Math 310 Spring 2021 Problem set 8

Problem (Ex3.1)(a) Let R be the set $\mathbb{Z}$. Define addition in R to be ordinary addition in $\mathbb{Z}$, and define multiplication in R by $ab = 0$ for all $a, b \in R$. Determine whether the set R, together with these addition and multiplication operations, is a ring. Prove your answer.

Answer. R is a ring. □

Proof. Suppose R is the set $\mathbb{Z}$. Define addition in R to be ordinary addition in Z and define multiplication as $ab = 0$ for all $a, b \in R$. Let a0, a1, a2, a3, a4, m0, m1, and d1 be the properties stated for the definition of a ring. Since R is the set $\mathbb{Z}$ and addition in R remains the same as in $\mathbb{Z}$, by Theorem 1.T.1(a) R has properties a0-a4. Remaining are properties m0, m1, and d1.

For the following let $a, b, c \in R$.

(m0) Closure for multiplication: $ab = 0$ by our definition of multiplication. Since $0 \in \mathbb{Z}$, R is closed under multiplication.

(m1) Associative multiplication: We know $(bc) \in R$ by the property (m0), so $a(bc) = 0$ by our definition of multiplication. By similar reasoning, $(ab)c = 0$. Thus $a(bc) = (ab)c$ and multiplication is associative.

(d1) Distributive property: We know $(b + c) \in R$ by property (a0), so $a(b + c) = 0$ by our definition of multiplication. $ab + ac = 0 + 0$ and $0+0 = 0$ by property (a2). Then $a(b+c) = ab+ac$. Using similar reasoning, $(a + b)c = ac + bc$ and the distributive property holds.

R satisfies all of these properties and is nonempty because $R = \mathbb{Z}$. Therefore R is a ring. □

Problem (Ex3.1)(b) Let R be the set $\mathbb{Z}$. Define addition in R to be ordinary addition in $\mathbb{Z}$, and define multiplication in R by $ab = 1$ for all $a, b \in R$. Determine whether the set R, together with these addition and multiplication operations, is a ring. Prove your answer.

Answer. R is not a ring. □

Proof. Suppose R is the set $\mathbb{Z}$. Define addition in R to be ordinary addition in Z and define multiplication as $ab = 1$ for all $a, b \in R$. Let a0, a1, a2, a3, a4, m0, m1, and d1 be the properties stated for the definition of a ring. Since R is the set $\mathbb{Z}$ and addition in R remains the same as in $\mathbb{Z}$, by Theorem 1.T.1(a) R has properties a0-a4. Remaining are properties m0, m1, and d1.

(d1) Distributive property: Let $a, b, c \in R$. Since $(b+c) \in R$ by property (a0), $a(b + c) = 1$ by our definition of multiplication. Next we can see that

$ab + ac = 1 + 1 = 2$, and $a(b + c) \neq ab + ac$. Thus the distributive property does not hold.

Since R doesn't satisfy (d1), it cannot be a ring by the definition of a ring. $\square$

Problem (Ex3.3)(b) Let $R = M_{2\times 2}(\mathbb{R})$ and $S = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}$.
(i) Prove that the subset S, with the addition and multiplication from the ring R, is a subring of R.

Proof of (i). Let $R = M_{2\times 2}(\mathbb{R})$ and $S = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}$. Say we have any two matrices in S, $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $B = \begin{pmatrix} j & k \\ 0 & l \end{pmatrix}$. (1) When we subtract A from B we get $\begin{pmatrix} a - j & b - k \\ 0 & d - l \end{pmatrix}$. All entries are in $\mathbb{R}$ because of closure for subtraction, and since the entry of row 2, column 1 is still 0, this matrix is an element of S. Thus S has closure under subtraction. (2) When we multiply, $A * B = \begin{pmatrix} aj + b * 0 & ak + bl \\ 0 * j + d * 0 & 0 * k + dl \end{pmatrix} = \begin{pmatrix} aj & ak + bl \\ 0 & dl \end{pmatrix}$. All entries are still in $\mathbb{R}$ because of closure for addition and multiplication, and the entry at row 2, column 1 is still zero, so this matrix is an element of S. Thus S has closure for multiplication. (0) Since we know that R is non-empty by definition of a ring, we also know that our S is nonempty. If we have our matrix in R, all we have to do is change the entry at row 2, column 1 to 0 and it would be in S.

