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Allocating equitable parliamentary seating using mathematical optimization: A case study on Dáil Éireann, the Irish parliament

by

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Abstract

The fair allocation of parliamentary seats to political parties is a challenge which arises in legislative chambers the world over. The application of mathematical optimization to address this type of problem is relatively novel. Existing research has focused primarily on the development of heuristic approaches to solve large chamber instances. Promising results have been found for simple problems, but no existing model could robustly incorporate the nuanced seating rules that are present in many parliamentary settings. This paper first evaluates several candidate exact approaches for small and medium-sized parliaments, finding that a Facility Location Model (FLM) is appropriate for chambers of under 200 seats. Secondly, a Location-Allocation Heuristic (LAH), already proposed in the literature for simple problems in larger settings, is revised to allow for the inclusion of intricate seating customs. These two models - exact and heuristic - are evaluated against historical electoral datasets for the Irish parliament. They are found to provide robust solutions, and persuasive evidence is presented for their practical adoption in alternative legislative chambers.

Acknowledgements

I would like to sincerely thank Sergio García Quiles for his supervision, guidance and direction throughout this project, particularly given the unusual remote-working circumstances brought about by COVID-19. My deepest thanks to my family for their continued support during the entire dissertation period. Also, my extreme gratitude to Finbar and Peter, who provided rigorous proofreading and excellent feedback.

Own Work Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Sam Gormley,
August 2020

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Chapter 1

Introduction

1.1 Motivation

Seat allocation in legislative chambers is a contentious issue. Existing processes for developing seating plans make it challenging to ensure complete objectivity due to the significant human input necessary. Typically, parliamentary seating arrangements are finalised through negotiation among political party whips. This procedure has the significant downside of larger parties disproportionately securing preferable arrangements for themselves, leaving smaller parties to fight it out for what remains. Many competing opinions and desires exist within different individual parliaments as to what constitutes a *better* seating arrangement for each party. Examples include:

- Prominence for speaking purposes - Spain (Garea, 2016)
- Prominence for visibility - Finland (News Now Staff, 2019)
- Association with (or disassociation from) neighbouring parties - Germany (The Economist, 2019)
- Access to microphones - Netherlands (Tuin, 2019)
- Access to aisle seats - Netherlands (Tuin, 2019)

However, the generally unifying desire among all legislatures is for all party members be seated together *compactly*, such that communication can flow freely between them.

An automated, rule-based process for the allocation of seats would remove the human factor from these decisions and help ensure that neutrality is maintained in the allocation process. Smaller parties would be protected from the overreach of those that are larger, and *fairness* would be evenly distributed across all parliamentary members.

In an era when democracy is under recurrent threat, parliamentary equity takes on even greater importance in restoring faith in political institutions.

1.2 The Irish Context

Since the foundation of the Irish state in 1922, its politics has been dominated by two parties: Fianna Fáil and Fine Gael. Executive power has been vested in one or the other, without interruption, for almost 100 years. Given the pre-established seating rules that exist within the Irish parliamentary chamber, this two-party system had, until recently, resulted in a relatively deterministic seating arrangement problem.

However, over the past decade, the Irish parliament has experienced a rate of political change unseen in its history. The collective share of the two aforementioned parties has undergone a rapid decline, with ground conceded to independent politicians and other parties alike. Independents and some smaller parties have recently taken to aligning themselves into *technical groups* to obtain shared parliamentary speaking rights. These trends have all resulted in a much more volatile parliamentary makeup across each term. Four different parties have held more than 20% of all parliament seats within the past decade, and the average number of parliamentary political groupings has increased from five to ten. The seat allocation problem in an Irish context has become far from deterministic.

The new seating arrangement for the Irish parliament is determined after the formation of a new government. An initial proposal is prepared by legislative clerks and submitted to the party whips. After deliberation and negotiation over the proposal, a finalised seating arrangement is determined.

Not only does this process experience the neutrality issue raised in Section 1.1, but it also presents a second opportunity for bias to influence the result. Implicitly or overtly, the political biases of the parliamentary clerks may serve to prejudice the *equity* of the proposed plan.

This paper proposes rule-based optimization algorithms for generating proposed seating arrangements without human intervention. In the application to the Irish parliament specifically, this proposal would ideally be binding, unless parties wished to exchange seats by mutual consent. This process would ensure a fair and neutral starting point, whereby alterations would only occur if mutually beneficial.

The paper is organised as follows. First, a literature review in Chapter 1 explores the existing research into parliamentary allocation problems, as well as the mathematically-related field of territory design. Chapter 2 sets out the mathematical concepts necessary in order to translate individual problem elements into an optimization model. Chapter 3 evaluates several candidate formulations, both exact and heuristic, balancing strict adherence to objectives against computational efficiency, and in Chapter 4, experimentation is conducted upon these formulations to determine those most promising. Chapter 5 provides a detailed visual comparison of the results of the best-performing exact approach and the results of the heuristic model against the historical seating arrangements of the Irish parliament. The paper concludes in Chapter 6 with recommendations for the substantive application of these models and possible avenues for further research.

1.3 Literature Review

The use of mathematical optimization to solve parliamentary seat allocation problems is an application which has received little attention before the last few years.

The literature consists of only two existing papers, as far as could be ascertained by this author. The first is a paper written by Roland Oliver Hales & Sergio García and published in TOP: the official journal of the Spanish Society of Statistics and Operations Research (Hales & García, 2019). The second is a bachelor's thesis written by David Tuin at the Eindhoven University of Technology (Tuin, 2019).

Hales & García (2019) set out to solve the seating allocation problem of the Spanish Congress of Deputies, motivated by recent arguments over the unequal allocation of prominent seats within the chamber. They drew on the wealth of literature in the field of territory design, which possesses many similar attributes to the issue of parliamentary seating. Their paper set out two conceivable exact formulations that could be adopted: a facility-location model and a minimum k-partitioning approach. Due to the magnitude of their application (341 parliamentary members and seven political parties), these exact techniques were both computationally unsuitable. These formulations, however, formed the basis upon which their two heuristic methodologies were grounded. These models, an iterative location-allocation formulation and a heuristic based on computational geometry, were heavily influenced by the work of Kalcsics *et al.* (2005) who suggested a unified approach to territory design. Hales & García (2019) concluded their research by stating that "their heuristics [were] able to produce visually appealing seating plans for basic cases, but problems [could] occur when there [were] additional requirements to be satisfied."

Tuin (2019), in his bachelor's thesis, looked to generate an exact formulation based on a network flow approach, which could be universally adapted to parliaments of all layouts and sizes. Inspired by disagreement in the Dutch parliament (Tweede Kamer) regarding seating allocation and the associated access to microphones and aisles, he began his analysis with the two national political chambers of the Netherlands. His proposed network flow model performed well for the magnitude of the Dutch Senate (Eerste Kamer), which had 75 members and 12 parties; however, it was computationally inadequate for the scale of the parliament, with 150 members and 13 parties.

As previously noted, the parliamentary seat allocation problem shares many similarities with the well-researched field of territory design, and its extensive sub-field of political districting. The field of political districting is particularly long-studied, with many regarding the paper by Vickrey (1961) as the earliest in the topic. Kalcsics *et al.* (2005) provide an in-depth review of the existing literature for applications of territory design problems, and solution approaches for solving these types of problems, while Ricca *et al.* (2011), in particular, provide an extremely comprehensive review of the history of approaches to political districting. These problems, broadly, involve the "grouping of small geographic areas, called *basic areas* into larger geographic clusters called *territories*, in such a way that the latter are acceptable according to relevant planning criteria." (Kalcsics *et al.*, 2005) These planning criteria, which form the objectives of the problem, are typically:

1. Balance (of *activity measures*)

2. *Compactness*

3. Contiguity/Connectedness

Each of these objectives can be understood as follows. The *activity measure* of a given *basic area* is its population within political districting or a metric such as sales potential within sales territory design. One of the prerequisites of territory design problems is the need for well-balanced territories for the *activity measure* under consideration. However, as basic areas are generally regarded as indivisible in most models, a perfect balance is usually unachievable. Instead, a parameter on the relative deviation is introduced, and a constraint developed which limits imbalance within a threshold. This particular modelling objective presents the most significant divergence from the parliamentary seat allocation problem, where instead the requirement is that each party (*territory*) be allocated with a specific number of seats (*basic areas*) with no deviation allowed (*perfect balance*). The problem is comparatively simpler in that all *basic areas* effectively have an equal *activity measure*; however, it involves the introduction of an additional index for the political parties, which is a relative complication.

"In the context of territory design, a territory is said to be geographically *compact* if it is somewhat round-shaped and undistorted. Although being a very intuitive concept, a rigorous definition of compactness does not exist" (Kalcsics *et al.*, 2005). Niemi *et al.* (1990) provide a comprehensive review of the various metrics proposed for measuring *compactness* across the political districting literature. As mentioned above, in the context of parliamentary seat allocation, *compactness* is more intuitively measured based on communication possibilities. It will be more precisely defined in Section 2.1.

Due to the often irregularly-shaped *basic areas* in territory design problems, objectifying *compactness* is often not sufficient to ensure the geographic connectedness of a territory. That being said, "it is particularly difficult to deal with and, sometimes, it is even discarded from the [political districting] models and considered *a posteriori*" (Ricca *et al.*, 2011). In the context of parliamentary seating, the connectedness of party members is equally important. However, the more regular nature of the seating layouts ensures that aiming for a *compactness* measure based on communication will generally result in a relatively contiguous solution. Tuin (2019), in his proposed model, introduced a series of network flow constraints, utilising source and sink nodes, which enforced strict contiguity to his solutions. His analysis showed, however, that this added significant increases to his model's run-time. For these reasons, contiguity is not strictly codified in the models that are developed in this paper, but it is assessed after-the-fact.

A further contrast between these two varying types of problems is the additional rules and traditions that exist within legislative chambers, which must be layered onto a basic model. These can be useful from the perspective of reducing computational complexity, as they limit a problem's feasible solutions. Occasionally, however, specific rules which involve the introduction of a large number of constraints may serve to overly-complicate the model, as will be later illustrated. Most importantly, this differentiation limits the applicability of specific territory design heuristics to the parliamentary seat allocation problem, as these methods were developed without the need for additional constraints in mind.

As outlined by Ricca *et al.* (2011), "Due to the difficulty of the [political districting] problem and to its multicriteria nature, the contributions in this research field in the last years were mainly concentrated on the production of heuristic and metaheuristic methods". However, it must be noted that territory design problems broadly are generally at a significantly larger scale than those of parliamentary seat allocation. For this reason, this paper delves into the practical application of exact methods, as well as heuristics, when determining appropriate models.

1.4 Paper Goals

This paper effectively picks up where Hales & García (2019) left off, with two primary objectives.

First, to perform a comprehensive analysis of various prospective exact formulations, scrutinising the trade-off between the measures of compactness employed and their associated computational efficiency. This investigation is carried out to determine the maximum threshold for the useful application of exact methods. The Irish parliament, along with many other international chambers, is significantly smaller than that of the Spanish Congress of Deputies and it may be tractable within a functional time-frame.

Second, to develop a heuristic solution which can be applied to chambers of a much larger size, that not only robustly approximates the optimal solution of its associated exact approach, but which also overcomes the main obstacle faced by Hales & García (2019), allowing for the layering of "additional requirements".

Problem Definition

Dáil Éireann, commonly referred to as the Dáil (pronounced *dawl*), is the lower house and principal chamber of the Irish legislature. The chamber consists of 169 utilisable seats, which are occupied by members of parliament known as *Teachtaí Dála (TDs)*.

A simplified visualisation of the chamber's seats is shown in Figure 2.1. All visualisations in this chapter have been designed by the author using Python (Python Software Foundation, 2001–2020), employing the Matplotlib (The Matplotlib development team, 2012–2020) and NetworkX (NetworkX developers, 2014–2020) packages. The coordinates for each seat were manually generated by superimposing a graph onto a publicly available blueprint of the chamber using an online graphing tool. The author generated the seat numbers for reference and modelling purposes.

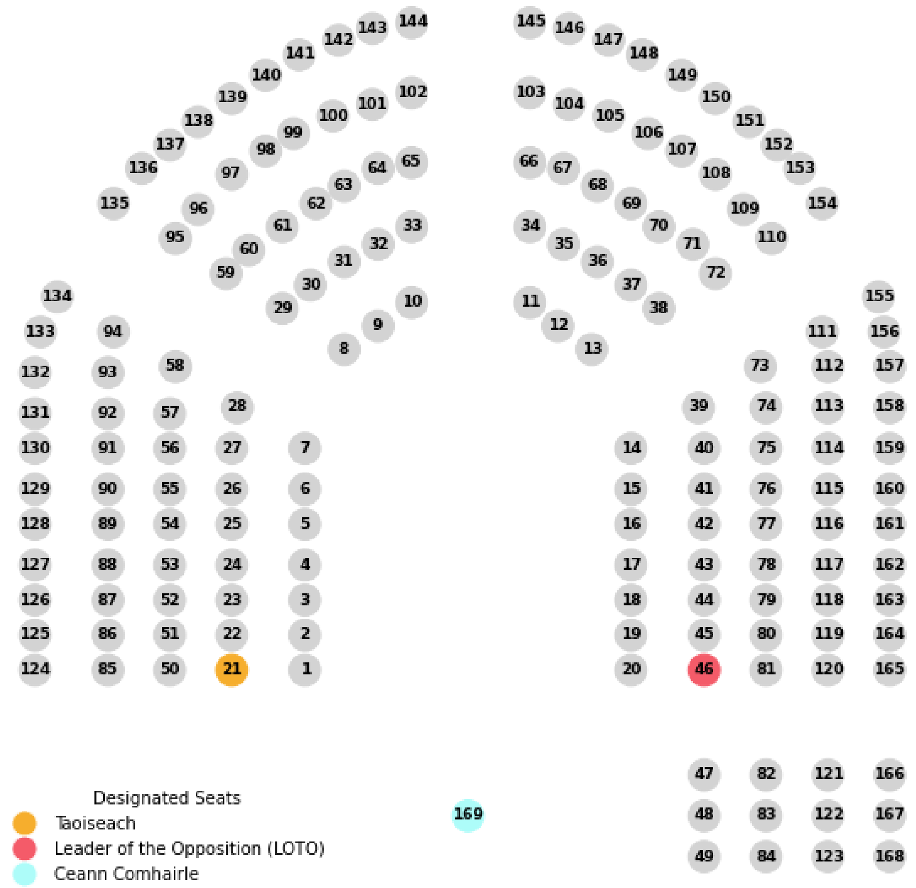

Figure 2.1: Seating Layout of the Dáil

The Dáil chamber has confrontational benches, but the end segment is curved to create a partial hemicycle. It is a combination of the two most prevalent parliamentary layout styles, with opposing benches in the British style and the semicircular nature of the European style. This layout is common in countries which once formed part of the British Empire and can also be found in the legislative chambers of Australia, India and New Zealand.

The chamber consists of five distinct sections, separated from one another by stairway aisles. There are five rows of seating circling the chamber, the upmost four of which are fronted by benches. Thus, the second row of seats, from seat 21 around to seat 46, is known as the *front bench*. These seats are generally regarded as the most prominent in the chamber. They are ordinarily occupied by the Government and the leaders and spokespeople of the larger opposition parties. The chair/speaker of the house is known as the *Ceann Comhairle*, and they occupy seat 169, as shown above.

