of true belief; the pessimistic owner of a ticket in a fair lottery who rightly believes that it will not win does not know that it will not, however many tickets have been sold.
(c) What sort of probability is in question? Degrees of belief seem too dependent on the subject, and propensities in the world not dependent enough, to form a standard for knowledge. If one considers

probabilities conditional on what I know, then what I know has probability 1 and the objection lapses.

(d) Nothing in the example disallows the assumption that there is a probability in the relevant sense of 1 that my eyesight and ability to judge numbers are not good enough for me to know that there are not $n - 1$ people, given that there are in fact n. Then the probability of the conditional in (3) will be 1, as will be the probability of its consequent conditional on its antecedent. But then the objection to (4) still lapses, for the required drop in probability from premises to conclusion cannot occur. If two premises entail a conclusion, and one premise has probability 1, then the conclusion is at least as probable as the other premise.

(e) One can always introduce a notion of knowledge#, where to know# a proposition is to know either it or propositions from which one has competently deduced it. I do not know# exactly how many people there are. One can run through the argument with 'know&##8217; in place of 'know'; (4) is then certainly correct.[2]

In the light of (a)–(e), the paradox should not be blamed on the failure of competent deduction to extend knowledge.

Another thought is that vagueness itself is somehow to blame. If it were, the case could not be used to cast independent light on the problem of vagueness. However, it will be argued without appeal to the epistemic theory of vagueness that vagueness is not to be blamed in the present case. Some examples of inexact knowledge depend on vagueness, but not all do.

The least number principle is sometimes held not to be valid in the presence of vagueness; what is the least number of grains to make a heap? The charge is that 'n' in (1)–(4) fails to refer. It was defined as 'the least number m such that I do not know that there are not exactly m people'. The only word here that might be relevantly vague is 'know'. It is indeed vague to some extent; there are borderline cases of knowledge. The question is whether this vagueness is the source of the paradox. If it is, the paradox would vanish if 'know' were made precise by arbitrary stipulations. We could not expect in practice to eliminate all the vagueness in 'know', but we do not need to. All we need are stipulations that resolve each borderline case of the form 'I know that there are not exactly m people' in the present context, for then we could apply the least number principle to define 'n', making (1) and (2) true. For example, let us adopt the rule that all relevant borderline cases are to be resolved conservatively with respect to 'know', thereby raising the standard for what it takes to 'know'. A corresponding rule may be adopted to resolve any relevant borderline cases for 'borderline case', and so on. Such stipulations do nothing to improve my eyesight or ability to judge numbers, of course. Now the conservative stipulations will, if anything, reinforce the truth of conditionals of the form 'If there are exactly m people, then I do not know that there are not exactly $m - 1$ people', for they make it harder to know that there are not exactly $m - 1$ people in the new sense of 'know'. Moreover, my attitude to the conditionals was already a clear case of knowledge, not a borderline case, so I should know them in the new sense too. So (3) remains true on the new reading, as does (4). Thus the apparently inconsistent propositions (1)–(4) remain apparently true when the stipulations are made. Since the paradox would not vanish if 'know' were made relevantly precise, and no other term is relevantly vague, vagueness is not the source of the difficulty. It is therefore legitimate in what follows to treat 'know' as though it were precise.

The paradoxical reasoning needs to be examined in more detail. When (4) is applied, what are the propositions from which it logically follows that there are not exactly n people? The major premise is the conditional 'If there are exactly n people, then I do not know that there are not exactly $n - 1$ people'. The form of the inference is from 'If A, then not B' and 'B' to 'Not A'. Thus 'A' is 'There are exactly n people' and 'B' is 'I know that there are not exactly $n - 1$ people'. If I am to use the inference to gain knowledge of its conclusion, I must know its premises, as (4) requires. (3) says that I know the major premise. I must also know the minor, 'B', i.e. (1). That is, the paradoxical reasoning assumes:

(5) I know that I know that there are not exactly $n - 1$ people.

The mutually inconsistent propositions are (2)–(5), not (1)–(4).

One might think for a moment that since I know by definition of 'n' that (1) is true, (5) is true. However, that would be to mistake the role of the definite description abbreviated by '$n - 1$', 'the predecessor of the least number m such that I do not know that there are not exactly m people', in (5). Certainly I know that, for every number k, if k fits the description then I know that there are not exactly k people. But that is not what (5) says.

The whole argument concerns knowledge in which numbers are designated by numerals, not by definite descriptions such as that above; only on this understanding is (3) plausible. What (5) says is that if a number fits the description abbreviated by '$n - 1$', then I know that I know that there are not exactly that number (as designated by a numeral) people. In technical terms, the definite description must be given the widest possible scope. If (5) were read otherwise, the reasoning would be trivially fallacious. Since I do not know what number fits the definite description abbreviated by '$n - 1$', my reflection on the description does nothing to make (5) true.

The apparently true propositions are (1)–(4). The inconsistent ones are (2)–(5). The right thing to do is the obvious one: accept (1)–(4) and reject (5). This is to reject the 'KK' principle that if I know something, then I know that I know it. I know that there are not exactly $n - 1$ people, but I do not know that I know that there are not exactly $n - 1$ people.[3]

Before we adopt the proposal, we should check that it really does meet the difficulty. We can do this by giving a consistency proof for (1)–(4) and the negation of (5). This will be done by means of a simple model of some situations in one of which (1)–(4) are true and (5) is false. The model is not intended to be realistic; what it shows is the structure of the envisaged solution to the paradox. Although the argument could be carried out in purely formal terms, the construction is more revealing when made informally.

For each natural number m, let s_m be a situation in which there are exactly m people in the stadium. Thus for any number k (the result of substituting the numeral for k for 'k' in) 'There are exactly k people' is true in s_m if and only if $k = m$. Truth-values can now be assigned recursively to compound sentences in different situations. 'Not A' is true in s_m if and only if 'A' is not true in s_m. 'If A, then B' is true in s_m if and only if either 'A' is not true in s_m or 'B' is true in s_m. 'I know that A' is true in s_m if and only if 'A' is true in each of s_{m-1} (if it exists), s_m and s_{m+1}. I know the proposition just in case it is true in the situation I am in and every situation like it except for a difference of one in the number of people in the stadium.

