Robust data-driven model predictive control

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***Abstract*— This paper presents a robust data-driven control approach for linear systems subject to bound constraints on inputs and outputs. The approach builds on a previously presented hierarchical direct data-driven control architecture for constrained systems. Robustness is achieved by using a robust model predictive controller (RMPC) instead of a deterministic model predictive controller (MPC) in the outer loop of the architecture. The RMPC controller generates a reference signal for the inner loop that robustly satisfies the constraints. To this end, it utilizes a model of the inner loop with uncertainty within an unknown bounded disturbance set. A major contribution of this work is the development of a method to calculate this set from an identified AutoRegressive model with eXogenous inputs (ARX). Numerical validations of the robust control architecture are performed, and the results are presented.**

I. Introduction

Design of control systems can be broadly classified into two categories: model-based and data-driven. Model-based control design techniques use an explicit model of the plant being controlled. This involves selecting a model that trades-off between complexity and accuracy. Model selection is followed by experiments to identify the parameters. These steps introduce several challenges. To avoid this, one can resort to a direct data-driven controller design methodology.

Data-driven controller design methods avoid explicitly identifying the plant model. They synthesize a controller directly from I/O data obtained from the plant. A review of several such methods can be found in [1]. One such method, virtual reference feedback tuning (VRFT) introduced in [2], has been used to design a stabilizing feedback controller within a hierarchical control architecture in [3] for LTI/LPV systems. The architecture employs an outer MPC controller, which utilizes the reference model selected for VRFT to generate a tracking signal. Performance bounds on the plant are translated into constraints on the optimization problem solved by the MPC controller. Since the reference model might not reasonably reflect the performance of the closed-loop plant, there is a possibility of constraint violation. To avoid this, one can use a robust MPC controller in the outer loop, which generates a tracking signal that robustly satisfies the system constraints. A review of the concepts related to robust MPC controllers can be found in [4]. An alternative is to use a robust reference governor, which modifies the reference signal supplied to the plant in a way that constraints are satisfied. These are reviewed in [5]. Both the techniques need a model of the plant being controlled, with the model explicitly incorporating uncertainties.

In this work, we propose a robust data-driven control approach, building on the hierarchical control architecture presented in [3]. The robust controller uses a model of the inner closed-loop, which consists of the plant and the VRFT-based feedback controller. This model is not the same as the reference model used for VRFT, but is separately identified using an ARX parameterization from closed-loop experiments. The model incorporates uncertainties as exogenous disturbance signals, assumed to lie within an unknown bounded polyhedral set. Since robust control methods explicitly incorporate uncertainty information in their formulation, identification of the polyhedral sets is necessary.

In this work, a novel method to do so is presented. Such a method falls under the umbrella of identification for robust control. This field includes a large volume of literature on set-membership techniques for parameter estimation. A review of these can be found in [6]. A method to obtain ellipsoidal parameterizations of these sets during ARX estimation is discussed in [7]. To the best of the authors' knowledge, no similar work has been done to calculate polyhedral sets of input exogenous disturbance signals from output data. Following the identification of a model and corresponding disturbance sets, an RMPC controller is designed, which replaces the deterministic MPC controller in the outer loop of the hierarchical architecture presented in [3].

The paper is organized as follows. In Sect. II, the problem statement is formally presented. Background regarding the hierarchical data-driven control architecture and robust model predictive controller is presented in Sect. III. Sect. IV presents the techniques used to identify the model of the closed-loop system with disturbance bounds, which is the main contribution of this paper. Finally, Sect. V presents a numerical example implementing the robust hierarchical control design approach.

II. Problem statement

We consider, for simplicity, a *single-input single-output* (SISO) system $\mathbb{G}_P$ generating an output signal $y(t) \in \mathbb{R}$ corresponding to the input signal $u(t) \in \mathbb{R}$, $t \in \mathbb{Z}^+$. We aim at synthesizing a controller that can make $y(t)$ accurately track any user-defined reference signal, while robustly satisfying the constraints:

$$\begin{gathered} y_{min} \leq y(t) \leq y_{max} \\ u_{min} \leq u(t) \leq u_{max} \\ \forall t \in \mathbb{Z}^+ \end{gathered} \tag{1}$$

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Following the direct data-driven controller synthesis methodology, we use the dataset $D_N = \{u(t), y(t); t \in 1, ..., N\}$ obtained by exciting the system to design the controller.