Since it satisfies (0), (1), and (2) for Theorem 3.6, S is a subring of R. $\square$

(ii) Determine whether or not S is a ring with identity.
(iii) Determine whether S is commutative.
(In parts (ii-iii) give a brief justification of your answer; a full proof is not needed in (ii-iii)).

Answers for (ii) and (iii).

(ii) R has an identity $1_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and 1_R is an element of S. 1_R must be an identity for all elements of the set R and since S is a subset of R, it holds in S.

(iii) S is not commutative. Using the same A and B from earlier, $A * B = \begin{pmatrix} aj & ak + bl \\ 0 & dl \end{pmatrix}$ and $B * A = \begin{pmatrix} ja & jb + kd \\ 0 & ld \end{pmatrix}$. $AB \neq BA$ so it's not commutative. $\square$

Problem (Ex3.4)(b)(a) Let S be a subring of a ring R. Prove that if R is commutative, then S is commutative.

Proof. Suppose S is a subring of a ring R and that R is commutative. Let a and b be any elements where $a, b \in S$. By the definition of subring S is a subset of R, meaning a and b are elements in R by the definition of subset. Since R is commutative, $ab = ba$. Therefore by the definition of commutative, S is commutative. □

Problem (Ex3.4)(b)(b) Let T and U be rings, and let $T \times U$ be the product ring. Prove that if T and U are commutative, then $T \times U$ is commutative.

Proof. Let T and U be rings, and let $T \times U$ be the product ring. Suppose T and U are commutative. By Theorem 3.1, $T \times U$ is a ring. Then by the definition of the product ring, the multiplication in $T \times U$ is defined by $(t, u)(t', u') = (tt', uu')$. Since T and U are commutative, $(tt', uu') = (t't, u'u)$. Then $(t't, u'u) = (t', u')(t, u)$ using our definition of multiplication in $T \times U$. Therefore $(t, u)(t', u') = (t', u')(t, u)$ and by definition of commutative, $T \times U$ is commutative. □

Problem (Ex3.6)(b) Let R be a ring and fix an element b of R. Show that the subset $S = \{rb \mid r \in R\}$, with the addition and multiplication from R, is a subring.

Proof. Let R be a ring with a fixed element b of R. and suppose there's a subset of R, $S = \{rb \mid r \in R\}$ with the addition and multiplication from R. For the following let j and k be elements of S. Then $j = rb$ and $k = r'b$ for $r, r' \in R$.

(1) Now $j + k = rb + r'b$. Then this can be rewritten as $(r + r')b$. Since there is closed addition in R, $(r + r1) \in R$, meaning $(r + r')b$ is in our set S. Thus $(j + k) \in S$.

(2) Next $j * k = rb * r'b = (rbr')b$. Since there's closure for multiplication in R, $(rbr') \in R$ and $(j * k)$ is in our set S.

(3) We have $0_R \in R$ by the definition of a Ring. Then $0_R * b = 0_R$ by Lemma 3.T.3(2). Since $0_R * b$ fits our definition for S, $0_R \in S$.

(4) $j + x = 0_R$ for some $x \in R$ by definition of inverses for addition. Then $x = 0_R - j = 0_R - rb$. Using the additive identity, $x = -rb$ or $x = (-r)b$. Because R is a ring and has inverses for addition, $(-r) \in R$ and $x \in S$.

Then by Theorem 3.2, S is a subring of R. □

Problem (Ex3.7) Let R be a ring. An element e of R is defined to be idempotent if $e^2 = e$.

Problem (Ex3.7)(b) Find four idempotent elements in the ring $M_{2\times 2}(\mathbb{Z}_2)$. (In part (b), no proof is needed; you only need to give your final answer.)

Answer. $\begin{pmatrix} [1] & [0] \\ [1] & [0] \end{pmatrix}, \begin{pmatrix} [1] & [0] \\ [0] & [1] \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ [1] & [1] \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ [0] & [1] \end{pmatrix}$ □

Problem (Ex3.7)(c) Find all of the idempotent elements of the ring $\mathbb{Z}_{12}$. (In part (c), no proof is needed; you only need to give your final answer.)

Answer. $[0], [1], [4], [9]$ □