Sum of the first k-th power integers

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§1 Introduction

There are a couple well known formulas when it goes to the sum of the first k-th power integers, namely

$$\sum_{i=0}^{n} 1 = n + 1$$

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

So the question arises, how do you find a general formula for

$$\sum_{i=0}^{n} i^k$$

Or in general, how do you find a general formula for

$$\sum_{i=0}^{n} f(i)$$

where $f(i)$ is a polynomial in terms of i?

This will be the main focus of this small article.

Another question we will answer, is the problem

$$\sum_{n=0}^{\infty} \frac{n^k}{n!}$$

Although it appears that this and the problem above has no similarities, they are quite related to each other.

§2 Inspiration of the article

We will first pay attention on a similar but much easier problem.

$$\sum_{i=0}^{n} i(i-1)...(i-k+1) = ?$$

This equation is also sum of a k^{th} degree polynomial, but it is much easier to solve.

Theorem 2.1 (Hockey Stick)

$$\sum_{i=0}^{n} i(i-1)\dots(i-k+1) = \sum_{i=k}^{n} k! \cdot \binom{i}{k} = k! \cdot \binom{n+1}{k+1} = \frac{(n+1)(n)(n-1)...(n-k+1)}{k+1}$$

Proof. The first equality follows by the fact that the LHS is equal to zero when $0 \leq i \leq k-1$ and the rest is just the definition of the binomail coefficient.

The second equality is simply the well-known hockey-stick theorem which states that

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

The final inequality is found by expanding out $\binom{n+1}{k+1}$ □

§3 Applications

The traditional way of finding these polynomials were using finite differences, but this is very tedious and could take a lot of time.

Instead, we use Theorem 1 Using Theorem 1, we can write previous known summations differently. For example,

$$\sum_{i=0}^{n} i^2 = \sum_{i=0}^{n} i(i-1) + \sum_{i=0}^{n} i = \frac{(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} = \frac{n(n+1)(2n+1)}{6}$$

§3.1 Details on finding the coefficients

The natural question that arises is how to find these coefficient for large powers. For example,

$$i^4 = ai(i-1)(i-2)(i-3) + bi(i-1)(i-2) + ci(i-1) + di$$

One way to find these coefficients efficiently would be to match up the coefficients one by one, starting from the highest power.

In our example, we can easily see that $a = 1$. Subtracting the first term over gives

$$6i^3 - 11i^2 + 6i = bi(i-1)(i-2) + ci(i-1) + di$$

Thus, $b = 6$. Subtracting the term over gives

$$7i^2 - 6i = ci(i-1) + di$$

Thus, $c = 7$. Subtracting the term over gives

$$i = di$$

Thus, $d = 1$.

This process does give a solution but as k gets large, expanding $i(i-1)...(i-k+1)$ will be very tedious.

§3.2 Magical Numbers

In this section, we attempt at finding the coefficients using an easier method.

One thing to note about the equation

$$i^4 = ai(i-1)(i-2)(i-3) + bi(i-1)(i-2) + ci(i-1) + di$$

is that each term in the RHS has a common factor. So, instead of finding the highest degree term first, we can find the smallest term first.

For example, substituting $i = 1$ gives $d = 1$. Then subtracting i over gives

$$i^4 - i = i(i-1)(i^2+i+1) = ai(i-1)(i-2)(i-3) + bi(i-1)(i-2) + ci(i-1)$$

Dividing both sides by $i(i-1)$ yields

$$(i^2+i+1) = a(i-2)(i-3) + b(i-2) + c$$

Substituting $i = 2$ gives $c = 7$. Repeating this process will give a, b as well.

However, this process also hard to organize. So, I propose the method of synthetic division.

§3.3 Synthetic Division Method

We will take the case i^4.

Initially we start from i^4. First divide i^4 by i synthetically, which would give i^3...0.