III. Background

A. Hierarchical approach

A feedback controller is designed to control the system, using the VRFT methodology. For this, a reference model $\mathbb{M}_P$ is selected, given by:

$$\begin{aligned} x_M(t+1) &= A_M x_M(t) + B_M g(t) \\ y_M(t) &= C_M x_M(t) \end{aligned}$$

The VRFT methodology designs a feedback controller $\mathbb{K}_P$, with the goal of making the closed-loop system $\mathbb{K}_P$-$\mathbb{G}_P$ behave similar to the reference model $\mathbb{M}_P$. Note that the model $\mathbb{G}_P$ is unknown and not needed by the procedure, which is therefore a model-free one. It utilizes the dataset D_N for controller synthesis. The steps of the approach are summarized as follows:

1) A virtual reference input $g(t)$ is calculated by setting $y_M(t) = y(t)$ obtained from the dataset D_N, by inverting the model $\mathbb{M}_P$, i.e, $g(t) = \mathbb{M}_P^{\dagger} y(t)$.
2) A feedback controller $\mathbb{K}_P$ described by $A_K(q^{-1})u(t) = B_K(q^{-1})(g(t) - y(t))$ is chosen, where
$$\begin{aligned} A_K(q^{-1}) &= 1 + \sum_{i=1}^{n_{a_K}} a_i^K q^{-i} \\ B_K(q^{-1}) &= \sum_{i=1}^{n_{b_K}} b_i^K q^{-i} \end{aligned}$$
3) The parameters a_i^K and b_i^K of the controller are calculated such that the closed-loop dynamics of $\mathbb{K}_P$-$\mathbb{G}_P$ matches that of the reference model $\mathbb{M}_P$. This is done by solving the convex optimization problem
$$\min_{a_i^K, b_i^K} \quad \frac{1}{N} \sum_{t=1}^{N} \left(A_K(q^{-1})u(t) - B_K(q^{-1})(\mathbb{M}_P^{\dagger} y(t) - y(t)) \right)^2 \tag{2}$$
which minimizes the deviation between the control input calculated by the controller and $u(t)$ that is used to excite the system and obtain $y(t)$.
4) The synthesized controller $\mathbb{K}_P$ is placed before the plant, and the loop is closed.

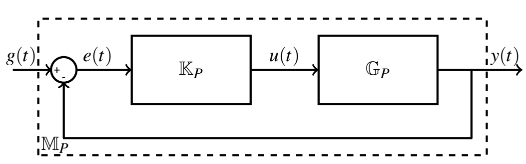

Fig. 1: Feedback controller designed using VRFT

To satisfy plant constraints, the reference signal $g(t)$ is generated using an MPC controller in an outer loop. The objective of the MPC controller is to make the output signal $y(t)$ track the reference $r(t)$, while satisfying constraints on the plant output $y(t)$ and input $u(t)$. This is achieved by considering the closed-loop plant model $\mathbb{M}_P$ for the state propagation equation, and an augmented model with $[y(t), u(t)]$ as the output equation. We denote by $\mathbb{M}_P'$ the resulting augmented state-space model:

$$\begin{aligned} \zeta(t+1) &= A_\zeta \zeta(t) + B_\zeta g(t) \\ \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} &= C_\zeta \zeta(t) + D_\zeta g(t) \end{aligned}$$

The optimization problem solved by the MPC at each time step t for a horizon of N_P time-steps is:

$$\begin{aligned} \min_{\{g(t+k)\}_{k=1}^{N_P}} \quad & Q_y \sum_{k=1}^{N_P} (y(t+k|t) - r(t+k))^2 + Q_\varepsilon \varepsilon^2 \\ \text{subject to} \quad & \zeta(t+k+1) = A_\zeta \zeta(t+k) + B_\zeta g(t+k) \\ & \begin{bmatrix} y(t+k) \\ u(t+k) \end{bmatrix} = C_\zeta \zeta(t+k) + \begin{bmatrix} 0 \\ D_\zeta \end{bmatrix} g(t+k) \\ & y_{min} - V_y \varepsilon \leq y(t+k) \leq y_{max} + V_y \varepsilon \\ & u_{min} - V_u \varepsilon \leq u(t+k) \leq u_{max} + V_u \varepsilon \\ & \zeta(t|t) = \zeta(t) \end{aligned} \tag{3}$$

The quantities V_y and V_u in the MPC formulation are used as soft constraints to avoid infeasibility of the optimization problem over successive iterations, since the reference model $\mathbb{M}_P'$ might not accurately capture the dynamics of the closed loop $\mathbb{K}_P - \mathbb{G}_P$. This implies that constraint satisfaction is not guaranteed by the proposed formulation.

