Identification of exogenous disturbance signal sets

XYZ, A.Bemporad

***Abstract*— This work deals with uncertain linear models of dynamical systems, with the uncertainty modeled as an exogenous disturbance signal acting on the output. A method to identify the set within which this uncertainty lies is presented.**

I. Introduction

Model predictive control (MPC) schemes can be used to design controllers for systems with constraints. The schemes choose control inputs by making predictions of the future evolution of the system. They use a model of the system being controlled to make these predictions. Very often, the model does not represent the system exactly, resulting in the predictions not being accurate. This can result in the system constraints being violated. To avoid this, robust model predictive control schemes have been proposed in literature. A review of the considerations to be made while developing such robust schemes can be found in [1].

To improve predictions, the models are appended with uncertainty descriptions. They are explicit characterizations of the uncertainty present within the model. Such a characterization helps in establishing bounds on the predictions made by the model, such that the real performance of the system lies within these bounds. The robust model predictive controller scheme should ensure constraint satisfaction for all possible predictions of the system evolution within these bounds. A popular approach for this is to use a linear model of the system being controlled, and use a parametric uncertainty description, keeping the parameters of the model within a bounded set $\mathbb{S}$. In [2], approaches to solve MPC problems with such uncertainty descriptions using LMIs are presented. Alternatively, one can use a non-parametric uncertainty description by modeling the uncertainty as an unknown disturbance, lying within a bounded set $\mathbb{W}$. Robust tube-MPC approach is presented in [3], which guarantees robust constraint satisfaction for bounded disturbance uncertainty description.

Using the parametric uncertainty description requires identifying the set $\mathbb{S}$ in which the parameters lie. A review of the methods to do so can be found in [4]. To the best of the authors' knowledge however, no work has been presented to identify the sets $\mathbb{W}$ in which the bounded disturbances lie in case of non-parametric uncertainty description. This work aims to present one such method, along with a simple robust MPC scheme in which this uncertainty description can be employed. The approach uses an ARX model that is identified by performing experiments on the plant, similar to what is done in [5] to identify the sets $\mathbb{S}$ of parameter variability. Following the assumption of zero-measurement noise, the main idea is that the uncertainty in the identified model is equivalent to prediction error in ARX identification. Since this prediction error can be extracted out from ARX identification and can be shown to be driven by a noise signal, obtaining the sets in which this noise signal lies is equivalent to identifying the set $\mathbb{W}$ in which the disturbances lie, thus quantifying model uncertainty.

The robust MPC scheme we present here is based on reference governor approaches, a review of which is presented in [6]. The model used by the MPC scheme is obtained from ARX identification of the plant, which

*This work was not supported by any organization

Sampath Kumar Mulagaleti and Alberto Bemporad are with the IMT School for Advanced Studies Lucca, Piazza San Francesco 19, 55100 Lucca, Italy. Email: {s.mulagaleti,alberto.bemporad}@imtlucca.it

II. Problem Statement

We consider a *multi-input multi-output* plant $\mathbb{G}_P$, generating an output signal $y(t) \in \mathbb{R}^{n_y}$ corresponding to the input signal $y(t) \in \mathbb{R}^{n_u}, t \in \mathbf{Z}^+$. We aim at synthesizing a controller that can make the output track a user-defined reference signal $r(t) \in \mathbb{R}^{n_y}$, while robustly respecting the polyhedral constraints $Hy(t) \leq h, t \in \mathbf{Z}^+$. Towards this end, we first perform an experiment to identify a model of the plant, the details of which are given in the next section.

