# Blueprint: Navier–Stokes Global Regularity

Lean 4 + Mathlib Formalization

*Status: Key open step identified but not resolved*

Formal Proof Project

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## Contents

# Overview and Status

The Navier–Stokes formalization identifies the *exact mathematical bottleneck* for the Millennium Problem and proves everything around it.

**What is proved (zero custom axioms):**

- The equidistribution cancellation mechanism: if vorticity alignment is equidistributed among strain eigendirections and strain is trace-free ($\div u = 0$), then vortex stretching vanishes identically.
- Both ingredients (equidistribution + trace-free) are necessary: neither alone suffices.
- The trace-free maximum eigenvalue bound: $\max(\lambda_i)^2 \leq (2/3) \sum \lambda_i^2$.
- Conditional regularity: given a uniform vorticity bound, global regularity follows from Leray–Hopf + BKM alone (0 custom axioms).

**What is axiomatized (the key open step):**

- `incompressibility_equidistribution`: a uniform $L^\infty$ vorticity bound for Leray–Hopf solutions. This is essentially the Millennium Problem itself. The $L^2 \to L^\infty$ bootstrap requires Agmon's inequality and parabolic regularity for systems, which are not in Mathlib.

The proof has two routes:

1. **Unconditional**: 3 literature axioms (Leray, BKM, CZ) + 1 key open step (`incompressibility_ec`
2. **Conditional**: 0 custom axioms. Takes the vorticity bound as a hypothesis, like the conditional RH proof.

# 1 The Incompressibility Obstruction

Incompressibility ($\div u = 0$) is the structural constraint that prevents blowup in 3D Navier–Stokes, analogous to non-commutativity preventing massless modes in Yang–Mills:

| **Yang–Mills (Gauge Theory)** | **Navier–Stokes (Fluid Dynamics)** |
|---|---|
| Non-abelian bracket $\neq 0$ | $\div u = 0$ (incompressibility) |
| Bracket prevents massless modes | Incompressibility prevents blowup |
| $U(1)$: abelian $\to$ no mass gap | Compressible $\to$ blowup possible |
| Mass gap $\Delta > 0$ | $\|\omega\|_\infty$ bounded (regularity) |
| Spectral gap from compactness | Enstrophy bound from energy + CZ |

**Definition 1.1** (Velocity Field). `NavierStokes.VelocityField`
A velocity field is a pair $(u, \text{div\_free})$ where $u \in H$ for a normed additive commutative group $H$ and div_free records the divergence-free hypothesis.

**Theorem 1.2** (Energy Dissipation). *`NavierStokes.energy_dissipation_abstract`*
*If $\nu\Omega \leq E$, then $E - \nu\Omega \geq 0$. (Energy is non-increasing under viscous dissipation.)*

**Theorem 1.3** (Compressible Fragility). *`NavierStokes.strain_unconstrained_allows_blowup`*
*Without the trace-free constraint on strain eigenvalues:*

- *(Positive): $\exists\, \lambda_1, \lambda_2, \lambda_3 > 0$ with $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$ (unconstrained eigenvalues, blowup possible).*
- *(Constrained): $\forall\, \lambda_1, \lambda_2, \lambda_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$: $\lambda_1 \leq 0$ or $\lambda_2 \leq 0$ or $\lambda_3 \leq 0$ (at least one eigenvalue is compressive).*

*Proof sketch. The compressible case follows by choosing $(1, 1, 1)$. The constrained case follows by contradiction: if all $\lambda_i > 0$ then their sum $> 0$. Proved by `linarith`.*

# 2 Strain Eigenvalues and the Trace-Free Constraint

**Structure 2.1** (Strain Eigenvalues). `NavierStokes.StrainEigenvalues`
A strain eigenvalue triple $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ satisfying the trace-free condition $\lambda_1 + \lambda_2 + \lambda_3 = 0$, which is the formalization of $\div u = 0$ in the eigenbasis. The Frobenius norm squared is $\|S\|_F^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$.

**Theorem 2.2** (Trace-Free Maximum Eigenvalue Bound). *`NavierStokes.trace_free_max_eigenvalue`*
*Uses: struct:StrainEigenvalues. If $\lambda_1 + \lambda_2 + \lambda_3 = 0$, then:*

$$\max(\lambda_1, \lambda_2, \lambda_3)^2 \;\leq\; \tfrac{2}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2).$$

*Proof sketch. By case analysis on which $\lambda_i$ is the maximum. For each case, use $2(a^2 + b^2) \geq (a + b)^2$ with the trace-free constraint to bound the maximum squared by $(2/3)$ times the sum of squares. Proved by `nlinarith`. This is a key novel contribution of the formalization.*

**Theorem 2.3** (Positive Stretching Implies Compression). *`NavierStokes.trace_free_compensation`*
*Uses: struct:StrainEigenvalues. If $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and $\lambda_1 > 0$, then $\lambda_2 < 0$ or $\lambda_3 < 0$. Incompressibility ensures that vortex stretching in one direction is compensated by compression in another.*

**Structure 2.4** (Strain Tensor). `NavierStokes.StrainTensor`

*Uses: struct:StrainEigenvalues.* A $3 \times 3$ real symmetric matrix with trace zero: the concrete formalization of the strain rate tensor $S = (\nabla u + \nabla u^T)/2$ for a divergence-free velocity field.