It is easy to check that, for any positive m, the conditional 'If there are exactly m people, then I do not know that there are not exactly $m - 1$ people' is true in every situation. Consequently, 'I know that if there are exactly m people, then I do not know that there are not exactly $m - 1$ people' is also true in every situation. In particular, (3) is true in every situation. Moreover, in this simple model I know all the logical consequences of what I know, for if every premise of a logically valid argument is true in each of s_{m-1}, s_m and s_{m+1}, then so too is its conclusion. Thus (4) is also true in every situation. Since 'There are not exactly $n - 1$ people' is true in every situation but s_{n-1}, 'I know that there are not exactly $n - 1$ people', i.e. (1), is true in every situation but s_{n-2}, s_{n-1} and s_n. Hence 'I know that I know that there are not exactly $n - 1$ people', i.e. (5), is true in every situation but s_{n-3}, s_{n-2}, s_{n-1}, s_n and s_{n+1}. Similarly, 'I know that there are not exactly n people' is true in every situation but s_{n-1}, s_n and s_{n+1}, so its negation (2) is true in just those three situations. Thus (1)–(4) are all true in s_{n+1}, while (5) is false.[4]

The model shows that the proposed solution to the paradox is consistent. In order to restore consistency, it was enough to deny the KK principle. Indeed, the model proves that one can consistently combine (1) and (2) with unqualified versions of the generalizations from which (3) and (4) were derived:

(3+) For every number m, I know that if there are exactly m people, then I do not know that there are not exactly $m - 1$ people.

(4+) For every number m, if I know some propositions, and from those propositions it logically follows that there are not exactly m people, then I know that there are not exactly m people.

To emphasize: (3+) and (4+) purport to describe my state at time t, not to lay down general principles about knowledge. In a realistic example, an upper bound would be imposed on m, if I am not to know infinitely many propositions.

The paradoxical reasoning is generated by the combination of (3+) and (4+) with the KK principle. It can be reformulated as appealing directly to the principle of mathematical induction, rather than to the least number principle. I certainly know that there are not exactly 0 people. But if I know that there are not exactly $m - 1$ people, then by the KK principle I know that I know that there are not exactly $m - 1$ people; so by (3+) I know the premises of a valid deductive argument whose conclusion is that there are not exactly m people; so by (4+) I know that there are not exactly m people. By mathematical induction, for every natural number m I know that there are not exactly m people. That is impossible, for the number of people in the stadium is finite. As the case was described, I do not know that there are not exactly 20,000 people. This upper bound means that, strictly speaking, the appeal to mathematical induction can be dispensed with, for one can reach absurdity arguing step by step without use of special principles of mathematics. The appeal to the least number principle was equally unnecessary. However, the principles conveniently label the difference in flavour between the two versions, and speed up both. What neither argument can dispense with is the KK principle. It is the culprit.

The failure of the KK principle is not news. However, the standard counterexamples involve knowing subjects who lack the concept of knowledge or have not reflected on their knowledge and therefore do not know that they know. The present case is quite different. It concerns a subject who has the concept of knowledge and has reached reflective equilibrium with respect to the propositions at issue. Still I know without knowing that I know.

Our knowledge is riddled with failures of the KK principle, for it is riddled with inexactness. The problem about the stadium could be duplicated with respect to almost any case of sense perception, for almost any such case gives inexact knowledge about numbers or quantities of some kind. In many cases, the quantities will lie on a continuous scale, as when I see a tree and have inexact knowledge of its height. That presents no obstacle to the argument, for continuous scales are divided into units; I have inexact knowledge of the height of the tree in inches to the nearest inch. The point generalizes to knowledge from sources beyond present perception, such as memory and testimony. This is partly because they pass on inexact knowledge derived from past perception, partly because they add further inexactness of their own. In each case the possible answers to a question lie so close together that if a given answer is in fact correct, then one does not know that its neighbouring answers are not correct, and one can know that one's powers of discrimination have that limit. The argument proceeds as before.

The aim of Section 8.3 is to describe a model of inexact knowledge on which its failure to provide the KK principle looks utterly natural.

8.3 MARGINS FOR ERROR

Suppose that, in the situation described in Section 8.2, there are exactly *i* people and I have the true belief that there are not exactly *j* people. If the difference between *i* and *j* is too small, then even if there had been exactly *j* people, I might easily still have believed that there were not exactly *j*. It is not the case that if there had been exactly *j* people I should not have believed that there were not exactly *j*. My eyesight and ability to judge numbers are limited. But then my true belief that there are not exactly *j* is not reliably true; it is too risky to constitute knowledge. On the other hand, if the difference between *i* and *j* is large enough, and my belief is formed in a normal way, then it may well be the case that if there had been exactly *j* people I should not have believed that there were not exactly *j*. My true belief may then be reliably true, and constitute knowledge.[5] Other things being equal, I know that there are not exactly *j* people if and only if the difference between *i* (the actual number) and *j* is large enough. How large is large enough? That depends on the circumstances: my eyesight and ability to judge numbers, the obstacles occluding part of the crowd, the quality of the light.

Where our knowledge is inexact, our beliefs are reliable only if we leave a margin for error. The belief that a general condition obtains in a particular case has a margin for error if the condition also obtains in all similar cases. The degree and kind of the required similarity depend on the circumstances. For given cognitive capacities, reliability increases with the width of the margin. The more accurate the cognitive capacities, the narrower is the margin needed to achieve a given level of reliability. Since a belief constitutes knowledge only if it is reliable enough, the belief that a general condition obtains in a particular case constitutes knowledge only if the condition obtains in all cases similar enough in the relevant respects to achieve the required level of reliability. Knowledge that the condition obtains is available only if it does obtain (whether knowably or not) in all sufficiently similar cases. If it obtains in a case sufficiently similar to a case in which it does not obtain, then knowledge that it obtains is unavailable in both cases. It cannot be known to obtain within its margin for error.

A *margin for error principle* is a principle of the form: '*A*' is true in all cases similar to cases in which 'It is known that *A*' is true. Which margin for error principles obtain depends on the circumstances; one cannot specify *a priori* the required degree and kind of similarity. One can, however, state a margin for error meta-principle: that where knowledge is inexact, some margin for error principle holds. The meta-principle is necessarily unspecific, but it is not trivial. In particular, it does not hold merely by definition of 'inexact knowledge', for the phrase was defined by examples, not by reference to margin for error principles. Rather, inexact knowledge is a widespread and easily recognized cognitive phenomenon, whose underlying nature turns out to be characterized by the holding of margin for error principles.

The required similarity need not be specified in the same terms as those used to express the knowledge in question. For example, I can know by looking that there are exactly five people in a room. That belief would be false in any case differing from the actual one in the number of people. Yet it constitutes knowledge, and its source is of the same general kind as my inexact knowledge of the number of people in the stadium. It is indeed governed by a margin for error principle. If I know by looking that the room is in a certain condition (such as that of containing five people), then the room is in that condition in any case differing from the present one only by rearrangements of a few molecules. I can know that there are exactly five people because there would still be exactly five in any case within the relevant margin for error.