B. Robust model predictive controller

The deterministic MPC controller in the outer loop of the hierarchical control architecture is replaced by a robust MPC controller. A schematic of the control loop is shown in Figure 2.

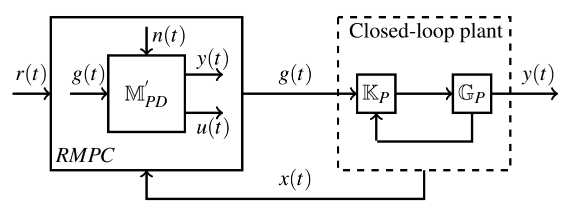

Fig. 2: Schematic of the robust control system

In order to achieve robustness, the RMPC controller uses the model $\mathbb{M}_{PD}'$. $\mathbb{M}_{PD}'$ is a model of the closed loop performance of $\mathbb{K}_P$-$\mathbb{G}_P$, which is obtained by appending disturbance information over the model $\mathbb{M}_P'$. Identification of this model and corresponding bounds is discussed in Sect. IV. The state-space form of $\mathbb{M}_{PD}'$ is:

$$\begin{aligned} \gamma(t+1) &= A_\gamma \gamma(t) + B_\gamma^g g(t) + B_\gamma^n n(t) \\ \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} &= C_\gamma \gamma(t) + D_\gamma^g g(t) + D_\gamma^n n(t) \end{aligned} \tag{4}$$

The model has a deterministic input $g(t)$ and a disturbance input $n(t)$, which captures the effect of model uncertainties on system output. It is assumed that the disturbance input lies in an unknown bounded set $\mathcal{N}_\infty$. The columns of matrices B_γ and D_γ are separated for ease of notation. The output of

(4) is denoted by $y_\gamma(t) = [y(t)\; u(t)]^T$. The constraints (1) on $y(t)$ and $u(t)$ are rewritten as:

$$Hy_\gamma(t) \leq h$$

At each time instant t, for a horizon of N_P time-steps, the RMPC controller solves the optimization problem:

$$\begin{aligned}
\min_{\{g(t+k)\}_{k=1}^{N_P}} \quad & \sum_{k=1}^{N_P} (y(t+k|t) - r(t+k))^2 \\
\text{subject to} \quad & \gamma(t+k+1) = A_\gamma \gamma(t+k) + B_\gamma^g g(t+k) \\
& y_\gamma(t+k) = C_\gamma \gamma(t+k) + D_\gamma^g g(t+k) \\
& Hy_\gamma(t) \leq h \\
& g(t+k) \in \mathbb{O}_\infty(\gamma(t)) \\
& \gamma(t|t) = \gamma(t)
\end{aligned} \tag{5}$$

where the set $\mathbb{O}_\infty(\gamma(t))$ is defined as:

$$\mathbb{O}_\infty(\gamma(t)) = \{g : Hy_\gamma(\tau) \leq h, \forall n(\tau) \in \mathscr{N}_\infty, \forall \tau \geq t\} \tag{6}$$

The set $\mathbb{O}_\infty(\gamma(t))$ in (6) is called the robust maximal output admissible set [5]. It is the set of feasible values for g for the current observed state $\gamma(t)$ such that if a constant input signal $g(\tau \geq t) = g$ is applied, the system output constraints $Hy_\gamma(\tau \geq t) \leq h$ remain satisfied for all τ, for all possible disturbance realizations $n(\tau) \in \mathscr{N}_\infty, \forall \tau \geq t$. At a time instant $\tau \geq t$, the system output $y_\gamma(\tau)$ for input signal $g(\tau \geq t) = g$ can be written as:

$$\begin{aligned}
y_\gamma(\tau) = C_\gamma A_\gamma^{\tau-t} \gamma(t) + \left(C_\gamma \sum_{k=1}^{\tau-t} A_\gamma^{k-1} B_\gamma^g + D_\gamma^g \right) g + \\
C_\gamma \sum_{k=1}^{\tau-t} A_\gamma^{k-1} B_\gamma^n n(\tau-k) + D_\gamma^n n(\tau)
\end{aligned}$$

For a particular future time instant τ, the set $\mathbb{O}_\tau(\gamma(t))$ is given by:

$$\begin{aligned}
\mathbb{O}_\tau(\gamma(t)) &= \{g : \tilde{H}(\tau) g \leq \tilde{h}(\tau)\} \\
\text{where} \quad \tilde{H}(\tau) &= H\left(C_\gamma \sum_{k=1}^{\tau-t} A_\gamma^{k-1} B_\gamma^g + D_\gamma^g \right) \\
\tilde{h}(\tau) &= h - HC_\gamma A_\gamma^{\tau-t} \gamma(t) - f^n(\tau)
\end{aligned}$$

