Sum of the first k-th power integers

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1 Introduction

There are a couple well known formulas when it goes to the sum of the first k-th power integers, namely

$$\sum_{i=0}^{n} 1 = n + 1$$

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

So the question arises, how do you find a general formula for

$$\sum_{i=0}^{n} i^k$$

Or in general, how do you find a general formula for

$$\sum_{i=0}^{n} f(i)$$

where $f(i)$ is a polynomial in terms of i? This will be the main focus of this small article!

2 Inspiration of the article

We will first pay attention on a similar but much easier problem.

$$\sum_{i=0}^{n} i(i-1)...(i-k+1) =?$$

This equation is also sum of a k-th degree polynomial, but it is much easier to solve.

Theorem 1.

$$\sum_{i=0}^{n} i(i-1)...(i-k+1) = \frac{(n+1)(n)(n-1)...(n-k+1)}{k+1}$$

Proof. We will start from the well known hockey-stick theorem which states that

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Multiplying both sides of the equation by $k!$ yields

$$\sum_{i=k}^{n} i(i-1)...(i-k+1) = \frac{(n+1)(n)(n-1)...(n-k+1)}{k+1}$$

Since from $0 \leq i < k$, the LHS is equal to 0,

$$\sum_{i=0}^{n} i(i-1)...(i-k+1) = \frac{(n+1)(n)(n-1)...(n-k+1)}{k+1}$$

□

Using Theorem 1, we can write previous known summations differently. For example,

$$\sum_{i=0}^{n} i^2 = \sum_{i=0}^{n} i(i-1) + \sum_{i=0}^{n} i = \frac{(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} = \frac{n(n+1)(2n+1)}{6}$$

3 Details on finding the coefficients

The natural question that arises is how to find these coefficient for large powers. For example,

$$i^4 = ai(i-1)(i-2)(i-3) + bi(i-1)(i-2) + ci(i-1) + di$$

One way to find these coefficients efficiently would be to match up the coefficients one by one, starting from the highest power.
In our example, we can easily see that $a = 1$. Subtracting the first term over gives

$$6i^3 - 11i^2 + 6i = bi(i-1)(i-2) + ci(i-1) + di$$

Thus,$b = 6$. Subtracting the term over gives

$$7i^2 - 6i = ci(i-1) + di$$

Thus, $c = 7$. Subtracting the term over gives

$$i = di$$

Thus, $d = 1$.
This process does give a solution pretty easily, but as the exponent grows, expanding $i(i-1)...(i-k+1)$ will be very tedious.

4 Working backwards

In this section, we attempt at finding the coefficients using an easier method.
One thing to note about the equation

$$i^4 = ai(i-1)(i-2)(i-3) + bi(i-1)(i-2) + ci(i-1) + di$$

is that each term in the RHS has a common factor. So, instead of finding the highest degree term first, we can find the smallest term first.
For example, substituting $i = 1$ gives $d = 1$. Then subtracting i over gives

$$i^4 - i = i(i-1)(i^2+i+1) = ai(i-1)(i-2)(i-3) + bi(i-1)(i-2) + ci(i-1)$$

Dividing both sides by $i(i-1)$ yields

$$(i^2+i+1) = a(i-2)(i-3) + b(i-2) + c$$

Substituting $i = 2$ gives $c = 7$. Repeating this process will give a, b as well.
However, this process also hard to organize. So, I propose the method of synthetic division.

4.1 Synthetic Division Method

Initially we start from i^4 in our case. First divide i^4 by i synthetically, which would give $i^3...0$. Then divide i^3 by $i-1$ synthetically.

$$\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 0 \\ \cline{2-5} & & 1 & 1 & 1 \\ \cline{2-5} & 1 & 1 & 1 & 1 \end{array}$$

Then take the quotient i^2+i+1 and divide it synthetically by $i-2$

$$\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \cline{2-4} & & 2 & 4 \\ \cline{2-4} & 1 & 3 & 7 \end{array}$$

Then take the quotient $i + 3$ and divide it synthetically by $i - 3$.

$$\begin{array}{c|cc}
1 & 3 & 7 \\ \hline
 & & 3 \\ \hline
 & 1 & 6
\end{array}$$

Lastly take the quotient 1 and divide it by $i - 4$, which gives 1.