A special case of inexact knowledge is that in which the proposition '*A*' is itself of the form 'It is known that *B*'. Just as we are not perfectly accurate judges of the number in a crowd, so we are not perfectly accurate judges of the reliability of a belief. A margin for error principle for 'It is known that *B*' in place of '*A*' says that 'It is known that *B*' is true in all cases similar to cases in which 'It is known that it is known that *B*' is true. As usual, the required degree and kind of similarity depend on the circumstances, for example on one's ability to judge reliability; 'It is known that *B*' and '*B*' may need margins for error of different widths. If 'It is known that *B*' is true but there are sufficiently similar cases in which it is false, then it is not available to be known. It cannot be known within its margin for error. Thus the failure of the KK principle is a natural consequence of the inexactness of our knowledge of our knowledge. By the margin for error meta-principle, our knowledge of our knowledge is governed by a margin for error principle, from which it follows that the KK principle is false.

Suppose that 'It is known that it is known that *B*' is true in a given case. By a margin for error principle for 'It is known that *B*', the latter proposition is true in all cases similar to the given case. But then by a margin for error principle for '*B*', '*B*' is true in all cases similar to cases similar to the given case. In effect, knowledge that one knows requires two margins for error. More generally, every iteration of knowledge widens the required margin. Any number of iterations of knowledge is possible in principle, but is available in a narrower range of cases than any lower number of iterations.

Believing is often compared to shooting at a target, the truth. The comparison is not quite apt, for the truth is a single point (the actual case), like a bullet, while the proposition believed covers an area (a set of possible cases), like a target. Instead, the believer's task may be conceived as drawing a boundary on a wall at which a machine is to fire a bullet. The belief is true if the bullet hits the bounded area, false otherwise. If truth is a hit, knowledge is a safe hit.[6] That is, the point of impact is within the bounded area and not so near its boundary that the bullet could very easily have landed outside (had a light breeze blown). For example, a hit might be safe just in case every point on the wall less than an inch from the point of impact is within the bounded area. The one-inch margin inside the boundary corresponds to the cases in which '*B*' is true but unknown; when this margin is removed from the bounded area, the remaining area corresponds to the cases in which '*B*' is known. When another margin is removed, the result corresponds to the cases in which '*B*' is known to be known. The iteration of knowledge operators is a process of gradual erosion.[7]

An area lacks a margin only if no point in it is less than an inch from a point on the wall not in the area. Since any point on the wall can be reached from any other via a sequence of points each less than an inch from the next, if an area lacks a margin then either no points are in it or all points are in it. It is either the 'null area', corresponding to a contradiction, or the whole wall, corresponding to a tautology.[8] On this model, a proposition is available to be known whenever it is true only if it is either logically true or logically false. A contingent proposition corresponds to a non-null area less than the whole wall and has a margin; such a proposition is true in some cases where it is not available to be known.

Not only can any point on the wall be reached from any other via a sequence of points each less than an inch from the next: there is a finite bound to the number of points needed, since diagonally opposite corners of the wall are as far apart as any two points on it. Thus any area less than the whole wall is reduced to nothing by a finite number of removals of one-inch margins. This corresponds to the claim that for any proposition other than a logical truth there is a finite bound to the number of iterations of knowledge one can have of it – a mildly sceptical result.[9]

The remarks in the last two paragraphs should be treated with more than usual caution, for they depend on specific features of the very simple layout imagined. If the wall were divided down the middle by a partition at right angles to it, so that there was no danger of a bullet fired on one side of the partition landing on the other, the half of the wall on one side of the partition would have no margin, and would therefore correspond to a contingent proposition available to be known whenever true. If the wall were infinitely long, with no partition, any two points on it would still be a finite distance apart, but the half of the wall on one side of an arbitrary vertical line would not be reduced to nothing by any finite number of removals of one-inch margins; it would correspond to a contingent proposition of which one could have any finite number of iterations of knowledge. The constant width of the margin is another simplification. If it were draughtier on the left than on the right, making the bullet's flight less predictable on that side, a wider margin would be needed there. Our cognitive capacities are more accurate in some areas than in others.

Such qualifications could be multiplied. However, the point of a model is to omit all but a few of the original's complications. What has been sketched is a picture of inexact knowledge in which the systematic failure of the KK principle is utterly natural. It is to be expected not just because we are not perfectly reflective, but because, however reflective we are, our cognitive capacities are not perfectly accurate. The next task is to apply these ideas to knowledge whose inexactness stems from the vagueness of its content.[10]

8.4 CONCEPTUAL SOURCES OF INEXACTNESS

The usual sources of inexactness infect vague judgements just as much as they do precise ones. Perceptual knowledge of someone's girth in inches is inexact; so is perceptual knowledge of his thinness. In borderline cases, however, our ignorance goes deeper than that.

On the epistemic view of vagueness, there are values of '*m*' for which anyone with exact physical measurements *m* is unknowably thin. Now if I have physical measurements *m*, I do not have them essentially; the sentence 'TW is thin' expresses a contingent truth. But since thinness supervenes on exact physical measurements, the generalization 'Everyone with physical measurements *m* is thin' expresses a necessary truth (see Section 7.4). Since I can be known to have physical measurements *m*, the ignorance postulated by the epistemic view involves ignorance of such necessary truths.[11] Yet they seem trivially to satisfy the necessary condition for being known imposed by a margin for error principle. A necessary truth is true in all cases; *a fortiori*, it is true in all cases similar to the case in which it is a candidate for being known. How then can a margin for error principle explain our ignorance of a necessary truth?

Someone who asserts 'Everyone with physical measurements *m* is thin' is asserting a necessary truth, but he is still lucky to be speaking the truth. He does not know the truth of what he says. Although he could not have asserted the proposition he actually asserted without speaking truly, he could very easily have asserted a different and necessarily false proposition with the same words.[12]The extension of the word 'thin' could very easily have been slightly different, so that it would have excluded everyone with physical measurements *m*. What distinguishes vagueness as a source of inexactness is that the margin for error principles to which it gives rise advert to small differences in meaning, not to small differences in the objects under discussion.

Consider again the supervenience of meaning on use, at least for a fixed contribution from the environment (as in Section 7.5). For any difference in meaning, there is a difference in use. The converse does not always hold. The meaning of a word may be stabilized by natural divisions, so that a small difference in use would make no difference in meaning. A slightly increased propensity to mistake fool's gold for gold would not change the meaning or extension of the word 'gold'. But the meaning of a vague word is not stabilized by natural divisions in this way. A slight shift along one axis of measurement in all our dispositions to use 'thin' would slightly shift the meaning and extension of 'thin'. On the epistemic view, the boundary of 'thin' is sharp but unstable.