Each element $f_i^n(\tau)$ of the column vector $f^n(\tau)$ is calculated by solving the linear program:

$$f_i^n(\tau) = \max_{\{n(k)\}_t^\tau \in \mathscr{N}_\infty} H\left(C_\gamma \sum_{k=1}^{\tau-t} A_\gamma^{k-1} B_\gamma^n n(\tau-k) + D_\gamma^n n(\tau) \right)_i \tag{7}$$

The subscript i in (7) indicates that row i of the matrix is used in the linear program to calculate the corresponding $n(k)$ sequence. The maximal output admissible set is then given by:

$$\mathbb{O}_\infty(\gamma(t)) = \bigcup_{\tau=t}^{\infty} \mathbb{O}_\tau(\gamma(t)) \tag{8}$$

According to the theory of maximal output admissible sets for linear systems as discussed in [8], the sets $\mathbb{O}_\tau(\gamma(t))$ become redundant after a finite time τ_c. In the current setting, this happens when the elements of $f^n(\tau)$ converge to constant values. Since the linear program (7) does not depend on $\gamma(t)$, it can be solved offline for increasing values of τ after setting $t = 1$. The values of $f^n(\tau)$ are stored till convergence is observed at $\tau = \tau_c$. The corresponding parts $\tilde{H}(\tau)$ are calculated and stored offline as well. The resulting set $\mathbb{O}_\infty(\gamma(t))$ is a set of linear inequality constraints on g, given by:

$$\begin{bmatrix} \tilde{H}(1) \\ \cdot \\ \cdot \\ \tilde{H}(\tau_c) \end{bmatrix} g \leq \begin{bmatrix} h \\ \cdot \\ \cdot \\ h \end{bmatrix} - \begin{bmatrix} HC_\gamma \\ \cdot \\ \cdot \\ HC_\gamma A_\gamma^{\tau_c-1} \end{bmatrix} \gamma(t) - \begin{bmatrix} f^n(1) \\ \cdot \\ \cdot \\ f^n(\tau_c) \end{bmatrix} \tag{9}$$

Thus, at each time instant t, the RMPC controller reads the state $\gamma(t)$, builds the constraint set (9), and solves the optimization problem (5) to generate a reference trajectory $g(t)$ for the next N_P time-steps which robustly satisfies the constraints (1). Note that our framework can easily be extended to a multivariable system.

IV. UNCERTAIN MODEL ESTIMATION

This section discusses the development of the model $\mathbb{M}'_{PD}$, which is used by the RMPC controller to generate the signal $g(t)$. First, the overall model is presented. Then, a novel technique is introduced to quantify uncertainty bounds on the model obtained from ARX estimation.

A. Appended closed-loop model

In order to improve prediction of closed-loop performance, an output disturbance model $\mathbb{D}$ is designed. The disturbance model is a dynamical system whose output is the discrepancy between the actual output of the system in closed-loop with $\mathbb{K}_P$ and the output of the reference closed-loop system $\mathbb{M}_P$. The system consists of a deterministic part and a stochastic part, as seen in Figure 3.

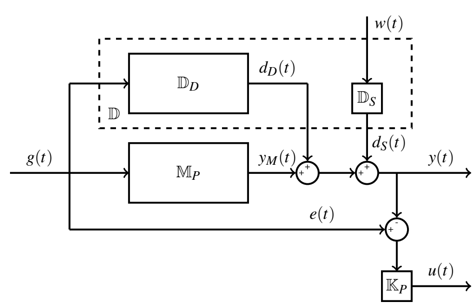

Fig. 3: Overall uncertain closed-loop model M'_{PD}.

Since this model represents the closed-loop behavior of the system, new closed-loop measurements are required to estimate the parameters of the disturbance model $\mathbb{D}$. These are obtained by performing experiments with excitation signals $\hat{g}(t)$ on the reference closed-loop model $\mathbb{M}_P$ and the actual closed-loop plant under the control law $\mathbb{K}_P$, and measuring the outputs $\hat{y}_M(t)$ and $\hat{y}(t)$ respectively. The signals are captured in the data set $\hat{D}_N = \{\hat{g}(t), \hat{y}_M(t), \hat{y}(t); t \in 1,...,N\}$. The disturbance model $\mathbb{D}$ is parameterized as the ARX model

