Robustly constrained data-driven control

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***Abstract*— This is a very skeletal overview.**

I. Problem Setup

Consider a *single-input single-output* system $\mathbb{G}_P$ generating an output signal $y(t) \in \mathbb{R}$ corresponding to the input signal $u(t) \in \mathbb{R}$ for the time $t \in \mathbb{Z}$. Consider the data $D_N = \{u(t), y(t); t \in 1, ..., N\}$ obtained by exciting the system.
A feedback controller is designed to control the system, using the VRFT methodology. For this, a reference model $\mathbb{M}_P$ is selected. The VRFT methodology designs a feedback controller $\mathbb{K}_P$, with the goal of making the closed-loop system $\mathbb{K}_P$-$\mathbb{G}_P$ behave similar to the reference model $\mathbb{M}_P$. To this end, the VRFT methodology utilizes the dataset $\mathbb{D}_N$. The desired closed-loop behavior is described by the LTI state-space model $\mathbb{M}_P$

$$\begin{aligned} x_M(t+1) &= A_M x_M(t) + B_M g(t) \\ y_d(t) &= C_M x_M(t) \end{aligned}$$

The VRFT methodology utilized to design the feedback controller $\mathbb{K}_P$ is now explained:

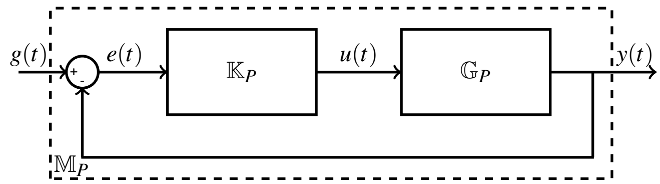
1) A virtual reference input $g(t)$ is calculated by setting $y_d(t) = y(t)$ obtained from the dataset $\mathbb{D}_N$, by the inverting the model $\mathbb{M}_P$. Let this mapping be defined by $g(t) = \mathbb{M}_P^{\dagger} y(t)$.
2) A feedback controller $\mathbb{K}_P$ described by $A_K(q^{-1})u(t) = B_K(q^{-1})(g(t) - y(t)$ is chosen, where

$$\begin{aligned} A_K(q^{-1}) &= 1 + \sum_{i=1}^{n_{a_K}} a_i^K q^{-i} \\ B_K(q^{-1}) &= \sum_{i=1}^{n_{b_K}} b_i^K q^{-i} \end{aligned}$$

   The parameters of the controller a_i^K and b_i^K are calculated by the VRFT methodology, such that the closed loop performance of $\mathbb{K}_P$-$\mathbb{G}_P$ matches open loop performance of the reference model $\mathbb{M}_P$.
3) This is done by solving the convex optimization problem

$$\min_{a_i^K, b_i^K} \quad \frac{1}{N} \sum_{t=1}^{N} \left| A_K(q^{-1})u(t) - B_K(q^{-1})(\mathbb{M}_P^{\dagger} y(t) - y(t)) \right|^2$$

   which minimizes the deviation between the control input calculated by the controller and $u(t)$ that is used to excite the system and obtain $y(t)$.
4) The synthesized controller $\mathbb{K}_P$ is placed before the plant, and the loop is closed. A reference step input is given to evaluate the controller performance.



The performance of this setup is improved by providing the reference signal $g(t)$ using an MPC controller in an outer loop. The objective of the MPC controller is to make the output signal $y(t)$ track the reference $r(t)$, while satisfying constraints on the plant output $y(t)$ and input $u(t)$. This is achieved by considering the closed-loop plant model $\mathbb{M}_P$ for the state propagation equation, and an augmented model with $[y(t), u(t)]$ as the output equation. This augmented state-space model is called $\mathbb{M}'_P$.

$$\begin{aligned} \zeta(t+1) &= A_\zeta \zeta(t) + B_\zeta g(t) \\ \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} &= C_\zeta \zeta(t) + D_\zeta g(t) \end{aligned}$$

The optimization problem solved by the MPC at each time step t for a horizon of N_P timesteps is shown below. The lower and upper bounds on $y(t)$ and $u(t)$ are $[y_{min}, y_{max}]$ and $[u_{min}, u_{max}]$ respectively.

