Classification of Differential Equations

An equation involving derivatives of one or more dependant variable with respect to one or more independent variables is called a differential equation.

- if the derivatives involved are ordinary derivatives $\rightarrow$ ordinary differential equations (ODEs)
- if the derivatives involved are partial derivatives $\rightarrow$ partial differential equations (PDEs)

The order of a differential equation is the order of the highest derivative in the differential equation. For higher order derivatives, we use the notation $x^{(n)}$, where the order n is surrounded by brackets.

Any n^{th} order ODE can be written in the form:

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0$$

Consider a function defined on an open interval I and possessing at leas n derivatives that are continuous on I. If the function satisfied the preceding equation, then the function is called an explicit solution to the equation.

Note that not every differential equation possesses a solution. However, most differential equations have infinitely many solutions. Classification of a differential equation is very important:

An n^{th} order differential equation is considered **linear** if it has the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = b(x)$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and $b(x)$ are some functions of x.

- if $b(x) = 0$, then the equation is **homogenous**
- if $b(x) \neq 0$, then the equation is **non-homogenous**

A solution of the above equation containing n-arbitrary constants is called a general solution. Any solution obtained from the general solution is called a particular solution. A solution that cannot be obtained from the general solution is called a singular solution.

Separable Equations

An equation of the form

$$\frac{dy}{dx} = f(x)g(y)$$

is called a separable differential equation. Equivalently, a separable can be given in the form $f(x)dx + g(y)dy = 0$. To solve these equations, you simply integrate both sides with respect to their respective variable after separating the x components from the y components.

$$\int f(x)dx = \int g(y)dy$$

Converting Equations into Separable Equations

As we have seen so far, separable differential equations are easy to solve. There are certain types of non-separable differential equations that can be converted into separable equations by using suitable substitutions.

The first type of such equations is the homogeneous equation. A differential equation $dy/dx = f(x, y)$ is considered homogeneous if $f(x, y)$ can be written as a function of y/x.

To solve these equations, substituting $y = ux$ (or $u = y/x$) turns the homogeneous equation into a solvable separable equation in x and u.

$$\begin{aligned} & y = ux \\ \implies & \frac{dy}{dx} = u + x\frac{du}{dx} \\ \implies & dy = xdu + udx \end{aligned}$$

We can also solve differential equations using substitution. For equations of the form

$$\frac{dy}{dx} = f(Ax + By + C)$$

where $A, B, C \in \mathbb{R}$, make the substitution $u = Ax + By + C$ to obtain a separable equation in u and x.

Linear Equations

An equation of the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

is called a linear equation. The equation has the solution:

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)q(x)dx + c\right)$$

where $\mu(x)$ is called the integrating function.

$$\mu(x) = e^{\int p(x)dx}$$

Bernoulli Equations

An equation of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

is called a Bernoulli Equation. If $n = 1, n = 0$ then the equation is just a regular linear equation.

To solve Bernoulli Equations, you must:

1. divide both sides by y^n
2. substitute $v = y^{1-n}$
3. solve the linear equation in v and x
4. make the back substitution

Exact Equations

Consider a function of two variables of the form $F(x, y) = c$. The total differential of F is given by

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = F_x dx + F_y dy$$

A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is called an exact equation if it is equivalent to the differential of a function $F(x,y) = c$.

If the differential equation $M(x,y)dx + N(x,y)dy = 0$ is exact,

$$\implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

$$\exists F(x,y) = c : dF = F_x dx + F_y dy : F_x = M, F_y = N$$

There are a couple methods to solve exact equations. First, you must verify that the equation is by verifying $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

1. Integrate M or N depending on which one is easier. Instead of getting a constant C, you will get a constant ϕ, which is a function of the other variable.
2. Differentiate with respect to the other variable.
3. Compare ϕ' with the function you didn't integrate in step 1.
4. Identify ϕ' and integrate to get ϕ.

Another method is called the grouping method:

1. Separate M and N into their components (distribute dx and dy).
2. Ignore the terms that can be easily integrated.
3. Look at the remaining terms and group similar terms into groups of two.
4. Deduce the original function who when differentiated using product rule will give you the two terms.
5. Integrate the easy terms and the grouped terms using the method above.

Integrating Factors

When multiplied with a non-exact equation, an integrating factor is a function that turns a non-exact equation into an exact equation.

$$M(x,y)dx + N(x,y)dy = 0$$

$$\therefore M_y \neq N_x$$

- if $f(x) = \frac{1}{N}(M_y - N_x)$ (only a factor of x), then $\mu = e^{\int f(x)dx}$ is an integrating factor
- if $g(y) = \frac{1}{M}(N_x - M_y)$ (only a factor of y), then $\mu = e^{\int g(y)dy}$ is an integrating factor

How to solve:

1. Check if $M_y \neq N_x$.
2. Choose one of the formulas above based on if the denominator will cancel out.
3. Find μ.
4. Multiply the original equation by μ.
5. Solve the new exact equation.