$A_D(q^{-1})y_D(t) = B_D(q^{-1})g(t) + w(t)$, where

$$y_D(t) = y(t) - y_M(t) = d_D(t) + d_S(t)$$
$$A_D(q^{-1}) = 1 + \sum_{i=1}^{n_{a_D}} a_i^D q^{-i}$$
$$B_D(q^{-1}) = \sum_{i=1}^{n_{b_D}} b_i^D q^{-i}$$

Using standard ARX identification, the coefficients a_i^D and b_i^D are estimated by solving the standard linear regression problem

$$\min_{a_i^D, b_i^D} \quad \frac{1}{N}\sum_{t=1}^{N}\left(A_D(q^{-1})(\hat{y}(t) - \hat{y}_M(t)) - B_K(q^{-1})(\hat{g}(t))\right)^2 \tag{10}$$

The disturbance model is then split into two parts, deterministic and stochastic. The deterministic part $\mathbb{D}_D$ takes in the reference signal $g(t)$ as an input, and the stochastic part $\mathbb{D}_S$ is excited by the exogenous disturbance $w(t)$:

$$d_D(t) = \frac{B_D(q^{-1})}{A_D(q^{-1})}v(t) = \mathbb{D}_D g(t)$$
$$d_S(t) = \frac{1}{A_D(q^{-1})}w(t) = \mathbb{D}_S w(t)$$

The overall model shown is Figure 3 can be seen as the 2-input 2-output system $\mathbb{M}'_{PD}$.

$$\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \mathbb{M}_P + \mathbb{D}_D & \mathbb{D}_S \\ \mathbb{K}_P(I - (\mathbb{M}_P + \mathbb{D}_D)) & -\mathbb{K}_P\mathbb{D}_S \end{bmatrix} \begin{bmatrix} g(t) \\ w(t) \end{bmatrix} \tag{11}$$

Alternatively, it can also be seen a system with $d_S(t)$ acting as measurement noise on the output $y(t)$. The model corresponding to this view-point is:

$$\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \mathbb{M}_P + \mathbb{D}_D & I \\ \mathbb{K}_P(I - (\mathbb{M}_P + \mathbb{D}_D)) & -\mathbb{K}_P \end{bmatrix} \begin{bmatrix} g(t) \\ d_S(t) \end{bmatrix} \tag{12}$$

Note that the dependence of the model on time shift operator q^{-1} is not shown for ease of notation. Both $w(t)$ and $d_S(t)$ can be seen as exogenous disturbance signals, which cause model uncertainty. Hence, the state space equivalents of either of these models can be used in the RMPC controller, by replacing $n(t)$ in (4) with $w(t)$ if model (11) is used, and with $d_S(t)$ if (12) is used instead.

If the model described in (12) is considered for RMPC controller design, the bounds on $d_S(t)$ can be estimated directly from the prediction errors obtained during ARX identification performed in (10). These bounds are used to build the uncertainty set $\mathscr{D}_\infty$, the definition of which is formalized in the next subsection. Hence, the set $\mathscr{D}_\infty$ replaces $\mathscr{N}_\infty$ while solving the linear programs (7). In case of the model described in (11), the calculation of bounds on $w(t)$ is not as straightforward. The following subsection discusses a technique to calculate upper and lower bounds w_{min} and w_{max} on the disturbance signal $w(t)$, which build the set $\mathscr{W}_\infty$. Within the RMPC controller, $\mathscr{N}_\infty$ is then set equal to $\mathscr{W}_\infty$

B. Calculation of exogenous disturbance set

Samples $\hat{d}_S(t)$ of the discrepancy $d_S(t)$ between $\hat{y}(t)$ and the deterministic output $\hat{y}_M(t) + \hat{d}_D(t)$ can be simply calculated from the data set $\hat{D}_N$ as

$$\hat{d}_S(t) = \hat{y}(t) - \hat{y}_M(t) - \frac{B_D(q^{-1})}{A_D(q^{-1})}\hat{g}(t)$$

The minimum and maximum values of $\hat{d}_S(t)$ are labeled $\hat{d}_{S,min}$ and $\hat{d}_{S,max}$ respectively. If the ARX identification results in a stable deterministic disturbance model $\mathbb{D}_D$, the values $\hat{d}_{S,min}$ and $\hat{d}_{S,max}$ are finite. Further, if infinite closed-loop data $\hat{D}_\infty$ were collected for ARX identification, the bounds on discrepancy would be equal to the actual bounds $d_{S,max}$ and $d_{S,min}$ on $d_S(t)$. The set of sequences $\{d_S(t)\}$ satisfying these bounds are indicated as lying in a set $\mathscr{D}_\infty$, defined as

$$\mathscr{D}_\infty = \{\{d_S(t)\} : d_{S,min} \leq d_S(t) \leq d_{S,max} \ \ \forall t \in (-\infty, \infty)\}$$