$$\begin{aligned} \min_{\{g(t+k)\}_{k=1}^{N_P}} \quad & Q_y \sum_{k=1}^{N_P} (y(t+k|t) - r(t+k))^2 + Q_\varepsilon \varepsilon^2 \\ \text{subject to} \quad & \zeta(t+k+1) = A_\zeta \zeta(t+k) + B_\zeta g(t+k) \\ & \begin{bmatrix} y(t+k) \\ u(t+k) \end{bmatrix} = C_\zeta \zeta(t+k) + \begin{bmatrix} 0 \\ D_\zeta \end{bmatrix} g(t+k) \\ & y_{min} - V_y \varepsilon \leq y(t+k) \leq y_{max} + V_y \varepsilon \\ & u_{min} - V_u \varepsilon \leq u(t+k) \leq u_{max} + V_u \varepsilon \\ & \zeta(t|t) = \zeta(t) \end{aligned} \tag{1}$$

The quantities V_y and V_u in the MPC formulation are used to avoid infeasibility of the optimization problem over successive iterations, since the reference model $\mathbb{M}'_P$ might not accurately capture the dynamics of the unknown plant $\mathbb{G}_P$. This implies that constraint satisfaction is not guaranteed by the proposed formulation.

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II. Disturbance sensor

In order to improve prediction of closed loop performance, a disturbance sensor $\mathbb{D}$ is designed. The disturbance sensor is a dynamical system whose output is the discrepancy between output of the actual closed loop system $\mathbb{K}_P$-$\mathbb{G}_P$ and reference closed loop system $\mathbb{M}_P$. The system consists of a deterministic part and a stochastic part, as seen in Fig. 1.

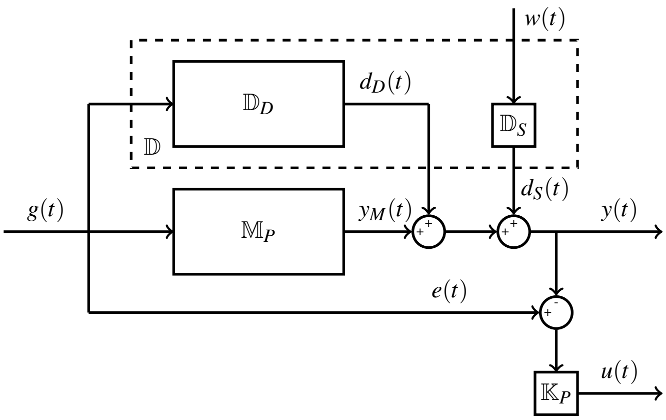

Fig. 1: Reference model appended with disturbance sensor

This can be seen as a 2-input 2-output system, whose transfer function is described as

$$\begin{bmatrix} Y(z) \\ U(z) \end{bmatrix} = \begin{bmatrix} \mathbb{M}_P + \mathbb{D}_D & \mathbb{D}_S \\ \mathbb{K}_P(I - (\mathbb{M}_P + \mathbb{D}_D)) & -\mathbb{K}_P \mathbb{D}_S \end{bmatrix} \begin{bmatrix} G(z) \\ W(z) \end{bmatrix} \tag{2}$$

Note that the dependence of transfer functions on time shift operator z (or q^{-1}) is not denoted for ease of notation. The signal $w(t)$ is interpreted as an external noise signal. Since this model represents the closed loop behavior of the system, new closed loop measurements are required to estimate the parameters of the disturbance sensor. These are obtained by performing experiments with excitaion signals $\hat{g}(t)$ on the closed loop model and plant, and measuring the outputs $\hat{y}_M(t)$ and $\hat{y}(t)$ respectively. The output is captured as the data sequence $\hat{D}_N = \{\hat{g}(t), \hat{y}_M(t), \hat{y}(t); t \in 1, ..., N\}$. The disturbance sensor $\mathbb{D}$ is parameterized as $A_D(q^{-1})y_D(t) = B_D(q^{-1})g(t) + w(t)$, where

$$y_D(t) = y(t) - y_M(t) = d_D(t) + d_S(t)$$
$$A_D(q^{-1}) = 1 + \sum_{i=1}^{n_{a_D}} a_i^D q^{-i}$$
$$B_D(q^{-1}) = \sum_{i=1}^{n_{b_D}} b_i^D q^{-i}$$