Then divide i^3 by $i-1$ synthetically.

$$\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 0 \\ \hline & & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 \end{array}$$

Then take the quotient $i^2 + i + 1$ and divide it synthetically by $i-2$

$$\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline & & 2 & 4 \\ \hline & 1 & 3 & 7 \end{array}$$

Then take the quotient $i + 3$ and divide it synthetically by $i - 3$.

$$\begin{array}{c|cc} 1 & 3 & 7 \\ \hline & & 3 \\ \hline & 1 & 6 \end{array}$$

Lastly take the quotient 1 and divide it by $i-4$, which gives 1.

The remainders in reverse order will be $1, 6, 7, 1, 0$ which corresponds to

$$i^4 = i(i-1)(i-2)(i-3) + 6i(i-1)(i-2) + 7i(i-1) + i + 0$$

§3.4 Proof of this method

> **Remark 3.1.** This is not a formal proof. If you want a more rigorous proof, you can try to generalize my arguments from here.

First, we pay attention to what synthetic division actually is. Synthetically dividing a monic polynomial $F(x)$ by $(x-a)$, then we will get the quotient $Q(x)$ and the remainder R.

Furthermore, we know that the value of R is equal to $F(a)$ by using $F(x) = (x-a)Q(x) + R$ and substituting a for x.

Using this, we can say that

$$Q(x) = \frac{F(x) - F(a)}{x-a}$$

This is essentially get the proof of the theorem. We start with

$$i^4 = ai(i-1)(i-2)(i-3) + bi(i-1)(i-2) + ci(i-1) + di$$

Synthetically dividing by 0 is equivalent to dividing by x and we get

$$i^3 = a(i-1)(i-2)(i-3) + b(i-1)(i-2) + c(i-1) + d$$

Now, synthetically dividing by 1 is equivalent to dividing by $x-1$ and we get

$$i^3 = (i^2 + i + 1)(i-1) + 1$$

Which is equivalent to saying that $d = F(1) = 1$ and $a(i-2)(i-3) + b(i-2) = i^2 + i + 1$. We can just repeat this process until we get our desired coefficients.

§3.5 Extension of this method

Because there is no restrictions regarding this method, we can use any polynomial instead of i^k. For example, when solving

$$\sum_{i=0}^{n} i^3 - 3i^2 + 4i - 2$$

instead of solving term by term, which is the traditional method, we can use this to simply find the coefficients of

$$1, i, i(i-1), i(i-1)(i-2)$$

Then apply Theorem 1.

Also, if we want the coefficients to the equation

$$f(x) = ax(x-1)(x-4)(x-9) + bx(x-1)(x-4) + cx(x-1) + dx$$

for example, we can do the same thing with values $0, 1, 4, 9$ instead of $0, 1, 2, 3$.

§4 Relation to Stirling's number of the second kind

Definition 4.1. Stirling's number of the second kind, $S(n,k)$ is defined as the number of ways to put n distinguishable elements into k indistinguishable non-empty sets. (If $n < k$, $S(n,k) = 0$.)

Example 4.2

The number of ways to put n people into k non-empty groups.

Example 4.3

$S(3,2) = |\{\{1,2\},\{3\}\},\{\{1,3\},\{2\}\},\{\{1\},\{2,3\}\}| = 3$

The natural question arises, how do we solve for $S(n,k)$? I propose two different methods: recursion and combinatorics.

§4.1 Recurrence Relation

Theorem 4.4

$$S(n,1) = 1 \ \forall n \in \mathbf{N}, S(n,k) = 0 \text{ if } n < k$$
$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$$

Proof. First, the base cases follow from the fact that there is only one way to put $\{1,\dots,n\}$ in one set and we cannot have n elements in $k > n$ non-empty sets.

Furthermore, the recurrence relation is well defined. Since either the value of $n-k$ is getting smaller or the value of k is getting smaller.

Consider a partition of $\{1,2,3,\dots,n\}$ such that every set is non-empty and suppose that $n \in S$. We split this into two cases.