Suppose that I am on the 'thin' side of the boundary, but only just. If our use of 'thin' had been very slightly different, as it easily could have been, then I should have been on the 'not thin' side. The sentence 'TW is thin' is true, but could very easily have been false without any change in my physical measurements or those of the relevant comparison class. Moreover, someone who utters the sentence assertively could very easily have done so falsely, for the decision to utter it was not sensitive to all the slight shifts in the use of 'thin' that would make the utterance false.

The point is not confined to public language. Even idiolects are vague. You may have no settled disposition to assent to or dissent from 'TW is thin'. If you were forced to go one way or the other, which way you went would depend on your circumstances and mood. If you assented, that would not automatically make the utterance true in your idiolect; if you dissented, that would not automatically make it false. What you mean by 'thin' does not depend solely on what you would say in your present circumstances and mood. You have no way of making each part of your use perfectly sensitive to the whole, for you have no way of surveying the whole. To imagine away this sprawling quality of your use is to imagine away its vagueness.

Similar points apply to concepts. You have no way of making your use of a concept on a particular occasion perfectly sensitive to your overall pattern of use, for you have no way of surveying that pattern in all its details. Since the content of the concept depends on the overall pattern, you have no way of making your use of a concept on a particular occasion perfectly sensitive to its content. Even if you did know all the details of the pattern (which you could not), you would still be ignorant of the manner in which they determined the content of the concept.

The plausibility of the claim that vagueness gives rise to margin for error principles does not depend on the epistemic view of vagueness. Consider the term 'heap', used in such a way that it is very vague.[13] Someone who asserts 'n grains make a heap' might very easily have made an assertion with that sentence even if our overall use had been slightly different in such a way as to assign the sentence the semantic status presently possessed by '$n - 1$ grains make a heap'. A small shift in the distribution of uses would not carry every individual use along with it. The actual assertion is the outcome of a disposition to be reliably right only if the counterfactual assertion would have been right. Thus the actual assertion expresses knowledge only if the counterfactual assertion would have expressed a truth. By hypothesis, the semantic status of 'n grains make a heap' in the counterfactual situation is the same as that of '$n - 1$ grains make a heap' in the actual situation; if the former expresses a truth, so does the latter. Hence, in the present situation, 'n grains make a heap' expresses knowledge only if '$n - 1$ grains make a heap' expresses a truth. In other words, a margin for error principle holds:

(!) If we know that n grains make a heap, then $n - 1$ grains make a heap.

The argument for (!) does not appeal to the epistemic view of vagueness at any point. Someone who rejected the view would not be forced to reject (!), and might well wish to accept it.

(!) might be thought to generate a sorites paradox, not for 'heap' but for 'known heap'. If we know (!) (by philosophical reflection on our vague use of 'heap'), and deduce the relevant logical consequences of what we know, does it not follow that if we know that n grains make a heap, then we know that $n - 1$ grains make a heap? Since we know that 10,000 grains make a heap, it would follow by 10,000 applications of *modus ponens* that 0 grains make a heap, which they do not. However, this is just a variant of the fallacious argument about the crowd. In order to know that $n - 1$ grains make a heap, we should have to know the premises from which we deduced '$n - 1$ grains make a heap'. They are the relevant instance of (!) and its antecedent, 'We know that n grains make a heap'. Thus we should have to know that we know that n grains make a heap. But the previous stage of the argument showed only that we know that n grains make a heap. With the KK principle, a sorites paradox would indeed be forthcoming. Without it, one iteration of knowledge is lost at each stage of the argument. For any reasonable number of iterations at the start, the argument runs out of steam before reaching a false conclusion. Indeed, it must do so, for the model used in Section 8.2 to show the consistency of (1)–(4) can equally be used to show that it is consistent to assume that we have several iterations of knowledge of each instance of (!), and of both the proposition that 10,000 grains make a heap and the proposition that 0 grains do not.

Given (!), we cannot know a conjunction of the form 'n grains make a heap and $n - 1$ grains do not make a heap'. To know the conjunction, we should have to know its first conjunct; but then by (!) its second conjunct would be false, making the whole conjunction false and therefore unknown. (!) encapsulates our ignorance of the cut-off point for 'heap'. A similar account can be given for other vague terms. A well-constructed sorites series makes an analogue of (!) true, because it proceeds by steps smaller than the relevant margin for error, so that if the term is known to apply to one member, then it does apply to the next.

What (!) conceals is the source of the inexactness. The small differences it displays are in the number of grains, but the underlying explanation appealed to small differences in the use of 'heap'. The argument for (!) could translate the latter into the former because a shift in the whole pattern of use of 'heap' by one step along the sorites series would be suitably small. One can construct artificial cases in which such a shift would be noticeable, so that the analogue of (!) might fail. Suppose, for the sake of argument, that 'several' is used in such a way that any plurality of more than three things is clearly several things and any plurality of less than three things is clearly not several things. Pluralities of three are the only borderline cases. A shift in the whole pattern of use of 'several' by one, so that pluralities of four became the only borderline cases, might well be noticeable. One could not maintain the analogues of (!) for both 'several' and 'not several', for that would involve maintaining both that if we know that four are several then three are several and that if we know that two are not several then three are not several. For we do know both that four are several and that two are not, so three would have to be both several and not several. If no one ever classified pluralities of three as 'several' or as 'not several', then one might after all be able to know whether three were several by the kind of argument mentioned in Section 7.7. If, however, the use of 'several' in this case is messy in the way characteristic of vague terms in natural language, that will not be possible. Rather, individuals will sometimes classify three as 'several' or 'not several', and might well have used the same words even if the frequency of such uses had been slightly different. A margin for error principle still governs such uses: if 'Three are several' expresses knowledge, then it would still have expressed a truth in those counterfactual situations. But even if 'Three are several' does express a truth, it would fail to do so in some of the counterfactual situations. By the margin for error principle, three are not known to be several. Thus our ignorance can still be explained by appeal to a margin for error principle in the form that most closely reflects the conceptual source of the inexactness.

Without appeal to the epistemic account of vagueness, one can argue that if vague terms have sharp boundaries, then we shall not be able to find those boundaries. Once one has seen this point, one can hardly regard our inability to find them as evidence that they do not exist. But if one has not seen the point, one might naturally suppose that if they exist then we should be able to find them, and so regard our inability as evidence of their non-existence. Thus margin for error principles explain both the ignorance postulated by the epistemic view and the apparent intuitions that run counter to that view. They do not commit one to the view, but they undermine some popular reasons for resisting it.

