Minimal flows of Aut(G)

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G-flow

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Definition

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Let X be a G - flow. A *factor* of X is a G - flow Y such that there is a surjective morphism $\pi : X \to Y$, a continuous map respecting the G action.

The idea is that X is 'at least complicated' as Y.

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The universal minimal flow $M(G)$ is a minimal flow that factors onto every other minimal flow of G.

Importance of Universal Minimal Flows

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Ellis-Hewitt Theorem (1960)

Every topological group G has a unique universal minimal flow up to isomorphism.

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Example

The universal minimal flow of S_∞ is all linear orderings of $\mathbb{N}$.

Random Graph

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Given vertices $1, 2, \ldots$ we flip a coin to decide whether two vertices will be connected to each other or not. This will "almost always" form the Random graph.

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Fact

The random graph is unique up to isomorphism.

Properties of the Random graph

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Extension Property

For any finite disjoint subsets A, B, there exists a vertex v outside of A, B such that v is connected to all vertices in A and not connected to any of the vertices in B.

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Partition Regularity

If you partition the random graph into finitely many pieces, at least one will be isomorphic to the random graph.

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Automorphism Group of a Graph G

An automorphism group of G is the set of all automorphisms of G that preserves edge structure.

Examples

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Example

S_n is the automorphism group of $\{1, 2, \ldots, n\}$
S_∞ is the automorphism group of $\mathbb{Z}$.

Aut(G)

We denote Aut(G) to be the automorphism group of the random graph.

Problem in Question

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Can we identify all minimal flows of Aut(G) up to isomorphism?

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Why?

Why is this important?

What do we know?

What do we know?

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Similar Structures

All non-equivalent minimal flows of S_∞ has been identified.

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Reducts of the random ordered graph (2013)

All 42 reducts of the random ordered graph has been identified. Of the 42, the reducts of the random graph are vLO (linear order), BR (betweeness relation), CO (circular order), SR (sepration relation).

Thus, we know all subgroups between $(G, \leq) < H <$Aut(G).

Strategy to solve for this problem

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Almost one-to-one factors

Suppose $G \curvearrowright X$ is a non trivial miinmal flow with G_δ orbit $\Omega = G \cdot x_0$. An almost one-to-one factor of Aut(G) is a minimal flow of G that is injective on Ω.

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Relationship with Minimal Flows

For every non-trivial minimal flow X of Aut(G), there exists a flow (LO, BR, CO, SR) X' such that there is a factor map $X' \to X$ that is injective on Ω.

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ICERs

These almost one-to-one factors equivalently create a invariant, closed, equivalence relation (ICER) of $(G, \leq)$.

So, by identifying all ICERs, we can identify all almost one-to-one factors.

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How to fix this problem?
We don't know!

THANKS!

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