Al-Farabi Kazakh National University

S. A. Aisagaliev, Zh. Kh. Zhunussova

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Reviewers:

doctor of physical and mathematical sciences, professor ***M.Akhmet***;
candidate of philological sciences, professor ***A.A.Moldagalieva***

Aisagaliev S.A., Zhunussova Zh.Kh.
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Some theoretical foundations of mathematical programming are expounded in the textbook: elements of convex analysis; convex, nonlinear, linear programming required for planning and production control for solution of the topical problems of the controlled processes in natural sciences, technology and economy.

The tasks for independent work with concrete examples, brief theory and solution algorithms of the problems, term tasks on sections of the mathematical programming are put in the appendix.

It is intended as a textbook for the students of the high schools training on specialties "applied mathematics", "mathematics", "mechanics", "economic cybernetics" and "informatics". It will be useful for the post-graduate students and scientific workers of the economic, mathematical, naturally-technical and economic specialties.

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FOREWORD

The main sections of mathematical programming, the numerical methods of function minimization of the finite number variables are expounded in the textbook. It is written on the basis of the lectures on optimization methods which have been delivered by the authors in Al-Farabi Kazakh National University.

In connection with transition to credit technology of education the book is written in the manner of scholastic-methodical complex, containing alongside with lectures problems for independent work with solutions of the concrete examples, brief theory and algorithm on sections of the course, as well as term tasks for mastering of the main methods of the optimization problems solution.

In the second half XX from necessity of practice appeared the new direction in mathematics - "Mathematical control theory" including the following sections: mathematical programming, optimal control with processes, theory of the extreme problems, differential and matrix games, controllability and observability theory, stochastic programming. Mathematical control theory was formed at period of the tempestuous development and creating of the new technology, spacecrafts, developing of the mathematical methods in economy, controlling by the different process in natural sciences. Aroused new problems could not be solved by classical methods of mathematics and required new approaches and theories. The different research-and-production problems were solved due to mathematical control theory, in particular: production organizing to achieve maximum profit with provision for insufficiency resources, optimal control by nucleus and chemical reactors, electrical power and robotic systems, control by moving of the ballistic rockets, spacecrafts and satellites and others. Methods of the mathematical control theory were useful for developing mathematics. Classic boundary problems of the differential equations, problems of the best function approach, optimal choice of the parameters in the iterative processes, minimization of the difficulties with equations are reduced to studying of the extreme problems.

Theory foundation and solution algorithms of the convex, nonlinear and linear programming are expounded on the lectures 1-17. Execution of the three term tasks for individual work of the students is provided for these sections.

Solution methods of the extreme problems are related to one of the high developing direction of mathematics. That is why to make a textbook possessed by completion and without any shortage is very difficult. Authors will be grateful for critical notations concerning the textbook.

INTRODUCTION

Lecture 1

THE MAIN DETERMINATIONS. STATEMENT OF THE PROBLEM

Let E^n be the euclidean space of vectors $u = (u_1, u_2, ..., u_n)$ and a scalar function $J(u) = J(u_1, ..., u_n)$ which is determined on a set U of the space E^n. It is necessary to find the maximum (minimum) of the function $J(u)$ on set *U*, where $U \subset E^n$ is the set.

Production problem. The products of the five types are made on the enterprise. The cost of the unit product of the each type accordingly c_1, c_2, c_3, c_4, c_5, in particular $c_1 = 3$, $c_2 = 5$, $c_4 = 1$, $c_5 = 8$. For fabrication specified products the enterprise has some determined recourses expressed by the following normative data:

Type of the product	Materials	Energy	Labor expenses	Recourses
1	$a_{11} = 3$	$a_{21} = 6$	$a_{31} = 1$	$b_1 = 50$
2	$a_{12} = 1$	$a_{22} = 3$	$a_{32} = 3$	$b_2 = 120$
3	$a_{13} = 4$	$a_{23} = 1$	$a_{33} = 1$	$b_3 = 20$
4	$a_{14} = 2$	$a_{24} = 4$	$a_{34} = 2$	
5	$a_{15} = 1$	$a_{25} = 5$	$a_{35} = 4$	

Here a_{11} is an amount of the materials required for fabricating the units of the 1-st type products; a_{12} is a consumption of the material for fabrication of the 2-nd type unit products and so on; a_{ij}, $i = 1, 2, 3$, $j = 1, 2, 3, 4, 5$ is an expenses of the material, energy, labor expenses for fabrication of the j type unit products.

It is required to find a such production plan output such that provides the maximal profit. So as the profit is proportional to total cost of marketed commodities, that hereinafter it is identified with total cost of marketed commodities.

Let u_1, u_2, u_3, u_4, u_5 be five products. The mathematical formalization of the problem function maximizations profit has the form

$$J(u) = J(u_1, u_2, u_3, u_4, u_5) = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 + c_5u_5 = \\ = 3u_1 + 5u_2 + 10u_3 + u_4 + 8u_5 \to \max \qquad (1)$$

under the conditions (restrictions of the resources)

$$\left.\begin{aligned} \overline{g}_1(u) = a_{11}u_1 + a_{12}u_2 + a_{13}u_3 + a_{14}u_4 + a_{15}u_5 \le b_1, \\ \overline{g}_2(u) = a_{21}u_1 + a_{22}u_2 + a_{23}u_3 + a_{24}u_4 + a_{25}u_5 \le b_2, \\ \overline{g}_3(u) = a_{31}u_1 + a_{32}u_2 + a_{33}u_3 + a_{34}u_4 + a_{35}u_5 \le b_3, \end{aligned}\right\} \qquad (2)$$

$$u_1 \ge 0, \ u_2 \ge 0, \ u_3 \ge 0, \quad u_4 \ge 0, \quad u_5 \ge 0, \qquad (3)$$

where a_{ij}, $i = \overline{1,3}$, $j = \overline{1,5}$ are normative coefficients which values presented above; b_1, b_2, b_3 are recourses of the enterprise. Since amount of the products are nonnegative numbers that necessary the condition (3).

If we introduce the notation

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{pmatrix};$$

$$c=\begin{pmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{pmatrix};\quad u=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{pmatrix};\ b=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix};\quad \bar{g}=\begin{pmatrix}\bar{g}_1\\\bar{g}_2\\\bar{g}_3\end{pmatrix},$$

then optimization problem (1) – (3) can be written as

$$J(u)=c'u\to\max; \qquad (1')$$

$$\bar{g}(u)=Au\le b; \qquad (2')$$

$$u\ge 0, \qquad (3')$$

where $c'=(c_1,c_2,c_3,c_4,c_5)$ is a vector-line; $(\,'\,)$ is a sign of transposition.

Let a vector-function be $g(u)=\bar{g}(u)-b=Au-b$, but a set $U_0=\{u\in E^5/u\ge 0\}$. Then problem (1) – (3) is written as:

$$J(u)=c'u\to\max; \qquad (4)$$

$$u\in U=\{u\in E^5/u\in U_0,\ g(u)\le 0\}\subset E^5. \qquad (5)$$

The problem (1) – (3) (or $(1')$ – $(3')$, or (4), (5)) belongs to type so called problem of the linear programming, since $J(u)$ is linear with respect to u function; the vector-function $g(u)$ also is linear with respect to u.

In the case above vector u has five components, vector-function $g(u)$ – three and matrix A has an order 3×5 was considered. In the case vector u has dimensionality n, $g(u)$ – m-a measured function, but matrix A has an order $m\times n$ and there are restrictions of the equality type (for instance $g_1(u)=A_1u-b_1$, where A_1 – a matrix of the order $m_1 \times n$, $b_1 \in E^{m_1}$) problem (4), (5) is written:

$$J(u) = c'u \to max; \tag{6}$$

$$u \in U,$$
$$U = \{u \in E^n / u \in U_0,\, g(u) = Au - b \le 0,\, g_1(u) = A_1 u - b_1 = 0\}, \tag{7}$$

where set $U_0 = \{u \in E^n / u \ge 0\}$, $c \in E^n$. Problem (6), (7) belongs to type of the general problem of the linear programming.

We suppose that $J(u)$ is convex function determined on the convex set U_0 (unnecessary linear); $g(u)$ is a convex function determined on the convex set U_0. Now problem (6), (7) is written as

$$J(u) \to max; \tag{8}$$

$$u \in U = \{u \in E^n / u \in U_0,\; g(u) \le 0,\, g_1(u) = A_1 u - b_1 = 0\}. \tag{9}$$

Problem (8), (9) belongs to type so called problem of the convex programming.

Let $J(u)$, $g(u)$, $g_1(u)$ be arbitrary functions determined on the convex set U_0. In this case, problem (8), (9) can be written as

$$J(u) \to \max; \tag{10}$$

$$u \in U = \{u \in E^n / u \in U_0,\; g(u) \le 0,\; g_1(u) = 0\}. \tag{11}$$

Optimization problem (10), (11) belongs to type so called problem of the nonlinear programming. In all presented problems of the linear, convex and nonlinear programming are specified the concrete ways of the prescription set U from E^n. If it is distracted from concrete way of the prescription set U from E^n, so optimization problem in finite-dimensional space possible to write as

$$J(u) \to \max; \quad u \in U, \quad U \subset E^n.$$

Finally, we note that problem of the function maximization $J(u)$ on a set U tantamount to the problem of the function minimization $-J(u)$ on set U. So further it is possible to limit by consideration of the problem

$$J(u) \to \min; \quad u \in U, \quad U \subset E^n, \tag{12}$$

for instance, in the problem (1) - (3) instead of maximization $J(u)$ minimization of the function $-3u_1 - 5u_2 - 10u_3 - u_4 - 8u_5 \to \min$.

Finally, the first part of the course "Methods of the optimization" is devoted to the solution methods of the convex, nonlinear, linear programming.

The question arises: whether the problem statement (12) correct in general case? It is necessary the following definitions from mathematical analysis for answer.

Definition 1. *The point $u_* \in U$ called the minimum point of the function $J(u)$ on a set U if inequality $J(u_*) \le J(u)$ is executed under all $u \in U$. The value $J(u_*)$ is called the least or minimum value of the function $J(u)$ on set U.*

The set $U_* = \left\{ u_* \in U \middle/ J(u_*) = \min\limits_{u \in U} J(u) \right\}$ contains all minimum points of the function $J(u)$ on set U.

It follows from the definition, that the global (or absolute) minimum of the function $J(u)$ on the set U is reached on the set $U_* \subset U$. We remind that a point $u_{**} \in U$ is called the local minimum point of the function $J(u)$, if inequality $J(u_{**}) \le J(u)$ is valid under all $u \in o(u_{**}, \varepsilon) \cap U$, where set

$$o(u_{**}, \varepsilon) = \left\{ u \in E^n \middle/ | u - u_{**} | < \varepsilon \right\} \subset E^n$$

- an open sphere with the centre in u_{**} and radius $\varepsilon > 0$; $|a|$ - Euclidean norm of the vector $a = (a_1, \dots, a_n) \in E^n$, i.e. $|a| = \sqrt{\sum_{i=1}^{n} a_i^2}$.

Example 1. Let $J(u) = \cos^2 \dfrac{\pi}{u}$, and $U = \{u \in E^1 / \ 1/2 \le u \le 1\}$. Then set $U_* = \{2/3\}$; b) $U = \{u \in E^1 / \ 1/4 \le u \le 2\}$, then set $U_* = \{2/7, 2/5, \ 2/3, \ 2\}$; c) $U = \{u \in E^1 / 2 < u < \infty\}$, then set $U_* = \varnothing$, where $\varnothing$ - empty set.

Example 2. The function $J(u) = \ln u$, set $U = \{u \in E^1 / 0 < u \le 1\}$. The set $U_* = \varnothing$.

Example 3. The function $J(u) = J_1(u) + c, \ \ c = const$, where function

$$J_1(u) = \begin{cases} |u-a|, если \ u < a; \\ u-b, если \ u > b; \\ c, \quad если \ a \le u \le b, \end{cases}$$

and set $U = E^1$. The set $U_* = [a,b]$.

Definition 2. It is spoken, that function $J(u)$ is bounded below on set U if the number M such that $J(u) \ge M$ under all $u \in U$ exists. The function $J(u)$ is not bounded below on set U if the sequence $\{u_k\} \subset U$ such that $\lim\limits_{k \to \infty} J(u_k) = -\infty$ exists.

Function $J(u)$ is bounded below on set U in the examples 1, 3, but function $J(u)$ is not bounded on U in the example 2.

Definition 3. We say, that function $J(u)$ is bounded from below on set U. Then value $J_* = \inf\limits_{u \in U} J(u)$ is called the lower bound of the function $J(u)$ on set *U*, if: 1) $J_* \le J(u)$ under all $u \in U$; 2) for arbitrary small number $\varepsilon > 0$ is found the point $u(\varepsilon) \in U$ such that value $J(u(\varepsilon)) < J_* + \varepsilon$. When the function $J(u)$ is not bounded from below on set *U*, lower bound $J_* = -\infty$.

We notice, that for example 1 value $J_* = 0$, but for examples 2, 3 values $J_* = -\infty$, $J_* = c$, accordingly. If set $U_* \ne \varnothing$, so

$J_* = \min_{u \in U} J(u)$ (refer to examples). Value J_* always exists, but $\min_{u \in U} J(u)$ does not always exist. Since lower bound J_* for function $J(u)$ determined on set U always exists, independently of that, whether set U_* is emptily or in emptily, problem (12) can be written in the manner of

$$J(u) \to \inf, \quad u \in U, \ \ U \subset E^n. \tag{13}$$

Since value $\min_{u \in U} J(u)$ does not exist, when set $U_* = \varnothing$, that correct form of the optimization problem in finite-dimensional space has the form of (13).

Definition 4. The sequence $\{u_k\} \subset U$ *is called* minimizing for function $J(u)$ determined on set *U,* if limit $\lim_{k \to \infty} J(u_k) = \inf_{u \in U} J(u) = J_*$.

As follows from definition lower bound, in the case $\varepsilon_k = 1/k$, $k = 1,2,3...$ we have the sequences $\{u(\varepsilon_k)\} = \ = \{u(1/k)\} = \{u_k\} \subset U \ \ k = 1,2,...,$ for which $J(u_k) < J_* + 1/k$

Thence follows that limit $\lim_{k \to \infty} J(u_k) = J_*$. Finally, minimizing sequence $\{u_k\} \subset U$ always exists.

Definition 5. It is spoken*,* that sequence $\{u_k\} \subset U$ converges to set $U_* \subset U$, if limit $\lim_{k \to \infty} \rho(u_k, U_*) = 0$, where $\rho(u_k, U_*)$ $= \inf_{u_* \in U_*} |u_k - u_*|$ - a distance from point $u_k \in U$ *till set* U_*.

It is necessary to note, if set $U_* \neq \varnothing$, so the minimizing sequence always exists which converges to set U_*. However statement about any minimizing sequence converges to set U_*, in general case, untrue.

Example 4. Let function $J(u) = u^4/(1 + u^6)$, $U = E^1$.

For the example set $U_* = \{0\}$, but the sequences $\{u_k = 1/k, k = 1,2,...\} \subset U$, $\{u_k = k, \, k = 1,2,...\} \subset U$ are minimizing. The first sequence converges to set U_*, the second infinitely is distanced from it. Finally, source optimization problem has the form of (13). We consider the following its solutions:

1-st problem. Find the value $J_* = \inf_{u \in U} J(u)$. In this case independently of that whether set U_* is emptily or in emptily, problem (13) has a solution.

2-nd problem. Find the value $J_* = \inf_{u \in U} J(u)$ and the point $u_* \in U_*$. In order to the problem (13) has a solution necessary the set $U_* \neq \varnothing$.

3-rd problem. Find the minimizing sequence $\{u_k\} \subset U$ which converges to set U_*. In this case necessary that set $U_* \neq \varnothing$.

Most often in practice it is required solution of the 2-nd problem (refer to production problem).

Lecture 2

WEIERSTRASS'S THEOREM

We consider the optimization problem

$$J(u) \to \inf, \quad u \in U, \ \ U \subset E^n. \tag{1}$$

It is necessary to find the point $u_* \in U_*$ and value $J_* = \inf_{u \in U} J(u)$. We notice, if set $U_* \neq \varnothing$, so $J_* = J(u_*) = \min_{u \in U} J(u)$.

The question arises: what requirements are imposed on the function $J(u)$ and on set U that the set $U_* = \left\{ u_* \in U \middle/ J(u_*) = \min_{u \in U} J(u) \right\}$ be in emptily? For answer it is necessary to enter the notions of compact set and half-continuously from below of the function $J(u)$ on set U.

The compact sets. Let $\{u_k\} \subset E^n$ be a certain sequence. We remind that: a) the point $v \in E^n$ is called the limiting point of the sequence $\{u_k\}$, if subsequence $\{u_{k_m}\}$ exists for which limit $\lim_{m \to \infty} u_{k_m} = v$; b) sequence $\{u_k\} \subset E^n$ is identified bounded , if the number $M \geq 0$ exists, such the norm $|u_k| \leq M$ for all $k = 1,2,3,...$; b) set $U \subset E^n$ is identified bounded , if the number $R \geq 0$ exists, such that norm $|u| \leq R$ under all $u \in U$; c) the point $v \in E^n$ is identified the limiting point of the set *U*, if any its ε - set neighborhood $o(v, \varepsilon)$ contains the points from U differenced from v; d) for any limiting point v of the set U is found the sequence $\{u_k\} \subset U$ for which $\lim_{k \to \infty} u_k = v$; e) set $U \subset E^n$ is identified closed, if it contains all their own limiting points.

Definition 1. *The set* $U \in E^n$ *is identified compact, if any sequence* $\{u_k\} \subset U$ *has at least one limiting point v, moreover* $v \in U$.

It is easy make sure the definition is equally to known from the course of the mathematical analysis statement about any bounded and closed set is compactly. In fact, according to Bolzano and Weierstrass theorem any bounded sequence has at least one limiting point (the set U is bounded), but from inclusion $v \in U$ follows the property of being closed of the set U.

Half-continuously from below. Let $\{u_k\} \subset E^n$ be a sequence. Then $\{J_k\} = \{J(u_k)\}$ is the number sequence. We notice that: a) numerical set $\{J_k\}$ is bounded from below, if the number α exists, such that $J_k \geq \alpha, \quad k = 1,2,3,...$; b) numerical set $\{J_k\}$ is not bounded from below, if $\{J_{k_m}\}$ subsequence exists such the limit $\lim_{m\to\infty} J_{k_m} = -\infty$.

Definition 2. *By the lower limit of the bounded from below numeric sequence* $\{J_k\}$ *is identified the value a denoted* $a = \underline{\lim}_{m\to\infty} J_k$ *if: 1) subsequence* $\{J_{k_m}\}$ *exists for which* $\lim_{m\to\infty} J_{k_m} = a$ *; 2) all other limiting points to sequences* $\{J_k\}$ *is not less the value a. If numeric sequence* $\{J_k\}$ *is not bounded from below, so the value* $a = -\infty$.

Example 1. Let $J_k = 1 + (-1)^k, \quad k = 0,1,2,...$. The value $a = 0$.

Definition 3. *It is spoken, the function* $J(u)$ *determined on set* $U \subset E^n$ *, half-continuous from below in the point* $u \in U$ *, if for any sequence* $\{u_k\} \subset U$ *for which the limit* $\lim_{k\to\infty} u_k = u$ *, the inequality* $\underline{\lim}_{k\to\infty} J(u_k) \geq J(u)$ *is executed. The function* $J(u)$ *is half-continuous from below on set U, if it is half-continuous from below in each point of the set U.*

Example 2. Let set $U = \{u \in E^1 / -1 \leq u \leq +1\}$, function $J(u)$ $= u^2$ under $0 < |u| < 1$, $J(0) = -1$. The function $J(u)$ is uncontinuous on set $0 < |u| \leq 1$, consequently, it is half-continuous from below on the set. We show, that function $J(u)$ is half-continuous from below in the point $u=0$. In fact, the sequence $\{1/k\},\ k=1,2,...$. belongs to the set U and the limit of the sequence is equal to zero. Numeric sequence $\{J(u_k)\}=\{1/k^2\}$, moreover limit $\lim_{k\to\infty} J(u_k)=\underline{\lim}_{k\to\infty} J(u_k)=0$. Consequently, $\underline{\lim}_{k\to\infty} J(u_k)=0>-1$. It means function $J(u)$ is half-continuous from below on set U.

Similarly possible to enter the notion of the half-continuity from above of the function $J(u)$ on set U. We notice, if the function $J(u)$ is uncontinuous in the point $u\in U$, so it is half-continuous in it as from below, so and from above.

The most suitable check way of the half-continuity from below of the function $J(u)$ on set U gives the following lemma.

Lemma. *Let function $J(u)$ be determined on closed set $U\subset E^n$. In order the function $J(u)$ to be half-continuous from below on set U, necessary and sufficiently that Lebesgue's set $M(c)=\{u\in E^n / J(u)\le c\}$ be closed under all $c\in E^1$.*

Proof. Necessity. Let function $J(u)$ be half-continuous from below on closed set U. We show, that set $M(c)$ is closed under all $c\in E^1$. We notice, that empty set is considered as closed. Let v be any limiting point of the set $M(c)$. From definition of the limiting point follows the existence of the sequence $\{u_k\}\subset M(c)$ which converges to the point v. From inclusion $\{u_k\}\subset M(c)$ follows that value $J(u_k)\le c$, $k=1,2,...$. With consideration of $\{u_k\}\subset U$ and half-continuity from below $J(u)$ on U follows $J(v)\le \underline{\lim}_{k\to\infty} J(u_k)\le c$. Consequently, the point $v\in M(c)$. The property of being closed of the set $M(c)$ is proved.

Sufficient. Let U be closed set, but set $M(c)$ is closed under all $c\in E^1$. We show, that function $J(u)$ is half-continuous from below on U. Let $\{u_k\}\subset U$ - a sequence which converges to the point $u \in U$. We consider the numeric sequence $\{J(u_k)\}$. Let the value $a = \underline{\lim}_{k \to \infty} J(u_k)$. By definition of the below limit exists the sequence $\{J(u_{k_m})\}$ for which $\lim_{m \to \infty} J(u_{k_m}) = a$. Consequently, for any sufficiently small $\varepsilon > 0$ is found the number $N = N(\varepsilon)$, such that $J(u_{k_m}) \le a + \varepsilon$ under $m > N$. Thence with consideration of $\lim_{m \to \infty} u_{k_m} = u$, $u_{k_m} \in M(a + \varepsilon)$, we have $J(u) \le a + \varepsilon$. Then with consideration of arbitrarily $\varepsilon > 0$ it is possible to write the following inequality: $J(u) \le \underline{\lim}_{k \to \infty} J(u_k) = a$, i.e. function $J(u)$ is half-continuous from below in the point $u \in U$. Since set U is closed, function $J(u)$ is half-continuous from below in any point $u \in U$. Lemma is proved.

We notice, in particular, that under $c = J_*$ from lemma follows that set $M(J_*) = \{u \in E^n / u \in U, \ J(u) \le J_*\} = U_*$ is closed.

Theorem 1. *Let function $J(u)$ be determined, finite and half-continuous from below on compact set $U \subset E^n$. Then $J_* = \inf_{u \in U} J(u) > -\infty$, set*

$$U_* = \left\{ u_* \in E^n \Big/ u_* \in U, \ J(u_*) = \min_{u \in U} J(u) \right\}$$

is inempty, compactly and any minimizing sequence converges to set U_*.

Proof. Let $\{u_k\} \subset U$ be any minimizing sequence, i.e. the limit of the numeric set $\lim_{k \to \infty} J(u_k) = J_*$. We notice, that such minimizing sequence always exists. Let u_* - any limiting point of the minimizing sequence. Consequently, subsequence $\{u_{k_m}\} \subset U$ exists for which $\lim_{m \to \infty} u_{k_m} = u_*$. With consideration of the compactness of the set U all limiting points of the minimizing sequence belongs to the set U.

As follows from definition of lower bound and half-continuity from below function $J(u)$ on set U the following inequalities are faithful

$$J_* \leq J(u_*) \leq \varliminf_{m\to\infty} J(u_{k_m}) = \lim_{k\to\infty} J(u_k) = J_* . \qquad (2)$$

Since the sequence $\{J(u_k)\}$ converges to value J_*, that any its subsequence also converges to value J_*. We have that value $J_* = J(u_*)$ from inequality (2). Consequently, set $U_* \neq \varnothing$ and $J_* = J(u_*) > -\infty$. Since the statement faithfully for any limiting point of the minimizing sequence, so it is possible to confirm that any minimizing sequence from U converges to set U_*.

We show that set $U_* \subset U$ is compact. Let $\{w_k\}$ - any sequence taking from set U_*. From inclusion $w_k \subset U$ follows that $\{w_k\} \subset U$. Then with consideration of compactness set U subsequence $\{w_{k_m}\}$ exists which converges to a point $w_* \in U$. Since $\{w_k\} \subset U$, so values $J(w_k) = J_*$, $k = 1,2,...$. Consequently, the sequence $\{w_k\} \subset U$ is minimizing. Then, with consideration of proved above, limiting point to this sequences $w_* \in U_*$. Finally, closeness of the set U_* is proved. Restriction of the set U_* follows from inclusion $U_* \subset U$. Compactness of the set U_* is proved. Theorem is proved.

Set U is not often bounded in the applied problems. In such cases the following theorems are useful.

Theorem 2. *Let function $J(u)$ be determined, finite and half-continuous from below on inempty closed set $U \subset E^n$. Let for certain point $v \in U$ Lebesgue's set*

$$M(v) = \{u \in E^n \mid u \in U,\ J(u) \leq J(v)\}$$

is bounded . Then $J_* > -\infty$, set U_* is inempty, compact and any minimizing sequence $\{u_k\} \subset M(v)$ converges to set U_*.

Proof. Since all condition of the lemma are executed, set $M(v)$ is closed. From restrictedness and closeness of the set $M(v)$ follows its compactness. The set $U = M(v) \bigcup M_1(v)$, where set $M_1(v)$ $= \{u \in E^n \,|\, u \in U, \quad J(u) > J(v)\}$, moreover on set $M_1(v)$ function $J(u)$ does not reach its lower bound J_*. Hereinafter proof of the theorem 1 is repeated with change set U on compact set $M(v)$. Theorem is proved.

Theorem 3. *Let function $J(u)$ be determined, finite and half-continuous from below on inempty closed set $U \subset E^n$. Let for any sequence $\{v_k\} \subset U$, $|v_k| \to \infty$, under $k \to \infty$ the equality $\lim\limits_{k\to\infty} J(v_k) = \infty$ is executed. Then $J_* > -\infty$, set U_* is inempty, compact and any minimizing sequence $\{u_k\} \subset U$ converges to set U_*.*

Proof. Since $\lim\limits_{k\to\infty} J(v_k) = \infty$, so the point $v \in U$ such that $J(v) > J_*$ exists. We enter Lebesgue's set $M(v) = \{u \in E^n \,|\, u \in U,$ $J(u) \le J(v)\}$. With consideration of lemma set $M(v)$ is closed. It is easy to show that set $M(v)$ is bounded. In fact, if set $M(v)$ is not bounded, then the sequence $\{w_k\} \subset M(v)$, such that $|w_k| \to \infty$ under $k \to \infty$ exists. By condition of the theorem for such sequences the value $J(w_k) \to \infty$ under $k \to \infty$. Since $J(w_k) \le J(v) < \infty$ it is not possible. Finally, set $M(v)$ is bounded and closed, consequently, it is compact.

Hereinafter proof of the theorem 1 is repeated for set $M(v)$. Theorem is proved.

Chapter I. CONVEX ROGRAMMING. ELEMENTS OF THE CONVEX ANALYSIS

Amongst methods of the optimization problems solution in finite-dimensional space the most completed nature has a method of the problem solution of the convex programming. In the convex analysis developing intensive last years characteristic of the convex sets and functions are studied which allow generalizing the known methods of the problems solution on conditional extreme.

Lecture 3

CONVEX SETS

In the applied researches often meet the problems of the convex programming in the following type:

$$
\begin{gathered}
J(u) \to \inf, \\
u \in U = \left\{ u \in E^n \big/ u \in U_0,\ g_i(u) \le 0,\ i = \overline{1, m}, \right. \qquad (1) \\
\left. g_i(u) = < a^i, u > -b_i = 0,\ i = \overline{m+1, s} \right\},
\end{gathered}
$$

where $J(u)$, $g_i(u)$, $i = \overline{1, s}$, - convex functions determined on convex set $U_0 \subset E^n$.

Definition 1. *Set* $U \subset E^n$ *called by convex, if for any* $u \in U$, $v \in U$ *and under all* α, $0 \le \alpha \le 1$, *the point* $u_\alpha = \alpha u + (1-\alpha)v =$ $= v + \alpha(u - v) \in U$.

Example1. We show that closed sphere

$$\bar{S}(u_0, R) = \{u \in E^n \,\big|\, |u - u_0| \le R\}$$

Is a convex set. Let the points $u \in \bar{S}(u_0, R)$, $v \in \bar{S}(u_0, R)$, i. e. norms $|u - u_0| \le R$, $|v - u_0| \le R$. We take a number α, $\alpha \in [0, 1]$, and define the point $u_\alpha = \alpha u + (1-\alpha)v$. Norm

$$|u_\alpha - u_0| = |\alpha u + (1-\alpha)v - u_0| = |\alpha(u - u_0) + (1-\alpha)(v - u_0)| \le$$
$$\le \alpha|u - u_0| + (1-\alpha)|v - u_0| \le \alpha R + (1-\alpha)R = R.$$

Consequently, the point $u_\alpha \in \bar{S}(u_0, R)$, and set $\bar{S}(u_0, R)$ is convex.

Example2. We show that hyperplane

$$\Gamma = \{u \in E^n \,|\, < c, u >= \gamma\}$$

is a convex set, where $c \in E^n$ is the vector, γ is the number. Let the points $u \in \Gamma$, $v \in \Gamma$. Consequently, scalar products $< c, u >= \gamma$, $< c, v >= \gamma$. Let $u_\alpha = \alpha u + (1-\alpha)v$, $\alpha \in [0, 1]$. Then scalar product

$$< c, u_\alpha >=< c, \alpha u + (1-\alpha)v >= \alpha < c, u > +$$
$$+ (1-\alpha) < c, v >= \alpha\gamma + (1-\alpha)\gamma = \gamma.$$

Thence follows that the point $u_\alpha \in \Gamma$ and set Γ is convex.

Example 3. We show that affine set $M = \{u \in E^n \,|\, Au = b\}$, where A is a constant matrix of the order $m \times n$, $b \in E^n$ is the vector which is convex. For the point $u_\alpha = \alpha u + (1-\alpha)v$, $u \in M$, $v \in M$, $\alpha \in [0,1]$ we have

$$Au_\alpha = A(\alpha u + (1-\alpha)v) = \alpha Au + (1-\alpha)Av = \alpha b + (1-\alpha)b = b$$

Thence follows that $u_\alpha \in M$ and set M is convex.

Let u^1, u^2, …, u^{n-r} be linear independent solutions of the linear homogeneous system $Au = 0$. Then set M can be presented in the manner of

$$M = \{u \in E^n \mid u = u_0 + v,\ v \in L\},\ \ L = \left\{u \in E^n \,\middle|\, v = \sum_{i=1}^{n-r} \alpha_i u^i\right\}$$

where vector $u_0 \in E^n$ is a partial solution of the nonhomogeneous system $Au = b$, L is a subspace to dimensionality $n - r$ formed by the vectors u^1, u^2, …, u^{n-r}, α_i, $i = \overline{1, n-r}$ are the numbers and dimension of the affine set M is taken equal to the dimension of the space L.

Definition 2. *By affine cover of the arbitrary set $U \subset E^n$ called the intersection of all affine sets containing set U and it is denoted by $\operatorname{aff} U$. Dimensionality of the set U is called the dimension its affine cover and it is denoted by $\dim U$.*

Since the intersection of any number convex sets is convex set, so the set $\operatorname{aff} U$ is convex. Instead of source problem $J(u) \to \inf$, $u \in U$, where U is an arbitrary set it is considered the approximate problem $J(u) \to \inf$, $u \in \operatorname{aff} U$, since solution of the last problem in the many cases more simpler, than source.

Definition 3. *The set A called by the sum of the sets $A_1, A_2, \ldots, A_m$, i.e. $A = A_1 + A_2 + \ldots A_m$ if it contains that and only that points $a = \sum_{i=1}^{m} a_i$, $a_i \in A_i$, $i = \overline{1, m}$. Set A is called by the set difference B and C i.e $A = B - C$, if it contains that and only that* *points* $a = b - c,\ b \in B,\ c \in C$. *The set* $A = \lambda D$, *where* λ *is a real number, if it contains the points* $a = \lambda d,\ d \in D$.

Theorem 1. *If the sets* $A_1, A_2, ..., A_m, B, C, D$ *are convex, so the sets* $A = A_1 + A_2 + ... + A_m$, $A = B - C$, $A = \lambda D$ *are convex.*

Proof. Let the points be $a = \sum_{i=1}^{m} a_i \in A$, $e = \sum_{i=1}^{m} e_i \in \ \in A$, $a_i, e_i \in A_i$, $i = \overline{1,m}$. Then the point $u_\alpha = \alpha a + (1-\alpha)e = \sum_{i=1}^{m} (\alpha a_i + (1-\alpha)e_i)$, $\alpha \in [0,1]$. Since the sets A_i, $i = \overline{1,m}$ are convex, so $\alpha a_i + (1-\alpha)e_i \in A_i$, $i = \overline{1,m}$. Consequently, the point $a_\alpha = \sum_{i=1}^{m} \overline{a}_i$, $\overline{a}_i = \alpha a_i + (1-\alpha)e_i \in A_i$, $i = \overline{1,m}$. Thence follows that point $a_\alpha \in A$. Convexity of the set A is proved.

We show that set $A = B - C$ is convex. Let the points $a = b - c \in A$, $e = b_1 - c_1 \in A$, where $b, b_1 \in B$, $c, c_1 \in C$. The point $a_\alpha = \alpha a + (1-\alpha)e = \alpha(b - c) + (1-\alpha)(b_1 - c_1) = [\alpha b + + (1-\alpha)b_1] - [\alpha c + (1-\alpha)c_1]$, $\alpha \in [0,1]$. Since the sets B and C are convex, the points $\overline{b} = \alpha b + (1-\alpha)b_1 \in B$, $\overline{c} = \alpha c + (1-\alpha)c_1 \in C$. Then the point $a_\alpha = \overline{b} - \overline{c} \in A$, $\overline{b} \in B$, $\overline{c} \in C$. It is followed that set A is convex. From $a = \lambda d_1 \in A$, $e = \lambda d_2 \in A$, $d_1, d_2 \in D$ follows that $a_\alpha = \alpha a + (1-\alpha)e = \alpha \lambda d_1 + (1-\alpha)\lambda d_2$, $\alpha \lambda d_1 + (1-\alpha)\lambda d_2$, $\alpha \in [0,1]$. Then $a_\alpha = \lambda \overline{d} \in A$, $\overline{d} = \alpha d_1 + (1- -\alpha)d_2 \in D$ Theorem is proved.

Let U be the set from E^n, $v \in E^n$ is a certain point. There is one and only one of the following possibilities:

a) The number $\varepsilon > 0$ is found such that set $o(\upsilon, \varepsilon) \subset U$. In this case point υ is an internal point of the set U. We denote through $\operatorname{int} U$ the ensemble of all internal points of the set U.

b) Set $o(\upsilon,\varepsilon)$ does not contain nor one point of the set U. The point υ is called by external with respect to U.

c) Set $o(\upsilon,\varepsilon)$ contains both points from set U, and points from set $E^n \setminus U$. The point υ is called by bound point of the set U. The set of all border points of the set U is denoted by $\Gamma\text{p}\,U$.

d) The point $\upsilon \in U$, but set $o(\upsilon,\varepsilon)$ does not contain nor one point from set U, except the points υ. The point υ is called the isolated point of the set U. Convex set does not contain the isolated points.

Definition 4. *The point $\upsilon \in U$ called comparatively internal point of the set U if the intersection $o(\upsilon,\varepsilon) \cap aff\, U \subset U$. We denote through riU the ensemble of all comparatively internal points of the set.*

Example 4. Let set

$$U = \{u \in E^1 / 0 \le u \le 1;\;\; 2 \le u \le 3\} \subset E^1.$$

For the example the sets

$$\operatorname{int} U = \{u \in E^1 / 0 < u < 1;\; 2 < u < 3\}.$$
$$affU = \{u \in E^1 / 0 \le u \le 3\},\; riU = \{u \in E^1 / 0 \le u < 1;\, 2 < u \le 3\}.$$

the following statements are faithful:

1) If A is a convex set, so closing $\overline{A}$ is convex too. In fact, if $a,\; b \in \overline{A}$, so subsequences $\{a_k\} \subset A,\; \{b_k\} \subset A,$ such that $a_k \to a,$ $b_k \to b$ under $k \to \infty$ exists. With consideration of convexity of the set A the point $\alpha\, a_k + (1-\alpha) b_k \in A,\;\; k = 1,2,3,\ldots$. Then limiting point $\overline{a} = \lim\limits_{k\to\infty} [\alpha\, a_k + (1-\alpha) b_k] = \alpha\, a + (1-\alpha) b \in \overline{A}$ under all $\alpha \in [0,1]$. Thence follows the convexity of the set $\overline{A}$.

2) If U is a convex set and $\operatorname{int} U \ne \varnothing$, so the point

$$v_\alpha = v + \alpha (u_0 - v) \in \operatorname{int} U,\;\; u_0 \in \inf U,\;\; v \in \overline{U},\;\;\; 0 < \alpha \le 1.$$

To prove.

3) If U is a convex set, so $\operatorname{int} U$ is convex too. To prove.

4) If U is a convex inempty set, so $riU \neq \varnothing$ and $ri\,U$ is convex. To prove.

5) If U is a convex set and $ri\,U \neq \varnothing$, so the point

$$v_\alpha = v + v_\alpha = v + \alpha(u_0 - v) \in riU,\ u_0 \in riU,\ v \in \overline{U},\ 0 < \alpha \leq 1$$

To prove.

Definition 5. *The point $u \in E^n$ called the convex combination of the points $u^1,\ u^2,...,u^m$ from E^n, if it is represented in the manner of $u = \sum_{i=1}^{m} \alpha_i u^i$, where the numbers $\alpha_i \geq 0,\ i = \overline{1,m}$ but their sum* $\alpha_1 + \alpha_2 + ... + \alpha_m = 1$.

Theorem 2. *The set U is convex if and only if, when it contains all convex combinations of any finite number their own points.*

Proof. Necessity. Let U be a convex set. We show, that it contains all convex combinations of the finite number of their own points. We use the method of induction to proof of the theorem. From definition of the convex set follows that statement faithfully for any two points from *U*. We suppose, that set *U* contains the convex combinations $m-1$ of their own points, i.e. the point $v = \sum_{i=1}^{m-1} \beta_i u^i \in U,\ \beta_i \geq 0,\ i = \overline{1, m-1},\ \beta_1 + ... + \beta_{m-1} = 1$. We prove, that it contains the convex combinations *m* their own points. In fact, expression $u = \alpha_1 u^1 + ... + \alpha_m u^m = (1 - \alpha_m) \sum_{i=1}^{m-1} \frac{\alpha_i}{1 - \alpha_m} u^i + \alpha_m u^m$.

We denote $\beta_i = \alpha_i / (1 - \alpha_m)$. We notice, that $\beta_i \geq 0$, $i = \overline{1, m-1}$, but sum $\beta_1 + ... + \beta_{m-1} = 1,$ since $\alpha_1 + ... + \alpha_{m-1} = 1 - \alpha_m,\ \alpha_i \geq 0,$ $\alpha_1 + \alpha_2 + ... + \alpha_m = 1$. Then the point $u = (1 - \alpha_m) v + \alpha_m u^m$, $v \in U,\ u^m \in U$. Thence with consideration of the set convexity U the point $u \in U$. Necessity is proved.

Sufficiency. Let the set U contains all convex combinations of any finite number its own points. We show that U - a convex set. In particular, for $m=2$ we have $u_\alpha = \alpha u^1 + (1-\alpha)u^2 \in U$ under any $u^1,\ u^2 \in U$ under all $\alpha \in [0,1]$. The inclusion means that U - a convex set. Theorem is proved.

Definition 6. By convex cover of arbitrary set $U \subset E^n$ called the intersection of all convex sets containing set U and it is denoted by CoU.

From given definition follows that CoU is least (on remoteness from set U) convex set containing the set U. We notice, that source problem $J(u) \to \inf$, $u \in U$, with arbitrary set $U \subset E^n$ can be replaced on the approximate problem $J(u) \to \inf$, $u \in CoU$. We note, that if U is closed and bounded(compact) set, that CoU is also boundedand closed (compactly).

Theorem 3. *The set CoU contains that and only that points which are convex combination of the finite number points from* U.

Proof. To proof of theorem it is enough to show that $CoU = W$, where W is set of the points which are convex combination of any finite number points from set U. The inclusion $W \subseteq CoU$ follows from theorem 2, since $W \subseteq CoU$ and set CoU is convex. On the other hand, if the points $u,\ v \in W$, i.e. $u = \sum_{i=1}^{m} \alpha_i u^i$,

$$\alpha_i \ge 0,\ i = \overline{1,m};\ \sum_{i=1}^{m} \alpha_i = 1;\ v = \sum_{i=1}^{p} \beta_i v^i,\ \beta_i \ge 0,\ i = \overline{1,p},\ \sum_{i=1}^{p} \beta_i = 1,$$

that under all $\alpha \in [0,1]$ the point $u_\alpha = \alpha u + (1-\alpha)v = \sum_{i=1}^{m} \overline{\alpha}_i u^i + \sum_{i=1}^{p} \overline{\beta}_i v^i$, where $\overline{\alpha}_i = \alpha\alpha_i \ge 0,\ i = \overline{1,m},\ \overline{\beta}_i = (1-\alpha)\beta_i \ge 0,\ i = \overline{1,p}$ moreover the sum $\sum_{i=1}^{m} \overline{\alpha}_i + \sum_{i=1}^{p} \overline{\beta}_i = 1$. Thence follows that $u_\alpha \in W$, consequently, the set W is convex. Since set $U \subseteq W$, so the inclusion $CoU \subseteq W$ exists. From inclusion $W \subseteq CoU$, $CoU \subseteq W$ follows that $CoU = W$. Theorem is proved.

Theorem 4. *Any point* $u \in CoU$ *can be presented as convex combination no more than* $n+1$ *points from* U.

Proof. As follows from theorem 3, the point $u \in CoU$ is represented in the manner of $\sum_{i=1}^{m} \alpha_i u^i$, $\alpha_i \geq 0$, $\sum_{i=1}^{m} \alpha_i = 1$. We suppose, that number $m > n+1$. Then $n+1-$dimensional vectors $\overline{u}_i = (u^i, 1)$, $i = \overline{1,m}$, $m > n+1$ are linearly dependent. Consequently, there are the numbers $\gamma_1, ..., \gamma_m$, not all equal to zero, such that sum $\sum_{i=1}^{m} \gamma_i \overline{u}_i = 0$. Thence follows, that $\sum_{i=1}^{m} \gamma_i u^i = 0$, $\sum_{i=1}^{m} \gamma_i = 0$. Using first equality, the point u we present in the manner of

$$u = \sum_{i=1}^{m} \alpha_i u^i = \sum_{i=1}^{m} \alpha_i u^i - t \sum_{i=1}^{m} \gamma_i u^i = \sum_{i=1}^{m} (\alpha_i - t\gamma_i) u^i = \sum_{i=1}^{m} \overline{\alpha}_i u^i,$$

where $\overline{\alpha}_i = \alpha_i - t\gamma_i \geq 0$, $\sum_{i=1}^{m} \overline{\alpha}_i = 1$ under enough small t. Let $\gamma_{i*} = \min \gamma_i$, amongst $\gamma_i > 0$, where index i, $1 \leq i \leq m$. We choose the number t from the condition $\alpha_{i*} / \gamma_{i*} = t$. Since $\sum_{i=1}^{m} \gamma_i = 0$, that such $\gamma_{i*} > 0$ always exists. Now the point

$$u = \sum_{i=1}^{m} \overline{\alpha}_i u^i = \sum_{\substack{i=1 \\ i \neq i*}}^{m} \overline{\alpha}_i u^i, \overline{\alpha}_i \geq 0, \ \sum_{\substack{i=1 \\ i \neq i*}}^{m} \overline{\alpha}_i = 1. \qquad (2)$$

Specified technique are used for any $m > n+1$. Iterating the given process, eventually get $m = n+1$. Theorem is proved.