III. Open-loop model identification

An ARX model of a open-loop dynamical system is identified, which is parameterized as follows:

$$A(q^{-1})y(t) = B(q^{-1})u(t) + w(t)$$

For this, the dataset $D_N = \{u(k), y(k); k \in 1, 2, .., N\}$ obtained from open-loop experiments is utilized. Assuming the model is invertible, it is rewritten as

$$y(t) = M(q^{-1})u(t) + D(q^{-1})w(t) \tag{1}$$

where the transfer functions are $M(q^{-1}) = B(q^{-1})/A(q^{-1})$ and $D(q^{-1}) = 1/A(q^{-1})$. Hence, the output $y(t)$ is the sum of outputs of two systems, a schematic of which is shown in Fig.1. If the dataset D_N is noise-free, the part $y_D(t)$ of

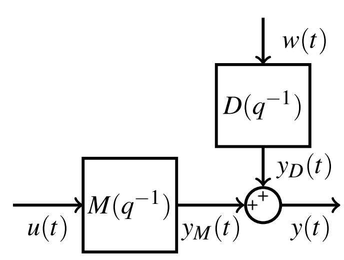

Fig. 1: ARX model

the output $y(t)$ that the model $M(q^{-1})$ does not capture can be attributed to model uncertainty. Hence, the uncertainty is modeled as an exogenous disturbance signal acting on the output. Robust model-based control schemes can utilize this model for controller synthesis, provided the uncertainty set the exogenous disturbance signal $w(t)$ belongs to is available. In the next section, one such control scheme is discussed. Following that, a method to obtain the set in which $w(t)$ lies is presented. It is noted that using a disturbance model $D(q^{-1})$ for controller synthesis is an approach similar to that followed for offset-free model predictive control, as discussed in [7]. However, since the focus is on extracting and utilizing the disturbance sets, no analysis of the offset-removing nature of the presented scheme is performed.

IV. Robust controller design

For controller synthesis, we first convert the transfer function in Eq.(1) to state space form, and obtain the following equations:

$$
\begin{bmatrix} x_M(t+1) \\ x_D(t+1) \end{bmatrix} = \begin{bmatrix} A_M & 0 \\ 0 & A_D \end{bmatrix} \begin{bmatrix} x_M(t) \\ x_D(t) \end{bmatrix} + \begin{bmatrix} B_M \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ B_D \end{bmatrix} w(t)
$$
$$
y(t) = \begin{bmatrix} C_M & C_D \end{bmatrix} \begin{bmatrix} x_M(t) \\ x_D(t) \end{bmatrix} + D_M u(t) + D_D w(t) \tag{2}
$$

where the states $x_M(t)$ and $x_D(t)$ belong to the system model M and the disturbance model D respectively. In a condensed way, they are written as:

$$
\begin{aligned} x(t+1) &= Ax(t) + B_U u(t) + B_W w(t) \\ y(t) &= Cx(t) + D_U u(t) + D_W w(t) \end{aligned} \tag{3}
$$

A robust reference governor can be designed to provide a control input $u(t)$ that makes the output $y(t)$ track a reference signal $r(t)$. At each time step t, the controller solves the optimization problem:

$$
\begin{aligned} \min_{\bar{u}} \quad & \sum_{k=1}^{N_P} (\hat{y}(t+k) - r(t+k))^2 \\ \text{subject to} \quad & \hat{x}(t+k+1) = A\hat{x}(t+k) + B_U \bar{u} \\ & \hat{y}(t+k) = C\hat{x}(t+k) + D_U \bar{u} \\ & \hat{x}(t) = x(t) \\ & (x(t), \bar{u}) \in \mathbb{O}_{N_P} \end{aligned} \tag{5}
$$

It reads the initial state $x(t)$ of the system, and calculates a constant control input $\bar{u}$ which is feasible with respect to the output admissible set $\mathbb{O}_{N_P}$ of the system Eq.(2) defined as:

$$
\mathbb{O}_{N_P} = \{(x(t), \bar{u}) : y \in \mathbb{Y}, u(t+k) = \bar{u}, \forall w \in \mathbb{W} \;\; \forall k \in \{1, 2, .., N_P\}\} \tag{6}
$$

