Sam Kirchner

Homework #7

Problem 1 Let $d : \mathbb{N} \to \mathbb{N}$, where $d(n)$ is the number of positive divisors of n.

1. Is the function d an injection? *Justify your answer.*
2. Is the function d a surjection? *Justify your answer.*

Answer.

1. d is not an injection because $d(2) = d(3) = 2$.
2. Let $d : \mathbb{N} \to \mathbb{N}$, where $d(n)$ is the number of positive divisors of n. We will prove that d is a surjection. Let x be an element of the codomain $\mathbb{N}$. We can use induction to show that $d(2^{k-1}) = k$ for every natural number k. Let $P(k)$ be the statement $d(2^{k-1}) = k$. First we will show the base case, $P(1)$. That is $d(2^0) = d(1)$. The only positive divisor is 1, thus $d(1) = 1$. Next we will prove the inductive step. We will assume $P(k)$ and prove $P(k+1)$. $P(k)$ is true, meaning that $d(2^{k-1}) = k$. Now let's multiply the preimage by 2. $d(2 * 2^{k-1}) = d(2^k)$. Notice that 2^{k-1} already has divisors 1 and 2 when $k > 1$, so when we multiply 2^{k-1} by 2, the product has only 1 new *divisor*, itself, which is 2^k. Therefore $d(2^k) = k + 1$ and we have proved $P(k)$ is true. From this we can see that the image of 2^{k-1} under d is k. Thus d is a surjective function.

$\square$

Problem 2 Let $s : \mathbb{N} \to \mathbb{N}$, where $s(n)$ is the sum of all positive divisors of n.

1. Is the function s an injection? *Justify your answer.*
2. Is the function s a surjection? *Justify your answer.*

Answer.

1. s is not an injection because $s(6) = s(11) = 12$.
2. s is not a surjection because there does not exist $n \in \mathbb{N}$ such that $s(n) = 2$. If we think about it, for each $k \in \mathbb{N}$, $k < k + 1 \leq s(k)$. Now if we check the first two natural numbers, $s(1) = 1$ and $s(2) = 3$. Then there cannot be $s(n) = 2$ as 1 and 3 are the first and second minimum in the range of s.

$\square$

Problem 3 Let $A = \{(m, n) \mid m, n \in \mathbb{Z}, n \neq 0\}$. Define $f : A \to \mathbb{Q}$ by $f(m, n) = \frac{m+n}{n}$.

1. Is the function f an injection? *Justify your answer.*
2. Is the function f a surjection? *Justify your answer.*

Answer.

1. f is not an injection because $f(1, 9) = s(2, 18) = \frac{10}{9} = \frac{20}{18}$.
2. Let $A = \{(m, n) \mid m, n \in \mathbb{Z}, n \neq 0\}$. Define $f : A \to \mathbb{Q}$ by $f(m, n) = \frac{m+n}{n}$. We will prove that f is a surjection. Let $x \in \mathbb{Q}$. By the definition of rational numbers $x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. $f(a - b, b) = \frac{a}{b}$. Thus f is a surjection.

$\square$

Problem 4 Prove that if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection.

Answer. TODO. $\square$