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Homework #6

Problem 1 Let A and B be subsets of some universal set U. Prove that $A - B$ and $A \cap B$ are disjoint sets.

Answer. Let A and B be subsets of some universal set U. To show that $A - B$ and $A \cap B$ are disjoint sets, we will prove that $(A - B) \cap (A \cap B) = \emptyset$. Using Theorem 5.20, $(A - B) \cap (A \cap B) = (A \cap B^c) \cap (A \cap B)$. Then by the properties of Theorem 5.18,

$$\begin{aligned}
(A \cap B^c) \cap (A \cap B) &= ((A \cap B^c) \cap A) \cap ((A \cap B^c) \cap B) && (1)\\
&= ((A \cap B^c) \cap A) \cap (A \cap (B^c \cap B)) && (2)\\
&= ((A \cap B^c) \cap A) \cap (A \cap (\emptyset)) && (3)\\
&= ((A \cap B^c) \cap A) \cap (\emptyset) && (4)\\
&= \emptyset. && (5)
\end{aligned}$$

We will prove line (3) by contradiction, that $B^c \cap B = \emptyset$. Assume there exists $x \in B^c \cap B$. Then we know $x \in B$ and $x \notin B$ by the definition of intersection and the complement. Thus we have a contradiction. Therefore there does not exist an element in the set, and $B^c \cap B = \emptyset$. Therefore $A - B$ and $A \cap B$ are disjoint sets.

$\square$

Problem 2 Let A and B be subsets of some universal set U. Prove that

$$A = (A - B) \cup (A \cap B).$$

Answer. Let A and B be subsets of some universal set U. We will prove that $A = (A - B) \cup (A \cap B)$. Using Theorem 5.20, $(A - B) \cup (A \cap B) = (A \cap B^c) \cup (A \cap B)$. Then by the properties of Theorem 5.18,

$$\begin{aligned}
(A \cap B^c) \cup (A \cap B) &= ((A \cap B^c) \cup A) \cap ((A \cap B^c) \cup B) && (6)\\
&= ((A \cup A) \cap (A \cup B^c)) \cap ((B \cup A) \cap (B \cup B^c)) && (7)\\
&= ((A \cup A) \cap (A \cup B^c)) \cap ((B \cup A) \cap U) && (8)\\
&= (A \cap (A \cup B^c)) \cap ((B \cup A) \cap U) && (9)\\
&= A \cap ((B \cup A) \cap U) && (10)\\
&= A \cap (B \cup A) && (11)\\
&= A. && (12)
\end{aligned}$$

We will prove line (12), that $A \cap (B \cup A) = A$. Assume $x \in A \cap (B \cup A)$. By the definition of intersection $x \in A$. Then $A \supseteq A \cap (B \cup A)$. Next assume $y \in A$. $y \in B \cup A$ by the definition of union. Then since $y \in A$ and $y \in B \cup A$, $y \in A \cap (B \cup A)$ by the definition of intersection. Then $A \subseteq A \cap (B \cup A)$. Since they are subsets of each other, $A = A \cap (B \cup A)$. Therefore $A = (A - B) \cup (A \cap B)$.

$\square$

Problem 3 Prove or disprove each of the following:

1. $A - (A \cap B^c) = A \cap B$.
2. $(A^c \cap B)^c \cap A = A - B$.
3. $(A \cup B) - A = B - A$.

Answer.

1. $A - (A \cap B^c) = A \cap B$.

   Let A and B be subsets of some universal set U. We will prove that $A - (A \cap B^c) = A \cap B$. Using Theorems 5.20 and 5.20,

$$\begin{aligned} A - (A \cap B^c) &= A \cap (A \cap B^c)^c && (13) \\ &= A \cap (A^c \cup B) && (14) \\ &= (A \cap A^c) \cup (A \cap B) && (15) \\ &= \emptyset \cup (A \cap B) && (16) \\ &= (A \cap B). && (17) \end{aligned}$$

   We will prove line (16) by contradiction, that $A \cap A^c = \emptyset$. Assume there exists $x \in A \cap A^c$. Then we know $x \in A$ and $x \notin A$ by the definition of intersection and the complement. Thus we have a contradiction. Therefore there does not exist an element in the set, and $A \cap A^c = \emptyset$. Therefore $A - (A \cap B^c) = A \cap B$.

2. (Counterexample): Let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$ be subsets of the universal set $U = \{1, 2, 3, 4, 5\}$. We have $A - B = \{3\}$. By De Morgan's Laws,

$$(A^c \cap B)^c \cap A = (A \cup B^c) \cap A.$$

   $B^c = \{3, 4, 5\}$. Then $A \cup B^c = \{1, 2, 3, 4, 5\}$, and finally $(A \cup B^c) \cap A = \{1, 2, 3\}$. Therefore, $(A^c \cap B)^c \cap A \neq A - B$.

3. Let A and B be subsets of some universal set U. We will prove that $(A \cup B) - A = B - A$. By the definition of the set difference and the properties of Theorem 5.18,

$$\begin{aligned}
(A \cup B) - A &= (A \cup B) \cap A^c && (18)\\
&= (A^c \cap A) \cup (A^c \cap B) && (19)\\
&= \emptyset \cup (A^c \cap B && (20)\\
&= (A^c \cap B) && (21)\\
&= B \cap A^c && (22)\\
&= B - A. && (23)
\end{aligned}$$

We will prove line (20) by contradiction, that $A^c \cap A = \emptyset$. Assume there exists $x \in A \cap A^c$. Then we know $x \in A$ and $x \notin A$ by the definition of intersection and the complement. Thus we have a contradiction. Therefore there does not exist an element in the set, and $A^c \cap A = \emptyset$. Thus we have proven that $(A \cup B) - A = B - A$.

□

Problem 4 Prove Theorem 5.25, Part (5):

$$A \times (B - C) = (A \times B) - (A \times C).$$

Answer. Let A, B, and C be subsets of some universal set U. We will prove that $A \times (B - C) = (A \times B) - (A \times C)$. Let $q \in A \times (B - C)$. Then there exists $x \in A$ and $y \in (B - C)$ such that $q = (x, y)$. By the definition of set difference, $y \in B$ and $y \notin C$. Then $q \in A \times B$, but $q \notin A \times C$ because $y \notin C$. Then by the definition of the set difference, $q \in (A \times B) - (A \times C)$. Thus $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

Next, let $r \in (A \times B) - (A \times C)$. By the set difference, $r \notin (A \times C)$. Then there exists $a \in A$ and $b \in B - C$ such that $r = (a, b)$. Now, $r \in A \times (B - C)$ because $a \in A$ and $b \in B - C$. Thus $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

Therefore $A \times (B - C) = (A \times B) - (A \times C)$.

□

Problem 5 For each natural number n, let

$$A_n = \{k \in \mathbb{N} \mid k \geq n\}.$$

1. $\bigcap_{i \in \mathbb{N}} A_i$

2. $\bigcup_{i \in \mathbb{N}} A_i$

Answer.

1. $\bigcap_{i \in \mathbb{N}} A_i = \emptyset$ because $n \notin A_n \cap A_{n+1}$.
2. $\bigcup_{i \in \mathbb{N}} A_i = \mathbb{N}$ because we start with set of natural numbers A_1 and any other A_n is a subsets of A_1.

$\square$