Executive power in Ireland is vested in the Government, which consists of the *Taoiseach* (the head of Government, pronounced *thee-shuck*) and the government ministers. The parties currently forming Government have all of their members seated towards the left side of the speaker, with the Taoiseach occupying the nearest seat in the front bench (seat 21). The opposition parties sit to the right side of the speaker, and it is customary for the leader of the largest non-government party to occupy the seat directly opposing the Taoiseach (seat 46). This member is customarily referred to as the Leader of the Opposition (LOTO).

Government ministers, of which there are usually 15, generally occupy seats in the front bench. However, if a government majority is small, or if a minority government exists, some may sit in the front row instead, rather than extending into opposition territory.

The number of sitting TDs in the Dáil has historically varied from as few as 138 to as many as 166. With 169 seats, this has meant that some unallocated seats have always been present. The current Dáil, the 33$^{\text{rd}}$, consists of 160 members.

Independent TDs tend to occupy seats in the upmost rows of the chamber, but no explicit rule exists for their positioning. However, unless they form part of a coalition government, they must sit to the right of the government parties. As will be discussed in Section 2.4, these members must be allocated as part of a post-optimization procedure, so there is a need to ensure that adequate unallocated seats exist in the correct location during the optimization process.

Although usually a political party member, the Ceann Comhairle is expected to observe strict impartiality in all matters. Therefore, as there is no requirement for them to be seated in proximity to any political group, they can be excluded from the problem.

Thus, the allocation problem can be defined as follows:

> "Designate an appropriate number of seats to each political party such that the *compactness* of each party is maximized, while ensuring that the Taoiseach and the Leader of the Opposition occupy their traditional positions, the non-government parties are seated to the right of those in government, government parties have sufficient seats in the frontmost two rows to accommodate their ministers, and sufficient unallocated seats exist to the right of the government in which to seat Independent TDs."

The elements of this problem will now be translated into mathematical concepts that can be implemented as part of an optimization model.

2.1 Compactness

The pioneering paper by Hales & García (2019) on the application of political districting techniques to parliamentary seat allocation defined *compactness* in a similar geometric fashion to that pre-existing in the literature.

Distances between adjacent nodes were calculated as Euclidean distances, with network distances being used for non-adjacent nodes. Their overarching idea was to ensure geometric *compactness* which could be easily judged by visual inspection.

This paper proposes a concept of compactness which is grounded in the idea of *communication distance*, rather than geometric distance. It defines *communication distance* as the relative ease by which two adjacent seats can communicate discretely with one another. Network distances are similarly adopted to calculate the *communication distance* between non-adjacent seats. The theory behind this is that communication is possible between non-adjacent members, whether by note or by successive verbal delivery. However, notes can go astray, and whispers can become distorted, and as such cumulative network distances provide a reasonable estimation of the relative distance between non-adjacent seats.

This paper also proposes the related concept of *territorial dominance*, which it defines as each party's aim to occupy as much of a defined section of parliament as possible, relative to its size. The rationale for the additional inclusion of *territorial dominance* into the concept of *compactness* is that *communication distance* alone is often not sufficient to ensure a clear distinction between parties.

The paper posits that in the allocation of parliamentary seating, parties hold two primary objectives:

1. To be seated such that the *communication distance* between each member and *every other member* is minimized.

2. To be seated such that their *territorial dominance* is maximised.

2.1.1 Communication Distance

As underlined by Hales & García (2019), the concept of adjacency in the context of a parliamentary chamber is somewhat arbitrary, and a strict definition needs to be enforced to ensure consistency. Their approach was to calculate the Euclidean distance between all pairs of seats using their two-dimensional coordinates and to determine a threshold above which seats were no longer deemed to be adjacent.

The concept of adjacency using *communication distance* differs from this, in that it relies on the subjective possibility of direct communication between one member and another, unimpeded by a third member. This form of adjacency can result from seats being located side-by-side, having an overlap across adjacent rows, or from being located horizontally or diagonally across an aisle.

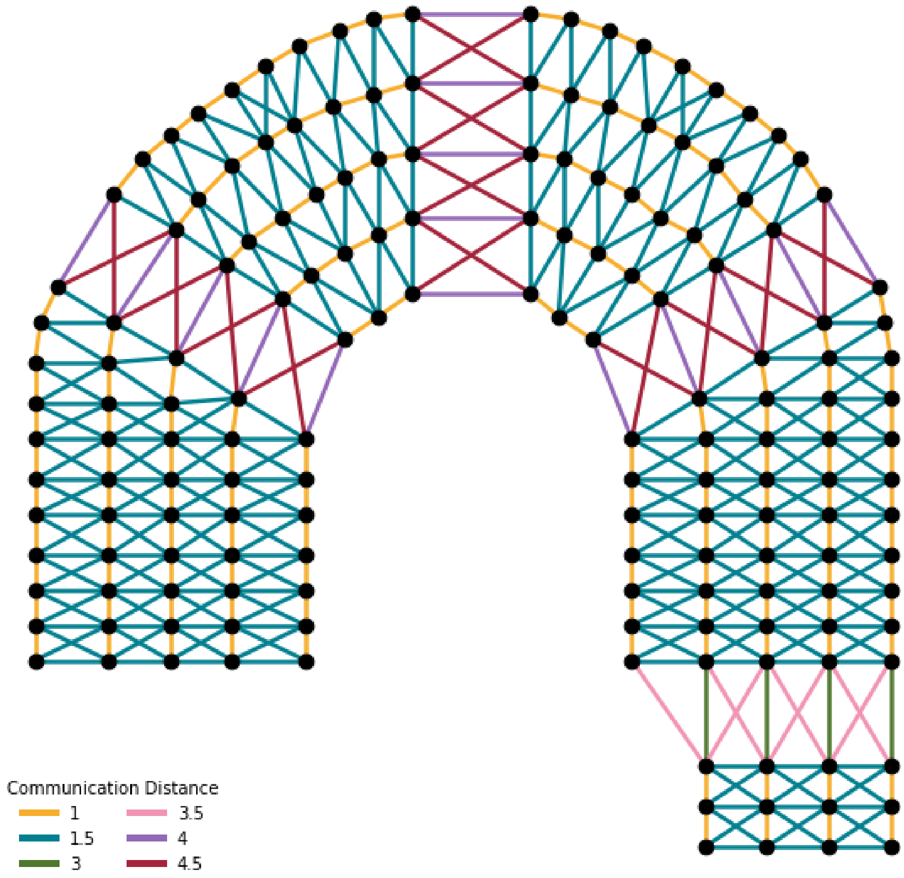

Figure 2.2: Communication Network of the Dáil

Figure 2.2 illustrates the adjacency graph of the proposed *communication distance* network within the Dáil. The colour of each edge corresponds to the *communication distance* between the incident nodes.

Many factors influence the ease by which members can communicate, which cannot be captured by two-dimensional geometric distances alone. Obstacles such as benches, aisles and even height differentials between rows can all serve to reduce the ease of communication. In order to numerically define *communication distances*, the relationship with the lowest possible amount of communication friction is set with a value of 1. This relationship corresponds with two seats which are located side-by-side. The next closest relationships are those in adjacent rows, where the seats have an overlap and are given a value of 1.5. Aisle-separated seats have larger distance values.

The use of *communication distances* has another advantage over existing methods, in that it enables mathematical optimization to be applied to parliaments in the British style. Hales & García (2019) admitted that a drawback of their definition of the adjacency matrix was that it was not utilisable where "seats are arranged in two banks separated by an aisle, [as] this would result in two disconnected graphs". Under the approach of *communication distance*, the ends of these banks could be easily connected by 'long' edges, generating a connected graph.

2.1.2 Territorial Dominance

Experimentation with the sole use of *communication distances* to model for *compactness* revealed some deficiencies. Notably, the disincentive for a party to spread itself across multiple chamber sections, rather than to fully occupy a single section was not sufficient. In order for a party to distinguish itself easily from others, it would ideally adhere to the natural boundaries of the chamber, where possible. This adherence should reduce the disputes that occur over a party's neighbours due to a fear of association (or misidentification), as the borders between parties are made more explicit overall.

The proposed solution to this shortcoming was to further increase the subjective communication distances across natural dividers in the chamber. In the case of the Dáil, this involved increasing the *width* of the aisles. These manual alterations involve some measure of trial and error, in order to ensure that the desired results are achieved. For example, it was found to be subjectively beneficial to increase the *width* of the rightmost aisle to a lesser extent than the other three. This refinement incentivised the filling of the rightmost two sections by a party before it would spread itself further around the chamber. This distinction in aisle widths can be seen in Figure 2.2.

Relatedly *territorial dominance* captures the desire for smaller parties to occupy a distinct placement in the chamber, rather than to be embedded within another party/parties. This inclination holds, even when this envelopment is associated with the possession of more prominent seats.

In terms of practical modelling considerations, the adjacency graph in Figure 2.2 is generated from an adjacency matrix, which is curated through the manual process outlined above. In order to obtain the corresponding shortest-path distance matrix, the Floyd-Warshall algorithm was employed using the SciPy package (SciPy developers, 2020) in Python.

2.2 Neutrality and Unbiasedness

As discussed in Chapter 1, the central philosophy of this paper is the ensuring of neutrality and unbiasedness in the allocation process. The automated models developed should be applicable to all possible electoral outcomes, without the need for manual constraint adjustment. *Fairness* is determined as the equal opportunity of all parties to obtain seats which are in keeping with existing parliamentary rules.

This approach differs significantly from that taken by Hales & García (2019), whereby they instead target strict equality of outcome in the allocation problem. Through the subjective imposition of additional *fairness* constraints, they attempted to generate a seating arrangement that strictly apportioned equally good seats to all parties. The results were mixed, and occasionally, these new constraints had unintended consequences, such as creating discontiguous party layouts. From a practical perspective, these subjective rules would also need to be mandated through parliamentary standing orders before they could be utilised.

This paper adopts an approach of non-interference in the application of constraints. It proposes *equity* based on a Rawlsian 'veil of ignorance', whereby all parties know that they have an equal *opportunity* to the apportionment of 'good' seats, even though some will be arbitrarily designated better seats than others by the model. This approach allows for the more critical assessment of the *equity* pertaining to existing rules and allows parliaments themselves to decide upon new regulations if deemed necessary. These new rules would apply to future parliaments, where the results of elections are unknown, and as such would remain credibly behind the 'veil'. Altman (1997) provides an insightful discussion on the difficulties of applying neutrality and unbiasedness to political districting problems, which has many parallels with this context.

The outcomes of the models later proposed will be evaluated through the lens of this laissez-faire approach to *equity*. The inherent *fairness* of the existing Dáil regulations will also be assessed.

2.3 Leftness/Rightness

While the requirement to seat non-government parties to the right of those in government sounds straightforward at first reading, a new concept of *leftness/rightness* must be developed to robustly model for this in a hemicycle-shaped parliament.

Hales & García (2019) encountered a somewhat similar requirement in the Spanish Congress of Deputies, whereby parties preferred to sit on the side of parliament which best aligned with their position on the left-right political spectrum. They addressed this custom by utilising the coordinate data of each seat to mandate that each party be restricted from certain areas in the chamber. This strategy involves significant manual adjustment for each election dataset, as the constraints are reliant on the number of seats apportioned to certain parties. This type of intervention contradicts with this paper's philosophy of ensuring a universally applicable model for all electoral outcomes.

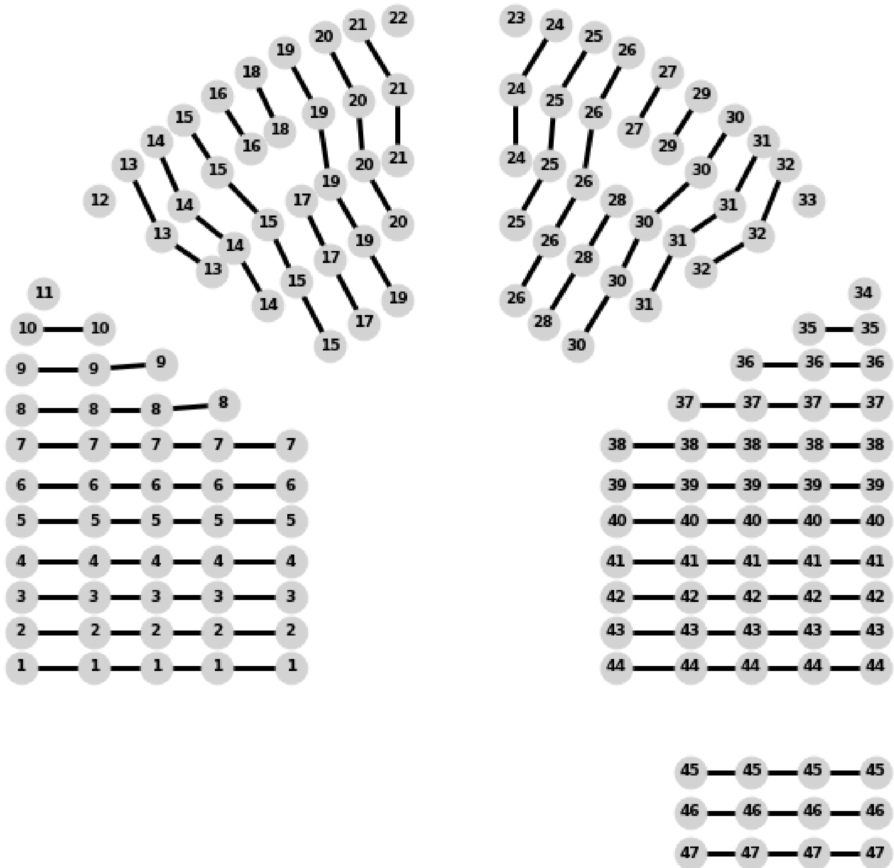

Figure 2.3: Columns of the Dáil Seats

To robustly model for this type of requirement, this paper introduces the concept of a seat *column*, which tracks the relative leftness/rightness of each seat within the chamber. These columns allow for the definition of a constraint by which the *column* of a seat allocated to a government party must not be greater than the *column* of any seat allocated to a non-government party. Figure 2.3 illustrates how this author has defined these *columns* for the Dáil. This approach is very easily adapted to accommodate alternative parliamentary layouts.

2.4 Independent Members

Independent TDs, who are not members of any political party, are a common feature of Irish politics and they play a more central role than in many other legislatures. This prominence is primarily as a result of the proportional representation voting system. In recent decades, Independent TDs have accounted for an average of about 10% of all parliamentary members.

The allocation of independent members of parliament was not a consideration of either Hales & García (2019) or Tuin (2019), as independents did not feature in the Spanish Congress of Deputies or either of the Dutch chambers. The most straightforward method of allocating these members would be to treat them as single-seat political parties, allowing the optimization algorithm to assign them arbitrarily to appropriate seats. Unfortunately, computational experimentation revealed that this method adds significant complexity to the model, as it introduces a significant range of new feasible solutions. These additional solutions must be explored and ruled out before an optimization algorithm can confirm the optimal solution. Thus, it can take much longer for the lower bound to converge with the best existing solution.

Accordingly, these independent members were removed from the optimization stage entirely, and instead, a constraint was introduced to ensure that an adequate number of appropriate unallocated seats were reserved. A post-optimization procedure was developed to assign the independent members to the suitable unallocated seats. The full detail of this post-op algorithm can be found in Appendix C.