Using standard ARX identification, the coefficients a_i^D and b_i^D are estimated by solving the optimization problem

$$\min_{a_i^D, b_i^D} \quad \frac{1}{N} \sum_{t=1}^{N} \left| A_D(q^{-1})(\hat{y}(t) - \hat{y}_M(t)) - B_K(q^{-1})(\hat{g}(t)) \right|^2$$

The disturbance sensor is then split into two parts, deterministic and stochastic. The deterministic part $\mathbb{D}_D$ takes in the reference signal $g(t)$ as an input, and the stochastic part $\mathbb{D}_S$ takes in the noise $w(t)$. The I/O behavior of these parts are separately written as

$$d_D(t) = \frac{B_D(q^{-1})}{A_D(q^{-1})} g(t) = \mathbb{D}_D g(t)$$
$$d_S(t) = \frac{1}{A_D(q^{-1})} w(t) = \mathbb{D}_S w(t)$$

Once all the parameters of the disturbance sensor are calculated, the model (2) can be used in a formulation similar to (1) to generate optimal input signal $g(t)$. However, the model has an additional noisy input $w(t)$, which will be transformed to process and measurement noises on the state space model corresponding to (2). Hence, it is desirable to calculate upper and lower bounds w_{min} and w_{max} on the noise signal $w(t)$, and explicitly provide them to the optimization problem in order to obtain as robust formulation.

III. Noise input bounds calculation

A realization of the discrepancy $d_S(t)$ between desired output $y(t)$ and deterministic output $y_M(t) + d_D(t)$ can be calculated from the data set $\hat{D}_N$ as

$$\hat{d}_S(t) = \hat{y}(t) - \hat{y}_M(t) - \frac{B_D(q^{-1})}{A_D(q^{-1})} \hat{g}(t)$$

The maximum and minimum values of this discrepancy are labeled $\hat{d}_{S,max}$ and $\hat{d}_{S,min}$ respectively. If the ARX estimation results in a stable deterministic disturbance sensor $\mathbb{D}_D$, the values $\hat{d}_{S,max}$ and $\hat{d}_{S,min}$ are finite. Further, if infinite closed loop data $\hat{D}_\infty$ is collected for ARX estimation, the bounds on discrepancy are equal to the actual bounds $d_{D,max}$ and $d_{D,min}$. The set of sequences $d_S(t)$ satisfying these bounds are indicated as lying in a set D_∞, defined as

$$D_\infty = \{d_S(t) : d_{S,min} \leq d_S(t) \leq d_{S,max} \;\; \forall t \in (-\infty, \infty)\}$$

From these bounds, we attempt to calculate the set W_∞ defined as

$$W_\infty = \left\{ w(t) : \forall t \in (-\infty, \infty), \begin{matrix} w_{min} \leq w(t) \leq w_{max} \\ \mathbb{D}_S w(t) \in D_\infty \end{matrix} \right\}$$

This is the set of all $w(t)$ sequences such that the output signal of $\mathbb{D}_S$ corresponding to any $w(t) \in W_\infty$ lies within the bounds $[d_{S,min}, d_{S,max}]$. It must be noted that the converse is not implied. That is, it does not mean that there cannot exist a noise sequence $w(t)$ not belonging to W_∞ but producing a corresponding output sequence belonging to D_∞.

$$\forall w(t) \in W_\infty : \mathbb{D}_S w(t) \in D_\infty \;\not\Longrightarrow\; \nexists w(t) \notin W_\infty : \mathbb{D}_S w(t) \in D_\infty$$