where $y \in \mathbb{Y}$ denotes the future output sequence $\{y(t+k) \in \mathbb{Y}, k = 1, .., N_P\}$ and $w \in \mathbb{W}$ denotes the future disturbance sequence $\{w(t+k) \in \mathbb{W}, k = 0, .., N_P\}$. It is the set of initial states $x(t)$ and a constant control input $\bar{u}$ such that the future output trajectory of the system does not violate the constraints defined by $\mathbb{Y}$ for any possible bounded disturbance sequence $w \in \mathbb{W}$, within the horizon time N_P. It can also be seen as an N_P-step reachable set from a single point initial state set $\{x(t)\}$ and a constant control input $\bar{u}$, subject to bounded disturbances in $\mathbb{W}$. More details are discussed in [8]. It is noted that the presented scheme is similar to open-loop min-max robust MPC, except that in the prediction model, disturbances are ignored, and are considered only while calculating input bounds.

At any future time instant $t+k$, given the initial state $x(t)$ and a constant control input $\bar{u}$, the output $y(t+k)$ is given by:

$$
\begin{aligned} y(t+k) = CA^k x(t) + \left(C \sum_{j=0}^{k-1} A^j B_U + D_U \right) \bar{u} + \\ C \sum_{j=0}^{k-1} A^j B_W w(t+k-1-j) + D_W w(t+k) \end{aligned} \tag{7}
$$

It is desired to constraint the output $y(t+k)$ at a time instant $t+k$ within the polyhedral set $Hy(t+k) \leq h, h \in I\!R^{n_c}$. The output constraint set $\mathbb{Y}$ represents a collection of these pointwise in time polyhedron constraints, and is written as:

$$
\mathbb{Y} = \left\{ y : \begin{bmatrix} H & . & . & 0 \\ . & H & . & . \\ . & . & . & . \\ 0 & . & . & H \end{bmatrix} \begin{bmatrix} y(t+1) \\ . \\ . \\ y(t+N_P) \end{bmatrix} \leq \begin{bmatrix} h \\ . \\ . \\ h \end{bmatrix} \right\} = \{y : \tilde{H} y \leq \tilde{h}\}
$$

Hence, the definition of set $\mathbb{O}_{N_P}$ in Eq.(8) can be rewritten as:

$$
\mathbb{O}_{N_P} = \{(x(t), \bar{u}) : \tilde{H} y \leq \tilde{h}, u(t+k) = \bar{u}, \forall w \in \mathbb{W} \;\; \forall k \in \{1, 2, .., N_P\}\} \tag{8}
$$

Using the form in Eq.(7), the constraints can be enumerated as shown in Eq.(4). It is written in a simplified notation as:

$$
\mathbb{O}_{N_P} = \left\{ (x(t), \bar{u}) : \tilde{H} \left(H_x x(t) + H_u \bar{u} + H_w w \right) \leq \tilde{h} \right\} \tag{9}
$$

Since the output feasibility should hold over all possible future disturbance sequences, we desire to calculate the set

$$
\mathbb{O}_{N_P} = \{(x(t), \bar{u}) : H_x x(t) + H_u \bar{u} \in \mathbb{Y} \sim D\mathbb{W} \sim .. \sim CA^{t+N_P-1} B\mathbb{W}\} \tag{10}
$$

where $\sim$ denotes P-subtraction of sets. Towards this end, we first rewrite (9) as:

$$
\begin{aligned} \mathbb{O}_{N_P} &= \left\{ (x(t), \bar{u}) : G_x(t) + G_u \bar{u} + G_w w \leq \tilde{h} \right\} \\ &= \left\{ (x(t), \bar{u}) : G_x(t) + G_u \bar{u} + \begin{bmatrix} G_w^1 \\ . \\ . \\ G_w^{n_c N_P} \end{bmatrix} w \leq \begin{bmatrix} h_Y^1 \\ . \\ . \\ h_Y^{n_c N_P} \end{bmatrix} \right\} \end{aligned} \tag{11}
$$

where $G_x = \tilde{H} H_x$, $G_u = \tilde{H} H_u$ and $G_w = \tilde{H} H_w$. Row i of the matrix G_w is denoted as G_w^i, and element i of the vector $\tilde{h}$ is denoted as h_Y^i. For polyhedral sets like in our case, performing P-subtractions results in the set:

$$
\begin{aligned} \mathbb{O}_{N_P} &= \{(x(t), \bar{u}) : G_x x(t) + G_u \bar{u}) \leq h_{YW}\} \\ h_{YW}^i &= h_Y^i - \sup_{w \in \mathbb{W}} G_w^i w \end{aligned} \tag{12}
$$

where h_{YW}^i is the element i of the vector h_{YW}.