2.5 Technical Groups

As outlined in Chapter 1, a greater level of complexity has been introduced to the parliamentary seat allocation problem of the Dáil in recent years, with the newfound introduction of *technical groups*. As parliamentary speaking rights are reserved for groups of 5 members or more, Independent TDs and small parties have recently tended to pool together in order to share these rights. These groups tend to prefer to be seated together as parties, even though they do not necessarily possess similar political ideology. An outstanding question exists as to whether these technical groups should be given equally favourable treatment as political parties in the allocation of seats. Strictly codifying the answer to this question, through the standing orders of the chamber, would serve to increase political transparency. Lacking a clear answer, this paper's models do not distinguish between political parties and technical groups.

Chapter 3

Modelling Approaches

After defining the problem being addressed in Chapter 2, the paper now moves on to proposing possible modelling formulations.

The choice of objective function is of utmost importance. A scoring metric must be selected which captures the essence of desired *compactness* as accurately as possible, while also allowing for an efficient formulation that can be solved in an appropriate time-frame.

First, the notation common to all formulations must be defined.

3.1 Notation

Sets

$$\begin{aligned} I &= \text{integer set of chamber seats.} \\ J &= \text{string set of political parties.} \end{aligned}$$

Fixed Data

$$\begin{aligned} c_i &= \textit{column} \text{ associated with each seat.} \\ r_i &= \textit{row} \text{ within which each seat is located.} \\ d_{ik} &= \textit{communication distance} \text{ between each pair of seats.} \end{aligned}$$

Variable Data

$$\begin{aligned} n^j &= \text{number of parliamentary members in party } j. \\ g^j &= \begin{cases} 1 & \text{if party } j \text{ is in government.} \\ 0 & \text{otherwise.} \end{cases} \\ t^j &= \begin{cases} 1 & \text{if party } j \text{ holds the office of Taoiseach.} \\ 0 & \text{otherwise.} \end{cases} \\ l^j &= \begin{cases} 1 & \text{if party } j \text{ holds the title of Leader of the Opposition.} \\ 0 & \text{otherwise.} \end{cases} \\ m^j &= \text{number of government ministers in party } j. \\ p &= \text{number of Independent TDs.} \end{aligned}$$

Decision Variables

$$z_i^j = \begin{cases} 1 & \text{if seat } i \text{ is assigned to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

3.2 Total Distance

The natural first candidate for the *compactness* scoring metric is the *communication distance* between each seat and every other seat allocated to the same party. The minimization of this metric encourages tightly-packed party arrangements. The associated model, defined as the *Total Distance* model, would be formulated as follows:

Objective Function

$$\min \sum_{\substack{i,k \in I:\\ i<k}} \sum_{j \in J} d_{ik} \cdot z_i^j \cdot z_k^j$$

Model Constraints

$$\sum_{j \in J} z_i^j \leq 1 \qquad \forall i \in I; \tag{3.1}$$

$$\sum_{i \in I} z_i^j = n^j \qquad \forall j \in J; \tag{3.2}$$

$$z_i^j \in \{0,1\} \qquad \forall i \in I,\ \forall j \in J; \tag{3.3}$$

The constraints can be understood in the following way:

- Equation (3.1) ensures that each seat is assigned to, at most, one party.
- Equation (3.2) ensures that each party must be allocated exactly the required number of seats for its parliamentary members.
- Equation (3.3) enforces that the z_i^j variables are binary.

This formulation is a quadratic problem, but one that can be easily linearised in the following manner. A new variable w_{ik}^j is introduced, such that:

$$w_{ik}^j = \begin{cases} 1 & \text{if seat } i \text{ and seat } k \text{ are both allocated to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

The new linear objective function as such becomes:

$$\min \sum_{\substack{i,k \in I:\\ i<k}} \sum_{j \in J} d_{ik} \cdot w_{ik}^j$$

The existing constraints remain unchanged, with the addition of linking constraints between the z_i^j and the w_{ik}^j variables:

$$z_i^j + z_k^j \leq w_{ik}^j + 1 \qquad \forall i,k \in I : i<k,\ \forall j \in J;$$

This problem formulation shares many elements with a proposed formulation for a table-allocation problem by Garcia *et al.* (2014). Their paper investigated the allocation of individuals to tables in order to maximize total *social benefit* at an event. As they showed, this type of quadratic linearisation is rather loose. However, through the implementation of *star inequalities* as defined by Sørensen (2004) in his paper on the *weighted clique* problem, it can be reasonably tightened. These *star inequalities* can be adapted to the *Total Distance* formulation as follows:

$$\sum_{\substack{k \in I:\\ i<k}} w_{ik}^j + \sum_{\substack{k \in I:\\ i>k}} w_{ik}^j \leq \left(n^j - 1\right) \cdot z_i^j \qquad \forall i \in I,\ \forall j \in J;$$

These constraints ensure that if seat i is not allocated to party j, then no pairs of the form (i, k) exist for this party and hence all $w_{ik}^{j} = 0$.

Other methods for tightening this type of formulation are explored by Garcia *et al.* (2014), such as imposing *triangle clique inequalities* and *symmetry-breaking inequalities* as cuts in a branch-and-cut framework. Due to the unpromising scalability of this formulation as illustrated by Garcia *et al.* (2014), even after implementing this branch-and-cut strategy, this paper does not attempt additional tightenings past the *star inequality* constraints.

The practical application of this method results in larger parties, who have a far greater number of seat connections, having an imbalanced contribution to the objective function. The following scaling factor should be introduced to maintain *equity* on a per-seat basis:

$$\sum_{a=1}^{n^j-1} a = \frac{\left(n^j - 1\right)\left(n^j\right)}{2}$$

Thus, the amended objective function would be:

$$\min \sum_{\substack{i,k \in I:\\ i<k}} \sum_{j \in J} \left(\frac{2d_{ik}}{n^j\left(n^j - 1\right)}\right) \cdot w_{ik}^{j}$$

Experimentation with the *Total Distance* formulation on a small scale showed promising results for the desired *compactness* of solutions. However, as will be shown in Chapter 4, the computational efficiency was severely deficient. The fully aggregated formulation of the *Total Distance* approach, including the incorporation of the Dáil-specific constraints, is outlined in Appendix A.

3.3 Communication Flows

The above-referenced paper by Garcia *et al.* (2014) inspired an alternative approach to the parliamentary allocation problem. They developed the concept of *communicability*, which measured "how well one person may actually communicate with other people". A *communicability matrix* can be developed by specifying for every pair of seats i and k, a value $f_{ik} \in [0, \alpha]$, where α is a fixed value representing perfect communication. Each element of this type of *communicability matrix* can be defined as the inverse of the corresponding element of the previously defined *communication* adjacency matrix.

The model formulation is quite similar to that of the *Total Distance* formulation, except that it now maximizes *communication flow* instead of minimizing *communication distance*. Also, crucially, it only calculates *communication flow* between adjacent seats within the same party, rather than calculating *communication distances* between all seats within the party. The hypothesis is, that while not capturing desired *compactness* quite as precisely as the *Total Distance* model, it should be much less computationally intensive, and quicker to solve.

The fundamental idea of *communication flow* was also adopted by Tuin (2019), although his approach was grounded in a network flow model, as opposed to a linear programming model. The *star inequality* tightening constraints implemented in the *Total Distance* formulation can also be applied here, and they have a marked impact on decreasing the run-time. The basic model is laid out below:

New Notation

f_{ik} = *communication flow* between each pair of adjacent seats.

Objective Function

$$\max \sum_{\substack{i,k \in I:\\ i<k}} \sum_{j \in J} f_{ik} \cdot z_i^j \cdot z_k^j$$

Modelling Constraints

$$\sum_{j \in J} z_i^j \leq 1 \qquad \forall i \in I;$$
$$\sum_{i \in I} z_i^j = n^j \qquad \forall j \in J;$$
$$z_i^j \in \{0, 1\} \qquad \forall i \in I, \ \forall j \in J;$$

These constraints and their explanations are identical to those seen in Section 3.2. Again, this is a quadratic model at heart, but one that can be easily linearised, with the introduction of a variable v_{ik}^j, such that:

$$v_{ik}^j = \begin{cases} 1 & \text{if seat } i \text{ and seat } k \text{ are adjacent, and both allocated to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

As the model is now maximizing, the linking constraints required are quite different to those seen in the *Total Distance* formulation:

$$v_{ik}^j \leq z_i^j \qquad \forall i, k \in I : i < k, \ \forall j \in J;$$
$$v_{ik}^j \leq z_k^j \qquad \forall i, k \in I : i < k, \ \forall j \in J;$$

The entire formulation is illustrated as a comprehensive model in Appendix A. Despite a seemingly promising objectification of *compactness*, experimental testing of the *Communication Flows* model revealed some serious flaws. As with the *Total Distance* model, it is necessary to scale the contributions of each party to the objective function. This scaling would need to be derived as the number of same-party adjacent seats summed over all of each party's seats. However, there is no obvious way for this to be derived. Where excess seats are present, and this form of scaling does not exist, the model often finds it advantageous to allocate smaller parties into non-contiguous arrangements to the benefit of larger parties. This result severely violates the aim of *fairness* across all seats and points to significant difficulty in the practical application of the model.

3.4 Minimum k-Partitioning

A third candidate for an exact approach to the problem was proposed in the paper by Hales & García (2019). They hypothesised that by minimizing the number of 'cut edges', i.e. the number of adjacent seats belonging to distinct parties, a *compact* solution would be indirectly encouraged.

They proposed the model as follows (some minor notational changes are made here for consistency):

New Notation

$$a_{ik} = \begin{cases} 1 & \text{if seat } i \text{ and seat } k \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Objective Function

$$\min \sum_{\substack{i,k \in I: \\ i<k}} \sum_{j \in J} a_{ik} \cdot z_i^j \cdot \left(1 - z_k^j\right)$$

Model Constraints

$$\sum_{j \in J} z_i^j \leq 1 \qquad \forall i \in I;$$
$$\sum_{i \in I} z_i^j = n^j \qquad \forall j \in J;$$
$$z_i^j \in \{0, 1\} \qquad \forall i \in I, \ \forall j \in J;$$

These constraints and their explanations are identical to those already seen in Section 3.2 and Section 3.3.

This proposed quadratic formulation can again be easily linearised, through the introduction of a variable u_{ik}^j, such that:

$$u_{ik}^j = \begin{cases} 1 & \text{if seat } i \text{ and seat } k \text{ are adjacent, but both are not allocated to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

The linking constraints are reminiscent of those seen in the *Total Distance* formulation, but with a slight adjustment:

$$z_i^j + \left(1 - z_k^j\right) \leq u_{ik}^j + 1 \qquad \forall i, k \in I : i < k, \ \forall j \in J;$$

Once again, the comprehensive formulation can be found in Appendix A. After experimental testing, it became clear that this formulation performs unpredictably in the presence of excess seats, as these excess seats are inadvertently grouped and treated as another party. This flaw was admittedly hinted at by Hales & García (2019), and they proposed a workaround whereby theoretical seats were added to parties for the optimization process before being subsequently removed. This tactic introduces some issues concerning bias, which will be explored in Section 3.7, where the Geometric Cutting Heuristic (GCH) based on this exact formulation is discussed.

It should be noted that the same *star inequalities* applied to earlier approaches were unable to be generated for this formulation, due to its logical basis. For this reason, the *Minimum k-Partitioning* formulation likely performs worse than would otherwise be expected when compared with the *Total Distance* and *Communication Flows* models.

3.5 Facility Location Model (FLM)

Another option for an exact formulation, and one that is well documented throughout the field of territory design, is to implement that of a capacitated facility location model. Indeed, FLMs have been proposed for solving political districting problems as far back as Hess *et al.* (1965), in what is generally considered as the earliest [Operational Research] paper in political districting (Ricca *et al.*, 2011).

An FLM was the second exact approach proposed by Hales & García (2019), and it formed the basis for their other heuristic approach - a Location-Allocation Heuristic (LAH). Instead of attempting to capture *compactness* by minimizing the distance between each seat and every other party seat, the FLM minimizes the distance between each seat and the party's chosen *centre seat*. This metric does perfectly capture desired *compactness*, and this is particularly visible for smaller parties where square arrangements are not explicitly encouraged over cross-shaped layouts. However, the sacrifice of a small measure of accuracy results in dramatic improvements to computational efficiency. The FLM also has the desirable feature of apportioning equal weight to each member, regardless of party size. This trait allows for a transparent objective function, without the need for complex weightings. It should be noted here that the concept of a *centre seat* has no substantive meaning in the context of a political party's seating, and is purely a modelling construct.

The FLM, including each of the Dáil-specific requirements outlined in Chapter 2, is defined as follows:

Decision Variables

$$x_{ik} = \begin{cases} 1 & \text{if seat } i \text{ is assigned to centre seat } k. \\ 0 & \text{otherwise.} \end{cases}$$

$$y_k^j = \begin{cases} 1 & \text{if centre seat } k \text{ is assigned to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

Objective Function

$$\min \sum_{i \in I} \sum_{k \in I} d_{ik} \cdot x_{ik}$$

Model Constraints

$$\sum_{k \in I} x_{ik} \leq 1 \qquad \forall i \in I; \tag{3.4}$$

$$\sum_{i \in I} x_{ik} = \sum_{j \in J} n^j \cdot y_k^j \qquad \forall k \in I; \tag{3.5}$$

$$\sum_{j \in J} y_k^j \leq 1 \qquad \forall k \in I; \tag{3.6}$$

$$\sum_{k \in I} y_k^j = 1 \qquad \forall j \in J; \tag{3.7}$$

$$x_{ik} \in \{0, 1\} \qquad \forall i, k \in I; \tag{3.8}$$

$$y_k^j \in \{0, 1\} \qquad \forall k \in I, \ \ \forall j \in J; \tag{3.9}$$

Dáil-specific Constraints

$$\sum_{k \in I} x_{ik} = \sum_{k \in I} y_k^j \qquad i = 21, \ \ \forall j \in J : t^j = 1; \tag{3.10}$$

$$\sum_{k \in I} x_{ik} = \sum_{k \in I} y_k^j \qquad i = 46, \ \ \forall j \in J : l^j = 1; \tag{3.11}$$

$$\sum_{k \in I} y_k^a \cdot c_k \leq \sum_{k \in I} y_k^b \cdot c_k \qquad \forall a \in J : g^a = 1, \ \ \forall b \in J : g^b \neq 1; \tag{3.12}$$

$$\sum_{\substack{i \in I: \\ r_i \leq 2}} x_{ik} \geq y_k^j \cdot m^j \qquad \forall k \in I, \ \ \forall j \in J; \tag{3.13}$$

$$\sum_{\substack{i \in I: \\ c_i \geq 23, r_i \geq 4}} \left(1 - \sum_{k \in I} x_{ik}\right) \geq p. \tag{3.14}$$

These constraints can be explained as follows:

- Equation (3.4) ensures that each seat is assigned to, at most, one *centre seat*.
- Equation (3.5) ensures that the number of seats assigned to each *centre seat* for each party is equal to the number of parliamentary members of that party.

- Equation (3.6) enforces that each seat can be a *centre seat* for, at most, one party.
- Equation (3.7) requires that precisely one *centre seat* is assigned to each party.
- Equation (3.8) and Equation (3.9) enforce that the x_{ik} and y_k^j variables are binary.
- Equation (3.10) and Equation (3.11) ensure that the appropriate seats are allocated to the parties that hold the roles of Taoiseach and Leader of the Opposition.
- Equation (3.12) enforces that the *centre seats* of all government parties must be to the left of the *centre seats* of all opposition parties. This constraint represents a slight, but beneficial, relaxation to the previous formulations where *all* seats were required to be non-overlapping. This flexibility allows for the suggestion of some otherwise infeasible (and occasionally preferable) results.
- Equation (3.13) ensures that each party is allocated sufficient seats in the front two rows to accommodate their government ministers.
- Equation (3.14) ensures that there are enough unallocated seats to the rear-right of the chamber to seat the Independent TDs in the post-optimization procedure.