Definition 7. *The convex cover drawing on the points* u^0, $u^1, ..., u^m$ *from* E^n *is called* m*-dimensional simplex, if vectors*

$\{u^1 - u^0\}$, $i = \overline{1,m}$ *are linear independent and it is denoted by* S_m. *The points* u^0, $u^1, \dots, u^m$ *are called the tops of the simplex.*

Set S_m is a convex polyhedron with dimension m and by theorem 3 it is represented in the manner of

$$S_m = \left\{ u \in E^n \Big/ u = \sum_{i=1}^{m} \alpha_i u^i, \ \ \alpha_i \geq 0, \ \ i = \overline{0,m}, \ \ \sum_{i=0}^{m} \alpha_i = 1 \right\}.$$

Lecture 4

CONVEX FUNCTIONS

The convex programming problem in general type is formulated such: $J(u) \to \inf, \quad u \in U,\ U \subset E^n$, where U - convex set from E^n; $J(u)$ - a convex function determined on convex set U.

Definition 1. *Let function $J(u)$ be determined on convex set U from E^n. Function $J(u)$ is called convex on set U, if for any points $u,\ v \in U$ and under all $\alpha,\ 0 \le \alpha \le 1$ is executed the inequality*

$$J(\alpha u + (1-\alpha)v) \le \alpha J(u) + (1-\alpha)J(v). \qquad (1)$$

If in the inequality (1) an equality possible under only $\alpha = 0$ and $\alpha = 1$, then function $J(u)$ is called strictly convex on convex set U. Function $J(u)$ is concave (strongly concave) if function $J(u)$ concave (strongly concave) on set U.

Definition 2. *Let function $J(u)$ be determined on convex set U. Function $J(u)$ is called strongly convex on set U if the constant $\kappa > 0$ exists, such that for any points $u,\ v \in U$ and under all $\alpha,\ 0 \le \alpha \le 1$ is executed the inequality*

$$J(\alpha u + (1-\alpha)v) \le \alpha J(u) + (1-\alpha)J(v) - \alpha(1-\alpha)\kappa\,|u-v|^2. \qquad (2)$$

Example 1. Let function $J(u) = \langle c,u \rangle$ be determined on convex set U from E^n. We show that $J(u) = \langle c,u \rangle$ - a convex function on U. In fact, $u,\ v \in U$ are arbitrary points, number $\alpha \in [0,1]$, then the point $u_\alpha = \alpha u + (1-\alpha)v \in U$ on the strength of convexity of the set U. Then

$$J(u_\alpha) = \langle c, u_\alpha \rangle = \alpha \langle c, u \rangle + (1-\alpha)\langle c, v \rangle = \alpha J(u) + (1-\alpha) J(v).$$

In this case relationship (1) is executed with sign of the equality. The symbol $\langle .,. \rangle$ - a scalar product.

Example 2. Let function $J(u) = |u|^2$ be determined on convex set $U \subset E^n$. We show that $J(u)$ - strongly convex function on set U. In fact, for $u,\ v \in U$ and under all $\alpha \in [0,1]$ the point $u_\alpha = \alpha u + (1-\alpha)v \in U$. The value

$$J(u_\alpha) = |\alpha u + (1-\alpha)v|^2 = \langle \alpha u + (1-\alpha)v,\ \alpha u + (1-\alpha)v \rangle =$$
$$= \alpha^2 |u|^2 + 2\alpha(1-\alpha)\langle u, v \rangle + (1-\alpha)^2 |v|^2 . \quad (3)$$

The scalar product

$$\langle u - v, u - v \rangle = |u - v|^2 = |u|^2 - 2\langle u, v \rangle + \ + |v|^2 .$$

Thence we have $2\langle u, v \rangle = |u|^2 + |v|^2 - |u - v|^2$. Having substituted the equality to the right part of expression (3), we get

$$J(u_\alpha) = \alpha |u|^2 + (1-\alpha)|v|^2 - \alpha(1-\alpha)|u - v|^2 =$$
$$= \alpha J(u) + (1-\alpha)J(v) - \alpha(1-\alpha)|u - v|^2 .$$

Finitely, relationship (2) is executed with sign of the equality, moreover number $\kappa = 1$.

Example 3. The function $J(u) = |u|^2$ determined on convex set $U \subset E^n$ is strongly convex. In fact, value

$$J(u_\alpha) = |\alpha u + (1-\alpha)v|^2 \le \alpha \,|\, u\,|^2 + (1-\alpha)\,|\, v\,|^2 = \alpha J(u) + (1-\alpha)J(v),$$

moreover equality possible under only $\alpha = 0$ and $\alpha = 1$.

In general case check of convexity or strong convexity of the function $J(u)$ on convex set U by the definitions 1, 2 rather complex. In such events the following theorems are useful.

The criteria to convexity of the even functions. Let $C^1(U)$, $C^2(U)$ are accordingly spaces of the continuously differentiated and twice continuously differentiated functions $J(u)$, determined on set U. We notice, that gradient

$$J'(u) = (\partial J(u)/\partial u_1, \ldots, \partial J(u)/\partial u_n) \in E^n$$

under any fixed $u \in U$.

Theorem 1. *In order the function $J(u) \in C^1(U)$ to be convex on the convex set $U \subset E^n$, necessary and sufficiently executing of the inequality*

$$J(u) - J(v) \ge \langle J'(v), u - v \rangle, \quad \forall u, \;\; v \in U. \qquad (4)$$

Proof. Necessity. Let function $J(u) \in C^1(U)$ be convex on *U*. We show, that inequality (4) is executed. From inequality (1) we have

$$J(v + \alpha(u - v)) - J(v) \le \alpha[J(u) - J(v)], \qquad \alpha \in [0,1], \quad \forall u, v \in U.$$

Thence on base of the formula of the finite increments we get

$$\alpha \langle J'(v + \theta\alpha(u - v)), u - v \rangle \le \alpha[J(u) - J(v)], \quad 0 \le \theta \le 1.$$

Both parts of the given inequality are divided into $\alpha > 0$ and turn to the limit under $\alpha \to +0$, with consideration of $J(u) \in C^1(U)$ we get the inequality (4). Necessity is proved.

Sufficiency. Let for function $J(u)\in C^1(U)$, U be a convex set inequality (4) is executed. We show, that $J(u)$ is convex on *U*. Since the set U is convex, the point $u_\alpha=\alpha u+(1-\alpha)v\in U$, $\forall u,\,v\in U,\,\alpha\in[0,1]$. Then from inequality (4) follows, that $J(u)-J(u_\alpha)\geq\langle J'(u_\alpha),u-u_\alpha\rangle,\quad J(v)-J(u_\alpha)\geq\langle J'(u_\alpha),v-u_\alpha\rangle,$ $\forall u,\ u_\alpha,\ v\in U$. We multiply the first inequality on α, but the second inequality - on the number $(1-\alpha)$ and add them. As a result we give $\alpha J(u)+(1-\alpha)J(v)\geq J(u_\alpha)$. Thence follows the convexity to functions $J(u)$ on set *U*. Theorem is proved.

Theorem 2. *In order the function* $J(u)\in C^1(U)$ *to be convex on convex set U necessary and sufficiently executing of the inequality*

$$\langle J'(u)-J'(v),u-v\rangle\geq 0,\ \ \forall u,\quad v\in U. \tag{5}$$

Proof. Necessity. Let the function $J(u)\in C^1(U)$ be convex on convex set *U*. We show, that inequality (5) is executed. Since inequality (4) faithfully for any $u,\,v\in U$, that, in particular, we have $J(v)-J(u)\geq\langle J'(u),v-u\rangle$. We add the inequality with inequality (4), as a result we get the inequality (5). Necessity is proved.

Sufficiency. Let for the function $J(u)\in C^1(U)$, U be a convex set, inequality (5) is executed. We show, that $J(u)$ is convex on U. For proving it is enough to show, that difference

$$\alpha J(u)+(1-\alpha)J(v)-J(u_\alpha)=$$
$$=\alpha J(u)+(1-\alpha)J(v)-J(\alpha u+(1-\alpha)v)\geq 0,\ \ \forall u,\quad v\in U.$$

Since $J(u)\in C^1(U)$, the following inequality faithfully:

$$J(u+h)-J(u)=\langle J'(u+\theta_1 h),h\rangle=\int_0^1\langle J'(u+th),h\rangle dt,$$
$$\forall u,\quad u+h\in U.$$

The first equality follows from formula of the finite increments, but the second - from theorem about average value, $0 \le \theta_1 \le 1$ Then difference

$$\alpha J(u) + (1-\alpha)J(u) - J(u_\alpha) = \alpha[J(u) - J(u_\alpha)] +$$

$$+ (1-\alpha)[J(v) - J(u_\alpha)] = \alpha \int_0^1 \langle J'(u_\alpha + t(u - u_\alpha)), u - u_\alpha \rangle dt +$$

$$+ (1-\alpha) \int_0^1 \langle J'(u_\alpha + t(v - u_\alpha)), v - u_\alpha \rangle dt.$$

Let $z_1 = u_\alpha + t(u - u_\alpha) = u_\alpha + t(1-\alpha)(u - v)$, $z_2 = u_\alpha + {}$ $+ t(v - u_\alpha) = u_\alpha + t\alpha(v - u)$. Then $z_1 - z_2 = t(u - v)$, but differences $u - u_\alpha = (1-\alpha)(z_1 - z_2)/t$, $v - u_\alpha = -\alpha(z_1 - z_2)/t$. Now previous inequality is written:

$$\alpha J(u) + (1-\alpha)J(v) - J(u_\alpha) = \alpha(1-\alpha) \int_0^1 \langle J'(z_1) - J'(z_2) z_1 - z_2 \rangle \frac{1}{t} dt.$$

Since the points $z_1,\ z_2 \in U$, that according to inequality (5) we have $\alpha J(u) + (1-\alpha)J(v) - J(u_\alpha) \ge 0$.

Theorem is proved.

Theorem 3. *In order the function $J(u) \in C^2(U)$ to be convex on convex set U, $\operatorname{int} U \ne \varnothing$, necessary and sufficiency executing of the inequality*

$$\langle J''(u)\xi, \xi \rangle \ge 0, \qquad \forall \xi \in E^n, \ \ \forall u \in U. \qquad (6)$$

Proof. Necessity. Let function $J(u) \in C^2(U)$ be convex on U. We show, that inequality (6) is executed. If the point $u \in \operatorname{int} U$, then the number $\varepsilon_0 > 0$ such that points $u + \varepsilon\xi \in U$ for any $\xi \in E^n$ and under all ε, $|\varepsilon| \leq \varepsilon_0$ is found. Since all conditions of theorem 2 are executed, that inequality faithfully

$$\langle J'(u+\varepsilon\xi) - J'(u), \varepsilon\xi \rangle \geq 0, \qquad \forall \xi \in E^n, \ |\varepsilon| \leq \varepsilon_0.$$

Thence, considering that

$$\langle J'(u+\varepsilon\xi) - J'(u), \varepsilon\xi \rangle = \langle J''(u+\theta\varepsilon\xi)\xi, \xi \rangle \varepsilon^2, \qquad 0 \leq \theta \leq 1,$$

and transferring to limit under $\varepsilon \to 0$, we get the inequality (6). If $u \in U$ is a border point, then the following sequence $\{u_k\} \subset \operatorname{int} U$ exists, such that $u_k \to 0$ under $k \to 0$. Hereinafter, by proved $\langle J''(u_k)\xi, \xi \rangle \geq 0$, $\forall \xi \in E^n$, $u_k \in \inf U$.

Transferring to limit and taking into account that $\lim\limits_{k\to\infty} J''(u_k) = J''(u)$ with considering of $J(u) \in C^2(U)$, we get the inequality (6). Necessity is proved.

Sufficiency. Let for function $J(u) \in C^2(U)$, U be a convex set, $\operatorname{int} U \neq \varnothing$, inequality (6) is executed. We show, that $J(u)$ is convex on U. So as the equality faithfully

$$\langle J'(u) - J'(v), u-v \rangle = \langle J''(v+\theta(u-v))u-v, u-v \rangle, \qquad 0 \leq \theta \leq 1,$$

that, denoting by $\xi = u - v$ and considering that $u_\theta = v + \theta(u - -v) \in U$, we get

$$\langle J'(u) - J'(v), u-v \rangle = \langle J''(u_\theta)\xi, \xi \rangle \geq 0, \qquad \forall \xi \in E^n, \qquad \forall u_\theta \in U.$$

Thence with considering of theorem 2, follows the convexity to function $J(u)$ on U. Theorem is proved.

We note, that symmetrical matrix

$$J''(u)=\begin{pmatrix} \partial^2 J(u)/\partial u_1^2 & \partial^2 J(u)/\partial u_1\partial u_2 & \dots & \partial^2 J(u)/\partial u_1\partial u_n \\ \partial^2 J(u)/\partial u_2\partial u_1 & \partial^2 J(u)/\partial u_2^2 & \dots & \partial^2 J(u)/\partial u_2\partial u_n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \partial^2 J(u)/\partial u_n\partial u_1 & \partial^2 J(u)/\partial u_n\partial u_2 & \dots & \partial^2 J(u)/\partial u_n^2 \end{pmatrix}-$$

a scalar product.

$\langle J''(u),\xi,\xi\rangle=\xi' J''(u)\xi$.

The theorems 1 - 3 for strongly convex function $J(u)$ on U are formulated in the manner specified below and its are proved by the similar way.

Theorem 4. *In order the function $J(u)\in C^1(U)$ to be strongly convex on the convex set U necessary and sufficiently executing of the inequality*

$$J(u)-J(v)\geq\langle J'(v),u-v\rangle+k\,|u-v|^2,\forall u,\quad v\in U. \qquad (7)$$

Theorem 5. *In order the function $J(u)\in C^1(U)$ to be strongly convex on the convex set U, necessary and sufficiently executing of the inequality*

$$\begin{gathered}\langle J'(u)-J'(v),u-v\rangle\geq\mu\,|u-v|^2,\\ \mu=\mu(k)=const>0,\ \forall u,\quad v\in U.\end{gathered} \qquad (8)$$

Theorem 6. *In order the function $J(u)\in C^2(U)$ to be strongly convex on the convex set U, $\operatorname{int}U\neq\varnothing$ necessary and sufficiently executing of the inequality*

$$\begin{gathered}\langle J''(u)\xi,\xi\rangle\geq\mu\,|\xi|^2,\quad\forall\xi\in E^n,\quad\forall u\in U,\\ \mu=\mu(k)=const>0.\end{gathered} \qquad (9)$$

The formulas (4) - (9) can be applied for defining of the convexity and strong convexity of the even functions $J(u)$ determined on the convex set $U\subset E^n$.

For studying of convergence of the successive approximation methods are useful the following lemma.

Definition 3. *It is spoken that gradient* $J'(u)$ *to function* $J(u) \in C^1(U)$ *satisfies to Lipschitz's condition on the set U, if*

$$|J'(u) - J'(v)| \leq L|u - v|, \quad \forall u, v \in U, \quad L = const \geq 0. \qquad (10)$$

Space of such functions is denoted by $C^{1,1}(U)$.

Lemma. *If function* $J(u) \in C^{1,1}(U)$, U *be convex set, that inequality faithfully*

$$|J(u) - J(v) - \langle J'(v), u - v\rangle| \leq L|u - v|^2 / 2, \qquad (11)$$

Proof. From equality

$$J(u) - J(v) - \langle J'(v), u - v\rangle = \int_0^1 \langle J'(v + t(u - v)) - J'(v), u - v\rangle dt$$

follows that

$$\left|J(u) - J(v) - \langle J'(v), u - v\rangle\right| \leq \int_0^1 \left|J'(v + t(u - v)) - J'(v)\right| \left|u - v\right| dt.$$

Thence with consideration of inequality (10) and after integration on t we get the formula (11). Lemma is proved.

The properties of the convex functions. The following statements are faithful.

1) If $J_i(u), i = \overline{1,m}$ are convex functions on the convex set, then function $J(u) = \sum_{i=1}^{m} \alpha_i J_i(u), \alpha_i \geq 0, \ i = \overline{1,m}$ is convex on set U.

In fact,

$$J(u_\alpha) = \sum_{i=1}^{m} \alpha_i J_i(u_\alpha) \leq \sum_{i=1}^{m} \alpha_i [\alpha J_i(u) + (1-\alpha) J_i(v)] =$$

$$= \alpha \sum_{i=1}^{m} \alpha_i J_i(u) + (1-\alpha) \sum_{i=1}^{m} \alpha_i J_i(v) = \alpha J(u) + (1-\alpha) J(v), \quad \forall u, v \in U.$$

2) If $J_i(u),\ i \in I$ is a certain family of the convex functions on the convex set U, then function $J(u) = \sup J_i(u),\ i \in I$ is convex on set U.

In fact, by the determination of the upper boundary index $i_* \in I$ and the number $\varepsilon > 0$, such that $J(u_\alpha) - \varepsilon \le J_{i_*}(u_\alpha)$, $i_* = i_*(\varepsilon, \alpha) \in I$ are found. Thence we have

$$J(u_\alpha) \le \varepsilon + \alpha\, J_{i_*}(u) + (1-\alpha) J_{i_*}(v),\ u_\alpha = \alpha\, u + (1-\alpha) v.$$

Consequently, $J(u_\alpha) \le \alpha\, J(u) + (1-\alpha) J(v)$, under $\varepsilon \to +0$.

3) Let $J(u)$ be a convex function determined on the convex set $U \subset E^n$. Then the inequality is correct

$$J\left(\sum_{i=1}^{m} \alpha_i u_i\right) \le \sum_{i=1}^{m} \alpha_i J(u_i),\ \alpha_i \ge 0,\ \ \sum_{i=1}^{m} \alpha_i = 1.$$

To prove by the method to mathematical induction.

4) If function $f(t),\ t \in [a,b]$ is convex and it is not decrease, but function $F(u)$ is convex on the convex set $U \subset E^n$, moreover the values $F(u) \in [a,b]$, then function $J(u) = f(F(u))$ is convex on U.

In fact, the value

$$\begin{aligned} J(u_\alpha) &= f(F(u_\alpha)) \le f(\alpha\, F(u) + (1-\alpha) F(v)) \le \\ &\le \alpha\, f(F(u)) + (1-\alpha) f(F(v)) = \alpha\, J(u) + (1-\alpha) J(v). \end{aligned}$$

We notice, that $t_1 = F(u) \in [a,b],\ t_2 = F(v) \in [a,b]$ and

$$\begin{aligned} f(\alpha\, F(u) + (1-\alpha) F(v)) &= f(\alpha\, t_1 + (1-\alpha) t_2) \le \\ &\le \alpha\, f(t_1) + (1-\alpha) f(t_2),\ \forall t_1, t_2 \in [a,b], \end{aligned}$$

with consideration of convexity of the function $f(t)$ on segment $[a,b]$.

Lecture 5

THE ASSIGMENT WAYS OF THE CONVEX SETS. THEOREM ABOUT GLOBAL MINIMUM. OPTIMALITY CRITERIA. THE POINT PROJECTION ON SET

Some properties of the convex functions and convex set required for solution of the convex programming problem and the other questions which are important for solution of the optimization problems in finite-dimensional space are studied separately in the previous lectures.

The assignment ways of the convex sets. At studying of the properties to convex function $J(u)$ and convex set U any answer to question as convex set U is assigned in space E^n was not given.

Definition 1. *Let $J(u)$ be a certain function determined on set U from E^n. The set*

$$epiJ = \{(u,\gamma) \in E^{n+1} / u \in U \subset E^n, \gamma \geq J(u)\} \subset E^{n+1} \quad (1)$$

is called the above-graphic (or epigraph) to function $J(u)$ on set U.

Theorem 1. *In order the function $J(u)$ to be convex on the convex set U necessary and sufficiently that set epi J to be convex.*

Proof. Necessity. Let function $J(u)$ be convex on the convex set U. We show, that set *epiJ* is convex. In order sufficiently to make sure in that for any $z = (u,\gamma_1) \in epi\ J$, $w = (v,\gamma_2) \in epi\ J$

and under all α, $0 \le \alpha \le 1$, the point $z_\alpha = \alpha z + (1-\alpha)w \in epi\,J$. In fact, the point $z_\alpha = \left(\alpha u + (1-\alpha)v, \alpha\gamma_1 + (1-\alpha)\gamma_2\right)$, moreover $\alpha u + (1-\alpha)v \in U$ with consideration of set convexity *U*, but the value $\alpha\gamma_1 + (1-\alpha)\gamma_2 \ge \alpha J(u) + (1-\alpha)J(w) \ge J(u_\alpha)$. Finally, the point $z_\alpha = \left(u_\alpha, \gamma_\alpha\right)$, where $u_\alpha = \alpha u + (1-\alpha)v \in U$, but value $\gamma_\alpha \ge J(u_\alpha)$. Consequently, the point $z_\alpha \in$ epi*J*. Necessity is proved.

Sufficiency. Let set $epi\ J$ be convex, U be a convex set. We show that function $J(u)$ is convex on U. If the points $u, v \in U$, so $z = \left(u, J(u)\right) \in epi\,J$, $w = \left(v, J(v)\right) \in epi\ J$. For any $\alpha, \alpha \in [0,1]$, the point $z_\alpha = \left(u_\alpha, \alpha J(u) + (1-\alpha)J(v)\right) \in epi\,J$ with consideration of convexity set $epi\ J$. From the inclusion follows that value $\gamma_\alpha = \alpha J(u) + (1-\alpha)J(v) \ge J(u_\alpha)$. It means that function $J(u)$ is convex on U. Theorem is proved.

Theorem 2. *If function* $J(u)$ *is convex on the convex set* $U \subset E^n$*, then set*

$$M(C) = \left\{u \in E^n / u \in U,\ J(u) \le C\right\}$$

is convex under all $C \in E^1$.

Proof. Let the points *u, v*$\in$*M(C),* i.e. $J(u) \le C$, $J(v) \le C$, u, $v \in U$. The point $u_\alpha = \alpha u + (1-\alpha)v \in U$ under all $\alpha, \alpha \in [0,1]$ with consideration of convexity the set *U*. Since function $J(u)$ is convex on *U*, consequently value $J(u_\alpha) \le \alpha J(u) + (1-\alpha)J(v)$, $\forall u$, $v \in U$. Thence with consideration of $J(u) \le C$, $J(v) \le C$, we get $J(u_\alpha) \le \alpha C + (1-\alpha)C = C$. In the end, the point $u_\alpha \in U$, value $J(u_\alpha) \le C$. Consequently, for any $u, v \in U$ the point $u_\alpha \in M(C)$ under all $C \in E^1$. It means that set $M(C)$ is convex. Theorem is proved.

We consider the following optimization problem as appendix:

$$J(u) \to \inf\ , \qquad (2)$$

$$u \in U = \{u \in E^n / u \in U_0,\ g_i(u) \le 0,\ i = \overline{1,m};$$

$$g_i(u) = \langle a_i, u\rangle - b_i = 0,\ i = \overline{m+1,s}\}, \qquad (3)$$

where $J(u)$, $g_i(u),\ i = \overline{1,m}$ are convex functions determined on convex set U_0; $a_i \in E^n$, $i = \overline{m+1,s}$ are the given vectors; b_i, $i = \overline{m+1,s}$ - the given numbers. We enter the following sets:

$$U_i = \{u_i \in E^n / u \in U_0, g_i(u) \le 0\},\ i = \overline{1,m},$$

$$U_{m+1} = \{u \in E^n / \langle a_i, u\rangle - b_i = 0, i = \overline{m+1,s}\}.$$

The sets U_i, $i = \overline{1,m}$ are convex, since $g_i(u), i = \overline{1,m}$ are convex functions determined on convex set U_0 (refer to theorem 2, $C = 0$), but set

$$U_{m+1} = \{u \in E^n / Au = b\} = \{u \in E^n / \langle a_i, u\rangle - b_i = 0, i = \overline{m+1,s}\}$$

where A is a matrix of the order $(s-m)\times n$; the vectors $a_i, i = \overline{m+1,s}$ are the lines of the matrix *A*; $b = (b_{m+1}, \ldots, b_s) \in E^{n-s}$ is affine set.

Now the problem (2), (3) possible write as

$$J(u) \to \inf, u \in U = \bigcap_{i=0}^{m+1} U_i \subset U_0\ . \qquad (4)$$

Thereby, the problem (2), (3) is problem of the convex programming, since the intersection of any number convex sets is a convex set.

Theorem about global minimum. We consider the problem of the convex programming (4), where $J(u)$ - a convex function determined on convex set U from E^n.

Theorem 3. *If* $J(u)$ *is a convex function determined on convex set* U *and* $J_* = \inf_{u\in U} J(u) > -\infty$, $U_* = \{u_* \in E^n / u_* \in U, J(u_*) = =\min_{u\in U} J(u)\} \neq \varnothing$, *then any point of the local minimum to function* $J(u)$ *on* U *simultaneously is the point of its global minimum on U, moreover the set* U_* *is convex. If* $J(u)$ *is strongly convex on U, then set* U_* *contains not more one point.*

Proof. Let $u_* \in U$ be a point of the local function minimum $J(u)$ on set *U*, i.e. $J(u_*) \le J(u)$ under all $u \in o(u_*, \varepsilon) \cap U$. Let $v \in U$ be an arbitrary point. Then the point $w = u_* + \alpha(v - u_*) \in o(u_*, \varepsilon) \cap U$ with consideration of convexity the set *U,* if $\alpha|v - u_*| < \varepsilon$. Consequently, inequality $J(u_*) \le J(w)$ is executed. On the other hand, since $J(u)$ - a convex function on *U*, that the inequality $J(w) = = J(u_* + \alpha(v - u_*)) = J(\alpha v + (1-\alpha)u_*) \le \alpha J(v) + (1-\alpha)J(u_*)$ exists and $\alpha > 0$ - sufficiently small number i.e. $\alpha \in [0,1]$. Now inequality $J(u_*) \le J(w)$ to write as $J(u_*) \le J(w) \le \alpha J(v) + (1- -\alpha)J(u_*)$. This implies, that $0 \le \alpha[J(v) - J(u_*)]$. Consequently, $J(u_*) \le J(v)$, $\forall\, v \in U$.

This means that in the point $u_* \in U$ the global minimum to function $J(u)$ on U is reached.

We show, that set U_* is convex. In fact, for any $u, v \in U_*$, i.e. and $J(u) = J(v) = J_*$ under all $\alpha, \alpha \in [0,1]$ the value $J(u_\alpha) \le J(\alpha u + (1-\alpha)v) \le \alpha J(u) + (1-\alpha)J(v) = J_*$. Thence follows that $J(u_\alpha) = J_*$, consequently, the point $u_\alpha \in U_*$. Convexity of the set is proved. We notice, that from $U_* \neq \varnothing$ follows $J_* = J(u_*)$.

If $J(u)$ be strongly convex function, then must be

$J(u_\alpha) < \alpha J(u) + (1-\alpha) J(v) = J_*$, $0 < \alpha < 1$. Contradiction is taken off, if $u = v$, i.e. the set U_* contains not more one of the point. Theorem is proved.

Finally, in the convex programming problems any point of the local minimum to function $J(u)$ on U will be the point its global minimum on *U,* i.e. it is a solution of the problem.

Optimization criteria. Once again we consider the convex programming problem (4) for case, when $J(u) \in C^1(U)$.

Theorem 4. *If $J(u) \in C^1(U)$ is an arbitrary function, U is a convex set, the set $U_* \neq \varnothing$, then in any point $u_* \in U_*$ necessary the inequality is executed*

$$\langle J'(u_*), u - u_* \rangle \geq 0, \ \forall u \in U. \tag{5}$$

If $J(u) \in C^1(U)$ is a convex function, U is a convex set, $U_* \neq \varnothing$, then in any point $u_* \in U_*$ necessary and sufficiently performing of the condition (5).

Proof. Necessity. Let the point $u_* \in U_*$. We show, that inequality (5) is executed for any function $J(u) \in C^1(U)$, U - a convex set (in particular, $J(u)$ - a convex function on U). Let $u \in U$ be an arbitrary point and number $\alpha \in [0,1]$. Then the difference $J(u_\alpha) - J(u_*) \geq 0$, where $u_\alpha = \alpha u + (1-\alpha) u_* \in U$. Thence follows that $0 \leq J(\alpha u + (1-\alpha) u_*) - J(u_*) = J(u_* + \alpha(u - - u_*)) - J(u_*) = \alpha \langle J'(u_*), u - u_* \rangle + o(\alpha)$, moreover $o(\alpha)/\alpha \to 0$ under $\alpha \to +0$. Both parts are divided into $\alpha > 0$ and transferring to the limit under $\alpha \to +0$ we get the inequality (5). Necessity is proved.

Sufficiency. Let $J(u) \in C^1(U)$ be a convex function, U be a convex set, $U_* \neq \varnothing$ and inequality (5) is executed. We show that point $u_* \in U_*$. Let $u \in U$ - an arbitrary point. Then, by theorem 2 (refer to section "Convexity criteria of the even functions"), we have $J(u)-J(u_*)\geq \langle J'(u_*),u-u_*\rangle \geq 0$ under all $u\in U$. This implies that $J(u_*)\leq J(u), u\in U$. Consequently, the point $u_*\in U_*$. Theorem is proved.

Consequence. If $J(u)\in C^1(U)$, U is a convex set, $U_*\neq\varnothing$ and the point $u_*\in U_*$, $u_*\in \operatorname{int}U$, then the equality $J'(u_*)=0$ necessary is executed.

In fact, if $u_*\in \operatorname{int}U$, then the number $\varepsilon_0>0$ such that for any $e\in E^n$ point $u=u_*+\varepsilon\, e\in U$ under all $\varepsilon,\ |\varepsilon|\leq\varepsilon_0$. is found. Then from (5) follows $\langle J'(u_*),\varepsilon\, e\rangle\geq 0,\ \ \forall e\in E^n$, under all $\varepsilon,\ |\varepsilon|\leq\varepsilon_0$. Thence we have $J'(u_*)=0$.

Formula (5) as precondition of the optimality for nonlinear programming problem and as necessary and sufficiently condition of the optimality for convex programming problem will find applying in the following lectures.

The point projection on set. As application to theorems 1, 2 we consider the point projection on convex closed set.

Definition 2. *Let U be certain set from E^n, but the point $v\in E^n$. The point $w\in U$ is called the projection of the point $v\in E^n$ on set U, if norm $|v-w|=\inf\limits_{u\in U}|v-u|$, and it is denoted by $w=P_U(v)$*

Example 1. Let the set be

$$U=\overline{S}(u_0,R)=\{u\in E^n\,/\,|u-u_0|\leq R\},$$

but the point $v\in E^n$. The point $w=P_U(v)=u_0+R(v-u_0)/|v-u_0|$, that follows from geometric interpretation.

Example 2. If the set

$$U=\Gamma=\{u\in E^n\,/\,\langle c,u\rangle=\gamma\},$$

then point projection $v\in E^n$ on U is defined by the formula

$w = P_U(v) = v + [\gamma - \langle c, v\rangle] c / |c|^2$. Norm of the second composed is equal to the distance from the point v till U .

Theorem 5. *Any point $v \in E^n$ has a single projection on convex closed set $U \subset E^n$, moreover for point $w = P_U(v)$ necessary and sufficiently executing of the condition*

$$\langle w - v, u - w\rangle \geq 0, \forall u \in U . \tag{6}$$

In particular, if U - affine set, that condition (6) to write as

$$\langle w - v, u - w\rangle = 0, \forall u \in U. \tag{7}$$

Proof. The square of the distance from the point v till $u \in U$ is equal to $J(u) = |u - v|^2$. Under fixed $v \in E^n$ function $J(u)$ - strongly convex function on convex closed set U . Any strongly convex function is strongly convex. Then by theorem 1 function $J(u)$ reaches the lower boundary in the single point w, moreover $w \in U$ with consideration of its completeness. Consequently, the point $w = P_U(v)$. Since the point $w \in U_* \subset U$ and gradient $J'(w) = 2(w - v)$, that necessary and sufficient optimality condition for the point $w, (w = u_*)$ according to formula (5) is written in the manner of $2\langle w - v, u - w\rangle \geq 0, \forall u \in U$. This implies the inequality (6).

If $U = \{u \in E^n / Au = b\}$ is affine set, then from $u_0, u \in U$, $u_0 \neq u$ follows that $2u_0 - u \in U$. In fact, $A(2u_0 - u) = 2Au_0 - - Au = 2b - b = b$ In particular, if $u_0 = w = P_U(v)$, then point $2w - u \in U$. Substituting to the formula (6) instead of the point $u \in U$ we get

$$\langle w - v, w - u\rangle \geq 0, \forall u \in U . \tag{8}$$

From inequality (6) and (8) follows the relationship (7). Theorem is proved.

Example 3. Let the set $U = \{u \in E^n / u \geq 0\}$, the point $v \in E^n$. Then projection $w = P_U(v) = (\bar{v}_1, \ldots, \bar{v}_n)$, where $\bar{v}_i = \max(0, v_i)$, $i = \overline{1,n}$. To prove.

Example 4. Let the set

$$U = \{u = (u_1, \ldots, u_n) \in E^n / \alpha_i \leq u_i \leq \beta_i, \, i = \overline{1,n}\} -$$

n-dimensional parallelepiped, but the point $v \in E^n$. Then components of the vector $w \in P_U(v)$ are defined by formula

$$w_i = \begin{cases} \alpha_i, \text{ если } v_i < \alpha_i \\ \beta_i, \text{ если } v_i > \beta_i \\ v_i, \text{ если } \alpha_i \leq v_i \leq \beta_i, i = \overline{1,n}. \end{cases}$$

In fact, from formula (6) follows that sum $\sum_{i=1}^{n} (w_i - v_i)(u_i - w_i) \geq 0$ under all $\alpha_i \leq v_i \leq \beta_i$, $i = \overline{1,n}$. If $v_i < \alpha_i$, that $w_i = \alpha_i, u_i \geq \alpha_i$, consequently the product $(w_i - v_i)(u_i - w_i) = (-v_i + \alpha_i)(u_i - {}$ ${}- \alpha_i) \geq 0$

Similarly, if $v_i > \beta_i$, that $w_i = \beta_i$, $u_i \leq \beta_i$ so $(w_i - v_i)(u_i - w_i)$ $= (-v_i + \beta_i)(u_i - \beta_i) \geq 0$. Finally, if $\alpha_i \leq v_i \leq \beta_i$, then $w_i = v_i$, so $(w_i - v_i)(u_i - w_i) = 0$. In the end, for the point $w = (w_1, \ldots, w_n) \in P_U(v)$ inequality (6) is executed.

Lecture 6, 7

SEPARABILITY OF THE CONVEX SETS

Problems on conditional extreme and method of the indefinite Lagrangian coefficients as application to the theory of the implicit functions in course of the mathematical analysis were considered. At the last years the section of mathematics is reached the essential development and the new theory called "Convex analysis" appeared. The main moment in the theory is a separability theory of the convex sets.

Definition 1. *It is spoken, that hyperplane $\langle c,u\rangle=\gamma$ with normal vector c, $|c|\neq 0$ divides (separates) the sets A and B from E^n if inequality are executed*

$$\sup_{b\in B}\langle c,b\rangle\le\gamma\le\inf_{a\in A}\langle c,a\rangle. \tag{1}$$

If $\sup\limits_{b\in B}\langle c,b\rangle<\inf\limits_{a\in A}\langle c,a\rangle$, that sets A and B are strongly separated, but if $\langle c,b\rangle<\langle c,a\rangle,\ a\in A, b\in B$, that they are strongly separated.

We notice that if hyperplane $\langle c,u\rangle=\gamma$ separates the sets *A* and *B*, that hyperplane $\langle\alpha\, c,u\rangle=\alpha\,\gamma$, where $\alpha\neq 0$ - any number also separates them, so under necessity possible to suppose the norm $|\,c\,|=1$.

Theorem 1. *If U - a convex set from E^n, but the point $v\notin \operatorname{int} U$, that hyperplane $\langle c,u\rangle=\gamma$ dividing set U and the point v exists. If U - a convex set, but the point $v\notin\overline{U}$, that set U and the point v is strongly separable.*

Proof. We consider the event, when the point $v \notin \overline{U}$. In this case by theorem 5 the point $v \in E^n$ has a single projection $w = P_{\overline{U}}(v)$ moreover $\langle w-v,\, u-w\rangle \geq 0, \quad \forall u \in \overline{U}$. Let vector $c = w - v$. Then we have $\langle c, u-v\rangle = \langle w-v, u-v\rangle = \langle w-v, u+w-w-v\rangle =$ $\langle w-v, w-v\rangle + \langle w-v, u-w\rangle \geq |c|^2 > 0$. Thence follows, that $\langle c,u\rangle > \langle c,v\rangle, \forall u \in \overline{U}$, i.e. set $\overline{U}$ and the point $v \in E^n$ are strongly separable. Consequently, the set U and the point v are strongly separable.

We notice, that, if $v \notin \operatorname{int} U$, that $v \in \Gamma p U$. Then by definition of the border point the sequence $\{v_k\} \not\subset \overline{U}$ such that $v_k \to v$ under $k \to \infty$ exists. Since the point $v_k \notin \overline{U}$, then by proved inequality $\langle c_k, u\rangle > \langle c_k, v_k\rangle,\ \forall u \in \overline{U},\ |c_k| = 1$ is executed. By Bolzano–Weierstrass' theorem from limited sequence $\{c_k\}$ possible to select subsequence $\{c_{k_m}\}$, moreover $c_{k_m} \to c$ under $m \to \infty$ and $|c| = 1$. For the elements of the subsequence the previous inequality is written as: $\langle c_{k_m}, u\rangle. > \langle c_{k_m}, v_{k_m}\rangle, \forall u \in \overline{U}$. Transferring to limit under $m \to \infty$ with consideration of $v_{k_m} \to v,$ we get $\langle c,u\rangle \geq \langle c,v\rangle, \forall u \in U$. Theorem is proved.

From proof of theorem 1 follows that through any border point v of the convex set U possible to conduct hyperplane for which $\langle c,u\rangle \geq \langle c,v\rangle, \forall u \in U$. Hyperplane $\langle c,u\rangle = \gamma$, where $\gamma = \langle c, v\rangle, v \in U$ is called to be supporting to set U in the point v, but vector $c \in E^n$ - a supporting vector of the set U in the point $v \in \overline{U}$.

Theorem 2. *If convex sets A and B from E^n have a no points in common, that hyperplane $\langle c,u\rangle = \gamma$ separating set A and B, as well as their closures $\overline{A}$ and $\overline{B}$, in the event of point $y \in \overline{A} \cap \overline{B}$,* *number* $\gamma = \langle c, y\rangle$ *exists.*

Proof. We denote $U = A - B$. As it is proved earlier (refer to theorem 1 from lecture 3), set U is convex. Since the intersection $A \cap B = \varnothing$, that $0 \notin U$. Then with consideration of theorem 1 hyperplane $\langle c, u\rangle = \gamma$, $\gamma = \langle c, 0\rangle = 0$, separating set U and the point 0, i.e. $\langle c, u\rangle \geq 0$, $\forall u \in U$ exists. Thence with consideration of $u = a - b \in U$, $a \in A$, $b \in B$, we get $\langle c, a\rangle \geq \langle c, b\rangle$, $\forall a \in A$, $\forall b \in B$. Finally, hyperplane $\langle c, u\rangle = \gamma$, where the number γ satisfies to inequality $\inf_{a \in A}\langle c, a\rangle \geq \gamma \geq \sup_{b \in B}\langle c, b\rangle$ separates the sets A and B. Let the points $a \in \overline{A}$, $b \in \overline{B}$. Then the subsequences $\{a_k\} \subset A$, $\{b_k\} \subset B$ such that $a_k \to a$, $b_k \to b$ under $k \to \infty$ exist, moreover with consideration of proved above the inequalities $\langle c, a_k\rangle \geq \gamma \geq \langle c, b_k\rangle$. are executed. Thence, transferring to limit under $k \to \infty$ we get $\langle c, a\rangle \geq \gamma \geq \langle c, b\rangle$ $\forall a \in \overline{A}, \forall b \in \overline{B}$.

In particular, if the point $y \in \overline{A} \cap \overline{B}$, that $\gamma = \langle c, y\rangle$. Theorem is proved.

Theorem 3. *If the convex closed sets* A *and* B *have a no points in common and one of them is limited, that hyperplane strongly separating sets* A *and* B *exists.*

Proof. Let the set $U = A - B$. We notice that point $0 \notin U$, since the intersection $A \cap B = \varnothing$, U - a convex set. We show, that set U is closed. Let u - a limiting point of the set U. Then sequence $\{u_k\} \subset U$ such that $u_k \to u$ under $k \to \infty$ exists, moreover $u_k = a_k - b_k$, $a_k \in A$, $b_k \in B$. If set A is limited, that sequence $\{a_k\} \subset A$ is limited, consequently, subsequence $\{a_{k_m}\}$ which converges to a certain point a exists, moreover with consideration of completeness of the set A the point $a \in A$. We consider the sequence $\{b_k\} \subset B$, where $b_k = a_k - u_k$. Since $a_{k_m} \to a$, $u_{k_m} \to u$ under $m \to \infty$ then $b_{k_m} \to a - u = b$ under $m \to \infty$. With consideration of completeness of the B the point $b \in B$. Finally, the point $u = a - b$, $a \in A$, $b \in B$, consequently, $u \in U$. Completeness of set U is proved.

Since the point $0 \notin U$, $U = \overline{U}$, then with consideration of theorem 1 hyperplane $\langle c,u\rangle = \gamma$ strongly separating the set U and the point 0 exists, i.e $\langle c,u\rangle > 0$, $\forall u \in U$. This implies, that $\langle c,a\rangle > \langle c,b\rangle,\ \forall a \in A,\ \forall b \in B$. Theorem is proved.

Theorem 4. *If the intersection of the inempty convex sets $A_0, A_1, \ldots, A_m$ - empty set, i.e. $A_0 \cap A_1 \cap \ldots \cap A_m = \varnothing$, that necessary vectors $c_0, c_1, \ldots, c_m$ from E^n, not all equal zero and the numbers $\gamma_0, \gamma_1, \ldots, \gamma_m$ such that*

$$\langle c_i, a_i\rangle \geq \gamma_i, \forall a_i \in A_i, i = \overline{0,m}; \tag{2}$$

$$c_0 + c_1 + \ldots + c_m = 0; \tag{3}$$

$$\gamma_0 + \gamma_1 + \ldots + \gamma_m = 0; \tag{4}$$

exist.

Proof. We enter the set $A = A_0 \times A_1 \times \ldots \times A_m$ - a direct product of the sets A_i, $i = \overline{0,m}$ with elements $a = (a_0, a_1, \ldots, a_m) \in A$, where $a_i \in A_i$, $i = \overline{0,m};$. We notice that set $A \subset E$, where $E = E^{n(m+1)}$. It is easy to show, that set A is convex. In fact, if $a^1 = = (a_0^1, a_1^1, \ldots, a_m^1) \in A$, $a^2 = (a_0^2, a_1^2, \ldots, a_m^2) \in A$, then under all α, $\alpha \in [0,1]$, the point $a_\alpha = \alpha a^1 + (1-\alpha)a^2 = (a_{0\alpha}, a_{1\alpha}, \ldots, a_{m\alpha}) \in A$, since $a_{i\alpha} = \alpha a_i^1 + (1-\alpha)a_i^2 \in A_i$, $i = \overline{0,m}$. We enter the diagonal set $B = \{\overline{b} = (b_0, b_1, \ldots, b_m) \in E / b_0 = b_1 = \ldots = b_m = b,\ b_i \in E^n, i = \overline{0,m}\}$. Set B is convex. It is not difficult to make sure, the intersection $A_0 \cap A_1 \cap \ldots \cap A_m = \varnothing$, if and only if, when intersection $A \cap B = \varnothing$.

Finally, under performing the condition of theorem the sets A and B are convex and $A \cap B = \varnothing$, i.e all conditions of theorem 2 are executed. Consequently, the vector $c=(c_0,c_1,\ldots,c_m)\in E,\ \ |c|\neq 0$ such that $\langle c,a\rangle\geq\langle c,b\rangle,\ \ \forall a\in A,\ \forall b\in B$ exists. The inequality can be written in the manner of

$$\sum_{i=0}^{m}\langle c_i,a_i\rangle\geq\sum_{i=0}^{m}\langle c_i,b_i\rangle=\left\langle\sum_{i=0}^{m}c_i,\ b\right\rangle,\ \forall a_i\in A,\ \forall b_i\in E^n. \quad (5)$$

As follows from inequality (5) linear function $J(b)=\left\langle\sum_{i=0}^{m}c_i,b\right\rangle$, $b\in E^n$ is limited, since in (5) $a_i\ \in A_i,\ \ i=\overline{0,m}$. The linear function $J(b)$ is limited if and only if, when $\sum_{i=0}^{m}c_i=0$. Thence follows fairness of the formula (3). Now the inequality (5) is written as $\sum_{i=0}^{m}\langle c_i,a_i\rangle\geq 0,\ \forall a_i\ \in A_i,\ i=\overline{0,m}$. We fix the vectors $a_i\ =\overline{a}_i\in A_i$, $i\neq k$, then $\langle c_k,a_k\rangle\geq-\sum_{i=0}^{m}\langle c_i,\overline{a}_i\rangle=const,\ \ \forall a_k\in A_k$. Finally, the value $\langle c_k,a_k\rangle\geq\inf_{a_k\in A_k}\langle c_k,a_k\rangle=\gamma_k>-\infty,\ \ \forall a_k\in A_k,k=\overline{1,m}$. We denote $\gamma_0=-(\gamma_1+\gamma_2+\ldots+\gamma_m)$. Then $\langle c_0,a_0\rangle\geq-\inf_{a_k\in A_k}\sum_{k=1}^{m}\langle c_k,a_k\rangle=$

$=-\sum_{k=1}^{m}\gamma_k=\gamma_0$.

