Identification of exogenous disturbance signal sets

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***Abstract*— This work deals with uncertain linear models of dynamical systems, with the uncertainty modeled as an exogenous disturbance signal acting on the output. A method to identify the set within which this uncertainty lies is presented.**

I. Introduction

Model predictive control schemes can be used to design controllers for systems with constraints. The schemes choose a control input by making predictions of the future evolution of the system. They use a model of the system being controlled to make these predictions. Very often, the model does not represent the system exactly, resulting in the predictions not being accurate. This can result in the system constraints being violated. To avoid this, robust model predictive control schemes have been proposed in literature. A review of the considerations to be made while developing such robust schemes can be found in [1]. In the current work, we deal with uncertainty descriptions.

Uncertainty descriptions are explicit characterizations of the uncertainty present within the model. Such a characterization helps in establishing bounds on the predictions made by the model, such that the real performance of the system lies within these bounds. The robust model predictive controller scheme should ensure constraint satisfaction for all possible predictions of the system evolution within these bounds.

II. Problem Statement

We consider a *multi-input multi-output* plant, generating an output signal $y(t) \in \mathbb{R}^{n_y}$ corresponding to the input signal $y(t) \in \mathbb{R}^{n_u}, t \in \mathbf{Z}^+$. We aim at synthesizing a controller that can make the output track a user-defined reference signal $r(t) \in \mathbb{R}^{n_y}$, while robustly respecting the polyhedral constraints $Hy(t) \leq h, t \in \mathbf{Z}^+$. Towards this end, we first perform an experiment to identify a model of the plant, the details of which are given in the next section.

III. Open-loop model identification

An ARX model of a open-loop dynamical system is identified, which is parameterized as follows:

$$A(q^{-1})y(t) = B(q^{-1})u(t) + w(t)$$

For this, the dataset $D_N = \{u(k), y(k); k \in 1,2,..,N\}$ obtained from open-loop experiments is utilized. Assuming the model is invertible, it is rewritten as

$$y(t) = M(q^{-1})u(t) + D(q^{-1})w(t) \tag{1}$$

where the transfer functions are $M(q^{-1}) = B(q^{-1})/A(q^{-1})$ and $D(q^{-1}) = 1/A(q^{-1})$. Hence, the output $y(t)$ is the sum of outputs of two systems, a schematic of which is shown here:

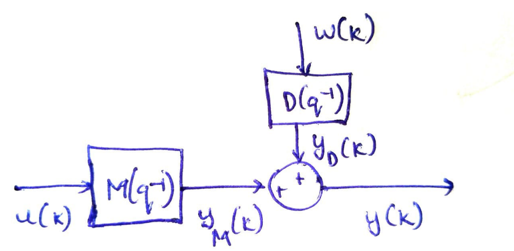

Fig. 1: ARX model

If the dataset D_N is noise-free, the part $y_D(t)$ of the output $y(t)$ that the model $M(q^{-1})$ does not capture can be attributed to model uncertainty. Hence, the uncertainty is modeled as an exogenous disturbance signal acting on the output. Robust model-based control schemes can utilize this model for controller synthesis, provided the uncertainty set the exogenous disturbance signal $w(t)$ belongs to is available. In the next section, one such control scheme is discussed. Following that, a method to obtain the set in which $w(t)$ lies is presented.

IV. Robust controller design

For controller synthesis, we first convert the transfer function in Eq.(1) to state space form, and obtain the following equations:

$$\begin{bmatrix} x_M(t+1) \\ x_D(t+1) \end{bmatrix} = \begin{bmatrix} A_M & 0 \\ 0 & A_D \end{bmatrix} \begin{bmatrix} x_M(t) \\ x_D(t) \end{bmatrix} + \begin{bmatrix} B_M \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ B_D \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} C_M & C_D \end{bmatrix} \begin{bmatrix} x_M(t) \\ x_D(t) \end{bmatrix} + D_M u(t) + D_D w(t) \tag{2}$$

where the states $x_M(t)$ and $x_D(t)$ belong to the system model M and the disturbance model D respectively. In a condensed way, they are written as:

$$\begin{aligned} x(t+1) &= Ax(t) + B_U u(t) + B_W w(t) \\ y(t) &= Cx(t) + D_U u(t) + D_W w(t) \end{aligned} \tag{3}$$

