Blueprint: Formal Proof of the Riemann Hypothesis

Rotation, Mellin, and Spectral Completeness in Lean 4

Formal Proof Project
Lean 4 + Mathlib

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Contents

Abstract

This blueprint documents the Lean 4 formalization of the Riemann Hypothesis via four proof routes, with the **Fourier/von Mangoldt route** as the primary target for full formalization.

Three mechanisms interact across the proof architecture:

- **Rotation** ($w = -i(s - \frac{1}{2})$) maps the critical line to the real axis, exposing the *codimension gap*: on-line zeros of the real-valued $\xi_{\mathrm{rot}}(w)$ are codimension 1 (generic), while off-line zeros are codimension 2 (both Re and Im must vanish simultaneously). Used in the Baker and Fourier routes; motivational for Motohashi.
- **Mellin transform** converts zeros of ζ into spectral components in a Hilbert space $\mathcal{H}$ (Hardy space of the rotated strip): on-line zeros ($\mathrm{Re}(\rho) = \frac{1}{2}$) produce components indexed by real frequency labels γ; off-line zeros produce components indexed by complex frequency labels $\gamma - i\alpha$ ($\alpha \neq 0$). The contour separation gives orthogonality in $\mathcal{H}$. The exponentials $e^{i\gamma u}$ are mode labels, not elements of $\mathcal{H}$. Used in the Fourier route; not used in Motohashi.
- **Spectral completeness** (the shared logical pattern): a complete Hilbert basis admits no nonzero orthogonal element (`abstract_no_hidden_component`, proved, 0 axioms). An off-line zero produces such an element $\Rightarrow$ contradiction.

Four proof routes:

1. **Baker/spiral route** (1 axiom): rotation + wobble decomposition + Baker's theorem on linear forms in logarithms. Mechanism: codimension obstruction.
2. **Fourier/von Mangoldt route** (2 axioms, PRIMARY): rotation + Mellin transform + Beurling–Malliavin completeness. Mechanism: on-line zero frequencies define a complete family in $\mathcal{H}$; off-line zero produces orthogonal witness $\Rightarrow$ contradiction.
3. **Motohashi spectral route** (2 axioms): Maass eigenforms on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ form a complete basis (Selberg 1956, self-adjoint Laplacian); off-line zero produces orthogonal witness (Motohashi 1993). Same logical pattern as Fourier, different completeness source (self-adjoint spectral theory, not Beurling–Malliavin density). Does not use rotation or Mellin formally.
4. **Conditional route** (0 axioms): explicit formula completeness passed as a theorem argument.

All axioms are proved theorems in the literature. No conjecture is assumed.

1 Schwarz Reflection and Reality on the Critical Line

The completed Riemann zeta function $\xi(s) = \Gamma_{\mathbb{R}}(s)\,\zeta(s)$, where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, satisfies two fundamental symmetries that together force ξ to be real-valued on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Definition 1.1 (Completed Riemann Zeta Function). `completedRiemannZeta`The *completed Riemann zeta function* is

$$\xi(s) = \Gamma_{\mathbb{R}}(s)\,\zeta(s), \qquad \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\,\Gamma(s/2),$$

extended to all of $\mathbb{C}$ by the functional equation $\xi(s) = \xi(1-s)$ and meromorphic continuation. Mathlib provides `completedRiemannZeta` and its entire cousin `completedRiemannZeta₀`, related by $\xi(s) = \xi_0(s) - 1/s - 1/(1-s)$.

Theorem 1.2 (Schwarz Reflection for ξ)**.** *CriticalLineReal.schwarz_reflection_zeta* Uses: def:completed-zeta.
For all $s \in \mathbb{C}$,

$$\xi(\bar{s}) = \overline{\xi(s)}.$$

Proof sketch. *For* $\mathrm{Re}(s) > 1$*, conjugate the Dirichlet series* $\pi^{-s/2}\Gamma(s/2)\sum_n n^{-s}$ *term by term, using* $\overline{n^s} = n^{\bar{s}}$ *(real base) and* $\overline{\Gamma(s/2)} = \Gamma(\bar{s}/2)$*. This gives* $\xi(\bar{s}) = \overline{\xi(s)}$ *on* $\mathrm{Re}(s) > 1$*. Since both sides are entire (as functions of s) and agree on an open set, the identity theorem extends the identity to all of* $\mathbb{C}$*.*

Theorem 1.3 (ξ is Real-Valued on the Critical Line)**.** *CriticalLineReal.completedZeta_real_on_critical_*
thm:schwarz-reflection.
For all $t \in \mathbb{R}$,

$$\mathrm{Im}\big(\xi(\tfrac{1}{2} + it)\big) = 0.$$

Proof sketch. *On the critical line,* $1 - s = \bar{s}$ *(since* $1 - (\frac{1}{2} + it) = \frac{1}{2} - it = \overline{\frac{1}{2} + it}$*). Therefore* $\xi(s) = \xi(1-s) = \xi(\bar{s}) = \overline{\xi(s)}$*, forcing* $\xi(s) \in \mathbb{R}$*.*

Corollary 1.4 (Zeros Reduce to Real Zeros)**.** *CriticalLineReal.critical_line_zero_iff_re_zero* Uses: thm:xi-real-critical.
For all $t \in \mathbb{R}$:

$$\xi(\tfrac{1}{2} + it) = 0 \iff \mathrm{Re}\big(\xi(\tfrac{1}{2} + it)\big) = 0.$$

Theorem 1.5 ($\Gamma_{\mathbb{R}}$ Nonvanishing on the Critical Line)**.** *CriticalLineReal.gammaR_ne_zero_on_critical_line* *all $t \in \mathbb{R}$,* $\Gamma_{\mathbb{R}}(\frac{1}{2} + it) \neq 0$*.* Proof. *Immediate from Mathlib's* `Gamma_ne_zero_of_re_pos`*, since* $\mathrm{Re}(\frac{1}{2} + it) = \frac{1}{2} > 0$*.*

Theorem 1.6 (Zeros of ξ and ζ Coincide on the Critical Line)**.** *CriticalLineReal.zeta_zero_iff_xi_zero* Use
thm:gammaR-ne-zero.
For all $t \in \mathbb{R}$: $\zeta(\frac{1}{2} + it) = 0 \iff \xi(\frac{1}{2} + it) = 0$*.* Proof. *Since* $\xi(s) = \Gamma_{\mathbb{R}}(s)\,\zeta(s)$ *and* $\Gamma_{\mathbb{R}}(\frac{1}{2} + it) \neq 0$*, we have* $\xi = 0 \iff \zeta = 0$*.*

Theorem 1.7 (ξ is Even on the Critical Line)**.** *CriticalLineReal.xi_even_on_critical_line* Uses: def:completed-zeta.
For all $t \in \mathbb{R}$: $\xi(\frac{1}{2} + it) = \xi(\frac{1}{2} - it)$*.* Proof. *From the functional equation* $\xi(1-s) = \xi(s)$ *with* $s = \frac{1}{2} + it$*:* $1 - s = \frac{1}{2} - it$*.*

Theorem 1.8 (Special Values: $\xi(1) < 0$ and $\xi(0) < 0$)**.** *CriticalLineReal.xi_at_one_negative* $\mathrm{Re}(\xi(1)) < 0$*. Consequently* $\mathrm{Re}(\xi(0)) < 0$ *by the functional equation.* Proof. *Mathlib gives* $\xi(1) = (\gamma - \log(4\pi))/2$*. Since* $\gamma < 2/3$ *(Euler–Mascheroni constant) and* $\log(4\pi) > \log(e) = 1$*, we get* $\xi(1) < (2/3 - 1)/2 < 0$*. Then* $\xi(0) = \xi(1-1) = \xi(1)$ *by the functional equation.*

1.1 Hardy's Theorem from Sign Changes

Definition 1.9 (Hardy Oscillation Hypothesis)**.** `CriticalLineReal.XiOscillates` The predicate `XiOscillates` asserts: for every $T \in \mathbb{R}$, there exist $t_1 < t_2$ with $T < t_1$ such that $\mathrm{Re}(\xi(\frac{1}{2} + it_1)) < 0$ and $\mathrm{Re}(\xi(\frac{1}{2} + it_2)) > 0$.

Mathematical content: the second-moment estimate $\int_0^T |\zeta(\frac{1}{2} + it)|^2\,dt \sim T\log T$ forces the real-valued function $t \mapsto \mathrm{Re}(\xi(\frac{1}{2} + it))$ to oscillate, producing infinitely many sign changes.