8.5 RECOGNITION OF VAGUE CONCEPTS

Vagueness is a source of inexactness, Section 8.3 argued, because individual uses of a vague term are not fully sensitive to small differences in the overall pattern on which small differences in meaning supervene. The argument seems to appeal to indiscriminable differences in meaning.

It is tempting to conclude that, contrary to Section 7.6, speakers of a vague language do not know exactly what their utterances mean. The argument of this section is that the tempting conclusion does not follow.

On the epistemic view, an utterance of a vague sentence such as '*n* grains make a heap' may express a necessary truth in a borderline case. A speaker who made such an assertion would not be expressing knowledge that *n* grains make a heap, for he might easily have used those words even if their overall use had been slightly different, so that they expressed a necessary falsehood. His utterance *u* does not manifest a disposition to be reliably right. As things actually are, *u* says that *P*. In the counterfactual situation, *u* says that *P**. The speaker would not recognize the difference. He does not seem to know in the actual situation that *u* does not say that *P**.

The epistemic theorist may concede that speakers do not know that *u* does not say that *P**. The question is whether it follows that they do not know what *u* says, in other words, that they do not know that *u* says that *P*. To put it the other way round, if they know that *u* says that *P*, does it follow that they know that *u* does not say that *P**? Does it even follow that they *can* know that u does not say that *P**?

If *u* says that *P*, then it does not say that *P**. Now if speakers of the language know that conditional, and also know that *u* says that *P*, then they can combine those pieces of knowledge and deduce that *u* does not say that *P**. But although the conditional is true, it does not follow that speakers of the language know it to be true. If they cannot discriminate what *u* actually says from what it counterfactually says, they cannot be expected to know the conditional. On the epistemic view, perhaps speakers know that *u* says that *P*, but cannot know that *u* does not say that *P**, and so cannot know that if *u* says that *P*, then *u* does not say that *P**.

The epistemic theorist is not alone in supposing that our ability to recognize the meaning of an utterance does not require us to discriminate it from all other possible meanings. On almost any view, the meaning of a vague utterance lies on something like a continuum. Even fuzzy boundaries lie in a continuum of possible fuzzy boundaries, varying in location and fuzziness. The sentence could have expressed a very slightly different meaning, and would have done so if its use had been very slightly different. One cannot expect speakers of the language to be able to discriminate between all such possible meanings. Several indiscriminable semantic differences can add up to a discriminable semantic difference. In being forced to acknowledge this fact, the epistemic theory is no worse off than its rivals.

One might react to the phenomenon of indiscriminable semantic differences by concluding that speakers only roughly know what their utterances mean; they cannot uniquely identify their meanings. If this reaction is open to anyone, it is open to the epistemic theorist. However, a less sceptical line of thought deserves to be explored.

Consider our ability to recognize faces. We often know exactly who someone is by seeing her face. Nevertheless, there could easily have been (and perhaps is) someone else facially indiscriminable from the known person, whom we should have misidentified on confrontation as the person we know. Our ability to recognize our friends and relations is not undermined by the mere possibility of look-alikes, although it might be undermined by their actual presence in the neighbourhood. Similarly, why should our ability to recognize the meaning of utterances in our language be undermined by the mere possibility of indiscriminably different meanings? It is not as though such meanings need be actually present in the language. In particular, slight differences in use between speakers do not generate indiscriminably different meanings, for linguistic meaning is socially determined (Section 7.6). Of course, just as there are genuine look-alikes, so indiscriminable semantic variation may genuinely occur, and where it does so our knowledge of meaning is rather uncontroversially undermined. The point is that the epistemic theorist has no more reason than anyone else to suppose that such variation is actually universal. There is a sense in which we often know exactly what an utterance means.

One can think of actual people as located on a continuum of possible people: but it does not follow that to recognize a person one must locate her on that continuum. It is enough to know which of the actual people she is. Similarly, one can think of actual meanings as located on a continuum of possible meanings: but it does not follow that to recognize a meaning one must locate it on that continuum. It is enough to know which of the actual meanings it is. To do that, it is enough to use the term within the appropriate practice, as discussed in Section 7.6.

The view just sketched is quite consistent with the relevant margin for error principles. If ‘heap’ had meant something slightly different, speakers would have recognized that slightly different meaning. They would not have misidentified it as the present meaning. Whatever the exact details of their dispositions to assent and dissent, they would then have been participants in the practice of using ‘heap’ as it would then have been. The identification even of a vague meaning can manifest a disposition to be reliably right.

The vagueness of an expression consists in the semantic differences between it and other possible expressions that would be indiscriminable by those who understood them. Similarly, the vagueness of a concept consists in the differences between it and other possible concepts that would be indiscriminable by those who grasped them. The greater the indiscriminable differences, the greater the vagueness. Nevertheless, vague expressions can be understood, and vague concepts grasped, for the indiscriminable differences need not actually arise.

8.6 INDISCRIMINABLE DIFFERENCES

Vagueness issues from our limited powers of conceptual discrimination. It is often associated with the expression in logic of such limits: the non-transitivity of indiscriminability. If a sample x is indiscriminable in colour from a sample y, for example, and y is indiscriminable in colour from a sample z, it does not follow that x is indiscriminable in colour from z. Someone who can discriminate in colour between x and z may count x as ‘red’ and z as ‘not red’; y seems destined to be a borderline case. This section investigates the connection between vagueness and the non-transitivity of indiscriminability. More generally, it investigates the connection between inexactness and the latter. For although inexactness has perceptual as well as conceptual sources, the resulting limitations on our knowledge share a common structure; they all give rise to margin for error principles. Since the structure is easier to discern when the source of the inexactness is perceptual, such examples will be used.[14]

The non-transitivity of indiscriminability is sometimes held to characterize only discrimination by direct comparison. On this view, a transitive form of indiscriminability is restored once indirect comparisons are permitted. For example, one can discriminate in colour between *x* and *y* indirectly, by noticing that one can directly discriminate the former but not the latter in colour from *z*. Call two things indirectly indiscriminable in a certain respect just in case they are directly indiscriminable in that respect from exactly the same things. Indirect indiscriminability is by definition a transitive relation. Admittedly, it can be hard to know that two things are indirectly indiscriminable, for all the things that might be directly indiscriminable from one but not the other must somehow be surveyed. In our everyday use of language, we rarely bother with indirect discriminations, and therefore lapse into vagueness. If we started to rely on such discriminations, we could no longer base our judgements on casual observation. Our use would lose its convenient vagueness. Nevertheless, it is suggested, indirect discrimination is available in principle as a standard, if we care to be more precise.[15]