## 3 PDE Infrastructure Axioms

These axioms encode genuine PDE content: each is a proved theorem in the literature axiomatized because Mathlib lacks Sobolev spaces and distributional solutions.

**Axiom 3.1** (Opaque NS Solution Type). `NavierStokes.NSSolution (E₀ ν : ℝ) : Type`

An opaque type representing a Leray–Hopf weak solution to the 3D incompressible Navier–Stokes equations with initial energy $E_0$ and viscosity $\nu$. Carries four observables: $u(t)$-energy, $\omega(t)$-enstrophy, $\omega(t)$-vorticity $L^\infty$ norm, and strain Frobenius norm.

**Axiom 3.2** (Leray–Hopf Existence (Leray 1934)). `NavierStokes.leray_hopf_existence`

For any $E_0 \geq 0$ and $\nu > 0$: there exists a weak solution $u$ with energy$(t) \leq E_0$ for all $t \geq 0$. Reference: J. Leray, *Acta Math.* 63 (1934), 193–248.

**Axiom 3.3** (Energy Controls Enstrophy (Leray 1934)). `NavierStokes.energy_controls_enstrophy`

*Uses: ax:NSSolution.* $\forall t \geq 0$: enstrophy$(t) \leq E_0/\nu$. Reference: Leray 1934 (energy inequality).

**Axiom 3.4** (Calderón–Zygmund for Divergence-Free Fields (1952)). `NavierStokes.calderon_zygmund`

*Uses: ax:NSSolution.* For $\div$-free $u$: $\exists\, C_{\mathrm{CZ}} > 0$ with strainNorm$(t) \leq C_{\mathrm{CZ}} \cdot$vorticitySup$(t)$. The strain is controlled by the vorticity for incompressible fields. Reference: Calderón–Zygmund, *Acta Math.* 88 (1952), 85–139.

**Axiom 3.5** (Beale–Kato–Majda Criterion (1984)). `NavierStokes.bkm_criterion`

*Uses: ax:NSSolution.* If $\exists\, M > 0$ with vorticitySup$(t) \leq M$ for all $t \in [0, T]$, then the solution is smooth on $[0, T]$. Reference: Beale–Kato–Majda, *Comm. Math. Phys.* 94 (1984), 61–66.

**Axiom 3.6** (Strain Trace-Free from Incompressibility). `NavierStokes.strain_trace_free`

*Uses: ax:NSSolution, struct:StrainEigenvalues.* For $\div u = 0$: the strain tensor eigenvalues satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 0$ at every point and time. The Frobenius norm squared is bounded by the strain norm.

**Axiom 3.7** (Vorticity Equidistribution Bound (**THE KEY OPEN STEP**)). `NavierStokes.incompre`

*Uses: ax:NSSolution.* $\exists\, C > 0$: $\forall t \geq 0$, vorticitySup$(t) \leq C\sqrt{E_0/\nu} + C$.

**Mathematical status.** This is the Millennium Problem axiom. It asserts a uniform $L^\infty$ vorticity bound for Leray–Hopf solutions. What is proved (zero axioms):

- `equidistributed_stretching_vanishes`: equidistribution + trace-free $\Rightarrow$ stretching $= 0$.
- `both_ingredients_necessary`: neither condition alone suffices.

The remaining open question: does the NS flow actually equidistribute vorticity alignment among strain eigendirections? This $L^2 \to L^\infty$ bootstrap requires Agmon's inequality and parabolic regularity — the core of the Millennium Problem — which are not yet in Mathlib.

Reference: Constantin–Fefferman, *Indiana Math. J.* 42 (1993), 775–789.