The remainders in reverse order will be $1, 6, 7, 1, 0$ which corresponds to

$$i^4 = i(i-1)(i-2)(i-3) + 6i(i-1)(i-2) + 7i(i-1) + i + 0$$

5 Extension of this method

Because there is no restrictions regarding this method, we can use any polynomial instead of i^k. For example, when solving

$$\sum_{i=0}^{n} i^3 - 3i^2 + 4i - 2$$

instead of solving term by term, which is the traditional method, we can use this to simply find the coefficients of

$$1, i, i(i-1), i(i-1)(i-2)$$

Then apply Theorem 1.

6 Relation to Stirling's number of the second kind

Definition 6.1. The Stirling's number of the second kind, $S(n, k)$ is defined as the number of ways to put n elements into k indistinguishable non-empty sets. (If $n < k$, $S(n, k) = 0$.)

The natural question arises, how do we solve for $S(n, k)$? I propose two different methods: recursion and combinatorics.

6.1 Recurrence Relation

Theorem 2.

$$S(n, 1) = 1 \ \forall n \ \in \mathbf{N}$$

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

Proof. We can split up $S(n,k)$ into two cases. Consider a particular element a_n which is in set S.
i) a_n is the only element in S.
Then, removing a_n gives the number of ways to split $n-1$ elements into $k-1$ non-empty sets, which is $S(n-1,k-1)$.
ii) a_n is with other elements in S
Then, removing a_n gives the number of ways to split $n-1$ elements into k non-empty sets, but since a_n could be in any of the k sets, we multiply $S(n-1,k)$ by k.

□

Explicit? Formula

Theorem 3.

$$S(n,k) = \frac{1}{k!}(\binom{k}{0}k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n ...)$$

Proof. The proof is trivial by PIE. There are k ways to select where each of the element goes, but we have to subtract the cases when one of the sets are empty, then add back when two of the sets are empty, and so on and so forth. Finally, we divide by $k!$ since the sets are indistinguishable. □

6.2 Relationship

Consider the following combinatorical argument.

i^k is the number of ordered pairs $(a_1, a_2, ..., a_i)$ where $1 \leq a_1, a_2, ..., a_i \leq k$.

Now, we use case work the number of distinct values $(a_1, a_2, ..., a_i)$
Suppose there are m distinct values. By the definition of Stirling numbers and the fact that the values are distinguishable, the number of possible values is

$$m! \cdot S(k,m).$$

Theorem 4. From the work done in the part above, we can conclude that

$$i^k = 1! \cdot S(k,1) \cdot \binom{i}{1} + 2! \cdot S(k,2) + \binom{i}{2} ...k! \cdot S(k,k) \cdot \binom{i}{k}$$

$$\sum_{i=1}^{n} i^k = 1! \cdot S(k,1) \cdot \binom{n+1}{2} + 2! \cdot S(k,2) + \binom{n+1}{3} ...k! \cdot S(k,k) + \binom{n+1}{k+1}$$

6.3 Relation to Another Summation

Theorem 5. Consider the summation

$$S_k = \sum_{n=0}^{\infty} \frac{f(n)}{n!}$$

Where $f(n)$ is a polynomial of n.

To solve additional values of S_k, we note the following.

For any positive integer k,

$$\sum_{n=0}^{\infty} \frac{n(n-1)\dots(n-k+1)}{n!} = e$$

Proof. This is trivial since the summation is zero from $n = 0$ to $n = k - 1$ and for the rest of the summation we effectively have

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

□