Post-Optimization Allocation of Independents

When the optimization process has been completed, and all parties have been designated seats, the Independent TDs must be then allocated to the seats that have been purposely left vacant for them.

A post-optimization procedure is run, by which empty seats on the right side of the chamber are filled in descending index order. This algorithm is detailed in Appendix C.

Scaling

Hales & García (2019) discuss the possible need to scale each party's contribution to the objective function for the FLM. They make the argument that larger parties are given slight favour over smaller groupings. They found that by scaling the objective function appropriately, that smaller parties did gain slight increases to their *compactness*, but usually at the expense of being embedded within larger parties. As outlined in Subsection 2.1.2, this form of envelopment is undesirable under this paper's definition of *territorial dominance*. Ultimately, Hales & García (2019) chose not to adopt their discussed scaling in their final models, and this paper will do likewise.

3.6 Location-Allocation Heuristic (LAH)

As described above, Hess *et al.* (1965), in their seminal paper, proposed an FLM model for the application of mathematical optimization to political districting. However, considering the scale of problems generally under consideration in this field, the combinatorial complexity of the FLM approach had limited practical application.

Hess *et al.* (1965) suggested the utilisation of a Location-Allocation approach to address this deficiency. This method involves the separation of the *location* stage of the problem, where the *centre seats* are determined, from the *allocation* stage, where seats are assigned to those *centre seats*. The *location* problem is a simple 1-median problem, while the *allocation* problem is formulated as a capacitated assignment problem.

The LAH approach was proposed by Hales & García (2019) as a mechanism by which to approximate their FLM model for larger parliaments. They outlined the central approach as follows (with some minor adjustments made here for notational consistency):

Algorithm 1: Location-Allocation Heuristic (Hales & García)

Step 1: Randomly designate one initial central seat, $\hat{o}^j \in I$ for each party $j \in J$;

Step 2: Solve the allocation problem (see below) to obtain the solution $\left\{ z_i^j : i \in I, \;\; j \in J \right\}$;

Step 3: Calculate the new central seats, given by

$$\tilde{o}^j = \underset{k=1,\ldots,n}{\arg\min} \sum_{i \in I} d_{ik} \cdot z_i^j$$

Step 4: If $\tilde{o}^j = \hat{o}^j$ for all $j \in J$ then output the solution and stop. Else, set $\hat{o}^j = \tilde{o}^j$ for all $j \in J$ and return to step 2.

The allocation problem, in the context of the Dáil, would be as follows:

New Notation

$\hat{o}^j$ = the *centre seat* of party j from the *location* stage.

$M = \left(\max_{e \in I} \{c_e\} - 1 \right)$, the maximum difference between *columns*.

Objective Function

$$\min \sum_{i \in I} \sum_{j \in J} d_{i\hat{o}^j} \cdot z_i^j$$

Constraints

$$\sum_{j \in J} z_i^j \leq 1 \qquad \forall i \in I;$$

$$\sum_{i \in I} z_i^j = n^j \qquad \forall j \in J;$$

$$z_i^j \in \{0, 1\} \qquad \forall i \in I, \;\; \forall j \in J;$$

$$z_i^j = 1 \qquad \forall i \in I : i = 21, \;\; \forall j \in J : t^j = 1;$$

$$z_i^j = 1 \qquad \forall i \in I : i = 46, \;\; \forall j \in J : l^j = 1;$$

$$\sum_{\substack{c \in J: \\ g^c = 1}} z_a^c \cdot c_a \leq \sum_{\substack{d \in J: \\ g^d \neq 1}} z_b^d \cdot c_b + \left(1 - \sum_{\substack{d \in J: \\ g^d \neq 1}} z_b^d \right) \cdot M \qquad \forall a, b \in I; \tag{3.15}$$

$$\sum_{\substack{i \in I: \\ r_i \leq 2}} z_i^j \geq m^j \qquad \forall j \in J;$$

$$\sum_{\substack{i \in I: \\ c_i \geq 23, r_i \geq 4}} \left(1 - \sum_{j \in J} z_i^j \right) \geq p$$

The allocation problem is highly similar to the FLM formulation, except that the *centre seats* are now fixed. This variation means that the constraint defined in the FLM whereby 'no *centre seat* of a government party can be to the right of a *centre seat* of a non-government party' can no longer be used (Equation (3.12)). Instead, the stricter Equation (3.15), must be introduced, which enforces that 'no government party can be allocated a seat which is to the right of an opposition party's seat'.

As outlined in Algorithm 1, the iterative procedure of solving the *location* and *allocation* problems continues until the *centre seats* converge. It is important to note that the starting *centre seats* are chosen randomly.

Hales & García (2019) applied the LAH model to the Spanish Congress of Deputies. They found that the model produced inferior results to those of their Geometric Cutting Heuristic (GCH), and in particular, that the approach failed to produce reliably contiguous solutions. Upon inspection by this author, however, it was noticed that only ten random-start algorithm iterations were performed, resulting in a very low probability that the model would produce satisfactory results. An investigation was conducted into why so few iterations were performed of what should be a computationally efficient heuristic.

Experimentation with the LAH formulation revealed that the *allocation* phase of the heuristic was indeed much slower than would be expected. The source of the inefficiency in the Dáil model was identified as Equation (3.15), with its removal resulting in a ten-fold decrease to the algorithm's run-time. This equation adds $|I| \times |I|$ constraints to the *allocation* problem, which amounts to almost 30,000 constraints for the Dáil. Hales & García (2019)'s model had an equation consisting of $|I| \times |J|$ constraints, which amounted to over 2,500 constraints for their application.

The *allocation* problem is still rapid in comparison to the FLM problem. However, it must be solved multiple times per LAH iteration, and ideally, a large number of iterations are conducted in order to maximize the probability that the heuristic approximates the FLM. This complication explains why Hales & García (2019) experienced sub-optimal results with this approach.

This paper determined that instead of imposing these constraints directly, an optimal strategy would be to remove them and to introduce a *filtering* mechanism whereby any solutions not adhering to the absent set of constraints would be discarded.

This strategy was improved with the realisation that a *nuanced* random-start approach could boost the number of feasible solutions. This boost would increase the probability that the LAH finds a better approximation of the FLM. As one seat is predetermined for the party holding the role of Taoiseach, and another for the party possessing the role of LOTO, these seats could be fixed as the starting *centres* for these parties. Similarly, as the other government and opposition parties sit on the left and right sides of the chamber, respectively, their starting *centres* could be restricted to be randomly selected only from their appropriate side. As will be illustrated in Chapter 4, these tactics ensured that an average of over 90% of solutions remained feasible, while decreasing the algorithm run-time by a factor of ten. This technique resulted in a substantial rise in the number of iterations possible, and a corresponding improvement in the heuristic's ability to approximate the exact model.

An outline of the revised LAH algorithm is provided in Algorithm 2. The rigorous mathematical formulation can be found in Appendix B.

Algorithm 2: Location-Allocation Heuristic (Condensed)

```
initialization;
best score := ∞;
while the number of desired iterations has not yet been reached do
    set the starting centre of the Taoiseach's party to seat 21;
    set the starting centre of the LOTO's party to seat 46;
    set the starting centres of other government parties randomly to the left of the chamber;
    set the starting centres of other opposition parties randomly to the right of the chamber;
    while the centres before and after the LAH process have not converged do
        solve the Allocation Problem to determine the best seats, given fixed centres;
        solve the Location Problem to determine the best centres, given fixed seats;
    determine if the incumbent LAH solution is feasible under the removed constraint;
    calculate the score of the incumbent solution;
    update the best solution if the incumbent score is lower and feasible;
allocate Independents to the best solution as per Appendix C;
```

3.7 Geometric Cutting Heuristic (GCH)

The most successful model derived by Hales & García (2019) was a heuristic grounded in graph partitioning and computational geometry and based on the *Minimum k-Partitioning* formulation. It was inspired by a heuristic developed by Kalcsics *et al.* (2005) for the territory design problem. The approach uses straight-line cuts to partition the connected graph into appropriate blocks of parties while attempting to cut as few edges as possible. It begins by randomly dividing parties into two groups and finding a geometric line that separates them while cutting the fewest possible edges. It then considers each subgroup independently and repeats the procedure iteratively until all parties have been divided.

As their paper concluded, this model could "produce visually appealing seating plans for basic cases, but problems [could] occur when there [were] additional requirements to be satisfied." As the original heuristic was generated for the application to territory design, it was not built with the need for additional constraints in mind. Hales & García (2019) worked around this limitation using a similar *filtering* mechanism as this paper adopts for the LAH model, whereby solutions violating non-present constraints are removed. This tactic was viable for the Spanish Congress of Deputies, where few explicit seating rules existed. After 1,000 iterations of the GCH algorithm, only six solutions remained feasible; however, each represented a visually appealing result. In the case of Dáil Éireann and other parliaments with a more extensive set of nuanced requirements, this strategy would be much less rewarding, as it would be highly unlikely that any feasible solutions would be produced.

A second challenge with the application of this model stems from the perspective of neutrality. This heuristic formulation, as well as the *Minimum k-Partitioning* formulation upon which it is based, requires that no excess seats exist within the chamber. Hales & García (2019) proposed a workaround whereby several artificial members were assigned to each party for the optimization stage, before being subsequently removed. This tactic requires manual human determination of how these artificial members are allocated, however, and allows for a possible route whereby bias could enter the model. For these two reasons, the GCH was not explored in further depth in this paper.

The paper now moves on to the computational evaluation of the four exact approaches and the LAH model discussed above.

Chapter 4

Computational Study

All computational experiments were performed using FICO Xpress Optimization (FICO, 2020) on a 1.4GHz Intel Dual-Core i7 processor with 8GB RAM. The details for accessing all model code and the datasets tested can be found in Appendix D.

The computational performance of each of the proposed formulations from Chapter 3 was assessed against a range of increasing parliament sizes. These *parliaments* were generated using the Dáil chamber as a template. The four smaller parliaments consisted of the first two, three, four and five rows of the Dáil, respectively. Four larger parliaments were also generated by adding additional hypothetical rows to the Dáil chamber, in the same pattern as the existing seat layout. These eight datasets spanned a range of parliament sizes from 49 to 402 seats, with each embodying the complexity of a realistic chamber layout.

For experimental consistency across all parliament sizes, data with a comparable level of complexity was required for each. Later results will show that the data from the 33$^{\text{rd}}$ Dáil was the most complex of the historical datasets, so this dataset was chosen as the template. New datasets were generated from the template for each of the eight parliament sizes, by scaling the party data per the parliament size.

4.1 Exact Formulations

Each of the four proposed exact formulations was tested using each of the eight parliament sizes and the corresponding scaled dataset. Each formulation consisted of all of the necessary modelling constraints, any tightening constraints, and each of the Dáil-specific constraints. The complete models for each approach are documented fully in Appendix A. The maximum run-time for each model was set to one hour. Table 4.1 illustrates the time taken, in seconds, for each of these formulations to solve to optimality for each parliament size. A result in parentheses indicates that the formulation was unable to solve to optimality within the maximum time allowed, and thus instead represents the optimality gap outstanding after that time. Subsequent datasets were not tested when this occurred.

	Seats							
Model	49	84	123	168	219	274	335	402
LP - Min. k-Partitioning	*(28%)*	-	-	-	-	-	-	-
LP - Total Distance	730	*(75%)*	-	-	-	-	-	-
LP - Comm. Flows	4	624	*(8%)*	-	-	-	-	-
FLM	5	27	563	2832	*(33%)*	-	-	-

Table 4.1: Run-time for each exact formulation, in seconds

These results indicate the inadequacy of the first three exact formulations at solving parliaments of any meaningful size within a reasonable time-frame. As illustrated in Chapter 3, the *Total Distance* model represented the most accurate adherence to the desired definition of *compactness*. However, it suffers acutely from combinatorial explosion and is not usable in its current state.

For further insight into the issue of computational complexity, Altman (1997) provides an in-depth commentary for the related field of political districting.

The FLM formulation, however, works well for a parliament of the size of the Dáil (168 seats). It manages to solve this parliament size in less than 1 hour, even for this challenging dataset. Other historical Dáil datasets were solved in significantly shorter times, as can be seen below. However, when this model was applied to a 219-seat parliament, it appeared unlikely to be solved within a meaningful time-frame. Accordingly, the application of the FLM model should be restricted to problems of about 200 seats or fewer, depending on the complexity of the dataset and the allowable run-time.

4.2 FLM and Dataset Complexity

With dataset complexity having been mentioned several times now, it is important to understand the factors which influence problem difficulty for the FLM model. Table 4.2 details the two key factors which affect run-times: the number of parties and the number of *excess* seats. These features, as well as the run-time for the FLM model, are documented across the six available historical datasets.

The number of parties here is inclusive of technical groups. *Excess* seats are defined as the number of empty seats in the chamber, plus the number of unaffiliated independents, as this represents the number of unallocated seats at the optimization stage of the model.

	Dáil					
	28^{th}	29^{th}	30^{th}	31^{st}	32^{nd}	33^{rd}
Number of Parties	5	6	6	5	10	10
Number of Empty Seats	3	3	3	3	11	9
Number of Unaligned Independents	10	14	5	15	2	0
Total *Excess* Seats	13	17	8	18	13	9
Run-time, including Dáil constraints (s)	755	285	1020	860	2105	2832

Table 4.2: Run-times of the FLM, in seconds, alongside dataset complexity

These results show that a greater number of parties generally corresponds with a longer run-time, as displayed by the 32^{nd} and 33^{rd} Dáils. This finding is consistent with that of both Hales & García (2019) and Tuin (2019).

Holding the number of parties constant, a higher number of *excess* seats typically coincides with a shorter run-time. This trend can be seen in the differences in run-times between the 33^{rd} and 32^{nd}, as well as the 30^{th} and 29^{th} Dáils. Hales & García (2019) and Tuin (2019) each conducted their analyses without any excess seats present, so this represents a new finding.

In order to ensure that the FLM model could be applied to parliaments with differing rules and traditions from the Dáil, it is necessary to evaluate the run-times of the model without the Dáil-specific constraints. Table 4.3 documents these results.

	Dáil					
	28^{th}	29^{th}	30^{th}	31^{st}	32^{nd}	33^{rd}
Run-time, w/ Dáil constraints (s)	755	285	1020	860	2105	2832
Run-time, w/o Dáil constraints (s)	1903	206	1900	468	2050	5649

Table 4.3: Run-times of the FLM w/ and w/o the Dáil specific constraints, in seconds

The results show that the Dáil-specific constraints have no conclusive impact on run-times. They neither reliably simplify the model through tighter constraints, nor overly complicate it through the addition of a large number of them. The run-times are longer for particular datasets, and shorter for others, indicating that some combination of these two impacts is at play, depending on the dataset in question. The model embodies the reliability necessary for its application to other parliamentary settings.

The FLM is a computationally robust candidate for parliamentary allocation problems consisting of approximately 200 seats or fewer. Whether this computational robustness is paired with visual consistency will be evaluated in Chapter 5. Now, however, the paper moves to the evaluation of the heuristic approach, proposed for problem sizes over 200 seats.

