Integration Packet Extension

Samuel Kim

March 12, 2025

§1 Taylor Series

Recall from Calculus that

Remark 1.1.

$$\begin{aligned}
\frac{1}{1-r} &= 1 + r + r^2 + \dots \ (|r| < 1)^* \\
\ln(1-x) = \int \frac{1}{1-x} &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \\
\ln(1+x) = \int \frac{1}{1+x} &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \\
\frac{\pi^2}{6} &= 1 + \frac{1}{4} + \frac{1}{9} + \dots \\
\frac{\pi^2}{6} - 2\frac{\pi^2}{24} = \frac{\pi^2}{12} &= 1 - \frac{1}{4} + \frac{1}{9} - \dots
\end{aligned}$$

Example 1.2

Consider

$$\int_0^\infty \frac{x}{1+e^x}$$

Multiplying top and the bottom by e^{-x}

$$\begin{aligned}
\int_0^\infty \frac{x}{1+e^x} &= \int_0^\infty \frac{xe^{-x}}{1+e^{-x}} \\
&= \int_0^\infty xe^{-x}(1 - e^{-x} + e^{-2x} - \dots) \\
&= \sum_{n=1}^\infty \int_0^\infty xe^{-nx}(-1)^{n-1} \\
&= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} \text{ by IBP} \\
&= \frac{\pi^2}{12}
\end{aligned}$$

Problem 1.3

Exercises:

1. Evaluate
$$\int_0^1 \frac{\ln(1+x^2)}{x}$$
2. Evaluate
$$\int_0^1 \frac{\ln(x)}{1-x}$$
3. Evaluate
$$\int_0^1 x\ln(x^2)\ln(1+x^2)$$
(Source: CMIMC Integration Bee 2021)
4. Evaluate
$$\int_0^{\frac{\pi}{2}} \ln(\sin(x))\sec(x)$$
5. Evaluate
$$\int_0^1 \frac{(x+1)\ln(x)}{x^3-1}$$
(Source: CMIMC Integration Bee 2022)

§2 Integral to Summation

A common problem is one involving $\lfloor x \rfloor$ or $\{x\} = x - \lfloor x \rfloor$.

Example 2.1

$$\int_0^1 \frac{1}{\lfloor 1-\log_2(1-x)\rfloor}$$

(Source: MIT Integration Bee 2014)

Let $u = 1-x$ and let $\lfloor 1-\log_2(x)\rfloor = n$. Then, solving for x gives

$$\begin{aligned} n &\le 1-\log_2(x) < n+1 \\ n-1 &\le -\log_2(x) < n \\ -n &< \log_2(x) \le -n+1 \\ 2^{-n} &< x \le 2^{-n+1} \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{1}{\lfloor 1-\log_2(1-x)\rfloor} &= \int_0^1 \frac{1}{\lfloor 1-\log_2(x)\rfloor} \\ &= \sum_{n=1}^{\infty}\int_{2^{-n}}^{2^{-n+1}} \frac{1}{n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n2^{-n}} \\ &= -\ln(1-\frac{1}{2}) = \ln(2) \end{aligned}$$

Problem 2.2

Exercises:

1. Evaluate
$$\int_0^{2025} \{x\}$$
2. Evaluate
$$\int_0^{2025} \{\sqrt{x}\}$$
(Source: MIT Integration Bee 2025)
3. Evaluate
$$\int_0^{2025} \frac{\lfloor x \rfloor}{\lceil \sqrt{x} \rceil}$$
(Source: MIT Integration Bee 2025)
4. Evaluate
$$\int_0^{\infty} e^{-\lfloor x \rfloor (1+\{x\})}$$
(Source: CMIMC Integration Bee 2023)
5. Evaluate
$$\int_1^{\infty} \frac{\lfloor x^2 \rfloor}{x^5}$$
(Source: CMIMC Integration Bee 2024)
6. Evaluate
$$\int_2^{\infty} \frac{\pi(x)}{x^3 - x}$$
where $\pi(x)$ is the number of primes less than or equal to x.
(Source: CMIMC Integration Bee 2022)

§3 Forcing Substitutions

§3.1 Trig Substitutions

When given a integral of a fraction of trig functions, one of the most common tricks you can use along side with bound swap is ”forcing” the substitution $u = \tan(x)$ (sometimes $u = \sec(x)$) by multiplying top and the bottom by something.

