# Project #2

**Problem 1** Consider the linear program ($LP_0$).

a. Give an optimal solution $x^*$ and the corresponding optimal tableau.

b. Write down the dual problem. Give the complementary optimal solution $y^*$ to the dual.

c. Can Xco marginally increase its profit $Z$ by purchasing more of resource $R_1$? If so, how much should they be willing to pay for one additional unit? What about $R_2$? Should Xco buy more? If so, at what unit price?

**Answer.**

a. Initial tableau:

$$\begin{array}{c} \\ z \\ x_4 \\ x_5 \\ x_6 \end{array}\left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & -2.5 & -2 & -4 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & \boxed{1} & 1 & 0 & 0 & 8 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 2 & 2 & 0 & 0 & 1 & 20 \end{array}\right] \tag{0}$$

$\frac{8}{1} < \frac{20}{2}$. By the minimum ratio test, element 1 is our pivot. $x_4$ leaves and $x_3$ enters the basis.

$$\begin{array}{c} \\ z \\ x_3 \\ x_5 \\ x_6 \end{array}\left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 1.5 & -2 & 0 & 4 & 0 & 0 & 32 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 0 & 8 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & -1 & \boxed{2} & 0 & -2 & 0 & 1 & 4 \end{array}\right] \tag{1}$$

$\frac{4}{2} < \frac{12}{1}$, thus $x_2$ enters the basis and $x_6$ leaves.

$$\begin{array}{c} \\ z \\ x_3 \\ x_5 \\ x_2 \end{array}\left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 0.5 & 0 & 0 & 2 & 0 & 1 & 36 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 0 & 8 \\ 0 & 2.5 & 0 & 0 & 1 & 1 & -0.5 & 10 \\ 0 & -0.5 & 1 & 0 & -1 & 0 & 0.5 & 2 \end{array}\right] \tag{2}$$

All row 0 coefficients are non-negative, therefore $x^* = (0, 2, 8)$ and the corresponding optimal tableau is (2).

b. The Dual Problem:

$$(DLP_0) \begin{cases} \text{Minimize} & W = yb, \\ \text{subject to:} & yA \geq c, \\ \text{and} & y \geq 0, \end{cases}$$

where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}, b = \begin{bmatrix} 8 \\ 12 \\ 20 \end{bmatrix}, \text{and } c = \begin{bmatrix} 2.5 & 2 & 4 \end{bmatrix}.$$

Using our optimal tableau in part a, we can get the shadow prices $y^* = (2, 0, 1)$. These prices are the coefficients of the slack variables in row 0. $y^*$ is the complementary solution to the dual.

c. Xco can marginally increase its profit $Z$ by purchasing more of resource $R_1$. For an additional unit they should pay no more than $y_1^* = 2$. If we were to increase $R_2$ by 1 unit, $Z$ would increase by 0. Therefore, it is not worth it to buy more units of $R_2$.

□

**Problem 2** Suppose that Xco revises the model program $(LP_0)$ to reflect changes in prices, supplies, etc.

a. We'll solve $(LP_1)$ by two different means. First solve the problem directly by the simplex method. Give the optimal solution $\bar{x}^*$ and the optimal tableau. How many iterations were required?

b. In the next two parts, we'll solve $(LP_1)$ by sensitivity analysis. Give the matrices $y^*$ and $S^*$ from the optimal tableau for $(LP_0)$. Use them to construct a revised tableau for $(LP_1)$.

c. Is the revised tableau from part (b) in proper form? If so, reoptimize. If not, put it in proper form and then reoptimize. Give the optimal solution and optimal tableau for $(LP_1)$.

d. Which solution method was faster? Direct application of the simplex method, or sensitivty analysis?

**Answer.**

a. Initial Tableau:

$$\begin{array}{c} \\ z \\ x_4 \\ x_5 \\ x_6 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & -3 & -2 & -4 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & \boxed{1} & 1 & 0 & 0 & 10 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 2 & 1 & 0 & 0 & 1 & 20 \end{array}\right] \qquad (0)$$

$$\begin{array}{c} \\ z \\ x_3 \\ x_5 \\ x_6 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 1 & -2 & 0 & 4 & 0 & 0 & 40 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 0 & 10 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & \boxed{2} & 0 & -1 & 0 & 1 & 10 \end{array}\right] \qquad (1)$$

$$\begin{array}{c} \\ z \\ x_3 \\ x_5 \\ x_2 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 1 & 0 & 0 & 3 & 0 & 1 & 50 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 0 & 10 \\ 0 & 2 & 0 & 0 & 0.5 & 1 & -0.5 & 7 \\ 0 & 0 & 1 & 0 & -0.5 & 0 & 0.5 & 5 \end{array}\right] \qquad (2)$$

After 2 iterations, we get the optimal solution $x^* = (0, 5, 10)$ and the corresponding optimal tableau (2).

b. From $(LP_0)$, we get the matrices

$$y^* = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \text{ and } S^* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -0.5 \\ -1 & 0 & 0.5 \end{bmatrix}.$$

We can use $y^*$ and $S^*$ to construct the revised tableau for $(LP_1)$.

$S^*\bar{b} = \begin{bmatrix} 10 \\ 12 \\ 0 \end{bmatrix}$.

$y^*\bar{b} = 40$.

$S^*\bar{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2.5 & 0 & 0.5 \\ -0.5 & 1 & -0.5 \end{bmatrix}$.