A robust reference governor can be designed to provide a control input $u(t)$ that makes the output $y(t)$ track a reference

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$$\mathbb{O}_{N_P} = \left\{ (x(t),\bar{u}) : \tilde{H} \left(\begin{bmatrix} CA \\ CA^2 \\ . \\ . \\ CA^{N_P} \end{bmatrix} x(t) + \begin{bmatrix} CB_U + D_U \\ CAB_U + D_U \\ . \\ . \\ C\sum_{j=0}^{N_P-1} A^j B_U + D_U \end{bmatrix} \bar{u} + \begin{bmatrix} CB_W & D_W & . & . & 0 \\ CAB_W & CB_W & D_W & . & 0 \\ . & & & & \\ . & & & & \\ CA^{N_P-1}B_W & CA^{N_P-2}B_W & . & . & D_W \end{bmatrix} \begin{bmatrix} w(t) \\ w(t+1) \\ . \\ . \\ w(t+N_P) \end{bmatrix} \right) \leq \tilde{h} \right\} \tag{4}$$

signal $r(t)$. At each time step t, the controller solves the optimization problem:

$$\begin{aligned} \min_{\bar{u}} \quad & \sum_{k=1}^{N_P} (\hat{y}(t+k) - r(t+k))^2 \\ \text{subject to} \quad & \hat{x}(t+k+1) = A\hat{x}(t+k) + B_U\bar{u} \\ & \hat{y}(t+k) = C\hat{x}(t+k) + D_U\bar{u} \\ & \hat{x}(t) = x(t) \\ & (x(t),\bar{u}) \in \mathbb{O}_{N_P} \end{aligned} \tag{5}$$

It reads the initial state $x(t)$ of the system, and calculates a constant control input $\bar{u}$ which is feasible with respect to the output admissible set $\mathbb{O}_{N_P}$ of the system Eq.(2) defined as:

$$\begin{aligned} \mathbb{O}_{N_P} = \{ (x(t),\bar{u}) : y \in \mathbb{Y}, u(t+k) = \bar{u}, \forall w \in \mathbb{W} \\ \forall k \in \{1,2,..,N_P\} \} \end{aligned} \tag{6}$$

where $y \in \mathbb{Y}$ denotes the future output sequence $\{y(t+k) \in \mathbb{Y}, k = 1,..,N_P\}$ and $w \in \mathbb{W}$ denotes the future disturbance sequence $\{w(t+k) \in \mathbb{W}, k = 0,..,N_P\}$. It is the set of initial states $x(t)$ and a constant control input $\bar{u}$ such that the future output trajectory of the system does not violate the constraints defined by $\mathbb{Y}$ for any possible bounded disturbance sequence $w \in \mathbb{W}$, within the horizon time N_P.
At any future time instant $t+k$, given the initial state $x(t)$ and a constant control input $\bar{u}$, the output $y(t+k)$ is given by:

$$\begin{aligned} y(t+k) = CA^k x(t) + \left(C\sum_{j=0}^{k-1} A^j B_U + D_U \right) \bar{u} + \\ C\sum_{j=0}^{k-1} A^j B_W w(t+k-1-j) + D_W w(t+k) \end{aligned} \tag{7}$$

It is desired to constraint the output $y(t+k)$ at a time instant $t+k$ within the polyhedral set $Hy(t+k) \leq h, h \in I\!R^{n_c}$. The output constraint set $\mathbb{Y}$ represents a collection of these pointwise in time polyhedron constraints, and is written as:

$$\mathbb{Y} = \left\{ y : \begin{bmatrix} H & . & . & 0 \\ . & H & . & . \\ . & . & . & . \\ 0 & . & . & H \end{bmatrix} \begin{bmatrix} y(t+1) \\ . \\ . \\ y(t+N_P) \end{bmatrix} \leq \begin{bmatrix} h \\ . \\ . \\ h \end{bmatrix} \right\} = \{ y : \tilde{H}y \leq \tilde{h} \}$$

