CMP_SC 3050: Heaps

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September 4, 2014

Heaps

A heap is a nearly complete binary tree:

- The tree is completely filled except at the lowest level
- At the lowest level, the tree must be filled from the left upto a point

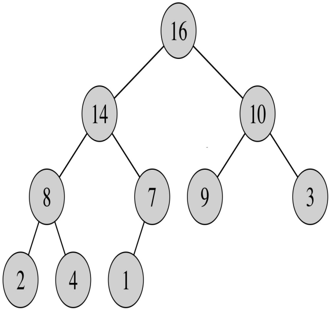
In addition, a heap must satisfy one of the following properties

1. Max-heap property: The value stored at every node must be greater than the value stored in its children

OR

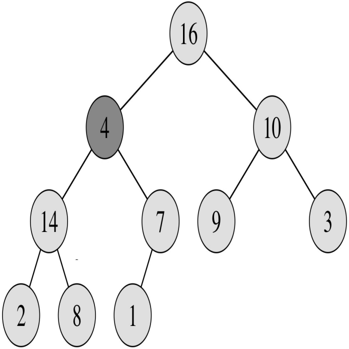
2. Min-heap property: The value stored at every node must be less than the value stored in its children

Example



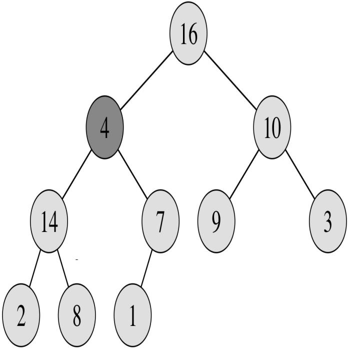
Convert a nearly complete binary tree to a max-heap

- First consider the case that at only one node the value stored is less than its children



Convert a nearly complete binary tree to a max-heap

- First consider the case that at only one node the value stored is less than its children
- Swap the value at this node with the larger value of its children



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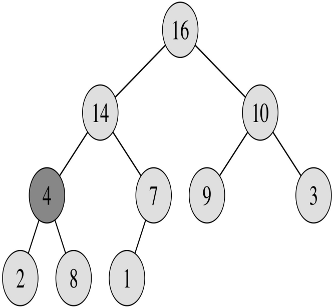

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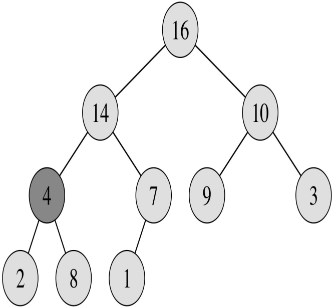

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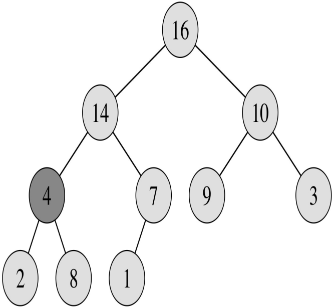

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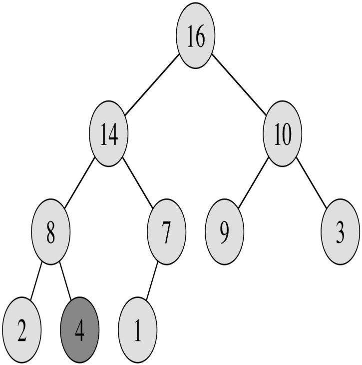

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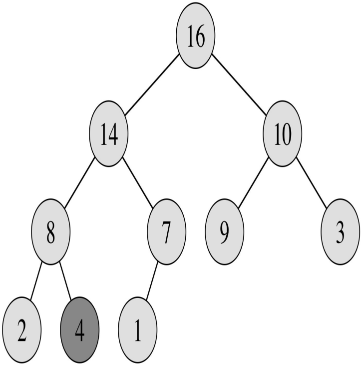