$y^*\bar{A} - \bar{c} = \begin{bmatrix} 3 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$.

$$\text{The Revised Tableau:} \quad \begin{array}{c} \\ z \\ x_3 \\ x_5 \\ x_2 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 0 & 0 & -1 & 2 & 0 & 1 & 40 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 0 & 10 \\ 0 & 2.5 & 0 & 0.5 & 1 & 1 & -0.5 & 12 \\ 0 & -0.5 & 1 & -0.5 & -1 & 0 & 0.5 & 0 \end{array}\right]$$

c. The revised tableau from part (b) is not in proper form. First, we most restore proper form. The Revised Tableau in Proper Form:

$$\begin{array}{c} \\ z \\ x_3 \\ x_5 \\ x_2 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 1 & 0 & 0 & 3 & 0 & 1 & 50 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 0 & 10 \\ 0 & 2 & 0 & 0 & 0.5 & 1 & -0.5 & 7 \\ 0 & 0 & 1 & 0 & -0.5 & 0 & 0.5 & 5 \end{array}\right]$$

Luckily enough, this is the optimal tableau for $(LP_1)$, with the solution $\bar{x}^* = (0, 5, 10)$.

d. Sensitivity analysis was faster.

□

**Problem 3** Suppose that Xco revises the model program $(LP_0)$ yet again.

a. Solve $(LP_2)$ by two different means. First treat the problem directly by the simplex method. You don't have to show your work. Just give the optimal solution $\bar{x}^*$ and the optimal tableau. How many iterations were required?

b. In the next two parts, you'll solve $(LP_2)$ by sensitivity analysis. Give the matrices $y^*$ and $S^*$ from the optimal tableau for $(LP_0)$. Use them to construct a revised tableau for $(LP_2)$.

c. Is the revised tableau from part (b) in proper form? If not, put it in proper form.

d. Is the tableau (now in proper form) feasible? If not, use the dual simplex method to restore it to feasibility.

e. Is the (now feasible) tableau optimal? If not, use the simplex method to find an optimal solution. Give the optimal solution and optimal tableau for $(LP_2)$.

**Answer.**

a. $\bar{x}^* = (6, 0, 7)$
Optimal Tableau for $(LP_2)$ after 2 iterations:

$$\begin{array}{c} \\ z \\ x_4 \\ x_1 \\ x_3 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 0 & 2.25 & 0 & 0 & 0.25 & 2 & 43 \\ \hline 0 & 0 & -5/4 & 0 & 1 & -1/4 & -1/2 & 1 \\ 0 & 1 & 1/2 & 0 & 0 & 1/2 & 0 & 6 \\ 0 & 0 & 3/4 & 1 & 0 & -1/4 & 1/2 & 7 \end{array}\right] \tag{2}$$

b. From $(LP_0)$, we get the matrices

$$y^* = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \text{ and } S^* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -0.5 \\ -1 & 0 & 0.5 \end{bmatrix}.$$

We can use $y^*$ and $S^*$ to construct the revised tableau for $(LP_2)$.
$S^*\bar{b} = \begin{bmatrix} 14 \\ 16 \\ -4 \end{bmatrix}$.
$y^*\bar{b} = 48$.

$$\begin{array}{c} \\ z \\ x_3 \\ x_5 \\ x_2 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 0.5 & 0 & 0 & 2 & 0 & 1 & 48 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 0 & 14 \\ 0 & 2.5 & 0 & 0 & 1 & 1 & -0.5 & 16 \\ 0 & -0.5 & 1 & 0 & -1 & 0 & 0.5 & \boxed{-4} \end{array}\right]$$

c. The revised tableau for $(LP_2)$ is already in proper form.

d. The tableau is not feasible however, because we have a negative element $(-4)$ on the RHS. We can use the dual simplex method to restore the tableau to feasibility.
The only negative value on the RHS is $(-4)$, so $x_2$ will be the LV. For the EV, we choose $x_1$ because $|\frac{0.5}{-0.5}| < |\frac{2}{-1}|$.

$$\begin{array}{c} \\ z \\ x_3 \\ x_5 \\ x_1 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1.5 & 44 \\ \hline 0 & 0 & 2 & 1 & -1 & 0 & 1 & 6 \\ 0 & 0 & 5 & 0 & -4 & 1 & 2 & \boxed{-4} \\ 0 & 1 & -2 & 0 & 2 & 0 & -1 & 8 \end{array}\right]$$

$x_5$ is the LV and $x_4$ is the EV as $(-4)$ is the only negative value in the pivot row.

$$\begin{array}{c} \\ z \\ x_3 \\ x_4 \\ x_1 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 0 & -1/4 & 0 & 0 & 1/4 & 1 & 43 \\ \hline 0 & 0 & 3/4 & 1 & 0 & -1/4 & 1/2 & 7 \\ 0 & 0 & -5/4 & 0 & 1 & -1/4 & -1/2 & 1 \\ 0 & 1 & 0.5 & 0 & 0 & 1/2 & 0 & 6 \end{array}\right]$$

e. The (now feasible) tableau is optimal. $\bar{x}^* = (6, 0, 7)$. The final tableau:

$$\begin{array}{c} \\ z \\ x_3 \\ x_4 \\ x_1 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RS \\ \hline 1 & 0 & -1/4 & 0 & 0 & 1/4 & 1 & 43 \\ \hline 0 & 0 & 3/4 & 1 & 0 & -1/4 & 1/2 & 7 \\ 0 & 0 & -5/4 & 0 & 1 & -1/4 & -1/2 & 1 \\ 0 & 1 & 0.5 & 0 & 0 & 1/2 & 0 & 6 \end{array}\right]$$

$\square$