Math 310 Spring 2021 Problem set 9

Problem (Ex3.10)(a) Prove that if R is an integral domain, then the only idempotent elements of R are 0_R and 1_R.

Proof. Suppose R is an integral domain. Using the definition of idempotent, an idempotent $e \in R$ must satisfy $e^2 = e$. Then using part (a3) of a Ring's definition we can rewrite this as $e^2 - e \overset{(a3)R}{=} 0_R$. By the definition of an integral domain R is a ring with identity, so we can use Lemma 3.T.3 (7) to rewrite it as $e^2 + (-1_R)e \overset{(7)R}{=} 0_R$. Let's rewrite this as $ee + (-1_R)e = 0_R$ and then we can use the distributive property of Rings (d1). We then have $e(e - 1_R) \overset{(d1)R}{=} 0_R$. Integral domains have the no zero divisor property (n0), meaning either e or $(e - 1_R)$ equals 0_R. If $e = 0_R$, then our equation is satisfied and 0_R is an idempotent element. If $e \neq 0_R$, then $(e - 1_R) = 0_R$ by (n0). Then using the inverse for addition in R (a3), $e = 1_R$ and 1_R is an idempotent element. Therefore the only idempotent elements of R are 0_R and 1_R. $\square$

Problem (Ex3.10)(b) Prove that if (R and S are integral domains and S is a subring of R), then $1_S = 1_R$.

Proof. Suppose R and S are integral domains and that S is a subring of R. Let's suppose that there exists an element $a \in S$ which is not the additive identity 0_R. By the definition of subrings, $a \in S$ and $a \in R$. Since they are both integral domains they are both rings with identity. Then the multiplicative identity (m3) says that there is an element 1_R such that $a1_R = a$ and there is an element 1_S such that $a1_S = a$. From this we have $a1_S = a1_R$. Theorem 3.7 says R satisfies the cancellation property (canc), so we have $1_S \overset{(canc)R}{=} 1_R$ because $a \neq 0_R$ and $1_S \in R$. Therefore $1_S = 1_R$. $\square$

Problem (Ex3.11)(b) Let T be the product ring $T = \mathbb{R} \times \mathbb{Z}$. Find an element of T that is neither a unit, nor a zero divisor, nor equal to 0_T.
(In part (b) a proof is not needed; you only need to give your final answer.)

Answer. (2,2) $\square$

Problem (Ex3.13)

(Note: Since problem (Ex3.13) asks you to prove part of Lemma 3.T.10, you may not use that lemma in any part of this problem.)

Problem (Ex3.13)(a) Let R, S, and T be rings, and let $f : R \to S$ and $g : S \to T$ be homomorphisms. Prove that the function $g \circ f : R \to T$ is a homomorphism.

Proof. Suppose R, S, and T are rings, and that $f : R \to S$ and $g : S \to T$ are homomorphisms. Also let there be the function $g \circ f : R \to T$. Suppose that there are elements a and b that are in R.

Now we can have $g \circ f(a + b)$. Using the definition of of composition, this is $g(f(a + b))$. Since we know f is a homomorphism, by it's definition $f(a+b) = f(a)+f(b)$ so then $g(f(a+b)) = g(f(a)+f(b))$. And then similarly g is a homomorphism so $g(f(a) + f(b)) = g(f(a)) + g(f(b))$. By definition of composition, this is $g\circ f(a)+g\circ f(b)$. So we have $g\circ f(a+b) = g\circ f(a)+g\circ f(b)$.

Next let's have $g \circ f(ab)$. By the definition of composition, this is $g(f(ab))$. Then since f is a homomorphism, $g(f(ab)) = g(f(a)f(b))$. Next because g is a homomorphism, we have $g(f(a)f(b)) = g(f(a))g(f(b))$. By the definition of composition $g(f(a))g(f(b)) = (g \circ f)(a)(g \circ f)(b)$. Thus $(g \circ f)(ab) = (g \circ f)(a)(g \circ f)(b)$.

By the definition of homomorphism, $g \circ f$ is also a homomorphism.