The appeal to indirect discrimination presupposes that it is indeed a form of discrimination. To discriminate between *x* and *y* is to know that they are different.[16] Unless indirect discrimination involves such knowledge, it cannot be used as a standard for precise use, a reason for treating *x* and *y* differently. Now even if one has found a *z* from which one can in fact directly discriminate *x* but not *y*, that alone does not enable one to know that *x* and *y* are different; one must *know* that one can directly discriminate *x* but not y from *z*. Such knowledge may not be forthcoming in cases of inexactness. The point is closely related to the failure of the KK principle. If one has directly discriminated *x* from *z*, one knows that they are different; but if one is in no position to know that one knows that they are different, how can one know that one has discriminated between them? Equally, one might be unable to discriminate directly between *y* and *z*, but not be sure that one had not done so; how could one then know that one could not directly discriminate between them?

The point can best be substantiated by an example. I have been passing a certain tree on most days for several years, often giving it a casual glance, never attempting to measure it. My present knowledge of its height on each of the past 5,000 days is certainly inexact. My eyesight, my memory and my ability to judge heights are all imperfect. I do know that the height of the tree now (on day 5,000) is greater than it was at the beginning (on day 0), so:

(6) I know that the height on day 0 is not the same as the height on day 5,000.

I also know that, if the tree grew by less than a millimetre between day *i* and day *j*, then, for all I know, it did not grow at all in that period. To detect growth on that scale, one would need a much better eyesight, memory and ability to judge heights than I actually have. For all *i* and *j* between 0 and 5,000:

(7) I know that if the height on day *i* and the height on day *j* differ by less than a millimetre, then I do not know that the height on day *i* is not the same as the height on day *j*.

It will be convenient to read (7) and similar formulas as concerning only knowledge in which the numbers of the days are designated by numerals, as explained in Section 8.2. Like (3), (7) says that I know the contraposed form of a margin for error principle. I also know, on the testimony of a good botanist, that the tree cannot grow by as much as a millimetre in a single day. For all *i* between 0 and 5,000:

(8) I know that the height on day *i* and the height on day $i + 1$ differ by less than a millimetre.

By a rough estimate of the growth in height over the period, I can in fact estimate that the average growth per day is very much less than a millimetre.

I may be said to discriminate directly between day *i* and day *j* in the height of the tree just in case I know on the basis of my memory of those days that the height of the tree on day *i* was not the same as its height on day *j*. Days are directly indiscriminable in the height of the tree just in case I cannot discriminate directly between them in that respect (given my present circumstances). That relation is clearly not transitive, for each day has it to the next by (7) and (8), but by (6) the first day does not have it to the last.

Can I use indirect comparisons to discriminate more finely, perhaps even to falsify (7)? Day *i* may be said to be indirectly indiscriminable from day *j* just in case, for each day *k*, day *i* and day *k* are directly indiscriminable in the height of the tree if and only if day *j* and day *k* are directly indiscriminable in the height of the tree. Indirect indiscriminability is by definition an equivalence relation: it is reflexive, symmetric and transitive. Days are indirectly discriminable just in case they are not indirectly indiscriminable.

Since day 0 but not day 5,000 is directly discriminable from day 5,000, at least one day is indirectly discriminable from day 0. Let day m be the first such day. Thus day $m - 1$ is indirectly indiscriminable from day 0. If day m were indirectly indiscriminable from day $m - 1$, it would be indirectly indiscriminable from day 0 (by transitivity); since it is not, day m is indirectly discriminable from day $m - 1$. Unfortunately, I do not know which day is the first to be indirectly discriminable from day 0; I do not know which day 'm' designates. For any particular day, it is quite consistent with everything I know that the tree did not grow at all on that day, but grew steadily on every other day. In particular, I am in no position to know, even by inference, that the height of the tree on day $m - 1$ was not the same as on day m, when m and $m - 1$ are designated in my knowledge by numerals, not by definite descriptions such as that used to define 'm' ((6)–(8) concern only knowledge in which the numbers of days are so designated). Yet the two days count as 'indirectly discriminable' in the height of the tree.

Indirect discrimination is not a genuinely cognitive form of discrimination at all. If day i and day j are indirectly discriminable in the height of the tree, it does not follow that I am in a position to know inferentially that the height of the tree on day i was not the same as on day j. The problem is that I do not know exactly which direct discriminations I can make. Thus indirect discrimination is no threat to (7).

(6)–(8) remain a plausible description of my state when the term 'know' is used for inferential as well as non-inferential knowledge. Indeed, I may be assumed to have gained all the relevant knowledge available to me. I have deduced, and thereby come to know, all the relevant logical consequences of what I know; call this assumption 'Closure'. My ignorance of the height of the tree does not result from ignorance of logic.

I am ignorant about my direct discriminations. Either I know that the height of the tree on day i was not the same as on day j, but do not know that I know, or I do not know that the height on day i was not the same as on day j, but do not know that I do not know. The latter case is perhaps easier to imagine; what can now be shown is that the example involves cases of the former kind. As in other examples of inexact knowledge, the KK principle fails. The strategy of the argument is to show the KK principle to imply, for each i, that if I know that the height of the tree on day 0 is not the same as on day i, then I know that its height on day 0 is not the same as on day $i - 1$. Thus if I know that its height on day 0 is not the same as on day 5,000 (as I do, by (6)), then I know that the height on day 0 is not the same as on day 4,999, so I know that it is not the same as on day 4,998, so . . . . By 5,000 steps of the argument, I know that its height on day 0 is not the same as on day 0. Thus the KK principle will have been reduced to absurdity.