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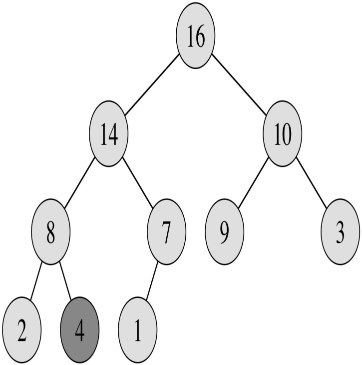
- What is the time complexity of Max-Heapify?
  - At each level the procedure spends a constant amount of time
  - Hence the running time is $O(h)$ if h is the height of the error



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- Now, there might be more than one errors
- Start at the bottom of the tree
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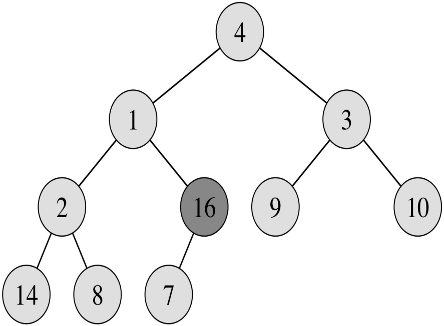
- Keep going up..
- This procedure is called BUILD-MAX-HEAP

Example: Build-Max-Heap



(a)

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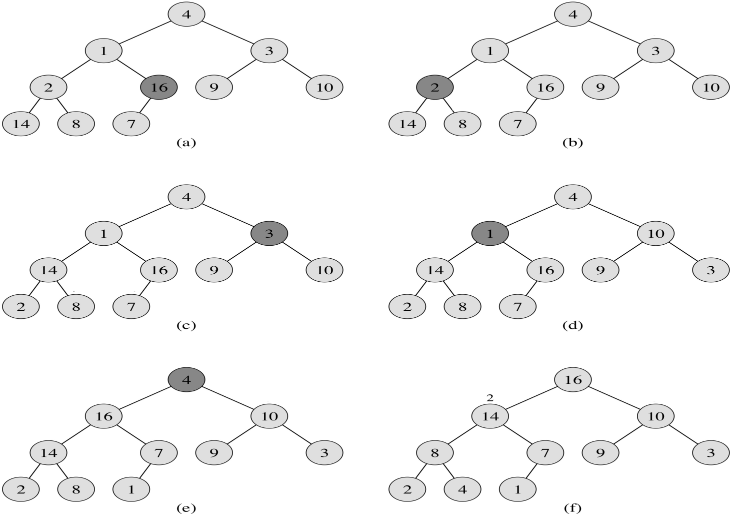

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- Max-Heapify takes $O(h)$ times on nodes at height h
- There are $\lceil \frac{n}{2^{h+1}} \rceil$ nodes of height h
- Therefore, the total cost of Build-Max-Heap is

$$\sum_{h=0}^{\lfloor \log n \rfloor} \lceil \frac{n}{2^{h+1}} \rceil O(h) = O(n) \sum_{h=0}^{\lfloor \log n \rfloor} O(\frac{h}{2^h})$$

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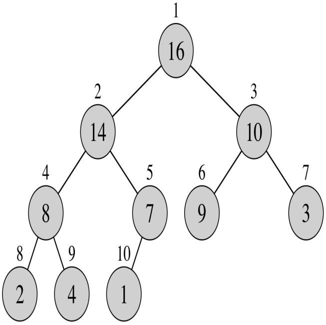
- Thus MAX-HEAPIFY takes $O(n)$ time
- You should know the result

But how do we implement heaps?