To calculate the bounds w_{min} and w_{max}, the optimization problems shown in **??** are solved for increasing

$$
\begin{array}{lc}
\max_{X_{min}} & w_{min}^{N} \\
\text{subject to} & d_S(k) = -\sum_{i=1}^{n_{a_D}} a_i^D d_S(k-i) + w(k), k = 1:N \\
& d_{S,min} \leq d_S(k) \leq d_{S,min}, k = 1:N-1 \\
& w_{min}^{N} \leq w(k), k = 1:N \\
& d_S(N) \leq d_{S,min}
\end{array}
\qquad
\begin{array}{lc}
\min_{X_{max}} & w_{max}^{N} \\
\text{subject to} & d_S(k) = -\sum_{i=1}^{n_{a_D}} a_i^D d_S(k-i) + w(k), k = 1:N \\
& d_{S,min} \leq d_S(k) \leq d_{S,min}, k = 1:N-1 \\
& w(k) \leq w_{max}^{N}, k = 1:N \\
& d_S(N) \geq d_{S,max}
\end{array}
$$

$$
\text{where } X_{min} = \left\{ \begin{array}{c} \{d_S(-n_{a_D}+1), \ldots, d_S(N)\}, \\ \{w(1), \ldots, w(N)\}, \\ w_{min}^{N} \end{array} \right\}
\qquad
\text{where } X_{max} = \left\{ \begin{array}{c} \{d_S(-n_{a_D}+1), \ldots, d_S(N)\}, \\ \{w(1), \ldots, w(N)\}, \\ w_{max}^{N} \end{array} \right\}
$$

Fig. 2: Linear programs solved to calculate the set W_∞

lengths of time horizon N. These equations solve for a sequence of inputs $\{w(k), k = 1:N\}$ such that the sequence of corresponding outputs at all instances except at time N lie within D_∞. This means that the chosen input sequence $w(k)$ can drive the system output out of bounds in N time steps. The initial conditions on $d_S(k)$ are left free to be chosen by the problem. This makes sure that the assumption of infinite data $\hat{D}_\infty$ holds.
Any sequence $\{w(k) > w_{min}^{N}, k = 1:N\}$ will not drive $d_S(k \geq N)$ to the bound $d_{S,min}$. Similarly, $\{w(k) < w_{max}^{N}, k = 1:N\}$ will not drive $d_S(k \geq N)$ to the bound $d_{S,max}$. Hence, in order to explain the whole set D_∞, we need to find the lowest possible value of w_{min}^{N} and the highest possible value of w_{max}^{N} such that $d_S(k)$ is driven to its bounds in finite time k. This is achieved by increasing the value of N, which would require lower and higher values of $w(k)$ to make $d_S(N)$ reach $d_{S,min}$ and $d_{S,max}$ respectively. This leads to w_{min}^{N} and w_{max}^{N} asymptotically reaching their true values w_{min} and w_{max} as N increases, thus making the set W_∞ completely explain the set D_∞. It is noted that the presented method provides a methodology to quantify uncertainty in any general ARX identified model.
After finding bounds on the noise, the appended model with disturbance sensor can be used in a robust control framework. In the current work, a robust reference governer is developed. The reference governer alters the reference input $g(t)$ from the MPC algorithm in (1) to generate a new reference signal $v(t)$, for which the closed loop plant satisfies output constraints.

IV. Robust reference governer

The robust reference governer is a signal regulator which alters the command input such that the system robustly satisfies constraints. It does so by utilizing knowledge of disturbances acting on the system. A schematic of the control system is shown in Fig. 3.

The *MPC* controller utilizes the model $\mathbb{M}'_P$, and the reference governer *RG* uses the model $\mathbb{M}'_{PD}$, which is

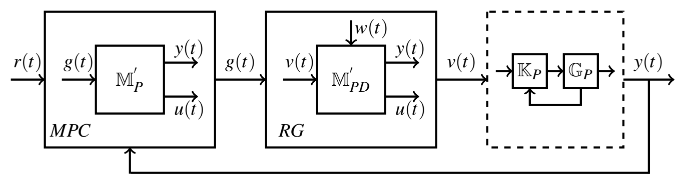

Fig. 3: Schematic of the control system

the state space representation of (2). This is written as:

$$
\gamma(t+1) = A_\gamma \gamma(t) + B_\gamma^v v(t) + B_\gamma^w w(t)
$$
$$
\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = C_\gamma \gamma(t) + D_\gamma^v v(t) + D_\gamma^w w(t)
$$