□

Problem (Ex3.13)(b) Let R, S, and T be rings, and let $f : R \to S$ and $g : S \to T$ be isomorphisms. Prove that the function $g \circ f : R \to T$ is an isomorphism.

Proof. Suppose R, S, and T are rings, and that $f : R \to S$ and $g : S \to T$ are isomorphisms. Suppose there's also the function $g \circ f : R \to T$.

Suppose there's an element $t \in T$. Since g is surjective by the definition of isomorphism, there must exist some $s \in S$ such that $g(s) = t$ by the definition of surjective. Similarly because f is surjective, there must exist some $r \in R$ such that $f(r) = s$. From this we have $g(f(r)) = t$. Then using the definition of composition, $t = g\circ f(r)$. This shows that $g\circ f$ is surjective.

Suppose there are elements a and b in R. Let $g \circ f(a) = g \circ f(b)$. Then by definition of composition, $g(f(a)) = g(f(b))$. Since g is injective by definition of isomorphism, $f(a) = f(b)$. Similarly, f is also injective and $a = b$. This shows that $g \circ f$ is injective.

By Ex3.13(a), $g\circ f$ is a homomorphism because g and f are both homomorphisms by the definition of isomorphism.

Therefore by definition of isomorphism, since $g \circ f$ is both bijective and a homomorphism, $g \circ f$ is an isomorphism.

□

Problem (Ex3.13)(c) Explain briefly why part (Ex3.13)(b) shows that the relation $\cong$ on the set of rings satisfies the transitive property.

Explanation. Saying $f : R \to S$ is an isomorphism means that R is isomorphic to S. In part B we showed that if $R \cong S$ and $S \cong T$, then $R \cong T$, which is the transitive property. $\square$

Problem (Ex3.14) Prove Thm 3.T.12(e): Let R and S be rings with $R \cong S$. Prove that if R is a field then S is a field.
(Note: You may use Theorem 3.T.12 parts (a) and (b) in your proof.)

Proof. Suppose R and S are rings with $R \cong S$ and R is a field. By the definition of isomorphic, let our isomorphism be $f : R \to S$. By Theorem 3.T.12 (a)/(b) we know that S is a commutative ring with identity. Next suppose there is an $r \in R$ where $r \neq 0_R$. Since R is a field, we know there exists an inverse of r by property (m4). Let's write this as r^{-1}. Now we can use our function to get $f(r)$ and $f(r^{-1})$ which are elements of S. When we multiply them together using our definition of isomorphism (iii) we get $f(r)f(r^{-1}) \overset{(iii)}{=} f(rr^{-1})$. Then by our property (m4) in R, $f(rr^{-1}) \overset{(m4)R}{=} f(1_R)$. Then we can use Theorem 3.10 (4) since R is ring with identity and f is surjective by definintion of isomorphism. $f(1_R) \overset{Thm3.10(4)}{=} 1_S$. If r is a unit in R, then $f(r)$ is a unit in S by Theorem 3.10 (5). Note: $r \neq 0_R$, meaning $f(r) \neq 0_R$ because $f(0_R) \overset{Thm3.10(1)}{=} 0_R$. Because f is surjective, every nonzero element in S is a unit. Therefore S is a field by Lemma 3.T.6. $\square$

Problem (Ex3.17) In each part of this problem, decide whether the ring R is isomorphic to the ring S.
(If yes, then give an example of a function $f : R \to S$ that is an isomorphism; if no, then give a very brief justification.)

Problem (Ex3.17)(a) $R = \mathbb{Z} \times \mathbb{Z}$ and $S = \mathbb{Z}$.
Answer. No, f couldn't be injective because there are more elements in R than S. (Correction: They are actually the same size. TODO: write a better explanation) $\square$

Problem (Ex3.17)(b) $R = \mathbb{R}$ and $S = \left\{ \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, r \in \mathbb{R} \right\}$ with matrix addition and multiplication (a subring of $M_{2\times 2}(\mathbb{R})$).
Answer. Yes, $f : R \to S$ could be defined as $f(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. $\square$