Thereby, values $\langle c_i,a_i\rangle\geq\gamma_i\quad\forall a_i\ \in A_i,\quad i=\overline{0,m}$, $\gamma_0+\gamma_1+\ldots+\gamma_m=0$. Faithfulness of the relationships (2), (4) is proved. Theorem is proved.

Convex cones. Theorems 1 - 5 are formulated and proved for convex sets from E^n. Separability theorems are often applied in the extreme problem theories for events when convex sets are convex cones.

Definition 2. *Set K from E^n is called by cone with top in zero, if it together with any its point $u \in K$ contains the points $\lambda u \in K$ under all $\lambda > 0$. If the set K is convex, that it is called the convex cone, if K is closed, that it is called by closed cone, if K is open, that it is called by open cone.*

Example 1. The set $K = \{u \in E^n / u \geq 0\}$ -a convex closed cone. In fact if $u \in K$, that point $\lambda u \in K$ under all $\lambda > 0, \lambda u > 0$.

Example 2. The set $K = \{u \in E^n / \langle a, u\rangle = 0\}$ - a convex closed cone. Since for any $\lambda > 0$, scalar product $\langle a, \lambda u\rangle = \lambda\langle a, u\rangle = 0,$ $u \in K$, consequently, vector $\lambda u \in K$.

Example 3. The set $K = \{u \in E^n / \langle a, u\rangle < 0\}$ - an open convex cone, $K = \{u \in E^n / \langle a, u\rangle \leq 0\}$ - closed convex cone, $K = \{u \in E^n\} =$ $= E^n$ - a convex cone.

We enter the set

$$K^* = \{c \in E^n / \langle c, u\rangle \geq 0,\ \forall u \in K\}. \tag{6}$$

We notice, that set $K \subset K^*$, since from $c = u \in K$ follows that $\langle c, u\rangle = |c|^2 \geq 0$. The set $K^* \neq \varnothing$, since it contains the element $c = 0$. The condition $\langle c, u\rangle \geq 0, \forall u \in K$ means that vector $c \in K^*$ forms an acute angle (including $\pi/2$) with vectors $u \in K$. To give geometric interpretation of the set K^*.

Finally, the set K^* from E^n is the cone, if $c \in K^*$, i.e. $\langle c, u\rangle \geq 0, \forall u \in K$, then for vector $\lambda c,\ \lambda > 0$ we have $\langle \lambda c, u\rangle = \lambda\langle c, u\rangle \geq 0$. Consequently, vector $\lambda c \in K^*$.

Definition 3. *The set K^* determined by formula (6) is called dual (or reciprocal) cone to cone K.*

It is easy to make sure, that for cone K from example 1 the dual cone $K^* = \{c \in E^n / c \geq 0\}$, for cone K from example 2 the reciprocal cone K $K^* = \{c \in E^n / c = \beta\alpha,\ \ \beta - number\}$, for example 3 the dual cones $K^* = \{c \in E^n / c = -\beta\alpha,\ \beta \geq 0\}$, $K^* = \{c \in E^n / c = = -\beta\alpha,\ \beta \geq 0\}$, $K^* = \{0\}$ accordingly.

Theorem 5. If the intersection of the inempty convex with vertex in zero cones $K_0, K_1, \ldots, K_m$ - empty set, that necessary the vectors $c_i \in K_i^*, i = \overline{0,m}$ not all are equal to zero and such that $c_0 + c_1 + \ldots + c_m = 0$ exist.

Proof. Since all condition of theorem 4 are executed, that the relationship (2) - (4) are faithful. We notice, that $K_i, i = \overline{0,m}$ - cones, then from $a_i \in K_i, i = \overline{0,m}$ follows that vectors $\lambda a_i \in K_i, i = \overline{0,m}$ under all $\lambda > 0$. Then inequality (2) is written so: $\langle c_i, \lambda a_i \rangle \geq \gamma_i$, $\forall a_i \in K_i$, $i = \overline{0,m}$, for any $\lambda > 0$. Thence we have $\langle c_i, a_i \rangle \geq \gamma_i / \lambda$, $i = \overline{0,m}$. In particular, when $\lambda \to +\infty$, we get $\langle c_i, a_i \rangle \geq 0$, $i = \overline{0,m}$, $\forall a_i \in A_i$. On the other hand, $\gamma_i = \inf\limits_{a_i \in A} \langle c_i, a_i \rangle = 0$, $i = \overline{0,m}$. From condition $\langle c_i, a_i \rangle \geq 0$, $\forall a_i \in A_i$, $i = \overline{0,m}$, follows that vectors $c_i \in K_i^*, i = \overline{0,m}$. Theorem is proved.

Theorem 6. *(Dubovicky–Milyutin's theorem). In order the intersection of the inempty convex with vertex in zero cones $K_0, K_1, \ldots, K_m$ to be an empty set, when all these cones, except, can be one open, necessary and sufficiently existence of the vectors $c_i \in K_i^*$, $i = \overline{0,m}$, not all are equal to zero and such that $c_0 + c_1 + \ldots + c_m = 0$.*

Proof. Sufficiency. We suppose inverse, i.e. that vectors $c_i \in K_i^*$, $i = \overline{0,m}$, $c_0 + c_1 + \ldots + c_m = 0$, though $K_0 \cap K_1 \cap \ldots \cap K_m \neq \varnothing$. Then the point $w \in K_0 \cap K_1 \cap \ldots \cap K_m$ exists. Since vectors $c_i \in K_i^*$, $i = \overline{0,m}$, that inequalities $\langle c_i, w \rangle \geq 0$, $i = \overline{0,m}$

are executed. Since sum $c_0 + c_1 + \ldots + c_m = 0$ and $\langle c_i, w\rangle \geq 0$, $i = \overline{0,m}$, then from equality $\langle c_0 + c_1 + \ldots + c_m, w\rangle = 0$ follows that $\langle c_i, w\rangle = 0$, $i = \overline{0,m}$. By condition of the theorem not all c_i, $i = \overline{0,m}$ are equal to zero, consequently, at least, two vectors c_i, c_j, $i \neq j$ differenced from zero exist. Since all cones are open, except can be one, that in particular, it is possible to suppose that cone K_i - an open set. Then K_i contains the set $o(w, 2\varepsilon) =$ $= \{u \in K_i / |u - w| < 2\varepsilon,\ \varepsilon > 0\}$. In particular, the point $u = w -$ $- \varepsilon c_i / |c_i| \in K_i$. Since $\langle c_i, u\rangle \geq 0$, $u \in K_i$, $c_i \in K_i^*$, that we have $\langle c_i, w - \varepsilon c_i / |c_i|\rangle = \langle c_i, w\rangle - \varepsilon |c_i| = -\varepsilon |c_i| \geq 0$, where $\varepsilon > 0$, $|c_i| \neq 0$. It is impossible. Consequently, the intersection $K_0 \cap K_1 \cap \ldots \cap K_m \neq \varnothing$. Necessity follows from theorem 5 directly. Theorem is proved.

Lecture 8

LAGRANGE'S FUNCTION. SADDLE POINT

Searching for the least value (the global minimum) to function *J(u)* determined on set U from E^n is reduced to determination of the saddle point to Lagrange's function. By such approach to solution of the extreme problems the necessity to proving of the saddle point existence of the Lagrange's function appears.

We consider the next problem of the nonlinear programming:

$$J(u) \to \inf, \tag{1}$$

$$u \in U = \{u \in E^n / u \in U_0,\ g_i(u) \le 0,\ i = \overline{1,m};$$

$$g_i(u) = 0,\ i = \overline{m+1,s}\}, \tag{2}$$

where U_0 is the given convex set from E^n; $J(u)$, $g_i(u)$, $i = \overline{1,m}$ are the functions determined on set U_0. In particular, $J(u)$, $g_i(u)$, $i = \overline{1,m}$ are convex functions determined on convex set U_0, but functions $g_i(u) = \langle a_i, u\rangle - b_i$, $a_i \in E^n$, $i = \overline{m+1,s}$, b_i, $i = \overline{m+1,s}$, are the given numbers. In this case problem (1), (2) is related to the convex programming problems. In the beginning we consider a particular type of the Lagrange's function for problem (1), (2).

Lagrange's Function. Saddle point. Function

$$\begin{aligned} &L\,(u,\lambda) = J(u) + \sum_{i=1}^{s} \lambda_i g_i(u),\ u \in U_0, \\ &\lambda \in \Lambda_0 = \{\lambda \in E^s / \lambda_1 \ge 0, \ldots, \lambda_m \ge 0\} \end{aligned} \tag{3}$$

is called by Lagrange's function to problem (1), (2).

Definition 1. *Pair* $(u_*, \lambda^*) \in U_0 \times \Lambda_0$, *i.e.* $u_* \in U_0$, $\lambda^* \in \Lambda_0$ *is called by saddle point to Lagrange's function (3) if the inequalities are executed*

$$L(u_*, \lambda) \le L(u_*, \lambda^*) \le L(u, \lambda^*), \quad \forall u \in U_0, \forall \lambda \in \Lambda_0. \qquad (4)$$

We notice that in the point $u_* \in U_0$ *minimum to function* $L(u, \lambda^*)$ *is reached on set* U_0, *but in point* $\lambda^* \in \Lambda_0$ *the maximum to function* $L(u_*, \lambda)$ *is reached on set* Λ_0. *By range of definition of the Lagrange's function is the set* $U_0 \times \Lambda_0$.

The main lemma. *In order the pair* $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ *to be a saddle point to Lagrange's function (3) necessary and sufficiently performing of the following conditions:*

$$L(u_*, \lambda^*) \le L(u, \lambda^*), \ \forall u \in U_0; \qquad (5)$$

$$\lambda_i^* g_i(u_*) = 0, \ \ i = \overline{1,s}, \ \ u_* \in U. \qquad (6)$$

Proof. Necessity. Let pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ - a saddle point. We show, that the condition (5). (6) are executed. Since pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ - saddle point to Lagrange's function (3), that inequalities (4) are executed. Then from the right inequality follows the condition (5). It is enough to prove fairness of the equality (6). The left inequality from (4) is written so:

$$J(u_*) + \sum_{i=1}^{s} \lambda_i g_i(u_*) \le \ J(u_*) + \sum_{i=1}^{s} \lambda_i^* g_i(u_*),$$
$$\lambda = (\lambda_1, \dots, \lambda_s) \in \Lambda_0 \subset E^s.$$

Consequently, the inequality exists

$$\sum_{i=1}^{s} (\lambda_i^* - \lambda_i) g_i(u_*) \ge 0, \ \forall \lambda \in \Lambda_0. \qquad (7)$$

At first, we show that $u_* \in U$, i.e. $g_i(u_*) \le 0,\ i=\overline{1,m},$ $g_i(u_*) = 0,\ i=\overline{m+1,s}$. It is easy to make sure in that vector

$$\lambda = (\lambda_1, \dots, \lambda_s) = \begin{cases} \lambda_i = \lambda_i^*, & i=\overline{1,s},\ i \ne j, \\ \lambda_j = \lambda_j^* + 1, & for\ j\ of\ 1 \le j \le m, \end{cases} \tag{8}$$

belongs to the set $\Lambda = \{\lambda \in E^s / \lambda_1 \ge 0, \dots, \lambda_m \ge 0\}$.

Substituting value $\lambda \in \Lambda_0$ from (8) to the inequality (7) we get $(-1) g_i(u_*) \ge 0$. Thence follows that $g_i(u_*) \le 0,\ \ i=\overline{1,m}$ (since it is faithful for any j of $1 \le j \le m$). Similarly the vector

$$\lambda = (\lambda_1, \dots, \lambda_s) = \begin{cases} \lambda_i = \lambda_i^*, & i=\overline{1,s},\ \ i \ne j, \\ \lambda_j = \lambda_j^* + g_j(u_*), & для\ j\ из\ \ m+1 \le j \le s, \end{cases}$$

also belongs to the set Λ_0. Then from inequality (7) we have $-|g_i(u_*)|^2 \ge 0,\ j=\overline{m+1,s}$. Consequently, the values $g_j(u_*) = 0$, $j=\overline{m+1,s}$. From relationship $g_j(u_*) \le 0,\ j=\overline{1,m}$, $g_j(u_*) = 0$, $j=\overline{m+1,s}$ follows that point $u_* \in U$.

We choose the vector $\lambda \in \Lambda_0$ as follows:

$$\lambda = (\lambda_1, \dots, \lambda_s) = \begin{cases} \lambda_i = \lambda_i^*,\ i=\overline{1,s},\ i \ne j, \\ \lambda_j = 0, \ \ для\ j\, из\ 1 \le j \le m. \end{cases}$$

In this case inequality (7) is written so: $\lambda_j^* g_j(u_*) = 0, j=\overline{1,m}$. Since the value $\lambda_j^* \ge 0, j=\overline{1,m}$ under proved above value $g_j(u_*) \le 0$, $j=\overline{1,m}$, consequently, the equality $\lambda_j^* g_j(u_*) = 0, j=\overline{1,m}$ exists. The equalities $\lambda_j^* g_j(u_*) = 0, j=\overline{m+1,s}$ follow from equality $g_j(u_*) = 0,\ \ j=\overline{m+1,s}$. Necessity is proved.

Sufficiency. We suppose, that the conditions (5), (6) for a certain pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ are executed. We show, that (u_*, λ^*) - saddle point to Lagrange's function (3). It is easy to make sure in that product $(\lambda_i^* - \lambda_i) g_i(u_*) \geq 0$ for any i, $1 \leq i \leq m$. In fact, from condition $u_* \in U$ follows that $g_i(u_*) \leq 0$, for any i, $1 \leq i \leq m$. If $g_i(u_*) = 0$, that $(\lambda_i^* - \lambda_i) g_i(u_*) = 0$. In the event of $g_i(u_*) < 0$ the value $\lambda_i^* = 0$, since the product $\lambda_i^* g_i(u_*) = 0$, consequently, $(\lambda_i^* - \lambda_i) g_i(u_*) = -\lambda_i g_i(u_*) \geq 0, \;\; \lambda_i \geq 0$. Then sum $\sum_{i=1}^{m} (\lambda_i^* - \lambda_i) g_i(u_*) + \sum_{i=m+1}^{s} (\lambda_i^* - \lambda_i) g_i(u_*) = \sum_{i=1}^{s} (\lambda_i^* - \lambda_i) g_i(u_*) \geq 0,$ since $g_i(u_*) = 0, \;\; i = \overline{m+1, s}, \;\; u_* \in U$. Thence follows that

$$\sum_{i=1}^{s} (\lambda_i^* g_i(u_*) \geq \sum_{i=1}^{s} (\lambda_i g_i(u_*),$$

$$J(u_*) + \sum_{i=1}^{s} \lambda_i g_i(u_*) \geq J(u_*) + \sum_{i=1}^{s} \lambda_i g_i(u_*).$$

The second inequality can be written in the manner of $L(u_*, \lambda) \leq L(u_*, \lambda^*)$. The inequality in the ensemble with (5) defines the condition

$$L(u_*, \lambda) \leq L(u_*, \lambda^*) \leq L(u, \lambda^*), \;\; \forall u \in U_0, \forall \lambda \in \Lambda_0 .$$

This means that pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ - saddle point to Lagrange's function (3). Theorem is proved.

The main theorem. *If pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ - a saddle point to Lagrange's function (3), that vector $u_* \in U$ is a solution of the problem (1), (2) i.e.*

$$u_* \in U_* = \left\{ u_* \in E^n \big/ u_* \in U, J(u_*) = \min_{u \in U} J(u) \right\}.$$

Proof. As follows from the main lemma for pair $\left(u_*,\lambda^*\right)\in U_0\times\Lambda_0$ the conditions (5), (6) are executed. Then value $J(u_*,\lambda^*)=J(u_*)+\sum_{i=1}^{s}\lambda_i^* g_i(u_*)=J(u_*)$, since the point $u_*\in U$. Now inequality (5) is written so:

$$J(u_*)\le J(u)+\sum_{i=1}^{s}\lambda_i^* g_i(u),\ u\in U_0. \tag{9}$$

Since the set $U\subset U_0$, that inequality (9), in particular, for any $u\in U\subset U_0$ faithfully, i.e.

$$J(u_*)\le J(u)+\sum_{i=1}^{s}\lambda_i^* g_i(u),\ \ \forall u\in U,\ \ \lambda^*\in\Lambda_0. \tag{10}$$

As follows from condition (2) if $u\in U$, that $g_i(u)\le 0,\ i=\overline{1,m}$, and $g_i(u)=0,\ i=\overline{m+1,s}$, consequently, the product $\lambda_i^* g_i(u_*)\le 0$ for any $i,\ 1\le i\le s$. We notice, that $\lambda^*=(\lambda_1^*,\dots,\lambda_s^*)\in\Lambda_0$, where $\lambda_1^*\ge 0,\dots,\lambda_m^*\ge 0$. Then from (10) follows $J(u_*)\le J(u),\ \forall u\in U$. It means that in the point $u_*\in U$ the global (or absolute) minimum of the function $J(u)$ is reached on set *U*. Theorem is proved.

We note the following:

1) The main lemma and the main theorem for the nonlinear programming problem (1), (2) were proved, in particular, its are true and for the convex programming problem.

2) Relationship between solutions of the problem (1), (2) and saddle point to Lagrange's function in type (3) is established. In general event for problem (1), (2) Lagrange's function is defined by the formula

$$L(u,\overline{\lambda}) = \lambda_0 J(u) + \sum_{i=1}^{s} \overline{\lambda}_i \, g_i(u_*), \; u \in U_0,$$
$$\overline{\lambda} \in \Lambda_0 = \left\{ \overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_s) \in E^{s+1} \Big/ \overline{\lambda}_0 \geq 0, \ldots, \overline{\lambda}_m \geq 0 \right\}. \quad (11)$$

If the value $\lambda_0 \geq 0$, that Lagrange's function (11) possible to present in the manner of $L\,(u,\overline{\lambda}) = \lambda_0 \, L\,(u,\lambda)$, where $\lambda_i = \overline{\lambda}_i / \lambda_0$, function $L\,(u,\lambda)$ is defined by formula (3). In this case the main lemma and the main theorem are true for Lagrange's function in type (11).

3) We consider the following optimization problems: $J(u) \to \inf, \; u \in U$, $J_1(u) \to \inf, \; u \in U$, where function $J_1(u) \leq J(u) + \sum_{i=1}^{s} \lambda_i^* g_i(u).$ As follows from proof of the main theorem the conditions $J_1(u) \leq J(u), \;\; \forall u \in U; \; J_1(u_*) = J(u_*),$ $u_* \in U$ are executed.

4) For existence of the saddle point to Lagrange's function necessary that set $U_* \neq \varnothing$. However this condition does not guarantee existence of the saddle point to Lagrange's function for problem $J(u) \to \inf, \; u \in U$. It is required to impose the additional requirements on function $J(u)$ and set U that reduces efficiency of the Lagrange's coefficients method. We notice, that Lagrange's coefficients method is an artificial technique solution of the optimization problems requiring spare additional condition to problem.

Kuhn-Tucker's theorem. The question arises: what additional requirements are imposed on function $J(u)$ and on set U that Lagrange's function (3) would have a saddle point?

Theorems in which becomes firmly established under performing of which conditions Lagrange's function has the saddle point are called Kuhn-Tacker's theorems (Kuhn and Tucker - American mathematicians).

It can turn out to be that source problem $J(u) \to \inf,\ u \in U$, $U_* \neq \varnothing$, has a solution, i.e. the point $u_* \in U_*$ exists, however Lagrange's function for the given problem has not saddle point.

Example. Let function $J(u) = u - 1$, but set $U = \{u \in E^1 / 0 \leq \leq u \leq 1;\ (u-1)^2 \leq 0\}$. The function $g(u) = (u-1)^2$, set $U_0 = = \{u \in E^1 / 0 \leq u \leq 1\}$, moreover functions $J(u)$ and $g(u)$ are convex on set U_0. Since set U consists of single element, i.e. $U = \{1\}$, that set $U_* = \{1\}$. Consequently, the point $u_* = 1$. Lagrange's function $L\,(u, \lambda) = (u-1) + \lambda(u-1)^2,\ \lambda \geq 0,\ u \in U_0$ for the problem has not saddle point. In fact, from formula (4) follows

$$L\,(u_*, \lambda) = 0 \leq L\,(u_*, \lambda^*) = 0 \leq L\,(u, \lambda^*) = (u-1) + \lambda^*(u-1)^2,$$
$$\lambda \geq 0,\quad 0 \leq u \leq 1.$$

Thence we have $0 \leq -\varepsilon + \lambda^* \varepsilon^2$, where $u - 1 = -\varepsilon,\ 0 \leq \varepsilon \leq 1$. Under sufficiently small $\varepsilon > 0$ does not exist the number $\lambda^* \geq 0$ such the given inequality is executed.

1-st case. We consider the next problem of the convex programming:

$$J(u) \to \inf; \tag{12}$$

$$u \in U = \left\{u \in E^n \middle/ u \in U_0,\ g_i(u) \leq 0,\ i = \overline{1,m}\right\}, \tag{13}$$

where $J(u),\ g_i(u),\ i = \overline{1,m}$ - convex functions determined on convex set U_0.

Definition 2. *If the point $u^i \in U_0$ such that value $g_i(u^i) < 0$ exists, that it is spoken that restriction $g_i(u) \leq 0$ on set is regularly. It is spoken that set U is regularly, if all restrictions $g_i(u) < 0,\ i = \overline{1,m}$ from the condition (13) are regular on set U_0.*

We suppose, that point $\bar{u} \in U_0$ exists such that

$$g_i(\bar{u}) < 0, \; i = \overline{1,m}. \tag{14}$$

The condition (14) is called by Sleyter's condition.

Let in the points $u^1, u^2, \ldots, u^k \in U_0$ all restriction are regularly, i.e. $g_j(u^i) < 0, \; j = \overline{1,m}, \; i = \overline{1,k}$. Then in the point $\bar{u} = \sum_{i=1}^{k} \alpha_i u^i \in U_0$, $\alpha_i \geq 0, \; i = \overline{1,k}, \; \alpha_1 + \ldots \alpha_k = 1$ all restrictions $g_j(u) \leq 0, \; j = \overline{1,m}$, are regular. In fact,

$$g_j(\bar{u}) = g_j\left(\sum_{i=1}^{k} \alpha_i u^i\right) \leq \sum_{i=1}^{k} \alpha_i g_j(u^i) < 0, \; j = \overline{1,m}.$$

Theorem 1. *If $J(u)$, $g_i(u)$, $i = \overline{1,m}$ - convex functions determined on convex set U_0, the set U is regularly and $U_* \neq \varnothing$, then for each point $u_* \in U_*$ necessary the Lagrangian multipliers $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \in \Lambda_0 = \{\lambda \in E^m / \lambda_1 \geq 0, \ldots, \lambda_m \geq 0\}$ such that pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ forms a saddle point to Lagrange's function $L(u, \lambda) = J(u) + \sum_{i=1}^{m} \lambda_i g_i(u), \; u \in U_0, \; \lambda \in \Lambda_0$ exist.*

As follows from condition of the theorem, for the convex programming problem (12), (13) Lagrange's function has a saddle point, if set U is regularly (under performing the condition that set $U_* \neq \varnothing$). We note, that in the example is represented above set U is not regular, since in the single point $u = 1 \in U_0$ restriction $g(1) = 0$. We notice, that if set is regularly, so as follows from Sleyter's condition (14), the point $\bar{u} \in U \subset U_0$. Proof of Theorem 1 is provided in the following lectures.

Lectures 9, 10

KUHN-TUCKER'S THEOREM

For problem of the convex programming the additional conditions imposed on convex set U under performing of which Lagrange's function has a saddle point are received. We remind that for solving of the optimization problems in finite-dimensional space by method of the Lagrangian multipliers necessary, except performing the condition $U_* \neq \varnothing$, existence of the saddle point to Lagrange's function. In this case, applying of the main theorem is correct.

Proof of the theorem 1. Proof is conducted on base of the separability theorem of the convex sets *A* and *B* in space E^{m+1}. We define as follows the sets *A* and *B*:

$$A = \{a = (a_0, a_1, \dots, a_m) \in E^{m+1} / a_0 \geq J(u),$$
$$a_1 \geq g_1(u) \dots, a_m \geq g_m(u), \ u \in U_0\}; \qquad (15)$$

$$B = \{b = (b_0, b_1, \dots, b_m) \in E^{m+1} / b_0 < J_*, \ b_1 < 0 \dots, b_m < 0\}, \quad (16)$$

where $J_* = \inf_{u \in U} J(u) = \min_{u \in U} J(u)$ since set $U_* \neq \varnothing$.

a) We show that sets *A* and *B* have a no points in common. In fact, from the inclusion $a \in A$ follows that $a_0 \geq J(u)$, $a_1 \geq g_1(u), \dots, a_m \geq g_m(u), \ u \in U_0$. If the point $u \in U \subset U_0$, that inequality $a_0 \geq J(u) \geq J_*$ is executed, consequently, the point $a \notin B$. If the point $u \in U_0 \setminus U$, then for certain number *i*, from $1 \leq i \leq m$ is $g_i(u) > 0$ the inequality is executed. Then $a_i \geq g_i(u) > 0$ and once again the point $a \notin B$. Finally, the intersection $A \cap B = \varnothing$.

b) We show that sets *A* and *B* are convex. We take two arbitrary points $a \in A, d \in A$ and the number $\alpha, \alpha \in [0,1]$. From inclusion $a \in A$ follows that the point $u \in U_0$ such that $a_0 \geq J(u)$, $a_i \geq g_i(u), \quad i = \overline{1,m}, \quad a = (a_0, a_1, \ldots, a_m)$ exists. Similarly from $d \in A$ we have $d_0 \geq J(v),\ d_i \geq g_i(v),\ i = \overline{1,m},$ where $v \in U_0$, $d = (d_0, d_1, \ldots, d_m)$. Then the point $a_\alpha = \alpha a + (1-\alpha)d =$ $= (\alpha\, a_0 + (1-\alpha)d_0, \alpha\, a_1 + (1-\alpha)d_1, \ldots, \alpha\, a_m + (1-\alpha)d_m), \alpha \in [0,1]$, moreover $\alpha\, a_0 + (1-\alpha)d_0 = a_{0\alpha} \geq \alpha J(u) + (1-\alpha)J(v) \geq J(u_\alpha)$, $i = \overline{1,m}$ with consideration of convexity $J(u)$ on U_0, where $u_\alpha = \alpha\, u + (1-\alpha)v$. Similarly

$$\alpha\, a_i + (1-\alpha)d_i = a_{i\alpha} \geq \alpha\, g_i(u) + (1-\alpha)g_i(v) \geq g_i(u_\alpha),$$

$$i = \overline{1,m},$$

with consideration of function convexity $g_i(u),\ i = \overline{1,m}$ on set U_0. Finally, the point $a_\alpha = (a_{0\alpha}, a_{1\alpha}, \ldots, a_{m\alpha}),\ a_{0\alpha} \geq J(u_\alpha),\ a_{i\alpha} \geq$ $\geq g_i(u_\alpha),\ i = \overline{1,m},$ moreover $u_\alpha \in U_0$. This implies that point $a_\alpha \in A$ under all $\alpha,\ 0 < \alpha < 1$. Consequently set *A* is convex. By the similar way it is possible to prove the convexity of the set *B*.

c) Since sets A and *B* are convex and $A \cap B = \varnothing$, that, by theorem 2 (the lecture 6), hyperplane $\langle c, w \rangle = \gamma$, where normal vector $c = (\lambda_0^*, \lambda_1^* \ldots, \lambda_m^*) \in E^{m+1},\ |c| \neq 0,\ w \in E^{m+1}$, separating sets *A* and *B*, as well as its closures $A = \overline{A},\ \overline{B} = \{b = (b_0,$ $b_1, \ldots, b_m) \in E^{m+1} / b_0 \leq J_*,\ b_1 \leq 0 \ldots, b_m \leq 0\}$ exists. Consequently, inequality are executed

$$\langle c, b \rangle = \sum_{i=0}^{m} \lambda_i^* b_i \leq \gamma \leq \langle c, a \rangle = \sum_{i=0}^{m} \lambda_i^* a_i,\ \forall a \in A,\ \forall b \in \overline{B}\ . \qquad (17)$$

We notice that if the point $u_* \in U_*$, that $J(u_*) = J_*$, $g_i(u_*) \le 0,\ i = \overline{1,m}$, consequently, the vector $y = (J_*, 0, \ldots, 0) \in$ $\in \overline{A} \cap \overline{B}$ and the value $\gamma = \langle c, y \rangle = \lambda_0^* J_*$.

Now inequalities (17) are written as

$$\lambda_0^* b_0 + \sum_{i=0}^{m} \lambda_i^* b_i \le \gamma = \lambda_0^* J_* \le \lambda_0^* a_0 + \sum_{i=0}^{m} \lambda_i^* a_i,$$

$$a \in A,\ b \in \overline{B}. \tag{18}$$

Thence, in particular, when vector $b = (J_* - 1, 0, \ldots, 0) \in \overline{B}$, from the left inequality we have $\lambda_0^*(J_* - 1) \le \lambda_0^* J_*$. Consequently, value $\lambda_0^* \ge 0$. Similarly, choosing vector $b = (J_*, 0, \ldots, 0, -1, 0 \ldots, 0)$ from the left inequality we get $\lambda_0^* J_* - \lambda_i^* \le \lambda_0^* J_*$. This implies that $\lambda_i^* \ge 0,\ i = \overline{1,m}$.

d) We take any point $u_* \in U_*$. It is easy to make sure in that point $b = (J_*, 0, \ldots, 0, g_i(u_*), 0 \ldots, 0) \in \overline{A} \cap \overline{B}$, since value $g_i(u_*) \le 0$. Substituting the point to the left and right inequalities (18), we get $\lambda_0^* J_* + \lambda_i^* g_i(u_*) \le \lambda_0^* J_* \le \lambda_0^* J_* + \lambda_i^* g_i(u_*)$. Thence we have $\lambda_i^* g_i(u_*) \le 0 \le \lambda_i^* g_i(u_*)$. Consequently, the values $\lambda_i^* g_i(u_*) = 0$, $i = \overline{1,m}$.

f) We show, that value $\lambda_0^* > 0$. In item c) it is shown that $\lambda_0^* \ge 0$. By condition of the theorem set U is regular, i.e. the Sleyter's condition is executed (14). Consequently, the point $\overline{u} \in U \subset U_0$ such that $g_i(\overline{u}) < 0,\ i = \overline{1,m}$ exists. We notice, that point $a = (J(\overline{u}), g_1(\overline{u}), \ldots,$ $g_m(\overline{u})) \in A$. Then from the right inequality (18) we have $\lambda_0^* J_* < \lambda_0^* J_*(\overline{u}) + \lambda_1^* g_1(\overline{u}) + \ + \lambda_m^* g_m(u)$.

We suppose opposite i.e. $\lambda_0^* = 0$. Since vector

$c=(\lambda_0^*,\lambda_1^*,\ldots,\lambda_m^*)\neq 0$, that at least one of the numbers $\lambda_i^*,\, 1\le i\le m$ are different from zero. Then from the previous inequality we have $0\le \lambda_i^* g_i(\bar{u})<0$. It is impossible, since $\lambda_i^*\neq 0,\ \lambda_i^*\ge 0$. Finally, the number λ_0^* is not equal to zero, consequently, $\lambda_0^*>0$. Not derogating generalities, it is possible to put $\lambda_0^*=1$.

g) Let $u\in U_0$ be an arbitrary point. Then vector $a=(J(u),$ $g_1(u),\ldots,g_m(u))\in A$, from the right inequality (18) we get $J_*\le J(u)+\sum_{i=1}^{m}\lambda_i^* g_i(u)=L\left(u,\lambda^*\right),\forall u\in U_0$. Since the product $\lambda_i^* g_i(u_*)=0,\ i=\overline{1,m}$, that $J_*+\sum_{i=1}^{m}\lambda_i^* g_i(u)=L\left(u_*,\lambda^*\right)$ $\le L\left(u,\lambda^*\right),\ \le L\left(u,\lambda^*\right),\ \forall u\in U_0$. From the inequality and from equality $\lambda_i^* g_i(u_*)=0,\ i=\overline{1,m},\ u_*\in U$, where $\lambda_i^*\ge 0,\ i=\overline{1,m}$ follows that pair $\left(u_*,\lambda^*\right)\in U_0\times\Lambda_0$ - saddle point to Lagrange's function. Theorem is proved.

2-nd case. Now we consider the next problem of the convex programming:

$$J(u)\to\inf, \tag{19}$$

$$u\in U=\left\{u\in E^n\big/\, u_j>0,\ j\in I,\ \ g_i(u)=\langle a_i,u\rangle-b_i\le 0,\ i=\overline{1,m};\right.$$

$$\left. g_i(u)=\langle a_i,u\rangle-b_i=0,\ i=\overline{m+1,s}\right\}, \tag{20}$$

where $I\subset\{1,2,\ldots,n\}$ - a subset; the set $U_0=\{u\in E^n/u_j\ge$ $\ge 0,j\in I\}$, $J(u)$ - a convex function determined on convex set U_0, $a_i\in E^n$, $i=\overline{1,s}$ - the vectors; $b_i,\, i=\overline{1,s}$ - the numbers. For problem (19), (20) Lagrange's function

$$L(u,\lambda) = J(u) + \sum_{i=1}^{m} \lambda_i g_i(u), \ u \in U_0, \tag{21}$$

$$\lambda = (\lambda_1, \ldots, \lambda_s) \in \Lambda_0 = \{\lambda \in E^s / \lambda_1 \ge 0, \ldots, \lambda_m \ge 0\}.$$

We notice that, if function $J(u) = c'u$ is linear, the task (19), (20) is called the general problem of the linear programming.

We suppose, that set $U_* \neq \varnothing$. It is appeared that for task of the convex programming (19), (20) Lagrange's function (21) always has saddle point without some additional requirements on convex set U. For proving of this it is necessary the following lemma.

Lemma. *If* $a_1, \ldots, a_p$ *- a finite number of the vectors from* E^n *and the numbers* $\alpha_i \ge 0, \ i = \overline{1,p}$*, that set*

$$Q = \left\{ a \in E^n \Big/ a = \sum_{i=1}^{p} \alpha_i a_i, \ \alpha_i \ge 0, \ i = \overline{1,p} \right\} \tag{22}$$

Is convex closed cone.

Proof. We show, that set Q is a cone. In fact, if $a \in Q$ and $\lambda > 0$ - an arbitrary number, that $\lambda a = \sum_{i=1}^{p} \lambda \alpha_i a_i = \sum_{i=1}^{p} \beta_i a_i$, where $\beta_i = \lambda_i \alpha_i \ge 0, \ i = \overline{1,p}$. This implies that vector $\lambda a \in Q$. Consequently, set Q is the cone.

We show, that Q is a convex cone. In fact, from $a \in Q, \ b \in Q$ follows that

$$\alpha a + (1-\alpha) b = \sum_{i=1}^{p} \left[\alpha \alpha_i a_i + (1-\alpha) \overline{\beta}_i a_i \right] =$$

$$= \sum_{i=1}^{p} (\alpha \alpha_i + (1-\alpha) \overline{\beta}_i) a_i = \sum_{i=1}^{p} \gamma_i a_i, \gamma_i = \alpha \alpha_i + (1-\alpha) \overline{\beta}_i \ge 0,$$

$i = \overline{1,m}$, where $\alpha \in [0,1]$, $b = \sum_{i=1}^{p} \overline{\beta}_i a_i, \ \overline{\beta}_i \ge 0, \ i = \overline{1,p}$.

Thence we have $\alpha a+(1-\alpha)b\in Q$ under all $\alpha, 0\le\alpha\le 1$. Consequently, set Q - a convex cone.

We show, that Q is convex closed cone. We prove this by the method of mathematical induction. For values $p=1$ we have $Q=\left\{a\in E^n \middle/ a=\alpha a_1,\ \alpha\ge 0\right\}$ - a half-line. Therefore, Q - closed set. We suppose, that set $Q_{p-1}=\left\{a\in E^n \middle/ a=\sum_{i=1}^{p}\alpha_i a_i,\ \alpha_i\ge 0\right\}$ is closed. We prove, that set $Q=Q_{p-1}+\beta a_p, \beta\ge 0$ is closed. Let $c\in E^n$ is limiting point of the set Q. Consequently, the sequence $\{c_m\}\subset Q$ exists, moreover $c_m\to c,\ m\to\infty$. The sequence $\{c_m\}$ is represented in the manner of $c_m=b_m+\beta_m a_p$, where $\{b_m\}\subset Q_{p-1},\ \{\beta_m\}$ - a numeric sequences. As far as set Q_{p-1} is closed, that $b_m\to b,\ m\to\infty$, under $b\in Q_{p-1}$. It is remains to prove that numeric sequence $\{\beta_m\}$ is bounded. We suppose opposite i.e. $\beta_{m_k}\to\infty$ under $k\to\infty$. Since $b_{m_k}/\beta_{m_k}=c_{m_k}/\beta_{m_k}-a_p$ and $\{b_{m_k}/\beta_{m_k}\}\subset Q_{p-1}$ the sequence $\{c_{m_k}\}$ is bounded, so far as $c_{m_k}\to c,\ k\to\infty$, then under $k\to\infty$ we have $-a_p\in Q_{p-1}$. Since set $Q_{p-1}\subseteq Q$ (in the event of $\beta=0,\ Q_{p-1}=Q$), that vector $-a_p\in Q$. It is impossible. Therefore, the sequence $\{\beta_m\}$ is bounded. Consequently, vector $c=b+\beta a_p\in Q$, where $\beta_{m_k}\to\beta, \beta\ge 0$ under $k\to\infty$. Theorem is proved.

We formulate and prove Farkas's theorem having important significant in theories of the convex analysis previously than prove the theorem about existence of saddle point to Lagrange's function for task (19), (20).

Farkas' theorem. *If cone K with vertex in zero is determined by inequalities*

$$K = \left\{ e \in E^n \middle/ \langle c_i, e\rangle \le 0,\ i = \overline{1,m};\ \langle c_i, e\rangle < 0,\ i = \overline{m+1,p}; \right.$$

$$\left. \langle c_i, e\rangle = 0,\ i = \overline{p+1,s},\ \ c_i \in E^n,\ i = \overline{1,s} \right\}, \qquad (23)$$

the dual to it cone $K^* = \left\{ c \in E^n / \langle c, e\rangle \ge 0, \forall e \in K \right\}$ *has the type*

$$K^* = \left\{ c \in E^n \middle/ c = -\sum_{i=1}^{s} \lambda_i c_i,\ \lambda_1 \ge 0, \ldots, \lambda_p \ge 0 \right\}. \qquad (24)$$

Proof. Let cone K is defined by formula (23). We show, that set of the vectors $c \in K^*$ for which $\langle c, e\rangle \ge 0, \forall e \in K$ is defined by formula (24), i.e. set

$$Q = \left\{ c \in E^n \middle/ c = -\sum_{i=1}^{s} \lambda_i c_i,\ \lambda_1 \ge 0, \ldots, \lambda_p \ge 0 \right\} \qquad (25)$$

complies with $K^* = \left\{ c \in E^n / \langle c, e\rangle \ge 0, \forall e \in K \right\}$. We represent $\lambda_i = \alpha_i - \beta_i$, $\alpha_i \ge 0$, $\beta_i \ge 0$, $i = \overline{p+1,s}$. Then set Q from (25) is written as

$$Q = \left\{ c \in E^n \middle/ c = \sum_{i=1}^{p} \lambda_i (-c_i) + \sum_{i=p+1}^{s} \alpha_i (-c_i) + \sum_{i=p+1}^{s} \beta_i (c_i), \right.$$

$$\left. \lambda_i \ge 0,\ i = \overline{1,p},\ \alpha_i \ge 0,\ i = \overline{p+1,s},\ \beta_i \ge 0,\ i = \overline{p+1,s} \right\}. \qquad (26)$$

As follows from expressions (26) and (22) the set Q is convex closed cone generated by the vectors $-c_1, \ldots, -c_p, -c_{p+1}, \ldots,$- $-c_s, c_{p+1}, \ldots, c_s$ (refer to the lemma).

We show, that $Q \subseteq K^*$. In fact, if $c \in Q$, i.e. $c = -\sum_{i=1}^{s} \lambda_i c_i$,

$\lambda_1 \geq 0,\ldots,\lambda_p \geq 0$, that product $\langle c,e\rangle = \; = -\sum_{i=1}^{s}\lambda_i\langle c_i,e\rangle \geq 0$ with consideration of correlations (23). Consequently, vector $c \in Q$ belongs to set

$$K^* = \left\{c \in E^n \big/ \left\langle c,e\right\rangle \geq 0,\ \forall e \in K\right\}.$$

This implies that $Q \subseteq K^*$.

We show, that $K^* \subseteq Q$. We suppose opposite, i.e. vector $a \in K^*$, however $a \notin Q$. Since set Q – convex closed cone and the point $a \notin Q$, that by theorem 1 (the lecture 6) the point $a \in E^n$ is strongly separable from set Q i.e. $\left\langle d,c\right\rangle > \left\langle d,a\right\rangle, \forall c \in Q$, where - d, $\left|d\right| \neq 0$ is a normal vector to the hyperplane $\left\langle d,u\right\rangle = \gamma = \left\langle d,a\right\rangle$. Thence we have

$$\left\langle d,c\right\rangle = -\sum_{i=1}^{s}\lambda_i\left\langle d,c_i\right\rangle > \left\langle d,a\right\rangle,\ \forall \lambda_i, i=\overline{1,s},$$
$$\lambda_1 \geq 0,\ldots,\lambda_p \geq 0 \cdot \qquad (27)$$

We choose the vector $\lambda = (\lambda_1 \geq 0,\ldots,\lambda_p \geq 0, \lambda_{p+1},\ldots,\lambda_s)$ so:

$$\lambda = (\lambda_1,\ldots,\lambda_s) = \begin{cases} \lambda_i = 0, & i \neq j,\ i=\overline{1,s} \\ \lambda_j = t,\ t > 0, & j\ из\ 1 \leq j \leq p. \end{cases}$$

For the vector λ inequality (27) is written in the form $-t\left\langle d,c_i\right\rangle > \left\langle d,a\right\rangle$.. We divide into $t > 0$ and $t \to +\infty$, as a result we get $\left\langle c_j,d\right\rangle \leq 0,\ j=\overline{1,p}.$ Hereinafter, we take the vector

$$\lambda = (\lambda_1,\ldots,\lambda_s) = \begin{cases} \lambda_i = 0, & i \neq j,\ i=\overline{1,s}, \\ \lambda_j = t\left\langle c_j,d\right\rangle,\ t > 0, & j\ из\ p+1 \leq j \leq s. \end{cases}$$

From inequality (27) we have $-t\left|\langle c_j,d\rangle\right|^2 > \langle d,a\rangle$. We divide into $t>0$ and, directing $t\to\infty$, we get $\langle c_j,d\rangle = 0,\ j=\overline{p+1,s}$. Finally, for the vector $d\in E^n$ inequalities $\langle c_j,d\rangle \le 0,\quad j=\overline{1,p},\ \langle c_j,d\rangle = 0,$ $j=\overline{p+1,s}$ are executed. Then from (23) follows that $d\in\overline{K}$, where $\overline{K}$ - a closure of the set K.

Since the vector $a\in K^*$, that the inequality $\langle a,e\rangle \ge 0, \forall e\in K$ exists. Thence, in particular, for $e=d\in\overline{K}$ follows $\langle a,d\rangle \ge 0$. However from (27) under $\lambda_i=0,\ i=\overline{1,s}$, we have $\langle a,d\rangle < 0$. We have got the contradiction. Consequently, vector $a\in K^*$ belongs to the set *Q*. From inclusions $Q\subseteq K^*$, $K^*\subseteq Q$ follows that $K^*=Q$. Theorem is proved.