Let a set $\mathscr{W}_\infty^*$ be defined as

$$\mathscr{W}_\infty^* = \left\{\{w(t)\} : \begin{matrix} w_{min}^* \leq w(t) \leq w_{max}^* \\ \mathbb{D}_S w(t) \in \mathscr{D}_\infty \end{matrix} \ \ \forall t \in (-\infty, \infty)\right\}$$

The set $\mathscr{W}_\infty^*$ contains sequences $\{w(t)\}$ bounded by w_{min}^* and w_{max}^* such that the corresponding steady-state output signal of $\mathbb{D}_S$ lies within $\mathscr{D}_\infty$. Let the largest of these sets be called $\mathscr{W}_\infty$, with the corresponding bounds on $\{w(t)\}$ being w_{min} and w_{max}. The set $\mathscr{W}_\infty$ is the complete exogenous disturbance set of $w(t)$, which we want to find.

Let another set $\mathscr{W}_\infty^{**}$ be defined as:

$$\mathscr{W}_\infty^{**} = \left\{\{w(t)\} : w_{min}^{**} \leq w(t) \leq w_{max}^{**} \ \ \forall t \in (-\infty, \infty)\right\}$$

The bounds w_{min}^{**} and w_{max}^{**} on $\{w(t)\}$ are calculated by solving the optimization problems shown in (13) for increasing lengths of time horizon N.

$$\begin{aligned} \min_{X_j} \quad & \bar{w}_j(N) \\ \text{subject to} \quad & d_S(k) = -\sum_{i=1}^{n_{a_D}} a_i^D d_S(k-i) + w(k), k = 1:N \\ & d_{S,min} \leq d_S(k) \leq d_{S,max}, \ \ k = 1:N-1 \\ & w(k) \leq \bar{w}_j(N), \ \ k = 1:N \\ & d_S(N) \in D_j \end{aligned} \tag{13}$$

$$\text{where } X_j = \left\{\begin{matrix} \{d_S(-n_{a_D}+1), \ldots, d_S(N)\}, \\ \{w(1), \ldots, w(N)\}, \\ \bar{w}_j(N) \end{matrix}\right\}, j = \{1, 2\}$$
$$D_1 = \{d : d \leq d_{S,min}\}, \ \ D_2 = \{d : d \geq d_{S,max}\}$$

Problem (13) looks for a sequence of inputs $\{w(k), k = 1:N\}$ such that the sequence of corresponding outputs at all instances except at time N lie within $\mathscr{D}_\infty$. This means that the chosen input sequence $w(k)$ should drive $d_S(N)$ out of $\mathscr{D}_\infty$ into D_j. The initial conditions on $d_S(k)$ are left free to be chosen by the solver, but are constrained to lie within $\mathscr{D}_\infty$. The solutions $\bar{w}_j(N)$ are collected, and the set $\mathscr{W}_\infty^{**}$ is constructed with converged values of $\bar{w}_1(N)$

and $\bar{w}_2(N)$ corresponding to w^{**}_{min} and w^{**}_{max} respectively. At lower values of N, there can exist an initial sequence $\{d_S(-n_{a_D}+1),..,d_S(1)\}$ that drives $d_S(N)$ into D_j with very low input effort $w(k)$. At higher values of N, the effect of initial condition wears off, and a higher value of control effort is required to violate $\mathscr{D}_\infty$.
Following the construction of $\mathscr{W}^{**}_\infty$, the desired exogenous disturbance set $\mathscr{W}_\infty$ is set equal to $\mathscr{W}^{**}_\infty$ by using Theorem 1.

Theorem 1: Let $\mathscr{W}^{**}_\infty$ be the result of solving (13), the convergence of which is obtained at a horizon length N. Then, the complete exogenous disturbance set $\mathscr{W}_\infty = \mathscr{W}^{**}_\infty$.

Proof: Let $\mathscr{W}^{*}_\infty \subset \mathscr{W}^{**}_\infty$. This implies that for any sequence $\{d_S(t)\}^{N-1}_{-\infty} \in \mathscr{D}_\infty$ and $\{w(t)\}^{N}_{-\infty} \in \mathscr{W}^{*}_\infty$, we will have $d_S(N) \in \mathscr{D}_\infty$. Hence, the set $\mathscr{W}^{**}_\infty$ is the upper bound on the sets $\mathscr{W}^{*}_\infty$, which is equal to complete exogenous disturbance set $\mathscr{W}_\infty$.