Example 3.1

$$\int_0^{\frac{\pi}{2}} \frac{1}{(\sin(x)+\cos(x))^4}$$

Multiply the numerator and denominator by $\sec^4(x)$ to get

$$\int_0^{\frac{\pi}{2}} \frac{\sec^4(x)}{\sec^4(x)(\sin(x)+\cos(x))^4} = \frac{\sec^4(x)}{(1+\tan(x))^4}$$

Let$u = \tan(x)$, we get

$$\int_0^{\infty} \frac{1+u^2}{(1+u)^4}$$

Which is simple using partial fraction decomposition.

Problem 3.2

Exercises:

1. Evaluate
$$\int_0^{\infty} \frac{1}{(\sqrt{\sin(x)}+\sqrt{\cos(x)})^4}$$
2. Evaluate
$$\int_0^{\frac{\pi}{2}} \frac{1}{4-3\cos^2(x)}$$
(Source: CMIMC Integration Bee 2021)
3. Evaluate
$$\int_0^{\pi} \frac{1}{3+\cos(x)}$$
4. Evaluate
$$\int_0^{\pi} \ln(1+a\cos(x)) \ |a|<1$$
5. Evaluate
$$\int_0^{\pi} \frac{1}{2+\cos^2(x)}$$
6. Evaluate
$$\int_0^{\frac{\pi}{2}} \frac{1}{\sin^4(x)+\cos^4(x)}$$

§3.2 Forcing $u = a^x$

Another common way to force a u substitution is by mulitplying the numerator, denominator by a constant to substitution $u = a^x$.

Example 3.3

$$\int_{-\infty}^{\infty} \frac{6^x}{4^x + 6^x + 9^x}$$

Divide the numerator and the denominator by 4^x to get

$$\int_{-\infty}^{\infty} \frac{(\frac{3}{2})^x}{(\frac{3}{2})^{2x} + (\frac{3}{2})^x + 1}$$

Let $u = (\frac{3}{2})^x \to du = \ln(\frac{3}{2}) \cdot (\frac{3}{2})^x$

$$I = \frac{1}{\ln(\frac{3}{2})} \cdot \int_0^{\infty} \frac{1}{u^2 + u + 1}$$

Evaluating $\frac{1}{u^2+u+1}$

$$\frac{1}{(u + \frac{1}{2})^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \arctan(\frac{2u+1}{\sqrt{3}})\Big|_0^{\infty} = \frac{2\pi}{3\sqrt{3}}$$

$$I = \frac{2\pi}{3\sqrt{3}} \cdot \frac{1}{\ln(3) - \ln(2)}$$

Problem 3.4

Exercises:

1. Evaluate
$$\int_0^{\infty} \frac{1}{2^x + 1}$$
2. Evaluate
$$\int_0^{\infty} \frac{3^x}{9^x + 1}$$
3. Evaluate
$$\int_0^{\infty} \frac{\ln 2 \cdot x}{2^x - 1}$$
4. Evaluate
$$\int_0^{\infty} \frac{1}{e^{2x} + e^x}$$
5. Evaluate
$$\int_1^{\infty} \frac{1 + 2x\ln(2)}{x\sqrt{x4^x - 1}}$$

(Source: CMIMC Integration Bee 2021)

§3.3 Forcing $u = x \pm \frac{1}{x}$

Remark 3.5.

$$(x - \frac{1}{x})^2 + 2 = (x + \frac{1}{x})^2 - 2 = x^2 + \frac{1}{x^2}$$

$$(x + \frac{1}{x})^3 - 3(x + \frac{1}{x}) = x^3 + \frac{1}{x^3}$$

$$(x - \frac{1}{x})^3 + 3(x - \frac{1}{x}) = x^3 - \frac{1}{x^3}$$

Example 3.6

$$\int_0^\infty \frac{1}{x^4 + x^2 + 1} = \frac{\frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}}$$

Notice that

$$\frac{2}{x^2} = \frac{d}{dx}(x - \frac{1}{x}) + \frac{d}{dx}(x + \frac{1}{x})$$

So,

$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{u^2 + 3} + 0 \\ &= \frac{\pi}{2\sqrt{3}} \end{aligned}$$

Problem 3.7

Exercises:

1. Evaluate
$$\int_0^\infty \frac{1}{1 + x^4}$$
2. Evaluate
$$\int_0^\infty e^{-x^2 - \frac{1}{x^2}}$$
3. Evaluate
$$\int_0^\infty e^{-x^2 - \frac{4}{x^2}}$$
4. Evaluate
$$\int_{-2}^{2} ((((x^2 - 2)^2 - 2)^2 - 2) - 2)$$

(Source MIT Integration Bee)

§4 Inverse Functions

When the integral is the sum of two functions, but both of them look impossible to calculate, suspect if it is in this form.