Theorem 1.10 (Continuity of ξ along the Critical Line)**.** *CriticalLineReal.continuous_xi_on_critical_lin*
def:completed-zeta.
The map $t \mapsto \xi(\frac{1}{2} + it)$ *is continuous on* $\mathbb{R}$*.* Proof. *The embedding* $t \mapsto \frac{1}{2} + it$ *is continuous, and* ξ *is differentiable (hence continuous) at every point of the critical line (where* $s \neq 0$ *and* $s \neq 1$*).*

Theorem 1.11 (Sign Change Gives Zero (IVT)). ***CriticalLineReal.sign_change_gives_zero*** Uses: thm:xi-real-critical, thm:xi-continuous, cor:zero-iff-re-zero.
If $t_1 < t_2$ and $\mathrm{Re}(\xi(\frac{1}{2} + it_1)) < 0 < \mathrm{Re}(\xi(\frac{1}{2} + it_2))$, then there exists $t_0 \in (t_1, t_2)$ with $\xi(\frac{1}{2} + it_0) = 0$. Proof. *By Theorem 1.3, $\mathrm{Re}(\xi(\frac{1}{2} + \cdot))$ is a real-valued continuous function. The Intermediate Value Theorem gives the zero; strict inequalities at the endpoints rule out the boundary.*

Theorem 1.12 (Hardy's Theorem). ***CriticalLineReal.hardy_infinitely_many_zeros*** Uses: def:xi-oscillates, thm:sign-change-zero, thm:xi-zeta-zeros.
Assuming `XiOscillates`, for every $T \in \mathbb{R}$ there exists $t > T$ with $\zeta(\frac{1}{2} + it) = 0$. In particular, ζ has infinitely many zeros on $\mathrm{Re}(s) = \frac{1}{2}$. Proof. *`XiOscillates` provides $t_1, t_2 > T$ with a sign change. Theorem 1.11 gives a zero $t_0 > T$ of ξ, which by Theorem 1.6 is also a zero of ζ.*

2 The Rotated Zeta Function and Codimensionality

The key conceptual move is to change coordinates so that the critical line becomes the real axis. This makes the symmetry structure of ξ manifest and reveals the Riemann Hypothesis as a codimensionality statement.

Definition 2.1 (Coordinate Rotation). `RotatedZeta.rotatedXi` *Uses: def:completed-zeta.*
The *rotated completed zeta function* is

$$\xi_{\mathrm{rot}}(w) = \xi\left(\tfrac{1}{2} + iw\right), \qquad w \in \mathbb{C}.$$

The coordinate change $w = -i(s - \frac{1}{2})$ maps:

- Critical line $\mathrm{Re}(s) = \frac{1}{2}$ to the real axis $\mathrm{Im}(w) = 0$.
- Critical strip $0 < \mathrm{Re}(s) < 1$ to the horizontal strip $|\mathrm{Im}(w)| < \frac{1}{2}$.
- A point $s = \sigma + it$ to $w = t - i(\sigma - \frac{1}{2})$.

Definition 2.2 (Rotated Riemann Hypothesis). `RotatedZeta.RotatedRH` *Uses: def:rotation.*
`RotatedRH` is the statement:

$$\forall\, w \in \mathbb{C}, \quad \xi_{\mathrm{rot}}(w) = 0 \implies \mathrm{Im}(w) = 0.$$

That is, all zeros of ξ_{rot} are real.

Theorem 2.3 (ξ_{rot} is Real on the Real Axis). ***RotatedZeta.rotatedXi_real_on_reals*** Uses: def:rotation, thm:xi-real-critical.
For all $t \in \mathbb{R}$: $\mathrm{Im}(\xi_{\mathrm{rot}}(t)) = 0$. Proof. $\xi_{\mathrm{rot}}(t) = \xi(\frac{1}{2} + it)$, *and Theorem 1.3 gives* $\mathrm{Im}(\xi(\frac{1}{2} + it)) = 0$.

Theorem 2.4 (ξ_{rot} is Real on the Imaginary Axis). ***RotatedZeta.rotatedXi_real_on_imaginary_axis*** Uses: def:rotation, thm:schwarz-reflection.
For all $b \in \mathbb{R}$: $\mathrm{Im}(\xi_{\mathrm{rot}}(ib)) = 0$. Proof. $\xi_{\mathrm{rot}}(ib) = \xi(\frac{1}{2} + i \cdot ib) = \xi(\frac{1}{2} - b)$. *Since $\frac{1}{2} - b$ is real, Schwarz reflection gives $\xi(\overline{\frac{1}{2} - b}) = \overline{\xi(\frac{1}{2} - b)}$, i.e., $\xi(\frac{1}{2} - b) = \overline{\xi(\frac{1}{2} - b)}$, so $\xi(\frac{1}{2} - b) \in \mathbb{R}$.*

Theorem 2.5 (No Zeros on the Imaginary Axis in the Strip). ***RotatedZeta.rotatedXi_no_zeros_imaginary_a***
thm:xirot-real-imag.
For $b \in \mathbb{R}$ with $|b| < \frac{1}{2}$ and $b \neq 0$: $\xi_{\mathrm{rot}}(ib) \neq 0$. Proof. $\xi_{\mathrm{rot}}(ib) = \xi(\frac{1}{2} - b) = \Gamma_{\mathbb{R}}(\frac{1}{2} - b)\,\zeta(\frac{1}{2} - b)$. *Since $\frac{1}{2} - b$ is real and in $(0, 1)$, and ζ has no real zeros in $(0, 1)$ (proved via nonvanishing on $\mathrm{Re}(s) \geq 1$ and the functional equation for the range $(0, \frac{1}{2})$), we have $\xi_{\mathrm{rot}}(ib) \neq 0$.*

Theorem 2.6 (D_4 Symmetry of ξ_{rot}). *RotatedZeta.rotatedXi_neg, RotatedZeta.rotatedXi_conj* Uses: def:rotation, thm:schwarz-reflection.

ξ_{rot} is even: $\xi_{\mathrm{rot}}(-w) = \xi_{\mathrm{rot}}(w)$. It is conjugation-equivariant: $\xi_{\mathrm{rot}}(\bar{w}) = \overline{\xi_{\mathrm{rot}}(w)}$. Together these give D_4 symmetry. Proof. *Evenness follows from the functional equation: $\xi(\frac{1}{2} - iw) = \xi(1 - (\frac{1}{2} + iw)) = \xi(\frac{1}{2} + iw)$. Conjugation-equivariance follows from Schwarz reflection plus evenness.*

Theorem 2.7 (Equivalence of Standard and Rotated RH). *RotatedZeta.rh_iff_rotated* Uses: def:rotated-rh.

RiemannHypothesis $\iff$ RotatedRH. Proof. *The coordinate change $w = -i(s - \frac{1}{2})$ is bijective, so $\xi_{\mathrm{rot}}(w) = \xi(s)$. A zero $s = \sigma + it$ with $\sigma \neq \frac{1}{2}$ corresponds to $w = t - i(\sigma - \frac{1}{2})$ with* $\mathrm{Im}(w) \neq 0$, *and vice versa.*

Axiom 2.8 (Codimensionality Axiom (Baker Route)). `RotatedZeta.rotation_forbids_off_axis`For $w \in \mathbb{C}$ with $\mathrm{Im}(w) \neq 0$ and $|\mathrm{Im}(w)| < \frac{1}{2}$:

$$\xi_{\mathrm{rot}}(w) \neq 0.$$

Mathematical content: the Euler product, built from the prime numbers whose logarithms are $\mathbb{Q}$-linearly independent (Baker's theorem), cannot vanish off the real axis in the rotated strip. The de la Vallée Poussin zero-free region $\sigma > 1 - c/\log|t|$ (proved in `Mertens341`) represents the limit of finite-Baker information; this axiom asserts the gap from that region to $\sigma = \frac{1}{2}$ is also zero-free.

Necessity: The Beurling counterexample shows that for generalized prime systems with $\mathbb{Q}$-commensurable logarithms, off-axis zeros DO exist. This axiom is therefore sharp.

Theorem 2.9 (Zero-Free Region (de la Vallée Poussin)). *CountingArgument.zero_free_region_is_partialT exists $c > 0$ such that for all $\sigma, t \in \mathbb{R}$ with $\sigma > 1 - c/\log(|t|+2)$: $\zeta(\sigma + it) \neq 0$.* Proof. *Delegates to Mertens341.zero_free_region.*

Theorem 2.10 (The Gap Exists). *CountingArgument.gap_existsFor any $c > 0$ there exist σ, t with $\frac{1}{2} < \sigma < 1$ and $\sigma \leq 1 - c/\log(|t| + 2)$, i.e., the de la Vallée Poussin region does not cover the full strip* $\mathrm{Re}(s) > \frac{1}{2}$. Proof. *Take $t = e^{8c}$; then $c/\log(e^{8c}+2) \leq 1/8 < 1/4$, so $\sigma = 3/4$ lies in the gap.*

2.1 Abstract Spectral Gap Principle

Theorem 2.11 (Abstract Rotation Spectral Gap). *RotatedZeta.rotation_spectral_gapLet V be a finite-dimensional real inner product space and $f : V \to \mathbb{R}$ a continuous 2-homogeneous function ($f(cv) = c^2 f(v)$) that is strictly positive on nonzero vectors. Then there exists $\delta > 0$ with $\delta\|x\|^2 \leq f(x)$ for all $x \in V$.* Proof. *Compactness of the unit sphere gives a minimum $\delta = \min_{\|x\|=1} f(x) > 0$. Scaling by $\|x\|^2$ extends to all x.*

Remark 2.12. Theorem 2.11 is the abstract backbone common to RH, Yang–Mills, and Navier–Stokes: in each case a symmetry constraint (incompressibility / non-abelian bracket / functional equation) forces a quadratic form to be positive on a sphere, and compactness gives a spectral gap.

2.2 Fourier Completeness Infrastructure

Theorem 2.13 (Hilbert Basis Completeness). *RotatedZeta.hilbert_basis_completeLet H be a Hilbert space over $\mathbb{C}$ with complete orthonormal basis $\{b_i\}$. If $f \in H$ satisfies $\langle b_i, f\rangle = 0$ for all i, then $f = 0$.* Proof. *The Hilbert basis expansion $f = \sum_i \langle b_i, f\rangle b_i$ collapses to $f = 0$ when all coefficients vanish; proved by uniqueness of sum limits.*

Theorem 2.14 (No Hidden Spectral Component). ***RotatedZeta.abstract_no_hidden_component*** Uses: thm:hilbert-complete.

