CMP_SC 3050: Asymptotic growth of functions

Rohit Chadha

September 3, 2014

Announcements

Homework 1 is on Blackboard. Due, September 9, 2014 at 12:29:29 pm.

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- Teaching Assistant: Kevin Melkowski
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  Office hours: Friday, 1:00 - 3:00 pm
- Teaching Assistant: Avimanyou Vatsa
  Email: akvhxd@mail.missouri.edu
  Office: 303 EBN
  Office hours: Friday, 3:00 pm - 5:00 pm
- Teaching Assistant: Eric C. Gaudiello
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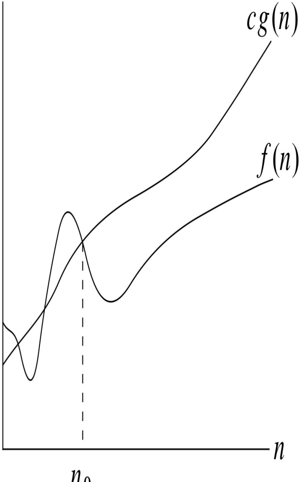
  Email: ecgprc@mail.missouri.edu
  Office hours: Tuesday, Thursday 3:30 pm - 4:30 pm

Asymptotic analysis

- Running times are functions of sizes of input
- Our focus is on how the running time grows as input size grows
- Such kind of analysis is called asymptotic analysis (analysis in the limit)
- In this analysis, we can ignore low-order terms and constant factors of functions
- The Θ notation is an example of this analysis (compares functions)
- The other example that you have seen in CMP_SC 2050 is the O (big-oh) notation

O-notation



$f(n) = O(g(n))$

Informal: Asymptotically, a function $f(n)$ is said to be $O(g(n))$ if in the limit $f(n)$ is smaller than some constant times $g(n)$

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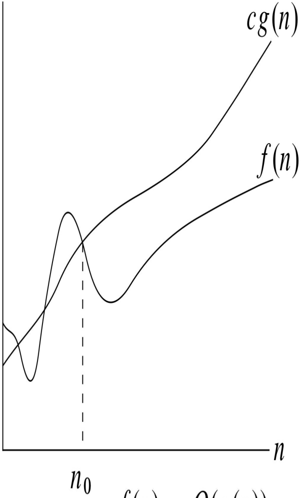

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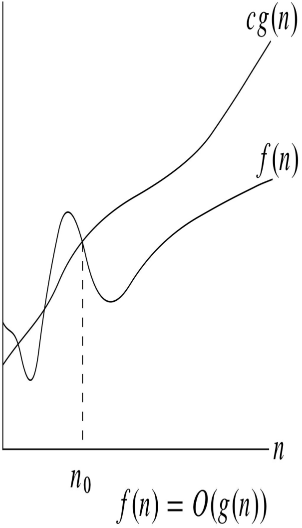

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$O \approx \leq$: Useful to think of O-notation as saying that the function f is less than the function g

Examples

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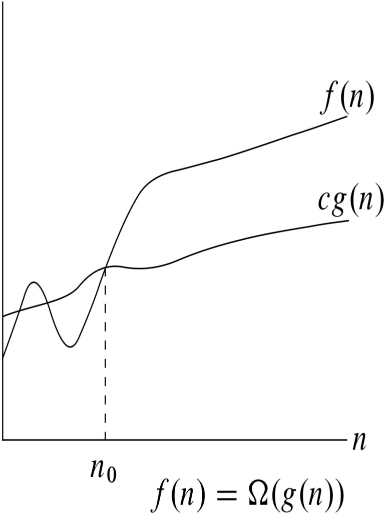
Ω-notation



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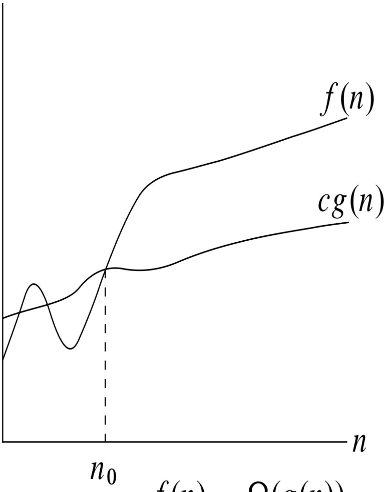