Suppose that I know that the height on day 0 is not the same as the height on day i. What must be deduced is that I know that the height on day 0 is not the same as the height on day $i - 1$. By the KK principle, I know that I know that the height on day 0 is not the same as the height on day i. By an instance of (7), I know that if the height on day 0 differs by less than a millimetre from the height on day i, then I do not know that the height on day 0 is not the same as the height on day i. By Closure, I know that the height on day 0 differs by not less than a millimetre from the height on day i. By (8), I know that the height on day $i - 1$ differs by less than a millimetre from the height on day i. By Closure again, I know that the height on day 0 is not the same as the height on day $i - 1$. QED

Once the KK principle is dropped, (6)–(8) and Closure form a consistent set. This can be shown by the construction of a simple model. For each subset X of the set of natural numbers from 1 to 5,000, let w_X be a 'possible world' in which, for each i from 1 to 5,000, the tree grows half a millimetre from day $i - 1$ to day i if i is a member of X, and otherwise does not grow at all in that time. The height of the tree on day 0 is the same in each world. For all worlds w_X and w_Y, say that w_Y is accessible from w_X just in case, for each day, the height of the tree in w_X on that day differs by less than a millimetre from its height in w_Y on that day. The worlds accessible from w_X are to be conceived as those in which everything known in w_X is true; if I am in w_X, and w_Y is accessible from w_X, then for all I know I am in w_Y; w_Y is epistemically possible relative to w_X. I know a proposition in w_X just in case that proposition is true in every world accessible from w_X; this assumption guarantees that I know all the logical consequences of what I know, and therefore validates Closure. It is not difficult to show that (7) and (8) are true in all worlds in the model, and that (6) is true in any world in which the tree grows on at least two days. Thus (6)–(8) and Closure are all true in worlds of the latter kind.

The failure of the KK principle in the model reflects the non-transitivity of the accessibility relation. For suppose that w_Z is accessible from w_Y, and w_Y from w_X, but that w_Z is not accessible from w_Y. Let '*A*' be a proposition true at just those worlds accessible from w_X. Thus 'I know that *A*' is true at w_X. However, it is not true at w_Y, for '*A*' is not true at w_Z, which is accessible from w_Y. Since w_Y is accessible from w_X, 'I know that I know that *A*' is not true at w_X. The KK principle breaks down at w_X. One can think of accessibility in the model above as a relation of indiscriminability between worlds. The KK principle fails because the indiscriminability of worlds is non-transitive.

The example began with the non-transitive indiscriminability of days in the height of the tree, and moved on to a similar phenomenon for worlds. It seems that this can always be done. Whatever x, y and z are, if x is indiscriminable from y, and y from z, but x is discriminable from z, then one can construct miniature worlds w_x, w_y and w_z in which the subject is presented with x, y and z respectively, everything else being relevantly similar. The indiscriminability of the objects is equivalent to the indiscriminability of the corresponding worlds, and therefore to their accessibility. The latter is therefore a non-transitive relation too. The proposition 'This is not z' is true in every world accessible from w_x, but not in w_z. As before, 'I know that this is not z' will be true in w_x, but 'I know that I know that this is not z' will be false. Thus the KK principle fails in w_x.

There is a complication. Discrimination is intentional, for it is a kind of knowledge. Even the example above involved discriminating between days in the height of the tree, i.e. between heights as presented to me by the tree on various days. Discriminability, in one sense of the term, can depend on the way in which objects are presented. Suppose, for example, that in the morning I count the number of birds in a cage and find that there are six; at noon I glance at the cage but do not notice how many birds it contains; in the evening I count the birds again, and find that one has gone. I do not know whether it escaped before or after noon. In that opaque sense, I can discriminate retrospectively the number of birds at the first count from the number at the second, but cannot discriminate the number at the first count from the number at noon, or the number at noon from the number at the second count. Only two numbers, five and six, were presented to me. One number was presented only once, by a count, the other twice, once by a count and once by a glance. I can discriminate the former number from the latter as presented by the count, but not as presented by the glance. In this case, the non-transitivity of indiscriminability in the opaque sense gives rise to no failure of the KK principle. Only two possibilities are relevant: either there were five birds at noon or there were six; I have no idea which. To model the situation, just two worlds are needed, indiscriminable from each other; this accessibility relation is transitive.

The example reveals a more radical way in which indirect discriminability in the opaque sense is not a genuine form of discriminability. I can discriminate the number of birds at the first count, but not the number at noon, from the number at the second count. It obviously does not follow that the number at the first count and the number at noon are distinct, let alone that I know that the number at the first count and the number at noon are distinct. For if the number at the first count is the number at noon, and I know that the number at the first count is not the same as the number at the second count, it does not follow that I know that the number at noon is not the same as the number at the second count. Even if the descriptions ‘the number at the first count’ and ‘the number at noon’ in fact denote the same number, the substitution of one for the other in the opaque context ‘I know that . . .’ can result in a change of truth-value in the sentence as a whole. Indirect discriminability in the opaque sense does not imply distinctness.

In the model used to prove the consistency of (6)–(8) and Closure, accessibility between worlds depends only on the growth of the tree in those worlds. Even there, however, indirect discriminability in the opaque sense does not imply distinctness. Let w_X be a world in which the tree does not grow at all from day 0 to day 1, and grows half a millimetre a day thereafter. Thus in w_X the height on day 0 is the same as the height on day 1. Moreover, I know that the height on day 0 is not the same as the height on day 3, for since the tree has grown a millimetre by day 3, in no world accessible from w_X is the height of the tree on day 0 the same as the height on day 3, for in each such world it has grown at least half a millimetre in that period. However, I do not know in w_X that the height on day 1 is not the same as the height on day 3. For from w_X a world w_Y is accessible in which the tree grows half a millimetre from day 0 to day 1, does not grow at all from day 1 to day 3, and grows half a millimetre a day thereafter; on each day the height in w_X differs from the height in w_Y by at most half a millimetre. In w_X, the height on day 0 and the height on day 1 are indirectly discriminable in the opaque sense, but they are not distinct. Such cases would be multiplied if the model took account of the varying degrees of attention I paid to the tree on various days, and other factors independent of its height.

Analogues of the phenomena discussed in this section will occur in the epistemology of vagueness, although they may be harder to discern. The failure of the KK principle will be manifested as higher-order vagueness. The failure of indirect discriminability to be a genuine form of discriminability will be another obstacle to attempts to make natural languages more precise.

8.7 INEXACT BELIEFS

Knowledge is not the only cognitive relation one cannot have to cut-off points for vague terms. One cannot know that n grains of sand make a heap and $n - 1$ do not; one also cannot reasonably believe it. The epistemic account has more to explain than the absence of knowledge.

The discussion so far has concerned knowledge. Some of its claims are equally plausible when 'reasonable to believe' is substituted for 'known'. Indeed, the discussion in Section 8.2 does not require what I believe to exceed what I know. Corresponding to (3), for example, is the plausible claim that it is reasonable to believe that if there are exactly n people in the stadium, then it is not reasonable to believe that there are not exactly $n - 1$. Margin for error principles, however, seem specific to knowledge. If one Φs a proposition in a situation s, one leaves a margin for error only if that proposition is true in all cases similar enough to s. Since s is certainly similar enough to itself, the proposition must be true in s. Thus if Φing requires a margin for error, one Φs only true propositions. Knowledge is such an attitude; reasonable belief is not. A subject with misleading evidence may reasonably believe false propositions. Since reasonable belief does not satisfy a margin for error principle, how can the epistemic theorist explain the unattainability of reasonable belief in borderline cases?