4.3 LAH Performance

The parameter for the desired number of iterations of the LAH model is set to 100 for experimental purposes, and Table 4.4 clearly outlines the computational benefits of adopting a heuristic approach. The run-times do increase with the size of the problem in a non-linear fashion. However, the growth rate is significantly lower than the FLM's, and a 402-seat problem can be tackled in under 20 minutes. Again, the results in parentheses represent the percentage optimality gap for the FLM after 1 hour.

	Seats							
Model	49	84	123	168	219	274	335	402
FLM	5	27	563	2832	*(33%)*	*(54%)*	*(64%)*	*(73%)*
LAH (100 iterations)	27	66	104	158	310	368	767	1032

Table 4.4: Run-times of the LAH compared with the FLM, in seconds

Satisfied with the computational efficiency of the LAH approach, an assessment must now be made as to its numerical approximation of the FLM. In Table 4.5, the scoring metrics of the FLM solution and the best LAH solution are compared for corresponding datasets. For the smaller four datasets, the FLM solution represents the optimal solution, and for the larger four, it represents the best solution found within the allowable run-time of 1 hour.

The number of eligible solutions for the LAH model is also assessed. These represent the number of iterations which are feasible even under the removed government-positioning constraint.

	Seats							
Model	49	84	123	168	219	274	335	402
FLM	61.5	146.5	246	379	*(613.5)*	*(1026)*	*(1482)*	*(2157)*
LAH (100 iterations)	62.5	148.5	252.5	388	575.5	793.5	1073	1423
LAH Gap (%)	1.63	1.37	2.64	2.37	-6.19	-22.66	-27.60	-34.03
LAH Eligible Solutions	100	92	96	99	99	97	83	91

Table 4.5: Scores for the LAH compared with the FLM

These results show that the numeric LAH solutions provide an approximation of the FLM solution within 3% of the optimal, even when assessing this computationally challenging dataset. For larger datasets, it could provide significantly superior solutions than the FLM found within the stated maximum run-time. The magnitude of this difference grows more substantial as the problem size increases.

The table also highlights the efficacy of the *nuanced random-start* approach, which ensures that an average of over 90% of all LAH solutions remains feasible under the removed constraint.

Chapter 5

Illustrative Results

The graphics in this chapter were generated using Python (Python Software Foundation, 2001–2020), employing the Matplotlib (The Matplotlib development team, 2012–2020) and pandas (the pandas development team, 2008–2020) packages.

The solutions from the FLM model are compared against the actual seating arrangements for the last six Dáil terms. This comparison allows for visual confirmation that the scoring metric used provides an appropriate *compactness* objective. The solutions from the LAH model are then compared against those of the FLM model, to confirm that their close numeric results are matched by analogous visual outcomes.

The data for the actual seating arrangements was kindly provided by the Houses of the Oireachtas Library and Research Service. Their records document the seating allocations for the Dáil beginning in March 2002 when electronic voting was introduced to the chamber. The data for each Dáil depicts the political situation at the beginning of its term, except for that of the 28$^{\text{th}}$ Dáil. It depicts the situation towards its end when the records began.

The data used for each Dáil varies slightly from the election results, as the Ceann Comhairle is excluded, parties with only one parliamentary member are combined with Independents, and certain small parties are merged with their aligned technical group.

The LAH model was set to perform 100 iterations, and as can be seen in Table 5.1, ran to completion for each dataset in under 5 minutes. At this speed, the iteration parameter could be increased to maximise the possibility that the model finds a solution even closer to that of the FLM. However, the purpose here is to show that the LAH model provides a reliable approximation of the FLM solution, even for a relatively low number of iterations. This provides evidence that 100 iterations would be sufficient for larger parliaments, where the time-per-iteration increases.

	Dáil					
Model	28$^{\text{th}}$	29$^{\text{th}}$	30$^{\text{th}}$	31$^{\text{st}}$	32$^{\text{nd}}$	33$^{\text{rd}}$
FLM	755	285	1021	1454	2105	2832
LAH (100 iterations)	69	127	115	148	266	148

Table 5.1: Run-times for Historical Dáil Datasets

Table 5.2 collates the scores from each of the datasets and calculates the percentage optimality gap from the LAH model when compared with the FLM solution. The robustness of the LAH model at numerically approximating the FLM solution is illustrated clearly here. Most solutions are within a 1% gap, while even the most challenging dataset, of the 33$^{\text{rd}}$ Dáil, is within 3% of the optimal.

	Dáil					
Model	28$^{\text{th}}$	29$^{\text{th}}$	30$^{\text{th}}$	31$^{\text{st}}$	32$^{\text{nd}}$	33$^{\text{rd}}$
Actual Seating	723.5	771.5	797.5	664.5	476.5	-
FLM	679.5	632	684.5	619	404.5	379
LAH (100 iterations)	684	636	690	619.5	407.5	388
LAH Gap (%)	0.66	0.63	0.80	0.08	0.74	2.37

Table 5.2: Scores for Historical Dáil Datasets

5.1 28th Dáil, March 2002

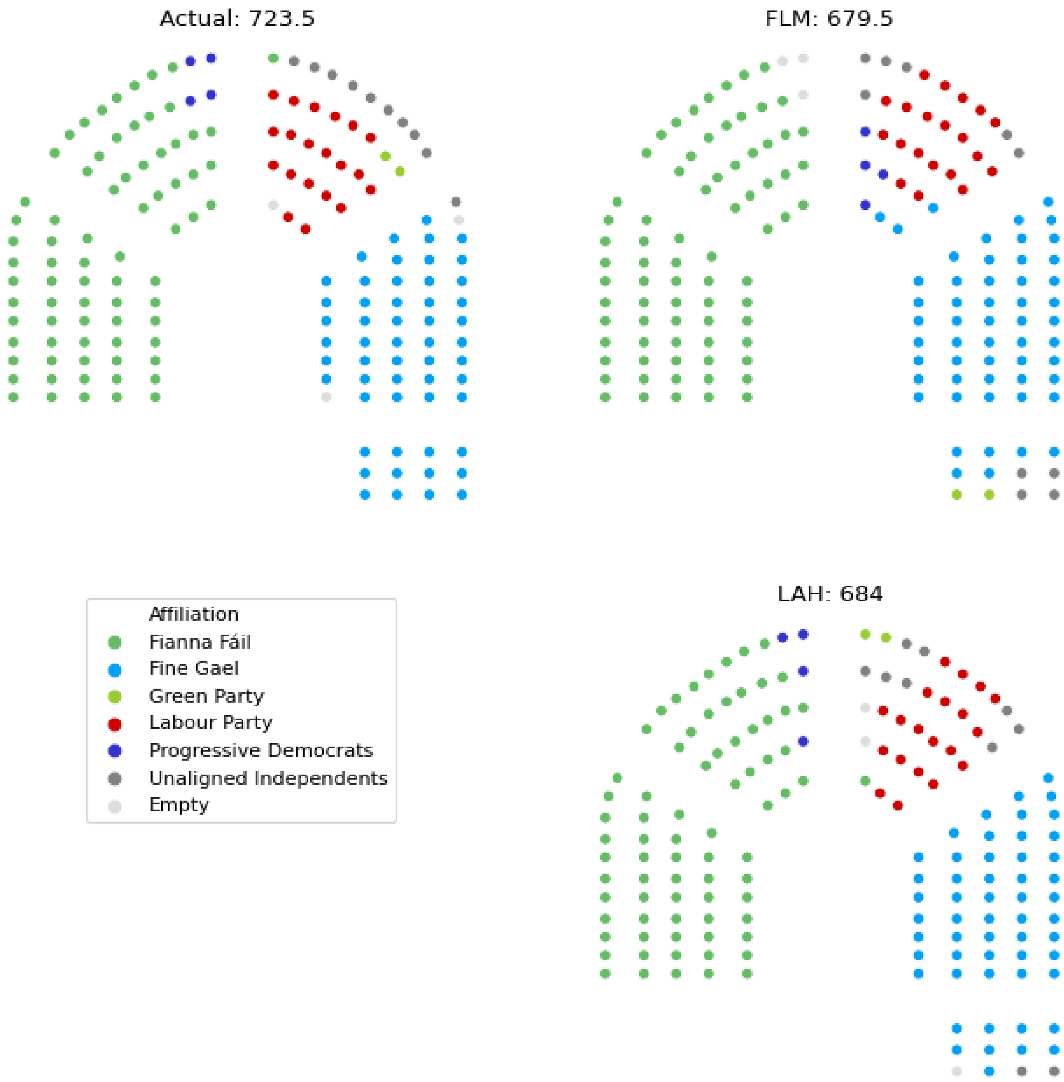

Figure 5.1: 28th Dáil Solutions

	Members	Government	Taoiseach	LOTO	Ministers
Fianna Fáil	75	Y	Y	-	14
Fine Gael	54	-	-	Y	-
Green Party	2	-	-	-	-
Labour Party	20	-	-	-	-
Progressive Democrats	4	Y	-	-	1
Unaligned Independents	10				
Empty Seats	3				

Table 5.3: Data for the 28$^{\text{th}}$ Dáil

In March 2002, towards the end of the 28$^{\text{th}}$ Dáil, Fianna Fáil and the Progressive Democrats (PDs) were governing in a coalition. Unusually, although the PDs had a government minister, they did not possess any front bench seats. This arrangement highlights that the ministerial constraint in this paper's models represents a custom, but not a strict rule. Fianna Fáil did not have a fully contiguous formation, with one seat stranded from the rest of the party. Fine Gael almost completely dominated two entire sections of the chamber, displaying compelling *territorial dominance*.

In order to fulfil the requirement that the PDs be given a front bench seat, the FLM model relocates them past the midway point and into the next section. This requirement has a knock-on effect on the Labour Party, giving them less prominent positioning. This outcome calls into question the inherent *fairness* of the ministerial constraint, as it has advantaged a smaller party to the detriment of a much larger one. This constraint may be requested for removal by the Dáil, or a more nuanced rule may need to be developed.

In the FLM solution, Fine Gael has seats now spread across three sections, instead of filling up two, which is less desirable from the perspective of *territorial dominance*. This shortfall highlights that further adjustments to the *width* of aisles in the adjacency matrix might be beneficial. The methodology for doing so was described in Subsection 2.1.2.

The LAH model suggests a solution that is visually closer to the actual seating than that of the FLM. Fianna Fáil and the PDs retain their positions, except that the discontiguous Fianna Fáil seat is relocated, and the PDs have a seat moved to the front bench to meet the ministerial constraint, which breaks their continuity. Altogether, it provides a suitable alternative to the FLP solution.

5.2 29th Dáil, June 2002

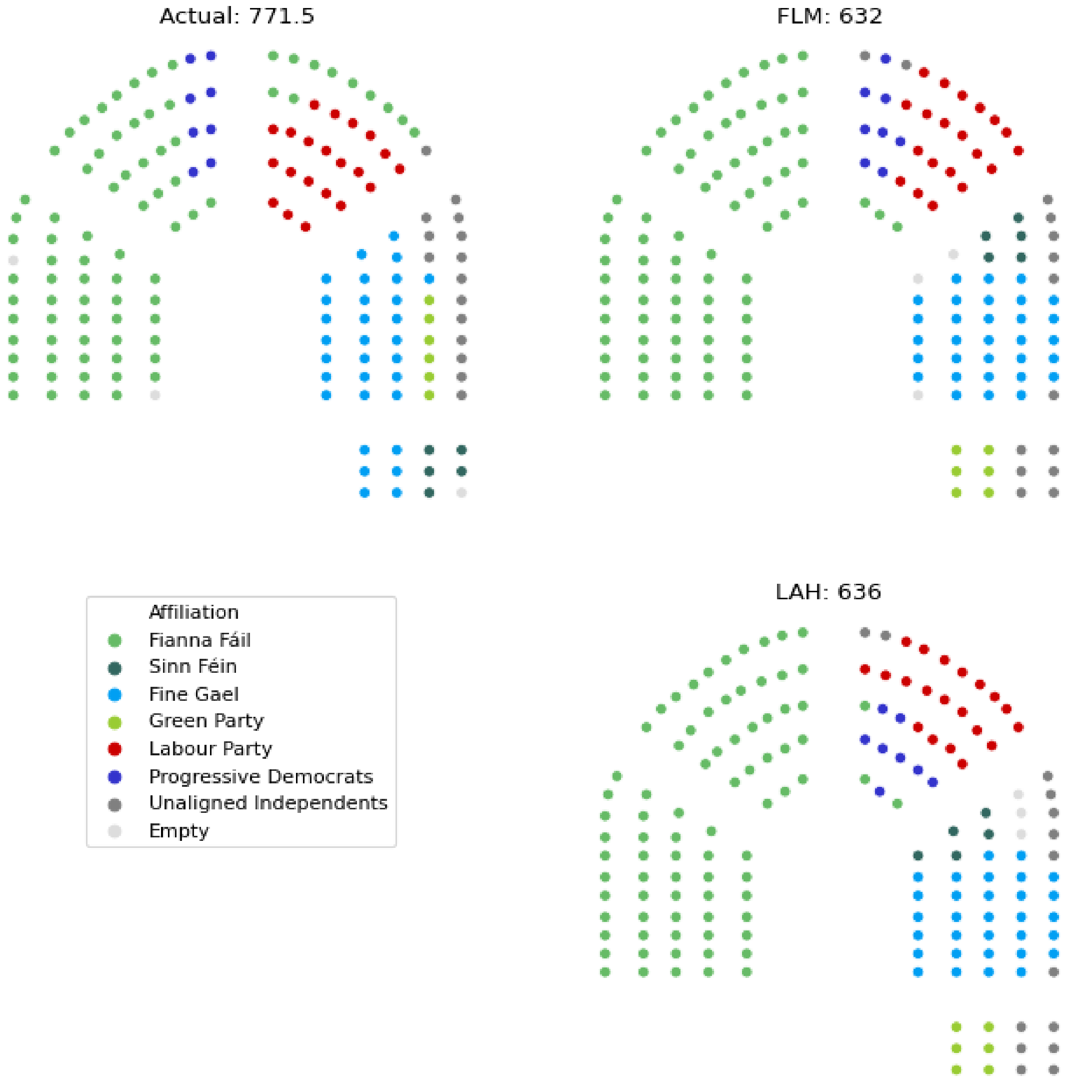

Figure 5.2: 29th Dáil Solutions

	Members	Government	Taoiseach	LOTO	Ministers
Fianna Fáil	81	Y	Y	-	13
Sinn Féin	5	-	-	-	-
Fine Gael	31	-	-	Y	-
Green Party	6	-	-	-	-
Labour Party	20	-	-	-	-
Progressive Democrats	8	Y	-	-	2
Unaligned Independents	14				
Empty Seats	3				

Table 5.4: Data for the 29$^{\text{th}}$ Dáil

After a general election in May 2002, Fianna Fáil and the PDs retained power and increased the size of their majority. They also maintained their relative seating positions, which resulted in an even more pronounced break in contiguity for Fianna Fáil. Fine Gael occupied an excessive number of front row seats relative to their parliamentary numbers, and at the expense of the Green Party and Sinn Féin, in what appeared to be a rather inequitable arrangement.

The FLM solution corrects for a number of these issues. Fianna Fáil gains a contiguous arrangement, with *territorial dominance* of two entire sections. Fine Gael are assigned a more *compact* arrangement with a number of prominent seats more proportionate to their size. The Green Party and Sinn Féin both benefit from being moved to more prominent locations. Only the Labour Party's outcome is ambiguous. On the one hand, they have lost some of their most prominent seats, mainly due to the PDs being given front bench seats per their two ministries. On the other hand, it could be argued that their *territorial dominance* has improved, as they are no longer encircled by Fianna Fáil to their rear.