Hence, the definition of set $\mathbb{O}_{N_P}$ in Eq.(8) can be rewritten as:

$$\begin{aligned} \mathbb{O}_{N_P} = \{ (x(t),\bar{u}) : \tilde{H}y \leq \tilde{h}, u(t+k) = \bar{u}, \forall w \in \mathbb{W} \\ \forall k \in \{1,2,..,N_P\} \} \end{aligned} \tag{8}$$

Using the form in Eq.(7), the constraints can be enumerated as shown in Eq.(4). It is written in a simplified notation as:

$$\mathbb{O}_{N_P} = \left\{ (x(t),\bar{u}) : \tilde{H} \left(H_x x(t) + H_u \bar{u} + \begin{bmatrix} H_w^1 \\ . \\ . \\ H_w^{n_y N_P} \end{bmatrix} w \right) \leq \begin{bmatrix} h_Y^1 \\ . \\ . \\ h_Y^{n_c N_P} \end{bmatrix} \right\} \tag{9}$$

where row i of the matrix associated to the future disturbance sequence w is denoted as H_w^i, and element i of the vector $\tilde{h}$ is denoted by h_Y^i. Since the output feasibility should hold over all possible future disturbance sequences, we desire to calculate the set

$$\mathbb{O}_{N_P} = \{ (x(t),\bar{u}) : H_x x(t) + H_u \bar{u} \in \mathbb{Y} \sim D\mathbb{W} \sim .. \sim CA^{t+N_P-1}B\mathbb{W} \} \tag{10}$$

where $\sim$ denotes P-subtraction of sets. For polyhedral sets like in our case, performing these operations result in the set:

$$\begin{aligned} \mathbb{O}_{N_P} = \{ (x(t),\bar{u}) : \tilde{H}(H_x x(t) + H_u \bar{u}) \leq h_{YW} \} \\ h_{YW}^i = h_Y^i - \sup_{w \in \mathbb{W}} H_w^i \vec{w} \end{aligned} \tag{11}$$

where h_{YW}^i is the element i of the vector h_{YW}.
To calculate this input feasible set, we need the disturbance sequence set $\mathbb{W}$. The calculation of this set is discussed in the next section.

V. Calculation of $\mathbb{W}$

As discussed earlier, the part $y_D(t)$ of the output $y(t)$ represents uncertainty modeled as an exogenous disturbance input. The signal $y_D(t)$ is generated by the model $D(q^{-1})$, whose state-space equations are written as

$$\begin{aligned} x_D(t+1) = A_D x_D(t) + B_D w(t) \\ y_D(t) = C_D x_D(t) + D_D w(t) \end{aligned} \tag{12}$$

A sample data set U_N of $y_D(t)$ can be obtained from D_N, by simulating the model $M(q^{-1})$ with the input signals $u(k)$, as:

$$U_N = \{ y_D(k) = y(k) - M(q^{-1})u(k); k \in 1,2,..,N \} \tag{13}$$

with the initial condition $x_M(0) = 0$.
Ideas: Say first N data points are collected. An outer bounding polyhedral set $\mathbb{Y}_N^D$ is calculated, that encompasses all the points in U_N. Now, every point in U_N lies in this set. As the number of data points increase, we will obtain for sure $\mathbb{Y}_{N+1}^D \subseteq \mathbb{Y}_N^D$. Assuming we have infinite data, we have the largest set $\mathbb{Y}_\infty^D$ in which every possible $y_D(t)$ will lie. Let $\mathbb{Y}_\infty^D$ be a polyhedron $H_D y_D(t) \leq h_D$. Then, starting at a timestep t, if we define $y_D \in \mathbb{Y}_D$ as the sequence $\{y_D(t+k) \in \mathbb{Y}_D, k = 1,..,N_P\}$, we can write the set $\mathbb{Y}_D$ as:

$$\mathbb{Y}_D = \left\{ y_D : \begin{bmatrix} H_D & . & . & 0 \\ . & H_D & . & . \\ . & . & . & . \\ 0 & . & . & H_D \end{bmatrix} \begin{bmatrix} y_D(t+1) \\ . \\ . \\ y_D(t+N_P) \end{bmatrix} \leq \begin{bmatrix} h_D \\ . \\ . \\ h_D \end{bmatrix} \right\} = \{ y_D : \tilde{H}_D y \leq \tilde{h}_D \}$$