Now we formulate theorem about existence of the saddle point to Lagrange's function (21) for problem of the convex programming (19), (20).

Theorem 2. *If $J(u)$ is a convex function on convex set U_0, $J(u)\in C^1(U_0)$ and set $U_*\ne\varnothing$ for problems (19), (20), then for each point $u_*\in U_*$ necessary the Lagrangian multipliers $\lambda^*=(\lambda_1^*,\ldots,\lambda_s^*)\in\Lambda_0=\{\lambda\in E^s / \lambda_1\ge 0,\ldots,\lambda_m\ge 0\}$ exist, such that pair $(u_*,\lambda^*)\in U_0\times\Lambda_0$ forms a saddle point to Lagrange's function (21) on set $U_0\times\Lambda_0$.*

Proof. Let the condition of the theorem is executed. We show the pair $(u_*,\lambda^*)\in U_0\times\Lambda_0$ - saddle point to Lagrange's function (21). Let $u_*\in U_*$ - an arbitrary point. We define the feasible directions coming from point u_* for convex set U. We notice that the vector $e=(e_1,\ldots,e_n)\in E^n$ is called by the feasible direction in the point u_*, if the number $\varepsilon_0>0$ such that $u=u_*+\varepsilon e\in U$ under all $\varepsilon,\ 0\le\varepsilon\le\varepsilon_0$ exists. From inclusion $u=u_*+\varepsilon e\in U$ with consideration of the expressions (20) we have

$$u_j^* + \varepsilon e_j \geq 0,\ \ j \in I;\ \ \langle a_i, u_* + \varepsilon e\rangle - b_i \leq 0,\ \ i = \overline{1, m};$$
$$\langle a_i, u_* + \varepsilon e\rangle - b_i = 0,\ \ i = \overline{m+1, p},\ \ \forall \varepsilon,\ \ 0 \leq \varepsilon \leq \varepsilon_0. \tag{28}$$

If the sets of the indexes $I_1 = \{j / u_j^* = 0, j \in I\}$, $I_2 = \{j / \langle a_i, u_*\rangle = b_i, 1 \leq i \leq m\}$ are entered, that conditions (28) are written so:

$$e_j \geq 0,\ \ j \in I_1;\ \langle a_i, e\rangle \leq 0,\ \ i \in I_2;$$
$$\langle a_i, e\rangle = 0,\ i = \overline{m+1, s}. \tag{29}$$

Finally, the set of the feasible directions in the point u_* according to correlation (29) is a cone

$$K = \{e \in E^n / -e_j = \langle -e^j, e\rangle \leq 0,\ \ j \in I_1;\ \ \langle a_i, e\rangle \leq 0,\ \ i \in I_2,$$
$$\langle a_i, e\rangle = 0,\ \ i = \overline{m+1, s}\}, \tag{30}$$

where $e^j = (0, \ldots, 0, 1, 0, \ldots, 0) \in E^n$ - a single vector. And inverse statement faithfully, i.e. if $e \in K$, that e - feasible direction.

Since $J(u)$ is a convex function on convex set $U \subset U_0$ and $J(u) \in C^1(U)$, that by theorem 4 (the lecture 5) in the point $u_* \in U_*$, necessary and sufficiently executing of the inequality $\langle J'(u_*), u - u_*\rangle \geq 0, \forall u \in U$. Thence with consideration of that $u - u_* = \varepsilon e,\ \ 0 \leq \varepsilon \leq \varepsilon_0,\ \ e \in K$, we have $\langle J'(u_*), e\rangle \geq 0, \forall e \in K$. Consequently, the vector $J'(u_*) \in K^*$. By Farkas's theorem dual cone K^* to cone (30) is defined by the formula (24), so the numbers $\mu_j^* \geq 0,\ \ j \in I_1;\ \lambda_i^* \geq 0,\ i \in I_2;\ \lambda_{m+1}^*, \ldots, \lambda_s^*$ such that

$$J'(u_*) = \sum_{j\in I_1} \mu_j^* e^j - \sum_{i\in I_2} \lambda_i^* a_i - \sum_{i=m+1}^{s} \lambda_i^* a_i \qquad (31)$$

exist.

Let $\lambda_i^* = 0$ for values $i \in \{1,2,\ldots,m\} \setminus I_2$. Then expression (31) is written as

$$J'(u_*) + \sum_{i=1}^{s} \lambda_i^* a_i = \sum_{j\in I_1} \mu_j^* e^j \,. \qquad (32)$$

We notice that $\lambda_i^* g_i(u_*) = 0,\ i = \overline{1,s}$, since $g_i(u_*) = 0,\ i \in I_2$ and $i = \overline{m+1,s}$. As follows from the expression (21) $\forall u \in U_0$ the difference

$$L\left(u, \lambda^*\right) - L\left(u_*, \lambda^*\right) = J(u) - J(u_*) + \sum_{i=1}^{s} \lambda_i^* \langle a_i, u - u_* \rangle . \qquad (33)$$

Since convex function $J(u) \in C^1(U)$, then according to the theorem 1 (the lecture 4) the difference $J(u) - J(u_*) \geq \langle J'(u_*),$ $u - u_* \rangle,\ \ \forall u \in U_0$. Now inequality (33) with consideration of correlations (32) can be written in the manner of

$$L\left(u, \lambda^*\right) - L\left(u_*, \lambda^*\right) \geq \left\langle J(u_*) + \sum_{i=1}^{s} \lambda_i^* a_i, u - u_* \right\rangle =$$

$$= \sum_{j\in I_1} \mu_j^* \langle e^j, u - u_* \rangle = \sum_{j\in I_1} \mu_j^* (u_j - u_j^*) = \sum_{j\in I_1} \mu_j^* u_j \geq 0,$$

since $\langle e^j, u - u_* \rangle = u_j - u_j^*,\ \ u_j^* = 0,\ j \in I_1$. This implies that $L\left(u_*, \lambda^*\right) \leq L\left(u, \lambda^*\right),\ \forall u \in U_0$. From the inequality and equality $\lambda_i^* g_i(u_*) = 0,\ i = \overline{1,s}$ follows that pair $\left(u_*, \lambda^*\right) \in U_0 \times \Lambda_0$ - saddle point to Lagrange's function (21). Theorem is proved.

3-rd case. We consider more general problem of the convex programming:

$$J(u) \to \inf, \qquad (34)$$

$$u \in U = \{u \in E^n / \, u \in U_0, \; g_i(u) \le 0, \; i = \overline{1,m}; \; g_i(u) = \langle a_i, u \rangle -$$

$$-b_i \le 0, \; i = \overline{m+1, p}, \quad g_i(u) = \langle a_i, u \rangle - b_i = 0, \; i = \overline{p+1, s}\}, \qquad (35)$$

where $J(u), \; g_i(u), \; i = \overline{1,m}$ are convex functions determined on convex set U_0; $a_i \in E^n, \; i = \overline{m+1, s}$ are the vectors; $b_i, \; i = \overline{m+1, s}$ are the numbers. Lagrange's function for tasks (34), (35)

$$\begin{aligned} &L(u, \lambda) = J(u) + \sum_{i=1}^{s} \lambda_i g_i(u), \; u \in U_0, \\ &\lambda \in \Lambda_0 = \{\lambda \in E^s / \, \lambda_1 \ge 0, \ldots, \lambda_m \ge 0\}. \end{aligned} \qquad (36)$$

Theorem 3. *If $J(u), \; g_i(u), \; i = \overline{1,m}$ are convex functions determined on convex set U_0, set $U_* \ne \varnothing$ for tasks (34), (35) and the points $\bar{u} \in riU_0 \cap U$ such that $g_i(\bar{u}) < 0, \; i = \overline{1,m}$ exists, then for each point $u_* \in U_*$ necessary the Lagrangian multiplier $\lambda^* = (\lambda_1^*, \ldots, \lambda_s^*) \in \Lambda_0 = \{\lambda \in E^s / \lambda_1 \ge 0, \ldots, \lambda_p \ge 0\}$ exist, such that the pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ forms a saddle point to Lagrange's function (36) on set $U_0 \times \Lambda_0$.*

Proof of theorem requires of using more fine separability theorems of the convex set, than theorems 1 - 6 (the lectures 6, 7). For studying more general separability theorems of the convex set and other Kuhn-Tucker's theorems is recommended the book: Rocafellar R. Convex analysis. Moskow, 1973.

We note the following:

1^0. The theorems 1-3 give the sufficient conditions of existence of the saddle point for the convex programming problem.

Example 1. Let function be $J(u) = 1 - u$, but set $U = \{u \in$

$\in E^1 / 0 \le u \le 1;\ (1-u)^2 \le 0\}$. For this example the functions $J(u),\ g(u) = (1-u)^2$ are convex on convex set $U_0 = \{u \in E^1 / 0 \le \le u \le 1\}$. The set $U = \{1\}$, consequently $U_* = \{1\}$, set $U_* = \{1\}$. The point $u_* = 1$ and the value $J(u_*) = 0$ are solution of the problem $J(u) \to \inf,\ u \in U$. Lagrange's function $L(u, \lambda) = (1-u) + \lambda(1-u)^2,\ u \in U_0, \lambda \ge 0$. The pair $(u_* = 1, \lambda^* \ge 0)$ - saddle point to Lagrange's function, since $L(u_*, \lambda^*) = 0 \le L(u, \lambda^*) = (1-u) + \lambda^*(1-u)^2,\ 0 \le u \le 1,\ \lambda^* g(u^*) = 0$.

We notice, that for the example neither Sleyter's theorem from theorem 1, nor condition of theorem 3 is not executed.

2°. In general event Lagrangian multipliers for the point $u_* \in U_*$ are defined ambiguous. In specified example the pair $(u_* = 1, \lambda^* \ge 0)$ is the saddle point to Lagrange's function under any $\lambda^* \ge 0$.

3°. Performing of the theorems 1 - 3 condition guarantees the existence of the saddle point to Lagrange's function. Probably, this circumstance for problem as convex, so and nonlinear programming is a slight learned area in theories of the extreme tasks.

Solution algorithm of the convex programming problem. The problems of the convex programming written in the manner of (34), (35) often are occurred in the applied researches. On base of the theories stated above we briefly give a solution sequence of the convex programming problem.

1^0. To make sure in that for task (34), (35) set $U_* \ne \varnothing$. In order to use the theorems 1-3 from lecture 2 (Weierstrass' theorem).

2^0. To check performing the conditions of the theorems 1-3 in depending on type of the convex programming problem to be a warranty of existence saddle points to Lagrange's function. For instance, if task has the form of (12), (13), then to show that set U is regularly; if task has the form of (19), (20), that necessary $J(u) \in C^1(U_0)$, but for task (34), (35) necessary existence of the point $\bar{u} \in riU_0 \cap U$ for which $g_i(\bar{u}) < 0,\ i = \overline{1,m}$.

3^0. To form Lagrange's function $L\left(u,\lambda\right)=J(u)+\sum_{i=1}^{s}\lambda_i g_i(u)$ with definitional domain $U_0 \times \Lambda_0$, where

$$\Lambda_0=\left\{\lambda \in E^s / \lambda_1 \geq 0,\ldots,\lambda_m \geq 0\right\}.$$

4^0. To find a saddle point $\left(u_*,\lambda^*\right) \in U_0 \times \Lambda_0$ to Lagrange's function from the conditions (the main lemma from lecture 8):

$$\begin{aligned} &L\left(u_*,\lambda^*\right) \leq L\left(u,\lambda^*\right), \ \forall u \in U_0, \\ &\lambda_i^* g_i(u_*)=0, \ i=\overline{1,s}, \qquad \lambda_1^* \geq 0,\ldots,\lambda_m^* \geq 0. \end{aligned} \tag{37}$$

a) As follows from the first condition, function $L\left(u,\lambda^*\right)$ reaches the minimum on set U_0 in the point $u_* \in U_* \subset U_0$. Since functions $J(u), \ g_i(u), \ i=\overline{1,m}$ are convex on convex set U_0, that function $L\left(u,\lambda^*\right)$ is convex on U_0. If functions $J(u) \in C^1(U_0), \ i=\overline{1,m}$, then according to the optimality criterion (the lecture 5) and theorems about global minimum (the lecture 5) the first condition from (37) can be replaced on $\left\langle L_{\mathrm{u}}\left(u_*,\lambda^*\right), u-u_*\right\rangle \geq 0, \ \forall u \in U_0$, where $L_{\mathrm{u}}\left(u_*,\lambda^*\right)=J'(u_*)+\sum_{i=1}^{s}\lambda_i^* g_i'(u_*)$. Now condition (37) are written as

$$\begin{aligned} &\left\langle L_u\left(u_*,\lambda^*\right), u-u_*\right\rangle \geq 0, \ \forall u \in U_0, \\ &\lambda_i^* g_i(u_*)=0, \ i=\overline{1,s}, \ \lambda_1^* \geq 0,\ldots,\lambda_m^* \geq 0. \end{aligned} \tag{38}$$

b) If we except $J(u) \in C^1(U_0), \ \ g_i(u) \in C^1(U_0), \ \ i=\overline{1,m}$, set $U_0=E^n$, then according to optimality criteria (lecture 5) the conditions (38) possible present in the manner of

$$L_u\left(u_*,\lambda^*\right)=0,\ \ \lambda_i^* g_i(u_*)=0,\ \ i=\overline{1,s},$$
$$\lambda_1^* \geq 0,\ldots,\lambda_m^* \geq 0\,. \qquad (39)$$

The conditions (39) are represented by the system $n+s$ of the algebraic equations comparatively $n+s$ unknowns $u_*=\left(u_1^*,\ldots,u_n^*\right)$, $\lambda^*=\left(\lambda_1^*,\ldots,\lambda_n^*\right)$. We notice, that if Lagrange's function has the saddle point, that system of the algebraic equations (39) has a solution, moreover must be $\lambda_1^* \geq 0,\ldots,\lambda_m^* \geq 0$. Conditions (39) are used and in that events, when $U_0 \subset E^n$, however in this case necessary to make sure the point $u_* \in U_0$.

5°. We suppose, that pair $\left(u_*,\lambda^*\right)\in U_0 \times \Lambda_0$ is determined. Then the point $u_* \in U$ and value $J_*=J(u_*)$ is the solution of the problem (34), (35) (refer to the main theorem from lecture 8).

Chapter II. **NONLINEAR PROGRAMMING**

There are not similar theorems for problem of the nonlinear programming as for problems of the convex programming guaranteed existence of the saddle point to Lagrange's function. It is necessary to note that if some way is installed that pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ is the saddle point to Lagrange's function then according to the main theorem the point $u_* \in U_*$ is the point of the global minimum in the problems of the nonlinear programming. We formulate the sufficient conditions of the optimality for the nonlinear programming problem by means of generalized Lagrange's function. We notice, that point $u_* \in U$ determined from sufficient optimality conditions, in general event is not a solution of the problem, but only "suspected" point. It is required to conduct some additional researches; at least, to answer on the question: will be the point $u_* \in U$ by point of the local minimum to function $J(u)$ on ensemble U ?

Lectures 11, 12.

STATEMENT OF THE PROBLEM. NECESSARY CONDITIONS OF THE OPTIMALITY

Statement of the problem. The following problem often is occurred in practice:

$$J(u) \to \inf \ , \tag{1}$$

$$u \in U = \{u \in E^n \ / \ u \in U_0, g_i(u) \le 0, i = \overline{1,m}; g_i(u) =$$
$$= 0, i = \overline{m+1,s}\}, \tag{2}$$

where $J(u)$, $g_i(u)$, $i = \overline{1,s}$ - the functions determined on convex ensemble U_0 from E^n.

Entering notations

$$\begin{aligned} U_i &= \{u \in E^n / \ g_i(u) \le 0\}, \ i = \overline{1,m}, \\ U_{m+1} &= \{u \in E^n / \ g_i(u) = 0, \ i = \overline{m+1,s}\}, \end{aligned} \tag{3}$$

ensemble U possible to present in the manner of

$$\begin{aligned} U_i &= \{u \in E^n / \ g_i(u) \le 0\}, \ i = \overline{1,m}, \\ U_{m+1} &= \{u \in E^n / \ g_i(u) = 0, \ i = \overline{m+1,s}\}, \end{aligned} \tag{3}$$

Now problem (1), (2) can be written in such form:

$$J(u) \to \inf, \ u \in U$$

We suppose that $J_* = \inf J(u) > -\infty$, ensemble

$U_* = \left\{ u_* \in E^n / u_* \in U, \ \ J(u_*) = \min_{u \in U} J(u) \right\} \neq \varnothing$. We notice that if ensemble $U_* \neq \varnothing$, so $J(u_*) = \min_{u \in U} J(u)$. It is necessary to find the point $u_* \in U_*$ and value $J_* = J(u_*)$.

For problem (1), (2) generalized Lagrange's function has the form

$$
\begin{gathered}
L\,(u, \bar{\lambda}) = \lambda_0 J(u) + \sum_{i=1}^{s} \lambda_i g_i(u), \ \ u \in U_0, \ \bar{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_s), \\
\bar{\lambda} \in \Lambda_0 = \left\{ \bar{\lambda} \in E^{s+1} / \lambda_0 \geq 0, \lambda_1 \geq 0, \ldots, \lambda_m \geq 0 \right\}.
\end{gathered} \tag{5}
$$

Let U_{01} - an open ensemble containing set $U_0 \subset U_{01}$.

Theorem 1 *(the necessary optimality conditions). If functions* $J(u) \in C^1(U_{01}), \ \ g_i(u) \in C^1(U_{01}), \ \operatorname{int} U_0 \neq 0, \ U_0$ *is a convex ensemble, but ensemble* $U_* \neq \varnothing$, *then for each point* $u_* \in U_*$ *the Lagrange's multipliers* $\bar{\lambda}^* = (\lambda_0^*, \lambda_1^*, \ldots, \lambda_s^*) \in \Lambda_0$ *necessary exist, such that the following condition is executed:*

$$
\left| \bar{\lambda}^* \right| \neq 0, \ \ \lambda_0^* \geq 0, \ \lambda_1^* \geq 0, \ldots, \lambda_m^* \geq 0, \tag{6}
$$

$$
\begin{aligned}
&\left\langle L_u(u_*, \bar{\lambda}^*), u - u_* \right\rangle = \\
&\qquad = \left\langle \lambda_0^* J'(u_*) + \sum_{i=1}^{s} \lambda_i^* g_i'(u_*), u - u_* \right\rangle \geq 0, \ \forall u \in U_0,
\end{aligned} \tag{7}
$$

$$
\lambda_i^* g_i(u_*) = 0, \ \ i = \overline{1, s}. \tag{8}
$$

Proof of the theorem is released on theorems 5, 6 (the lecture 7) about condition of the emptiness intersection of the convex cones (Dubovicky-Milutin's theorem) and it is represented below. We comment the conditions of the theorem 1.

We note the following:

a) In contrast to similar theorems in the convex programming problems is not become firmly established that pair $(u_*, \overline{\lambda}^*) \in U_0 \times \Lambda_0$ is an saddle point to Lagrange's function, i.e. the conditions of the main theorem are not executed.

b) Since pair $(u_*, \overline{\lambda}^*)$ in general event is not a saddle point, then from the condition (6) - (8) does not follow that point $u_* \in U$ - a solution of the problem (1), (2).

c) If the value $\lambda_0^* > 0$, that problem (1), (2) is identified nondegenerate. In this case it is possible to take $\lambda_0^* = 1$, since Lagrange's function is linear function comparatively $\overline{\lambda}$.

g) If $\lambda_0^* = 0$, that problem (1), (2) is called degenerate.

Number of the unknown Lagrangian multipliers, independently the problem (1), (2) is degenerate or nondegenerate, possible to reduce on unit by entering the normalization condition, i.e. condition (6) can be changed on

$$\left|\overline{\lambda}^*\right| = \alpha, \ \ \lambda_0^* \geq 0, \ \lambda_1^* \geq 0, \ldots, \lambda_m^* \geq 0, \qquad (9)$$

where $\alpha > 0$ - any given number, in particular, $\alpha = 1$.

Example 1. Let $J(u) = u - 1$,

$$U = \left\{u \in E^1 / 0 \leq u \leq 1; \ \ g(u) = (u-1)^2 \leq 0\right\}$$

Generalized Lagrange's function for the problem has the form $L\ (u, \overline{\lambda}) = \lambda_0 J(u) + \lambda(u-1)^2$, $u \in U_0 = \left\{u \in E^1 / 0 \leq u \leq 1\right\}$, $\lambda_0 \geq 0$, $\lambda \geq 0$. Ensemble $U = \{1\}$, consequently, ensemble $U_* = \{1\}$ moreover $J(u_*) = 0$, $g(u_*) = 0$. The condition (7) is written as $(\lambda_0^* J'(u_*) + \lambda^* g'(u_*))(u - u_*) \geq 0, \ \forall u \in U_0$. Thence we have $(\lambda_0^*)(u-1) \geq 0, \forall u \in U_0$, or $(-\lambda_0^*)(1-u) \geq 0, \ \ 0 \leq u \leq 1$. Since expression $(1-u) \geq 0$ under $0 \leq u \leq 1$, then for executing the inequality necessary that value $\lambda_0^* = 0$, so as $\lambda_0^* \geq 0$ by the condition (6). Finally, the source problem is degenerate. The condition (8) is executed for any $\lambda^* > 0$. Thereby, all conditions of the theorem 1 are executed in the point $(u_* = 1,\ \lambda_0^* = 0,\ \lambda^* > 0)$. We notice, the ordinary Lagrange's function $L\,(u,\lambda) = (u-1) + \lambda(u-1)^2$ for the problem has not saddle point.

Example 2. Let $J(u) = J(u_1,u_2) = u_1 + \cos u_2$,

$$U = \{(u_1,u_2) \in E^2 / g(u_1) = -u_1 \le 0\}.$$

Ensemble $U_0 \subset E^2$. Generalized Lagrange's function

$$L(u_1,u_2,\lambda_0,\lambda) = \lambda_0(u_1 + \cos u_2) - \lambda u_1,\ \ \lambda_0 \ge 0,\ \ \lambda \ge 0,$$

$u = (u_1,u_2) \in E^2$. Since ensemble $U_0 \subset E^2$, that condition (7) is written as $L_u(u_*,\overline{\lambda}^*) = 0$. Thence follows that $\langle g_i'(u_*), e\rangle = 0$, $i = \overline{m+1,s}$. Consequently, $\lambda_0^* = \lambda > 0,\ u_2^* = \pi k,\ k = 0,\pm 1,\pm 2,\ldots,$. From condition (9) follows that it is possible to take $\lambda_0^* = \lambda^* = 1$, where $\alpha = \sqrt{2}$. From condition (8) we have $u_1^* = 0$. Finally, necessary conditions of the optimality (6)-(8) are executed in the point $u_* = (u_1^*, u_2^*) = (0, k\pi),\ \lambda_0^* = 1,\ \lambda^* = 1$. To define in which points from $\{(0,\pi k)\}$ is reached minimum $J(u)$ on U the additional researches is required.. It is easy to make sure in the points $u_* = (0, \pi(2m+1)) \subset \{(0,\pi k)\}$ minimum $J(u)$ on U is reached, where $m = 0, \pm 1, \ldots$.

It is required to construct of the cones to prove the theorem 1: K_y - directions of the function $J(u)$ decrease in the point u_*; K_{b_i} - internal directions of the ensembles U_i, $i = \overline{0,m}$ in point u_*; $K_{m+1} = K_k$ - tangent directions for ensemble U_{m+1} in the point u_*.

Cone construction. We define the cones K_y, K_{b_i}, $i = \overline{1,m}$, K_k in the point $u_* \in U$.

Definition 1. *By direction of the function decrease* $J(u)$ *in the point* $u_* \in U$ *is identified vector* $e \in E^n$, $|e| \neq 0$ *if the numbers* $\varepsilon_0 > 0,\ \delta > 0$ *exist such that under all* $\bar{e} = o(\delta, e) = \{\bar{e} \in E^n / /|\bar{e} - e| < \delta\}$ *the following inequality*

$$J(u_* + \varepsilon\bar{e}) < J(u_*), \quad \forall\varepsilon,\ 0 < \varepsilon < \varepsilon_0 \tag{10}$$

is executed.

We denote through $K_y = K_y(u_*)$ the ensemble of all directions of the function decrease $J(u)$ in the point u_*. Finally, the ensemble

$$K_y = \{e \in E^n / |\bar{e} - e| < \delta, \quad J(u_* + \varepsilon\bar{e}) < J(u_*), \quad 0 < \varepsilon < \varepsilon_0\}. \tag{11}$$

As follows from the expression (11), ensemble K_y contains together with the point e and its δ-a neighborhood, consequently, K_y- open ensemble. The point $e \in K_y$. Since function $J(u) \in C^1(U_{01})$, that difference [refer to the formula (10), $\bar{e} \in K_y$]:

$$J(u_* + \varepsilon e) - J(u_*) = \langle J'(u_* + \theta\varepsilon e), e\rangle \varepsilon < 0, \quad \forall\varepsilon,\ 0 < \varepsilon < \varepsilon_0.$$

We notice, that open ensemble U_{01} contains the neighborhood of the point $u_* \in U$. Thence, dividing in $\varepsilon > 0$ and directing $\varepsilon \to +0$, we get $\langle J'(u_*), e\rangle < 0$. Consequently, ensemble

$$K_y = \{e \in E^n / \langle J'(u_*), e\rangle < 0\} \tag{12}$$

- open convex cone. By Farkas's theorem the dual cone to cone (12) is defined on formula

$$K_y^* = \{c_y \in E^n / c_y = -\lambda_0^* J'(u_*),\ \lambda_0^* \geq 0\}. \qquad (13)$$

Definition 2. *By internal direction of the ensemble* U_i*, in the point* $u_* \in U$ *is identified the vector* $e \in E^n$, $|e| \neq 0$*, if there are the numbers* $\varepsilon_0 > 0, \delta > 0$*, such that under all* $\bar{e} \in o(\delta, e)$ *inclusion* $u_* + \varepsilon\bar{e} \in U_i$, $\forall \varepsilon,\ 0 < \varepsilon < \varepsilon_0$ *exists.*

We denote through $K_{b_i} = K_{b_i}(u_*)$ the ensemble of all internal ensemble U_i directions in the point $u_* \in U$. Finally, the ensemble

$$K_{b_i} = \{e \in E^n / |\bar{e} - e| < \delta,\ u_* + \varepsilon\bar{e} \in U_i,\ \forall \varepsilon,\ 0 < \varepsilon < \varepsilon_0\}. \qquad (14)$$

We notice, that K_{b_i} - the open ensemble moreover the point $\bar{e} \in K_{b_i}$. For problem (1), (2) ensemble U_i, $i = \overline{1,m}$ is defined by the formula (3), i.e.

$$U_i = \{u \in E^n / g_i(u) \leq 0\}, \quad i = \overline{1,m}$$

Then ensemble K_{b_i} defined by formula (14) can be written as

$$K_{b_i} = \{e \in E^n / g_i(u_* + \varepsilon\bar{e}) < 0,\ \forall \varepsilon,\ 0 < \varepsilon < \varepsilon_0\}. \qquad (15)$$

Since the point $u_* \in U$, that it is possible two events: 1) $g_i(u_*) < 0$; 2) $g_i(u_*) = 0$. We consider the event, when $g_i(u_*) < 0$. In this case, on the strength of function continuity $g_i(u)$ on ensemble U_0 the number $\varepsilon_0 > 0$ such that value $g_i(u_* + \varepsilon e) < 0$ under all $\varepsilon,\ 0 < \varepsilon < \varepsilon_0$ and for any vector $e \in E^n$ is found. Then ensemble $K_{b_i} = E^n$ - open cone, but dual to it cone $K_{b_i} = \{0\}$.

In the event $g_i(u_*) = 0$ we have $g_i(u_* + \varepsilon e) - g_i(u_*) < 0$

[refer to formula (15)]. Thence with consideration of that function $g_i(u) \in C^1(U_{01})$, we get $\langle g_i'(u_* + \theta\varepsilon e), e\rangle \varepsilon < 0, \ \forall \varepsilon, \ 0 < \varepsilon < \varepsilon_0$. We divide into $\varepsilon > 0$ and direct $\varepsilon \to +0$, as a result the given inequality is written in the manner of $\langle g_i'(u_*), e\rangle < 0$. Then open ensemble K_{b_i} is defined by formula

$$K_{b_i} = \{e \in E^n / \langle g_i'(u_*), e\rangle < 0\}, \ \ i = \overline{1, m}. \qquad (16)$$

By Farkas's theorem the dual cone to cone (16) is written as:

$$K_{b_i}^* = \{c_i \in E^n / c_i = -\lambda_i^* g_i'(u_*), \ \ \lambda_i^* \geq 0\}, \ \ i = \overline{1, m}. \qquad (17)$$

We notice, that K_{b_i}, $i = \overline{1, m}$ - opened convex cones.

Definition 3. *By tangent direction to ensemble U_{m+1} in the point $u_* \in U$ is identified the vector $e \in E^n$, $|e| \neq 0$, if the number $\varepsilon_0 > 0$ and function $r(\varepsilon) = (r_1(\varepsilon), \ldots, r_n(\varepsilon)), \ 0 \leq \varepsilon \leq \varepsilon_0$ such that $r(0) = 0; \ r(\varepsilon)/\varepsilon \to 0$ under $\varepsilon \to +0$ and $u_* + \varepsilon e + r(\varepsilon) \in U_{m+1}$, $\forall \varepsilon, \ 0 < \varepsilon < \varepsilon_0$ exist.*

We denote through $K_{m+1} = K_{m+1}(u_*)$ the ensemble of all tangent directions of the ensemble U_{m+1} in the point $u_* \in U$. From given determinations follows that ensemble

$$K_{m+1} = \{e \in E^n / u_* + \varepsilon e + r(\varepsilon) \in U_{m+1}, \ \ \forall \varepsilon, \ 0 < \varepsilon < \varepsilon_0; \\ r(0) = 0, \ r(\varepsilon)/\varepsilon \to 0 \ \text{при} \ \varepsilon \to +0\}.$$

According to formula (3) the ensemble

$$U_{m+1} = \{u \in E^n / \ g_i(u) = 0, \ i = \overline{m+1, s}\},$$

consequently,

$$K_{m+1} = \{e \in E^n / g_i(u_* + \varepsilon e + r(\varepsilon)) = 0, \quad i = \overline{m+1,s},$$

$$\forall \varepsilon,\ 0 < \varepsilon < \varepsilon_0\}, \qquad (18)$$

where vector-function $r(\varepsilon)$ possesses by the properties $r(0) = 0$; $r(\varepsilon)/\varepsilon \to 0$ under $\varepsilon \to +0$. Since functions $g_i(u) \in C^1(U_{01})$, $i = \overline{m+1,s}$ and $g_i(u_*) = 0$, $i = \overline{m+1,s}$, $u_* \in U$, that differences $g_i(u_* + \varepsilon e + r(\varepsilon)) - g_i(u_*) = \langle g_i'(u_*), \varepsilon e + r(\varepsilon)\rangle + o_i(\varepsilon, u_*) = 0$, $i = \overline{m+1,s}$.. Thence, dividing in $\varepsilon > 0$ and directing $\varepsilon \to +0$, with consideration of that $r(\varepsilon)/\varepsilon \to 0$, $o_i(\varepsilon, u_*)/\varepsilon \to 0$ under $\varepsilon \to +0$, we get $\langle g_i'(u_*), e\rangle = 0$, $i = \overline{m+1,s}$. Consequently, ensemble (18)

$$K_{m+1} = \{e \in E^n / \langle g_i'(u_*), e\rangle = 0,\ i = \overline{m+1,s}\}. \qquad (19)$$

- closed convex cone. Dual cone to cone (19) by Farkash's theorem is defined by formula

$$K_{m+1}^* = \left\{ c \in E^n \middle/ c = -\sum_{i=m+1}^{s} \lambda_i^* g_i'(u_*) \right\}. \qquad (20)$$

Finally, we define K_0 - an ensemble internal directions of the convex ensemble U_0 in the point $u_* \in U \subset U_0$. It is easy to make sure in that, if $u_* = \operatorname{int} U_0$, that $K_0 = E^n$, consequently, $K^* = \{0\}$. If $u_* \in \Gamma p U_0$, that

$$K_0 = \{e \in E^n / e = \lambda(u - u_*),\ \forall u \in \operatorname{int} U_0,\ \lambda > 0\} \qquad (21)$$

- an open convex cone, but dual to it cone

$$K_0 = \{c_0 \in E^n / \langle c_0, u\rangle \geq \langle c_0, u_*\rangle \text{ при всех } u \in U_0\}. \qquad (22)$$

Cone's constructions are stated without strict proofs. Full details of these facts with proofs reader can find in the book: Vasiliev F. P. Numerical solution methods of the extreme problems. M.: Nauka, 1980.

Hereinafter, We denote by $K_i, i=\overline{1,m}$ the cones $K_{b_i}, i=\overline{1,m}$.

Lemma. *If $u_* \in U$ is a point of the function minimum $J(u)$ on ensemble U, that necessary the intersection of the convex cones*

$$K_y \cap K_0 \cap K_1 \cap \ldots \cap K_m \cap K_{m+1} = \varnothing. \qquad (23)$$

Proof. Let the point be $u_* \in U_* \subset U$. We show, that correlation (23) is executed. We suppose opposite, i.e. existence of the vector

$$e \in K_y \cap K_0 \cap K_1 \cap \ldots \cap K_m \cap K_{m+1}.$$

From inclusion $e \in K_y$ follows that $J(u_* + \varepsilon\overline{e}) < J(u_*)$ under all $\varepsilon,\ 0 < \varepsilon < \varepsilon_y,\ |e - \overline{e}| < \delta_y$, but from $e \in K_i, i = \overline{0,m}$ follows that $u_* + \varepsilon\overline{e} \in U_i$ under all $\varepsilon,\ 0 < \varepsilon < \varepsilon_i,\ |e - \overline{e}| < \delta_i,\ i = \overline{0,m}$. Let numbers be $\delta = \min(\delta_y, \delta_0, \ldots, \delta_m),\ \alpha = \min(\varepsilon_y, \varepsilon_0, \ldots, \varepsilon_m)$. Then inequality $J(u_* + \varepsilon\overline{e}) < J(u_*)$ is true and inclusion $u_* + \varepsilon\overline{e} \in \bigcap_{i=0}^{m} U_i$ exists under all $\varepsilon,\ 0 < \varepsilon < \alpha$ and $|e - \overline{e}| < \delta$.

Since vector $e \in K_{m+1}$, the point $u(\varepsilon) = u_* + \varepsilon e + r(\varepsilon) \in U_{m+1}$ under all $\varepsilon,\ 0 < \varepsilon < \varepsilon_{m+1},\ r(0) = 0;\ r(\varepsilon)/\varepsilon \to 0$ under $\varepsilon \to +0$. We choose the vector $\overline{e} = e + r(\varepsilon)/\varepsilon$. If $\varepsilon_{m+1} \le \alpha$ and $\varepsilon_{m+1} > 0$ - sufficiently small number, that norm $|\overline{e} - e| = |r(\varepsilon)/\varepsilon| < \delta$. Then the point

$$u_* + \varepsilon\overline{e} = u_* + \varepsilon e + r(\varepsilon) \in U_0 \cap U_1 \cap \ldots \cap U_m \cap U_{m+1} = U.$$

Finally, the point $u_* + \varepsilon\overline{e} \in U$ and $J(u_* + \varepsilon\overline{e}) < J(u_*)$. It is impossible, since the point $u_* \in U \subset U_0$ is solution of the problem (1), (2). The lemma is proved.

Previously than transfer to proof of the theorem, we note the following:

1) If $J'(u_*)=0$, that under $\lambda_0^* = 1, \lambda_1^* = \lambda_2^* = ... = \lambda_s^* = 0$ all condition of the theorem 1 are fulfilled. In fact, norm $\left|\overline{\lambda}^*\right| = 1 \neq 0$ is a scalar product

$$\left\langle L_u(u_*, \overline{\lambda}^*), u-u_* \right\rangle = \left\langle \lambda_0^* J'(u_*), u-u_* \right\rangle = 0,$$
$$\forall u \in U_0,\ \lambda_i^* g_i(u_*) = 0,\ i = \overline{1,s}.$$

2) If $g_i'(u_*)=0$ under a certain i, $1 \le i \le m$, then for values $\lambda_i^* = 1, \lambda_j^* = 0, j = \overline{1,s}, j \neq i$, all conditions of the theorems 1 are also executed. In fact, $\left|\overline{\lambda}^*\right| = 1$, $\left\langle L_u(u_*, \overline{\lambda}^*), u-u_* \right\rangle = \left\langle \lambda_i^* g_i'(u_*), \right.$ $\left.(u_*), u-u_* \right\rangle = 0,\ \forall u \in U,\ \lambda_i^* g_i(u_*) = 0,\ i = \overline{1,s}$.

3) Finally if the vectors $\left\{g_i'(u_*),\ i = \overline{m+1,s}\right\}$ are linearly dependent, also the conditions (6) - (8) of the theorem 1 are occurred. In fact, in this case the numbers $\lambda_{m+1}^*,\ \lambda_{m+2}^*, \ldots, \lambda_s^*$ not all equal to zero, such that

$$\lambda_{m+1}^* g_{m+1}'(u_*) + \lambda_{m+2}^* g_{m+2}'(u_*) + \ldots + \lambda_s^* g_s'(u_*) = 0\,.$$

exist.

We take $\lambda_0^* = \lambda_1^* = \ldots = \lambda_m^* = 0$. Then norm, $\left|\overline{\lambda}^*\right| \neq 0$,

$$\left\langle L_u(u_*, \overline{\lambda}^*), u-u_* \right\rangle = \left\langle \sum_{i=m+1}^{s} \lambda_i^* g_i'(u_*), u-u_* \right\rangle = 0,$$
$$\forall u \in U_0, \lambda_i^* g_i(u_*) = 0,\ i = \overline{1,s},$$

since $g_i(u_*),\ i = \overline{m+1,s}$.

From items 1 - 3 results that theorem 1 should prove for event, when $J'(u_*) \neq 0,\ g_i'(u_*) \neq 0$ under all $i = \overline{1,m}$, and vectors $\{g_i'(u_*), i = \overline{m+1,s}\}$ are linearly independent.

Proof of the theorem 1. By the data of the theorem the ensemble $u_* \in U$. Then, as follows from proved lemma, the intersection of the convex cones

$$K_y \cap K_0 \cap K_1 \cap \ldots \cap K_m \cap K_{m+1} = \varnothing \tag{24}$$

in the point $u_* \in U_*$, moreover all cones, except K_{m+1}, are open. We notice, that in the case of $J'(u_*) \neq 0,\ g_i'(u_*) \neq 0$, $i = \overline{1,m}$; $\{g_i'(u_*), i = \overline{m+1,s}\}$ are linearly independent, all cones K_y, $K_0, \ldots, K_{m+1}$ are inempty [refer to the formulas (12), (16), (19), (21)]. Then by Dubovicky-Milutin's theorem, for occurring of the correlation (24) necessary and sufficiently existence of the vectors $c_y \in K_y^*$, $c_0 \in K_0^*$, $c_1 \in K_1^*, \ldots, c_{m+1} \in K_{m+1}^*$, not all equal to zero and such that

$$c_y + c_0 + c_1 + \ldots + c_{m+1} = 0\,. \tag{25}$$

As follows from formulas (13), (17), (20) the vectors

$$c_y = -\lambda_0^* J'(u_*),\ \lambda_0^* \geq 0,\ \ c_i = -\lambda_i^* g_i'(u_*),\ \lambda_i^* \geq 0,\ i = \overline{1,m},$$

$$c_{m+1} = -\sum_{i=m+1}^{s} \lambda_i^* g_i'(u_*)\,.$$

We have $c_0 = -c_y - c_1 - \ldots - c_m - c_{m+1} = \lambda_0^* J'(u_*) + \sum_{i=1}^{s} \lambda_i^* g_i'(u_*)$ from equality (25). Since cone K_0^* is defined by formula (22), that

$$\langle c_0, u - u_0 \rangle = \left\langle \lambda_0^* J'(u_*) + \sum_{i=1}^{s} \lambda_i^* g_i'(u_*), u - u_0 \right\rangle =$$

$$= \langle L_u(u_*, \overline{\lambda}_*), u - u_0 \rangle \geq 0,\ \forall u \in U_0.$$

Thence follows fairness of the correlations (7). The condition (6) follows from that not all $c_y, c_0, c_1, \ldots, c_m$, are equal to zero. We notice, that if $g_i(u_*) < 0$ for a certain i from $1 \le i \le m$, that cone $K_i = E^n$, consequently, $K_i^* = \{0\}$. It means that $c_i = -\lambda_i^* g_i'(u_*) = 0,\ g_i'(u_*) \neq 0$. Thence follows that $\lambda_i^* = 0$. Thereby, the products $\lambda_i^* g_i'(u_*) = 0,\ i = \overline{1, s}$, i.e. the condition (8) is occurred. Theorem is proved.

Lecture 13

SOLUTION ALGORITHM OF THE NONLINEAR PROGRAMMING PROBLEM

We show a sequence of the solution of the nonlinear programming problem in the following type:

$$J(u) \to \inf, \tag{1}$$

$$u \in U = \{u \in E^n / u \in U_0,\ g_i(u) \le 0,\ i = \overline{1,m};$$

$$g_i(u) = 0,\ i = \overline{m+1,s}\}, \tag{2}$$

where functions $J(u) \in C^1(U_{01}),\ \ g_i(u) \in C^1(U_{01}),\ \ i = \overline{1,s},\ \ U_{01}$

- an open ensemble containing convex ensemble U_0 from E^n, in particular, $U_{01} = E^n$ on base of the correlations (6) - (8) from theorem 1 of the previous lecture. We notice, that these correlations are true not only for the point $u_* \in U_*$, but also for points of the local function minimum $J(u)$ on ensemble *U*. The question arises: under performing which conditions of the problem (1), (2) will be nondegenerate and the point $u_* \in U$ will be a point of the local minimum $J(u)$ on *U*?

1^0. It is necessary to make sure in that ensemble $U_* = \{u_* \in E^n / u_* \in U,\ \ J(u_*) = \min\limits_{u \in U} J(u)\} \ne \varnothing$ for which to use the theorems 1-3 (the lecture 2).

2^0. To form the generalized Lagrange's function for problems (1), (2):

$$L\,(u,\overline{\lambda}) = \lambda_0 J(u) + \sum_{i=1}^{s} \lambda_i g_i(u), \quad u \in U_0;$$

$$\overline{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_s) \in \Lambda_0 = \left\{\overline{\lambda} \in E^{s+1} \,/\, \lambda_0 \ge 0, \lambda_1 \ge 0, \ldots, \lambda_m \ge 0\right\}.$$

3^0. To find the points $u_* = (u_1^*, \ldots, u_n^*) \in U$, $\overline{\lambda}^* = (\lambda_0^*, \ldots, \lambda_s^*) \in \Lambda_0$ from the following conditions:

$$\left|\overline{\lambda}^*\right| = \alpha, \ \lambda_0^* \ge 0, \ \lambda_1^* \ge 0, \ldots, \lambda_m^* \ge 0, \qquad (3)$$

$$\left\langle L_u(u_*, \overline{\lambda}^*), u - u_* \right\rangle =$$

$$= \left\langle \lambda_0^* J'(u_*) + \sum_{i=1}^{s} \lambda_i^* g_i'(u_*), u - u_* \right\rangle \ge 0, \ \forall u \in U_0, \qquad (4)$$

$$\lambda_i^* g_i'(u_*) = 0, \ i = \overline{1, s}, \qquad (5)$$

where $\alpha > 0$ - the number, in particular $\alpha = 1$.

a) If the point $u_* \in \operatorname{int} U_0$ or $U_0 = E^n$, that condition (4) can be replaced on

$$L_u(u_*, \overline{\lambda}^*) = \lambda_0^* J'(u_*) + \sum_{i=1}^{s} \lambda_i^* g_i'(u_*) = 0. \qquad (6)$$

In this case we have a system $n+1+s$ of the algebraic equations (3), (5), (6) to determinate $n+1+s$ of the unknowns $u_1^*, \ldots, u_n^*; \lambda_0^*, \ldots, \lambda_s^*$.

b) If after solution of the algebraic equations system (3), (5), (6) [or formulas (3) - (5)] it is turned out that value $\lambda_0^* > 0$, that problem (1), (2) is identified by nondegenerate. The conditions (3) in it can be changed by more simpler condition $\lambda_0^* = 1, \lambda_1^* \ge 0, \lambda_2^* \ge 0, \ldots, \lambda_m^* \ge 0$. If in the nondegenerate problem a pair (u_*, λ^*), $\overline{\lambda}_s^* = (\lambda_1^*, \ldots, \lambda_s^*)$ forms saddle point of the Lagrange's function

$$L\ (u,\lambda) = J(u) + \sum_{i=1}^{s} \lambda_i g_i(u),\ u \in U_0,$$

$$\lambda \in \Lambda_0 = \left\{\lambda \in E^s / \lambda_1 \geq 0, \ldots, \lambda_m \geq 0\right\},$$

that $u_* \in U$ is the point of the global minimum.