- A heap of size n can be implemented as an array of size n



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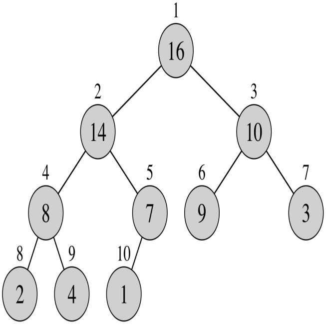

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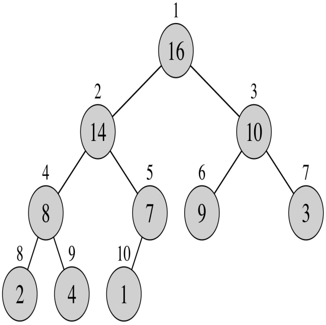

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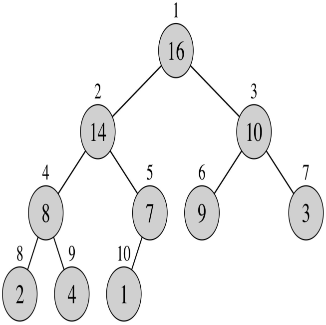

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

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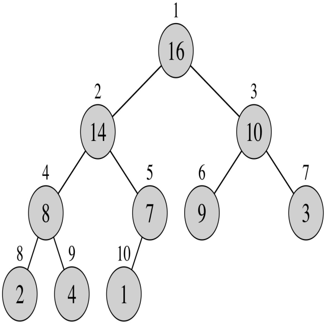

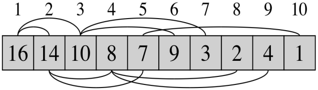

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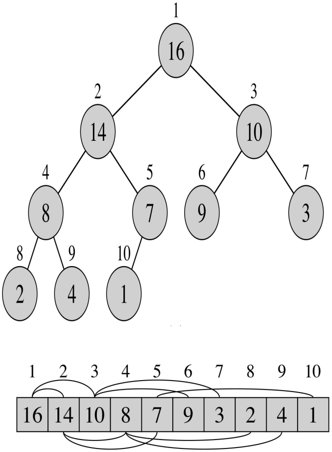
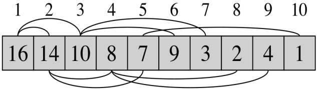
- We did this for heaps, but we can do the same thing for every nearly complete binary tree
  - Infact, we can consider any array as a nearly complete binary tree!



Arrays as nearly complete binary trees



$$
\begin{array}{lcl}
\text{Root of the tree} & = & A[1] \\
\text{Left child of } A[i] & = & A[2i] \\
\text{Right child of } A[i] & = & A[2i+1] \\
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$$

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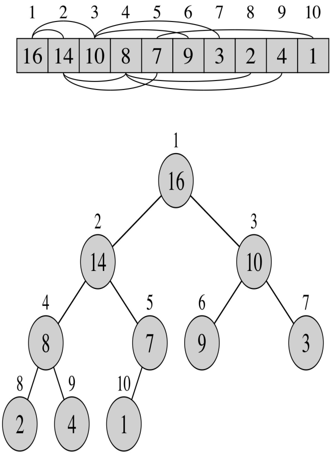

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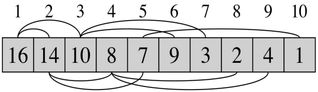

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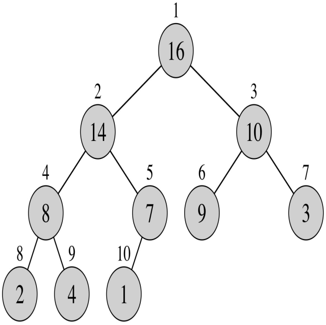

Pseudocode for MAX-HEAPIFY

```
MAX-HEAPIFY(A, i, n)
  l = LEFT(i)
  r = RIGHT(i)
  if l ≤ n and A[l] > A[i]
      largest = l
  else largest = i
  if r ≤ n and A[r] > A[largest]
      largest = r
  if largest ≠ i
      exchange A[i] with A[largest]
      MAX-HEAPIFY(A, largest, n)
```

Makes the subtree rooted at $A[i]$ a heap if the subtrees rooted at $A[\text{LEFT}(i)]$ and $A[\text{RIGHT}(i)]$ are heaps

Pseudocode for Build-Max-Heap

```
BUILD-MAX-HEAP(A, n)
  for i = ⌊n/2⌋ downto 1
      MAX-HEAPIFY(A, i, n)
```