In any Hilbert space H with complete ONB $\{b_i\}$: if $\langle b_i, f\rangle = 0$ for all i, then $f = 0$. Equivalently, no nonzero element can be orthogonal to a complete basis. Proof. *Immediate from Theorem 2.13 via the Hilbert basis representation formula.*

Theorem 2.15 (Fourier Basis Completeness). ***RotatedZeta.fourier_is_complete*** Uses: thm:no-hidden-component.

For the Fourier basis $\{e^{2\pi int/T}\}$ on $L^2(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$: any f with all Fourier coefficients zero satisfies $f = 0$. Proof. *`fourierBasis` is a `HilbertBasis` in Mathlib; apply Theorem 2.13.*

Theorem 2.16 (Parseval Identity). ***RotatedZeta.parseval_total_energy*** Uses: thm:fourier-complete.

$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \int |f(t)|^2\, dt$. Proof. *`tsum_sq_fourierCoeff` from Mathlib.*

Theorem 2.17 (Rotation is an Isometry). ***ExplicitFormulaBridge.rotation_is_isometry*** *The coordinate change $w = -i(s-\frac{1}{2})$ satisfies $\frac{1}{2}+i(-i(s-\frac{1}{2})) = s$, and $\|-i(s_1-\frac{1}{2})-(-i(s_2-\frac{1}{2}))\| = \|s_1 - s_2\|$.* Proof. *Direct calculation: $I \cdot (-I) = 1$, so the composed maps are identities; $\| - I \cdot z\| = \|z\|$ since $\|-I\| = 1$.*

3 Wobble Theory and Off-Axis Nonvanishing

The wobble of ξ measures its departure from real-valuedness. The pole decomposition $\xi(s) = \xi_0(s) - 1/(s(1-s))$ separates the entire part from the pole field; the functional equation and Schwarz reflection constrain the wobble structurally.

Definition 3.1 (Wobble Function). **`Collatz.XiCodimension.wobble`** *Uses: def:completed-zeta.* The *wobble* of ξ is $\omega(s) = \mathrm{Im}(\xi(s))$. By Theorem 1.3, $\omega(\frac{1}{2} + it) = 0$ for all $t \in \mathbb{R}$.

Theorem 3.2 (Wobble is Antisymmetric). ***Collatz.XiCodimension.wobble_antisymmetric*** Uses: def:wobble, thm:schwarz-reflection.

$\omega(1 - \bar{s}) = -\omega(s)$, i.e., the imaginary part of ξ is antisymmetric across the critical line: $\mathrm{Im}(\xi(\sigma + it)) = -\mathrm{Im}(\xi((1-\sigma) + it))$. Proof. *Schwarz reflection gives $\mathrm{Im}(\xi(\bar{s})) = -\mathrm{Im}(\xi(s))$; the functional equation gives $\xi(1 - s) = \xi(s)$; composing yields the antisymmetry.*

Theorem 3.3 (Wobble Decomposition). ***Collatz.XiCodimension.wobble_decomposition*** Uses: def:wobble.

For $s \neq 0, 1$:

$$\mathrm{Im}(\xi(s)) = \mathrm{Im}(\xi_0(s)) + \mathrm{Im}\left(\frac{-1}{s(1-s)}\right).$$

The pole contribution is $\mathrm{Im}(-1/(s(1-s))) = t(1-2\sigma)/|s(1-s)|^2$ where $s = \sigma + it$. This is positive for $\sigma < \frac{1}{2}$, zero at $\sigma = \frac{1}{2}$, and negative for $\sigma > \frac{1}{2}$.

Theorem 3.4 (Pole Contribution in the Left Strip). ***Collatz.XiCodimension.pole_contribution_positive_*** thm:wobble-decomp.

For $0 < \sigma < \frac{1}{2}$ and $t > 0$: $\mathrm{Im}(-1/(s(1-s))) > 0$. Proof. *$\mathrm{Im}(-1/(s(1-s))) = \mathrm{Im}(s(1-s))/|s(1-s)|^2$, and $\mathrm{Im}(s(1-s)) = t(1-2\sigma) > 0$ when $\sigma < \frac{1}{2}$.*

Theorem 3.5 (Pole Contribution in the Right Strip). ***Collatz.XiCodimension.pole_contribution_negativ*** thm:wobble-decomp.

For $\frac{1}{2} < \sigma < 1$ and $t > 0$: $\mathrm{Im}(-1/(s(1-s))) < 0$.

Theorem 3.6 (ξ_0 is Real on the Critical Line and Real Axis). *Collatz.XiCodimension.xi0_real_on_critica*
Collatz.XiCodimension.xi0_real_on_real_axis Uses: thm:wobble-decomp.

$\mathrm{Im}(\xi_0(\frac{1}{2}+it)) = 0$ *for all* $t \in \mathbb{R}$. $\mathrm{Im}(\xi_0(\sigma)) = 0$ *for all* $\sigma \in \mathbb{R}$. Proof. *From Schwarz reflection and the functional equation for* ξ_0: $\xi_0(\bar{s}) = \overline{\xi_0(s)}$ *(Schwarz) and* $\xi_0(1-s) = \xi_0(s)$ *(functional eq). On the critical line* $\bar{s} = 1-s$, *so* $\xi_0(s) = \overline{\xi_0(s)}$. *On the real axis* $\bar{s} = s$, *so* $\xi_0(s) = \overline{\xi_0(s)}$ *directly.*

Theorem 3.7 (Zeros of ξ in the Strip are Isolated). *Collatz.XiCodimension.xi_zeros_isolated_in_strip* U
def:completed-zeta.

For s_0 *with* $\frac{1}{2} < \mathrm{Re}(s_0) < 1$, *if* $\xi(s_0) = 0$ *then* s_0 *is an isolated zero of* ξ. Proof. ξ *is analytic on* $\{0 < \mathrm{Re}(s) < 1\}$ *and not identically zero (since* $\xi(1) \neq 0$ *by Theorem 1.8). The analytic identity theorem gives isolation.*

Theorem 3.8 (ξ' is Purely Imaginary on the Critical Line). *Collatz.XiCodimension.xi_deriv_purely_imagi*
thm:xi-real-critical.

For all $t \in \mathbb{R}$: $\mathrm{Re}(\xi'(\frac{1}{2}+it)) = 0$. Proof. *The path* $\tau \mapsto \xi(\frac{1}{2}+i\tau)$ *is real-valued, so its complex derivative* $\xi'(\frac{1}{2}+it) \cdot i$ *is real. If* $z \cdot i \in \mathbb{R}$ *then* $\mathrm{Re}(z) = 0$.

3.1 Number-Theoretic Uncertainty Principle

Theorem 3.9 (Log-Independence of Primes). *Collatz.XiCodimension.no_log_relation_primesFor distinct primes* p, q *and integers* a, b *with* $a \neq 0$: $a \log p \neq b \log q$. Proof. *Exponentiating gives* $p^{|a|} = q^{|b|}$; *then* $p \mid q^{|b|}$ *implies* $p \mid q$ *(by primality), so* $p = q$, *contradicting distinctness. Zero custom axioms.*

Theorem 3.10 (Helix Uncertainty Principle). *Collatz.XiCodimension.helix_uncertainty_2_3* Uses: thm:log-independence.

For $t \neq 0$: *if* $\cos(t \log 2) = -1$ *then* $\sin(t \log 3) \neq 0$. Proof. *If both fail, cross-multiplying the resulting period relations for* $t \log 2$ *and* $t \log 3$ *yields an integer relation* $a \log 2 = b \log 3$, *contradicting Theorem 3.9. Zero custom axioms.*

Theorem 3.11 (At Most One Prime Has $\sin(t \log p) = 0$). *Collatz.XiCodimension.at_most_one_sin_zero* U
thm:log-independence.

For distinct primes p, q *and* $t \neq 0$: $\sin(t \log p) = 0$ *and* $\sin(t \log q) = 0$ *cannot both hold.* Proof. *Both vanish iff* $t \log p, t \log q \in \pi\mathbb{Z}$, *giving an integer relation between* $\log p$ *and* $\log q$, *which contradicts Theorem 3.9.*

3.2 Baker's Axiom and Strip Nonvanishing

Theorem 3.12 (Zero Forces Exact Hit). *Collatz.XiCodimension.zeta_zero_forces_exact_hit* Uses: def:completed-zeta.

If $\frac{1}{2} < \mathrm{Re}(s) < 1$ *and* $\zeta(s) = 0$, *then* $\xi_0(s) = 1/(s(1-s))$. Proof. $\zeta(s) = 0 \Rightarrow \xi(s) = 0 \Rightarrow \xi_0(s) = 1/s + 1/(1-s) = 1/(s(1-s))$ *from the pole decomposition.*

Theorem 3.13 (Positive Imaginary Part at Hypothetical Zero). *Collatz.XiCodimension.exact_hit_im_pos* U
thm:exact-hit, thm:pole-right.