To calculate this input feasible set, we need the disturbance sequence set $\mathbb{W}$. The calculation of this set is discussed in the next section.

$$\mathbb{O}_{N_P} = \left\{ (x(t),\bar{u}) : \tilde{H} \left(\begin{bmatrix} CA \\ CA^2 \\ . \\ . \\ CA^{N_P} \end{bmatrix} x(t) + \begin{bmatrix} CB_U + D_U \\ CAB_U + D_U \\ . \\ . \\ C\sum_{j=0}^{N_P-1} A^j B_U + D_U \end{bmatrix} \bar{u} + \begin{bmatrix} CB_W & D_W & . & . & 0 \\ CAB_W & CB_W & D_W & . & 0 \\ . & & & & \\ . & & & & \\ CA^{N_P-1}B_W & CA^{N_P-2}B_W & . & . & D_W \end{bmatrix} \begin{bmatrix} w(t) \\ w(t+1) \\ . \\ . \\ w(t+N_P) \end{bmatrix} \right) \leq \tilde{h} \right\} \tag{4}$$

V. Calculation of $\mathbb{W}$

As discussed earlier, the part $y_D(t)$ of the output $y(t)$ represents uncertainty modeled as an exogenous disturbance input. The signal $y_D(t)$ is generated by the model $D(q^{-1})$, whose state-space equations are written as

$$\begin{aligned} x_D(t+1) &= A_D x_D(t) + B_D w(t) \\ y_D(t) &= C_D x_D(t) + D_D w(t) \end{aligned} \tag{13}$$

A sample data set U_N of $y_D(t)$ can be obtained from D_N, by simulating the model $M(q^{-1})$ with the input signals $u(k)$, as:

$$U_N = \{y_D(k) = y(k) - M(q^{-1})u(k); k \in 1,2,..,N\} \tag{14}$$

After collecting N data points of $y_D(k)$, an outer bounding convex polyhedral set $\mathbb{Y}_D^N$ is constructed that encompasses the whole dataset U_N. As the number of data points in the experiment increase, we obtain $\mathbb{Y}_D^N \subseteq \mathbb{Y}_D^{N+1}$ as $\mathbb{Y}_D^N = conv(U_N)$ and $\mathbb{Y}_D^{N+1} = conv(U_{N+1}) = conv(U_N \cup \{y_D(N+1)\})$, where $conv(X)$ is the convex hull of the set of elements within X.

Assumption 1: In the limit of infinite data used to build U_N, the outer encompassing sets $\mathbb{Y}_D^N$ converge to a steady set $\mathbb{Y}_D^\infty$.

Following assumption 1, $\mathbb{Y}_D^\infty$ is the largest set within which every possible exogenous disturbance $y_D(t)$ lies. That is, $y_D(t) \in \mathbb{Y}_D^\infty \,\forall \mathbf{Z}^+$. In a polyhedral form, this set defining the point wise in time values of $y_D(t)$ is written as $\mathbb{Y}_D^\infty = \{y_D(t) : H_D y_D(t) \leq h_D \forall t \in \mathbf{Z}^+\}$.