4°. We consider the necessary problem of the nonlinear programming:

$$J(u) \to \inf \tag{7}$$

$$u \in U = \left\{u \in E^n \middle/ g_i(u) = 0,\ g_i(u) \in C^1(E^n),\ i = \overline{1,m}\right\}. \tag{8}$$

The problem (7), (8) is a particular event of the problem (1), (2). The ensemble $U_0 = E^n$. The point in the problem (7), (8) is named by normal point of the minimum, if the vectors $\left\{g_i'(u_*), i = \overline{1,m}\right\}$ are linearly independent. We notice, that if $u_* \in U$ - normal point, that problem (7), (8) are nondegenerate. In fact, for problem (7), (8) the equality

$$\lambda_0^* J'(u_*) + \sum_{i=1}^{m} \lambda_i^* g_i'(u_*) = 0 \tag{9}$$

exists.

If $\lambda_0^* = 0$, that on the strength of linearly independence of the vectors $\left\{g_i'(u_*), i = \overline{1,m}\right\}$ we would get $\lambda_i^* = 0,\ i = \overline{1,m}$. Then the vector $\overline{\lambda^*} = 0$ that contradicts to (3).

Let $u_* \in U$ - a normal point for problem (7), (8). Then it is possible to take $\lambda_0^* = 1$, and Lagrange's function has the form

$$L\ (u,\lambda) = J(u) + \sum_{i=1}^{m} \lambda_i g_i(u),\ u \in E^n, \lambda = (\lambda_1, \ldots, \lambda_m) \in E^m$$

Theorem. *Let functions* $J(u),\ g_i(u),\ i=\overline{1,m}$ *are determined and twice continuously differentiable in neighborhoods of the point* $u_* \in U$ *. In order the point* $u_* \in U$ *to be a point of the local minimum* $J(u)$ *on ensemble* U *, i.e.* $J(u_*) \le J(u),\ \forall u,\ u \in o(u_*,\varepsilon) \cap U$ *sufficiently that quadratic form* $y' \dfrac{\partial^2 L(u_*,\lambda^*)}{\partial u^2} y$ *to be positively determined on hyperplane*

$$\left(\frac{\partial g(u_*)}{\partial u}\right)^* y = \begin{pmatrix} \partial g_1(u_*)/\partial u_1 & \dots & \partial g_1(u_*)/\partial u_n \\ \partial g_2(u_*)/\partial u_1 & \cdots & \partial g_2(u_*)/\partial u_n \\ \dots & \dots & \dots \\ \partial g_m(u_*)/\partial u_1 & \dots & \partial g_m(u_*)/\partial u_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix} = 0. \quad (10)$$

Proof. Let quadratic form

$$y' \frac{\partial^2 L(u_*,\lambda^*)}{\partial y^2} y =$$

$$=(y_1,\dots,y_n) \begin{pmatrix} \partial^2 L(u_*,\lambda^*)/\partial u_1^2 & \dots & \partial^2 L(u_*,\lambda^*)/\partial u_1 \partial u_n \\ \partial^2 L(u_*,\lambda^*)/\partial u_2 \partial u_1 & \cdots & \partial^2 L(u_*,\lambda^*)/\partial u_2 \partial u_n \\ \dots & \dots & \dots \\ \partial^2 L(u_*,\lambda^*)/\partial u_n \partial u_1 & \dots & \partial^2 L(u_*,\lambda^*)/\partial u_n^2 \end{pmatrix} \times$$

$$\times \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix} > 0, \quad |y| \ne 0,$$

on hyperplane (10). We show that $u_* \in U$ - a point of the local function minimum $J(u)$ on ensemble U. We notice, that u_* - a normal point and it is determined from the conditions $L_u(u_*, \lambda^*) = 0,\ \lambda_i^* g_i(u_*) = 0,\ i = \overline{1,m}$, i.e. $g_i(u_*) = 0,\ i = \overline{1,m}$. Finally, pair (u_*, λ^*) is known.

Since functions

$$J(u) \in C^2(o(u_*, \delta)),\ g_i(u) \in C^2(o(u_*, \delta)),$$

that continuous function $y'(\partial^2 L(u_*, \lambda^*)/\partial u^2)y$ comparatively variable y reaches the lower bound on compact ensemble $V = \{y \in E^n / |y| = 1,\ (\partial g(u_*)/\partial u)^* y = 0\}$. Let the number

$$\gamma = \min y' \frac{\partial^2 L(u_*, \lambda^*)}{\partial u^2} y, \quad y \in V$$

We enter the ensemble

$$A_\varepsilon = \{u \in E^n / u = u_* + \varepsilon y,\ |y| = 1,\ (\partial g(u_*)/\partial u)^* y = 0\}, \quad (11)$$

where * - a transposition sign for matrixes; $\varepsilon > 0$ - sufficiently small number. If the point $u \in A_\varepsilon$, that quadratic form

$$(u - u_*)' L_{uu}(u_*, \lambda^*)(u - u_*) = \varepsilon^2 y' L_{uu}(u_*, \lambda^*) y \geq \varepsilon^2 \gamma, \quad (12)$$

where $L_{uu}(u_*, \lambda^*) = \partial^2 L(u_*, \lambda^*)/\partial u^2$ on the strength of correlation (11). For the points $u \in A_\varepsilon$ difference

$$L(u, \lambda^*) - L(u_*, \lambda^*) = \langle L_u(u_*, \lambda^*), u - u_* \rangle + \frac{1}{2}(u - u_*)' \times$$
$$\times L_{uu}(u_*, \lambda^*)(u - u_*) + o(|u - u_*|^2) \geq \frac{\gamma \varepsilon^2}{2} + o(\varepsilon^2) =$$

$$= \varepsilon^2\left(\frac{\gamma}{2}+\frac{o(\varepsilon^2)}{\varepsilon^2}\right) \geq \frac{\gamma}{4}\varepsilon^2, \quad 0<\varepsilon<\varepsilon_1, \qquad (13)$$

and since $L_u(u_*, \lambda^*) = 0$ inequality (12) faithfully and $o(\varepsilon^2)/\varepsilon^2 \to 0$ under $\varepsilon \to 0$.

We enter the ensemble

$$B_\varepsilon = \left\{u \in E^n \middle/ \left|u-u_*\right| = \varepsilon, \ g_i(u)=0, \ i=\overline{1,m}\right\} \subset U. \qquad (14)$$

Since hyperplane (10) is tangent to manifold $g_i(u)=0, i=\overline{1,m}$ in the point u_*, then for each point $\widetilde{u} \in B_\varepsilon$ the point $u \in A_\varepsilon$ such that norm $\left|\widetilde{u}-u\right| \leq K\varepsilon^2$, $K = const > 0$ is found. In fact, if

$$u \in A_\varepsilon: \ g_i(u)-g_i(u_*) = g_i(u) = \left\langle g_i'(u_*), u-u_*\right\rangle +$$
$$+\frac{1}{2}(u-u_*)' g_{iuu}(u_*)\ (u-u_*) + o_i\left(\left|u-u_*\right|^2\right) =$$
$$= \frac{1}{2}\varepsilon^2 y' g_{iuu}(u_*) y + o_i(\varepsilon), \quad |y|=1; \ i=\overline{1,m}; \qquad (15)$$

$$\widetilde{u} \in B_\varepsilon: \ g_i(\widetilde{u})-g_i(u_*) = 0 = \left\langle g_i'(u_*), \widetilde{u}-u_*\right\rangle + \frac{1}{2}(\widetilde{u}-u_*)' \times$$
$$\times g_{iuu}(u_*)(\widetilde{u}-u_*) + o_i\left(\left|\widetilde{u}-u_*\right|^2\right) = \left\langle g_i'(u_*), \widetilde{u}-u+u-u_*\right\rangle +$$
$$+\frac{1}{2}(\widetilde{u}-u+u-u_*)' g_{iuu}(u_*)(\widetilde{u}-u+u-u_*) + \overline{o}_i(\varepsilon^2) =$$
$$= \left\langle g_i'(u_*), \widetilde{u}-u\right\rangle + \frac{1}{2}(\widetilde{u}-u)' g_{iuu}(u_*)(\widetilde{u}-u_*) + \frac{1}{2}(u-u_*)' \times$$
$$\times g_{iuu}(u_*)(u-u_*) + (\widetilde{u}-u)' g_{iuu}(u_*)(u-u_*) +$$
$$+\overline{o}_i(\varepsilon^2), \ i=\overline{1,m}; \qquad (16)$$

From (11), (14) and (15), (16) follows that norm $|\tilde{u}-u| \le K\varepsilon^2$, $\tilde{u} \in B_\varepsilon,\ u \in A_\varepsilon$.

Since the function $L_u(u,\lambda^*)$ continuously differentiable by u in neighborhoods of the point u_*, and derivative $L_u(u_*,\lambda^*)=0$, that difference $L_u(u,\lambda^*)-L_u(u_*,\lambda^*)=L_u(u,\lambda^*)= L_{uu}(u_*,\lambda^*)\times$ $\times(u-u_*)+o(\|u-u_*\|)$ in neighborhoods of the point u_*. This implies that, in particular, $u \in A_\varepsilon$, norm $\|L(u,\lambda^*)\| \le K_1\varepsilon$, if $0<\varepsilon<\varepsilon_1$, $\varepsilon_1>0$ - sufficiently small number.

For the points $u \in A_\varepsilon, \tilde{u} \in B_\varepsilon$ the difference $L(\tilde{u},\lambda^*)-$ $-L(u,\lambda^*)=\langle L_u(u,\lambda^*),\tilde{u}-u\rangle+\dfrac{1}{2}(\tilde{u}-u)'L_{uu}(u_*,\lambda^*)(\tilde{u}-u)+$ $+o(\|\tilde{u}-u_*\|^2)$, consequently, norm $|L(\tilde{u},\lambda^*)-L(u,\lambda^*)|$ $\le KK_1\varepsilon^3+|o_1(\varepsilon^3)| \le K_2\varepsilon^3$ under sufficiently small $\varepsilon_1>0$, $0<\varepsilon<\varepsilon_1$. Then the difference $(u \in A_\varepsilon, \tilde{u} \in B_\varepsilon)$

$$L(\tilde{u},\lambda^*)-L(u_*,\lambda^*)=L(u,\lambda^*)-L(u_*,\lambda^*)+L(\tilde{u},\lambda^*)-$$
$$-L(u,\lambda^*) \ge L(u,\lambda^*)-L(u_*,\lambda^*)-|L(\tilde{u},\lambda^*)-L(u,\lambda^*)| \ge$$
$$\ge \frac{\gamma}{4}\varepsilon^2-K_2\varepsilon^3 \ge 0,\ 0<\varepsilon<\varepsilon_1, \tag{17}$$

on the strength of correlations (13). So as values $L(\tilde{u},\lambda^*)=J(\tilde{u})$, $L(u_*,\lambda^*)=J(u_*)$, since $g_i(\tilde{u})=g_i(u_*)=0,\ i=\overline{1,m}$ then from (17) we have $J(u_*) \le J(\tilde{u})$. Consequently, $u_* \in U$ - the point of the local function minimum $J(u)$ on ensemble U. The Theorem is proved.

Example. Let function be $J(u)=-u_1^2+\sqrt{5}u_1u_2+u_2^2$, ensemble

$U = \{u = (u_1, u_2) \in E^2 / u_1^2 + u_2^2 = 1\}$. To find the function minimum $J(u)$ on ensemble U. For the example the ensemble $U_0 = E^2$, functions $J(u) \in C^2(E^2)$, $g_i(u) = u_1^2 + u_2^2 - 1 \in C^2(E^2)$, ensemble $U_* \neq \varnothing$, since $U \subset E^2$ - compact ensemble. Necessary optimality conditions

$$L_u(u^*, \lambda^*) = 0: \quad \begin{cases} -2u_1^* + \sqrt{5}u_2^* + 2\lambda^* u_1^* = 0, \\ 2u_2^* + \sqrt{5}u_1^* + 2\lambda^* u_2^* = 0, \end{cases}$$

$$g(u_*) = 0: (u_1^*)^2 + (u_2^*)^2 = 1,$$

where function $L(u, \lambda) = J(u) + \lambda g(u),\ u \in E^2, \lambda \in E^1$. Thence we find the points u_1^*, u_2^*, λ^*:

1) $\lambda^* = 3/2,\ u_1^* = -\sqrt{5/6},\ u_2^* = \sqrt{1/6}$;
2) $\lambda^* = 3/2,\ u_1^* = \sqrt{5/6},\ u_2^* = -\sqrt{1/6}$;
3) $\lambda^* = -3/2,\ u_1^* = \sqrt{5/6},\ u_2^* = \sqrt{1/6}$;
4) $\lambda^* = -3/2,\ u_1^* = -\sqrt{5/6},\ u_2^* = -\sqrt{1/6}$.

It is necessary to find in which point $(\lambda^*, u_1^*, u_2^*)$ from 1-4 the minimum $J(u)$ on U is reached. First of all we select the points where local minimum $J(u)$ on U is reached. We note, that problem is nondegenerate and matrix $L_{uu}(u_*, \lambda^*)$ and vector $g_u(u_*)$ are equal to

$$L_{uu}(u_*, \lambda^*) = \begin{pmatrix} -2 + 2\lambda^* & \sqrt{5} \\ \sqrt{5} & 2 + 2\lambda^* \end{pmatrix}, \quad g_u(u_*) = g'(u_*) = \begin{pmatrix} 2u_1^* \\ 2u_2^* \end{pmatrix}.$$

For the first point $\lambda^* = 3/2,\ u_1^* = -\sqrt{5/6},\ u_2^* = \sqrt{1/6}$ quadric form is $y' L_{uu}(u_*, \lambda^*) y = y_1^2 + 2\sqrt{5} y_1 y_2 + 5 y_2^2$, a hyperplane equation $-2\sqrt{5/6}\,y_1 + 2\sqrt{1/6}y_2 = 0$. Thence we have $y_2 = \sqrt{5}y_1$. Substituting the values $y_2 = \sqrt{5}y_1$ to the quadric form we get $y' L_{uu}(u_*, \lambda^*)y = 36y_1^2 > 0,\ y_1 \neq 0$. Consequently, $(\lambda^* = 3/2,$ $u_1^* = -\sqrt{5/6},\ u_2^* = \sqrt{1/6})$ - point of the local minimum $J(u)$ on U. By the similar way it is possible to make sure in that $\left(\lambda^* = 3/2,\ u_1^* = \sqrt{5/6},\ u_2^* = -\sqrt{1/6}\right)$ - a point of the local minimum, but the points $\left(\lambda^* = -3/2,\ u_1^* = \sqrt{5/6},\ u_2^* = \sqrt{1/6}\right)$ and $\left(\lambda^* = -3/2,\ u_1^* = -\sqrt{5/6},\ u_2^* = -\sqrt{1/6}\right)$ are not the points of the local minimum $J(u)$ on U. In order to find minimum $J(u)$ on U necessary to calculate the values of the function $J(u)$ in the points 1) and 2). It can be shown, that. $J(-\sqrt{5/6}, \sqrt{1/6}) = J(\sqrt{5/6},$ $-\sqrt{1/6}) = -3/2$. Consequently, in the points 1) and 2) the global minimum $J(u)$ on U is reached.

5^0. Now we consider the problem (1), (2) in the case $U_0 = E^n$, $J(u) \in C^2(E^n),\ g_i(u) \in C^2(E^n),\ i = \overline{1,s}$. We suppose, that problem (1), (2) is nondegenerate and the points $\left(u_*, \lambda_0^* > 0, \lambda^*\right)$ are determined by algorithm 1^0 - 3^0. We select amongst constraints $g_i(u) \leq 0,\ i = \overline{1,m},\ g_i(u) = 0,\ i = \overline{m+1,s}$ for which $g_i(u_*) = 0$, $i \in I$, where index ensembles

$$I = \left\{ i \,/\, i = \overline{m+1,s},\ g_i(u_*) = 0,\ 1 \leq i \leq m \right\}.$$

If the problem (1), (2) is nondegenerate, that vectors $\{g_i'(u_*), i \in I\}$ are linearly independent. According to specified above theorem from item 4^0 the point $u_* \in U$ is the point of the local minimum to functions $J(u)$ on U, if quadratic form $y' L_{uu}(u_*, \lambda^*)y \geq 0,\ |y| \neq 0$ on hyperplane $\left(\partial g_i(u_*)/\partial u\right)^* \times \times y = 0,\ i \in I$.

Lecture 14

DUALITY THEORY

On base of the Lagrange's function the main and dual problem are formulated and relationship between its solutions is established. The dual problems for the main, general and canonical forms writing of the linear programming problem are determined. Since dual problem - a problem of the convex programming regardless of that whether the main problem is the problem of the convex programming or not, then in many events reasonable to study the dual problem and using the relationship between its solutions to return to the source problem. Such acceptance often is used at solution of the linear programming problem.

We consider the nondegenerate nonlinear programming problem in the following type:

$$J(u) \to \inf, \tag{1}$$

$$u \in U = \{u \in E^n / u \in U_0,\ g_i(u) \le 0,\ i = \overline{1,m};$$

$$g_i(u) = 0,\ i = \overline{m+1,s}\}, \tag{2}$$

For problem (1), (2) Lagrange's function

$$L(u,\lambda) = J(u) + \sum_{i=1}^{s} \lambda_i g_i(u),\ u \in U_0;$$

$$\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_s) \in \Lambda_0 = \{\lambda \in E^s / \lambda_1 \ge 0, \ldots, \lambda_m \ge 0\}. \tag{3}$$

The main task. We enter the function [refer to. the formula (3)]:

$$X(u) = \sup_{\lambda \in \Lambda_0} L\,(u, \lambda), \quad u \in U_0. \tag{4}$$

We show, that function

$$X(u) = \begin{cases} J(u) \text{ при } & \text{всех } u \in U, \\ +\infty \text{ при } & \text{всех } u \in U_0 \setminus U. \end{cases} \tag{5}$$

In fact, if $u \in U$, that $g_i(u) \le 0$, $i = \overline{1, m}$, $g_i(u) = 0$, $i = \overline{m+1, s}$, consequently,

$$X(u) = \sup_{\lambda \in \Lambda_0} \left\{ J(u) + \sum_{i=1}^{s} \lambda_i g_i(u) \right\} =$$

$$= \sup_{\lambda \in \Lambda_0} \left\{ J(u) + \sum_{i=1}^{m} \lambda_i g_i(u) \right\} = J(u),$$

since $\lambda_1 \ge 0, \ldots, \lambda_m \ge 0, \sum_{i=1}^{m} \lambda_i g_i(u) \le 0$ under all $u \in U$, moreover $\lambda = 0 \in \Lambda_0$. If $u \in U_0 \setminus U$, that possible for a certain number i from $1 \le i \le m$, $g_i(u) > 0$ and for a certain j from $m+1 \le j \le s$, $g_j(u) \ne 0$. In the both events by choice a sufficient big $\lambda_i > 0$ or $\lambda_j = k g_j(u)$, $k > 0$ - sufficiently large number, the value $X(u)$ can be done by arbitrary large number.

Now the source problem (1), (2) can be written in the equal type:

$$X(u) \to \inf, \quad u \in U_0, \tag{6}$$

on the strength of correlations (4), (5). We notice that $\inf_{u \in U_0} X(u) = = \inf_{u \in U} J(u) = J_*$, consequently, if the ensemble $U_* \ne \varnothing$, that

$$U_* = \left\{ u_* \in U \middle/ \min_{u \in U} J(u) = J(u_*) = J_* \right\} =$$
$$= \left\{ u_* \in U_0 \middle/ X(u_*) = J(u_*) = J_* = \min_{u \in U} X(u) \right\}.$$

The source problem (1), (2) or tantamount its problem (6) is named by the main task.

Dual problem. On base of the Lagrange's function (3) we enter function

$$\psi(\lambda) = \inf_{u \in U} L\ (u, \lambda),\ \lambda \in \Lambda_0. \tag{7}$$

Optimization problem of the following type:

$$\psi(\lambda) \to \sup,\ \lambda \in \Lambda_0, \tag{8}$$

is called by dual problem to problem (1), (2) or tantamount its problem (6), but Lagrange's multipliers $\lambda = (\lambda_1, \ldots, \lambda_s) \in \Lambda_0$ - dual variables with respect to variables $u = (u_1, \ldots, u_n) \in U_0$. We denote through

$$\Lambda^* = \left\{ \lambda^* \in E^s \middle/ \lambda^* \in \Lambda_0,\ \psi(\lambda^*) = \max_{\lambda \in \Lambda_0} \psi(\lambda) \right\}$$

We notice, that if $\Lambda^* \neq \varnothing$, that $\psi(\lambda^*) = \sup_{\lambda \in \Lambda_0} \psi(\lambda) = \psi^*$.

Lemma. The values $J_* = \inf_{u \in U_0} X(u),\ \psi^* = \sup_{\lambda \in \Lambda_0} \psi(\lambda)$ for main (6) and dual (8) problems accordingly satisfy to the inequalities

$$\psi(\lambda) \le \psi^* \le J_* \le X(u),\ \forall u \in U_0, \forall \lambda \in \Lambda_0. \tag{9}$$

Proof. As follows from formula (7), function $\psi(\lambda) =$ $= \inf_{u \in U_0} L\ (u, \lambda) \le L\ (u, \lambda),\ \forall u \in U_0, \forall \lambda \in \Lambda_0$. Thence we have

$$\psi^* = \sup_{\lambda \in \Lambda_0} \psi(\lambda) \le \sup_{\lambda \in \Lambda_0} L\ (u, \lambda) = X(u),\ \forall u \in U_0, \tag{10}$$

on the strength of correlations (4). From correlations (10) transferring to lower bound on u we get $\psi^* \le \inf_{u \in U_0} X(u) = J_*$. Thence and from determinations of the lower bound the inequalities (9) follow. Lemma is proved.

Theorem 1. In order to execute the correlations

$$U_* \neq \varnothing,\ \Lambda^* \neq \varnothing,\ X(u_*) = J_* = \psi^* = \psi(\lambda^*), \tag{11}$$

necessary and sufficiently that Lagrange's function (3) has saddle point on ensemble $U_0 \times \Lambda_0$. The ensemble of the saddle points to function $L\ (u, \lambda)$ on $U_0 \times \Lambda_0$ coincides with ensemble $U_* \times \Lambda^*$.

Proof. Necessity. Let for the points $u_* \in U_*,\ \lambda^* \in \Lambda^*$ correlations (11) are complied. We show, that pair (u_*, λ^*) - saddle point to Lagrange's function (3) on ensemble $U_0 \times \Lambda_0$. Since $\psi^* = \psi(\lambda^*) = \inf_{u \in U_0} L\ (u, \lambda^*) \le L\ (u_*, \lambda^*) \le \sup_{\lambda \in \Lambda_0} L\ (u_*, \lambda) = X(u_*) =$ $= J_*$, that on the strength of correlations (11) we have

$$L\ (u_*, \lambda^*) = \inf_{u \in U_0} L\ (u, \lambda^*) = \sup_{\lambda \in \Lambda_0} L\ (u_*, \lambda),$$
$$u_* \in U_*, \lambda_* \in \Lambda_0. \tag{12}$$

From inequality (12) follows that

$$L\ (u_*, \lambda) \le L\ (u_*, \lambda^*) \le L\ (u, \lambda^*),\ \ \forall u \in U_0, \lambda \in \Lambda_0. \tag{13}$$

It means that pair $(u_*, \lambda^*) \in U_* \times \Lambda^*$ - saddle point. Moreover ensemble $U_* \times \Lambda^*$ belongs to the ensemble of the saddle points to Lagrange's function, since $u_* \in U_*,\ \lambda^* \in \Lambda^*$ - arbitrary taken points from ensemble U_*, Λ^* accordingly. Necessity is proved.

Sufficiency. Let pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ be a saddle point to

Lagrange's function (3). We show that correlations (11) are fulfilled.

As follows from determination of the saddle point which has the form (13) inequality $L(u_*,\lambda) \le L(u_*,\lambda^*)$, $\forall \lambda \in \Lambda_0$ faithfully. Consequently,

$$X(u_*) = \sup_{\lambda \in \Lambda_0} L(u_*,\lambda) = L(u_*,\lambda^*) \ . \qquad (14)$$

Similarly from the right inequality (13) we have

$$\psi(\lambda^*) = \inf_{u \in U_0} L(u,\lambda^*) = L(u_*,\lambda^*). \qquad (15)$$

From inequalities (14), (15) with consideration of correlation (9) we get

$$L(u_*,\lambda^*) = \psi(\lambda^*) \le \psi^* \le J_* \le X(u_*) = L(u_*,\lambda^*) \, .$$

Thence we have $\psi(\lambda^*) = \psi^* = J_* = X(u_*)$. Consequently, ensemble $U_* \ne \varnothing$, $\Lambda^* \ne \varnothing$ and moreover ensemble of the saddle points to function (3) belongs to the ensemble $U_* \times \Lambda^*$. The theorem is proved.

The following conclusions can be made on base of the lemma and theorem 1:

1°. The following four statements are equivalent: a) or $(u_*,\lambda^*) \in U_0 \times \Lambda_0$ - the saddle point to Lagrange's function (3) on ensemble $U_0 \times \Lambda_0$; b) or correlations (11) are executed; c) or the points $u_* \in U_0, \lambda^* \in \Lambda_0$ such that $X(u_*) = \psi(\lambda^*)$ exist; d) or equality

$$\max_{\lambda \in \Lambda_0} \inf_{u \in U_0} L(u,\lambda) = \min_{u \in U_0} \sup_{\lambda \in \Lambda_0} L(u,\lambda)$$

is equitable.

2°. If $(u_*,\lambda^*), (a_*,b^*) \in U_0 \times \Lambda_0$ are saddle points of the Lagrange's function (3) on $U_0 \times \Lambda_0$, that $(u_*,b_*), (a_*,\lambda^*)$ - also saddle points to function (3) on $U_0 \times \Lambda_0$, moreover $L(u_*,b_*) =$

$$= L\ (a_*,\lambda^*) = L\ (u_*,\lambda^*) = L\ (a_*,b_*) = \psi(\lambda^*) = X(u_*) = J_* = \psi^*.$$

However inverse statement, i.e. that from $L\ (u_*,\lambda^*) = L\ (a,b)$ follows $(a,b) \in U_0 \times \Lambda_0$ - saddle point in general event untrue.

3°. Dual problem (8) it is possible to write as

$$-\psi(\lambda) \to \inf,\ \lambda \in \Lambda_0. \tag{16}$$

Since function $L\ (u,\lambda)$ is linear by λ on convex ensemble Λ_0, that optimization problem (16) is the convex programming problem, $-\psi(\lambda)$ is convex on Λ_0 regardless of that, the main task (1), (2) would be convex or no. We notice, that in general event the dual problem to dual problem does not comply with source, i.e. with the main problem. There is such coincidence only for tasks of the linear programming.

We consider the problem of the linear programming as applications to duality theories.

The main task of the linear programming has the form

$$\begin{gathered} J(u) = \langle c,u \rangle \to \inf, \\ u \in U = \{u \in E^n / u \ge 0,\ Au - b \le 0\}, \end{gathered} \tag{17}$$

where $c \in E^n$, $b \in E^m$ are the vectors; A is the matrix of the order $m \times n$; the ensemble

$$U_0 = \{u \in E^n / u = (u_1 \ge 0, \ldots, u_n \ge 0) \ge 0\}$$

Lagrange's function for task (17) is written as

$$\begin{gathered} L\ (u,\lambda) = \langle c,u \rangle + \langle \lambda, Au - b \rangle = \langle c + A^*\lambda, u \rangle - \langle b,\lambda \rangle, \\ u \in U_0,\ \lambda = (\lambda_1,\ldots,\lambda_m) \in \Lambda_0 = \{\lambda \in E^m / \lambda_1 \ge 0,\ldots,\lambda_m \ge 0\}. \end{gathered} \tag{18}$$

As follows from formulas (17), (18), function

$$\psi(\lambda) = \inf_{u \in U_0} L\,(u,\lambda) = \begin{cases} -\langle b,\lambda\rangle, \text{если } c + A^*\lambda \geq 0 \\ -\infty, \quad \text{если}\,(c + A^*\lambda)_i < 0,\ \lambda \in \Lambda_0. \end{cases}$$

We notice, that under $c + A^*\lambda \geq 0$ lower bound which is equal to $-\langle b,\lambda\rangle$ is got under $u = 0 \in U_0$. If $(c + A^*\lambda)_i < 0$, that it is possible to choose $u_i \to +\infty$, but all $u_j = 0,\ j = \overline{1,n},\ i \neq j$, herewith $\psi(\lambda) \to -\infty,\ \lambda_* \in \Lambda_0$. Finally, the dual task to task (18) has the form

$$\begin{gathered} -\psi(\lambda) = \langle b,\lambda\rangle \to \inf, \\ \lambda \in \Lambda = \left\{\lambda \in E^m \big/ \lambda \geq 0,\ c + A^*\lambda \geq 0\right\}. \end{gathered} \tag{19}$$

The dual task to task (19) complies with source task (17).

By introducing of the additional variables $u_{n+i} \geq 0,\quad i = \overline{1,m}$, optimization problem (17) can be written in the following type:

$$\left.\begin{gathered} J(u) \to \inf \\ [Au]_i - b_i + u_{n+i} = 0,\ \ u \geq 0,\ \ u_{n+i} \geq 0,\ i = \overline{1,m} \end{gathered}\right\}. \tag{20}$$

General task of the linear programming has the form

$$\begin{gathered} J(u) = \langle c,u\rangle \to \inf, \\ u \in U = \left\{u \in E^n \big/ u_j \geq 0,\ j \in I,\ Au - b \leq 0,\ \overline{A}u - \overline{b} \leq 0\right\}, \end{gathered} \tag{21}$$

where $c \in E^n$, $b \in E^m$, $\overline{b} \in E^s$ are the vectors; $A, \overline{A}$ are the matrixes accordingly to the orders $m \times n,\ s \times n$; index ensemble $I \subseteq \{1,2,\ldots,n\}$. The ensemble

$$U_0 = \left\{u = (u_1,\ldots,u_n) \in E^n \big/ u_j \geq 0,\ j \in I\right\}$$

Lagrange’s function for task (21) is written as

$$L(u,\lambda)=\langle c,u\rangle+\langle \mu, Au-b\rangle+\langle \overline{\mu},\overline{A}u-\overline{b}\rangle=$$

$$=\langle c+A^*\mu+\overline{A}^*\overline{\mu},u\rangle-\langle b,\mu\rangle-\langle \overline{b},\overline{\mu}\rangle,$$

$$u\in U_0,\ \lambda=(\mu,\overline{\mu})\in\Lambda_0=\{\lambda=(\mu,\overline{\mu})\in E^m\times E^s/\mu\geq 0\}.$$

Function

$$\psi(\lambda)=\inf_{u\in U_0} L(u,\lambda)=$$

$$=\begin{cases}-\langle b,\mu\rangle-\langle \overline{b},\overline{\mu}\rangle, & \textit{if} \quad (c+A^*\mu+\overline{A}^*\overline{\mu})_i\geq 0,\ i\in I;\\ & \qquad (c+A^*\mu+\overline{A}^*\overline{\mu})_j=0,\ j\notin I;\\ -\infty & \qquad \textit{under rest}.\end{cases}$$

The dual task to task (21) is written as

$$\begin{gathered}-\psi(\lambda)=\langle b,\mu\rangle+\langle \overline{b},\overline{\mu}\rangle\to\inf;\ (c+A^*\mu+\overline{A}^*\overline{\mu})_i\geq 0,\ i\in I;\\ (c+A^*\mu+\overline{A}^*\overline{\mu})_j=0,\ j\notin I;\ \lambda=(\mu,\overline{\mu})\in E^n\times E^s,\ \mu\geq 0.\end{gathered} \tag{22}$$

It is possible to show that dual task to task (22) complies with (21).

By introducing of the additional variables $u_{n+i}\geq 0,\ i=\overline{1,m}$ and representations $u_i=q_i-v_i,\ q_i\geq 0, v_i\geq 0,\ i\notin I$ the problem (21) possible write as

$$\begin{gathered}J(u)=\langle c,u\rangle\to\inf,\quad [Au]_i-b_i+u_{n+i}=0,\ \ \overline{A}u-\overline{b}=0,\\ u_j\geq 0,\ j\in I,\quad u_i=q_i-v_i,\ q_i\geq 0,\ v_i\geq 0, i\notin I.\end{gathered} \tag{23}$$

Canonical task of the linear programming has the form

$$\begin{gathered}J(u)=\langle c,u\rangle\to\inf,\\ u\in U=\{u\in E^n/u\geq 0,\ Au-b=0\},\end{gathered} \tag{24}$$

where $c \in E^n$, $b \in E^s$ are the vectors; A is the matrix of the order $s \times n$; the ensemble

$$U_0 = \{u \in E^n / u = (u_1, \dots, u_n) \geq 0\}$$

Lagrange's function for task (24) is written so:

$$L(u, \lambda) = \langle c, u \rangle + \langle \lambda, Au - b \rangle = \langle c + A^* \lambda, u \rangle - \langle b, \lambda \rangle,$$

$$u \in U_0, \ \lambda \in \Lambda_0 = \{\lambda \in E^s\}.$$

Function

$$\psi(\lambda) = \inf_{u \in U_0} L(u, \lambda) = \begin{cases} -\langle b, \lambda \rangle, \textit{если} \ \ c + A^* \lambda \geq 0, \\ -\infty, \quad \textit{если} \ \ c + A^* \lambda < 0. \end{cases}$$

Then the dual problem to the problem (24) has the form

$$-\psi(\lambda) = \langle b, \lambda \rangle \to \inf; \quad c + A^* \lambda \geq 0, \ \ \lambda \in E^s. \qquad (25)$$

It is easy to make sure in the dual task to task (25) complies with task (24). Finally, we note that main and the general task of the linear programming by the way of introducing some additional variables are reduced to the canonical tasks of the linear programming [refer to formulas (20), (23)].

Chapter III. **LINEAR PROGRAMMING**

As it is shown above, the main and the general problem of the linear programming are reduced to the canonical problems of the linear programming. So it is reasonable to develop the general solution method of the canonical linear programming problem. Such general method is a simplex-method. Simplex-method for nondegenerate problems of the linear programming in canonical form is stated below.

Lectures 15, 16

STATEMENT OF THE PROBLEM. SIMPLEX-METHOD

We consider the linear programming problem in the canonical form

$$J(u) = \langle c, u \rangle \to \inf, \quad u \in U = \{u \in E^n / u \ge 0, \; Au - b = 0\}, \quad (1)$$

where $c \in E^n$, $b \in E^m$ are the vectors; A is the matrix of the order $m \times n$. Matrix $A = \|a_{ij}\|$, $i = \overline{1, m}$, $j = \overline{1, n}$ can be represented in the manner of

$$A = \left(a^1, a^2, ..., a^n\right), \quad a^j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{pmatrix}, \quad j = \overline{1,n}.$$

The vectors a^j, $j = \overline{1,n}$ are identified by the condition vectors, but vector $b \in E^m$ - a vector of the restrictions. Now the equation $Au = b$ can be written in the manner of $a^1 u_1 + a^2 u_2 + ... + a^n u_n = b$. Since ensemble $U_0 = \{u \in E^n / u \geq 0\}$ and $U = \{u \in E^n / Au = b\}$ - affine ensemble which are convex, that (1) is a problem of the convex programming. We notice that if ensemble

$$U_* = \left\{u_* \in E^n / u_* \in U, \;\; J(u_*) = \min_{u \in U} \langle c, u \rangle \right\} \neq \varnothing,$$

that Lagrange's function for problem (1) always has saddle point, any point of the local minimum simultaneously is the point of the global minimum and necessary and sufficiently condition of the optimality is written as $\langle J'(u_*), u - u_* \rangle = \langle c, u - u_* \rangle \geq 0, \;\; \forall u \in U_0$. We suppose, that ensemble $U_* \neq \varnothing$. It is necessary to find the point $u_* \in U_*$ and value $J_* = \inf_{u \in U} J(u) = J(u_*) = \min_{u \in U} J(u)$.

Simplex-method. For the first time solution of the problem (1) was considered on simplex

$$U = \left\{ u \in E^n \Big/ u \geq 0, \sum_{i=1}^{n} u_i = 1 \right\}$$

so solution method of such linear programming problem was called by simplex-method. Then method was generalized for event of ensemble U specified in the problem (1), although initial name of the method is kept.

Definition 1. *The point $u \in U$ is identified by extreme (or angular), if it is not represented in the manner of*

$u = \alpha u^1 + (1-\alpha)u^2,\ 0 < \alpha < 1, u^1, u^2 \in U$. *From given definition follows that extreme point is not an internal point of any segment belonging to ensemble U.*

Lemma 1. *Extreme point $u \in U$ has not more m positive coordinates.*

Proof. Not derogating generalities, hereinafter we consider that first components $k,\ k \le m$ of the extreme point are positive, since by the way of recalling the variables always possible to provide the given condition.

We suppose opposite, i.e. that extreme point $u \in U$ has $m+1$ positive coordinates

$$\left(u = \left(u_1 > 0,\, u_2 > 0, ..., u_{m+1} > 0,\ 0, ..., 0\right)\right).$$

We compose the matrix $A_1 = (a^1, a^2, ..., a^{m+1})$ of the order $m \times (m+1)$ from condition vectors corresponding to positive coordinates of the extreme point. We consider the homogeneous linear equation $A_1 z = 0$ for vector $z \in E^{m+1}$. The equation has a nonzero solution $\tilde{z},\ \tilde{z} \ne 0$. We define *n*-vector $\tilde{u} = (\tilde{z}, 0)$ and consider two vectors: $u^1 = u + \varepsilon \tilde{u},\ u^2 = u - \varepsilon \tilde{u}$, where $\varepsilon > 0$ - sufficiently small number. We notice that vectors $u^1,\ u^2 \in U$ under $0 < \varepsilon < \varepsilon_1$, where $\varepsilon_1 > 0$ is sufficiently small number. In fact, $Au^1 = Au + \varepsilon A\tilde{u} = Au + A_1 \tilde{z} = Au = b,\ u^1 = u + \varepsilon \tilde{u} \ge 0$ under sufficiently small $\varepsilon_1 > 0$ similarly $Au^2 = b,\ u^2 \ge 0$. Then the extreme point $u = (1/2)u^1 + (1/2)u^2,\ \alpha = 1/2$. It opposites to the definition of the extreme point. The lemma is proved.

Lemma 2. *Condition vectors corresponding to positive coordinates of the extreme point are linear independent*

Proof. Let $u = (u_1 > 0,\, u_2 > 0, ..., u_k > 0,\ 0, ..., 0) \in U,\ k \le m$ be an extreme point. We show that vectors $a^1, a^2, ..., a^k,\ k \le m$ are linear independent. We suppose opposite, i.e. that there are the numbers $\lambda_1, ..., \lambda_k$ not all equal to zero such that $\lambda_1 a^1 + \lambda_2 a^2 + ... +$

$+\lambda_k a^k = 0$ (the vectors $a^1,...,a^k$ are linear dependent). From inclusion $u \in U$ follows that $a^1u_1 + a^2u_2 + ... + a^ku_k = b,$ $u_i > 0,\ i = \overline{1,k}$. We multiply the first equality on $\varepsilon > 0$ and add (subtract) from the second equality as a result we get $a^1(u_1 \pm \varepsilon\lambda_1) + a^2(u_2 \pm \varepsilon\lambda_2) + ... + a^k(u_k \pm \varepsilon\lambda_k) = b$. We denote by $u^1 = (u_1 + \varepsilon\lambda_1,...,u_k + \varepsilon\lambda_k, 0,...,0) \in E^n$, $u^2 = (u_1 - \varepsilon\lambda_1,...,$ $-\varepsilon\lambda_k, 0,...,0) \in E^n$. There is a number $\varepsilon_1 > 0$ such that $u^1 \in U,$ $u^2 \in U$ under all $\varepsilon,\ 0 < \varepsilon < \varepsilon_1$. Then vector $u = (1/2)u^1 +$ $+(1/2)u^2 \in U$, $u^1 \in U,\ u^2 \in U$. We obtained the contradiction in that $u \in U$ - extreme point. Lemma is proved.

From lemmas 1, 2 follows that:

a) The number of the extreme points of the ensemble U is finite and it does not exceed the sum $\sum_{k=1}^{m} C_n^k$, where C_n^k - a combinations number from n-elements on k. In fact, the number of the positive coordinates of the extreme points is equal to $k,\ k \le m$ (on the strength of lemma 1), but the number of the linear independent vectors corresponding to positive coordinates of the extreme point is equal to C_n^k (on the strength of lemma 2). Adding on k within from 1 till m we get the maximum possible number of the extreme points.

b) The ensemble $U = \{u \in E^n / u \ge 0,\ Au - b = 0\}$ is convex polyhedron with final number of the extreme points under any matrix A of the order $m \times n$.

Definition 2. *The problem of the linear programming in canonical form (1) is identified nondegenerate if the number of the positive coordinates of he feasible vectors not less than rank of the matrix A, i.e. in the equation* $a^1u_1 + a^2u_2 + ... + a^nu_n = b$ *under* $u_i > 0,\ i = \overline{1,n}$ *the number differenced from zero summands not less, than rank of the matrix A.*

Lemma 3. *Let* $rangA = m,\ (m < n)$. *If in the nondegenerate* *problem the feasible vector has exactly m positive coordinates, that u - an extreme point of the ensemble U.*

Proof. Let the feasible vector $u=(u_1>0,\ u_m>0,\ 0,...,0)\in U$ has exactly *m* positive coordinates. We show, that *u* is an extreme point of the ensemble U.

We suppose opposite i.e. there are the points $u^1, u^2\in U,\ u^1\neq u^2$, and number $\alpha,\ 0<\alpha<1$ such that $u=\alpha u^1+(1-\alpha)u^2$ (the point $u\in U$ is not an extreme point). From given presentations follows that $u^1=(u_1^1,...,u_m^1,0,...,0)$, $u^2=(u_1^2,...,u_m^2,0,...,0)$. Let the point $u(\varepsilon)=u+\varepsilon(u^1-u^2)$, $u^1\neq u^2$. Consequently, $u(\varepsilon)=(u_1+\varepsilon(u_1^1-u_1^2),\ u_2+\varepsilon(u_2^1-$ $-u_2^2),...,u_m+\varepsilon(u_m^1-u_m^2),0,...,0)$. We notice, that $Au(\varepsilon)=Au+$ $+\varepsilon(Au^1-Au^2)=b$ under any ε. We assume, there is negative amongst the first *m* coordinates of the vector u^1-u^2. Then by increasing $\varepsilon>0$ from 0 to ∞ we find the number $\varepsilon_1>0$ such that one from *m* first coordinates of the vector $u(\varepsilon_1)$ becomes equal to zero, but all rest will be nonnegative. But it isn't possible in the nondegenerate problem. Similarly if $u^1-u^2>0$ that reducing ε from 0 till $-\infty$ we get the contradiction. Lemma is proved.

We notice, that from $rangA=m$ does not follow that problem of the linear programming (1) will be nondegenerate.

Example 1. Let the ensemble

$$U=\{u=(u_1,u_2,u_3,u_4)\in E^4 / u_j\geq 0,\ j=\overline{1,4};$$

$$3u_1+u_2+u_3+u_4=3,\ u_1-u_2+2u_3+u_4=1\}.$$

In this case the matrix

$$A=\begin{pmatrix}3 & 1 & 1 & 1\\ 1 & -1 & 2 & 1\end{pmatrix}=(a^1,a^2,a^3,a^4),\qquad rangA=2.$$

By the extreme points of the ensemble U are $u^1 = (1,0,0,0)$, $u^2 = (0,5/3,\ 4/3,\ 0)$, $u^3 = (0,1,0,2)$. Here u^2, u^3 are nondegenerate extreme points, u^1 is degenerate extreme point. Since there is feasible vector u^1, number of the positive coordinates less than $rangA$, the problem of the linear programming (1) is not nondegenerate. The number of the extreme points are equal to 3, that does not exceed the amount $c_4^1 + c_4^2 = 10$; the vectors corresponding to the positive coordinates of the extreme point $u^2 - (a^2, a^3)$, $u^3 - (a^2, a^4)$ are linear independent.