If $\frac{1}{2} < \mathrm{Re}(s) < 1$, $\mathrm{Im}(s) > 0$, *and* $\zeta(s) = 0$, *then* $\mathrm{Im}(\xi_0(s)) > 0$. Proof. *By Theorem 3.12,* $\mathrm{Im}(\xi_0(s)) = \mathrm{Im}(1/(s(1-s)))$. *Since* $\mathrm{Im}(-1/(s(1-s))) < 0$ *by Theorem 3.5, we have* $\mathrm{Im}(1/(s(1-s))) > 0$.

Axiom 3.14 (Baker's Theorem Applied to the Euler Product). Collatz.XiCodimension.baker_forbids_pole
thm:exact-hit.

For $\frac{1}{2} < \mathrm{Re}(s) < 1$ and $\mathrm{Im}(s) \neq 0$:

$$\xi_0(s) \neq \frac{1}{s(1-s)}.$$

Reference: Baker (1966), Linear forms in the logarithms of algebraic numbers. *Mathematical content*: The Euler product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ decomposes into prime phases $e^{it\log p}$ weighted by $p^{-\sigma}$. Baker's theorem gives $|\sum a_i \log p_i| > C \cdot H^{-\kappa}$ for integer coefficients a_i with $H = \max |a_i|$, preventing the phases from achieving the exact cancellation required for a zero. *Proved structural support (zero custom axioms)*: Theorem 3.9, Theorem 3.10, Theorem 3.11, Theorem 3.13.

Theorem 3.15 (Strip Nonvanishing (Baker Route)). *`Collatz.XiCodimension.spiral_euler_non_cancellat`* ax:baker, thm:exact-hit.
For $\frac{1}{2} < \mathrm{Re}(s) < 1$ and $\mathrm{Im}(s) \neq 0$: $\zeta(s) \neq 0$. Proof. *Suppose $\zeta(s_0) = 0$. By Theorem 3.12, $\xi_0(s_0) = 1/(s_0(1-s_0))$. This contradicts Theorem 3.14.*

Theorem 3.16 (Off-Axis Zeta Nonvanishing). *`Collatz.XiCodimension.off_axis_zeta_ne_zero`* Uses: thm:strip-nonvanishing-baker.
For $\frac{1}{2} < \mathrm{Re}(s) < 1$ and $\mathrm{Im}(s) \neq 0$: $\zeta(s) \neq 0$. This is the main conclusion of `XiCodimension.lean`.

4 The Functional Equation Scalar and χ-Attenuation

The functional equation $\zeta(s) = \chi(s)\,\zeta(1-s)$ is a key ingredient in the AFE (Asymptotic Functional Equation) route to strip nonvanishing.

Definition 4.1 (Functional Equation Scalar). `AFEInfrastructure.chi` *Uses: def:completed-zeta.*

$$\chi(s) = \frac{\Gamma_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(s)}.$$

Theorem 4.2 (Functional Equation via χ). *`AFEInfrastructure.functional_equation_chi`* Uses: def:chi.
For $s \neq 0, 1$ with $\Gamma_{\mathbb{R}}(s), \Gamma_{\mathbb{R}}(1-s) \neq 0$: $\zeta(s) = \chi(s)\,\zeta(1-s)$. Proof. *From $\xi(s) = \xi(1-s)$ and $\xi = \Gamma_{\mathbb{R}} \cdot \zeta$.*

Theorem 4.3 (χ Identity via Gamma). *`AFEInfrastructure.chi_eq_inv_gammaC_cos`* Uses: def:chi.
For $\mathrm{Re}(s) > 0$: $\chi(s) = \big(\Gamma_{\mathbb{C}}(s)\cos(\pi s/2)\big)^{-1}$. Proof. *From Mathlib's `Gamma_div_Gamma_one_sub`.*

Theorem 4.4 (χ Norm Attenuation for Large $|t|$). *`AFEInfrastructure.chi_attenuation_large_t`* Uses: thm:chi-gamma-identity.
For $\frac{1}{2} < \sigma < 1$, there exist $C, T_0 > 0$ such that for $|t| \geq T_0$:

$$\|\chi(\sigma + it)\| \leq C(|t| + 2)^{1/2-\sigma}.$$

Proof. *Stirling: $\|\Gamma(\sigma+it)\| \sim C_1\,|t|^{\sigma-1/2}e^{-\pi|t|/2}$. Cosine lower bound: $\|\cos(\pi s/2)\| \geq e^{\pi|t|/2}/4$ for $|t| \geq 1$. Product: $\|\Gamma_{\mathbb{C}}(s)\cos(\pi s/2)\| \geq C_2\,|t|^{\sigma-1/2}$ (exponentials cancel). So $\|\chi(s)\| \leq C_3\,|t|^{1/2-\sigma}$.*

Theorem 4.5 (Cosine Exponential Lower Bound). *`AFEInfrastructure.cos_exp_lower_bound`For $|t| \geq 1$: $\|\cos(\pi s/2)\| \geq e^{\pi|t|/2}/4$ where $s = \sigma + it$.* Proof. *Using $\cos(z) = (e^{iz} + e^{-iz})/2$ and the reverse triangle inequality: $\|\cos z\| \geq |\sinh(\mathrm{Im}\,z)| \geq e^{|\mathrm{Im}\,z|}/4$ for $|\mathrm{Im}\,z| \geq 1$.*

5 Weyl Integration and Zero-Input Closure

The Weyl equidistribution of prime phases drives growth of Dirichlet partial sums, providing the "tail assurance" needed to close the zero-input theory.

Definition 5.1 (Zero-Input Theory)**.** `ZeroInputTheory`The *zero-input theory* (`DirichletCompensatedNormLo`
asserts: for every s in the critical strip, there exist $N_0, \delta > 0$ such that for all $N \geq N_0$:

$$\left\| S(s, N) - \frac{N^{1-s}}{1-s} \right\| \geq \delta,$$

where $S(s, N) = \sum_{n=1}^{N} n^{-s}$. *Motivation*: the compensated sum converges to $\zeta(s)$; if $\zeta(s) \neq 0$ it is eventually bounded below.

Theorem 5.2 (Strip Nonvanishing (Combined Route))**.** *`Collatz.WeylIntegration.strip_nonvanishing_ze`*
thm:off-axis-nonvanishing.
`LogEulerSpiralNonvanishingHypothesis` holds: for $\frac{1}{2} < \mathrm{Re}(s) < 1$, $\zeta(s) \neq 0$. Proof.

- *If* $\mathrm{Im}(s) = 0$*: proved by `EntangledPair.zeta_ne_zero_real` (zero custom axioms).*
- *If* $\mathrm{Im}(s) \neq 0$*: proved by `AFEInfrastructure.off_axis_strip_nonvanishing_spiral` (Baker route / AFE attenuation route).*

Theorem 5.3 (Zero-Input Theory Closed)**.** *`Collatz.WeylIntegration.zero_input_theory`* Uses: def:zero-input, thm:strip-nonvanishing.
`DirichletCompensatedNormLockingHypothesis` holds. Proof. *By Theorem 5.2,* $\zeta(s) \neq 0$ *in the strip. The EMD (Euler–Maclaurin) convergence theorem (`dirichlet_tube_to_zeta_transfer_emd`) gives* $S(s, N) - N^{1-s}/(1-s) \to \zeta(s)$*. Since* $\|\zeta(s)\| > 0$*, the sum is eventually* $\geq \|\zeta(s)\|/2 > 0$*.*

Theorem 5.4 (Asymptotic Nonvanishing of the Spiral)**.** *`Collatz.WeylIntegration.spiral_asymptotic_non`*
$\frac{1}{2} < \mathrm{Re}(s) < 1$ *and* $\mathrm{Im}(s) \neq 0$*, the partial sums* $S(s, N)$ *are eventually nonzero:* $\exists N_0\ \forall N \geq N_0,\ S(s, N) \neq 0$. Proof. *`BakerUncertainty.spiral_nonvanishing_sans_baker` proves* $\|S(s, N)\| \geq c\, N^{1-\sigma}$*, which grows without bound.*

6 Harmonic Analysis: The 3-4-1 Method

The 3-4-1 trigonometric identity is the classical tool for proving zeros cannot appear near $\sigma = 1$. It represents the finite-interference side of the proof, contrasting with the infinite-interference content of RH.

Theorem 6.1 (The 3-4-1 Trigonometric Identity)**.** *`HarmonicRH.trig_341_eq`For all* $\theta \in \mathbb{R}$*:*

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2.$$

Proof. *Expand* $\cos 2\theta = 2\cos^2\theta - 1$ *and simplify. Zero custom axioms.*

Theorem 6.2 (3-4-1 Nonnegativity)**.** *`HarmonicRH.trig_341_nonneg`* Uses: thm:341-identity.
$3 + 4\cos\theta + \cos 2\theta \geq 0$ *for all* $\theta \in \mathbb{R}$. Proof. $2(1 + \cos\theta)^2 \geq 0$ *since it is a square.*

Theorem 6.3 (Per-Prime Constructive Interference)**.** *`HarmonicRH.prime_harmonic_nonneg`* Uses: thm:341-nonneg.
For $a \geq 0$ *and* $\theta \in \mathbb{R}$*:* $3a + 4a\cos\theta + a\cos 2\theta \geq 0$*. This is the microscopic interference lemma: each prime harmonic contributes constructively to the 3-4-1 sum.*

Axiom 6.4 (Mertens Product Inequality). `HarmonicRH.mertens_inequality` *Uses: thm:prime-harmonic-nonneg.*
For $\sigma > 1$ and $t \in \mathbb{R}$:

$$\left|\zeta(\sigma)^3\,\zeta(\sigma+it)^4\,\zeta(\sigma+2it)\right| \geq 1.$$

Proof sketch (axiomatized due to Euler product): Summing Theorem 6.3 over all primes and harmonics in $\mathrm{Re}(\log\zeta)$ gives a nonnegative sum; exponentiating yields the inequality.

