# Identification of exogenous disturbance signal sets

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***Abstract*— This work deals with uncertain linear models of dynamical systems, with the uncertainty modeled as an exogenous disturbance signal acting on the output. A method to identify the set within which this uncertainty lies is presented.**

## I. Introduction

An ARX model of a dynamical system is identified, which is parameterized as follows:

$$A(q^{-1})y(k) = B(q^{-1})u(k) + w(k)$$

For this, the dataset $D_N = \{u(k), y(k); k \in 1,2,..,N\}$ obtained from open-loop experiments is utilized. Assuming the model is invertible, it is rewritten as

$$y(k) = M(q^{-1})u(k) + D(q^{-1})w(k) \tag{1}$$

where the transfer functions are $M(q^{-1}) = B(q^{-1})/A(q^{-1})$ and $D(q^{-1}) = 1/A(q^{-1})$. Hence, the output $y(k)$ is generated as the sum of two systems, a schematic of which is shown here:

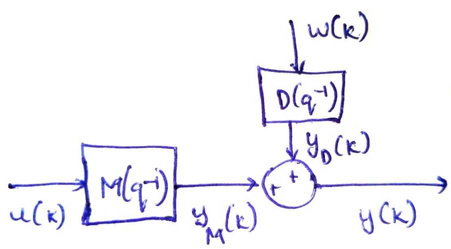


Fig. 1: ARX model

If the dataset $D_N$ is noise-free, the part $y_D(k)$ of the output $y(k)$ that the model $M(q^{-1})$ does not capture can be attributed to model uncertainty. Hence, the uncertainty is modeled as an exogenous disturbance signal acting on the output. Robust model-based control schemes can utilize this model for controller synthesis, provided the uncertainty set the exogenous disturbance signal $w(k)$ belongs to is available. In the next section, one such control scheme is presented, and a method to obtain the set in which $w(k)$ lies is presented.

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## II. Robust controller design

For controller synthesis, we first convert the transfer function in Eq.(1) to state space form, and obtain the following equations:

$$\begin{bmatrix} x_M(k+1) \\ x_D(k+1) \end{bmatrix} = \begin{bmatrix} A_M & 0 \\ 0 & A_D \end{bmatrix} \begin{bmatrix} x_M(k) \\ x_D(k) \end{bmatrix} + \begin{bmatrix} B_M \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ B_D \end{bmatrix} w(k)$$

$$y(k) = \begin{bmatrix} C_M & C_D \end{bmatrix} \begin{bmatrix} x_M(k) \\ x_D(k) \end{bmatrix} + D_M u(k) + D_D w(k) \tag{2}$$

where the states $x_M(k)$ and $x_D(k)$ belong to the system model $M$ and the disturbance model $D$ respectively. In a condensed way, they are written as:

$$\begin{aligned} x(k+1) &= Ax(k) + B_U u(k) + B_W w(k) \\ y(k) &= Cx(k) + D_U u(k) + D_W w(k) \end{aligned} \tag{3}$$

where the output vector $y(k) \in I\!R^{n_y}$. A robust reference governor can be designed to provide a control input $u(k)$ that makes the output $y(k)$ track a reference signal $r(k)$. At each time step $t$, the controller solves the optimization problem:

$$\begin{aligned} \min_{\bar{u}} \quad & \sum_{k=1}^{N_P} (\hat{y}(t+k) - r(t+k))^2 \\ \text{subject to} \quad & \hat{x}(t+k+1) = A\hat{x}(t+k) + B_U \bar{u} \\ & \hat{y}(t+k) = C\hat{x}(t+k) + D_U \bar{u} \\ & \hat{x}(t) = x(t) \\ & (x(t), \bar{u}) \in \mathbb{O}_{N_P} \end{aligned} \tag{4}$$

It reads the initial state $x(t)$ of the system, and calculates a constant control input $\bar{u}$ which is feasible with respect to the output admissible set $\mathbb{O}_{N_P}$ of the system Eq.(2) defined as:

$$\begin{aligned} \mathbb{O}_{N_P} = \{(x(t), \bar{u}) : y \in \mathbb{Y}, u(t+k) = \bar{u}, \forall w \in \mathbb{W} \\ \forall k \in \{1,2,..,N_P\}\} \end{aligned} \tag{5}$$

where $y \in \mathbb{Y}$ denotes the future output sequence $\{y(t+k) \in \mathbb{Y}, k = 1,..,N_P\}$ and $w \in \mathbb{W}$ denotes the future disturbance sequence $\{w(t+k) \in \mathbb{W}, k = 0,..,N_P\}$. It is the set of initial states $x(t)$ and a constant control input $\bar{u}$ such that the future output trajectory of the system does not violate the constraints defined by $\mathbb{Y}$ for any possible bounded disturbance sequence $w \in \mathbb{W}$, within the horizon time $N_P$.