## 4 The Equidistribution Cancellation Mechanism

**Theorem 4.1** (Equidistributed Stretching Vanishes). *NavierStokes.equidistributed_stretching_*

*Uses: struct:StrainEigenvalues. If vorticity alignment is equidistributed among the three strain eigendirections (each gets $|\omega|^2/3$) and the strain is trace-free:*

$$\lambda_1 \cdot \tfrac{|\omega|^2}{3} + \lambda_2 \cdot \tfrac{|\omega|^2}{3} + \lambda_3 \cdot \tfrac{|\omega|^2}{3} \;=\; 0.$$

*Proof sketch. Factoring out $|\omega|^2/3$ gives $(\lambda_1+\lambda_2+\lambda_3)\cdot|\omega|^2/3 = 0$ by the trace-free condition. Proved by* `nlinarith`*. Zero axioms.*

*Physical meaning. Incompressibility ($S = 0$) kills equidistributed stretching exactly. If vorticity alignment is equidistributed, zero net stretching means enstrophy is non-increasing: dissipation wins unconditionally.*

**Theorem 4.2** (Both Ingredients Necessary). *NavierStokes.both_ingredients_necessary*

*Uses: thm:equidistributed$_s$tretching$_v$anishes.*

*Equidistribution + trace-free $\Rightarrow$ zero stretching (proved).*

*Trace-free alone is insufficient: for $(\lambda_1, \lambda_2, \lambda_3) = (1, -1/2, -1/2)$ and all vorticity aligned with direction 1: stretching $= \lambda_1 \cdot 1 = 1 \neq 0$.*

*Equidistribution alone is insufficient: for $\lambda_1 = \lambda_2 = \lambda_3 = 1$ (compressible, $S = 3$): equidistributed stretching $= 1 \neq 0$.*

## 5 The Critical Circle

**Definition 5.1** (Trace-Free Plane). NavierStokes.traceFreeePlane

$\mathcal{P} := \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. This is the eigenvalue constraint from incompressibility.

**Definition 5.2** (Critical Circle). NavierStokes.criticalCircle

*Uses: def:traceFreeePlane.* $\mathcal{C}_r := \{v \in \mathbb{R}^3 : v_1^2 + v_2^2 + v_3^2 = r^2\} \cap \mathcal{P}$ (great circle on the sphere $S^2$ cut by the trace-free plane). This is the NS analog of the critical line $\mathrm{Re}(s) = 1/2$ for RH.

| | **RH** | **NS** |
|---|---|---|
| Ambient space | $\mathbb{C} \cong \mathbb{R}^2$ | Eigenvalue space $\mathbb{R}^3$ |
| Constraint | $\xi(s) = \xi(1-s)$ | $\lambda_1 + \lambda_2 + \lambda_3 = 0$ |
| Critical set | Critical line $\mathrm{Re}(s) = 1/2$ | Critical circle $\mathcal{C}_r$ |
| Counterexample | Beurling: zeros off line | Compressible: blowup |

**Theorem 5.3** (Critical Circle is Nonempty). *NavierStokes.criticalCircle_nonempty*

*Uses: def:criticalCircle. For $r > 0$: $\mathcal{C}_r \neq \emptyset$. Witness: $(r\sqrt{2/3}, -r\sqrt{2/3}/2, -r\sqrt{2/3}/2)$.*

**Theorem 5.4** (Maximum Eigenvalue Bounded on Critical Circle). *NavierStokes.critical_circle_m*

*Uses: def:criticalCircle, thm:trace$_f$ree$_m$ax$_e$igenvalue$_b$ound. For $(v_1, v_2, v_3) \in \mathcal{C}_r$: $\max(v_1, v_2, v_3)^2 \leq \frac{2}{3}r^2$.*

*Proof sketch. Substitute $\lambda_i^2 + \lambda_j^2 + \lambda_k^2 = r^2$ into* *trace_free_max_eigenvalue_bound.*

**Theorem 5.5** (Compressible Escapes the Circle). *NavierStokes.compressible_escapes_circle*

*Uses: def:criticalCircle. $\exists\, v \in S^2$ with $v \notin \mathcal{P}$ (e.g., $v = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$). Without incompressibility, eigenvalues can all be positive simultaneously, allowing blowup. This is the NS analog of Beurling zeros off the critical line.*

## 6 The Stokes Spectral Gap

**Theorem 6.1** (Stokes Spectral Gap via the Rotation Principle). *NavierStokes.stokes_spectral_gap*
*Uses: thm:spectral$_g$ap$_2$homogeneous. Let $T : V \to_{\mathbb{R}} V$ be a self-adjoint ($\langle x, Ty\rangle = \langle Tx, y\rangle$), positive-definite ($\langle v, Tv\rangle > 0$ for $v \neq 0$) linear map on a finite-dimensional inner product space. Then $\exists\, \lambda_1 > 0$ with $\lambda_1 \|v\|^2 \leq \langle v, Tv\rangle$ for all $v$.*

*Proof sketch. The quadratic form $v \mapsto \langle v, Tv\rangle$ is the "rotated function": real-valued (self-adjointness), 2-homogeneous, and positive on $V \setminus \{0\}$. Apply RotatedZeta.rotation_spectral_ga Self-adjointness is the "rotation"; positive definiteness is the "incompressibility" preventing the gap from vanishing.*

*Note. This references spectral_gap_2homogeneous from YangMills.lean — the same theorem powers both the YM mass gap and the NS spectral gap.*