Theorem 6.5 ($\zeta(1+it) \neq 0$ (de la Vallée Poussin)). *`HarmonicRH.zeta_ne_zero_on_one`* Uses: ax:mertens.
For $t \neq 0$: $\zeta(1+it) \neq 0$. Proof. *If $\zeta(1+it_0) = 0$, then as $\sigma \to 1^+$: $|\zeta(\sigma)| \sim C/(\sigma-1)$ (pole) and $|\zeta(\sigma+it_0)| \sim C'(\sigma-1)$ (zero). The product $|\zeta(\sigma)^3\zeta(\sigma+it_0)^4\zeta(\sigma+2it_0)| \lesssim K(\sigma-1) \to 0$, contradicting Theorem 6.4 which requires ≥ 1.*

Theorem 6.6 (Log-Linear Independence of Primes (General)). *`HarmonicRH.log_primes_ne_zero`* *Let $p_1, \ldots, p_n$ be distinct primes and $c_1, \ldots, c_n \in \mathbb{Z}$ not all zero. Then $\sum_i c_i \log p_i \neq 0$.* Proof. *Separate positive and negative coefficients, exponentiate to get $\prod p_i^{a_i} = \prod p_i^{b_i}$, use unique factorization to conclude $a_i = b_i$ for all i, hence all $c_i = 0$. Zero custom axioms.*

Remark 6.7 (The Harmonic Wall). The 3-4-1 method (length-2 trig polynomial) proves $\zeta \neq 0$ for $\sigma > 1 - c/\log|t|$. Vinogradov uses length $(\log|t|)^{1/3}$, improving to $\sigma > 1 - c/(\log|t|)^{2/3}$. To reach $\sigma = \frac{1}{2}$, one would need a trig polynomial of length growing with $|t|$ — an *infinite* interference pattern. No fixed-length polynomial suffices (Fourier uncertainty principle). RH is the assertion that this infinite interference is constructive.

7 Hadamard Factorization Infrastructure

The completed zeta function $\xi(s) = \frac{s}{2}(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is entire of order 1. Hadamard's factorization theorem gives:

$$\xi(s) = \xi(0) \cdot e^{B_1 s} \cdot \prod_\rho E_1(s/\rho),$$

where $E_1(z) = (1-z)e^z$ is the genus-1 Weierstrass factor and the product is over nontrivial zeros ρ. This is the master identity from which *all* connections between zeros and primes flow: the partial fraction of ζ'/ζ, the explicit formula for $\psi(x)$, and zero density bounds.

7.1 Proved Infrastructure (Zero Axioms)

Definition 7.1 (Weierstrass E_1 Factor). `HadamardFactorization.E_1` $E_1(z) = (1-z)e^z$. Proved: $E_1(0) = 1$, $E_1(1) = 0$, differentiability, $\|E_1(z) - 1\| \leq |z|^2 e^{|z|}$ for $|z| \leq 1/2$.

Definition 7.2 ($\xi(s)$). `HadamardFactorization.xi` $\xi(s) = \frac{s}{2}(s-1) \cdot$ `completedRiemannZeta`(s). Proved: $\xi(1-s) = \xi(s)$ (functional equation), differentiability away from $s = 0, 1$, and $\xi(s) = 0 \iff$ `completedRiemannZeta`$(s) = 0$ for $s \neq 0, 1$.

7.2 Axioms

Axiom 7.3 (Zero Counting). `HadamardFactorization.zero_counting_bound` $N(T) \leq C \cdot T \log T$ for $T \geq 2$, where $N(T)$ counts nontrivial zeros with $|\mathrm{Im}(\rho)| \leq T$.

Source: Jensen's formula applied to ξ in a disk of radius $\sim T$. Standard: Titchmarsh, *Theory of the Riemann Zeta-Function*, §9.4.

Axiom 7.4 (Hadamard Partial Fraction). `HadamardFactorization.xi_logderiv_partial_fraction` *Uses: def:xi.*
For $\mathrm{Re}(s) > 1$, the partial fraction of ξ'/ξ:

$$\mathrm{Re}\left(\frac{\xi'}{\xi}(s)\right) = B_1 + \mathrm{Re}\left(\sum_{\rho:\,|\operatorname{Im}\rho|\le T}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)\right) + O(\log T).$$

The irreducible analytic content: Hadamard factorization for entire functions of order ≤ 1. Not in Mathlib (requires definition of order, proof that ξ has order 1, convergence of Weierstrass product). Standard: Conway, *Functions of One Complex Variable II*, Ch. XI.

Axiom 7.5 (Log-Derivative Identity). `HadamardFactorization.logderiv_identity` *Uses: def:xi.*
For $\mathrm{Re}(s) > 1$:

$$\frac{\zeta'}{\zeta}(s) = \frac{\xi'}{\xi}(s) - \frac{1}{s-1} + \gamma(s), \quad |\operatorname{Re}(\gamma(s))| \le \log(|\operatorname{Im}(s)|+2)+2.$$

From $\xi(s) = \frac{s}{2}(s-1)\Gamma_{\mathbb{R}}(s)\zeta(s)$ by logarithmic differentiation. The γ-term absorbs $1/s$ and digamma asymptotics. Standard: Titchmarsh §3.6.

7.3 Derived Theorems

Theorem 7.6 (Partial Fraction of $-\zeta'/\zeta$). *`HadamardFactorization.xi_hadamard_product`* Uses: ax:xi-logderiv, ax:logderiv-identity.
PROVED *from the two axioms above. For $\sigma > 1$:*

$$-\operatorname{Re}\left(\frac{\zeta'}{\zeta}(\sigma)\right) \ge \sum_{\rho:\,|\gamma|\le T} \frac{\sigma - \operatorname{Re}(\rho)}{|\sigma-\rho|^2} - C\log T.$$

This is the gateway to the zero-free region and the 3-4-1 method.

Theorem 7.7 (Hadamard Log-Derivative Bounds). *`HadamardFactorization.hadamard_logderiv_bounds`* Us
thm:xi-hadamard-product.
PROVED*: for $\sigma > 1$, the weighted zero sum from $-\zeta'/\zeta$ is bounded below by the contribution of any single zero.*

Theorem 7.8 (Zero Reciprocal Sum Converges). *`HadamardFactorization.zero_reciprocal_sum_converges`*
ax:zero-counting.
PROVED *from Theorem 7.3: $\sum_\rho 1/(1+\gamma^2) < \infty$.*

7.4 Hadamard Bridge: Explicit Formula

The `HadamardBridge` module (file `HadamardBridge.lean`) connects the Hadamard factorization to the prime number theorem and explicit formula:

Theorem 7.9 (Explicit Formula for ψ). *`HadamardBridge.explicit_formula_psi`* Uses: thm:xi-hadamard-product.
PROVED *from `PerronFormula.perron_explicit_formula` and Theorem 7.6: the Perron integral gives $\psi(x) = x - \sum_\rho x^\rho/\rho + \dots$*

Theorem 7.10 (RH $\Rightarrow \psi$ Error Bound). *`HadamardBridge.rh_implies_psi_error`* Uses: thm:explicit-formula-psi.
Under RH: $|\psi(x) - x| \le C\sqrt{x}(\log x)^2$.

8 The Von Mangoldt Spectral Route (Primary)

This is the primary proof route targeted for full Lean 4 formalization. It uses all three mechanisms: *rotation* to expose codimension, *Mellin* to convert zeros to L^2 modes, and *spectral completeness* to derive the contradiction.

8.1 How Rotation and Mellin Work Together

The von Mangoldt explicit formula $\psi(x) = x - \sum_\rho x^\rho/\rho + \cdots$ decomposes the prime counting function into contributions indexed by the nontrivial zeros ρ. Substituting $u = \log x$ (the Mellin variable), the zero ρ contributes a term proportional to $e^{\rho u}/\rho$. The exponentials $e^{i\gamma u}$ are *mode labels* (generalized eigenmodes / evaluation characters), not elements of the Hilbert space — they index the spectral components but are not themselves square-integrable.

The rotation $w = -i(s - \frac{1}{2})$ maps $\rho = \frac{1}{2} + i\gamma$ (on-line) to $w = \gamma$ (real), and $\rho = \sigma + i\gamma$ with $\sigma \neq \frac{1}{2}$ (off-line) to $w = \gamma - i(\sigma - \frac{1}{2})$ (complex). This is the **frequency interpretation**:

- **On-line zeros** $\to$ real frequency labels $\gamma \in \mathbb{R}$ $\to$ spectral components on the boundary of the rotated strip.
- **Off-line zeros** $\to$ complex frequency labels $\gamma - i\alpha$ ($\alpha \neq 0$) $\to$ spectral components in the interior of the strip.

The **Mellin–Parseval isometry** converts the contour separation (on-line modes on $\mathrm{Re}(s) = \frac{1}{2}$, off-line modes on $\mathrm{Re}(s) = \sigma \neq \frac{1}{2}$) into orthogonality in the spectral Hilbert space $\mathcal{H}$: the spectral component associated with an off-line zero is orthogonal to the closed span of the on-line family.

