Proof of the generating function of central binomial coefficients

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1 Introduction

This mini article deals with an alternate proof for the generating function of the central binomial coefficient,

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

2 Anti- Chebyshev expansion

In this section, we define the anti-chebyshev expansion to be

$$\cos^n(x) = a_n \cos(nx) + a_{n-1}\cos((n-1)x) + ... + a_0$$

For example,

$$\cos^2(x) = \frac{1}{2}\cos(2x) + \frac{1}{2}$$

It is not hard to prove the general formula for the coefficients as

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Thus,

$$\left(\frac{e^{ix}+e^{-ix}}{2}\right)^n = \frac{1}{2^n}\left(\binom{n}{n}e^{nix} + \binom{n}{n}e^{-nix} + \binom{n}{n-1}e^{(n-2)ix} + \binom{n}{n-1}e^{-(n-2)ix} + ...\right)$$

$$= \frac{1}{2^{n-1}}\left(\binom{n}{n}\cos nx + \binom{n}{n-1}\cos{(n-2)x} + ...\right)$$

Note that if n is even, the constant term of the anti-chebyshev expansion will be $\frac{\binom{2n}{n}}{2^n}$ not $\frac{\binom{2n}{n}}{2^{n-1}}$ since in the binomial expansion of $\cos^n(x)$, there is no e^{0ix} and e^{-0ix}

3 Relations to an integral

Now, consider

$$\int_0^{\frac{\pi}{2}} \cos^{2n}(x)$$

Because integral of $\cos(2x), \cos(4x), ...$ from 0 to $\frac{\pi}{2}$ are all 0, the only term remaining in the anti-chebyshev expansion will be

$$\int_0^{\frac{\pi}{2}} a_0 = \frac{\binom{2n}{n}}{2^{2n}} = \frac{\pi}{2} \cdot \frac{\binom{2n}{n}}{4^n}$$

4 Making it a summation

Thus, to make the summation of integrals into the generating function form in the intro, we need to adjust the summation into

$$\sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2}} \frac{2}{\pi} \cdot \cos^{2n}(x) \cdot (4k)^n$$

Which would be

$$\sum_{n=0}^{\infty} \binom{2n}{n} k^n$$

Computing the summation of integrals is straightforward. Swapping the order of summation and the integral gives

$$\frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty} \cos^{2n}(x)(4k)^n$$

Which is a geometric series equal to

$$\frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}} \frac{1}{1 - 4k\cos^2(x)}$$

To compute this integral, we can multiply the top and the bottom of the fraction be $\sec^2(x)$, making it

$$\frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}} \frac{\sec^2(x)}{\sec^2(x) - 4k}$$

Using $\sec^2(x) = 1 + \tan^2(x)$,

$$\frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}} \frac{\sec^2(x)}{\tan^2(x) + 1 - 4k}$$

Substituting $u = \tan(x)$,

$$\frac{2}{\pi} \cdot \int_0^{\infty} \frac{1}{u^2 + 1 - 4k}$$

Using

$$\int_0^{\frac{\pi}{2}} \frac{1}{u^2 + a^2} = \frac{\pi}{2a}$$

The integral is equal to

$$\frac{2}{\pi} \cdot \frac{\pi}{2} \cdot \frac{1}{\sqrt{1-4k}} = \frac{1}{\sqrt{1-4k}}$$

Proving that

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$