Theorem 4.1

$$\int_a^b f(x) + \int_{f(a)}^{f(b)} f^{-1}(x) = bf(b) - af(a)$$

Proof. Let $x = f(u)$ for the second integral. Then,

$$\int_a^b f(x) + xf'(x) = (xf(x))\,|_a^b = bf(b) - af(a)$$

□

Example 4.2

$$\int_0^1 x^n + x^{\frac{1}{n}}$$

Obviously, we could just calculate it individually, but notice this is exactly the form we are looking for!

So the answer is $1f(1) - 0f(0) = 1$

Problem 4.3

Exercises:

1. Evaluate
$$\int_1^2 (e^{1-\frac{1}{(x-1)^2}} + 1) + (1 + \frac{1}{\ln(x-1)})$$
(Source: Berkeley Integration Bee)
2. Evaluate
$$\int_0^1 \sqrt{(x-1)^3+1} + x^{\frac{2}{3}} - (1-x)^{\frac{3}{2}} - \sqrt[3]{1-x^2}$$
(Source: Stanford Math Tournament)
3. Evaluate
$$\int_1^2 2^{x-1} + \ln_2(2x)$$
(Source: MIT Integration Bee)
4. Evaluate
$$\int_0^1 2^{\sqrt{x}}(\ln^2(2)) + \ln^2(1+x)$$
(Source: CMIMC Integration Bee 2023)

§5 Guess the Function!

When solving an indefinite integral and looks impossible with any method, often the way to go is try to reverse engineer or guess the original integral.

Example 5.1

$$\int \left(\frac{x-1}{x^2+1}\right)^2 e^x$$

(Source: CMIMC Integration Bee 2021)

When I was taking this test, after trying out different substitutions and methods for a couple minutes, I realized nothing worked. Since I knew that

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

I tried finding

$$\frac{d}{dx}\frac{e^x}{x^2+1}$$

Voila! The answers turned about to be exactly this.

Example 5.2

$$\int (2\ln(x)+1)e^{\ln^2(x)}$$

(Source: MIT Integration Bee 2023)

Once again, the problem looks impossible, so we might as well try

$$\frac{d}{dx}e^{\ln^2(x)}$$

We get

$$\frac{2e^{\ln^2(x)}\ln(x)}{x}$$

Realize that if took the derivative of

$$e^{\ln^2(x)} \cdot x$$

the x in the denominator would cancel out and give $2\ln(x)e^{\ln^2(x)}$.

$$\frac{d}{dx}xe^{\ln^2(x)} = (2\ln(x)+1)e^{\ln^2(x)}$$

Voila! With a bit of trial and error we can find the integral.

Problem 5.3

Exercises:

1. Evaluate
$$\int_0^2 x^{x^2+1}(2\ln(x)+1)$$
(Source: MIT Integration Bee 2019)
2. Evaluate
$$\int \frac{\cos(x)-\sin(x)}{e^x}$$
3. Evaluate
$$\int e^x\cos^2(x)+e^x\cos(x)\sin(x)-e^x\sin^2(x)$$
(Source: MIT Integration Bee 2019)
4. Evaluate
$$\int x(\frac{1}{2}+\ln(x))\ln(\ln(x))$$
(Source: MIT Integration Bee 2025)
5. Evaluate
$$\int \sin(x)\sin(\sin(x))\sin(\cos(x))+\cos(x)\cos(\sin(x))\cos(\cos(x))$$
(Source: MIT Integration Bee 2025)

§6 "Well Known" Integrals

These are integrals where you can't find them in 5 minutes, so you just memorize the result.

1. $$\int_0^{\frac{\pi}{2}}\ln(\sin(x))=\int_0^{\frac{\pi}{2}}\ln(\cos(x))=-\frac{\pi}{2}\ln(2)$$
2. $$\int_0^{\infty}\frac{\arctan(a\cdot\tan(x))}{\tan(x)}=\frac{\pi}{2}\ln(1+a)$$
3. $$\int_0^{\infty}\frac{x}{\tan(x)}=\int_0^{\infty}\frac{\arctan(1\cdot\tan(x))}{\tan(x)}=\frac{\pi}{2}\ln(2)$$
4. $$\int_0^{\frac{\pi}{2}}\ln(\sin(x))\ln(\cos(x))=\frac{\pi}{2}\ln^2(2)-\frac{\pi^3}{48}$$
5. $$\int_0^{\frac{\pi}{2}}\ln(\sin(x))^2=\int_0^{\frac{\pi}{2}}\ln(\cos(x))^2=\frac{\pi}{2}\ln^2(2)+\frac{\pi^3}{24}$$
6. $$\int_0^{\frac{\pi}{2}}\ln(a^2\sin^2(x)+b^2\cos^2(x))=\pi\ln(\frac{a+b}{2})$$
7. $$\int_0^{\pi}\ln(1+a\cos(x))=\pi\ln(\frac{1+\sqrt{1-a^2}}{2})$$