Since $y_D(t)$ lies within $\mathbb{Y}_D^\infty$ at each time instant t, we can define a sequence $y_D \in \mathbb{Y}_D$ starting from time instant t and going N_P time steps into the future $\{y_D(t+k) \in \mathbb{Y}_D, k = 1,..,N_P\}$ as:

$$\mathbb{Y}_D = \left\{ y_D : \begin{bmatrix} H_D & . & . & 0 \\ . & H_D & . & . \\ . & . & . & . \\ 0 & . & . & H_D \end{bmatrix} \begin{bmatrix} y_D(t+1) \\ . \\ . \\ y_D(t+N_P) \end{bmatrix} \leq \begin{bmatrix} h_D \\ . \\ . \\ h_D \end{bmatrix} \right\} \tag{16}$$

$$= \{y_D : \tilde{H}_D y \leq \tilde{h}_D\}$$

At any future time instant $t+k$ from the current time instant t, given the model (13) and initial condition $x_D(t)$, the output $y_D(t+k)$ can be written as:

$$y_D(t+k) = C_D A_D^k x_D(t) + C_D \sum_{j=0}^{k-1} A_D^j B_D w(t+k-1-j) + D_D w(t+k) \tag{17}$$

Similar to before, we construct a set $\mathbb{O}_{N_P}^D$ defined as:

$$\begin{aligned} \mathbb{O}_{N_P}^D &= \{(x_D(t),w) : y_D \in \mathbb{Y}_D \quad \forall k \in \{1,2,..,N_P\}\} \\ &= \{(x_D(t),w) : \tilde{H}_D y_D \leq \tilde{h}_D \quad \forall k \in \{1,2,..,N_P\}\} \end{aligned} \tag{18}$$

This is the set of initial conditions $x_D(t)$ and exogenous disturbance input sequence $\{w(t+k), k \in 0,1,..,N_P\}$, such that the output of the system (13) always stays inside the set defined by (16). This set too is an N_P-step reachable set, from a given point $x_D(t)$ and every possible input sequence w. Using the prediction of the output given by (17) in (18), the set $\mathbb{O}_{N_P}^D$ can be rewritten as (15), which is rewritten in (19) in a condensed form:

$$\mathbb{O}_{N_P}^D = \{(x_D(t),w) : \tilde{H}_D(H_x^D x_D(t) + H_w w) \leq \tilde{h}_D\} \tag{19}$$

Hence, given an initial condition $x_D(t) = x_D$, we can calculate the set $\mathbb{W}(x_D)$ as:

$$\begin{aligned} \mathbb{W}(x_D) &= \{w : \tilde{H}_D H_w w \leq \tilde{h}_D - \tilde{H}_D H_x^D x_D\} \\ &= \{w : Gw \leq g_x(x_D)\} \end{aligned} \tag{20}$$

This is the largest set in which the disturbance input sequence $\{w(t+k), k \in 0,1,..,N_P\}$ can lie in, such that the corresponding output sequence $\{y_D(t+k), k \in 1,..,N_P\}$ always lies within (16). In other words, it is the set in which the input sequence of the N_P-step reachable set $\mathbb{O}_{N_P}^D$ must lie such that it is output-admissible with respect to $\mathbb{Y}_D$, given an initial state point $x_D(t)$.

Assumption 2: The set $\mathbb{O}_{N_P}$ as defined in (10) is non-empty for $\mathbb{W} = \mathbb{W}(x_D(t))$ for all possible $x_D(t)$.

To calculate the input feasible set $\mathbb{O}_{N_P}$ by solving the equations in (12), we need the initial state $x(t)$ of the system (2). This initial state includes $x_D(t)$. We can use this to first calculate the corresponding disturbance sequence set $\mathbb{W}(x_D)$, and set $\mathbb{W} = \mathbb{W}(x_D)$. Following this, we calculate the set $\mathbb{O}_{N_P}$ as shown in (12). Assuming assumption 2 holds, the MPC scheme (5) to calculate the optimal control input $\bar{u}$. This procedure is summarized in Algorithm 1.