At any future time instant $t+k$, given the initial state $x(t)$ and a constant control input $\bar{u}$, the output $y(t+k)$ is given

$$\mathbb{O}_{N_P} = \left\{ (x(t),\bar{u}) : \tilde{H} \left( \begin{bmatrix} CA \\ CA^2 \\ . \\ . \\ CA^{N_P} \end{bmatrix} x(t) + \begin{bmatrix} CB_U + D_U \\ CAB_U + D_U \\ . \\ . \\ C\sum\limits_{j=0}^{N_P-1} A^j B_U + D_U \end{bmatrix} \bar{u} + \begin{bmatrix} CB_W & D_W & . & . & 0 \\ CAB_W & CB_W & D_W & . & 0 \\ . & & & & \\ . & & & & \\ CA^{N_P-1}B_W & CA^{N_P-2}B_W & . & . & D_W \end{bmatrix} \begin{bmatrix} w(t) \\ w(t+1) \\ . \\ . \\ w(t+N_P) \end{bmatrix} \right) \leq \tilde{h} \right\} \tag{6}$$

by:

$$y(t+k) = CA^k x(t) + \left( C\sum_{j=0}^{k-1} A^j B_U + D_U \right)\bar{u} + \\ C\sum_{j=0}^{k-1} A^j B_W w(t+k-1-j) + D_W w(t+k) \tag{7}$$

It is desired to constraint the output $y(t+k)$ at a time instant $t+k$ within the polyhedral set $Hy(t+k) \leq h, h \in IR^{n_c}$. The output constraint set $\mathbb{Y}$ represents a collection of these pointwise in time polyhedron constraints, and hence is written as:

$$\mathbb{Y} = \left\{ y : \begin{bmatrix} H & . & . & 0 \\ . & H & . & . \\ . & . & . & . \\ 0 & . & . & H \end{bmatrix} \begin{bmatrix} y(t+1) \\ . \\ . \\ y(t+N_P) \end{bmatrix} \leq \begin{bmatrix} h \\ . \\ . \\ h \end{bmatrix} \right\} = \{ y : \tilde{H}y \leq \tilde{h} \}$$

Hence, the definition of set $\mathbb{O}_{N_P}$ in Eq.(8) can be rewritten as:

$$\mathbb{O}_{N_P} = \{ (x(t),\bar{u}) : \tilde{H}y \leq \tilde{h}, u(t+k) = \bar{u}, \forall w \in \mathbb{W} \\ \forall k \in \{1,2,..,N_P\} \} \tag{8}$$

Using the form in Eq.(7), the constraints can be enumerated as shown in Eq.(6). It is written in a simplified notation as:

$$\mathbb{O}_{N_P} = \left\{ (x(t),\bar{u}) : \tilde{H} \left( H_x x(t) + H_u \bar{u} + \begin{bmatrix} H_w^1 \\ . \\ . \\ H_w^{n_y N_P} \end{bmatrix} w \right) \leq \begin{bmatrix} h_Y^1 \\ . \\ . \\ h_Y^{n_c N_P} \end{bmatrix} \right\} \tag{9}$$

where row $i$ of the matrix associated to the future disturbance sequence $w$ is denoted as $H_w^i$, and element $i$ of the vector $\tilde{h}$ is denoted by $h_Y^i$. Since the output feasibility should hold over all possible future disturbance sequences, we desire to calculate the set

$$(x(t),\bar{u}) \in \mathbb{Y} \sim D\mathbb{W} \sim .. \sim CA^{t+N_P-1}B\mathbb{W} \tag{10}$$

We do this by performing a P-subtraction, resulting in the set:

$$\mathbb{O}_{N_P} = \{ (x(t),\bar{u}) : \tilde{H}(H_x x(t) + H_u \bar{u}) \leq h_{YW} \} \\ h_{YW}^i = h_Y^i - \sup_{w \in \mathbb{W}} H_w^i \vec{w} \tag{11}$$

where $h_{YW}^i$ is the element $i$ of the vector $h_{YW}$.
To calculate this input feasible set, we need the disturbance sequence set $\mathbb{W}$. The calculation of this set is discussed in the next section.