From this, we can expand in the same way by leaving the w sequence as it is. This will lead to a very similar set of equations. which at each time instant k is indicated as lying in a polyhedral set

$$y_D(k) \in \mathbb{Y}_N^D = \{y_D : H_D y_D \leq h_D\} \tag{14}$$

In the limit of infinite data D_N, the set $\mathbb{Y}_N^D$ approaches the actual exogenous disturbance output set $\mathbb{Y}_\infty^D$. Using this set, which is the output constraint set of the system described by Eq.(12), we calculate $\vec{\mathbb{W}}(x_D(t))$ at each time instant t. It is the set in which the sequence of disturbance inputs $\{w(t+k) : k = 0 : N_P\}$ should lie in, such that the output constraint $\{y_D(t+k) \in \mathbb{Y}_\infty^D : k = 0 : N_P\}$ of the disturbance model are respected.
To calculate $\vec{\mathbb{W}}(x_D(t))$, we write the predicted output $y_D(t+k)$ of the disturbance model given the initial state $x_D(t)$ as:

$$\begin{aligned} y_D(t) &= C_D x_D(t) + D_D w(t) \text{ if } k = 0 \\ y_D(t+k) &= C_D A_D^k x_D(t) + C_D \sum_{j=0}^{k-1} A_D^j B_D w(t+k-1-j) + D_D w(t+k) \\ &\qquad \text{if } k > 0 \end{aligned} \tag{15}$$

Collecting the disturbance input sequences $\{w(t+k) : k = 1 : N_P\}$ in a vector $\vec{w}$, the set $\vec{\mathbb{W}}(x_D(t))$ can be written as:

$$\vec{\mathbb{W}}(x_D(t)) = \left\{ \vec{w} : H_D \left(\begin{bmatrix} C_D \\ C_D A_D \\ . \\ . \\ C_D A_D^{N_P} \end{bmatrix} x_D(t) + \begin{bmatrix} D_D & 0 & . & . & 0 \\ C_D B_D & D_D & . & . & 0 \\ C_D A_D B_D & C_D B_D & D_D & . & 0 \\ . & & & & . \\ C_D A_D^{N_P-1} & . & . & . & D_D \end{bmatrix} \vec{w} \right) \leq \begin{bmatrix} h_D \\ h_D \\ . \\ . \\ h_D \end{bmatrix} \right\} \tag{16}$$

Hence, at each time step t, the state of the disturbance model $x_D(t)$ can be read, and the corresponding disturbance input set $\vec{\mathbb{W}}(x_D(t))$ can be calculated. Following this, the linear programs in Eq.(11), which are rewritten as follows are solved:

$$h_{YW}^i = h_Y^i - \sup_{\vec{w} \in \vec{\mathbb{W}}(x_D(t))} H_w^i \vec{w} \tag{17}$$

Hence, the output admissible set $\mathbb{O}_{N_P}$ is obtained, which is used in the controller solving the optimization problem described in Eq.(5) to calculate a robust optimal control input $\bar{u}$.
Following this, write it in an algorithmic form, and write conditions for which the set $\mathbb{O}_{N_P}$ will be non-empty. (Something about the size of $\mathbb{W}$ set .)

VI. Using the projected set

Instead of solving so many LPs before a QP, we can project the set $\mathbb{W}$. For this, we need the set within which the state of the disturbance block can lie. This is possible to be obtained if we use a non-minimum realization as the state space model. When we have that, we can project within that subset and get a much more conservative thing. Have to write a theorem that projection is a good idea : The obtained set is larger than the original. Assumption that origin lies with W or something, so that it is feasible. Proof by doing minkowski difference : Sup over a larger feasible set leads to a larger number : But that should not break the feasibility of MPC.

References

[1] A. Bemporad and M. Morari, "Robust model predictive control: A survey," in *Robustness in identification and control* (A. Garulli and A. Tesi, eds.), (London), pp. 207–226, Springer London, 1999.