The columns of matrices B_γ and D_γ are separated for ease of notation. The outputs of the system are called $y_\gamma(t) = [y(t)u(t)]^T$. The constraints on these outputs are written as:

$$
\left. \begin{array}{r} y_{min} \leq y(t) \leq y_{max} \\ u_{min} \leq u(t) \leq u_{max} \end{array} \right\} \quad H y_\gamma(t) \leq h
$$

At each time instant t, the reference governer solves the quadratic program:

$$
\begin{array}{ll}
\min_{v} & \frac{1}{2} \| v - g(t) \|_2^2 \\
\text{subject to} & (x(t), v) \in \mathbb{O}_\infty
\end{array} \tag{3}
$$

where the set $\mathbb{O}_\infty$ is defined as:

$$
\mathbb{O}_\infty = \{(x(t), v) : x(t) = \gamma(t),\; H y_\gamma(\tau) \leq h \;\; \forall \tau \geq t\}
$$

This is called an output-invariant set. It is set of feasible values for v for the current observed state $x(t) = \gamma(t)$ such that if a constant input signal $v(\tau \geq t) = v$ is applied, the system output constraints $H y_\gamma(\tau \geq t) \leq h$ remain satisfied for all τ. At a time instant $\tau \geq t$, the system output $y_\gamma(\tau)$ for input signal $v(\tau \geq t) = v$ can be written as:

$$
y_\gamma(\tau) = C_\gamma A_\gamma^{\tau-t} \gamma(t) + \left(C_\gamma \sum_{k=1}^{\tau-t} A_\gamma^{k-1} B_\gamma^v + D_\gamma^v \right) v + C_\gamma \sum_{k=1}^{\tau-t} A_\gamma^{k-1} B_\gamma^w w(\tau-k) + D_\gamma^w w(\tau)
$$

For a particular future time instant τ, the set $\mathbb{O}_\infty(\tau)$ is given by:

$$\mathbb{O}_\infty(\tau) = \{(x(t),v) : x(t) = \gamma(t),\ \tilde{H}(\tau)v \leq \tilde{h}(\tau)\}$$
$$\text{where}\quad \tilde{H}(\tau) = H\left(C_\gamma \sum_{k=1}^{\tau-t} A_\gamma^{k-1} B_\gamma^v + D_\gamma^v\right)$$
$$\tilde{h}(\tau) = h - HC_\gamma A_\gamma^{\tau-t}\gamma(t) - f^w(\tau)$$

Each element $f_i^w(\tau)$ of the column vector $f^w(\tau)$ is calculated by solving the linear program:

$$f_i^w(\tau) = \max_{\{w(k)\}_t^\tau \in W_\infty} H\left(C_\gamma \sum_{k=1}^{\tau-t} A_\gamma^{k-1} B_\gamma^w w(\tau-k) + D_\gamma^w w(\tau)\right)_i$$

Subscript i in the above problem indicates that row i of the matrix is used in the linear program to calculate the corresponding $w(k)$ sequence. According to the theory of output invariant sets for linear systems with polyhedral constraints on noise and outputs, the set $\mathbb{O}_\infty = \mathbb{O}_\infty(\tau \to \infty)$ is reached when the elements of $f^w(\tau)$ converge to a constant value. Since the linear problem to be solved for $f_i^w(\tau)$ does not depend on values at current time instant t, it can be solved offline for increasing values of τ. When convergence is observed at time τ_c, the value $f^w(\tau_c)$ is stored. This results in problem (3) being simplified to

$$\begin{aligned} \min_{v} \quad & \frac{1}{2}\|v - g(t)\|_2^2 \\ \text{subject to} \quad & \tilde{H}(\tau_c)v \leq \tilde{h}(\tau_c) \end{aligned} \tag{4}$$

It must be noted that the value $\tilde{h}(\tau_c)$ depends on the current observed state $\gamma(t)$.

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