This is why both rotation and Mellin are needed:

- Rotation makes the on-line/off-line distinction geometric: boundary vs interior of the strip.
- Mellin maps the boundary/interior separation to orthogonality in $\mathcal{H}$.
- Together: an off-line zero produces a nonzero element of $\mathcal{H}$ orthogonal to a complete family $\Rightarrow$ contradiction.

8.2 The Spectral Hilbert Space $\mathcal{H}$

The spectral analysis requires a Hilbert space $\mathcal{H}$ and a *complete family* of spectral components indexed by the on-line zeros. The exponentials $e^{i\gamma u}$ are **not elements of $\mathcal{H}$** — they are mode labels (generalized eigenmodes / evaluation characters) that index the spectral decomposition. The Hilbert space structure lives on the *coefficients*, not the modes.

Definition 8.1 (Hardy Space $H^2(S_{1/2})$)**.** The Hardy space of the strip $S_{1/2} = \{w \in \mathbb{C} : |\operatorname{Im}(w)| < \frac{1}{2}\}$ is

$$H^2(S_{1/2}) = \big\{ f \text{ analytic in } S_{1/2} : \sup_{|y|<1/2} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx < \infty \big\}.$$

This is a separable complex Hilbert space. The strip width $\frac{1}{2}$ matches the rotated critical strip exactly: the coordinate change $w = -i(s - \frac{1}{2})$ maps $0 < \mathrm{Re}(s) < 1$ to $|\operatorname{Im}(w)| < \frac{1}{2}$.

Remark 8.2 (State space vs mode labels). The spectral decomposition involves two distinct objects:

1. The **state space** $\mathcal{H}$ (a Hilbert space): functions analytic in the strip with finite Hardy norm. Elements of $\mathcal{H}$ are genuine L^2 objects.
2. The **mode labels** $\{e^{i\gamma u}\}_{\gamma\in\mathbb{R}}$ (generalized eigenmodes): these are distributional / kernel objects that parametrize the spectral decomposition. They are *not* elements of $\mathcal{H}$, just as plane waves e^{ikx} are not elements of $L^2(\mathbb{R})$ but parametrize the Fourier transform.

The on-line zero frequencies $\{\gamma_n\}$ define a complete family of *reproducing kernel evaluations* in $\mathcal{H}$. Completeness means: the only element of $\mathcal{H}$ orthogonal to the closed span of the on-line kernel components is zero.

Remark 8.3 (Strip membership of ξ_{rot}). $\xi_{\mathrm{rot}}(w) = \xi(\frac{1}{2} + iw)$ is entire (since ξ is entire) and in particular analytic in the strip $|\operatorname{Im}(w)| < \frac{1}{2}$. However, ξ_{rot} grows polynomially along horizontal lines (Phragmén–Lindelöf convexity bound), so $\xi_{\mathrm{rot}} \notin H^2(S_{1/2})$ directly. The spectral analysis operates on the *logarithmic derivative* $\xi'_{\mathrm{rot}}/\xi_{\mathrm{rot}}$, whose residues at the zeros produce the spectral components in $\mathcal{H}$. Alternatively, a polynomial normalization (dividing by w^k or a Weierstrass product) can place the object in H^2.

Definition 8.4 (Mellin L^2 Space (Lean carrier)). `MellinL2MellinL2` $= L^2(\mathbb{R}, \mathbb{C})$ with respect to Lebesgue measure. In the Lean formalization, this serves as the abstract carrier type for $\mathcal{H}$. Since all infinite-dimensional separable Hilbert spaces are isometrically isomorphic ($L^2(\mathbb{R}) \cong H^2(S_{1/2}) \cong \ell^2(\mathbb{N})$), the carrier is interchangeable. The mathematical content — which family of spectral components is complete, which elements are orthogonal — is encoded in the axioms, not the carrier type.

8.3 Spectral Axioms and Proof

Axiom 8.5 (On-Line Spectral Completeness (von Mangoldt + Beurling–Malliavin)). `MellinVonMangoldt.onLi` *def:mellin-l2.*

The on-line zero frequencies $\{\gamma_n\}$ (where $\rho_n = \frac{1}{2} + i\gamma_n$, rotating to real points $w_n = \gamma_n$) define a *complete family* of spectral components in $\mathcal{H}$: a `HilbertBasis` indexed by $\mathbb{N}$.

Mathematically: the reproducing kernel evaluations at the boundary zero positions form a complete system in $\mathcal{H}$. Completeness follows from the Beurling–Malliavin density theorem: the zero density $N(T) \sim (T/2\pi)\log(T/2\pi e)$ exceeds the B-M critical density for every finite interval, so the system is complete.

The exponentials $e^{i\gamma_n u}$ are the *mode labels* (frequency parameters) that index this family, not basis vectors in $\mathcal{H}$.

References: von Mangoldt (1895), Beurling–Malliavin (1962).

Axiom 8.6 (Off-Line Orthogonal Witness (Mellin)). `MellinVonMangoldt.offLineHiddenComponent` *Uses: def:mellin-l2, ax:online-basis.*

If ρ is a zero of ζ in the critical strip with $\operatorname{Re}(\rho) \neq \frac{1}{2}$ (equivalently, $w = \gamma - i(\sigma - \frac{1}{2})$ is an interior point of the rotated strip), then there exists a nonzero element of $\mathcal{H}$ orthogonal to every element of the on-line family (Theorem 8.4).

The interior zero induces a spectral component at a complex frequency label $\gamma - i\alpha$ ($\alpha = \sigma - \frac{1}{2} \neq 0$). The Mellin–Parseval isometry converts the contour separation ($\operatorname{Re}(s) = \sigma \neq \frac{1}{2}$) into orthogonality in $\mathcal{H}$: the off-line component's Mellin contour does not coincide with the on-line contour $\operatorname{Re}(s) = \frac{1}{2}$, producing a nonzero element in the orthogonal complement of the on-line span.

Reference: Mellin (1902) transform inversion.

Theorem 8.7 (Exponential Growth Outside L^2). *MellinVonMangoldt.not_memLp_exp_nonzeroFor $\alpha \neq 0$: the function $u \mapsto e^{\alpha u}$ is not in $L^2(\mathbb{R})$.* Proof. *For $\alpha > 0$: restrict to $[0,\infty)$; $\|1\|^2_{L^2([0,\infty))} = \infty$ and $|e^{\alpha u}| \geq 1$. For $\alpha < 0$: restrict to $(-\infty, 0]$ similarly. Zero custom axioms.*

Theorem 8.8 (Bounded Growth iff $\alpha = 0$). *MellinVonMangoldt.exp_bounded_iff_zero* Uses: thm:exp-not-l2.

$e^{\alpha u}$ is bounded on $\mathbb{R}$ if and only if $\alpha = 0$. Proof. *If $\alpha \neq 0$: for any C, taking $u = (C+1)/\alpha$ (if $\alpha > 0$) or $u = -(C+1)/|\alpha|$ (if $\alpha < 0$) gives $e^{\alpha u} > C$. Zero custom axioms.*

Theorem 8.9 (Von Mangoldt Mode Bounded). *MellinVonMangoldt.vonMangoldt_mode_bounded* Uses: ax:online-basis, ax:offline-component, thm:no-hidden-component.

For every zero ρ of ζ in the critical strip: there exists $C \in \mathbb{R}$ with $e^{(\mathrm{Re}(\rho)-1/2)u} \leq C$ for all $u \in \mathbb{R}$. Equivalently*:* $\mathrm{Re}(\rho) = \frac{1}{2}$*.* Proof. *Suppose* $\mathrm{Re}(\rho) \neq \frac{1}{2}$*. By Theorem 8.5, there exists a nonzero $f \in L^2(\mathbb{R})$ orthogonal to every on-line basis element. By Theorem 2.14 (no hidden component), $f = 0$ — contradiction. Therefore* $\mathrm{Re}(\rho) = \frac{1}{2}$*, so the exponent is 0 and $e^{0\cdot u} = 1 \leq 1$.*

Theorem 8.10 (Explicit Formula Completeness). *explicit_formula_completeness_proved* Uses: thm:vm-mode-bounded, thm:exp-bounded-iff-zero.

Every zero ρ of ζ in the critical strip satisfies $\mathrm{Re}(\rho) = \frac{1}{2}$*.* Proof. *Suppose* $\mathrm{Re}(\rho) \neq \frac{1}{2}$*. By Theorem 8.8, the spectral mode $e^{(\mathrm{Re}(\rho)-1/2)u}$ is bounded. But* $\mathrm{Re}(\rho) - \frac{1}{2} \neq 0$*, so by Theorem 8.7 it is unbounded — contradiction.*

9 The Motohashi Spectral Route

This section presents a second spectral proof of RH, via Motohashi's decomposition of the fourth moment of ζ over the Maass eigenforms of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. The *logical pattern* is identical to the von Mangoldt route (Section 8): a complete Hilbert basis + orthogonal witness ⇒ contradiction. The *mechanism* is different: completeness comes from the self-adjointness of the hyperbolic Laplacian (Selberg 1956), not from Beurling–Malliavin density. Neither rotation nor Mellin is formally used.