Algorithm 1 Robust MPC scheme

1: **Repeat** every sampling instant: $t = 0,1,\dots$
2: Read current state $x(t)$, set $x_D = x_D(t)$
3: Calculate $\mathbb{W}(x_D)$ according to (20), set $\mathbb{W} = \mathbb{W}(x_D)$
4: Solve the linear programs in (12) to obtain $\mathbb{O}_{N_P}$
5: Use $\mathbb{O}_{N_P}$ in (5) to obtain control input $\bar{u}$
6: Apply $\bar{u}$ to the plant
7: **Until** end of run

It should be noted that according to Algorithm 1, at each time step t, $n_c N_P$ number of linear programs in (12) for the worst case disturbance sequences of w should be solved, before solving the quadratic program (5). This online computation might be infeasible in practical applications. In such applications, it is desirable to calculate the sets $\mathbb{W}$ and $\mathbb{O}_{N_P}$ offline.

To calculate these sets offline, we first define the set $\mathbb{X}^D = \{x_D : G_x^D x_D \leq g_x^D\}$. This is the set of all the initial conditions that the disturbance block with state-space model

$$\mathbb{O}^D_{N_P} = \left\{ (x_D(t), w) : \tilde{H}_D \left(\begin{bmatrix} C_D A_D \\ C_D A_D^2 \\ . \\ . \\ C_D A_D^{N_P} \end{bmatrix} x_D(t) + \begin{bmatrix} C_D B_D & D_D & . & . & 0 \\ C_D A_D B_D & C_D B_D & D_D & . & 0 \\ . & & & & . \\ . & & & & . \\ C_D A_D^{N_P-1} & . & . & . & D_D \end{bmatrix} \begin{bmatrix} w(t) \\ w(t+1) \\ . \\ . \\ w(t+N_P) \end{bmatrix} \right) \leq \tilde{h}_D \right\} \tag{15}$$

Algorithm 2 Robust MPC scheme with projected set

1: Compute $\mathbb{X}^D$ from $\mathbb{Y}^\infty_D$
2: Perform set projection to compute $\mathbb{W}_\Omega$ according to (21)
3: Set $\mathbb{W} = \mathbb{W}_\Omega$, compute $\mathbb{O}_{N_P}$ according to (12)
4: **Repeat** every sampling instant: $t = 0, 1, \dots$
5: Read current state $x(t)$
6: Use $\mathbb{O}_{N_P}$ in (5) to obtain control input $\bar{u}$
7: Apply $\bar{u}$ to the plant
8: **Until** end of run

(13) can take. If the state-space model (13) is a non-minimal realization of the I/O model $y_D(t) = D(q^{-1})w(t)$, the state vector $x_D(t)$ at any time instant t is a collection of the previous outputs $y_D(t)$. This is because the model $D(q^{-1}) = 1/A(q^{-1})$ means that there is no dependence of current output $y_D(t)$ on past input sequence $\{w(k), k < t\}$. Following assumption 1, if we can build the set $\mathbb{Y}^\infty_D$ in which the output $y_D(t)$ can lie at any instant of time t, construction of the corresponding set $\mathbb{X}^D$ becomes trivial, and follows along the lines of (16). Upon construction of $\mathbb{X}^D$, we can define the disturbance sequence set $\mathbb{W}_\Omega$ as:

$$\begin{aligned} \mathbb{W}_\Omega &= \text{proj}_w(\mathbb{O}^D_{N_P}) \\ &= \{w : \exists x_D(t) \in \mathbb{X}^D : \tilde{H}_D(H^D_x x_D(t) + H_w w) \leq \tilde{h}_D\} \end{aligned} \tag{21}$$

This is the set in which the input sequence of the N_P-step reachable set $\mathbb{O}^D_{N_P}$ must lie such that it is output-admissible with respect to $\mathbb{Y}_D$, given an initial state lying anywhere in the feasible set $\mathbb{X}^D$. By setting $\mathbb{W} = \mathbb{W}_\Omega$ in (12), the set $\mathbb{O}_{N_P}$ can be constructed offline. This set can be used to calculate the optimal control input $\bar{u}$ by solving (5). These steps are summarized in Algorithm 2.