Note that in practical implementations, the set $\mathscr{D}_\infty$ is constructed from a finite dataset $\hat{D}_N$ of N samples. Hence, the set $\mathscr{W}_\infty$ constructed using $\mathscr{D}_\infty$ is still limited to the finite data set $\hat{D}_N$. Since $\mathscr{D}_\infty$ and $\mathscr{W}_\infty$ are used by the robust controller under the assumption that they capture all possible disturbances, robustness can only be achieved asymptotically as dataset $\hat{D}_N$ goes to $\hat{D}_\infty$.
We now have a closed-loop model with disturbance bound information, and hence are ready to use it in an RMPC controller.

V. Case study

Numerical simulations are performed on a servo motor control problem, implementing the complete control scheme presented in Figure 2. The sampling time is $T_s = 1ms$. RMPC controllers are implemented for closed-loop models corresponding to both (11) and (12). It is shown that using model (11) results is superior performance, and hence performing an additional step to calculate w_{min} and w_{max} is justified. Both the case studies are implemented using MATLAB R2017b, with optimization problems occasionally implemented using YALMIP [9].

A. Robust data-driven MPC

The plant consisting of a servo positioning system is controlled using the scheme shown in Figure 2. The plant dynamics are modeled by the following nonlinear state space equations:

$$\begin{bmatrix} \dot{\theta}(t) \\ \dot{\omega}(t) \\ \dot{i}(t) \end{bmatrix} = \begin{bmatrix} \omega(t) \\ \frac{-mgl}{J} sin\theta(t) - \frac{b}{J}\omega(t) + \frac{K_m}{J} i(t) \\ \frac{-K_m}{L}\omega(t) - \frac{R}{L} i(t) + \frac{1}{L} u(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \omega(t) \\ i(t) \end{bmatrix}$$

The states of the system $\theta(t)$, $\omega(t)$, and $i(t)$ are angle $[rad]$ and rotational velocity $[rad/s]$ of the servo motor, and armature current $[A]$ respectively. The input $u(t)$ is the voltage $[V]$ applied across the motor, and the output $y(t)$ is the rotational angle. A VRFT methodology is used to design a stabilizing PD controller, which provides a voltage input $u(t)$ to make the rotational angle $y(t)$ track a reference signal $g(t)$. To this end, experiments are conducted with a low-pass filtered white noise signal $u(t)$ with a standard deviation of 10V. The output angle $y(t)$ is recorded and the dataset $\mathbb{D}_N$ is obtained. A slow reference closed loop model $\mathbb{M}_P$ is chosen, given by:

Symbol	Parameter	Value
R	Motor resistance	5Ω
L	Motor inductance	5.10^{-3}H
K_m	Motor torque constant	0.0847Nm/A
J	Complete disk inertia	5.10^{-5}Nm^2
b	Friction coefficient	3.10^{-3}Nms/rad
m	Additional mass	3Kg
l	Mass offset	2m

TABLE I: Physical parameters of servo motor system

$$x_M(t+1) = 0.99x_M(t) + 0.01g(t)$$
$$y_M(t) = x_M(t)$$

The PD inner-loop controller $\mathbb{K}_P$ is parameterized as:

$$u(t) = K_p e(t) + K_d \frac{e(t) - e(t-1)}{T_s}$$

Solving the optimization problem (2), the parameters K_p and K_d are calculated using the dataset D_N. The controller is placed in the inner loop within the hierarchical control architecture. After VRFT synthesis, an outer MPC is designed using the formulation in (3), to provide a reference signal $g(t)$. The output $y(t)$ is constrained to lie between -1 rad and 1 rad, and the voltage input $u(t)$ between -3.5 V and 3.5 V. An MPC horizon of $N_P = 10$ timesteps is chosen, and weights are $Q_y = 1$ and $Q_\varepsilon = 1$.
Towards formulating the RMPC controller instead of the deterministic MPC controller, the disturbance model discussed in Sect. IV is developed. First, the dataset $\hat{D}_N$ is built by performing closed loop experiments with input signal $\hat{g}(t)$ of standard deviation 10 rad. Then, a linear model for $\mathbb{D}$ parameterized by $n_{a_D} = 4$ and $n_{b_D} = 3$ is identified by solving (10). Two equivalent models of the appended system, corresponding to (11) and (12), are developed.