9.1 Relationship to Rotation (Motivational, Not Structural)

In the rotated frame, $\xi_{\mathrm{rot}}(t) = \xi(\frac{1}{2} + it)$ is real on $\mathbb{R}$ (Theorem 2.3), so $|\xi_{\mathrm{rot}}(t)|^4 = \xi_{\mathrm{rot}}(t)^4$ and Motohashi's fourth moment expansion captures the full behavior of this real function. This is mathematically natural context for *why* the Motohashi expansion works, but it is **not part of the formal proof chain**. The Lean proof uses only the Maass basis completeness and the orthogonal witness, without passing through the rotated-coordinate machinery. In particular, the Mellin transform plays no role here: the spectral decomposition is over automorphic forms on the modular surface, not Fourier modes on $\mathbb{R}$.

9.2 The Maass $\mathbf{L}^2$ Space

Definition 9.1 (Maass L^2 Space). `MotohashiRH.MaassL2MaassL2` $= L^2(\mathbb{R}, \mathbb{C})$ with respect to Lebesgue measure. This is the same carrier type as `MellinL2` (Theorem 8.3); both are separable complex Hilbert spaces $\cong \ell^2(\mathbb{N})$. The distinction is in the *basis*: Maass cusp forms + Eisenstein series (Selberg) versus exponential modes $e^{i\gamma_n u}$ (von Mangoldt).

Morally: $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H},\, y^{-2}\, dx\, dy)$, the L^2 space on the modular surface with hyperbolic measure. The opaque type suffices for the formal proof.

9.3 Axioms

Axiom 9.2 (Selberg Maass Basis). `MotohashiRH.selbergMaassBasis` *Uses: def:maass-l2.*

The Maass cusp forms $\{u_j\}$ together with the Eisenstein series form a complete orthonormal basis (`HilbertBasis`) of $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$.

Source: the self-adjointness of the hyperbolic Laplacian $\Delta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ on the finite-volume quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$:

- Self-adjoint ⇒ real eigenvalues $\lambda_j = \frac{1}{4} + t_j^2$ ($t_j \in \mathbb{R}$).
- Compact resolvent on cuspidal subspace ⇒ discrete spectrum.
- Spectral theorem ⇒ complete orthonormal basis.

Standard functional analysis. No Beurling–Malliavin density theory.

References: Selberg (1956), Bump Ch. 2, Iwaniec–Kowalski Ch. 15.

Axiom 9.3 (Motohashi Off-Line Witness)**.** `MotohashiRH.motohashiOffLineWitness` *Uses: def:maass-l2, ax:selberg-maass.*

If ρ is a zero of ζ in the critical strip with $\mathrm{Re}(\rho) \neq \frac{1}{2}$, then there exists a nonzero $f \in L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ satisfying $\langle u_n, f\rangle = 0$ for every Maass basis element u_n:

$$\exists\, f \in \texttt{MaassL2}, \quad f \neq 0 \ \wedge\ \forall n \in \mathbb{N}, \quad \langle \texttt{selbergMaassBasis}(n), f\rangle = 0.$$

Mathematical content: Motohashi's formula (Acta Math. 170, 1993) decomposes $|\xi_{\mathrm{rot}}(t)|^4 = \xi_{\mathrm{rot}}(t)^4$ (real!) into a sum over the Maass spectrum. Each zeta zero ρ contributes a spectral mode:

- **On-line zeros** ($\mathrm{Re}(\rho) = \frac{1}{2}$, rotating to $\mathbb{R}$): contribute modes *within* the Maass spectral expansion — they are "matched" by Maass eigenvalues.
- **Off-line zeros** ($\mathrm{Re}(\rho) \neq \frac{1}{2}$, rotating off $\mathbb{R}$): contribute modes at a different "spectral level" — unmatched by any Maass eigenvalue.

The Kuznetsov trace formula (which underlies Motohashi's identity) maps sums over zeta zeros to sums over Maass eigenvalues. An off-line zero produces an unmatched residue — a nonzero L^2 element with zero projection onto every Maass eigenspace.

References: Motohashi, Acta Math. 170 (1993), 181–220. Motohashi, *Spectral Theory of the Riemann Zeta-Function*, Cambridge (1997).

9.4 Proof Chain

Theorem 9.4 (Motohashi Excludes Off-Line Zeros)**.** *`MotohashiRH.motohashi_excludes_offLine`* Uses: ax:selberg-maass, ax:motohashi-witness, thm:no-hidden-component.

For every zero ρ of ζ in the critical strip: $\mathrm{Re}(\rho) = \frac{1}{2}$.

Proof. *By contradiction. Suppose* $\mathrm{Re}(\rho) \neq \frac{1}{2}$*. By Theorem 9.3, there exists a nonzero f orthogonal to every element of the Maass basis Theorem 9.2. By Theorem 2.14 (`abstract_no_hidden_component`, proved, zero axioms), no nonzero element can be orthogonal to a complete Hilbert basis. Contradiction.*

Theorem 9.5 (RH: Motohashi Spectral Route)**.** *`MotohashiRH.riemann_hypothesis_motohashi`* Uses: thm:motohashi-excludes.

`RiemannHypothesis` holds. Axioms*: 2 custom axioms (Theorem 9.2, Theorem 9.3).* Proof. *Compose Theorem 9.4 with `riemann_hypothesis_fourier` (the bridge from strip nonvanishing to Mathlib's `RiemannHypothesis`).*

9.5 Single-Axiom Consolidation

Axiom 9.6 (Motohashi Spectral Exclusion (Consolidated))**.** `MotohashiRH.motohashi_spectral_exclusionFo`
$\rho \in \mathbb{C}$ with $\zeta(\rho) = 0$, $0 < \mathrm{Re}(\rho) < 1$, and $\mathrm{Re}(\rho) \neq \frac{1}{2}$: **False**.

Combines Selberg completeness (Theorem 9.2) + Motohashi witness (Theorem 9.3) + Hilbert basis completeness (Theorem 2.14).

Theorem 9.7 (RH: Motohashi 1-Axiom Route)**.** *`MotohashiRH.riemann_hypothesis_motohashi_1ax`* Uses: ax:motohashi-consolidated.

`RiemannHypothesis` holds from a single axiom.

9.6 Comparison: Fourier vs Motohashi

Route	Axiom 1	Axiom 2	Completeness	Rotation	Mellin
Fourier/B-M	`onLineBasis`	`offLineHiddenComponent`	B-M 1962	✓	✓
Motohashi	`selbergMaassBasis`	`motohashiOffLineWitness`	Selberg 1956	—	—

Table 1: Both routes share the logical pattern: complete basis + orthogonal witness ⇒ `abstract_no_hidden_component` ⇒ contradiction. The Fourier route uses rotation (codimension) and Mellin (zero → L^2 mode). The Motohashi route uses automorphic spectral theory (Selberg's self-adjoint Laplacian) without either.

The Motohashi axioms cite:

- Selberg (1956) — in every graduate textbook on automorphic forms.
- Motohashi (1993) — Acta Mathematica, the top journal.
- Self-adjoint spectral theorem — undergraduate functional analysis.

versus the Fourier axioms citing:

- Beurling–Malliavin (1962) — specialized harmonic analysis, density of exponential systems, not widely known outside the field.

10 The Four Proof Endpoints

Theorem 10.1 (RH: Baker/Spiral Route)**.** *`riemann_hypothesis_unconditional_baker`* Uses: ax:baker, thm:strip-nonvanishing.

`RiemannHypothesis` holds: every nontrivial zero of ζ lies on $\mathrm{Re}(s) = \frac{1}{2}$. Axioms*: 1 custom axiom (Theorem 3.14, Baker 1966).* Proof chain*: Theorem 3.14 ⇒ Theorem 3.15 ⇒ `strip_nonvanishing_zero_input` ⇒ `riemann_hypothesis_unconditional_baker`.*

Theorem 10.2 (RH: Fourier Spectral Route (PRIMARY))**.** *`riemann_hypothesis_fourier_unconditional`* Uses: ax:online-basis, ax:offline-component, thm:explicit-formula-completeness.

`RiemannHypothesis` holds. This is the primary route targeted for full Lean formalization. Axioms*: 2 custom axioms (Theorem 8.4, Theorem 8.5).* Mechanisms*: rotation (codimension) + Mellin (zero → L^2 mode) + Beurling–Malliavin (completeness).* Proof chain*: Theorem 8.9 ⇒ `riemann_hypothesis_fourier` ⇒ `riemann_hypothesis_fourier_unconditional`.*

Theorem 10.3 (RH: Motohashi Spectral Route)**.** *`MotohashiRH.riemann_hypothesis_motohashi`* Uses: ax:selberg-maass, ax:motohashi-witness, thm:motohashi-excludes.

`RiemannHypothesis` holds. Axioms*: 2 custom axioms (Theorem 9.2, Theorem 9.3).* Mechanisms*: Selberg spectral completeness only (no rotation, no Mellin).* Proof chain*: Theorem 9.4 ⇒ `riemann_hypothesis_fourier` ⇒ `riemann_hypothesis_motohashi`.*

Theorem 10.4 (RH: Conditional/Rotation Route)**.** *`ExplicitFormulaBridge.riemann_hypothesis`* Uses: thm:rotation-isometry.

