Sam Kirchner

Homework #5

Problem 1 Prove by mathematical induction that for each natural number n, 6 divides $(n^3 - n)$.

Answer. Using the principle of mathematical induction, we will prove that for each natural number n, 6 divides $(n^3 - n)$. Let $P(n)$ be the statement 6 divides $(n^3 - n)$ with $n \in \mathbb{N}$.

First, we will prove the basis step, $P(1)$. $P(1)$ is the statement 6 divides $(1^3 - 1)$, simplified as 6 divides 0. This is true since $0 = 6 \cdot 0$. Therefore $P(1)$ is true.

Next, we will prove the inductive step. We will assume $P(k)$ is true and prove that $P(k+1)$ is also true. Since $P(k)$ is true we have 6 divides $(k^3 - k)$. By the definition of divides, that is $(k^3 - k) = 6j$ with $j \in \mathbb{Z}$. Then also, $k^3 = 6j + k$. Now using algebra and substitution,

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 2k \\ &= 6j + k + 3k^2 + 2k \\ &= 6j + 3k^2 + 3k \\ &= 6j + 3k(k+1).\end{aligned}$$

Notice that we have k and $(k+1)$ as factors of $3k(k+1)$. Since $k \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, we know $k \in \mathbb{Z}$. By the properties of $\mathbb{Z}$ we have two possible cases, k is even or k is odd.

(Case 1): k is even. By the definition of even, $k = 2m$ with $0 < m \in \mathbb{Z}$. Now we can substitute $2m$ for k,

$$\begin{aligned}(k+1)^3 - (k+1) &= 6j + 3k(k+1) \\ &= 6j + 3 \cdot 2m(2m+1) \\ &= 6j + 6m(2m+1) \\ &= 6(j + m(2m+1)).\end{aligned}$$

By the closure properties of $\mathbb{Z}$, $(j + m(2m+1)) \in \mathbb{Z}$. So by the definition of divides, 6 divides $((k+1)^3 - (k+1))$. Therefore $P(k+1)$ is true.

(Case 2): k is odd. By the definition of odd, $k = 2n + 1$ with $0 \leq n \in \mathbb{Z}$.

Now we can substitute $2n+1$ for k,

$$\begin{aligned}(k+1)^3-(k+1) &= 6j+3k(k+1)\\ &= 6j+3(2n+1)(2n+2)\\ &= 6j+3(2n+1)2(n+1)\\ &= 6j+6(2n+1)(n+1)\\ &= 6(j+(2n+1)(n+1)).\end{aligned}$$

By the closure properties of $\mathbb{Z}$, $(j+(2n+1)(n+1)) \in \mathbb{Z}$. So by the definition of divides, 6 divides $((k+1)^3-(k+1))$. Therefore $P(k+1)$ is true.

As we can see, in either case $P(k+1)$ is true. Therefore by the principle of mathematical induction, for each natural number n, 6 divides (n^3-n).

$\square$

Problem 2 Prove that for each odd natural number n with $n \geq 3$,

$$(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{4})(1-\frac{1}{5})\cdots(1+\frac{(-1)^n}{n}) = 1.$$

Answer. Let $P(n)$ with $n \in \mathbb{N}$ be the statement

$$(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{4})(1-\frac{1}{5})\cdots(1+\frac{(-1)^n}{n}) = 1.$$

Using the extended principle of mathematical induction, we will prove that for each odd natural number n with $n \geq 3$, $P(n)$.

First, we will prove the basis step, $P(3)$. $P(3)$ is the statement $(1+\frac{1}{2})(1-\frac{1}{3}) = 1$, which simplified is

$$\begin{aligned}1-\frac{1}{3}+\frac{1}{2}-\frac{1}{6} &= 1\\ 1 &= 1.\end{aligned}$$

We can see that the statement $P(3)$ is true.

Next, we will prove the inductive step. We will assume $P(k)$ is true and prove that $P(k+2)$ is also true. $P(k)$ is the following statement,

$$(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{4})(1-\frac{1}{5})\cdots(1+\frac{(-1)^k}{k}) = 1.$$

We will multiply both sides of this equation by $((1+\frac{1}{k+1})(1-\frac{1}{k+2}))$ and use algebra to show $P(k+2)$.