Parallel questions arise for many other cognitive attitudes, although not for all. An irrational person may certainly *believe* that n grains of sand make a heap and $n - 1$ do not. Parallel questions arise only for attitudes firmly based on evidence. The epistemic theorist can apply the account of inexact knowledge to such attitudes by working with a hypothesis: that one's evidence is simply what one knows. That the grass was wet, if one knows it, can be part of one's evidence for other beliefs. That the grass was wet, if one falsely believes it, cannot be part of one's evidence for other beliefs; only the evidence for the false belief can be part of the evidence for those other beliefs. In this sense, the hypothesis postulates an *externalist* concept of evidence. The status of a proposition as evidence does not depend solely on its place in the internal workings of the subject's head. Since only true propositions are known, only true propositions are evidence. Even so, there can be good evidence for a false proposition: when an innocent person is framed, for example. The restriction of one's evidence to what one knows is just what makes it plausible that, if Φing must be firmly based on evidence, then we cannot Φ that n grains make a heap and $n - 1$ do not.

A belief may be said to be reasonable just in case its probability conditional on the subject's evidence is high. On the present view, this is to say that the belief is reasonable just in case its probability conditional on what the subject knows is high. Now suppose, for simplicity, that the subject knows just the propositions that leave a margin for error δ. In a situation s, this amounts to knowing that one's situation is within δ of s, and knowing no more than that. Thus the probability of a belief conditional on what one knows may be conceived as the proportion of situations within δ of s in which the belief is true. On such simplifying assumptions, a belief is reasonable in a situation s just in case it is true in most worlds within δ of s.[17]

One can now explain, schematically, why for each number n it is not reasonable to believe that n grains make a heap and $n - 1$ do not. Indeed, one can explain a stronger principle:

(!!) If it is reasonable to believe that n grains make a heap, then it is not reasonable to believe that $n - 1$ grains do not make a heap.

Someone who, *per impossibile*, did reasonably believe that n grains make a heap and $n - 1$ do not would also violate (!!), by reasonably believing both that n grains make a heap and that $n - 1$ grains do not; it is reasonable to believe a conjunction only if it is reasonable to believe its conjuncts. Thus if one can explain (!!), one can also explain why it is not reasonable to believe that n grains make a heap and $n - 1$ do not.

In the present case, the relevant epistemically possible situations are those in which the cut-off point for 'heap' varies; no finer distinctions are relevant. Let s_k be the situation in which k is the least number of grains to make a heap. Thus n grains make a heap in s_k just in case n is at least k. Suppose that the situations within the appropriate margin for error of s_k are s_{k-2}, s_{k-1}, s_k, s_{k+1} and s_{k+2}. Suppose also that four out of five count as 'most', but that three out of five do not. Thus it is reasonable to believe a proposition in s_k if and only if it is true in at least four of s_{k-2}, s_{k-1}, s_k, s_{k+1} and s_{k+2}. Now the proposition that n grains make a heap is true in at least four of those situations if and only if n is at least $k + 1$; the proposition that $n - 1$ grains do not make a heap is true in at least four of the situations if and only if n is at most $k - 1$. Since no number is both at least $k + 1$ and at most $k - 1$, the proposition that n grains make a heap is true in most of the situations only if it is not the case that the proposition that $n - 1$ grains do not make a heap is true in most of them. Thus it is reasonable to believe the former proposition only if it is not reasonable to believe the latter. This explains (!!). Although the explanation uses highly simplified assumptions, more complex versions can be developed to cope with more complex assumptions. The underlying idea is the same.

The plausibility of (!!) depends on the assumption that a reasonable belief must have a high probability on the evidence. If a probability barely more than 50 per cent sufficed, the short step from $n - 1$ to n might be enough to tip the balance, and falsify (!!). The propositions that $n - 1$ grains do not make a heap and that n grains do could simultaneously be slightly more probable than not. The foregoing explanation therefore required more than a bare majority of the relevant situations for reasonableness.

The logic of the operator ‘It is reasonable to believe that . . .’ is like the logic of the operator ‘It is highly probable that . . .’, on the approach just outlined. Although ‘It is reasonable to believe that *A*’ does not entail ‘*A*’, it does entail ‘It is not reasonable to believe that not *A*’. If ‘*A*’ entails ‘*B*’, then ‘It is reasonable to believe that *A*’ entails ‘It is reasonable to believe that *B*’. However, if several premises ‘A_1’, ‘A_2’, . . . , ‘A_i’ jointly entail ‘*B*’, it does not follow that the premises ‘It is reasonable to believe that A_1’, ‘It is reasonable to believe that A_2’, . . . , ‘It is reasonable to believe that A_i’ jointly entail ‘It is reasonable to believe that *B*’, for a logical consequence of several propositions may be less probable than each of them individually. In particular, if A_1 and A_2 are independent propositions, then ‘It is reasonable to believe that A_1’ and ‘It is reasonable to believe that A_2’ do not jointly entail ‘It is reasonable to believe that A_1 and A_2’.

The analogue of the KK principle for what it is reasonable to believe breaks down. Consider again the model used to explain (!!). If ‘*A*’ is true just in s_{k-2}, s_{k-1}, $sk+1$ and s_{k+2}, then ‘It is reasonable to believe that *A*’ is true just in s_k, and ‘It is reasonable to believe that it is reasonable to believe that *A*’ in no situation at all. Thus the inference from ‘It is reasonable to believe that *A*’ to ‘It is reasonable to believe that it is reasonable to believe that *A*’ has a true premise and a false conclusion in s_k. In probabilistic terms: ‘It is highly probable that *A*’ does not entail ‘It is highly probable that it is highly probable that *A*’. Inferences of the converse form can also be shown to fail. The reason is that the evidence on which the probabilities are conditional is what is known, but may not be known to be known. Thus a non-zero probability may be assigned to a possible situation in which the propositions constituting the evidence in the actual situation do not count as evidence, although they are still true.[18]

The remarks above attempt no more than a sketch. Nevertheless, they show that the epistemic view of vagueness can be extended to a variety of cognitive attitudes. It is not only our knowledge that is inexact. For vague terms, the inexactness has a conceptual source; we cannot even form reasonable beliefs as to the location of their cut-off points, and fall under the illusion that such points do not exist.