The LAH solution provides a reasonably close approximation of the FLM solution; however, some issues are present. Primarily, the arrangement of the PDs and the Labour Party is unlikely to be to either's satisfaction. The Labour Party lose some of their more prominent seats, while the PDs lose significant *territorial dominance*, being encased between Labour and Fianna Fáil. If the LAH model was adopted, however, it seems likely that this proposed seating arrangement would lead to the trading of seats between the PDs and Labour, as this would be to their mutual benefit. A finalised arrangement close to that of the FLM might then be possible.

5.3 30th Dáil, June 2007

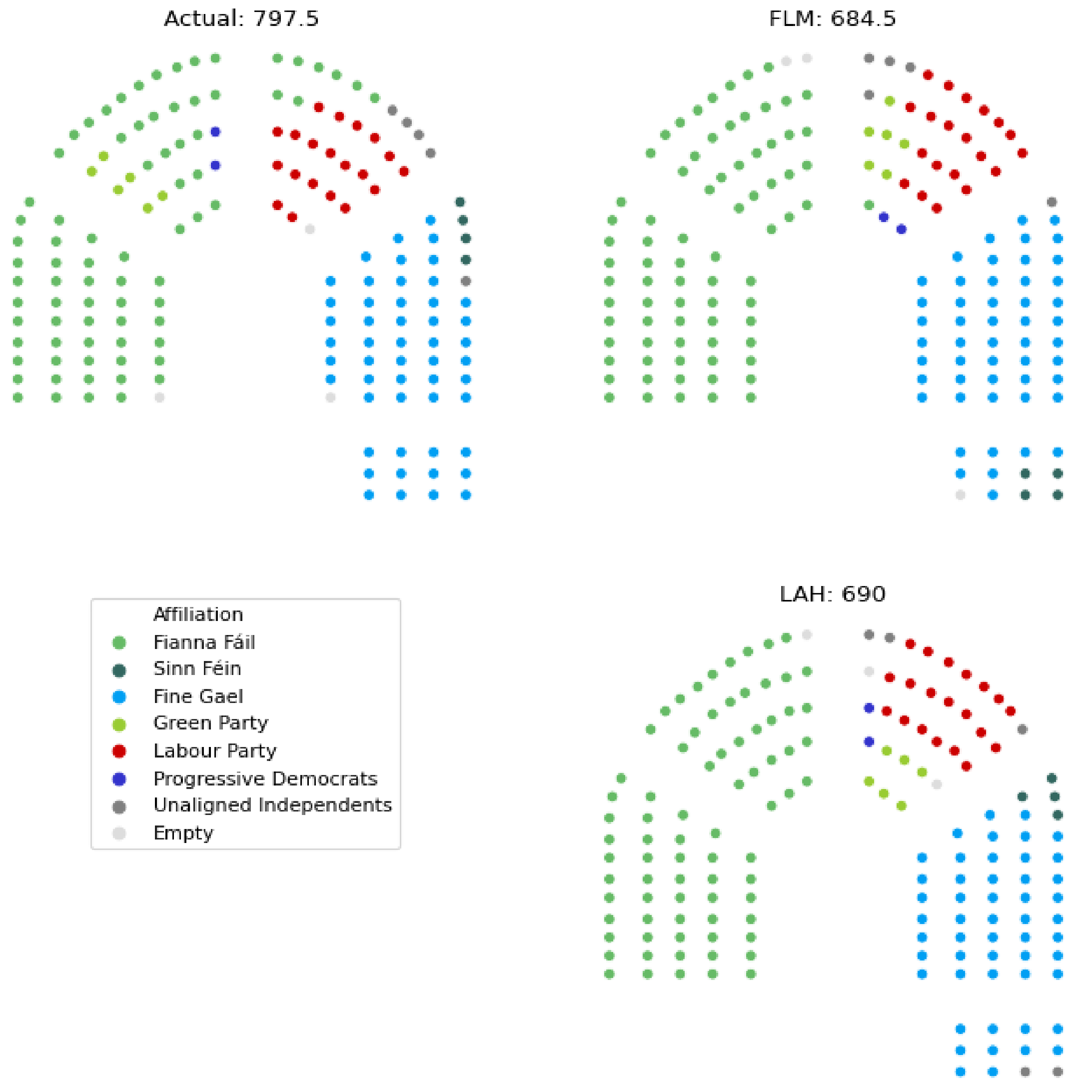

Figure 5.3: 30th Dáil Solutions

	Members	Government	Taoiseach	LOTO	Ministers
Fianna Fáil	77	Y	Y	-	12
Sinn Féin	4	-	-	-	-
Fine Gael	51	-	-	Y	-
Green Party	6	Y	-	-	2
Labour Party	20	-	-	-	-
Progressive Democrats	2	Y	-	-	1
Unaligned Independents	5				
Empty Seats	3				

Table 5.5: Data for the 30^{th} Dáil

The general election of 2007 saw a decline in the seats returned for both Fianna Fáil and the PDs. This decline required the introduction of the Green Party as a coalition partner, in order to retain a majority in parliament. Unusually, both the Green Party and the PDs opted for seating which provided sufficient front bench seats for their ministers, but to the evident sacrifice of *territorial dominance*, both being embedded fully within Fianna Fáil. It could be argued that this lack of distinction from Fianna Fáil was partially responsible for both parties losing all of their parliamentary seats at the subsequent election. This paper's models implicitly penalise for embedded solutions such as this and will avoid suggesting them, where possible. Fine Gael and Labour possessed seating arrangements largely similar to those seen in Section 5.1.

The FLM solution reallocates the Green Party and the PDs into positions that give them *territorial dominance* while ensuring they possess sufficient seats for their ministers. The Labour Party once again are the party most heavily penalised by this, losing some of their prominent seats. This fallout lends further credence towards a reevaluation of the *equity* of the ministerial constraint.

The LAH solution is broadly similar to that of the FLM, with some minor positional changes for Sinn Féin, the Greens and the PDs. The Labour Party are more *compact* but at the cost of more of their prominent seats.

5.4 31st Dáil, March 2011

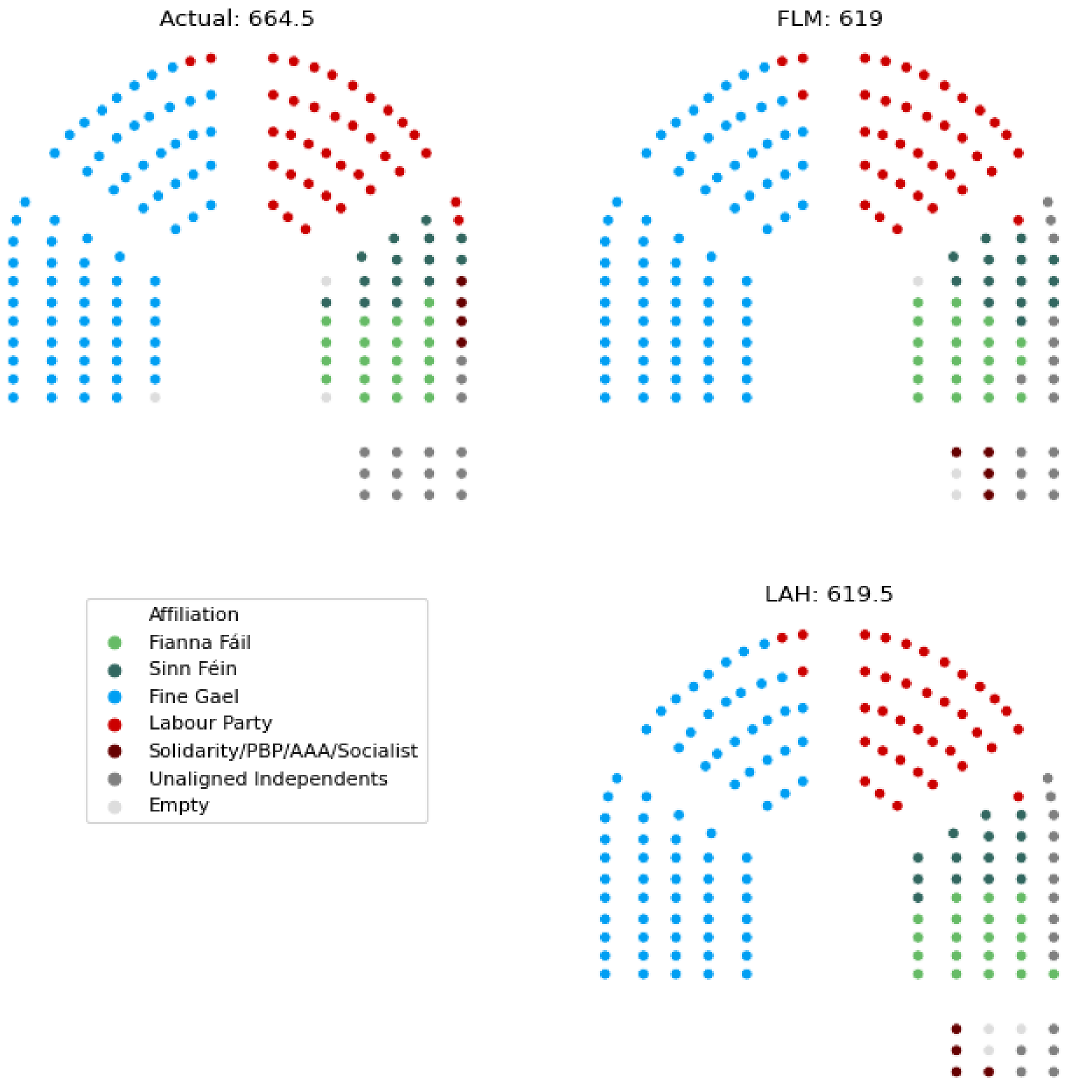

Figure 5.4: 31st Dáil Solutions

	Members	Government	Taoiseach	LOTO	Ministers
Fianna Fáil	20	-	-	Y	-
Sinn Féin	14	-	-	-	-
Fine Gael	75	Y	Y	-	10
Labour Party	37	Y	-	-	5
Solidarity/PBP/AAA/Socialist	4	-	-	-	-
Unaligned Independents	15				
Empty Seats	3				

Table 5.6: Data for the 31$^{\text{st}}$ Dáil

The election of the 31$^{\text{st}}$ Dáil in 2011 marked the beginning of a period of unprecedented change within Irish parliamentary politics. Fianna Fáil, which had been the largest political party since the foundation of the state, lost more than 70% of its seats after perceived mishandling of the Irish financial crisis. Fine Gael and Labour, who both principally benefited from this fallout, formed a coalition government which commanded the largest parliamentary majority in the state's history. Sinn Féin also made notable gains, increasing their seat count from 4 to 14.

The FLM solution is very similar to the actual seating arrangement. Solidarity/PBP are provided with more prominent seats, however, and Sinn Féin obtains a more compact layout.

The LAH solution is also close to that of the FLM, both numerically and visually, except that Sinn Féin are awarded marginally more prominent seating.

5.5 32nd Dáil, May 2016

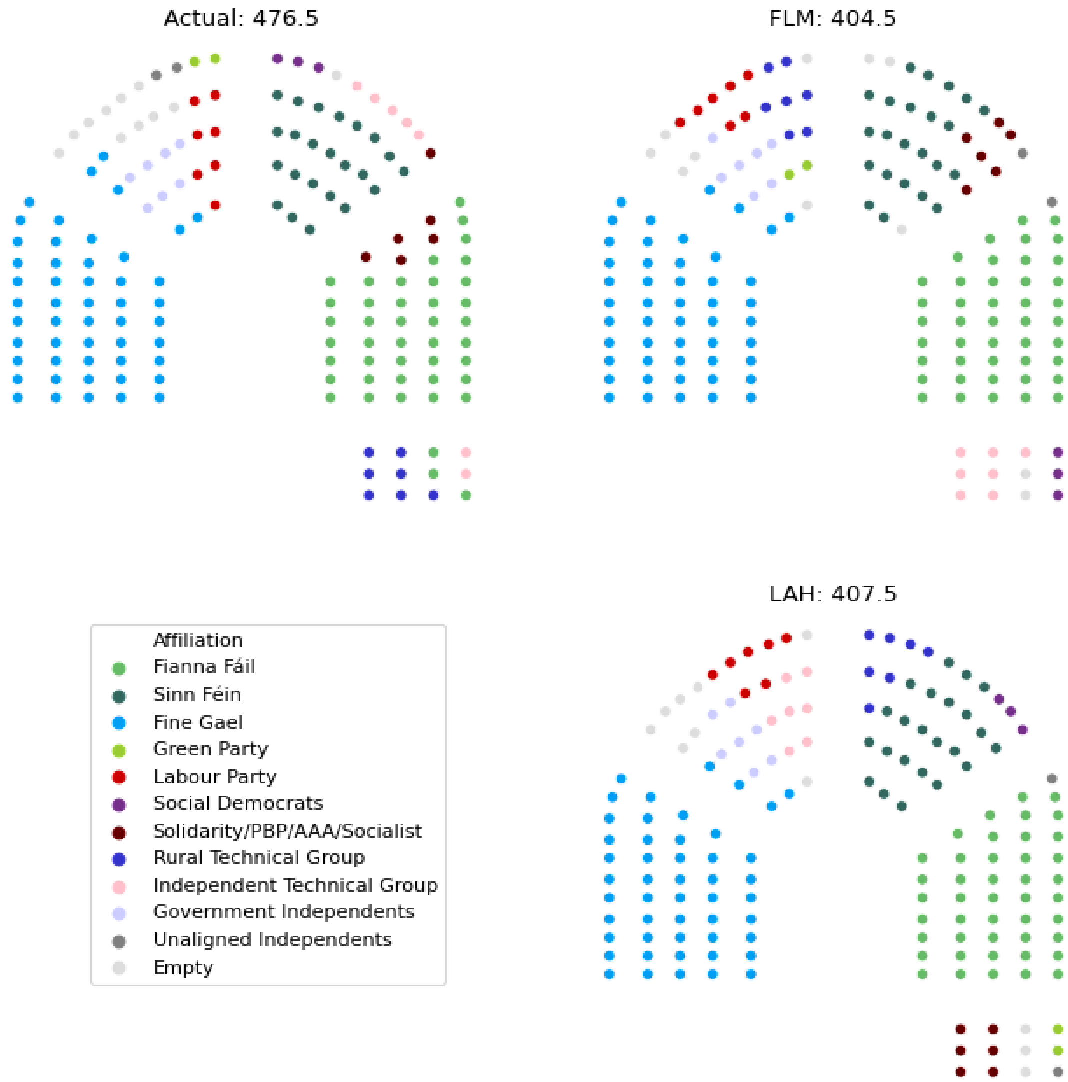

Figure 5.5: 32nd Dáil Solutions

	Members	Government	Taoiseach	LOTO	Ministers
Fianna Fáil	44	-	-	Y	-
Sinn Féin	23	-	-	-	-
Fine Gael	49	Y	Y	-	13
Green Party	2	-	-	-	-
Labour Party	7	-	-	-	-
Social Democrats	3	-	-	-	-
Solidarity/PBP/AAA/Socialist	6	-	-	-	-
Government Independents	7	Y	-	-	2
Rural Technical Group	7				
Independent Technical Group	7				
Unaligned Independents	2				
Empty Seats	11				

Table 5.7: Data for the 32$^{\text{nd}}$ Dáil

The election of the 32$^{\text{nd}}$ Dáil in 2016 brought about another series of marked parliamentary changes. The incumbent Fine Gael - Labour government had been deeply unpopular for their austerity policies, and they lost a significant portion of their seats. Fianna Fáil regained some ground from their collapse in support five years earlier, while Sinn Féin continued to make significant progress, this time increasing their seat count from 14 to 23. A change to parliamentary speaking rules brought about the introduction of technical groups, as discussed in Section 2.5.