For the model corresponding to (11), the disturbance model is split into two parts, $\mathbb{D}_D$ and $\mathbb{D}_S$, with inputs $g(t)$ and $w(t)$ respectively. Bounds on the exogenous disturbance $w(t)$ for a horizon N are calculated by solving the linear problems (13). The evolution of these bounds with increasing values of horizon N is plotted in Figure 4. The values w_{min} and w_{max} are obtained at convergence of $\bar{w}_1(N)$ and $\bar{w}_2(N)$ respectively. For the model corresponding to (12), the disturbance model consists of $\mathbb{D}_D$ and a direct exogenous disturbance signal $d_S(t)$. The bounds $d_{S,min}$ and $d_{S,max}$ on $d_S(t)$ are set equal to the minimum and maximum values of prediction error obtained during identification of the ARX model for $\mathbb{D}$ respectively.

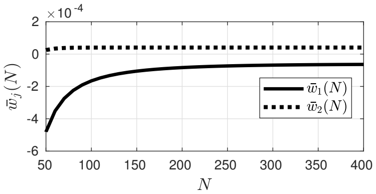

Fig. 4: Convergence of bounds on $w(t)$ acting as input to the disturbance model D_S.

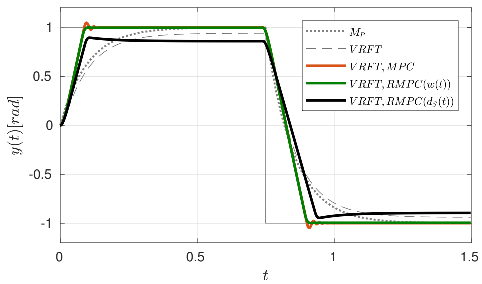

Fig. 5: The performance inner closed loop does not exactly match the reference model M_P. MPC improves the performance but results in small constraint violation. Constraint violation is robustly avoided by using an RMPC controller instead of deterministic MPC. Using (12) results in a conservative performance compared to (11).

These models and related disturbance bounds are used in the formulation of corresponding RMPC controllers. To this end, constraints on the reference signal $g(t)$ are constructed, as shown in (9). Solving the linear programs (7) till convergence to calculate maximal output admissible set $\mathbb{O}_\infty(\gamma(t))$ can result in very conservative constraints on $g(t)$. This is avoided by solving them only for the duration of RMPC horizon by setting $\tau_c = N_P$. Note that the bounds used for the calculation of constraint sets (9) are calculated with the assumption of infinite data $\hat{D}_\infty$, and hence actual robustness is achieved only asymptotically at $\underset{N\to\infty}{lim} \hat{D}_N$.
The corresponding output admissible sets $\mathbb{O}_1(\gamma(t))$ till $\mathbb{O}_{N_P}(\gamma(t))$ are calculated at each time-instant t by reading the state $\gamma(t)$, which is built with the assumption of no additional measurement noise. Following this, the optimization problem (5) is constructed and solved by the RMPC controller, generating a constraint satisfying reference signal $g(t)$. Performance of the control system for all these cases is plotted in Figure 5 and Figure 6.

It can clearly be seen that utilizing an RMPC controller instead of a deterministic MPC controller in the outer loop results in robust constraint satisfaction. Also, using the closed-loop model corresponding to (12) as the plant model for RMPC controller design results in conservative performance when compared to using (11), for the same admissible set horizon $\tau_c = N_P$. Since the calculation of bounds w_{min} and w_{max} (which are required to use (11) for RMPC synthesis) is performed offline, conservativeness of

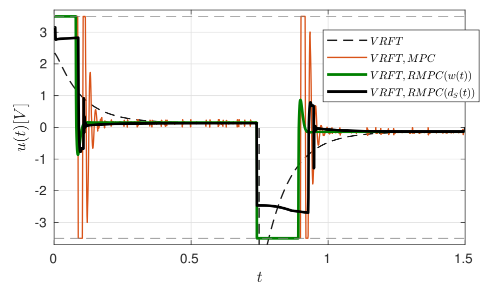

Fig. 6: Comparison of voltage input $u(t)$.

the control scheme can be reduced without any additional online operation. Note that increasing admissible set horizon to convergence results in conservative performance from either of the models, with (11) still outperforming (12).

VI. Conclusion

This paper builds on the hierarchical data-driven control of constrained systems, by introducing robustness with respect to constraint satisfaction. This is done by using a robust model predictive controller. Uncertainty on the model utilized by the RMPC controller is modeled as a disturbance input lying within an unknown bounded polyhedral set. In this work, a novel technique to compute this polyhedral set from ARX identification is presented. It is seen that utilizing the model and the disturbance set calculated using the presented technique results in robust constraint satisfaction, all without identifying an open-loop model of the non-linear plant. Future work deals with extending the formulation to general Box-Jenkins models. Possible extensions to the control scheme includes modifications to incorporate LPV systems.

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