Example 2. Let ensemble be

$$U = \{u = (u_1, u_2, u_3, u_4) \in E^4 / u_j \geq 0,\ j = \overline{1,4};$$

$$3u_1 + u_2 + u_3 + u_4 = 3,\quad -u_1 - u_2 + 2u_3 + u_4 = 1\}.$$

Here $rangA = 2$, the extreme points $u^1 = (1/2,\ 0,0,3/2)$, $u^2 = (5/7,0,\ 6/7,\ 0)$, $u^3 = (0,\ 5/3,\ 0,\ 4/3)$, $u^4 = (0,\ 1,\ 0,\ 2)$. The problem (1) is nondegenerate. We notice that in nondegenerate problem the number of the extreme points no more than C_n^m.

Lemma 4. *Any point $u \in U$ can be represented as convex linear combination points of the ensemble U, i.e.* $u = \sum_{k=1}^{s} \alpha_k u^k$, $\alpha_k \geq 0$, $\sum_{k=1}^{s} \alpha_k = 1$, $u^1, u^2, \ldots, u^s$ *- an extreme points of the ensemble U.*

Proof. We prove the lemma for event, when U is convex bounded closed ensemble. In fact, for order the ensemble $U_* \neq \varnothing$, necessary and sufficiency the ensemble U is compactly. It means that U is convex closed bounded ensemble. We prove the lemma by method of mathematical induction for events when $u \in \Gamma p U$ and $u \in \operatorname{int} U$.

Let $u \in \Gamma p U$, $u \in U$. If $n = 1$, that ensemble U is a segment;

consequently, statement of the lemma faithfully, since any point of the segment ($u \in \Gamma p U$, $u \in \operatorname{int} U$) can be represented in the manner of the convex combination of the extreme points (the end of the segment). Let the lemma be true for ensemble $U \subset E^{n-1}(n \geq 2)$. We conduct supporting hyperplane to ensemble $U \subset E^n$ through the point $\bar{u} \in \Gamma p U$, i.e. $\langle c,u \rangle \geq \langle c,\bar{u} \rangle,\ \forall u \in U$. We denote by $U_1 = U \cap \Gamma$, where $\Gamma = \{u \in E^n / \langle c,u \rangle = \langle c,\bar{u} \rangle\}$ - ensemble points of the supporting hyperplane. We notice that ensemble U_1 is convex, bounded and closed, moreover $U_1 \subset E^{n-1}$. Let $u^1, u^2,...,u^{s_1}$ - extreme points of the ensemble U_1, then by hypothesis the point $u = \sum_{k=1}^{s_1} \beta_k u^k \in E^{n+1}$, $\beta_k \geq 0,\ k = \overline{1, s_1}$, $\sum_{k=1}^{s_1} \beta_k = 1$. It is remains to show, the points $u^1, u^2,...,u^{s_1}$ are extreme points and ensembles U. Let $u^i = \alpha w + (1-\alpha)v,\quad w, v \in U$, $0 < \alpha < 1$. We show that $u^i = w = v$ for any extreme point $u^i \in U_1$. In fact, since $\langle c,w \rangle \geq \langle c,\bar{u} \rangle$, $\langle c,v \rangle \geq \langle c,\bar{u} \rangle$ and $\langle c,u^i \rangle = \langle c,\bar{u} \rangle$, that $\langle c,u^i \rangle = \alpha \langle c,w \rangle + \quad + (1-\alpha)\langle c,v \rangle \geq \langle c,\bar{u} \rangle = \langle c,u^i \rangle$. Thence follows that $\langle c,w \rangle = \ = \langle c,v \rangle = \langle c,\bar{u} \rangle$, i.e. $w, v \in U_1$. However the point u^i - an extreme point U_1, consequently, $u^i = w = v$. Therefore the points $u^1,u^2,...,u^{s_1}$ are extreme points of the ensemble U. For border point $u \in \Gamma p U$. Lemma is proved.

Let the point $u \in \operatorname{int} U$. We conduct through $u \in U$ a line l which crosses the borders of the ensemble U in the points $a \in U,\ b \in U$. At the point $u \in \operatorname{int} U$ is representable in the manner of. $u = \alpha a + (1-\alpha)b,\ 0 < \alpha < 1$. The points $a \in \Gamma p U$, $b \in \Gamma p U$

on the strength of proved $a = \sum_{k=1}^{s_1} \mu_k v^k$, $\mu_k \geq 0$, $k = \overline{1, s_1}$, $\sum_{k=1}^{s_1} \mu_k = 1$, $b = \sum_{k=1}^{s_2} \beta_k w^k$, $\beta_k \geq 0$, $k = \overline{1, s_2}$, $\sum_{k=1}^{s_2} \beta_k = 1$, where $v^1, ..., v^{s_1}$; $w^1, ..., w^{s_2}$ are extreme points of the ensemble U. Then the point $u = \alpha\, a + (1-\alpha) b = \sum_{k=1}^{s_1} \alpha\, \mu_k v^k + \sum_{k=1}^{s_1} (1-\alpha) \beta_k w_k$, moreover $\alpha_k = \alpha\, \mu_k \geq 0$, $k = \overline{1, s_1}$, $\overline{\alpha}_k = (1-\alpha)\beta_k \geq 0$, $k = \overline{1, s_2}$, $\sum_{k=1}^{s_1} \alpha_k + \sum_{k=1}^{s_2} \overline{\alpha}_k = \alpha + (1-\alpha) = 1$. Lemma is proved.

Lemma 5. *Let U be convex bounded closed ensemble from E^n, i.e. ensemble $U_* \neq \varnothing$. Then minimum to function $J(u)$ on ensemble U is reached in the extreme point of the ensemble U. If minimum $J(u)$ on U is reached in several extreme points $u^1, ..., u^k$ of the ensemble U, that $J(u)$ has same minimum value in any point* $\overline{u} = \sum_{i=1}^{k} \alpha_i u^i$, $\alpha_i \geq 0$, $\sum_{i=1}^{k} \alpha_i = 1$.

Proof. Let minimum $J(u)$ on U is reached in the point $u_* \in U_*$. If $u_* \in U$ - an extreme point, that lemma is proved.

Let $u_* \in U$ be certain border or internal point of the ensemble U. Then on the strength of lemma 4 we have $u_* = \sum_{i=1}^{s} \alpha_i u^i$, $\alpha_i \geq 0$, $\sum_{i=1}^{s} \alpha_i = 1$, where $u_1, ..., u^s$ - extreme points of the ensemble U . The value $J(u_*) = \langle c, u_* \rangle = \sum_{i=1}^{s} \alpha_i \langle c, u^i \rangle = \sum_{i=1}^{s} \alpha_i J_i$, where $J_i = = J(u^i) = \langle c, u^i \rangle$, $i = \overline{1, s}$.

Let $J_0 = \min_{1 \le i \le s} J_i = J(u^{i_0})$. Then $J(u_*) \ge J_0 \sum_{i=1}^{s} \alpha_i = J_0 = J(u^{i_0})$. Thence we have $J(u_*) = J(u^{i_0})$, consequently, minimum $J(u)$ on U is reached in the extreme point u^{i_0}.

Let the value $J(u_*) = J(u^1) = J(u^2) = ... = J(u^k)$, where $u^1,...,u^k$ - extreme points of the ensemble U. We show that $J(\overline{u}) = J\left(\sum_{i=1}^{k} \alpha_i u^i\right) = J(u_*)$, where $\alpha_i \ge 0, \sum_{i=1}^{k} \alpha_i = 1$. In fact, the value $J(\overline{u}) = \sum_{i=1}^{k} \alpha_i J(u^i) = \sum_{i=1}^{k} \alpha_i J(u_*) = J(u_*)$. Lemma is proved.

We notice that lemmas 4, 5 are true for any problem of the linear programming in the canonical form of the type (1) with restrict closed convex ensemble U. From lemmas 1 - 5 follows that solution algorithm of the linear programming problem in canonical form must be based on transition from one extreme point of the ensemble U to another, moreover under such transition the value of function $J(u)$ in the following extreme point less, than previous. Such algorithm converges to solution of the problem (1) through finite number steps, since number of the extreme points of the ensemble U doesn't exceed the number $\sum_{k=1}^{m} C_n^k$ (in the event of the nondegenerate problem C_n^m). Under each transition from extreme point $u^i \in U$ to the extreme point $u^{i+1} \in U$ necessary to make sure in that, whether the given extreme point $u^i \in U$ be solution of the problem (1). For this a general optimality criterion which easy checked in each extreme point must be existed. Optimality criterion for nondegenerate problem of the linear programming in canonical form is brought below.

Let the problem (1) be nondegenerate and the point $u_* \in U$ - a solution of the problem (1). According to lemma 5 the point $u_* \in U$

is extreme. Since the problem (1) is nondegenerate, that extreme point $u_* \in U$ has exactly m positive coordinates. Not derogating generalities, we consider the first m components of the vector $u_* \in U$ are positive, i.e.

$$u_* = \left(u_1^*, u_2^*, ..., u_m^*, 0, ..., 0\right), \; u_1^* > 0, \; u_2^* > 0, ..., u_m^* > 0 .$$

The vector $c \in E^n$ and matrix A we present in the manner of $c = \left(c_Б, c_Н\right), \; A = \left(A_Б, A_Н\right),$ where $c_Б = (c_1, c_2, ..., c_m) \in E^m$, $c_Н = (c_{m+1}, c_{m+2}, ..., c_n) \in E^{n-m}$, $A_Б = (a^1, a^2, ..., a^m)$, $A_Н =$ $= (a^{m+1}, a^{m+2}, ..., a^n)$. We notice, that according to lemma 2 condition vectors $a^1, a^2, ..., a^m$ corresponding to the positive coordinates of the extreme point $u_* \in U$ are linear independent, i.e. matrix $A_Б$ is nonsingular, consequently, there is the inverse matrix $A_Б^{-1}$.

Lemma 6 *(optimality criterion). In order the extreme point $u_* \in U$ to be a solution of the nondegenerate problem of the linear programming in canonical form (1), necessary and sufficiency to execute the inequality*

$$c_Н' - c_Б' A_Б^{-1} A_Н \geq 0. \tag{2}$$

Proof. Necessary. Let the extreme point be $u_* = \left(u_Б, u_Н^*\right) \in U$, where $u_Б^* = \left(u_1^*, u_2^*, ..., u_m^*\right), \; u_Н^* = (0, ..., 0)$ - a solution of the nondegenerate problem (1). We show, that inequality (2) is executed. Let $u = \left(u_Б, u_Н\right) \in U$, where $u_β = (u_1, ..., u_m)$, $u_Н =$ $= (u_{m+1}, ... u_n)$ - an arbitrary point. We define the ensemble of the feasible directions $\{l\}$ to the point $u_* \in U$. We notice that vector $l \in E^n, l \neq 0$ is identified by feasible direction in the point $u_* \in U$, if there is the number $ε_0 > 0$ such that vector $u = u_* + \varepsilon l \in U$

under all $0 \le \varepsilon \le \varepsilon_0$. We present the vector $l \in E^n$ in the manner of $l = \left(l_Б, l_H\right)$, where $l_Б = \left(l_1, ..., l_m\right) \in E^m, l_H = \left(l_{m+1}, ..., l_n\right) \in E^{n-m}$. From inclusion $u_* + \varepsilon l \in U$ follows that $u_Б^* + \varepsilon l_Б \ge 0,\ u_H^* + \varepsilon l_H = \varepsilon l_H \ge 0,\ A(u_* + \varepsilon l) = A_Б\left(u_Б^* + \varepsilon l_Б\right) + A_H \varepsilon l_H = b$. Since $u_* \in U$, that $Au_* = Au_Б^* = b$ consequently from the last equality we have $A_Б l_Б + A_H l_H = 0$. Thence follows the vector $l_Б = -A_Б^{-1} A_H l_H$. Since under sufficiency small $\varepsilon > 0$ inequality $u_Б^* + \varepsilon l_Б \ge 0$ is executed, the feasible directions in the point u_* are defined by the correlations

$$l_H \ge 0, \quad l_Б = -A_Б^{-1} A_H l_H. \tag{3}$$

Finally, having chosen the arbitrary vectors $l_H \in E^{n-m}, l_H \ge 0$ possible to find $l_Б = -A_Б^{-1} A_H l_H$ and construct the ensemble of the feasible directions L, which each element has the form $l = \left(-A_Б^{-1} A_H l_H, l_H \ge 0\right) \in E^n$, i.e. $L = \left\{l \in E^n / l = \left(l_Б, l_H\right), l_Б = -A_Б^{-1} A_H l_H,\ l_H \ge 0\right\} \subset E^n$. Now any point $u \in U$ can be represented in the manner of $u = u_* + \varepsilon l,\quad l \in L$, $\varepsilon > 0,\ 0 \le \varepsilon \le \varepsilon_0,\ \varepsilon_0 = \varepsilon_0(l)$. Consequently, $u - u_* = \varepsilon l,\quad l \in L$. Since the function $J(u) = \langle c, u \rangle \in C^1(U)$, then in the point $u_* \in U$ necessary inequality $\left\langle J'\left(u_*\right), u - u_*\right\rangle \ge 0, \forall u \in U$ (the lecture 5) is executed. Thence with provision for that $J'\left(u_*\right) = c,\ u - u_* = \varepsilon l$, $l \in L$ we get $\langle c, \varepsilon l\rangle = \left[\left\langle c_Б, l_Б\right\rangle + \left\langle c_H, l_H\right\rangle\right]\varepsilon \ge 0$. Since the number $\varepsilon > 0$, that we have $\left\langle c_Б, l_Б\right\rangle + \left\langle c_H, l_H\right\rangle \ge 0$. Substituting the value $l_Б = -A_Б^{-1} A_H l_H$ from formula (3), we get

$$\left\langle c_Б, -A_Б^{-1} A_H l_H\right\rangle + \left\langle c_H, l_H\right\rangle = \left(c'_H - c'_Б A_Б^{-1} A_H\right) l_H \ge 0$$

under all $l_H \ge 0$. Thence follows the inequality (2). Necessity is proved.

Sufficiency. Let the inequality (2) be executed. We show, that $u_* \in U_* \subset U$ - an extreme point of the ensemble U. Since the function $J(u) \in C^1(U)$ is convex on convex ensemble U, that necessary and sufficiency the execution of the inequality $J(u) - J(v) \geq \langle J'(v), u - v \rangle, \ \forall u, v \in U$ (the lecture 4). Hence in particular, $v = u_* \in U$, then we have

$$J(u) - J(u_*) \geq \langle J'(u_*), u - u_* \rangle = \langle c, u - u_* \rangle = \langle c, \varepsilon l \rangle =$$
$$= \varepsilon\left(\langle c_Б, l_Б \rangle + \langle c_H, l_H \rangle\right) = \varepsilon\left(c_H^{'} - c_Б^{'} A_Б^{-1} A_H\right) l_H \geq 0, \quad \forall l \in L,$$
$$\forall u \in U.$$

Then $J(u_*) \leq J(u), \ \forall u \in U$. Consequently, minimum is reached in the point $u_* \in U$ on U. According to the lemma 5 $u_* \in U$ - an extreme point. Lemma is proved.

It is easy to check the optimality criterion (2) by simplex table formed for the extreme point $u_* \in U_0$.

Basis	c		c_1	…	c_j	…	c_{j_0}	…	c_n	
$A_Б$	$C_б$	b	a^1		a^j	…	a^{j_0}	…	a^n	θ
a^1	c_1	u_1^*	u_{11}	…	u_{1j}	…	u_{1j_0}	…	u_{1n}	
…	…	…	…	…	…	…	…	…	…	…
a^{i_0}	c_{i_0}	$u_{i_0}^*$	$u_{i_0 1}$	…	$u_{i_0 j}$	…	$u_{i_0 j_0}$	…	$u_{i_0 n}$	θ_0
…	…	…	…	…	…	…	…	…	…	…
a^i	c_i	u_i^*	u_{i1}	…	u_{ij}	…	u_{ij_0}	…	u_{in}	θ_i
…	…	…	…	…	…	…	…	…	…	…
a^m	c_m	u_m^*	u_{m1}	…	u_{mj}	…	u_{mj_0}	…	u_{mn}	
	z		z_1	…	z_j	…	z_{j_0}	…	z_n	
	$z-c$		0	…	$z_j - c_j$	…	$z_{j_0} - c_{j_0}$	…	$z_n - c_n$	

The condition vectors $a^1, a^2, ..., a^m$ corresponding to positive coordinates of the extreme point $u_* = \left(u_1^*, ..., u_m^*, 0, ..., 0\right)$ are leaded in the first column of the table. Matrix $A_Б = \left(a^1, ..., a^m\right)$. Coordinates of the vector $c = \left(c_1, ..., c_n\right)$ corresponding to positive coordinates of the extreme point are given in the second column; in the third - corresponding to a^i positive coordinates of the extreme point. In the following columns the decomposition coefficients of the vector a^j, $j = \overline{1,n}$ on base vector $a_i, i = \overline{1,n}$ are reduced. Finally, in the last column the values θ_i which will be explained in the following lecture are shown. We consider the values specified in the last two lines more detail.

Since vectors $a^1, a^2, ..., a^m$ are linear independent (the lemma 2), that they form the base in the Euclidean space E^m, i.e. any vector a^j, $j = \overline{1,n}$ can be singularity decomposed by this base. Consequently,

$$a^j = \sum_{i=1}^{m} a^i u_{ij} = A_Б u^j, u^j = \left(u_{1j}, u_{2j}, ..., u_{mj}\right) \in E^m, j = \overline{1,n}.$$

Thence we have $u^j = A_Б^{-1} a^j, j = \overline{1,n}$. We denote through $z_j = \sum_{i=1}^{m} c_i u_{ij}$, $j = \overline{1,n}$. We notice, that $z_j = c_j, j = \overline{1,m}$, since $u^j = = (0,...,0,1,0,...,0)$, $j = \overline{1,m}$ .. Then vector

$$z - c = \left(\left(z - c\right)_Б, \left(z - c\right)_H = \left(0, \left(z - c\right)_H\right)\right),$$

where $\left(z - c\right)_Б = \left(z_1 - c_1, z_2 - c_2, ..., z_m - c_m\right) = \left(0,...,0\right)$,

$$\left(z - c\right)_H = \left(z_{m+1} - c_{m+1}, ..., z_n - c_n\right).$$

Since the values $z_j = \sum_{i=1}^{m} c_i u_{ij} = c'_Б u^j = c'_Б A_Б^{-1} a^j, j = \overline{1,n}$, that $z_j - c_j = c'_Б A_Б^{-1} a^j - \ c_j, \ j = \overline{1,n}$. Consequently, vector $(z-c)' =$ $= (0, c'_Б A_Б^{-1} A_Н - \ - c'_Н)$.

Comparing the correlation with optimality criterion (2), we make sure in the extreme point $u_* \in U$ to be a solution of the problem necessary and sufficiency that values $z_j - c_j \le 0, j = \overline{1,n}$. Finally, by sign of the values in the last line of the simplex-table it is possible to define whether the point $u_* \in U$ a solution of the nondegenerate problem (1).

Lecture 17

DIRECTION CHOICE. NEW SIMPLEX-TABLE CONSRUCTION. THE INITIAL EXTREME POINT CONSRUCTION

In the linear programming problem of the canonical form minimum to linear function is reached in the extreme point of the convex polyhedron *U*. A source extreme point in the simplex-method is defined by transition from one extreme point to the following, moreover value of the linear function in the next extreme point less, than in previous. It is necessary to choose the search direction from the extreme point and find the following, in the last to check the optimality criterion etc. Necessity in determination of the initial extreme point to use the simplex-method for problem solution is appeared.

We consider the nondegenerate problem of the linear programming in canonical form

$$J(u) = \langle c, u \rangle \to \inf,\ u \in U = \{u \in E^n / u \geq 0,\ Au - b = 0\}. \qquad (1)$$

Let $\bar{u} \in U$ be an extreme point and in the point $\bar{u} \in U$ the minimum $J(u)$ on U isn't reached, i.e. optimality criterion $z_j - c_j \leq 0, j = \overline{1,n}$ is not executed. Then necessary to go from given extreme point to other extreme point $\bar{\bar{u}} \in U$, where value $J(\bar{\bar{u}}) < J(\bar{u})$. It is necessary to choose the direction of the motion from given extreme point $\bar{u}$.

Direction choice. Since $\bar{u} \in U$ is an extreme point, not derogating generalities, it is possible to consider that the first m components of the vector $\bar{u}$ are positive, i.e. $\bar{u} = (\bar{u}_1, \bar{u}_2, ..., \bar{u}_m,$ $0,...,0)$, $\bar{u}_i > 0$, $i = \overline{1,m}$. According to formula (2) (the lecture 16), feasible directions in the point $\bar{u}$ are defined from correlations $l_H \geq 0, l_Б = -A_Б^{-1} A_H l_H$. Derivative of function $J(u)$ on feasible direction l in the point $\bar{u}$ is equal to

$$\partial J(\bar{u}) / \partial l = \langle c, l \rangle = \langle c_Б, l_Б \rangle + \langle c_H, l_H \rangle = \left(c_H' - c_Б' A_Б^{-1} A_H\right) l_H =$$

$$= -\sum_{j=m+1}^{n} \left(z_j - c_j\right) l_j,\ l_j \geq 0,\ j = \overline{m+1, n}.$$

Necessary to choose the feasible direction l^0 in the point $\bar{u}$ from minimum condition $\partial J(\bar{u}) / \partial l$ on ensemble of the feasible directions L.

The source direction $l^0 \in L$ is chosen by the following in the simplex-method:

a) Index $j_0 \in I_1$ is defined, where $I_1 = \{ j / m+1 \leq j \leq n,$ $z_j - c_j > 0 \}$ from condition $z_{j_0} - c_{j_0} = \max_{j \in I_1} (z_j - c_j)$. Since in the point $\bar{u} \in U$ minimum $J(u)$ on U is not reached, i.e. the inequality $z_j - c_j \leq 0$ under all j, $1 \leq j \leq n$ aren't executed, that ensemble $I_1 \neq \varnothing$.

b) Vector $l_H^0 \geq 0$, $l_H^0 \in E^{n-m}$ is chosen so that $l_H^0 = (0,...,0,1,0,...,0)$, i.e. j_0 - a component of the vector l_H^0 is equal to 1, but all rest components are equal to zero.

Finally, the motion direction l^0 in the point $\bar{u} \in U$ is defined by the correlations:

$$l^0 \in L,\ l^0 = \left(l_Б^0, l_H^0\right),\ l_H^0 = (0,...,0,\ 1,\ 0,...0),$$

$$l_Б^0 = -A_Б^{-1} A_Н l_Н^0 = -A_Б^{-1} a^{j_0} = -u^{j_0} =$$
$$= (-u_{1j_0}, -u_{2j_0}, ..., -u_{mj_0}) \qquad (2)$$

We notice, that derivative of the function $J(u)$ in the point $\overline{u}$ in the line of l^0 is equal to $\partial J(\overline{u})/\partial l^0 = \langle c, l^0 \rangle = -(z_{j_0} - c_{j_0}) < 0$.

It is follows to note that in general event $\partial J(\overline{u})/\partial l^0$ isn't the least value $\partial J(\overline{u})/\partial l$ on ensemble L. However such direction choice l^0 allows to construct a transition algorithm from one extreme point to other.

We consider the points ensemble from U along chosen direction l^0. These points $u(\theta) = \overline{u} + \theta l^0 \in U$, $\theta \geq 0$. Since $\overline{u} = (\overline{u}_1 > 0, ..., \overline{u}_m > 0, 0, ..., 0) \in U$ - an extreme point, $\theta l^0 = = \theta(l_Б^0, l_Н^0) = \theta(-u^{j_0})$, that the points $u(\theta) = (\overline{u}_1 - \theta u_{1j_0}$, $\overline{u}_2 - \theta u_{2j_0}, ..., \overline{u}_m - \theta u_{mj_0}, 0, ..., 0, \theta, 0, ..., 0)$, $\theta \geq 0$. We enter the indexes ensemble $I_2 = \{ i / 1 \leq i \leq m, u_{ij_0} > 0 \}$. We define the values $\theta_i = \overline{u}_i / u_{ij_0}$, $i \in I_2$. Let

$$\theta_0 = \min_{i \in I_2} \theta_i = \min_{i \in I_2} (\overline{u}_i / u_{ij_0}) = \theta_{i_0} > 0, i_0 \in I_2 .$$

Then for values $\theta = \theta_0$ vector $u(\theta_0) = (\overline{u}_1 - \theta_0 u_{1j_0}, ..., \overline{u}_{i_0-1} - -\theta_0 u_{i_0-1 j_0}, 0, \overline{u}_{i_0+1} - \theta_0 u_{i_0+1 j_0}, ..., \overline{u}_m - \theta_0 u_{mj_0}, 0, ..., 0, \theta_0, 0, ..., 0)$. We notice that $u(\theta_0) \geq 0$, $Au(\theta_0) = A\overline{u} + \theta_0 A l_0 = A\overline{u} + A_Б l_Б^0 + + A_Н l_Н^0 = A\overline{u} = b$, consequently, point $u(\theta_0) \in U$. On the other hand, vector $u(\theta_0)$ has exactly *m* positive coordinates. It means that $u(\theta_0) = \overline{\overline{u}}$ is the extreme point of the ensemble *U*. We calculate value $J(u(\theta_0)) = J(\overline{u} + \theta_0 l^0) = \langle c, \overline{u} + \theta_0 l^0 \rangle = \langle c, \overline{u} \rangle + \theta_0 \langle c, l^0 \rangle =$

$= J(\overline{u}) - \theta_0(z_{j_0} - c_{j_0})$. Thence we have $J(u(\theta_0)) - J(\overline{u}) = -\theta_0(z_{j_0} - c_{j_0}) < 0$. Then $J(u(\theta_0)) = J(\overline{\overline{u}}) < \; < J(\overline{u})$, i.e. in the extreme point $u(\theta_0) = \overline{\overline{u}}$ value $J(u)$ less, than in extreme point $\overline{u}$.

We note the following:

1) If ensemble of the indexes $I_2 = \varnothing$, i.e. $u_{ij_0} < 0, i = \overline{1,m}$, then for any $\theta > 0$ the point $u(\theta) \in U$, moreover $J(u(\theta)) = J(\overline{u}) - \theta(z_{j_0} - c_{j_0}) \to -\infty$ under $\theta \to +\infty$. In this case source problem has a no solution. However, if ensemble U is compact, that it isn't possible.

2) It can turn out so that $\theta_0 = \theta_{i_0} = \theta_{\bar{i}_0}$, $i_0 \in I_2$, $\bar{i}_0 \in I_2$. In this case vector $u(\theta_0)$ has $m-1$ positive coordinates. It means that source problem is degenerate. In such events possible appearance "thread", i.e. through determined number steps once again render in the extreme point $u(\theta_0)$. There are the different methods from "recirculations". We recommend the following books on this questions: Gabasov R., Kirillova F. M. Methods of the optimization. Minsk: BGU, 1975, Karmanov V.G. Mathematical programming. M.: Science, 1975; Moiseev N.N., Ivanilov U.P., Stolyarova E.M. Methods of the optimization. M.: Science, 1978.

To make sure in the extreme point $u(\theta_0) = \overline{\overline{u}} \in U$ is a solution of the problem (1), need to build the simplex-table for the extreme point $\overline{\overline{u}}$ with aim to check the optimality criterion (2) (lecture 16) in the point $\overline{\overline{u}}$.

Construction of the new simplex-table. The simplex-table built for the extreme point $u_* \in U$ in the previous lecture, in particular for the extreme point $\overline{u} \in U$ is true. In the simplex-table column j_0 and line i_0 and values $\theta_i, i \in I_2$ are indicated.

We build the simplex-table for the extreme point $u(\theta_0) = \overline{\overline{u}} \in U$ on base of simplex-table for the point $u_* = \overline{u}$. We notice, that

$\overline{u} \in U$ by the base vectors were condition vectors $a^1,...,a^m$ for the point corresponding to positive coordinates of the extreme point $\overline{u}$.

By the base vectors will be

$$a^1, a^2, ..., a^{i_0-1}, a^{j_0}, a^{i_0+1}, a^m, \qquad (2^*)$$

for extreme the point $u(\theta_0) = \overline{\overline{u}}$.

as condition vectors corresponding to positive coordinates. Thereby, the first column of the new simplex-table differs from previous, that instead of vector a^{i_0} is written the vector a^{j_0}. In the second column instead c_{i_0} is written c_{j_0}, in one third column nonnegative components of the vector $\overline{\overline{u}}$ are written, since $a^1\overline{\overline{u}}_1 + a^2\overline{\overline{u}}_2 + ... + a^{i_0-1}\overline{\overline{u}}_{i_0-1} + a^{j_0}\theta_0 + a^{i_0+1}\overline{\overline{u}}_{i_0+1} + ... + a^m\overline{\overline{u}}_m = b,$ where $\overline{\overline{u}}_i = \overline{\overline{u}}_i - \theta_0\overline{\overline{u}}_{ij_0}$, $i = \overline{1,m},\ i \neq i_0$. In the rest columns of the new simplex-table must be decompositions coefficients of the vector $a^j, j = \overline{1,n}$, on new base (2*). There were the decompositions $a^j = \sum_{i=1}^{m} a^i u_{ij}, j = \overline{1,n}$. in the previous base $a^1,...,a^m$. Thence follows that $a^j = \sum_{\substack{i=1\\ i\neq i_0}}^{m} a^i u_{ij} + a^{i_0}u_{i_0 j}, j = \overline{1,n}$. hen under $j = j_0$ we have

$$a^{j_0} = \sum_{\substack{i=1\\ i\neq i_0}}^{m} a^i u_{ij_0} + a^{i_0}u_{i_0 j_0}, a^{i_0} = -\sum_{\substack{i=1\\ i\neq i_0}}^{m} a^i \frac{u_{ij_0}}{u_{i_0 j_0}} + \frac{1}{u_{i_0 j_0}} a^{j_0}. \qquad (3)$$

Substituting value a^{i_0} from formula (3) we get

$$a^j = \sum_{\substack{i=1\\ i\neq i_0}}^{m} a^i u_{ij} + a^{i_0}u_{i_0 j} =$$

$$=\sum_{\substack{i=1\\ i\neq i_0}}^{m} a^i\left(u_{ij}-\frac{u_{ij_0}u_{i_0 j}}{u_{i_0 j_0}}\right)+\frac{u_{i_0 j}}{u_{i_0 j_0}}a^{j_0},\quad j=\overline{1,n}. \qquad (3^*)$$

The formula (3*) presents by the decompositions of the vectors $a^j, j=\overline{1,n}$ on new base (2*). From expression (3*) follows that in the new simplex-table coefficients $\left(u_{ij}\right)_{нов}$ are determined by formula $\left(u_{ij}\right)_{нов}=\dfrac{u_{ij}u_{i_0j_0}-u_{ij_0}u_{i_0j}}{u_{i_0j_0}}, i\neq i_0$, but in line i_0 of the new simplex-tables must be $(u_{i_0 j})_{нов}=u_{i_0 j}/u_{i_0 j_0},\quad j=\overline{1,n}.$ Since vector a^{j_0} in base then in column j_0 of the new simplex-table all $(u_{ij_0})_{нов}=0,\ i\neq i_0$, $(u_{i_0 j_0})_{нов}=1$. Finally, coefficients $(u_{ij})_{нов}$, $i=\overline{1,m}$, $j=\overline{1,n}$, are calculated by the known coefficients u_{ij}, $i=\overline{1,m}$, $j=\overline{1,n}$ of the previous simplex-table. Hereinafter, the last two lines of the new simplex-table are calculated by the known $\left(c_Б\right)_{нов}=\left(c_1,\ldots,c_{i_0-1},c_{j_0},c_{i_0+1},\ldots,c_m\right)$ and $(u_{ij})_{нов}$, $i=\overline{1,m}$, $j=\overline{1,n}$. If it turns out $(z_j-c_j)_{нов}\leq 0,\ j=\overline{1,n}$, that $\overline{\overline{u}}\in U$ - a problem solution. Otherwise transition to the following extreme point of the ensemble U and so on are fulfilled.

Construction of the initial extreme point. As follows from lemmas 2, 3, the extreme point can be determined from system of the algebraic equations $A_Б u_Б=b,\quad u_Б>0$, where $A_Á=$ $=(a^{j_1},a^{j_2},\ldots,a^{j_m}),\ a^{j_k},\ i=\overline{1,m}$, - linear independent vectors, column of the matrix A $u_Б=(u_{j_1},u_{j_2},\ldots,u_{j_m})$. However such determination way of the extreme point $\overline{u}=(0,\ldots,0,u_{j_1},0,\ldots,0,$ $u_{j_m},0,\ldots,0)$ is complete enough, when matrix A has the great dimensionality.

1. Let the main task of the linear programming is given

$$J(u) = \langle c,u \rangle \to \inf, \quad u \in U = \{u \in E^n / u \geq 0,\ Au - b \leq 0\}, \quad (4)$$

where $c \in E^n, b \in E^n$ - the vectors; A - the matrix of the order $m \times n$. We suppose that vector $b > 0$. By entering of the additional variables $u_{n+1} \geq 0, i = \overline{1,m}$ problem (4) can be represented to the canonical form

$$J(u) = \langle c,u \rangle \to \inf, \quad [Au]_i + u_{n+i} = b_i, \quad i = \overline{1,m}, \quad (5)$$

$$u = (u_1,...,u_n) \geq 0, \quad u_{n+i} \geq 0, \quad i = \overline{1,m}.$$

Entering some denotes $\bar{c} = (c,0) \in E^{n+m}$, $\bar{u} = (u, u_{n+1},..., u_{n+m}) \in E^{n+m}$, $\bar{A} = (A, I_m) = (A, a^{n+1}, a^{n+2},..., a^{n+m})$, $a^{n+k} =$ (0,...,0,1, 0,...,0), $k = \overline{1,m}$, where I_m - a unit matrix of the order $m \times m$, problem (5) is written in the manner of

$$J(\bar{u}) = \langle \bar{c},\bar{u} \rangle \to \inf, \quad \bar{u} \in \bar{U} = \{\bar{u} \in E^{n+m} / \bar{u} \geq 0,\ \bar{A}\bar{u} = b\}. \quad (6)$$

For problem (6) initial extreme point $\tilde{u} = (0,...,0,b_1,...,b_m) \in E^{n+m}$, since condition vectors $a^{n+1},...,a^{n+m}$ (the columns of the unit matrix I_m) corresponding to positive coordinates of the extreme point $\tilde{u}$, $b > 0$ are linear independent.

2. **Danzig's** method (two-phase method). We consider canonical problem of the linear programming

$$J(u) = \langle c,u \rangle \to \inf, \quad u \in U = \{u \in E^n / u \geq 0,\ Au - b = 0\}, \quad (7)$$

where $c \in E^n, b \in E^n$; A - a matrix of the order $m \times n$. We suppose that vector $b > 0$ (if $b_i \neq 0, i = \overline{1,m}$, that being multiplied by the first line, where $b_i < 0$, always possible to provide to $b_i > 0, i = \overline{1,m}$).

The following canonical problem on the first stage (phase) is solved:

$$\sum_{i=1}^{m} u_{n+i} \to \inf,$$
$$[Au]_i + u_{n+i} = b_i,\ i=\overline{1,m},\ \ u \ge 0,\ u_{n+i} \ge 0,\ i=\overline{1,m}. \tag{8}$$

By initial extreme point for problem (8) is vector $\tilde{u} = (0,...,0,b_1,...,b_m) \ge 0$. Hereinafter problem (8) is solved by the simplex -method and its solution $\tilde{u}_* = \left(\tilde{u}_1^*, \tilde{u}_2^*, ..., \tilde{u}_n^*, 0, ..., 0\right)$ is found. We notice, that if the source problem (7) has a solution and it is nondegenerate, that vector $\overline{u}_* = \left(\tilde{u}_1^*, \tilde{u}_2^*, ..., \tilde{u}_n^*\right) \in E^n$ has exactly m positive coordinates and it is the initial extreme point for the problem (7), since lower bound in (8) is reached under $u_{n+i} = 0, i = \overline{1,m}$.

On the second stage the problem (7) with initial extreme point $\overline{u}_* \in U$ is solved by the simplex-method.

3. Charnes' method (M-method). Charnes' method is a generalization of Danzig's method by associations of two stages of the problem solution.

So called M-problem of the following type instead of source problem (7) is considered:

$$\langle c,u\rangle + M\sum_{i=1}^{m} u_{n+i} \to \inf, \tag{9}$$

$$[Au]_i + u_{n+i} = b_i, u \ge 0, u_{n+i} \ge 0, i = \overline{1,m},$$

where $M > 0$ - feasible large number. For M-problem (9) the initial extreme point $\tilde{u} = (0,...,0,b_1,b_2,...,b_m) \in E^{n+m}, b > 0$, and it is solved by the simplex-method. If the source problem (7) has a solution, that M- problem has such solution: $\widetilde{u}_* = \left(\widetilde{u}_1^*, ..., \widetilde{u}_n^*, 0, ..., 0\right)$, where components $\tilde{u}^*_{n+i} = 0, i = \overline{1,m}$. Vector $u_* = \left(\tilde{u}_1^*, \tilde{u}_2^*, ..., \tilde{u}_n^*\right) \in E^n$ is the solution of the problem (7).

We notice, that for M-problem values $z_j - c_j = \alpha_j M + \, + \beta_j$, $j = \overline{1,n}$. So in the simplex-table for lines z_j, $z_j - c_j$ are conducted two lines: one for coefficients α_j, other for β_j. Index j_0, where $z_{j_0} - c_{j_0} = \max_j (z_j - c_j)$, $z_j - c_j > 0$ is defined by the value of the coefficients α_j, $\alpha_j > 0$.

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TASKS FOR INDEPENDENT WORK

For undertaking of the practical laboratory lessons it is reasonable to have a short base to theories required for solution of the problems and examples. To this effect in 1981 Methodical instructions on course "Methods of the optimization" prepared by S.A.Aysagaliev and T.N.Biyarov were released in al-Farabi Kazakh national university. Problems and bases to the theories on sections of the course "Methods of the optimization" on base of the mentioned workbook are brought in the appendix.

P.1.1. Multivariable function minimization in the absence thereof restrictions

Statement of the problem. Let scalar function $J(u)=J(u_1,...,u_n)$ be determined in all space E^n. To solve the following optimization problem:

$$J(u)\to \inf, \quad u\in E^n$$

The point $u_*\in E^n$ is identified by the point of the minimum $J(u)$ on E^n, if $J(u_*)\le J(u)$, $\forall u\in E^n$. The variable $J(u_*)$ is identified by least or minimum value of the function $J(u)$ on E^n. We notice that absolute (global) minimum $J(u)$ on E^n is reached in the point $u_*\subset E^n$.

The point $u_0\in E^n$ is identified by the point of the local minimum $J(u)$ on E^n, if $J(u_0)\le J(u)$ under all $|u-u_0|<\varepsilon$, $\varepsilon>0$ is sufficiently small number. Usually first define the points of the local minimum and then amongst of them find the points of the global minimum.

Following theorems known from the course of the mathematical analysis:

Theorem 1. *If function $J(u)\in C^1(E^n)$, then in the point $u_0\in E^n$ the equality $J'(u_0)=0$ (necessary first-order condition) is executed.*

Theorem 2. *If function* $J(u) \in C^2(E^n)$, *then in the point* $u_0 \in E^n$ *the following conditions:* $J'(u_0) = 0,\ \ J''(u_0) \geq 0$ *(necessary condition of the second order) are executed.*

Theorem 3. *In order to the point* $u_0 \in E^n$ *be a point of the local minimum to function* $J(u) \in C^2(E^n)$, *necessary and sufficiency*

$$J'(u_0) = 0,\ J''(u_0) > 0.$$

We notice, that problem $J(u) \to \sup,\ u \in E^n$ is tantamount to the problem $-J(u) \to \inf, u \in E^n$.

To find points of the local and global minimum function:

1. $J(u) = J(u_1,u_2,u_3) = u_1^2 + u_2^2 + u_3^2 - u_1u_2 + u_1 - 2u_3,$

$$u = (u_1, u_2, u_3) \in E^3.$$

2. $J(u_1,u_2) = u_1^3u_2^2(6 - u_1 - u_2),\ \ u = (u_1,u_2) \in E^2.$
3. $J(u_1,u_2) = (u_1 - 1)^2 - 2u_2^2,\ \ \ u = (u_1,u_2) \in E^2.$
4. $J(u_1,u_2) = u_1^4 + u_2^4 - 2u_1^2 + 4u_1u_2 - 2u_2^2,\ \ u = (u_1,u_2) \in E^2.$
5. $J(u_1,u_2) = (u_1^2 + u_2^2)e^{-(u_1^2+u_2^2)},\ u = (u_1,u_2) \in E^2.$
6. $J(u_1,u_2) = \dfrac{1 + u_1 - u_2}{\sqrt{1 + u_1^2 + u_1^2}},\ \ \ u = (u_1,u_2) \in E^2.$
7. $J(u_1,u_2,u_3) = u_1 + \dfrac{u_2^2}{4u_1} + \dfrac{u_3^2}{u_2} + \dfrac{2}{u_3},\ u_1 > 0,\ u_2 > 0,\ \ u_3 > 0.$
8. $J(u_1,u_2) = u_1^2 - u_1u_2 + u_2^2 - 2u_1 + u_2,\ \ u = (u_1,u_2) \in E^2.$
9. $J(u_1,u_2) = \sin u_1 \sin u_2 \sin(u_1 + u_2),\ 0 \leq u_1,\ \ u_2 \leq \pi.$
10. $J(u_1,\dots,u_n) = u_1u_2^2\dots u_n^n(1 - u_1 - 2u_2 - \dots - nu_n),\ \ u_i \geq 0,\ \ \ i = \overline{1,n}.$

P.1.2. Convex ensembles and convex functions

Let U be a certain ensemble E^n, but function $J(u) = J(u_1,...,u_n)$ is determined on ensemble U. The ensemble U is identified by convex, if the point $\alpha u + (1-\alpha)v \in U$, $\forall u,v \in U$ and under all $\alpha, \alpha \in [0,1]$. Function $J(u)$ is convex on convex ensemble U, if

$$J(\alpha u + (1-\alpha)v) \leq \alpha J(u) + (1-\alpha)J(v), \\ \forall u,v \in U, \quad \alpha \in [0,1]. \tag{1}$$

Function $J(u)$ is strictly convex on U, if in the equality (1) possible under only $\alpha = 0,\ \ \alpha = 1$. Function $J(u)$ is concave (strictly concave) on U, if function $-J(u)$ is concave (strictly concave) on U. Function $J(u)$ is strictly concave on U, if

$$J(\alpha u + (1-\alpha)v) \leq \alpha J(u) + (1-\alpha)J(v) - \alpha(1-\alpha)\kappa|u-v|^2, \\ \kappa > 0, \ \ \forall u,v \in U, \quad \alpha \in [0,1] \tag{2}$$

Theorem 1. *Let U be a convex ensemble of E^n. Then in order to the function $J(u) \in C^1(U)$ be a convex on U necessary and sufficiency to execute one of the following inequalities:*

$$J(u) - J(v) \geq \langle J'(v), u-v \rangle, \forall u,v \in U, \tag{3}$$

or

$$\langle J'(u)\text{-}J'(v), u\text{-}v \rangle \geq 0, \forall u,v \in U. \tag{4}$$

If $\operatorname{int} U \neq \varnothing$ и $J(u) \in \mathrm{C}^2(U)$*, then for convexity of $J(u)$ on U necessary and sufficiency that*

$$\langle J''(u)\xi, \xi \rangle \geq 0, \quad \forall \xi \in E^n, \quad \forall u \in U.$$

Theorem 2. *In order to the function $J(u) \in C^1(U)$ be strictly convex on convex ensemble U necessary and sufficiency execution one of the following two conditions:*

1) $J(u)-J(v)\geq\langle J'(v),u\text{-}v\rangle+\kappa|u-v|^2,\quad \kappa>0,\ \forall u,v\in U;$ (5)

2) $\langle J'(u)-J'(v),u-v\rangle\geq\mu|u-v|^2,\quad \mu=2\kappa>0,\ \forall u,v\in U.$
(6)

If int $U\neq\varnothing$ and $J(u)\in C^2(U)$*, then for strictly convexity of* $J(u)$ *on* U *necessary and sufficiency that*

$$\langle J''(u)\xi,\xi\rangle\geq\mu\left|\xi^2\right|,\ \mu=2\kappa>0,\quad \forall u,v\in U.$$

To solve the following problems on base of the determination (1), (2) and correlations (3) - (6):

1. To prove that intersection of any number convex ensembles are convex.

Is this statement for unions of ensembles faithfully? To prove that closing of the convex ensemble is convex.

2. a) Is ensemble $U\subset E^n$ convex, if for any points $u,v\in U$ the point $(u+v)/2\in U$? b) Is closed ensemble U convex (under performing of the previous condition)?