Proposition 1: Using Algorithm 2 instead of Algorithm 1 to compute $\bar{u}$ results in conservative controller performance.

Proof: Let $\mathbb{O}^1_{N_P}$ and $\mathbb{O}^2_{N_P}$ be the two polyhedral sets constructed in Algorithm 1 and Algorithm 2 respectively. By construction, $\mathbb{W}(x_D(t)) \subseteq \mathbb{W}_\Omega \, \forall x_D(t) \in \mathbb{X}^D$. This means that in (12),

$$\sup_{w \in \mathbb{W}(x_D(t))} G^i_w w \leq \sup_{w \in \mathbb{W}_\Omega} G^i_w w \quad \forall x_D(t) \in \mathbb{X}^D \tag{22}$$

Since G_x and G_u in (12) are the same for $\mathbb{O}^1_{N_P}$ and $\mathbb{O}^2_{N_P}$, $\mathbb{O}^2_{N_P} \subseteq \mathbb{O}^1_{N_P} \forall x_D(t) \in \mathbb{X}^D$. If Assumption 2 holds, the set $\mathbb{O}^2_{N_P}$ is non-empty. Hence, a feasible solution $\bar{u}$ exists when (5) is solved, but it belongs to a smaller set resulting in a conservative solution. ■

This conservativeness arises because on propagating the system (17) with initial condition $x_D(t) \in \mathbb{X}^D$, the corresponding output sequence set is $y_D \in \mathbb{Y}^\Omega_D \supseteq \mathbb{Y}_D$.

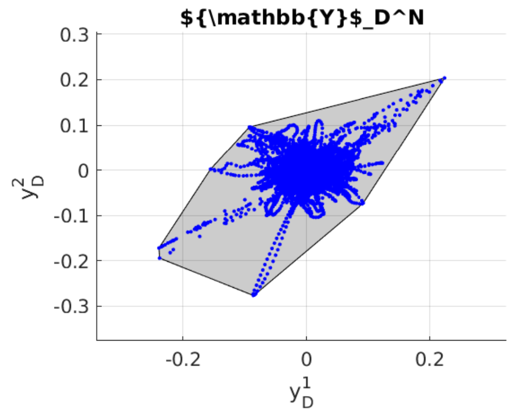

Fig. 2: Set $\mathbb{Y}^N_D$ obtained from U_N

VI. Numerical Simulations

Numerical simulations are performed on the following uncertain linear system to evaluate the approaches mentioned:

$$\mathbb{G}_P : \begin{aligned} \dot{x} &= A_u x + B_u u \\ y &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \end{aligned} \tag{23}$$

with the parameters varying randomly every sampling instant between:

Parameter	Range
A_u	$\begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}$ - $\begin{bmatrix} -0.9 & 1.1 \\ 2.1 & -2.9 \end{bmatrix}$
B_u	$\begin{bmatrix} 1 & 1 \\ 0.9 & 1.1 \end{bmatrix}$ - $\begin{bmatrix} 1.01 & 1.01 \\ 0.91 & 1.11 \end{bmatrix}$

Experiments are performed to identify a discrete time MIMO-ARX model of the system, by building the dataset D_N with $N = 10000$ sampled at $0.1s$. The models $M(q^{-1})$ and $D(q^{-1})$ are computed, as shown in (1), from which non-minimum realization state space models in (2) are extracted. A horizon of $N_P = 5$ is set in (5), and the matrices G_x and G_u in (12) are computed, which are required for the calculation of $\mathbb{O}_{N_P}$. Then, the dataset U_N of the samples of the signal $y_D(t)$ is constructed. From this dataset, $\mathbb{Y}^N_D$ is extracted, which is the outer bounding polyhedral set of each sample within U_N. This is shown in Figure **??**.
It is assumed that $\mathbb{Y}^\infty_D = \mathbb{Y}^N_D$, following which the set $\mathbb{Y}_D$ is calculated. It is the set in which the future sequence $\{y_D(t+k), k \in 1,..,N_P\}$ starting from current time instant t and going $N_P = 5$ time-steps into the future lies. This lead to the construction of $\mathbb{O}^D_{N_P}$. This is the set of states $x_D(t)$ and future disturbance input sequence $\{w(t+k), k \in 0,..,N_P\}$, such that the output set $\mathbb{Y}_D$ is respected. The state $x_D(t)$ of the

non-minimum realization state-space model (13) is updated using a dead-beat observer at each time-step t, and the state $x_M(t)$ is updated only using the dynamics of the model. Thus, we have everything necessary to implement Algorithm 1.