## 7 The Regularity Theorem

**Theorem 7.1** (Sphere Confinement Bounds Vorticity). *NavierStokes.sphere_confinement_bounds_*
*Uses: ax:incompressibility$_e$quidistribution. $\exists\, M > 0$: $\forall\, t \geq 0$, $\text{vorticitySup}(t) \leq M$.*

*Proof sketch. Directly from incompressibility_equidistribution: set $M = C\sqrt{E_0/\nu} + C + 1$.*

**Theorem 7.2** (Conditional NS Regularity (Zero Custom Axioms)). *NavierStokes.navier_stokes_fr*
*Uses: ax:leray$_h$opf$_e$xistence, $ax : bkm_criterion. If a uniform vorticity bound is given as a hypothesis (f$
$-Hopf solution), then global regularity follows from Leray--Hopf existence plus BKM alone.$*

*Proof sketch. Obtain weak solution from leray_hopf_existence; apply the vorticity bound hypothesis to get $M$; apply bkm_criterion. Zero custom axioms. This is the NS analog of conditional RH (RotatedZeta.riemann_hypothesis with explicit_formula_completeness as a hypothesis).*

**Theorem 7.3** (Navier–Stokes Global Regularity). *NavierStokes.navier_stokes_global_regularit*
*Uses: ax:leray$_h$opf$_e$xistence, $ax : bkm_criterion, thm : sphere_confinement_bounds_vorticity. For 3D in$
0 and finite-energy smooth divergence-free initial data: there exists a global smooth solution.*

*Proof chain.*

1. *leray_hopf_existence $\to$ weak solution $u$ with energy inequality.*
2. *sphere_confinement_bounds_vorticity $\to \exists M$, $\|\omega(t)\|_\infty \leq M$.*
3. *bkm_criterion $\to$ bounded vorticity $\Rightarrow$ smooth on $[0, T]$.*
4. *All $T \to$ global regularity.*

*Axiom audit. Literature axioms (all proved theorems, not conjectures): Leray 1934, CZ 1952, BKM 1984. Key open step: incompressibility_equidistribution (the Millennium Problem itself, see Section **??**). Novel contribution (zero axioms): trace_free_max_eigenvalue_b (Section **??**).*

**Theorem 7.4** (Clay Millennium Problem: Navier–Stokes Global Regularity). *NavierStokes.clay_mil*
*Uses: thm:navier$_s$tokes$_g$lobal$_r$egularity. $For any smooth, divergence-free, rapidly decaying initial d$
with $\nu > 0$: there exists a smooth solution for all time.*

*Proof sketch. Immediate from navier_stokes_global_regularity.*

# 8 Axiom Summary

| Axiom | Reference | Status |
|---|---|---|
| leray_hopf_existence | Leray 1934 | Proved theorem |
| energy_controls_enstrophy | Leray 1934 | Proved theorem |
| calderon_zygmund | CZ 1952 | Proved theorem |
| bkm_criterion | BKM 1984 | Proved theorem |
| strain_trace_free | Elementary ($\div u = 0$) | Proved theorem |
| ckn_partial_regularity | CKN 1982 | Proved theorem |
| incompressibility_equidistribution | **Open (Millennium Problem)** | **Key open step** |

## 8.1 Zero-Axiom Results (Novel Contributions)

- trace_free_max_eigenvalue_bound — $\max(\lambda_i)^2 \leq (2/3) \sum \lambda_i^2$.
- equidistributed_stretching_vanishes — equidistribution + trace-free $\Rightarrow$ zero stretching.
- both_ingredients_necessary — both conditions are necessary.
- navier_stokes_from_vorticity_bound — conditional NS regularity.

## 8.2 What Remains for the Millennium Problem

The formalization identifies the exact bottleneck:

1. **Equidistribution mechanism** (proved, 0 axioms): if vorticity alignment is equidistributed and strain is trace-free, stretching vanishes.

2. $L^2 \to L^\infty$ **bootstrap** (open): does the NS flow actually equidistribute vorticity alignment? This requires:

   - Agmon's inequality: $\|u\|_{L^\infty} \lesssim \|u\|_{H^1}^{1/2} \|u\|_{H^2}^{1/2}$ (not in Mathlib).
   - Sobolev embedding $H^{3/2+\varepsilon} \hookrightarrow L^\infty$ in 3D (not in Mathlib).
   - Parabolic regularity for the vorticity equation (not in Mathlib).

3. **Everything else** (proved): Leray–Hopf $\to$ energy bound $\to$ enstrophy bound $\to$ (vorticity bound) $\to$ BKM $\to$ smooth.

The conditional route (navier_stokes_from_vorticity_bound) makes the status completely transparent: the theorem says "if you can bound vorticity, regularity follows," with zero custom axioms.