During government formation talks in 2016, no majority could be established. However, an agreement was reached between Fine Gael and Fianna Fáil for a confidence-and-supply arrangement. This arrangement would allow Fine Gael to form a minority government, as Fianna Fáil would abstain on legislative votes while remaining the largest opposition party. Fine Gael managed to agree on a coalition government with the help of several Independent TDs.

The FLM much improves the compactness of Fianna Fáil and Solidarity/PBP and ensures the contiguity of the Independent Technical Group. The Green Party is awarded seats which, considering the party's small size, seem overly prominent. Under the Rawlsian 'veil of ignorance' approach, this outcome is deemed *equitable*; however, it could lead to calls for the introduction of a rule ensuring the priority of larger parties when allocating prominent seats.

The lack of rule-based distinction between technical groups and parliamentary parties is on display in the FLM results. The Independent Technical Group occupies much more prominent seats than the Labour Party, where both have an equal number of seats. A result such as this may also lead to calls for a constraint giving preference to political parties over technical groups in the allocation of prominent seating.

The LAH differs only somewhat arbitrarily from the FLM in this case, and it is also a perfectly suitable arrangement.

5.6 33rd Dáil, June 2020

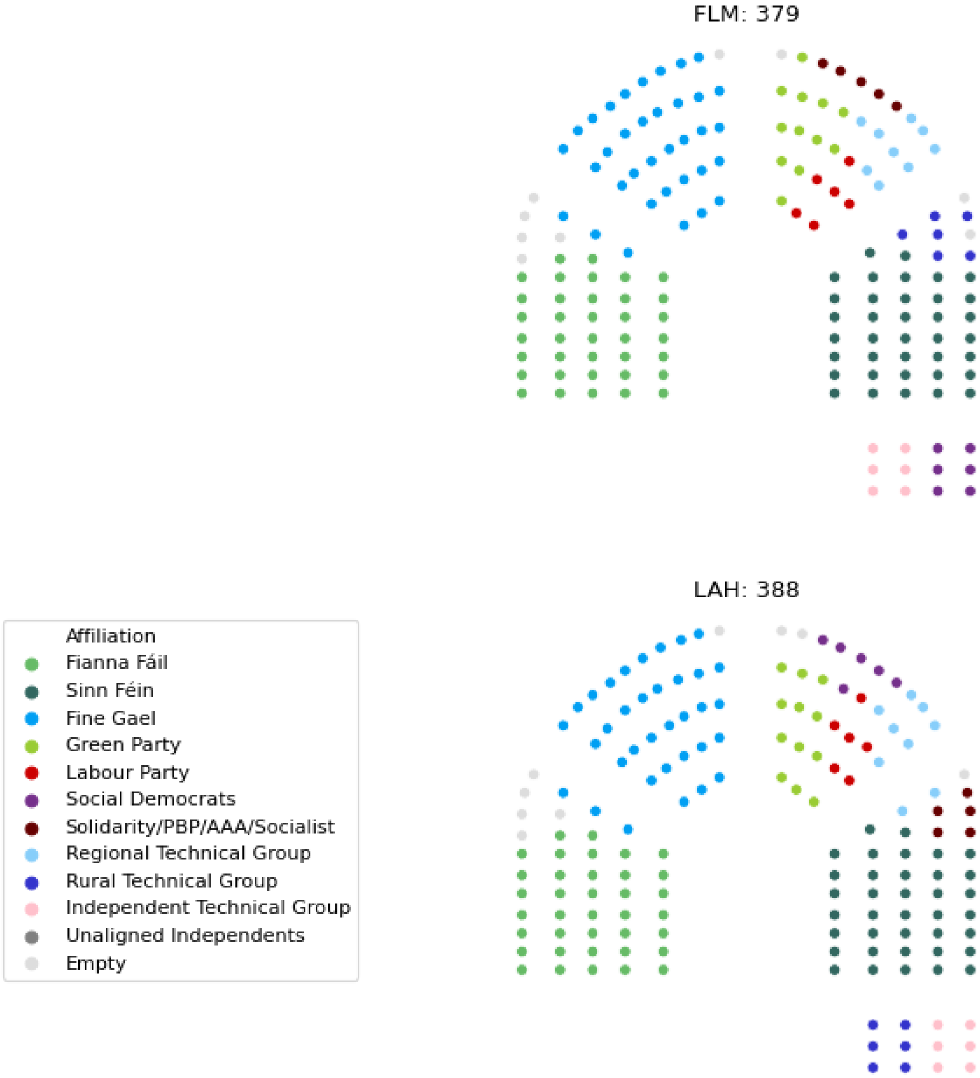

Figure 5.6: 33rd Dáil Solutions

	Members	Government	Taoiseach	LOTO	Ministers
Fianna Fáil	37	Y	Y	-	6
Sinn Féin	37	-	-	Y	-
Fine Gael	35	Y	-	-	6
Green Party	12	Y	-	-	3
Labour Party	6	-	-	-	-
Social Democrats	6	-	-	-	-
Solidarity/PBP/AAA/Socialist	5	-	-	-	-
Regional Technical Group	9				
Rural Technical Group	6				
Independent Technical Group	6				
Unaligned Independents	0				
Empty Seats	9				

Table 5.8: Data for the 33rd Dáil

The general election for the 33rd Dáil was held in February 2020, and it marked the first clear departure of Ireland from a two-party to a three-party political system. Fianna Fáil and Fine Gael both continued their downward trend of combined vote share, and they were surprisingly overtaken by Sinn Féin in the popular vote.

After many months of political negotiation, impacted by COVID-19, a government was finally formed in June 2020 consisting of Fianna Fáil, Fine Gael and the Green Party. This coalition, in itself, was a watershed moment. Fianna Fáil and Fine Gael, the two parties which formed on opposite sides of the Irish Civil War almost 100 years ago, entered government together for the first time.

As of this writing, due to the social distancing requirements brought about by COVID-19, Dáil Éireann has had to vacate its chamber in Leinster House to occupy a much larger space in the Convention Centre Dublin. As such, no formalised seating plan has yet been determined, and it remains to be seen when the parliament will return to its natural home. The suggested seating arrangements for the current Dáil are included here for completeness.

Territorial dominance has taken on even greater importance for the government parties, with each determined to retain their identity in this coalition government. The FLM manages to allocate a clear visual distinction between most parties.

The LAH solution is quite close to that of the FLM, except perhaps where the Regional Technical Group has its members unfortunately divided across an aisle. While this LAH solution represents the only occasion where an optimality gap larger than 1% was observed, it remains a very suitable potential solution.

It will be interesting to see how these solutions match up with the finalised arrangement when that time comes to pass.

Chapter 6

Conclusions

6.1 Summary

This paper has achieved the two primary objectives it set out in Chapter 1, namely:

1. To perform a comprehensive analysis of various prospective exact formulations, scrutinising the trade-off between the measures of compactness employed and their associated computational efficiency.

2. To develop a heuristic solution which can be applied to chambers of a much larger size, that not only robustly approximates the optimal solution of its associated exact approach, but which also overcomes the main obstacle faced by Hales & García (2019), allowing for the layering of "additional requirements".

It has explored four candidate exact approaches, finding that the *Total Distance* model most precisely captured the desired measure of *compactness*, but that the FLM provided the optimal balance between objective accuracy and computational performance. It also developed a robust heuristic which allowed for the imposition of additional constraints, through a refinement of the LAH model proposed by Hales & García (2019). This outcome was achieved by targeting the source of computational inefficiency in the original model and resolving it by implementing a *filtering* technique paired with a *nuanced random-start* approach.

The paper revised the definition of *compactness* that was present in the literature, to better capture the demands of the parliamentary seat allocation problem. It introduced the concepts of *communication distance* and *territorial dominance* to support this new definition. It proposed a philosophically differing approach to *equity*, whereby *fairness* was characterised as the equal *opportunity* of each party in the allocation process rather than by a strict determination of the equality of outcomes. The paper generated the concept of *leftness/rightness*, grounded in the assignment of *columns* to seats, to allow for the robust modelling of relative positioning. It developed a post-optimization algorithm by which to manage independent members of parliament, which had not been previously considered in the literature. Finally, it visually confirmed the appropriateness of the two suggested models, the FLM and the LAH, using historical data from Dáil Éireann.

6.2 Applied Usage

The FLM has been demonstrated to be a highly appropriate model for solving the parliamentary seat allocation problem within Dáil Éireann. Once the appropriate seating rules have been definitively codified within the standing orders, the model is ready for practical application. It could then be used repeatedly for any electoral outcome without the need for intervention of a modelling specialist unless the parliament decided to make changes to the seating rules.

The same FLM model is also appropriate for parliaments of any other shape, provided that they have fewer than approximately 200 seats. The subjective generation of a new *communication distance* adjacency matrix would be the most demanding necessary alteration. Its design would likely require some trial and error, mainly to ensure that *territorial dominance* was appropriately encouraged. The applicable parliamentary seating rules would also need to be codified through the generation of suitable constraints.

The LAH is recommended for parliaments of a larger scale. The only potentially challenging aspect of its application would be where a seating rule resulted in the introduction of an exponential number of constraints, as was seen for the government positioning constraint in the Dáil. In this scenario, removal of the constraints, paired with the introduction of a *filtering* mechanism, should overcome the difficulty. A *nuanced random-start* technique might also help to improve outcomes, if applicable.

The LAH has the advantageous quality of being able to be programmed for as long as the user has available time. This maximizes the probability that the model finds a result that best approximates the optimal solution.

6.3 Further Research and Improvements

In terms of future examination of the parliamentary seat allocation problem, this author believes that the most promising avenue of potential improvement would be the generation of a heuristic that even more accurately targets the desired definition of *compactness*. This might represent a heuristic based on the *Total Distance* exact formulation.

Refinements could be made to the existing model's application to Dáil Éireann, through collaboration with parliamentary clerks. The inherent *fairness* of the ministerial constraint could be improved through nuanced revision. A rule might be considered appropriate for the favouring of larger parties over smaller parties when allocating prominent seats. Similarly, it might be deemed necessary for the preference of political parties over technical groups. Guidelines governing the seat allocation of Independents could be more rigorously codified. All of these changes would need approval by parliament for their application, but their development would be aided significantly by collaborating with the clerks.

Input from politicians themselves would also help to define *territorial dominance* more stringently. A better understanding of the desire for distinguishment within chambers would allow for a more intelligent construction of the *communication distance* adjacency matrix, and thus an improvement to modelling outcomes.

6.4 Concluding Thoughts

Overall, the parliamentary seat allocation problem is a very worthwhile application of mathematical optimization. Much like the field of political districting, ensuring *fairness* in political processes is crucial to secure faith in democratic institutions. In an era when democracy is under ever greater challenge and threat, this is now more important than ever.

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Appendix A

Exact Formulations

Notation

Sets

$$
\begin{aligned}
I &= \text{integer set of chamber seats.}\\
J &= \text{string set of political parties.}
\end{aligned}
$$

Fixed Data

$$
\begin{aligned}
c_i &= \textit{column} \text{ associated with each seat.}\\
r_i &= \textit{row} \text{ within which each seat is located.}\\
d_{ik} &= \textit{communication distance} \text{ between each pair of seats.}\\
f_{ik} &= \textit{communication flow} \text{ between each pair of adjacent seats.}\\
a_{ik} &= \begin{cases} 1 & \text{if seat } i \text{ and seat } k \text{ are adjacent.}\\ 0 & \text{otherwise.} \end{cases}\\
M &= \left(\max_{e \in I} \{c_e\} - 1 \right) \text{, the maximum difference between } \textit{columns}.
\end{aligned}
$$

Variable Data

$$
\begin{aligned}
n^j &= \text{number of parliamentary members in party } j.\\
g^j &= \begin{cases} 1 & \text{if party } j \text{ is in government.}\\ 0 & \text{if party } j \text{ is in opposition.} \end{cases}\\
t^j &= \begin{cases} 1 & \text{if party } j \text{ holds the office of Taoiseach.}\\ 0 & \text{otherwise.} \end{cases}\\
l^j &= \begin{cases} 1 & \text{if party } j \text{ holds the title of Leader of the Opposition.}\\ 0 & \text{otherwise.} \end{cases}\\
m^j &= \text{number of government ministers in party } j.\\
p &= \text{number of independent TDs.}
\end{aligned}
$$

A.1 Total Distance - Linear Formulation

Decision Variables

$$w_{ik}^{j} = \begin{cases} 1 & \text{if seat } i \text{ and seat } k \text{ and both allocated to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

$$z_{i}^{j} = \begin{cases} 1 & \text{if seat } i \text{ is assigned to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

Objective Function

$$\min \sum_{\substack{i,k \in I: \\ i<k}} \sum_{j \in J} d_{ik} \cdot w_{ik}^{j}$$

Model Constraints

$$\sum_{j \in J} z_i^j \leq 1 \qquad \forall i \in I; \tag{A.1}$$

$$\sum_{i \in I} z_i^j = n^j \qquad \forall j \in J; \tag{A.2}$$

$$z_i^j + z_k^j \leq w_{ik}^j + 1 \qquad \forall i,k \in I : i < k, \ \ \forall j \in J; \tag{A.3}$$

$$z_i^j \in \{0,1\} \qquad \forall i \in I, \ \ \forall j \in J; \tag{A.4}$$

Tightening Constraint

$$\sum_{\substack{k \in I: \\ i<k}} w_{ik}^j + \sum_{\substack{k \in I: \\ i>k}} w_{ki}^j \leq \left(n^j - 1\right) \cdot z_i^j \qquad \forall i \in I, \ \ \forall j \in J; \tag{A.5}$$

Dáil-specific Constraints

$$z_i^j = 1 \qquad i = 21, \ \ \forall j \in J : t^j = 1; \tag{A.6}$$

$$z_i^j = 1 \qquad i = 46, \ \ \forall j \in J : l^j = 1; \tag{A.7}$$

$$\sum_{\substack{c \in J: \\ g^c = 1}} z_a^c \cdot c_a \leq \sum_{\substack{d \in J: \\ g^d \neq 1}} z_b^d \cdot c_b + \left(1 - \sum_{\substack{d \in J: \\ g^d \neq 1}} z_b^d\right) \cdot M \qquad \forall a,b \in I; \tag{A.8}$$

$$\sum_{\substack{i \in I: \\ r_i \leq 2}} z_i^j \geq m^j \qquad \forall j \in J; \tag{A.9}$$

$$\sum_{\substack{i \in I: \\ c_i \geq 23, \, r_i \geq 4}} \left(1 - \sum_{j \in J} z_i^j\right) \geq p. \tag{A.10}$$

Commentary

- Equation (A.1) ensures that each seat is assigned to, at most, one party.
- Equation (A.2) ensures that each party must be allocated exactly the required number of seats for its parliamentary members.
- Equation (A.3) ensures that w_{ik}^j is set to 1 when the z_i^j and z_k^j are both 1.
- Equation (A.4) enforces that the z_i^j variables are binary.
- Equation (A.5) are tightening constraints which ensure that if seat i is not allocated to party j, then no pairs of the form (i, k) exist for this party and hence all $w_{ik}^j = 0$.
- Equation (A.6) and Equation (A.7) ensure that the appropriate seats are allocated to the parties that hold the roles of Taoiseach and Leader of the Opposition.
- Equation (A.8) enforces that no government party can be allocated a seat which is in a column to the right of an opposition party's seat.
- Equation (A.9) ensures that each party must have at least as many seats in the front 2 rows as it has government ministers.
- Equation (A.10) ensures that there are enough unallocated seats on the right side of the chamber in the backmost two rows to seat the independent TDs.