8.
$$\int_0^\pi \ln(1-2a\cos(x)+a^2) = 2\pi\ln(a)$$

9.
$$\int_0^\pi \frac{\ln(1+a\cos(x))}{\cos(x)} = \pi\cdot\arcsin(a)$$

10.
$$int_0^{\frac{\pi}{2}} \frac{\ln(1+a\cos(x))}{\cos(x)} = \frac{1}{2}\cdot\left(\frac{\pi^2}{4}-\arccos(a)^2\right)$$

11.
$$\int_0^\infty \frac{\ln(x)}{x^2+a^2} = \frac{\pi}{2|a|}\ln(|a|)$$

12.
$$\int_0^1 \frac{\ln(x+1)}{x^2+1} = \frac{\pi}{8}\ln(2)$$

13.
$$\int_0^\infty \frac{\ln^2(x)}{x^2+1} = \frac{\pi^3}{8}$$

14.
$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+a^2} = \pi\ln(1+a)$$

15.
$$\int_0^\infty \frac{\cos(ax)}{x^2+1} = \frac{\pi}{2e^a}$$

16.
$$\int_0^\infty \frac{\sin(ax)}{x(x^2+1)} = \frac{\pi}{2}-\frac{\pi}{2e^a}$$

17.
$$\int_0^\infty \frac{\cos(ax)}{(x^2+1)^2} = \frac{\pi(a+1)}{4e^a}$$

18.
$$\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2}$$

19.
$$\int_0^\infty \cos(x^2) = \int_0^\infty \sin(x^2) = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

20.
$$\int_0^\infty \frac{1}{x^n+1} = \frac{\frac{\pi}{n}}{\sin(\frac{\pi}{n})}$$

21.
$$\int_0^\infty \frac{\ln(x)}{x^n+1} = -(\frac{\pi}{n})^2\cot(\frac{\pi}{n})\csc(\frac{\pi}{n})$$

22.
$$\int_0^{\frac{\pi}{2}} (\tan(x))^n = \frac{\frac{\pi}{2}}{\cos(\frac{\pi}{2}n)} : -1<n<1$$

23.
$$\int_0^\infty \frac{x^n}{1+x^{2n}} = \frac{\pi}{2n}\sec(\frac{\pi}{2n})$$

24.
$$\int_0^{2\pi} \frac{1}{a+b\cos(x)+c\sin(x)} = \frac{2\pi}{\sqrt{a^2-b^2-c^2}}$$

25.
$$I(a,b) = \int_0^\infty \frac{\sin(ax)\sin(bx)}{x^2} = \frac{\pi(|a+b| - |a-b|)}{4}$$

26.
$$I(a) = \int_0^\infty e^{-ax} \cdot \frac{\sin(x)}{x} \implies I(a) = -\tan^{-1}(a) + \frac{\pi}{2}$$

27.
$$I(a) = \int_0^1 \frac{x^a - 1}{\ln x} \implies I(a) = \ln(a+1)$$

28.
$$I(a) = \int_0^\pi e^{a\cos(x)} \cdot \cos(a\sin(x)) \implies I(a) = \pi$$

29.
$$\int_0^{\frac{\pi}{2}} \sin^n(x)\cos^m(x) = \begin{cases} \frac{(n-1)!!(m-1)!!}{(n+m)!!} \text{ when } n \text{ and } m \text{ are both even} \\ \\ \frac{(n-1)!!(m-1)!!}{(n+m)!!} \cdot \frac{\pi}{2} \text{ when both } n \text{ and } m \text{ are odd} \end{cases}$$

$\star\star\star$ Note that both 0!! and (-1)!! are equal to 1

30. If $f'(x)$ is continuous and the integral converges (note that there are other more general versions with weaker conditions),

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} = (f(\infty) - f(0)) \ln \frac{a}{b}$$

31.
$$\Gamma(z) = (z-1)! = \int_0^\infty x^{z-1} e^{-x}\, dx$$

32.
$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)$$