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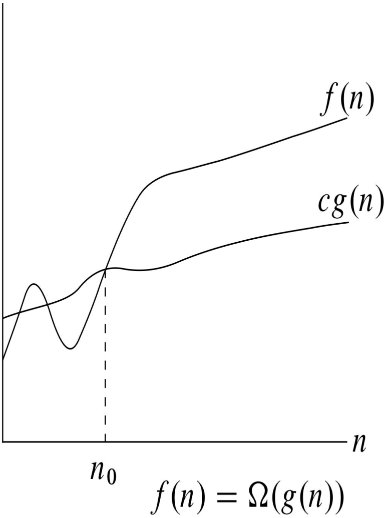

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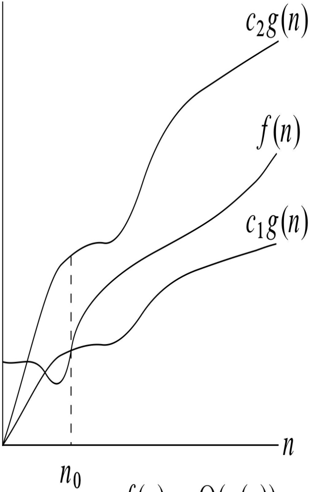

$f(n) = \Theta(g(n))$

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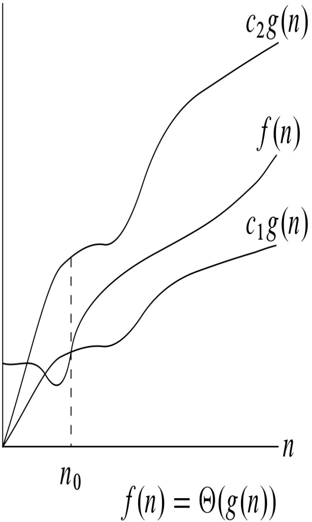

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- $f(n)$ is $O(f(n))$. $f(n)$ is also $\Omega(f(n))$. $f(n)$ is also $\Theta(f(n))$.

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Transpose Symmetry

- $f(n)$ is $O(g(n))$ if and only if $g(n)$ is $\Omega(f(n))$

A non-property

There are functions $f(n)$ and $g(n)$ which cannot be compared in asymptotic notation

That is neither $f(n)$ is $O(g(n))$ nor $f(n)$ is $\Omega(g(n))$

An example is $n^{1+\sin n}$ since $1 + \sin n$ oscillates between 0 and 2

Asymptotic bounds for polynomial functions

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- The function f is $\Theta(n^d)$ and hence $O(n^d)$
- Also, note that f is $O(n^{d+1})$ but not $\Theta(O^{d+1})$

Asymptotic bounds for logarithmic function

Logarithms: Recall $\log_b n$ (logarithm of n to the base b) is the number y such that $b^y = n$

- Base does not matter. That is for each $b > 1$, the function

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  In other words, $\log_b n$ grows much slowly than n^x

Asymptotic bounds for exponential function

Exponential:

- For each $r > 1$ and $x > 0$, the function

$$r^n \text{ is } \Omega(n^x)$$

- For each $r > 1$ and $x > 0$, the function

$$r^n \text{ is NOT } \Theta(n^x).$$

In other words, r^n grows much faster than n^x

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- $O(\log n)$: Example: Binary search
- $O(n \log n)$: Example: Merge Sort
- Cubic time:These are algorithms whose worst case behavior is $O(n^3)$

Summary

Study asymptotic behavior of running times (Chapter 3 of the book)

$$\begin{array}{ccc} O & \approx & \leq \\ \Omega & \approx & \geq \\ \Theta & \approx & = \end{array}$$

- While we will focus mainly on running times, we can also analyze the extra space used by an algorithm using the same asymptotic notation