## III. Calculation of $\vec{\mathbb{W}}$

Since the model uncertainty is captured as an exogenous disturbance signal, and the measurement data in $D_N$ is noise-free, the set in which this exogenous disturbance signal lies can be extracted from $D_N$. To this end, we first write the disturbance model separately as:

$$x_D(k+1) = A_D x_D(k) + B_D w(k) \\ y_D(k) = C_D x_D(k) + D_D w(k) \tag{12}$$

and the system model as:

$$x_M(k+1) = A_M x_M(k) + B_M u(k) \\ y_M(k) = C_M x_M(k) + D_M u(k) \tag{13}$$

The system model Eq.(13) is simulated with input signal $u(k)$ obtained from dataset $D_N$, with the initial condition $x_M(0) = 0$. The output of this simulation is $y_M(k)$. The output of the exogenous disturbance block $y_D(k)$ is hence calculated as $y(k) - y_M(k)$, which at each time instant $k$ is indicated as lying in a polyhedral set

$$y_D(k) \in \mathbb{Y}_N^D = \{ y_D : H_D y_D \leq h_D \} \tag{14}$$

In the limit of infinite data $D_N$, the set $\mathbb{Y}_N^D$ approaches the actual exogenous disturbance output set $\mathbb{Y}_\infty^D$. Using this set, which is the output constraint set of the system described by Eq.(12), we calculate $\vec{\mathbb{W}}(x_D(t))$ at each time instant $t$. It is the set in which the sequence of disturbance inputs $\{w(t+k) : k = 0 : N_P\}$ should lie in, such that the output constraint $\{y_D(t+k) \in \mathbb{Y}_\infty^D : k = 0 : N_P\}$ of the disturbance model are respected.
To calculate $\vec{\mathbb{W}}(x_D(t))$, we write the predicted output $y_D(t+k)$ of the disturbance model given the initial state $x_D(t)$ as:

$$y_D(t) = C_D x_D(t) + D_D w(t) \text{ if } k = 0 \\ y_D(t+k) = C_D A_D^k x_D(t) + C_D \sum_{j=0}^{k-1} A_D^j B_D w(t+k-1-j) + D_D w(t+k) \\ \text{if } k > 0 \tag{15}$$

Collecting the disturbance input sequences $\{w(t+k) : k = 1 : N_P\}$ in a vector $\vec{w}$, the set $\vec{\mathbb{W}}(x_D(t))$ can be written as:

$$\vec{\mathbb{W}}(x_D(t)) = \left\{ \vec{w} : H_D \left( \begin{bmatrix} C_D \\ C_D A_D \\ . \\ . \\ C_D A_D^{N_P} \end{bmatrix} x_D(t) + \begin{bmatrix} D_D & 0 & . & . & 0 \\ C_D B_D & D_D & . & . & 0 \\ C_D A_D B_D & C_D B_D & D_D & . & 0 \\ . & & & & . \\ C_D A_D^{N_P-1} & . & . & . & D_D \end{bmatrix} \vec{w} \right) \leq \begin{bmatrix} h_D \\ h_D \\ . \\ . \\ h_D \end{bmatrix} \right\} \tag{16}$$

Hence, at each time step $t$, the state of the disturbance model $x_D(t)$ can be read, and the corresponding disturbance input set $\vec{\mathbb{W}}(x_D(t))$ can be calculated. Following this, the linear programs in Eq.(11), which are rewritten as follows are solved:

$$h_{YW}^i = h_Y^i - \sup_{\vec{w} \in \vec{\mathbb{W}}(x_D(t))} H_w^i \vec{w} \tag{17}$$

Hence, the output admissible set $\mathbb{O}_{N_P}$ is obtained, which is used in the controller solving the optimization problem described in Eq.(4) to calculate a robust optimal control input $\bar{u}$.