3. Let $u_0\in E^n$, but number $r\in E^1, r>0$. Is ensemble $V=\{u\in E^n\,/\,|u-u_0|\leq r\}\setminus\{u_0\}$ (the ball without the centre) convex?

4. To show the convexity of the following function on E^1:

а) $J(u)=e^u$; *б*) $J(u)=|u|$;

в) $J(u)=\begin{cases} a(u\text{-}c), & a>0,\ u\leq c,\\ b(u\text{-}c), & b>0,\ u\leq c;\end{cases}$

г) $J(u)=\begin{cases} 0, & u\leq c,\\ a(u\text{-}c), & u\geq c,\ a>0.\end{cases}$

5. To prove that function $J(u)=1/u$ is strictly convex under $u>0$ and strictly concave under $u<0$ [using only determinations to strict convexity (the concavity)].

6. Let function $\varphi(v)$ is continuous and $\varphi(v)\geq 0,\quad -\infty<v<\infty$.

Then function $J(u)=\int\limits_{u}^{\infty}(v-u)\varphi(v)dv$ is convex on E^1. To prove.

7. Let $J(u_1,u_2)=\dfrac{1}{u_2}\int\limits_{u_1}^{\infty}(\xi-u_1)\varphi(\xi)d\xi,\quad u_2>0,\ \varphi(\xi)\geq 0$, $-\infty<\xi<+\infty$. Is $J(u_1,u_2)$ convex function on ensemble $U=\{u=(u_1,u_2)/u_1\in E^1,\ u_2>0)\}$?

8. To prove that $J(u)=\sum\limits_{\mathrm{i=1}}^{n}\alpha_i J_i(u)$ - a convex function on convex ensemble U of E^n, if negative coefficients α_i correspond to the concave functions $J_i(u)$, but positive α_i - convex functions $J_i(u)$.

9. To show, if $J(u)$ is convex and ensemble of values u satisfying to condition $J(u)=b,\ \ \forall b,b\in E^1$ is convex that $J(u)$ certainly is a linear function.

10. Function $J(u)$ is convex on U, if and only if function $g(\alpha)=J(u+\alpha(v-u))$ by one variable $\alpha,\ 0\leq\alpha\leq 1$ is convex under any $u,v\in U$. If $J(u)$ is strictly convex on U, that $g(\alpha),\ 0\leq\alpha\leq 1$ is strictly convex. To prove.

11. Function $J(u)$ determined on convex ensemble U from E^n is identified quasiconvex, if $J(\alpha u+(1-\alpha)v)\leq\max\{J(u),J(v)\}$, $\forall u,v\in U$ and under all $\alpha,\ 0\leq\alpha\leq 1.$ Is any convex function quasiconvex, conversely? To prove that $J(u)$ is quasiconvex on U if and only if ensemble $M(v)=\{u\in U/J(u)\leq J(v)\}$ is convex under all $v\in U$.

12. To check in each of the following exercises whether the function $J(u)$ convex (concave) on given ensemble *U*, or indicate such points from *U* in the neighborhood of which $J(u)$ is not neither convex, nor concave:

$a)\ J(u)=u_1^6+u_2^2+u_3^2+u_4^2+10u_1+5u_2-3u_4-20,\ U\equiv E^4;$

$б)\ J(u)=e^{2u_1+u_2},\ u\in E^2;$

в) $J(u) = -u_1^5 + 0{,}5u_3^2 - 7u_1 - u_3 + 6,$

$$U = \{u = (u_1, u_2, u_3) \in E^3 / u_i \le 0,\ i = 1,2,3\};$$

г) $J(u) = 6u_1^2 + u_2^3 + 6u_3^2 + 12u_1 - 8u_2 + 7,\quad U = \{u \in E^3 / u \ge 0\}.$

13. Let $J(u) = |Au - b|^2 = \langle Au - b, Au - b\rangle$, where A is the matrix of the order $m \times n,\ b \in E^m$ - the vector. To prove that $J(u)$ is convex on E^n. If A^*A is nondegenerate, that $J(u)$ is strictly convex on E^n. To find $J'(u), J''(u)$.

14. Let $J(u) = 0{,}5\langle Au, u\rangle - \langle b, u\rangle$, where $A = A^*$ is the matrix of the order $n \times n,\ b \in E^n$ - the vector. To prove that: 1) if $A = A^* \ge 0$, that $J(u)$ is convex on convex ensemble *U* from E^n; 2) if $A=A^*>0$, that $J(u)$ is strictly convex on *U*, moreover $\kappa = \lambda_1 / 2,$ where $\lambda_1 > 0$ - the least proper number of the matrix *A* .

15. To prove that ensemble *A* is convex iff, when for any numbers $\lambda \ge 0;\ \mu \ge 0$ the equality $(\lambda + \mu)A = \lambda A + \mu A$ is executed.

16. To prove that if $A_1, \ldots, A_m$ - a convex ensemble, that

$$Co\left(\bigcup_{i=1}^{m} A_i\right) = \left\{u \in E^n \middle/ u = \sum_{i=1}^{n} \alpha_i u_i,\ \alpha_i \ge 0,\ \sum_{i=1}^{n} \alpha_i = 1\right\}.$$

17. Let $J(u)$ be a continuous function on convex ensemble *U,* moreover for any $u, v \in U$ the inequality $J\left(\dfrac{u+v}{2}\right) \le [J(u) + J(v)]/2.$ is executed. To prove that $J(u)$ is convex on *U*.

18. To realize, under what values of the parameter α the following functions are convex: a) $J(u_1, u_2) = \alpha u_1^2 u_2^2 + (u_1 + u_2)^4$; b) $J(u) = \alpha u_1^2 u_2^2 + (u_1^2 + u_2^2)^2$.

19. To find on plane of the parameters (α, β) areas, where function $J(u_1, u_2) = u_1^\alpha u_2^\beta,\ u_1 > 0,\ u_2 > 0$ is convex (strictly convex) and concave (strictly concave).

20. To find on plane E^2 areas, where function $J(u_1, u_2) = e^{u_1 u_2}$ is convex, and areas in which it is concave.

21. Let $J_i(u),\ i = \overline{1,m}$ be convex, nonnegative, monotonous increasing on E^1 functions. To prove that function $J(u) = \prod_{i=1}^{m} J_i(u).$ possesses by these characteristics.

22. Let $J(u)$ be a function determined and convex on convex ensemble *U*. To prove that: a) ensemble $\Gamma = \left\{ v \in E^n / \sup_{u \in U} \left[\langle v, u \rangle - \right.\right.$ $\left.\left. - J(u) \right] \right\}$ is inempty and convex; b) function $J^*(u) = \sup_{u \in U} \left[\langle v, u \rangle - \right.$ $\left. - J(u) \right]$ is identified by conjugate to $J(u)$. Will $J(u)$ convex on *U*?

23. Let function $J(u)$ be determined and convex on convex ensemble U from E^n. To prove that for any (including for border) points $v \in U$ the inequality $\underline{\lim}_{u \to v} J(u) \le J(v)$ (the semicontinuity property from below) is executed.

24. Let function $J(u)$ be determined and convex on convex closed bounded ensemble *U*. Is it possible to confirm that: a) $J(u)$ is upper lower and bounded on U;

b) $J(u)$ reaches upper and lower border on ensemble *U*?

25. If $J(u)$ is convex (strictly convex) function in E^n and matrix $A \ne 0$ of the order $n \times n,\ b \in E^n$, that $J(Au + b)$ is also convex (strictly convex). To prove.

P.1.3. Convex programming. KUNA-TUCKER'S theorem

We consider the next problem of the convex programming:

$$J(u) \to \inf \tag{1}$$

at condition

$$u \in U = \{u \in E^n / u \in U_0;\ g_i(u) \le 0,\ i = \overline{1,m};$$

$$g_i(u) = \langle a_i, u \rangle - b_i \le 0,\ \ i = \overline{m+1, p};$$

$$g_i(u) = \langle a_i, u \rangle - b_i = 0,\ \ i = \overline{p+1, s}\}, \tag{2}$$

where $J(u),\ g_i(u),\ i = \overline{1,m}$ are convex functions determined on convex ensemble U_0 from E^n; $i = \overline{m+1, s},\ a_i \in E^n$ are the vectors; $b_i,\ i = \overline{m+1, s}$ are the numbers.

***Theorem* 1** *(the sufficient existence conditions of the saddle point).*

Let $J(u),\ g_i(u),\ \mathrm{i} = \overline{1,\mathrm{m}}$ *be convex functions determined on convex ensemble* U_0 *the ensemble* $U_* \ne \varnothing$ *and let the point* $\bar{u} \in ri\,U_0 \cap U$ *exists such that* $g_i(\bar{u}) < 0,\ i = \overline{1,m}.$ *Then for each point* $u_* \in U_*$ *Lagrange coefficients* $\lambda^* = (\lambda_1^*, \ldots, \lambda_s^*) \in \Lambda_0 = \{\lambda \in E^s / \lambda_1 \ge 0, \ldots, \lambda_p \ge 0\},$ *exist such that pair* $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ *forms saddle point to Lagrange's function*

$$L\ (u, \lambda) = J(u) + \sum_{i=1}^{s} \lambda_i g_i(u),\ \ u \in U_0,\ \ \lambda \in \Lambda_0, \tag{3}$$

on ensemble $U_0 \times \Lambda_0$, *i.e. the inequality*

$$L(u_*, \lambda) \le L(u_*, \lambda^*) \le L(u, \lambda^*),\ \ \forall u \in U_0,\ \ \forall \lambda \in \Lambda_0 \tag{4}$$

is executed.

Lemma. *In order to pair* $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ *be saddle point of the Lagrange's function (3), necessary and sufficiency performing the following conditions:*

$$
\begin{gathered}
L\,(u_*,\lambda^*) \le L\,(u,\lambda^*),\ \ \forall u \in U_0, \\
\lambda^*{}_i g_i(u_*) = 0,\ \ i=\overline{1,s},\ \ u_* \in U_* \subset U,\ \lambda^* \in \Lambda_0,
\end{gathered} \tag{5}
$$

i.e. inequalities (4) is tantamount to the correlations (5).

Theorem 2 *(the sufficient optimality conditions). If pair* $(u_*, \lambda^*) \in$ $\in U_0 \times \Lambda_0$ *is saddle point to Lagrange's function (3), that vector* $u_* \in U_*$ *is solution of the problem (1), (2).*

Problem algorithm of the convex programming based on the theorems 1,2 and lemma are provided in the lectures 10. We illustrate solution rule of the convex programming problem in the following example.

Example. To maximize the function

$$
-8u_1^2 - 10u_2^2 + 12u_1u_2 - 50u_1 + 80u_2 \to \sup \tag{6}
$$

at condition

$$
u_1 + u_2 = 1,\ 8u_1^2 + u_2^2 \le 2,\ u_1 \ge 0,\ u_2 \ge 0. \tag{7}
$$

Solution. Problem (6), (7) is tantamount to the problem

$$
J(u) = 8u_1^2 + 10u_2^2 - 12u_1u_2 + 50u_1 - 80u_2 \to \inf \tag{8}
$$

at condition

$$
u_1 + u_2 = 1,\ 8u_1^2 + u_2^2 \le 2,\ u_1 \ge 0,\ u_2 \ge 0. \tag{9}
$$

Problem (8), (9) possible to write as

$$
J(u) = 8u_1^2 + 10u_2^2 - 12u_1u_2 + 50u_1 - 80u_2 \to \inf, \tag{10}
$$

$$
\begin{gathered}
u \in U = \{u = (u_1,u_2) \in E^2 / u \in U_0, g_1(u) = 8u_1^2 + u_2^2 - 2 \le 0, \\
g_2(u) = u_1 + u_2 - 1 = 0\},\ \ U_0 = \{u = (u_1,u_2) \in \\
\in E^2 / u_1 \ge 0, u_2 \ge 0\}.
\end{gathered} \tag{11}
$$

We notice that ensemble $U_0 \subset E^2$ is convex, function $J(u)$ is convex on ensemble U_0, since symmetrical matrix

$$J''(u) = \begin{pmatrix} 16 & -12 \\ -12 & 20 \end{pmatrix} > 0, \quad \langle J''(u)\xi, \xi \rangle > 0, \quad \forall \xi \in E^2, \ \forall u \in U_0.$$

It is easy to check that functions $g_1(u), g_2(u)$ are convex on ensemble U_0. Finally, problem (10), (11) is a problem of the convex programming. Entering ensembles

$$U_1 = \{u \in E^2 / g_1(u) \le 0\}, \quad U_2 = \{u \in E^2 / g_2(u) = 0\},$$

problem (10), (11) possible to write as

$$J(u) \to \inf, \quad u \in U = U_0 \cap U_1 \cap U_2.$$

We notice that ensembles U_0, U_1, U_2 are convex, consequently, ensemble U is convex. Further we solve problem (10), (11) on algorithm provided in the lecture 10.

1^0. We are convinced of ensemble

$$U_* = \left\{ u_* \in U \big/ J(u_*) = \min_{u \in U} J(u) \right\} \ne \varnothing.$$

In fact, ensembles U_0, U_1, U_2 are convex and closed, moreover ensemble U is bounded, consequently, ensemble U is bounded and closed, i.e. U - a compact ensemble in E^n. As follows from correlation (10) function $J(u)$ is continuous (semicontinuous from below) on compact ensemble U. Then according to the theorem 1 (lecture 2) ensemble $U_* \ne \varnothing$. Now problem (10), (11) possible to write as $J(u) \to \min, u \in U \subset E^2$.

2^0. We show that Sleytera's condition is executed. In fact, the point

$$\bar{u} = (0,1) \in riU_0 \cap U, \quad (aff\, U_0 = U_0, \ aff\, U_0 \cap U_0 = U_0),$$

moreover $g_1(\bar{u}) = -1 < 0$. Consequently, by theorem 1 Lagrange's function for problem (10), (11)

$$L\,(u,\lambda) = 8u_1^2 + 10u_2^2 - 12u_1u_2 + 50u_1 - 80u_2 +$$
$$+ \lambda_1(8u_1^2 + u_2^2 - 2) + \lambda_2(u_1 + u_2 - 1), \qquad (12)$$
$$u = (u_1, u_2) \in U_0,\ \lambda = (\lambda_1, \lambda_2) \in \Lambda_0 = \{\lambda \in E^2 / \lambda_1 \geq 0\}$$

has saddle point.

3^0. Lagrange's function has the form (12), area of its determination $U_0 \times \Lambda_{0,}$, moreover $\lambda_1 \geq 0,$ but λ_2 can be as positive, as negative.

4^0. We define saddle point $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ to Lagrange's function (12) on base of (5). From inequality $L\,(u_*, \lambda^*) \leq L\,(u, \lambda^*),\ \ \forall u \in U_0$ follows that convex function $L\,(u_*, \lambda^*),\ \ u \in U_0$ (λ^* - fixed vector) reaches the least value on convex ensemble U_0 in the point u_*. Then according to optimality criterion (lecture 5, theorem 4) necessary and sufficiently executing of the inequality

$$\left\langle L_u(u_*, \lambda^*), u - u_* \right\rangle \geq 0,\ \ \forall u \in U_{0,} \qquad (13)$$

where

$$L\,(u_*, \lambda^*) = \begin{pmatrix} 16u_1^* - 12u_2^* + 50 + 16\lambda_1^* u_1^* + \lambda_2^{**} \\ 20u_2^* - 12u_1^* - 80 + 2\lambda_1^* u_2^* + \lambda_2^* \end{pmatrix},\ \ u_* = (u_1^*, u_2^*),$$

$\lambda^* = (\lambda_1^*, \lambda_2^*)$

Conditions $\lambda_i^* g_i(u_*) = 0,\ \ i = 1,2$ (the conditions of the complementing elasticity) are written:

$$\lambda_1^*\left(8u_1^{*2} + u_2^{*2} - 2\right) = 0,\ \ \lambda_2^*(u_1^* + u_2^* - 1) = 0,\ \ \lambda_1^* \geq 0. \qquad (14)$$

a) We suppose that $u_* \in \operatorname{int} U_0$. In this case from inequality (13) we have $L_u(u_*, \lambda^*) = 0.$ Saddle point is defined from conditions

$$16u_1^* - 12u_2^* + 50 + 16\lambda_1^* u_1^* + \lambda_2^* = 0,$$
$$20u_2^* - 12u_1^* - 80 + 2\lambda_1^* u_2^* + \lambda_2^* = 0,$$

$$\lambda_1^*(8u_1^{*2} + u_2^{*2} - 2) = 0, \quad u_1^* + u_2^* - 1 = 0, \quad \lambda_2^* \neq 0, \lambda_1^* \geq 0. \qquad (15)$$

We notice that from the second equation (14) and $u_* \in U$ follows that $u_1^* + u_2^* - 1 = 0,\ \ \lambda_2^* \neq 0$. We solve the system of the algebraic equations (15). It is possible several events:

1) $\lambda_1^* > 0, \lambda_2^* \neq 0$. In this case the point u_* is defined from solution of the system $8u_1^{*2} + u_2^{*2} = 2,\ u_1^* + u_2^* = 1$. Thence we have $u_1^* = 0{,}47$, $u_2^* = 0{,}53$. Then value λ_1^* is solution of the equation $126{,}2 + 6{,}46\lambda_1^* = 0$. Thence we have $\lambda_1^* = -\dfrac{126{,}2}{6{,}46} < 0.$

It is impossible, since $\lambda_1^* > 0$.

2) $\lambda_1^* = 0,\ \lambda_2^* \neq 0$. The point u_* is defined from equations $u_1^* + u_2^* = 1$, $16u_1^* - 12u_2^* + 50 + \lambda_2^* = 0$, $20u_2^* - 12u_1^* - 80 + \lambda_2^* = 0$. Thence we have $u_2^* = 158/60,\ \ u_1^* = -98/60 < 0$. The point $u_* \notin U_0$. So the case is excluded. Thereby, conditions (13), (14) are not executed in the internal points of the ensemble U_0. It remains to consider the border points of the ensemble U_0.

b) Since the point $u_* \in U$, that remains to check the conditions (13), (14) in the border points (1,0), (0,1) of the ensemble U_0. We notice that border point $(1{,}0) \notin U$, since in the point restriction $8u_1^2 + u_2^2 - 2 \leq 0.$ is not executed. Then the conditions (13), (14) follows to check in the singular point $u_* = (0{,}1) \in U_0$. In the point $g_1(u_*) = -1 < 0,$ consequently, value $\lambda_1^* = 0.$ and equalities (14) are executed. Since derivative $L_u(u_*, \lambda^*) = (38 + \lambda_2^*, -60 + \lambda_2^*),$ that inequality (13) is written as $(38 + \lambda_2^*)u_1 + (-60 + \lambda_2^*)(u_2 - 1) \geq 0$ under all $u_1 \geq 0,$ $u_2 \geq 0$. We choose $\lambda_2^* = 60$, then the inequality to write so: $98u_1 \geq 0,\ u_1 \geq 0,\ u_2 \geq 0$. Finally, the point $u_* = (u_1^* = 0, u_2^* = 1)$ is the saddle point to Lagrange's function (12).

5^0. Problem (10), (11) has the solutions $u_1^* = 0, \quad u_2^* = 1, \; J(u_*) = -70$.

To solve the following problems:

1. $J(u) = -u_1^2 - u_2^2 + 6 \to \sup; \quad u_1 + u_2 \le 5, \;\; u_1 \ge 0.$
2. $J(u) = -2u_1^2 - 2u_1 - 4u_2 + 3u_3 + 8 \to \sup;$

   $8u_1 - 3u_2 + 3u_3 = 40, \;\; -2u_1 + u_2 - u_3 = -3, \; u_2 \ge 0.$
3. $J(u) = -5u_1^2 - u_2^2 + 4u_1u_2 + 5u_1 - 4u_2 + 3 \to \sup,$

   $u_1^2 + u_2^2 + 2u_1 - 2u_2 \le 4, u_1 \ge 0, u_2 \ge 0$.
4. $J(u) = -3u_2^2 + 11u_1 + 3u_2 + u_3 + 27 \to \sup,$

   $u_1 - 7u_2 + 3u_3 \le -7, \quad 5u + 2u_2 - u_3 \le 2.$
5. $J(u) = -u_1 - u_2 - u_3 + 30u_4 + 8u_5 - 56u_6 \to \sup,$

   $2u_1 + 3u_2 - u_3 - u_4 + u_5 - 10u_6 \le 20,$

   $u_1 + 2u_2 + 3u_3 - u_4 - u_5 + 7u_6 = -11,$

   $x_1 \ge 0, \;\; x_2 \ge 0, \;\; x_3 \ge 0.$
6. The following problem: $J(u) = \langle c,u \rangle + 0{,}5\langle Du,u \rangle \to \sup$ at conditions $Au \le b, \, u \ge 0$, where $D = D^* \ge 0,$ is identified by the problem of the quadratic programming. To define the solutions of the problem.
7. To define the least distance from origin of coordinates till ensemble $u_1 + u_2 \ge 4, \, 2u_1 + u_2 \ge 5$.
8. To formulate Kuna-Tucker's conditions and dual problem for the following problem: $J(u) = 4u_1^2 + 4u_1u_2 + 2u_2^2 + 3u_1 + e^{u_1+2u_2} \to \inf$ at condition $u_1 + 2u_2 = 0, \;\; u_1^2 + u_2^2 \le 10, \; \sqrt{u_2} \le 1/2, u_2 \ge 0$
9. To find $J(u = pu_1^2 + qu_1u_2 \to \sup)$ at condition $u_1 + ru_2^2 \le 1,$ $u_1 \ge 0, \;\; u_2 \ge 0$, where - $p, \, q, \, r$ - parameters. Under what values of the parameters $p, \, q, \, r$ solution exists?
10. To solve the geometric problem: $J(u) = (u_1 - 3)^2 + u_2^2 \to \sup$ at condition $-u_1 + (u_2 - 1)^2 \ge 0, \;\; u_1 \ge 0, \;\; u_2 \ge 0$.

    Are Kuna–Tucker's conditions satisfied in the optimum point?

P.1.4. NONLINEAR PROGRAMMING

We consider the next problem of the nonlinear programming:

$$J(u) \to \inf, \tag{1}$$

$$u \in U, \tag{2}$$

$$U = \left\{u \in E^n \,/\, u \in U_0,\, g_i(u) \le 0,\, i = \overline{1,m};\, g_i(u) = 0,\, i = \overline{m+1,s}\right\},$$

where function $J(u) \in C^1(U_{01})$, $g_i(u) \in C^1(U_{01})$, $i = \overline{1,s}$, U_{01}- open ensemble contained the convex ensembles U_0 from E^n, in particular, $U_{01} = E^n$.

Theorem 1 *(the necessary optimality condition). If the functions* $J(u) \in C^1(E^n)$, $g_i(u) \in C^1(E^n)$, $i = \overline{1,s}$, $\operatorname{int} U_0 \ne \varnothing$, U_0 *is a convex ensemble, but ensemble* $U_* \ne \varnothing$, *then for each point* $u_* \in U$ *necessary exist the Lagrange's coefficients*

$$\overline{\lambda}^* = (\lambda_0^*, \lambda_1^*, \ldots, \lambda_s^*) \in \Lambda_0 = \left\{\overline{\lambda} \in E^{s+1} \,/\, \lambda_0 \ge 0, \lambda_1 \ge 0, \ldots, \lambda_m \ge 0\right\},$$

such the following conditions are executed:

$$\left|\overline{\lambda}^*\right| \ne 0,\ \lambda_0^* \ge 0,\ \lambda_1^* \ge 0,\ \ldots,\ \lambda_m^* \ge 0, \tag{3}$$

$$\left\langle L_u(u_*, \overline{\lambda}^*) u - u_* \right\rangle =$$

$$= \left\langle \lambda_0^* J'(u_*) + \sum_{i=1}^{S} \lambda_i^* g_i'(u_*), u - u_* \right\rangle \ge 0,\ \forall u \in U_0, \tag{4}$$

$$\lambda_i^* g_i(u_*) = 0,\ i = \overline{1,s},\quad u_* \in U. \tag{5}$$

Lagrange's function for problem (1), (2) has the form

$$L(u, \overline{\lambda}) = \lambda_0 J(u) + \sum_{i=1}^{S} \lambda_i g_i(u),\ u \in U_0,\ \overline{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_s) \in \Lambda_0 =$$

$$= \left\{\overline{\lambda} \in E^{S+1} \,/\, \lambda_0 \ge 0,\ \lambda_1 \ge 0, \ldots, \lambda_m \ge 0\right\}.$$

At solution of the problem (1), (2) necessary to consider separately two events: 1) $\lambda_0^* = 0$ (degenerate problem); 2) $\lambda_0^* > 0$ (nondegenerate problem). In this case possible to take $\lambda_0^* = 1$.

We suppose that the points $\left(u_*, \lambda_0^* \geq 0, \lambda^* = (\lambda_1^* \geq 0, \ldots, \lambda_m^* \geq \geq 0, \lambda_{m+1}^*, \ldots, \lambda_s^*)\right)$ from conditions (3), (4), (5) are found. The point $u_* \in U$ is identified by normal, if the vectors $\left(g_i'(u_*), i \in I, g_{m+1}'(u_*), \ldots, g_s'(u_*)\right)$ are linear independent, where ensemble $I = \{i \in \{1, \ldots, m\} / g_i(u_*) = 0\}$. If $u_* \in U$ - a normal point, that problem (1), (2) is nondegenerate, i.e. $\lambda_0^* = 1$. In the event $U_0 = E^n$ is executed the following theorem.

Theorem 2. *Let functions $J(u)$, $g_i(u)$, $i = \overline{1, s}$, be definite, continuous and twice continuously differentiable in the neighborhood of the normal point $u_* \in U$. In order to the normal point $u_* \in U$ be a point of the local minimum $J(u)$ on ensemble U, i.e. $J(u_*) \leq J(u), \forall u$, $u \in D(u_*, \varepsilon) \cap U$ sufficiently that quadric form $y'\left(\partial^2 L\, (u_*, \lambda^*) / \partial u^2\right)y$, $|y| \neq 0$ be positive definite on the hyperplane*

$$\left(\frac{\partial g_i(u_*)}{\partial u}\right)^* y = 0, \;\; i \in I, \;\; \left(\frac{\partial g_i(u_*)}{\partial u}\right)^* y = 0, \quad i = \overline{m+1, s}.$$

We consider the solutions of the following example on base of the theorems 1, 2 and solution algorithm of the nonlinear programming (the lectures 13).

Example 1. To find the problem solution

$$3 + 6u_1 + 2u_2 - 2u_1 u_2 - 2u_2^2 \to \sup \tag{6}$$

at conditions

$$3u_1 + 4u_2 \leq 8, \;\; -u_1 + 4u_2^2 \leq 2, \quad u_1 \geq 0, \;\; u_2 \geq 0. \tag{7}$$

Solution. Problem (6), (7) is tantamount to the problem

$$J(u) = 2u_2^2 + 2u_1 u_2 - 6u_1 - 2u_2 - 3 \to \inf, \quad (8)$$

$$u = (u_1, u_2) \in U = \{u \in E^2 / u \in U_0,\ g_1(u) \le 0,\ g_2(u) \le 0\}, \quad (9)$$

where $U_0 = \{u \in E^2 / u_1 \ge 0, u_2 \ge 0\}$; $g_1(u) = 3u_1 + 4u_2 - 8$; $g_2(u) = -u_1 + 4u_2^2 - 2$.

Function $J(u)$ is not a convex function on ensemble U_0, consequently, we have the nonlinear programming problem.

1^0. We show that ensemble $U_* \ne \varnothing$. Let $U_1 = \{u \in E^2 / g_1(u) \le 0\}$, $U_2 = \{u \in E^2 / g_2(u) \le 0\}$.

Then $U = U_0 \cap U_1 \cap U_2$ - closed bounded ensemble, consequently, it is convex. Function $J(u)$ is continuous on ensemble U. Thence in effect Weierstrass theorems we have $U_* \ne \varnothing$.

2^0. Generalized Lagrange's function for problem (8), (9) has the form

$$L(u_1, u_2, \lambda_0, \lambda_1 \lambda_2) = \lambda_0(2u_2^2 + 2u_1u_2 - 6u_1 - 2u_2 - 3) +$$
$$+ \lambda_1(3u_1 + 4u_2 - 8) + \lambda_2(-u_1 + 4u_2^2 - 2), \quad u \in (u_1, u_2) \in U_0,$$
$$\bar{\lambda} = (\lambda_0, \lambda_1, \lambda_2) \in \Lambda_0 = \{\bar{\lambda} \in E^3 / \lambda_0 \ge 0, \lambda_1 \ge 0, \lambda_2 \ge 0\}.$$

3^0. According to the conditions (3) - (5) quadruple $\left(u_*, \bar{\lambda}^* = \left(\lambda_0^*, \lambda_1^*, \lambda_2^*\right)\right) \in U_0 \times \Lambda_0$ is defined from correlations

$$|\bar{\lambda}^*| \ne 0,\ \lambda_0^* \ge 0,\ \lambda_1^* \ge 0,\ \lambda_2^* \ge 0, \quad (10)$$

$$\left\langle L_u\left(u_*, \bar{\lambda}^*\right), u - u_*\right\rangle \ge 0, \quad \forall\, u \in U_0, \quad (11)$$

$$\lambda_1^* g_1(u_*) = 0,\ \lambda_2^* g_2(u_*) = 0,\ u_* \in U, \quad (12)$$

where derivative

$$L_u(u_*,\overline{\lambda}^*) = \begin{pmatrix} \lambda_0^*(2u_2^*-6)+3\lambda_1^*-\lambda_2^* \\ \lambda_0^*(4u_2^*-2u_1^*-2)+4\lambda_1^*+8\lambda_2^*u_2^* \end{pmatrix}.$$

a) We expect that $u_* \in \operatorname{int} U_0$. Then from (11) follows that $L_u(u_*,\overline{\lambda}^*)=0$, consequently, pair $(u_*,\overline{\lambda}^*)$ is defined from solution of the following algebraic equations:

$$\lambda_0^*(2u_2^*-6)+3\lambda_1^*-\lambda_2^*=0,\ \lambda_0^*(4u_2^*-2u_1^*-2)+4\lambda_1^*+8\lambda_2^*\ u_2^*=0.$$

$$\lambda_1^*(3u_1^*+4u_2^*-8)=0,\ \lambda_2^*(-u_1^*+4u_2^{*2}-2)=0, \tag{13}$$

where $u_1^*>0,\ u_2^*>0,\ \lambda_0^*\ge 0,\ \lambda_1^*\ge 0,\ \lambda_2^*\ge 0$. We consider the event, when $\lambda_0^*=0$. Then $\lambda_2^*=3\lambda_1^*,\ 4\lambda_1^*(1+6u_2^*)=0$. If $\lambda_1^*=0$, that $\lambda_2^*=0$, consequently, condition (10) is broken, since $\lambda_0^*=0,\ \lambda_1^*=0,\ \lambda_2^*=0$. So it is possible only $\lambda_1^*>0$. Then $u_2^*=-1/6$. It is impossible, since $u_2^*>0$. Thence follows that $\lambda_0^*>0$, i.e. problem (8), (9) is nondegenerate, so it is possible to take $\lambda_0^1=1$ and conditions (13) are written so:

$$2u_2^*-6+3\lambda^{1*}-\lambda_2^*=0,\quad 4u_2^2-2u_1^*-2+4\lambda_1^*+8\lambda_2^*u_2^*=0,$$
$$\lambda_1^*(3u_1^*+4u_2^*-8)=0,\ \lambda_2^*(-u_1^*+4u_2^*-2)=0\ . \tag{14}$$

Now we consider the different possibilities:

1) $\lambda_1^*>0,\ \lambda_2^*>0.$ In this case the point $u_*=(u_1^*,u_2^*)$ is defined from system $3u_1^*+4u_2^*-8=0,\quad -u_1^*+4u_2^{*2}-2=0,\quad u_1^*>0,\quad u_2^*>0$ [refer to formula (14)]. Thence we have $u_1^*=1{,}43,\quad u_2^*=0{,}926,$ $J(u_1^*,u_2^*)=-9{,}07$. However $\lambda_1^*=1{,}216,\ \lambda_2^*=-0{,}5<0$, so the given point $u_*=(1{,}43;0{,}926)$ can not be solution of the problem (8), (9).

2) $\lambda_1^*=0,\ \lambda_2^*>0.$ In this case from (14) we have $2u_2^*-6-\lambda_2^*=0,$ $4u_2^*-2u_1^*-2+8\lambda_2^*u_2^*=0,\quad -u_1^*+4u_2^{*2}-2=0.$

Thence we have $u_1^* = 117,\ u_2^* = 5{,}454,\ \lambda_2^* = 4{,}908 > 0$. However the point $u_* = (117;\ 5{,}454) \notin U$, since the inequality $3u_1^* + 4u_2^* \le 8.$ aren't executed.

3) $\lambda_1^* > 0,\ \lambda_2^* = 0.$ Equations (14) are written so: $2u_2^* - 6 + 3\lambda_1^* = 0,$ $4u_2^* - 2u_1^* - 2 + 4\lambda_1^* = 0,\ 3u_1^* + 4u_2^* - 8 = 0.$

Thence we have $u_1^* = 28/9,\ u_2^* = -1/3 < 0,\ \lambda_1^* = 16/9 > 0$. The point $u_* = (28/9, -1/3) \notin U.$

4) $\lambda_1^* = 0, \lambda_2^* = 0.$ In this case from equation (14) we have $2u_2^* - 6 = 0,\ 4u_2^* - 2u_1^* - 2 = 0.$

Thence we have $u_2^* = 3, u_1^* = 5.$ However the point $u_* = (5;\ 3) \notin U$, since inequality $-u_1^* + 4u_2^* - 2 \le 0.$ are not executed. Thereby, the point $u_* \notin \operatorname{int} U_0$.

b) We expect that point $u_* \in \Gamma p U_0$. Here possible the following events:

1) $u = (0, u_2) \in \Gamma p U_0, u_2 \ge 0$; 2) $u = (u_1, 0) \in \Gamma p U_0,\ u_1 \ge 0.$ For the first type point of the restriction $g_1 = 4u_2 - 8 \le 0,$ $g_2 = 4u_2^2 - 2 \le 0$, consequently, $0 \le u_2 \le 1/\sqrt{2}$. Then. $u_* = \left(u_1^* = 0,\ u_2 = 1/\sqrt{2}\right),\ \ J(u_*) = -3{,}4$. For the second type of the border point of the restriction $g_1 = 3u_1 - 8 \le 0,\ \ g_2 = -u_1 - 2 \le 0$. Then $u_* = (8/3,\ 0)$, but value $J(u_*) = -19$.

Finally, problem solution (8), (9): $u_* = \left(u_1^* = 8/3,\ u_2^* = 0\right),$ $J(u_*) = -19.$

Example 2. It is required from wire of the given length l to do the equilateral triangle and square which total area is maximum.

Solution. Let $u_1,\ u_2$ be a sum of the lengths of the triangle sides of the square accordingly. Then sum $u_1 + u_2 = l$, but side of the triangle has a length $u_1/3$, side of the square - $u_2/4$ and total area is

$$S(u_1, u_2) = \frac{\sqrt{3}}{36}u_1^2 + \frac{1}{16}u_2^2.$$

Now optimization problem can be formulated so: to minimize the function

$$J(u_1, u_2) = -\frac{\sqrt{3}}{36}u_1^2 - \frac{1}{16}u_2^2 \to \inf \qquad (15)$$

at conditions

$$u_1 + u_2 = l, \ u_1 \geq 0, \ u_2 \geq 0. \qquad (16)$$

Entering indications $g_1(u_1, u_2) = u_1 + u_2 - l$, $g_2(u_1, u_2) = -u_1$, $g_3(u_1, u_2) = -u_2$ problem (15), (16) to write as

$$J(u_1, u_2) = -\frac{\sqrt{3}}{36}u_1^2 - \frac{1}{16}u_2^2 \to \inf \qquad (17)$$

at conditions

$$u \in U = \{u \in E^2 / g_1(u) = u_1 + u_2 - l = 0, \ g_2(u_1, u_2) = -u_1 \leq 0,$$
$$g_3(u_1, u_2) = -u_2 \leq 0\}, \qquad (18)$$

where possible $U_0 = E^n$. Unlike previous example the conditions $u_1 \geq 0, \ u_2 \geq 0$ are enclosed in restrictions $g_2(u_1, u_2)$, $g_3(u_1, u_2)$. Such approach allows using the theorem 2 for study properties of the functions $J(u_1, u_2)$ in the neighborhood of the point $u_* = (u_1^*, u_2^*) \in U$. Problem (17), (18) is the nonlinear programming problem since function $J(u_1, u_2)$ is not convex on ensemble $U_0 = E^n$.

1^0. Ensemble U is bounded and closed, consequently, it is compact. Function $J(u_1, u_2) \in C^2(E^2)$, so ensemble $U_* \neq \varnothing$

2^0. Generalized Lagrange's function for problem (17), (18) to write as

$$L\,(u,\overline{\lambda}) = L\,(u_1,u_2,\lambda_0,\lambda_1,\lambda_2,\lambda_3) = \lambda_0(-\frac{\sqrt{3}}{36}u_1^2 - \frac{1}{16}u_2^2) +$$
$$+\lambda_1(u_1+u_2-l)+\lambda_2(-u_1)+\lambda_3(-u_2),\quad u=(u_1,u_2)\in E^2,$$
$$\overline{\lambda} = (\lambda_0,\lambda_1,\lambda_2,\lambda_3)\in\Lambda_0 = \{\overline{\lambda}\in E^4 / \lambda_0\geq 0, \lambda_2\geq 0, \lambda_3\geq 0\}.$$

3^0. Since ensemble $U_0 = E^2$, that conditions (3) - (5) are written:

$$L\,(u_*,\overline{\lambda}^*) = \begin{pmatrix} -\frac{\sqrt{3}}{18}u_1^*\lambda_0^* + \lambda_1^* - \lambda_2^* \\ -\frac{1}{8}u_2^*\lambda_0^* + \lambda_1^* - \lambda_3^* \end{pmatrix} = 0, \qquad (19)$$

$$\lambda_1^*(u_1^*+u_2^*-l)=0,\ \lambda_2^*(-u_1^*)=0,\ \lambda_3^*(-u_2^*)=0, \qquad (20)$$

$$\left|\overline{\lambda}^*\right| \neq 0,\ \lambda_0^*\geq 0,\ \lambda_2^*\geq 0,\ \lambda_3^*\geq 0. \qquad (21)$$

a) We consider the event $\lambda_0^* = 0$. In this case, as follows from expression (19), $\lambda_1^* = \lambda_2^* = \lambda_3^*$. If $\lambda_2^* > 0$, that $\lambda_3 > 0$, consequently, $u_1^* = 0,\ u_2^* = 0$ [refer to formula (20)]. It is impossible, since $u_2^* + u_2^* = l$. It means, $\lambda_1^* = \lambda_2^* = \lambda_3^* = 0$. The equalities opposites to the condition (21). Then the source problem (17), (18) is nondegenerate. Consequently, value $\lambda_0^* = 1$.

b) Since problem (17), (18) is nondegenerate, the conditions (19) - (21) are written:

$$\underline{-\frac{\sqrt{3}}{18}u_1^* + \lambda_1^* - \lambda_2^* = 0,\ -\frac{1}{8}u_2^* + \lambda_1^* - \lambda_3^* = 0,\ u_1^* + u_2^* - l = 0,}$$

$$\underline{\lambda_1^* = 0,\ \lambda_2^* u_1^* = 0,\ \lambda_3^* u_2^* = 0,\ \lambda_2^* \geq 0,\ \lambda_3^* \geq 0.} \qquad (22)$$

We consider the different events:

1. $\lambda_2^* = 0,\ \lambda_3^* = 0$. Then $u_1^* = \dfrac{9l}{9+4\sqrt{3}}$, $u_2^* = \dfrac{4l\sqrt{3}}{9+4\sqrt{3}}$, $\lambda_1^* = \dfrac{l\sqrt{3}}{18+8\sqrt{3}}$.

2. $\lambda_2^* > 0,\ \lambda_3^* = 0$. In this case we have $u_1^* = 0,\ u_2^* = l,$ $\lambda_1^* = l/8,\ \lambda_2^* = l/8$.

3. $\lambda_2^* = 0,\ \lambda_3^* > 0$. Then from systems of the equations (22) we get $u_1^* = l,\ u_2^* = 0,\ \lambda_1^* = \sqrt{3}\cdot l/18,\ \lambda_3^* = \sqrt{3}\cdot l/18$. The event $\lambda_2^* > 0,\ \lambda_3^* > 0$ is excluded, since $u_1^* = 0,\ u_2^* = 0$ and condition $u_1^* + u_2^* = l.$ is not executed.

4^0. Quadric form $y'L_{uu}(u_*,\lambda^*)y = -\dfrac{\sqrt{3}}{18}y_1^2 - \dfrac{1}{8}y_2^2$.

Hyperplane equations for events 1 - 3 are accordingly written:

1) $\dfrac{\partial g_1(u_*)}{\partial u_1}y_1 + \dfrac{\partial g_1(u_*)}{\partial u_2}y_2 = y_1 + y_2 = 0$. Thence we have $y_1 = -y_2$. Then $y'L_{uu}(u_*,\lambda^*)y = -\left(4\sqrt{3}+9\right)y_1^2 < 0$, i.e. the point $(u_1^* = 9l/(9+4\sqrt{3}),\ u_2^* = 4l\sqrt{3}/(9+4\sqrt{3}))$ is not the point of the local minimum.

$$\frac{\partial g_1(u_*)}{\partial u_1}y_1 + \frac{\partial g_1(u_*)}{\partial u_2}y_2 = y_1 + y_2 = 0,$$
$$\frac{\partial g_2(u_*)}{\partial u_1}y_1 + \frac{\partial g_2(u_*)}{\partial u_2}y_2 = -y_1 = 0.$$

Thence we have $y_1 = 0,\ y_2 = 0;\ y'L_{uu}(u_*,\lambda^*)y = 0,\ |y| = 0$. The sufficient conditions of the local minimum are degenerated.

3) $\dfrac{\partial g_1(u_*)}{\partial u_1}y_1 + \dfrac{\partial g_1(u_*)}{\partial u_2}y_2 = y_1 + y_2 = 0$, $\dfrac{\partial g_3(u_*)}{\partial u_1}y_1 + \dfrac{\partial g_3(u_*)}{\partial u_2}y_2 = -y_2 = 0$.

The data of the equation have the solutions $y_1 = 0, y_2 = 0$. Again the sufficient optimality conditions do not give the onedigit answer. Solution of the problem is found by comparison of function values $J(u_1, u_2)$ in the last two events. In the second event value $J(u_1^*, u_2^*) = -l^2/16$, but in the third event $J(u_1^*, u_2^*) = -\sqrt{3}l^2/36.$ Then the problem solution is the point ($u_1^* = 0,\ u_2^* = l$), i.e. from the whole wire is made only square.

To solve the following problems:

1. To find $J(u_1, u_2, u_3) = u_1 u_2 u_3 \to \inf$ at conditions:

a) $u_1 + u_2 - u_3 - 3 = 0$, b) $u_1 - u_2 - u_3 - 8 = 0$;

c) $u_1 u_2 + u_1 u_3 + u_2 u_3 = a, \quad u_i \geq 0, \quad i = \overline{1,3}.$

2. To prove the equality

$$\frac{u_1^n + u_2^n}{2} \geq \left(\frac{u_1 + u_2}{2}\right)^n, \ n \geq 1,\ u_1 \geq 0,\ u_2 \geq 0.$$

3. To find the sides of the maximum rectangle area inserted in circle $u_1^2 + u_2^2 = R^2$.

4. To find the most short distance from the point (1,0) till ellipse $4u_1^2 + 9u_2^2 = 36$.

5. To find the distance between parabola $u_2 = u_1^2$ and line $u_1 - u_2 = 5$.

6. To find the most short distance from ellipse $2u_1^2 + 3u_2^2 = 12$ till line $u_1 + u_2 = 6$.

7. To find $J(u) = u'Au \to \sup,\ A = A^*$ at condition $u'u = 1$. To show, if $u_* \in E^n$ - the problem solution, that $J(u_*)$ is equal to the most characteristic root of the matrix A.