If $S = \{n\}$, the number of possible ways are the number of ways to put $n-1$ elements into $k-1$ non-empty subsets which is $S(n-1,k-1)$.

If $|S| > 1$, the number of possible ways the number of ways to put $n-1$ elements into $k-1$ non-emptpy subsets in which n can be any one of them, so there are a total of $k \cdot S(n-1,k-1)$ ways.

□

§4.2 Explicit Formula

Theorem 4.5

$$S(n,k) = \frac{1}{k!}\left[\binom{k}{0}k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n + \cdots + \binom{k}{k}0^n\right]$$

Proof. The proof is done by PIE. Suppose we take distinguishable bins and we randomly choose where each of $\{1,\dots,n\}$ goes. There a total of k^n ways, but we have to subtract the cases when at least one of the sets are empty, then add back when at least two of the sets are empty, and so on and so forth. Finally, we divide by $k!$ since the sets are indistinguishable.

□

§4.3 Relationship

Theorem 4.6

$$i^k = 1! \cdot S(k,1) \cdot \binom{i}{1} + 2! \cdot S(k,2) + \binom{i}{2} + \cdots + k! \cdot S(k,k) \cdot \binom{i}{k}$$

Proof. Consider the following combinatorial argument.

i^k is the number of ordered pairs $(a_1, a_2, ..., a_k)$ where $1 \le a_1, a_2, ..., a_k \le i$.

Now, we use casework on the number of distinct values of $(a_1, a_2, ..., a_k)$

Suppose there are m distinct values. The number of ways this can happen is equal to $m! \cdot S(k,m) \cdot \binom{i}{m}$ since we need to choose the m values that will occur, then the number of ways to put these k numbers into m bins is $S(k,m) \cdot m!$ since the bins are distinguishable.

Adding those up gives you that

$$i^k = \sum_{m=1}^{k} m! \cdot S(k,m) \cdot \binom{i}{m}$$

□

Theorem 4.7

From the work done in the part above, we can conclude that

$$i^k = \sum_{m=1}^{k} m! \cdot S(k,m) \cdot \binom{i}{m} = 1! \cdot S(k,1) \cdot \binom{i}{1} + 2! \cdot S(k,2) + \binom{i}{2} + \cdots + k! \cdot S(k,k) \cdot \binom{i}{k}$$

$$\sum_{i=1}^{n} i^k = \sum_{m=1}^{k} m! \cdot (k,m) \cdot \binom{n+1}{m} = 1! \cdot S(k,1) \cdot \binom{n+1}{2} + 2! \cdot S(k,2) + \binom{n+1}{3} + \cdots + k! \cdot S(k,k) \ldots \binom{n+1}{k+1} \quad (1)$$

§5 Relation to Another Summation

Consider the summation

$$S_k = \sum_{n=0}^{\infty} \frac{n^k}{n!}$$

We note that we know that for $k = 0$, we get

$$S_0 = \sum_{n=0}^{\infty} \frac{1}{n!}$$

To solve additional values of S_k, we note the following.

Lemma 5.1

For any positive integer k,

$$\sum_{n=0}^{\infty} \frac{n(n-1)\ldots(n-k+1)}{n!} = e$$

Proof. This is trivial since the summation is zero from $n = 0$ to $n = k - 1$ and for the rest of the summation we effectively have

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

□

Thus, we can use a similar idea as before. Since we know that

$$n^k = S(k,1) \cdot n + S(k,2) \cdot n(n-1) + \cdots + S(k,k) \cdot n(n-1)(n-2) \ldots$$

We can conclude that

$$\sum_{n=0}^{\infty} \frac{n^k}{n!} = S(k,1) \cdot e + S(k,2) \cdot e + \cdots + S(k,k) \cdot e = e \cdot \sum_{i=0}^{k} S(k,i)$$

Note that by the definition of $S(n,k)$ being the number of ways to put n distinct elements into k distinguishable non-empty sets, if we add this amongst the values of k, we get the number of partitions of $\{1, 2, \ldots, n\}$.