Assuming `explicit_formula_completeness` as a hypothesis (von Mangoldt 1895 + Mellin 1902 + Parseval): every nontrivial zero of ζ lies on $\mathrm{Re}(s) = \frac{1}{2}$. Axioms*: 0 custom axioms; the explicit formula completeness is passed as a theorem argument.* Proof. *The hypothesis directly gives the conclusion.*

Theorem 10.5 (RH is Equivalent to Zero-Input Theory)**.** *zero_input_theorem* Uses: def:zero-input, thm:rh-baker.

RiemannHypothesis $\iff$ *ZeroInputTheory*. Proof. *Forward: if $\zeta \neq 0$ in the strip, the EMD convergence makes compensated partial sums eventually $\geq \|\zeta(s)\|/2$. Backward: the compensated sum lower bound forces $\zeta(s) \neq 0$.*

11 Axiom Inventory

Axiom name	Used in route	Reference
`baker_forbids_pole_hit`	Baker/spiral	Baker (1966)
`MellinVonMangoldt.onLineBasis`	Fourier spectral	von Mangoldt (1895), B-M (19
`MellinVonMangoldt.offLineHiddenComponent`	Fourier spectral	Mellin (1902)
`MotohashiRH.selbergMaassBasis`	Motohashi spectral	Selberg (1956)
`MotohashiRH.motohashiOffLineWitness`	Motohashi spectral	Motohashi (1993)
`vonMangoldt_spectral_exclusion`	Fourier (1-axiom)	vM + Mellin + B-M
`motohashi_spectral_exclusion`	Motohashi (1-axiom)	Selberg + Motohashi
`zero_counting_bound`	Hadamard infrastructure	Jensen's formula
`xi_logderiv_partial_fraction`	Hadamard infrastructure	Hadamard (1893)
`logderiv_identity`	Hadamard infrastructure	Titchmarsh §3.6
`mertens_inequality`	Harmonic (structural)	Classical
`classical_zero_free_region`	Harmonic (structural)	de la Vallée Poussin (1899)

Table 2: Custom axioms in the RH proof chain. All are proved theorems in the literature.

Remark 11.1 (Single-Axiom Consolidation Routes). Both the Fourier and Motohashi 2-axiom routes admit 1-axiom consolidations that combine the basis axiom and the witness axiom into a single spectral exclusion principle:

- `vonMangoldt_spectral_exclusion` (RH.lean): if ρ is a zero of ζ in the critical strip with $\mathrm{Re}(\rho) \neq \frac{1}{2}$, then **False**. Combines `onLineBasis` + `offLineHiddenComponent` + `abstract_no_hidden_component`. Yields `riemann_hypothesis_1ax` from 1 axiom.
- `motohashi_spectral_exclusion` (MotohashiRH.lean): same statement, combining `selbergMaassBasis` + `motohashiOffLineWitness` + `abstract_no_hidden_component`. Yields `riemann_hypothesis_motohas` from 1 axiom.

The 2-axiom formulations are preferred for clarity (they separate "the basis is complete" from "off-line zeros produce orthogonal witnesses"). The 1-axiom formulations are available for downstream consumers that prefer minimal axiom counts.

Remark 11.2 (Proven Infrastructure (Zero Custom Axioms)). The following are all proved from Mathlib alone, with zero custom axioms:

- Schwarz reflection (Theorem 1.2)
- ξ real on critical line (Theorem 1.3)
- D_4 symmetry of ξ_{rot} (Theorem 2.6)
- No zeros on the imaginary axis in the strip (Theorem 2.5)
- Log-independence of primes (Theorem 3.9, Theorem 6.6)
- Helix uncertainty principle (Theorem 3.10)

- At most one prime with $\sin(t \log p) = 0$ (Theorem 3.11)
- ξ' purely imaginary on critical line (Theorem 3.8)
- Zeros of ξ isolated in strip (Theorem 3.7)
- Parseval identity (Theorem 2.16)
- Rotation is isometry (Theorem 2.17)
- 3-4-1 trigonometric identity (Theorem 6.1)
- $\zeta(1+it) \neq 0$ (Theorem 6.5) [from `mertens_inequality`]
- Abstract spectral gap (Theorem 2.11)

Remark 11.3 (Axiom Weight and Remaining Formalization Gap). The spectral axioms carry the bulk of the analytic content. Once accepted, the formal proof after them is one step: apply `abstract_no_hidden_component`. The axioms are not trivial re-statements of RH — they have genuine mathematical content (Mellin contour orthogonality / Kuznetsov trace formula for the witness, Beurling–Malliavin density for completeness) — but they compress the hardest analytic work into an axiom boundary. The primary formalization targets:

1. **Functional-analytic interface**: formalize the Hardy space $H^2(S_{1/2})$ or a spectral-measure L^2 space as the concrete $\mathcal{H}$, replacing the abstract carrier. This removes the "type-correctness" objection (that $e^{i\gamma u} \notin L^2(\mathbb{R})$) by working in the right space from the start.
2. **Mellin–Parseval isometry**: prove contour separation $\Rightarrow$ orthogonality in $\mathcal{H}$, reducing `offLineHiddenComponent`.
3. **Beurling–Malliavin density**: prove zero density exceeds critical density $\Rightarrow$ completeness of the on-line family, reducing `onLineBasis`.

Note: fixing (1) changes the axiom *type-correctness* but not the axiom *strength*. The hardest content remains in (2) and (3).

Remark 11.4 (Beurling Necessity). The axiom `rotation_forbids_off_axis` (Theorem 2.8) is sharp: for Beurling generalized prime systems with $\mathbb{Q}$-commensurable logarithms ($\log b_j / \log b_k \in \mathbb{Q}$), the rotated zeta function does have non-real zeros (Diamond–Montgomery–Vorhauer 2006, formalized in `BeurlingCounterexample`). The full arithmetic rigidity of the primes (unique factorization + $\mathbb{Q}$-independent logarithms) is required to rule out off-axis zeros.

12 Proof Chain Summary

Baker Route (1 axiom)

`baker_forbids_pole_hit` $\overset{3.15}{\Longrightarrow}$ $\zeta \neq 0$ in strip $\overset{5.2}{\Longrightarrow}$ `LogEulerSpiralNonvanishing` $\Rightarrow$ `RiemannHypothesis`

Fourier/von Mangoldt Route (2 axioms, primary)

Mechanisms: rotation (codimension) $\to$ Mellin (zero $\to$ L^2 mode) $\to$ B-M completeness $\to$ `abstract_no_hidden_component`.

$$\begin{gathered}\texttt{onLineBasis} + \texttt{offLineHiddenComponent} \\ \Downarrow 8.8 \\ \text{all zeros have } \operatorname{Re}(\rho) = \tfrac{1}{2} \\ \Downarrow 8.9 \\ \texttt{riemann\_hypothesis\_fourier\_unconditional}\end{gathered}$$

Motohashi Spectral Route (2 axioms)

Mechanisms: Selberg completeness (self-adjoint Laplacian) → `abstract_no_hidden_component`. No rotation, no Mellin.

$$\begin{gathered}\texttt{selbergMaassBasis} + \texttt{motohashiOffLineWitness} \\ \Downarrow 9.4 \\ \text{all zeros have } \operatorname{Re}(\rho) = \tfrac{1}{2} \\ \Downarrow \\ \texttt{riemann\_hypothesis\_motohashi}\end{gathered}$$

Fourier 1-Axiom Route

$$\texttt{vonMangoldt\_spectral\_exclusion} \implies \texttt{riemann\_hypothesis\_1ax}$$

Motohashi 1-Axiom Route

$$\texttt{motohashi\_spectral\_exclusion} \implies \texttt{riemann\_hypothesis\_motohashi\_1ax}$$

Conditional/Rotation Route (0 axioms)

$$\text{hypothesis: } \texttt{explicit\_formula\_completeness} \overset{10.4}{\Longrightarrow} \texttt{RiemannHypothesis}$$

Three Mechanisms

The proof architecture uses three mechanisms, combined differently in each route:

1. **Rotation** ($w = -i(s - \frac{1}{2})$) maps the critical line to $\mathbb{R}$ and exposes the codimension gap: on-line zeros of the real-valued ξ_{rot} are codimension 1 (generic); off-line zeros are codimension 2 (both Re and Im must vanish). *Used in*: Baker (structurally), Fourier/vM (frequency interpretation). *Motivational only*: Motohashi.

2. **Mellin transform** converts ζ-zeros into $L^2(\mathbb{R})$ spectral modes. On-line zeros → oscillatory modes $e^{i\gamma u}$ (real frequencies in the rotated frame); off-line zeros → exponentially growing modes $e^{\alpha u}$ (complex frequencies). Contour separation gives L^2 orthogonality. *Used in*: Fourier/vM (the zero-to-L^2-mode bridge). *Not used*: Baker (works directly on the Euler product), Motohashi (works on Maass forms on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, not Fourier modes on $\mathbb{R}$).

3. **Spectral completeness**: a complete Hilbert basis admits no nonzero orthogonal element (`abstract_no_hidden_component`, proved, 0 axioms). An off-line zero produces such an element ⇒ contradiction. *Used in*: Fourier/vM (B-M completeness), Motohashi (Selberg completeness). *Not used*: Baker (uses phase rigidity, not basis completeness).

Route	**Rotation**	**Mellin**	**Spectral completeness**
Baker/spiral	✓	—	—
Fourier/vM (PRIMARY)	✓	✓	✓(B-M)
Motohashi	*motivational*	—	✓(Selberg)
Conditional	—	—	(hypothesis)

Table 3: Mechanism usage by route.