In-order to implement Algorithm 2, we first need the set $\mathbb{X}^D$ in which the state $x_D(t)$ lies at any time-instant t. Since the model (13) is a non-minimum realization, construction of $\mathbb{X}^D$ from $\mathbb{Y}_D^\infty$ becomes trivial. After calculating $\mathbb{X}^D$, the set $\mathbb{W}_\Omega$ is calculated by projecting $\mathbb{O}_{N_P}^D$ onto the w coordinates. This set-projection operation was performed using the *projection* function within the multi-parameteric toolbox [9]. Following this, we have everything necessary to implement Algorithm 2.

Alternative to the algorithms presented, one can just use the set $\mathbb{Y}_D$ as output additive disturbance, not acting through the model $D(q^{-1})$, and design a robust controller with respect to that set. For this, the output $y_M(t)$ is desired to track the reference $r(t)$, and is constrained to lie within a reduced set $\tilde{\mathbb{Y}} = \mathbb{Y} \sim \mathbb{Y}_D$. The optimization problem equivalent to (5) solved by the controller at each time-step t in this case is:

$$
\begin{aligned}
\min_{\bar{u}} \quad & \sum_{k=1}^{N_P} (\hat{y}_M(t+k) - r(t+k))^2 \\
\text{subject to} \quad & \hat{x}(t+k+1)_M = A_M \hat{x}_M(t+k) + B_M \bar{u} \\
& \hat{y}_M(t+k) = C_M \hat{x}_M(t+k) + D_M \bar{u} \\
& \hat{x}_M(t) = x_M(t) \\
& \hat{y}_M(t+k) \in \tilde{\mathbb{Y}}, k = 1,..,N_P
\end{aligned}
\tag{24}
$$

If the state-space model of $M(q^{-1})$ is a non-minimum realization, the state $x_M(t)$ is updated using a deadbeat observer, which places the past plant outputs $y(t)$ in a column vector.

REFERENCES

[1] A. Bemporad and M. Morari, "Robust model predictive control: A survey," in *Robustness in identification and control* (A. Garulli and A. Tesi, eds.), (London), pp. 207–226, Springer London, 1999.

[2] M. V. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361 – 1379, 1996.

[3] D. Mayne, M. Seron, and S. Rakovi, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219 – 224, 2005.

[4] E. Walter and H. Piet-Lahanier, "Estimation of parameter bounds from bounded-error data: a survey," *Mathematics and Computers in Simulation*, vol. 32, no. 5, pp. 449 – 468, 1990.

[5] A. Mohammadi, M. Diehl, and M. Zanon, "Estimation of uncertain arx models with ellipsoidal parameter variability," in *2015 European Control Conference (ECC)*, pp. 1766–1771, July 2015.

[6] E. Garone, S. Di Cairano, and I. Kolmanovsky, "Reference and command governors for systems with constraints," *Automatica*, vol. 75, pp. 306–328, Jan. 2017.

[7] G. Pannocchia and J. B. Rawlings, "Disturbance models for offset-free model-predictive control," *AIChE Journal*, vol. 49, no. 2, pp. 426–437.

[8] F. Borrelli, A. Bemporad, and M. Morari, *Predictive control for linear and hybrid systems*. Cambridge University Press, 2017.

[9] M. Kvasnica, P. Grieder, and M. Baotić, "Multi-Parametric Toolbox (MPT)," 2004.

.