Makes the nearly complete binary tree stored in $A[1 \,..\, n]$ a heap

An application of Heaps: Sorting

Given an array, the Heapsort algorithm on an array A of size n acts as follows:

- Convert array A into a heap A (use $\textsc{Build-Max-Heapify}$)

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- Call $\textsc{Max-Heapify}$ to make $A[1\,..\,n-1]$ a heap

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- The $A[1..n-1]$ is also almost a heap except that the root may be smaller than its children
- Call MAX-HEAPIFY to make $A[1..n-1]$ a heap
- Repeat until only one node remains remains

Pseudocode and analysis of heap sort

HEAPSORT(A, n)	**cost**	**times**
BUILD-MAX-HEAP(A, n)		
for $i = n$ **downto** 2		
exchange $A[1]$ with $A[i]$		
MAX-HEAPIFY($A, 1, i-1$)		

Pseudocode and analysis of heap sort

HeapSort(A, n)	**cost**	**times**
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Heapsort takes $O(n \log n)$ time

A new data structure: Max Priority Queues

- Like stacks, queues and lists, maintains a dynamic set
- Each element has a *key*. May have other data

Operations

$\textsc{Insert}(A, x)$:	inserts element x into priority queue A
$\textsc{Maximum}(A)$:	returns element of A with largest key
$\textsc{Extract-Max}(A)$:	removes and returns element of A with largest key
$\textsc{Increase-Key}(A, x, k)$:	increases value of element x's key to k

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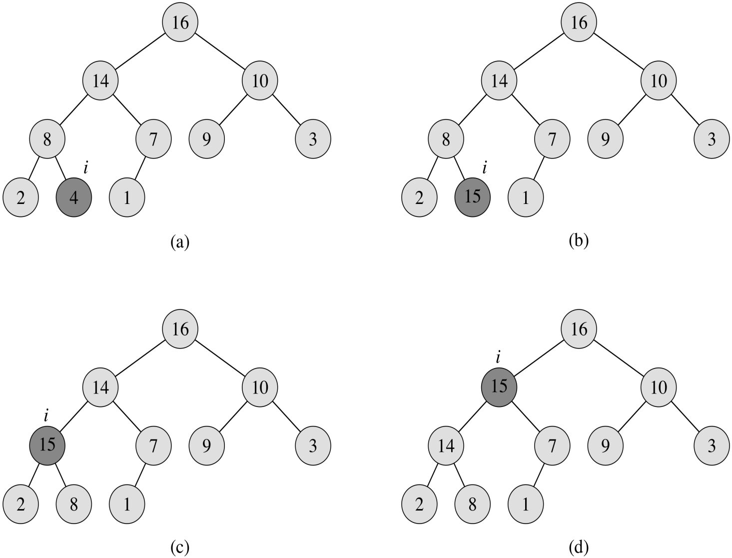
Can be implemented using heaps

Example INCREASE-KEY(A, x, k): Increase key 4 to 15



INCREASE-KEY(S, x, k)

HEAP-INCREASE-KEY(A, i, key)

if $key < A[i]$
  error "new key is smaller than current key"
$A[i] = key$
while $i > 1$ and $A[\text{PARENT}(i)] < A[i]$
  exchange $A[i]$ with $A[\text{PARENT}(i)]$
  $i = \text{PARENT}(i)$

- Make sure k is bigger than x's current key
- Update x's key value to k
- Traverse the heap upward comparing x to its parent and swapping keys until x's key is smaller than its parent's key

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- Make sure k is bigger than x's current key
- Update x's key value to k
- Traverse the heap upward comparing x to its parent and swapping keys until x's key is smaller than its parent's key
- Time complexity is $O(\log n)$

Conclusion

- We learnt a new useful data structure: Heaps
- Heaps can be used to sort in $O(n \log n)$ time
- Heaps can implement Priority Queues with insertion, deletion and changing keys operations all taking $O(\log n)$ time
- We just finished Chapter 6 of the book

Next class: We start with graph algorithms (Chapter 22)