A.2 Communication Flows - Linear Formulation

Decision Variables

$$v_{ik}^j = \begin{cases} 1 & \text{if seat } i \text{ and seat } k \text{ are adjacent and both allocated to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

$$z_i^j = \begin{cases} 1 & \text{if seat } i \text{ is assigned to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

Objective Function

$$\max \sum_{\substack{i,k \in I: \\ i<k}} \sum_{j \in J} f_{ik} \cdot v_{ik}^j$$

Model Constraints

$$\sum_{j \in J} z_i^j \leq 1 \qquad \forall i \in I; \tag{A.11}$$

$$\sum_{i \in I} z_i^j = n^j \qquad \forall j \in J; \tag{A.12}$$

$$v_{ik}^j \leq z_i^j \qquad \forall i,k \in I : i<k, \ \ \forall j \in J; \tag{A.13}$$

$$v_{ik}^j \leq z_k^j \qquad \forall i,k \in I : i<k, \ \ \forall j \in J; \tag{A.14}$$

$$z_i^j \in \{0,1\} \qquad \forall i \in I, \ \ \forall j \in J; \tag{A.15}$$

Tightening Constraint

$$\sum_{\substack{k \in I: \\ i<k}} v_{ik}^j + \sum_{\substack{k \in I: \\ i>k}} v_{ki}^j \leq \left(n^j - 1\right) \cdot z_i^j \qquad \forall i \in I, \ \ \forall j \in J; \tag{A.16}$$

Dáil-specific Constraints

$$z_i^j = 1 \qquad i = 21, \ \ \forall j \in J : t^j = 1; \tag{A.17}$$

$$z_i^j = 1 \qquad i = 46, \ \ \forall j \in J : l^j = 1; \tag{A.18}$$

$$\sum_{\substack{c \in J: \\ g^c = 1}} z_a^c \cdot c_a \leq \sum_{\substack{d \in J: \\ g^d \neq 1}} z_b^d \cdot c_b + \left(1 - \sum_{\substack{d \in J: \\ g^d \neq 1}} z_b^d\right) \cdot M \qquad \forall a,b \in I; \tag{A.19}$$

$$\sum_{\substack{i \in I: \\ r_i \leq 2}} z_i^j \geq m^j \qquad \forall j \in J. \tag{A.20}$$

$$\sum_{\substack{i \in I: \\ c_i \geq 23, r_i \geq 4}} \left(1 - \sum_{j \in J} z_i^j\right) \geq p. \tag{A.21}$$

Commentary

- Equation (A.11) ensures that each seat is assigned to, at most, one party.
- Equation (A.12) ensures that each party must be allocated exactly the required number of seats for its parliamentary members.
- Equation (A.13) and Equation (A.14) enforce that if either z_i^j or $z_k^j = 0$, then the corresponding $v_{ik}^j = 0$.
- Equation (A.15) enforces that the z_i^j variables are binary.
- Equation (A.16) are tightening constraints which ensure that if seat i is not allocated to party j, then no pairs of the form (i, k) exist for this party and hence all $v_{ik}^j = 0$.
- Equation (A.17) and Equation (A.18) ensure that the appropriate seats are allocated to the parties that hold the roles of Taoiseach and Leader of the Opposition.
- Equation (A.19) enforces that no government party can be allocated a seat which is in a column to the right of an opposition party's seat.
- Equation (A.20) ensures that each party must have at least as many seats in the front 2 rows as it has government ministers.
- Equation (A.21) ensures that there are enough unallocated seats on the right side of the chamber in the backmost two rows to seat the independent TDs.

A.3 Minimum k-Partitioning - Linear Formulation

Decision Variables

$$u_{ik}^j = \begin{cases} 1 & \text{if seat } i \text{ and seat } k \text{ are adjacent, but both not allocated to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

$$z_i^j = \begin{cases} 1 & \text{if seat } i \text{ is assigned to party } j. \\ 0 & \text{otherwise.} \end{cases}$$

Objective Function

$$\min \sum_{\substack{i,k \in I: \\ i<k}} \sum_{j \in J} a_{ik} \cdot u_{ik}^j$$

Model Constraints

$$\sum_{j \in J} z_i^j \leq 1 \qquad \forall i \in I; \tag{A.22}$$

$$\sum_{i \in I} z_i^j = n^j \qquad \forall j \in J; \tag{A.23}$$

$$z_i^j + \left(1 - z_k^j\right) \leq u_{ik}^j + 1 \qquad \forall i, k \in I : i < k, \ \forall j \in J; \tag{A.24}$$

$$z_i^j \in \{0, 1\} \qquad \forall i \in I, \ \forall j \in J; \tag{A.25}$$

Dáil-specific Constraints

$$z_i^j = 1 \qquad i = 21, \ \forall j \in J : t^j = 1; \tag{A.26}$$

$$z_i^j = 1 \qquad i = 43, \ \forall j \in J : l^j = 1; \tag{A.27}$$

$$\sum_{\substack{c \in J: \\ g^c = 1}} z_a^c \cdot c_a \leq \sum_{\substack{d \in J: \\ g^d \neq 1}} z_b^d \cdot c_b + \left(1 - \sum_{\substack{d \in J: \\ g^d \neq 1}} z_b^d\right) \cdot M \qquad \forall a, b \in I; \tag{A.28}$$

$$\sum_{\substack{i \in I: \\ r_i \leq 2}} z_i^j \geq m^j \qquad \forall j \in J. \tag{A.29}$$

$$\sum_{\substack{i \in I: \\ c_i \geq 23, r_i \geq 4}} \left(1 - \sum_{j \in J} z_i^j\right) \geq p. \tag{A.30}$$

Commentary

- Equation (A.22) ensures that each seat is assigned to, at most, one party.
- Equation (A.23) ensures that each party must be allocated exactly the required number of seats for its parliamentary members.
- Equation (A.24) ensures that u^j_{ik} is set to 1 only when $z^j_i = 1$ and $z^j_k = 0$.
- Equation (A.25) enforces that the z^j_i variables are binary.
- Equation (A.26) and Equation (A.27) ensure that the appropriate seats are allocated to the parties that hold the roles of Taoiseach and Leader of the Opposition.
- Equation (A.28) enforces that no government party can be allocated a seat which is in a column to the right of an opposition party's seat.
- Equation (A.29) ensures that each party must have at least as many seats in the front 2 rows as it has government ministers.
- Equation (A.30) ensures that there are enough unallocated seats on the right side of the chamber in the backmost two rows to seat the independent TDs.

Location-Allocation Heuristic

Notation

Sets

$$I = \text{integer set of chamber seats.}$$
$$J = \text{string set of political parties.}$$

Data

$$c_i = \textit{column} \text{ associated with each seat.}$$
$$d_{ik} = \textit{communication distance} \text{ between each pair of seats.}$$
$$g^j = \begin{cases} 1 & \text{if party } j \text{ is in government.} \\ 0 & \text{if party } j \text{ is in opposition.} \end{cases}$$
$$t^j = \begin{cases} 1 & \text{if party } j \text{ holds the office of Taoiseach.} \\ 0 & \text{otherwise.} \end{cases}$$
$$l^j = \begin{cases} 1 & \text{if party } j \text{ holds the title of Leader of the Opposition.} \\ 0 & \text{otherwise.} \end{cases}$$

New Heuristic Variables

$$\alpha = \text{desired number of iterations (parameter).}$$
$$\tilde{\alpha} = \text{current iteration.}$$
$$q^* = \text{value of the best solution.}$$
$$\tilde{q} = \text{value of the incumbent solution.}$$
$$o^{*j} = \text{centre seat of party } j \text{ in the best solution.}$$
$$\tilde{o}^j = \text{centre seat of party } j \text{ in the incumbent solution.}$$
$$\hat{o}^j = \text{centre seat of party } j \text{ in the current solution.}$$
$$s^{*j}_i = \begin{cases} 1 & \text{if seat } i \text{ is assigned to party } j \text{ in the best solution.} \\ 0 & \text{otherwise} \end{cases}$$
$$\tilde{s}^j_i = \begin{cases} 1 & \text{if seat } i \text{ assigned to party } j \text{ in the incumbent solution.} \\ 0 & \text{otherwise} \end{cases}$$
$$\hat{s}^j_i = \begin{cases} 1 & \text{if seat } i \text{ is assigned to party } j \text{ in the current solution.} \\ 0 & \text{otherwise} \end{cases}$$
$$\beta = \text{number of ineligible seat pairings for the incumbent solution.}$$

Algorithm 3: Location-Allocation Heuristic (Complete)

initialization;
$\alpha :=$ (set desired number of iterations);
$\tilde{\alpha} := 0$;
$q^* := \infty$;

Outer loop for the number of desired iterations;
while $\tilde{\alpha} < \alpha$ **do**
    $\tilde{\alpha} = \tilde{\alpha} + 1$;

    Set the rules for the random selection of starting centres;
    for $j \in J : t^j = 1$ **do**
        $\tilde{o}^j := 21$;
    for $j \in J : l^j = 1$ **do**
        $\tilde{o}^j := 46$;
    for $j \in J : t^j = 0$ *and* $g^j = 1$ **do**
        $\tilde{o}^j :=$ (function which randomly picks a seat on the left side of the chamber);
    for $j \in J : l^j = 0$ *and* $g^j = 0$ **do**
        $\tilde{o}^j :=$ (function which randomly picks a seat on the left side of the chamber);

    for $j \in J$ **do**
        $\hat{o}^j := 0$;

    Inner loop to run Location-Allocation iteratively until centres converge;
    while $\tilde{o}^j \neq \hat{o}^j$ **do**
        for $j \in J$ **do**
            $\hat{o}^j := \tilde{o}^j$;

        Solve Allocation Problem;
        see Section B.1

        Solve Location Problem using solution of Allocation Problem;
        for $i \in I, j \in J$ **do**
            $\tilde{s}_i^j := z_i^j$;
        for $j \in J$ **do**
            $\tilde{o}^j := \arg\min_{k \in I} \sum_{i \in I} d_{ik} \cdot \tilde{s}_i^j$

    Determine if the incumbent solution is eligible under the removed constraint;
    $\beta := 0$;
    for $a \in J : g^a = 1,\ \ b \in J : g^b = 0$ **do**
        if $c_{\tilde{o}^a} > c_{\tilde{o}^b}$ **then**
            $\beta = \beta + 1$;

    Calculate the value of the incumbent solution;
    $\tilde{q} := \sum_{i \in I} \sum_{j \in J} d_{i\tilde{o}^j} \cdot \tilde{s}_i^j$;

    Update the best solution if the incumbent is lower and eligible;
    if $\tilde{q} < q^*$ **and** $\beta = 0$ **then**
        $q^* := \tilde{q}$;
        for $i \in I,\ \ j \in J$ **do**
            ${s^*}_i^j := \tilde{s}_i^j$;
        for $j \in J$ **do**
            $o^{*j} := \tilde{o}^j$;

Allocate Independents to suitable unallocated seats in the best solution;
see Algorithm 4 for details;

B.1 Allocation Subproblem

Notation

$$
\begin{aligned}
r_i &= \text{row within which each seat is located.}\\
n^j &= \text{number of parliamentary members in party } j.\\
m^j &= \text{number of government ministers in party } j.\\
p &= \text{number of independent TDs.}\\
z_i^j &= \begin{cases} 1 & \text{if seat } i \text{ is assigned to party } j.\\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Objective Function

$$\min \sum_{i \in I} \sum_{j \in J} d_{i\hat{o}^j} \cdot z_i^j$$

Constraints

$$\sum_{j \in J} z_i^j \leq 1 \qquad \forall i \in I; \tag{B.1}$$

$$\sum_{i \in I} z_i^j = n^j \qquad \forall j \in J; \tag{B.2}$$

$$z_i^j \in \{0, 1\} \qquad \forall i \in I,\ \forall j \in J; \tag{B.3}$$

$$z_i^j = 1 \qquad \forall i \in I : i = 21,\ \forall j \in J : t^j = 1; \tag{B.4}$$

$$z_i^j = 1 \qquad \forall i \in I : i = 46,\ \forall j \in J : l^j = 1; \tag{B.5}$$

$$\sum_{\substack{i \in I: \\ r_i \leq 2}} z_i^j \geq m^j \qquad \forall j \in J; \tag{B.6}$$

$$\sum_{\substack{i \in I: \\ c_i \geq 23, r_i \geq 4}} \left(1 - \sum_{j \in J} z_i^j\right) \geq p \tag{B.7}$$

Commentary

- Equation (B.1) ensures that each seat is assigned to, at most, one party.
- Equation (B.2) ensures that each party must be allocated exactly the required number of seats for its parliamentary members.
- Equation (B.3) enforces that the z_i^j variables are binary.
- Equation (B.4) and Equation (B.5) ensure that the appropriate seats are allocated to the parties that hold the roles of Taoiseach and Leader of the Opposition.
- Equation (B.5) ensures that each party must have at least as many seats in the front 2 rows as it has government ministers.
- Equation (B.7) ensures that there are enough unallocated seats on the right side of the chamber in the backmost two rows to seat the independent TDs.

Appendix C

Post-Optimization Allocation of Independents

Notation

$p =$ number of independent members.

$\gamma =$ current independent member being seated.

$\delta =$ current highest indexed seat which is empty and on the right side.

Algorithm 4: Post-Optimization Allocation of Independents

```
initialization;
If applying to FLM formulation, first extract z_i^j variables;
for i, k ∈ I,  j ∈ J do
    if x_ik + y_k^j = 1 then
        z_i^j := 1;

γ := 1;
Continue to fill seats for as many independents as exist;
while γ ≤ p do
    γ := γ + 1;
    δ := argmax_{i∈I: Σ_{h∈J} z_i^h = 0 and c_i ≥ 23} {i};
    z_δ^{'Unaligned Independents'} := 1;
```

Appendix D

Code and Data Repository

All five optimization models developed throughout this paper, as well as all of the datasets generated, can be found in the following Github repository.

Glossary

Ceann Comhairle The Speaker/Chair of the Dáil. 6, 24

Dáil Éireann The Irish Assembly/Parliament. 5, 20, 37, 38

FLM Facility Location Model. 1, 15–19, 21–24, 26, 28, 30, 32, 34, 36, 37, 50

GCH Geometric Cutting Heuristic. 15, 20

independent TD which is not affiliated to any political party. 1, 6, 10, 11, 17, 19, 22, 24, 26, 28, 30, 32, 34, 36–38, 40, 42, 44, 46, 48–50, 52

LAH Location-Allocation Heuristic. 1, 17, 19, 20, 23, 24, 26, 28, 30, 32, 34, 36–38

LOTO Leader of the Opposition. 6, 19, 26, 28, 30, 32, 34, 36

Taoiseach The Head of Government. 6, 11, 17, 19, 26, 28, 30, 32, 34, 36, 40, 42, 44, 46, 47, 49

TD *Teachta Dála* - Member of the Dáil. 5, 6, 10, 11, 17, 34, 40, 42, 44, 46, 49

technical group A collection of Independent TDs and/or small parties that have aligned themselves to share speaking rights. 1, 10, 22, 24, 34