$$(1+\frac{1}{2})(1-\frac{1}{3})\cdots(1+\frac{(-1)^k}{k})(1+\frac{1}{k+1})(1-\frac{1}{k+2}) = 1(1+\frac{1}{k+1})(1-\frac{1}{k+2})$$

$$\begin{aligned}
&= 1(1+\frac{1}{k+1})(1-\frac{1}{k+2}) \\
&= 1(1-\frac{1}{k+2}+\frac{1}{k+1}-\frac{1}{(k+1)(k+2)}) \\
&= 1(1-\frac{(k+1)+(k+2)-1}{(k+1)(k+2)}) \\
&= 1(1+\frac{-(k+1)+(k+2)-1}{(k+1)(k+2)}) \\
&= 1(1+\frac{1-1}{(k+1)(k+2)}) \\
&= 1(1+0) \\
&= 1.
\end{aligned}$$

This shows that the statement $P(k+2)$ is also true. Therefore by the extended principle of mathematical induction, for each odd natural number n with $n \geq 3$,

$$(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{4})(1-\frac{1}{5})\cdots(1+\frac{(-1)^n}{n}) = 1.$$

$\square$

Problem 3 Prove that for each natural number n,

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{n\cdot(n+1)\cdot(n+2)} = \frac{1}{4}-\frac{1}{2(n+1)(n+2)}.$$

Answer. Let $P(n)$ with $n \in \mathbb{N}$ be the statement

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{n\cdot(n+1)\cdot(n+2)} = \frac{1}{4}-\frac{1}{2(n+1)(n+2)}.$$

Using the principle of mathematical induction, we will prove that for each natural number n, $P(n)$.

First, we will prove the basis step, $P(1)$. $P(1)$ is the statement $\frac{1}{1\cdot 2\cdot 3} = \frac{1}{4} - \frac{1}{2(1+1)(1+2)}$, which simplified is

$$\begin{aligned}
\frac{1}{6} &= \frac{1}{4}-\frac{1}{2\cdot 2\cdot 3} \\
\frac{1}{6} &= \frac{3}{12}-\frac{1}{12} \\
\frac{1}{6} &= \frac{2}{12} \\
\frac{1}{6} &= \frac{1}{6}.
\end{aligned}$$

We can see that the statement $P(1)$ is true.

Next, we will prove the inductive step. We will assume $P(k)$ is true and prove that $P(k+1)$ is also true. $P(k)$ is the following statement,

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{k\cdot(k+1)\cdot(k+2)}=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}.$$

We will add $\frac{1}{(k+1)(k+2)(k+3)}$ to both sides of this equation and use algebra to show $P(k+1)$.

$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{k\cdot(k+1)\cdot(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$$\begin{aligned}
&=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}\\
&=\frac{1}{4}+\frac{-(k+3)+2}{2(k+1)(k+2)(k+3)}\\
&=\frac{1}{4}+\frac{-(k+1)}{2(k+1)(k+2)(k+3)}\\
&=\frac{1}{4}-\frac{1}{2(k+2)(k+3)}
\end{aligned}$$

This shows that the statement $P(k+1)$ is also true. Therefore by the principle of mathematical induction, for each natural number n,

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{n\cdot(n+1)\cdot(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}.$$

.

∎

Problem 4 Assume that $f_1, f_2, ..., f_n, ...$ are the Fibonacci numbers. Prove that for each natural number n,

$$f_1+f_3+f_5+\cdots+f_{2n-1}=f_{2n}.$$

The left-hand side is the sum of all Fibonacci numbers with odd indices before f_{2n}.

Answer. Assume that $f_1, f_2, ..., f_n, ...$ are the Fibonacci numbers. Let $P(n)$ with $n\in\mathbb{N}$ be the statement

$$f_1+f_3+f_5+\cdots+f_{2n-1}=f_{2n}.$$

Using the principle of mathematical induction, we will prove that for each natural number n, $P(n)$.

First, we will prove the basis step, $P(1)$. $P(1)$ is the statement $f_1 = f_{2(1)}$. We know this is true because $f_1 = 1$ and $f_2 = 1$ by the definition of the Fibonacci numbers. Therefore $P(1)$ is true.

Next, we will prove the inductive step. We will assume $P(k)$ is true and prove that $P(k+1)$ is also true. $P(k)$ is the following statement,

$$f_1 + f_3 + f_5 + \cdots + f_{2k-1} = f_{2k}.$$

We can add the next odd Fibonacci number f_{2k+1} to both sides of this equation to show $P(k+1)$.

$$\begin{aligned} f_1 + f_3 + f_5 + \cdots + f_{2k-1} + f_{2k+1} &= f_{2k} + f_{2k+1} \\ &= f_{2k+2} \\ &= f_{2(k+1)}. \end{aligned}$$

$f_{2k} + f_{2k+1} = f_{2k+2}$ by the definition of the Fibonacci numbers. From this. we can see that $P(k+1)$ must also be true. Therefore by the principle of mathematical induction, for each natural number n,

$$f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}.$$

□