8. To find the parameters of the cylindrical tank which under given area of the surface S has the maximum volume.

9. a) $J(u) = u_1 + u_2 + u_3 \to \inf, \quad \dfrac{1}{u_1} + \dfrac{1}{u_2} + \dfrac{1}{u_3} = 1;$

b) $J(u) = u_1 u_2 u_3 \to \inf,$

$u_1 + u_2 + u_3 = 6,\ u_1 u_2 + u_1 u_3 + u_2 u_3 = 12;$

c) $J(u) = \dfrac{1}{u_1} + \dfrac{1}{u_2} \to \inf, \quad \dfrac{1}{u_1^2} + \dfrac{1}{u_2^2} = 1;$

d) $J(u) = 2u_1 + 3u_2^2 + u_3^2 \to \inf,$

$u_1 + u_2 + u_3 = 8,\ u_i \geq 0,\ i = \overline{1,3};$

e) $J(u) = u_1^2 + u_2^2 + u_3^2 \to \inf,$

$u_1 + u_2 + u_3 \leq 12,\ u_i \geq 0,\ i = \overline{1,3};$

f) $J(u) = u_1 u_2 + u_1 u_3 + u_2 u_3 \to \inf,\ u_1 + u_2 + u_3 \leq 4;$

g) $J(u) = u_1^2 u_2 + u_2^2 u_1 + u_1 u_2 u_3 \to \inf,$

$u_1 + u_2 + u_3 \leq 15,\ u_i \geq 0,\ i = \overline{1,3};$

l) $J(u) = u_1^2 - 2u_1 u_2 + u_3^2 \to \inf,$

$u_1 + 2u_2 + u_3 = 1,\ 2u_1 - u_2 + u_3 = 5,\ u_i \geq 0,\ i = \overline{1,3}.$

10. To find the conditional extreme to function $J(u) = (u_1 - 2)^2 + (u_2 - 3)^2$ at condition $u_1^2 + u_2^2 \leq 52$.

P.1.5. LINEAR PROGRAMMING. SIMPLEX-METHOD

General problem of the linear programming (in particular, basic task) is reduced to the linear programming problem in canonical form of the following type:

$$J(u) = c'u \to \inf, \tag{1}$$

$$u \in U = \{u \in E^n \,/\, u \geq 0, Au = b\}, \tag{2}$$

where $A = \left(a^1, a^2, ..., a^m, a^{m+1}, ..., a^n\right)$ - matrix of the order $m \times n$; $a^i \in E^m$, $i = \overline{1,n}$; vectors are identified by the condition vectors, but $b \in E^m$ - by the restriction vector.

Simplex-method is a general method of the problem solution of the linear programming in canonical form (1), (2). Since the general and the basic problems of the linear programming are reduced to type (1), (2), it is possible to consider that simplex-method is the general method of the problem solution of the linear programming.

The base of the linear programming theory is stated in the lectures 15 - 17. We remind briefly a rule of the nondegenerate problem solution of the linear programming in canonical form.

1^0. To build the initial extreme point $u^0 \in U$ ensemble U. We notice, if $U_* \neq \varnothing$, that lower border to linear function (1) on U is reached in the extreme point ensemble U, moreover in the nondegenerate problem the extreme point has exactly m positive coordinates i.e. $u^0 = \left(u_1^0 > 0,\ldots,u_m^0 > 0,0,\ldots,0\right)$, and vectors $a^1,\ldots,a^m$, corresponding to the positive coordinates of the extreme point are linear independent. The extreme point $u^0 \in U$ is defined in the general event by M-method (Charnes' method), but on a number of events - on source data ensemble U directly.

2^0. Simplex-table is built for extreme the point $u^0 \in U$. Vectors $a^i, i = \overline{1,m}$ are presented in the first column, in the second - elements of the vector c with corresponding to lower indexes, in the third - positive coordinates of the extreme point $u^0 \in U$ and in the rest column – decomposition coefficients of the vectors a^i, $i = \overline{1,n}$ on base $\left(a^1,\ldots,a^m\right)$ i.e. $a^j = \sum_{i=1}^{m} a^i u_{ij} = A_Б u^j$, where $A_Б = \left(a^1,\ldots,a^m\right)$ - nonsingular matrix; $u^j = \left(u_{1j},\ldots,u_{mj}\right)$, $j = \overline{1,n}$. Values $z_j = \sum_{i=1}^{m} c_i u_{ij}$, $j = \overline{1,n}$ are brought in penultimate line, but in the last line – values $z_j - c_j$, $j = \overline{1,n}$. The main purpose of the simplex-table is to check the optimality criterion for the point $u^0 \in U$. If it is turn out to be that values $z_j - c_j \leq 0$, $j = \overline{1,n}$, the extreme point $u^0 \in U$ - a solution of the problem (1), (2), but otherwise transition to the following extreme point $u^1 \in U$ – is realized, moreover value $J\left(u^1\right) < J\left(u^0\right)$.

30. The extreme point $u^1 \in U$ and corresponding to it simplex-table is built on base of the simplex-table of the point $u^0 \in U$. The index j_0 is defined from condition $z_{j0} - c_{j0} = \max(z_j - c_j)$ amongst $z_j - c_j > 0$. Column j_0 of the simplex-table of the point $u^0 \in U$ is identified pivotal column, and vector is entered a^{j_0} to the number of base in simplex-table of the point $u^1 \in U$ instead of vector a^{i_0}. The index i_0 is defined from condition $\min \frac{u_i^0}{u_{ij}} = \theta_{i_0}$ amongst $u_{ij_0} > 0$. The extreme point

$$u^1 = (u_1^0 - \theta_0 u_{1j_0}, \ldots, u_{i_0-1}^0 - \theta_0 u_{i_0-1j_0}, 0,\ u_{i_0+1}^0 - \theta u_{i_0+1j_0}, \ldots, u_m^0 - \theta_0 u_{mj_0},$$
$$0, \ldots, 0, \theta_0, 0, \ldots, 0),\ \theta_0 = \theta_{i_0}.$$

The base consists of the vectors $a^1, \ldots, a^{i_0-1}, a^{j_0}, a^{i_0+1}, \ldots, a^m$. Decomposition coefficients of the vectors a^j, $j = \overline{1,n}$ on the base are defined by the formula

$$(u_{ij})_{нов} = u_{ij} - \frac{u_{ij_0} u_{i_0 j}}{u_{i_0 j_0}},\ i \neq i_0,\ j \neq j_0;$$
$$(u_{i_0 j})_{нов} = \frac{u_{i_0 j}}{u_{i_0 j_0}},\ j = \overline{1,n}. \tag{3}$$

Hereinafter values $(z_j)_{нов}$, $(z_j - c_j)_{нов}$ are calculated by the known coefficients $(u_{ij})_{нов}$, $i = \overline{1,m},\ j = \overline{1,n}$. Thereby, new simplex-table is built for point $u^1 \in U$. Further optimality criterion is checked. If it is turn out to be that $(z_j - c_j)_{нов} \leq 0,\ j = \overline{1,n}$, so the extreme point $u^1 \in U$ - solution of the problem (1), (2), but otherwise transition to the new extreme point $u^2 \in U$ realized and etc.

Example 1. To solve the linear programming problem

$$J(u_1, u_2, u_3, u_4, u_5, u_6) = 6u_1 - 3u_2 + 5u_3 - 2u_4 - 4u_5 + 2u_6 \to \inf,$$

$u_1 + 2u_2 + u_4 + 3u_5 = 17,\ 4u_2 + u_3 + u_5 = 12,$

$u_2 + 8u_4 - u_5 + u_6 = 6,$

$u_1 \geq 0,\ u_2 \geq 0,\ u_3 \geq 0,\ u_4 \geq 0,\ u_5 \geq 0,\ u_6 \geq 0.$

Matrix A, vectors b and c are equal to

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 3 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 8 & -1 & 1 \end{pmatrix} = \left(a^1, a^2, a^3, a^4, a^5, a^6\right),$$

$$a^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ a^2 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix},\ a^3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ a^4 = \begin{pmatrix} 1 \\ 0 \\ 8 \end{pmatrix},\ a^5 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix},\ a^6 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$b = \begin{pmatrix} 17 \\ 12 \\ 6 \end{pmatrix},\quad c' = (6, -3, 5, -2, -4, 2).$$

Solution. The initial extreme point $u^0 = \left(17, 0, 12, 0, 0, 6\right)$ easy is defined for the example. The condition vectors corresponding to positive coordinates of the extreme point - a^1, a^3, a^6.

Base	$\bar{c}$	b	6	-3	5	-2	-4	2	0
			a^1	a^2	a^3	a^4	a^5	a^6	

I $\qquad$ *u^0=(17, 0, 12, 0, 0, 6)*

A^1	6	17	1	2	0	1	3	0	$\frac{17}{2}$	
A^3	5	12	0	(4)	1	0	1	0	$\frac{12}{4}$	$\Rightarrow i_0 = 3$
A^6	2	6	0	1	0	8	-1	1	$\frac{6}{1}$	

$z_j - c_j$	0	37	0	24	25	0		$J(u^0) = 174$

$$z_{j_0} - c_{j_0} = 37,$$

$$j_0 = 2$$

II $\qquad\qquad$ *u^1=(11, 3, 0, 0, 0, 3)*

a^1	6	11	1	0	-1/2	1	5/2	0	11	
a^2	-3	3	0	1	1/4	0	1/4	0	-	
a^6	2	3	0	0	-1/4	(8)	-5/4	1	3/8	$\Rightarrow i_0 = 6$
$z_j - c_j$			0	0	$-\frac{37}{4}$	24	$\frac{63}{4}$	0		$J(u^1) = 63$

$$z_{j_0} - c_{j_0} = 24, \quad j_0 = 4$$

III $\qquad\qquad$ *$u^2=(\frac{85}{8}, 3, 0, \frac{3}{8}, 0, 0)$*

a^1	6	$\frac{85}{8}$	1	0	$-\frac{15}{32}$	0	$(\frac{85}{32})$	$-\frac{1}{8}$	4	
a^2	-3	3	0	1	$\frac{1}{4}$	0	$\frac{1}{4}$	0	-	
a^4	-2	$\frac{3}{8}$	0	0	$-\frac{1}{32}$	1	$-\frac{5}{32}$	$\frac{1}{8}$	-	$i_0 = 1$
$z_j - c_j$			0	0	$-\frac{17}{2}$	0	$\frac{39}{2}$	-3		$J(u^2) = 54$

$$z_{j_0} - c_{j_0} = \frac{39}{2}, \quad j_0 = 5$$

IV $\qquad\qquad$ *u^3=(0, 2, 0, 1, 4, 0)*

a^5	-4	4	$\frac{32}{85}$	0	$-\frac{15}{85}$	0	1	$-\frac{4}{85}$		$z_j - c_j \le 0$ $j = \overline{1,6}$
a^2	-3	2	$-\frac{8}{85}$	1	$\frac{25}{85}$	0	0	$\frac{1}{85}$		

a^4	-2	1	$\frac{1}{17}$	0	$-\frac{5}{85}$	1	0	$\frac{10}{85}$			
$z_j - c_j$			$-\frac{624}{85}$	0	$-\frac{430}{85}$	0	0	$-\frac{177}{85}$			$J(u^3) = -24$

Example 2.

$$J(u) = -2u_1 - 3u_2 + 5u_3 - 6u_4 + 4u_5 \to \inf;$$

$$2u_1 + u_2 - u_3 + u_4 = 5;\ u_1 + 3u_2 + u_3 - u_4 + 2u_5 = 8;$$

$$-u_1 + 4u_2 + u_4 = 1, \quad u_j \geq 0, \quad j = \overline{1,5}.$$

Corresponding M-problem has the form

$$J(\overline{u}) = -2u_1 - 3u_2 + 5u_3 - 6u_4 + 4u_5 + Mu_6 + Mu_7 \to \inf;$$

$$2u_1 + u_2 - u_3 + u_4 + u_6 = 5;\ u_1 + 3u_2 + u_3 - u_4 + 2u_5 = 8;$$

$$-u_1 + 4u_2 + u_4 + u_7 = 1, \quad u_j \geq 0, \quad j = \overline{1,7}.$$

For M-problem matrix A, the vectors b and c are equal

$$A = \begin{pmatrix} 2 & 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 1 & -1 & 2 & 0 & 0 \\ -1 & 4 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} = (a^1, a^2, a^3, a^4, a^5, a^6, a^7)',$$

$$b = \begin{pmatrix} 5 \\ 8 \\ 1 \end{pmatrix}, \quad c' = (-2, -3, 5, -6, 4, M, M).$$

Solution. The initial extreme point $u^0 = (0, 0, 0, 0, 4, 5, 1)$. We present value $z_j - c_j$ in the manner of $z_j - c_j = \alpha_j M + \beta_j$, $j = \overline{1,n}$, i.e. instead of one line for $z_j - c_j$ are entered two: in the first values β_j are written, in the second - α_j. Since M – sufficiently great positive number, then $z_j - c_j > z_k - c_k$, if $\alpha_j > \alpha_k$. But if $\alpha_j = \alpha_k$, that $\beta_j > \beta_k$.

Base	$\overline{c}$	B	-2	-3	5	-6	4	M	M
			a^1	a^2	a^3	a^4	a^5	a^6	a^7

I $J(\bar{u}) = 16 + 6M$ $\qquad\qquad$ $\bar{u}^0 = (0,0,0,0,4,5,1)$

A^6	M	5	2	1	-1	1	0	1	0
A^5	4	4	1/2	3/2	1/2	-1/2	1	0	0
A^7	M	1	-1	(4)	0	1	0	0	$1 \Rightarrow$
$z_j - c_j$		β_j	4	9	-3	4	0	0	0
		α_j	1	5	-1	2	0	0	0

II $J(\bar{u}^1) = (55/4) + (19/4)M$ $\qquad\qquad$ $\bar{u}^1 = (0,14,0,0,29/8,19/4,0)$

a^6	M	19/4	(9/4)	0	-1	3/4	0	$1 \Rightarrow$
a^5	4	29/8	7/8	0	1/2	-7/8	1	0
a^2	M	1/4	-1/4	1	0	1/4	0	0
$z_j - c_j$		β_j	25/4	0	-3	7/4	0	0
		α_j	9/4	0	-1	3/4	0	0

III $J(\bar{u}^2) = 5/9$ $\qquad\qquad$ $\bar{u}^2 = (19/9,7/9,0,0,16/9,0,0)$

a^1	-2	19/9	1	0	-4/9	3/9	0
a^5	4	16/9	0	0	8/9	-21/18	1
a^2	-3	7/9	0	1	-1/9	3/9	0
$z_j - c_j$		β_j	0	0	-2/9	-1/3	0
		α_j	0	0	0	0	0

As follows from the last simplex-table by solution of the M-problem is a vector $\bar{u}^2 = (19/9,7/9,0,0,16/9,0,0)$. Then solution of the source problem is vector $u^2 = (19/9,7/9,0,0,16/9)$, $J(u^2) = 5/9$.

To solve the following linear programming problems:

1. $J(u) = 2u_1 + u_2 + 2u_3 + 3u_4 \to \sup;$

$$3u_1 - u_3 - u_4 \le 6;$$

$$u_2 - u_3 + u_4 \le 2;$$

$$-u_1+u_2+u_3\le 5;\quad u_j\ge 0,\quad j=\overline{1,4}.$$

2. $J(u)=u_1-u_2+u_3-3u_4+u_5-u_6-3u_7\to\inf;$

$$3u_3+u_5+u_6=6;$$
$$u_2+2u_3-u_4=10;$$
$$u_1+u_6=0;$$
$$u_3+u_6+u_7=6;\quad u_j\ge 0,\quad j=\overline{1,7}.$$

3. $J(u)=u_1-2u_2+u_3+2u_4-u_5\to\inf;$

$$u_1+2u_2+u_3\le 2;$$
$$2u_1-u_2+u_4=0;$$
$$u_1+3u_2+u_5=6;$$
$$u_1\ge 0;\quad u_2\ge 0,\quad u_3\ge 0,\quad u_4\ge 0.$$

4. $J(u)=2u_1+3u_2+2u_3+u_4\to\sup;$

$$2u_1+2u_2-3u_3+u_4\le 6;$$
$$-u_2+u_3-u_4\ge -2;$$
$$u_1-u_2+2u_3=5;$$
$$u_1\ge 0;\quad u_2\ge 0,\quad u_3\ge 0.$$

5. $J(u)=u_1-2u_2-u_3\to\sup;$

$$3u_1+u_2+u_3\ge 4;$$
$$2u_1+3u_2\ge 6;$$
$$-u_1+2^0u_2\le -1, u_j\ge 0,\ \ j=\overline{1,3}.$$

6. $J(u)=u_1+2u_2-u_3\to\inf;$

$$u_1-u_2\ge 1;$$
$$u_1+2u_2+u_3=8;$$
$$-u_1+3u_2\ge 3,\ \ u_j\ge 0,\ \ j=\overline{1,3}.$$

7. $J(u)=-2u_1-u_2+u_3\to\sup;$

$$u_1 + 2u_2 + u_3 = 4;$$
$$u_1 + u_2 \leq 2;$$
$$u_1 + u_3 \geq 1,\ u_j \geq 0,\ j = \overline{1,3}.$$

8. The steel twig by length 111 cm have gone into blanking shop. It is necessary to cut them on stocking up on 19, 23 and 30 cm in amount 311, 215 and 190 pieces accordingly. To build the model on base of which it is possible to formulate the extreme problem of the variant choice performing the work under which the number cut twig are minimum.

9. There are two products which must pass processing on four machines (I, II, III, IV) in process production. The time of the processing of each product on each of these machines is specified in the following form:

Machine	A	B
I	2	j
II	4	2
III	3	1
IV	1	4

The machines I, II, III and IV can be used accordingly during 45, 100, 300 and 50 hours. Price of the product A - 6 tenge per unit, product B - 4 tenge. In what correlation do follows to produce the product A and B to get maximum profit? To solve the problem in expectation that product A is required in amount not less 22 pieces.

10. Refinery disposes by two oil grade - A and B, moreover petrol and fuel oil are got under processing. Three possible production processes are characterized by the following scheme:

a) 1 unit of the sort A + 2 unit of the sort B $\rightarrow$ 2 unit of the fuel oil + 3 unit of petrol;

b) 2 unit of the sort A + 1 unit of the sort B $\rightarrow$ 5 unit of the fuel oil + 1 unit of petrol;

c) 2 unit of the sort A + 2 unit of the sort B $\rightarrow$ 2 unit of the fuel oil + 1 unit of petrol;

We assume the price of the fuel oil 1 tenge per unit, but the price of the petrol 10 tenge per unit. To find the most profitable production plan if there are 10 units to oil of the sort A and 15 units oil of the sort B.

TASKS ON MATHEMATICAL PROGRAMMING

Three term tasks for students of the 2-nd, 3-rd courses (5,6 terms accordingly) educating on the specialties “Applied mathematics”, “Mathematics”, “Mechanics”, “Informatics”, “Economical cybernetics” are worked out by the following parts:

convex ensembles and convex function (1-th task),

convex and nonlinear functions (2-nd task),

linear programming (3-rd task),

Further we prefer variants of the tasks on course “Mathematical programming”.

Task 1

To check the function $J(u)$ is convex (concave) on the ensemble U, or indicate such points from U in neighborhood of which $J(u)$ isn’t neither convex or concave (1 – 89-variants).

1. $J(u) = u_1^6 + u_2^2 + u_3^2 + u_4^2 + 10u_1 - 5u_2 - 3u_4 - 20;\ U = E^4.$

2. $J(u) = e^{2u_1 + u_2};\ U = E^2.$

3. $J(u) = -u_1^3 - u_2^3 - u_3^2 + 10u_1 - u_2 + 15u_3 + 10;\ U = E^3.$

4. $J(u) = u_1^2 + u_2^2 + \frac{1}{2}u_3^2 + u_1u_2 - u_3 + 10;\ U = E^3.$

5. $J(u) = -u_1^2 - u_2^2 - 2u_3^2 + u_1u_2 + u_1u_3 + u_2u_3 +$
$+ 5u_2 + 25;\ U = E^3.$

6. $J(u) = -u_2^5 + \frac{1}{2}u_3^2 + 7u_1 - u_3 + 6;\ U = \{u \in E^3 : u \leq 0\}.$

7. $J(u) = 3u_1^2 + u_2^2 + 2u_3^2 + u_1u_2 + 3u_1u_3 + u_2u_3 +$
$+ 3u_2 - 6;\ U = E^3.$

8. $J(u)=5u_1^2+\frac{1}{2}u_2^2+4u_3^2+u_1u_2+2u_1u_3+2u_2u_3+$

$+u_3+1;\ U=E^3.$

9. $J(u)=-2u_1^2-\frac{1}{2}u_2^2-5u_3^2+\frac{1}{2}u_1u_2+2u_1u_3+u_2u_3+$

$+3u_1-2u_2+6;\ \ U=E^3.$

10. $J(u)=u_1^3+2u_3^2+10u_1+u_2-5u_3+6;$

$U=\{u\in E^3 : u\le 0\}.$

11. $J(u)=5u_1^4+u_2^6+u_3^2-13u_1+7u_3-8;\ U=E^3.$

12. $J(u)=-3u_1^2-2u_2^2-u_3^2+3u_1u_2+u_1u_3+$

$+2u_2u_3+17;\ U=E^3.$

13. $J(u)=4u_1^3-u_2^4-\frac{1}{2}u_3^4+3u_1+8u_2+11;$

$U=\{u\in E^3 : u\le 0\}.$

14. $J(u)=8u_1^3-12u_3^2-3u_1u_3+6u_2+17;$

$U=\{u\in E^3 : u\ge 0\}.$

15. $J(u)=-2u_1^2-2u_2^2-4u_3^2+2u_1u_2+2u_1u_3+$

$+2u_2u_3+16;\ U=E^3.$

16. $J(u)=2u_1^2+u_2^2+\frac{1}{2}u_3^2+2u_1u_2+8u_3+12;\ U=E^3.$

17. $J(u)=-\frac{1}{2}u_2^7+\frac{1}{2}u_3^4+2u_2u_3+11u_1+6;$

$U=\{u\in E^3 : u\le 0\}.$

18. $J(u)=\frac{5}{2}u_1^2+u_2^2+4u_3^2+\frac{3}{2}u_1u_2+2u_1u_3+\frac{1}{2}u_2u_3+$

$+8u_3+13;\quad U=E^3.$

19. $J(u) = -3u_1^2 + \frac{1}{2}u_2^3 + 2u_1u_2 + 5u_1u_3 + 7u_1 + 16;$

$U = \{u \in E^3 : u \le 0\}.$

20. $J(u) = -2u_1^2 - u_2^2 - \frac{3}{2}u_3^2 + u_1u_2 + \frac{1}{2}u_1u_3 +$

$+ 2u_2u_3 + 10;\ U = E^3.$

21. $J(u) = 2u_1^2 + \frac{3}{2}u_3^2 + \frac{5}{2}u_1u_3 + 12u_2 + 18;\ U = E^3.$

22. $J(u) = 6u_1^2 + u_2^3 + 6u_3^2 + 12u_1 - 8u_2 + 7;$

$U = \{u \in E^3 : u \ge 0\}.$

23. $J(u) = -u_1^2 - \frac{3}{2}u_2^2 - 2u_3^2 + u_1u_2 + u_1u_3 -$

$- 2u_2u_3 + 8u_2;\ U = E^3.$

24. $J(u) = -4u_1^2 - \frac{5}{2}u_2^2 - u_3^2 - 4u_1u_2 + 11u_3 + 14;\ U = E^3.$

25. $J(u) = \frac{7}{2}u_1^2 + \frac{4}{2}u_2^3 - \frac{1}{2}u_3^3 + 13u_1 + 7u_3 - 9;$

$U = \{u \in E^3 : u \le 0\}.$

26. $J(u) = -\frac{5}{6}u_1^3 - \frac{1}{4}u_2^5 - \frac{3}{2}u_3^3 + 22u_2 + 17;$

$U = \{u \in E^3 : u \le 0\}.$

27. $J(u) = -\frac{5}{6}u_1^3 - 2u_2^2 + \frac{3}{2}u_3^2 + 2u_1u_2 + 3u_1u_3 + u_2u_3;$

$U = \{u \in E^3 : u \le 0\}.$

28. $J(u) = 2u_1^2 + 4u_2^2 + u_3^2 - u_1u_2 + 9u_1u_3 + u_2u_3 - 9;\ U = E^3.$

29. $J(u) = \frac{3}{2}u_1^2 + \frac{5}{2}u_2^2 - \frac{9}{2}u_3^2 - 3u_1u_3 + 7u_2u_3;\ U = E^3.$

30. $J(u) = -\frac{7}{6}u_1^3 + \frac{5}{2}u_2^2 + \frac{5}{12}u_3^4 + \frac{1}{2}u_4^3 - 3u_2;$

$U = \{u \in E^3 : u \geq 0\}.$

31. $J(u) = -3u_1^2 + \frac{1}{2}u_2^3 + 2u_1u_2 + 5u_1u_3 + 7u_1 + 16;$

$U = \{u \in E^3 : u \leq 0\}.$

32. $J(u) = \frac{5}{2}u_1^2 + u_2^2 + 4u_3^2 + \frac{3}{2}u_1u_2 + 2u_1u_3 +$

$+ \frac{1}{2}u_2u_3 + 8u_3 + 13; \qquad U = E^3.$

33. $J(u) = -\frac{1}{2}u_2^7 + \frac{1}{2}u_3^4 + 2u_2u_3 + 11u_1 + 6;$

$U = \{u \in E^3 : u \leq 0\}.$

34. $J(u) = 2u_1^2 + u_2^2 + \frac{1}{2}u_3^2 + 2u_1u_2 + 8u_3 + 12;\ U = E^3.$

35. $J(u) = -2u_1^2 - 2u_2^2 - 4u_3^2 + 2u_1u_2 - 2u_1u_3 +$

$+ 2u_2u_3 + 16;\ U = E^3.$

36. $J(u) = -\frac{7}{6}u_1^3 + \frac{5}{2}u_2^2 + \frac{5}{12}u_3^4 + \frac{1}{2}u_4^3 - 3u_2;$

$U = \{u \in E^4 : u \geq 0\}.$

37. $J(u) = \frac{3}{2}u_1^2 + \frac{5}{2}u_2^2 - \frac{9}{2}u_3^2 - 3u_1u_3 + 7u_2u_3;\ U = E^3.$

38. $J(u) = 2u_1^2 + 2u_2^2 + u_3^2 - u_1u_2 + 9u_1u_3 + u_2u_3 - 9;\ U = E^3.$

39. $J(u) = -\frac{5}{6}u_1^3 - 2u_2^2 + \frac{3}{2}u_3^2 + 2u_1u_2 + 3u_1u_3 + u_2u_3;$

$U = \{u \in E^3 : u \leq 0\}.$

40. $J(u) = -\frac{5}{6}u_1^3 - \frac{1}{4}u_2^5 - \frac{3}{2}u_3^3 + 22u_2 + 17;$

$U = \{u \in E^3 : u \leq 0\}.$

41. $J(u) = \frac{7}{2}u_1^2 - \frac{4}{3}u_2^3 - \frac{1}{2}u_3^3 + 13u_1 + 7u_3 - 9;$

$U = \{u \in E^3 : u \leq 0\}.$

42. $J(u) = -4u_1^2 - \frac{5}{2}u_2^2 - u_3^2 - 4u_1u_2 + 11u_3 + 14;\ U = E^3.$

43. $J(u) = -u_1^2 - \frac{3}{2}u_2^2 - 2u_3^2 + u_1u_2 + u_1u_3 -$

$- 2u_2u_3 + 8u_2;\ U = E^3.$

44. $J(u) = 6u_1^2 + u_3^2 + 6u_3^2 + 12u_1 - 8u_2 + 7;$

$U = \{u \in E^3 : u \geq 0\}.$

45. $J(u) = 2u_1^2 + \frac{3}{2}u_3^2 + \frac{5}{2}u_1u_3 + 12u_2 + 18;\ U = E^3.$

46. $J(u) = -2u_1^2 - u_2^2 - \frac{3}{2}u_3^2 + u_1u_2 + \frac{1}{2}u_1u_3 +$

$+ 2u_2u_3 + 10;\ U = E^3.$

47. $J(u) = u_1^6 + u_2^2 + u_3^2 + u_4^2 + 10u_1 - 3u_4 - 20;\ U = E^4.$

48. $J(u) = e^{2u_1 + u_2};\ U = E^2.$

49. $J(u) = -u_1^3 - u_2^3 - u_3^3 + 10u_1 - u_2 + 15u_3 + 10;\ U = E^3.$

50. $J(u) = u_1^2 + u_2^2 + \frac{1}{2}u_3^2 + u_1u_2 - u_3 + 10;\ U = E^3.$

51. $J(u) = -u_1^2 - u_2^2 - 2u_3^2 + u_1u_2 + u_1u_3 + u_2u_3 +$

$+ 5u_2 + 25;\ U = E^3.$

52. $J(u) = -u_2^5 + \frac{1}{2}u_3^2 + 7u_1 - u_3 + 6;\ U = \{u \in E^3 : u \leq 0\}.$

53. $J(u) = 3u_1^2 + u_2^2 + 2u_3^2 + u_1u_2 + 3u_1u_3 +$
$+ u_2u_3 + 3u_2 - 6;\ U = E^3.$

54. $J(u) = 5u_1^2 + \frac{1}{2}u_2^2 + 4u_3^2 + u_1u_2 + 2u_1u_3 + 2u_2u_3 + u_3 + 1;$
$U = E^3.$

55. $J(u) = u_1^3 + 2u_3^3 + 10u_1 + u_2 - 5u_3 + 6;$
$U = \{u \in E^3 : u \le 0\}.$

56. $J(u) = 5u_1^4 + u_2^6 + u_3^2 - 13u_1 + 7u_3 - 8;\ U = E^3.$

57. $J(u) = -3u_1^2 - 2u_2^2 - u_3^2 + 3u_1u_2 + u_1u_3 +$
$+ 2u_3u_2 + 17;\ U = E^3.$

58. $J(u) = 4u_1^3 - u_2^4 - \frac{1}{2}u_3^4 + 3u_1 + 8u_2 + 11;$
$U = \{u \in E^3 : u \le 0\}.$

59. $J(u) = 3u_1^3 - 12u_3^2 - 3u_1u_3 + 6u_2 + 17;$
$U = \{u \in E^3 : u \ge 0\}.$

60. $J(u) = -2u_1^2 - 2u_2^2 - 4u_3^2 + 2u_1u_2 + 2u_1u_3 +$
$+ 2u_2u_3 + 16;\ U = E^3.$

61. $J(u) = u_1^6 + u_2^2 + u_3^2 + u_4^2 + 10u_1 - 3u_4 + 20;\ U = E^4.$

62. $J(u) = e^{2u_1 - u_2};\ U = E^2.$

63. $J(u) = u_1^3 + u_2^3 - u_3^3 - 10u_1 - u_2 + 15u_3 + 10;\ U = E^3.$

64. $J(u) = -u_1^2 + u_2^2 + \frac{1}{2}u_3^2 - u_1u_2 - u_3 + 10;\ U = E^3.$

65. $J(u) = u_1^2 + u_2^2 - 2u_3^2 + u_1u_2 - u_1u_3 + u_2u_3 +$
$+ 5u_2 + 25;\ U = E^3.$

66. $J(u) = u_2^5 + \frac{1}{2}u_3^2 - 7u_1 - u_3 + 6;\ U = \{u \in E^3 : u \le 0\}.$

67. $J(u) = 3u_1^2 - u_2^2 - 2u_3^2 + u_1u_2 + 3u_1u_3 + u_2u_3 +$
$+ 3u_2 + 6;\ U = E^3.$

68. $J(u) = 5u_1^2 - \frac{1}{2}u_2^2 - u_3^2 + u_1u_2 + 2u_1u_3 + 2u_2u_3 +$
$+ u_3 - 1;\ U = E^3.$

69. $J(u) = -2u_1^2 - \frac{1}{2}u_2^2 - 5u_3^2 + \frac{1}{2}u_1u_2 + 2u_1u_3 + u_2u_3 +$
$+ 3u_1 - 2u_2 + 6;\quad U = E^3.$

70. $J(u) = u_1^3 + u_3^3 - 10u_1 + u_2 + 5u_3 - 6;$
$U = \{u \in E^3 : u \leq 0\}.$

71. $J(u) = 5u_1^4 + u_2^6 + u_3^2 + 13u_1 + 7u_3 + 8;\ U = E^3.$

72. $J(u) = -3u_1^2 + 2u_2^2 + 2u_3^2 - 3u_1u_2 - u_1u_3 +$
$+ 2u_2u_3 + 17;\ U = E^3.$

73. $J(u) = 4u_3 + u_2^4 - \frac{1}{2}u_3^4 - 3u_1 + 8u_2 - 11;$
$U = \{u \in E^3 : u \leq 0\}.$

74. $J(u) = 8u_1^3 + 12u_3^2 + 3u_1u_3 - 6u_2 - 17;$
$U = \{u \in E^3 : u \geq 0\}.$

75. $J(u) = 2u_1^2 + 2u_2^2 - 4u_3^2 + u_1u_2 - 2u_2u_3 - 16;\ U = E^3.$

76. $J(u) = 2u_1^2 + u_2^2 + \frac{1}{2}u_3^2 + 2u_1u_2 + 8u_3 + 12;\ U = E^3.$

77. $J(u) = +\frac{1}{2}u_2^7 + \frac{1}{2}u_3^4 + 2u_2u_3 + 11u_1 + 6;$
$U = \{u \in E^3 : u \leq 0\}.$

78. $J(u)=+\frac{5}{2}u_1^2+u_2^2-4u_3^2-\frac{3}{2}u_1u_2+2u_1u_3+$

$+\frac{1}{2}u_2u_3+8u_3+13; \quad U=E^3.$

79. $J(u)=u_1^2-\frac{1}{2}u_2^3+2u_1u_2+5u_1u_3+7u_1+16;$

$U=\{u\in E^3 : u\le 0\}.$

80. $J(u)=2u_1^2+u_2^2-\frac{3}{2}u_3^2+u_1u_2-\frac{1}{2}u_1u_3+$

$+2u_2u_3+10;\ U=E^3.$

81. $J(u)=u_1^2-\frac{3}{2}u_3^2+\frac{5}{2}u_1u_3-12u_2+18;\ U=E^3.$

82. $J(u)=6u_1^2-u_2^3-6u_3^2+12u_1+8u_2+17;$

$U=\{u\in E^3 : u\ge 0\}.$

83. $J(u)=u_1^2-\frac{3}{2}u_2^2+2u_2^2-u_1u_2+u_1u_3+2u_2u_3+8u_2;\ U=E^3.$

84. $J(u)=\frac{7}{2}u_1^2+\frac{4}{3}u_2^3+\frac{1}{2}u_3^3-13u_1+7u_3+9;$

$U=\{u\in E^3 : u\le 0\}.$

85. $J(u)=\frac{5}{6}u_1^3+\frac{1}{4}u_2^5-\frac{3}{2}u_3^3-22u_2+10;$

$U=\{u\in E^3 : u\le 0\}.$

86. $J(u)=-\frac{5}{6}u_1^3+2u_2^2-\frac{3}{2}u_3^2+2u_1u_2-3u_1u_3+u_2u_3;$

$U=\{u\in E^3 : u\le 0\}.$

87. $J(u)=2u_1^2-4u_2^2-u_3^2-u_1u_2+9u_1u_3-u_2u_3+8;\ U=E^3.$

88. $J(u)=\frac{3}{2}u_1^2-\frac{5}{2}u_2^2+\frac{9}{2}u_3^2-3u_1u_3+7u_2u_3;\ U=E^3.$

89. $J(u)=\frac{7}{6}u_1^3-\frac{5}{2}u_2^2+\frac{5}{12}u_3^4-\frac{1}{2}u_4^3-3u_2;$

$U=\{u\in E^3 : u\geq 0\}.$

Task 2

To evaluate the task of the convex or nonlinear programming (1 – 87– variants):

1. $J(u)=3\dot{u}_1-2u_2-\frac{1}{2}u_1^2-u_2^2+u_1u_2\to \max,$

   $2u_1+u_2\leq 2,\ u_1\geq 0,\qquad u_1+2u_2\leq 2,\ u_2\geq 0.$

2. $J(u)=3u_1-2u_2-\frac{1}{2}u_1^2-u_2^2+u_1u_2\to \max,$

   $u_1\leq 3,\ u_2\leq 6,\qquad u_1\geq 0,\ u_2\geq 0.$

3. $J(u)=-4u_1+8u_2-u_1^2-\frac{3}{2}u_2^2+2u_1u_2\to \max,$

   $u_1+u_2\leq 3,\ u_1\geq 0,\qquad u_1-u_2\leq 1,\ u_2\geq 0.$

4. $J(u)=-4u_1+8u_2-u_1^2-\frac{3}{2}u_2^2+2u_1u_2\to \max,$

   $-u_1+u_2\leq 1,\ u_1\geq 0,\qquad u_1\leq 4,\ u_2\geq 0.$

5. $J(u)=-4u_1+8u_2-u_1^2-\frac{3}{2}u_2^2+2u_1u_2\to \max,$

   $3u_1+5u_2\leq 15,\ u_1-u_2\leq 1,\qquad u_1\geq 0,\ u_2\geq 0.$

6. $J(u)=3u_1-2u_2-\frac{1}{2}u_1^2-u_2^2+u_1u_2\to \max,$

   $-u_1+2u_2\leq 2,\ u_1\geq 0,\qquad 2u_1-u_2\leq 2,\ u_2\geq 0.$

7. $J(u)=-u_1+6u_2-u_1^2-3u_2^2+3u_1u_2\to \max,$

   $4u_1+3u_2\leq 12,\ u_1\geq 0,\qquad -u_1+u_2\leq 1,\ u_2\geq 0.$

8. $J(u) = -u_1 + 6u_2 - u_1^2 - 3u_2^2 + 3u_1u_2 \to \max,$

$u_1 + u_2 \le 3,\ u_1 \ge 0, \qquad -2u_1 + u_2 \le 2,\ u_2 \ge 0.$

9. $J(u) = -u_1 + 6u_2 - u_1^2 - 3u_2^2 + 3u_1u_2 \to \max,$

$u_1 - u_2 \le 0,\ u_1 \ge 0, \qquad u_2 \le 5,\ u_2 \ge 0.$

10. $J(u) = 6u_2 - u_1^2 - \dfrac{3}{2}u_2^2 + 2u_1u_2 \to \max,$

$3u_1 + 4u_2 \le 12,\ u_1 \ge 0, \qquad -u_1 + u_2 \le 2,\ u_2 \ge 0.$

11. $J(u) = 6u_2 - u_1^2 - \dfrac{3}{2}u_2^2 + 2u_1u_2 \to \max,$

$-u_1 + 2u_2 \le 2,\ u_1 \ge 0, \qquad u_1 \le 2,\ u_2 \ge 0.$

12. $J(u) = 6u_2 - u_1^2 - \dfrac{3}{2}u_2^2 + 2u_1u_2 \to \max,$

$3u_1 + 4u_2 \le 12,\ u_1 \ge 0, \qquad -u_1 - 2u_2 \le -2,\ u_2 \ge 0.$

13. $J(u) = 8u_1 + 12u_2 - u_1^2 - \dfrac{3}{2}u_2^2 \to \max,$

$-2u_1 - u_2 \le -4,\ u_1 \ge 0, \qquad 2u_1 + 5u_2 \le 10,\ u_2 \ge 0.$

14. $J(u) = 8u_1 + 12u_2 - u_1^2 - \dfrac{3}{2}u_2^2 \to \max,$

$-u_1 + 2u_2 \le 2,\ u_1 \ge 0, \qquad u_1 \le 6,\ u_2 \ge 0.$

15. $J(u) = 8u_1 + 12u_2 - u_1^2 - \dfrac{3}{2}u_2^2 \to \max,$

$-3u_1 + 2u_2 \le 0,\ u_1 \ge 0, \qquad 4u_1 + 3u_2 \le 12,\ u_2 \ge 0.$

16. $J(u) = 3u_1 - 2u_2 - \dfrac{1}{2}u_1^2 - u_2^2 + u_1u_2 \to \max,$

$-2u_1 - u_2 \le -2,\ u_1 \ge 0, \qquad 2u_1 + 3u_2 \le 6,\ u_2 \ge 0.$

17. $J(u) = 6u_1 + 4u_2 - u_1^2 - \dfrac{1}{2}u_2^2 - u_1u_2 \to \max,$

$u_1 + 2u_2 \le 2,\ u_1 \ge 0, \qquad -2u_1 + u_2 \le 0,\ u_2 \ge 0.$

18. $J(u) = 6u_1 + 4u_2 - u_1^2 - \frac{1}{2}u_2^2 - u_1u_2 \to \max,$

$2u_1 + u_2 \le 2,\ u_1 \ge 0, \qquad u_2 \le 1,\ u_2 \ge 0.$

19. $J(u) = 6u_1 + 4u_2 - u_1^2 - \frac{1}{2}u_2^2 - u_1u_2 \to \max,$

$3u_1 + 2u_2 \le 6,\ u_1 \ge 0, \qquad -3u_1 - u_2 \le -3,\ u_2 \ge 0.$

20. $J(u) = 8u_1 + 6u_2 - 2u_1^2 - u_2^2 \to \max,$

$-u_1 + u_2 \le 1,\ u_1 \ge 0, \quad 3u_1 + 2u_2 \le 6,\ u_2 \ge 0.$

21. $J(u) = 8u_1 + 6u_2 - 2u_1^2 - u_2^2 \to \max,$

$-u_1 + u_2 \le 1,\ u_1 \ge 0, \quad u_1 \le 3,\ u_2 \ge 0.$

22. $J(u) = 8u_1 + 6u_2 - 2u_1^2 - u_2^2 \to \max,$

$-u_1 + u_2 \le 2,\ u_1 \ge 0, \quad 3u_1 + 4u_2 \le 12,\ u_2 \ge 0.$

23. $J(u) = 2u_1 + 2u_2 - u_1^2 - 2u_2^2 + 2u_1u_2 \to \max,$

$4u_1 + 3u_2 \le 12,\ u_1 \ge 0, \qquad u_2 \le 3,\ u_2 \ge 0.$

24. $J(u) = 2u_1 + 2u_2 - u_1^2 - 2u_2^2 + 2u_1u_2 \to \max,$

$2u_1 + u_2 \le 4,\ u_1 \ge 0, \qquad -u_1 + u_2 \le 2,\ u_2 \ge 0.$

25. $J(u) = 2u_1 + 2u_2 - u_1^2 - 2u_2^2 + 2u_1u_2 \to \max,$

$2u_1 - u_2 \le 2,\ u_1 \ge 0, \qquad u_2 \le 4,\ u_2 \ge 0.$

26. $J(u) = 4u_1 + 4u_2 - 3u_1^2 - u_2^2 + 2u_1u_2 \to \max,$

$4u_1 + 5u_2 \le 20,\ u_1 \ge 0, \qquad u_1 \le 4,\ u_2 \ge 0.$

27. $J(u) = 4u_1 + 4u_2 - 3u_1^2 - u_2^2 + 2u_1u_2 \to \max,$

$3u_1 + 6u_2 \le 18,\ u_1 \ge 0, \qquad u_1 - 4u_2 \le 4,\ u_2 \ge 0.$

28. $J(u) = 4u_1 + 4u_2 - 3u_1^2 - u_2^2 + 2u_1u_2 \to \max,$

$3u_1 + 4u_2 \le 12,\ u_1 \ge 0, \qquad u_1 - 2u_2 \le 2,\ u_2 \ge 0.$

29. $J(u) = 12u_1 + 4u_2 - 3u_1^2 - u_2^2 \to \max,$

$u_1 + u_2 \le 6,\ u_1 \ge 0, \quad -\frac{1}{2}u_1 + \frac{1}{2}u_2 \le -1,\ u_2 \ge 0.$

30. $J(u) = \frac{11}{2}u_1 - \frac{1}{6}u_2 - u_1^2 - \frac{2}{3}u_2^2 + \frac{1}{2}u_1u_2 \to \max,$

$2u_1 - u_2 \le 2,\ u_1 \ge 0, \qquad -u_1 + 2u_2 \le 2,\ u_2 \ge 0.$

31. $J(u) = 18u_1 + 12u_2 - 2u_1^2 - u_2^2 - 2u_1u_2 \to \max,$

$u_1 + u_2 \le 4,\ u_1 \ge 0, \qquad u_1 + \frac{1}{2}u_2 \ge 1,\ u_2 \ge 0.$

32. $J(u) = -6u_1 + 16u_2 - \frac{1}{2}u_1^2 - \frac{5}{2}u_2^2 + 2u_1u_2 \to \max,$

$5u_1 + 2u_2 \le 10,\ u_1 \ge 0, \qquad 3u_1 + 2u_2 \ge 6,\ u_2 \ge 0.$

33. $J(u) = 11u_1 + 8u_2 - 2u_1^2 - u_2^2 - u_1u_2 \to \max,$

$u_1 - u_2 \le 0,\ u_1 \ge 0, \qquad 3u_1 + 4u_2 \le 12,\ u_2 \ge 0.$

34. $J(u) = 8u_2 - 4u_1^2 - 2u_2^2 + 4u_1u_2 \to \max,$

$u_1 \le 4,\ u_2 \le 3, \qquad u_1 \ge 0,\ u_2 \ge 0.$

35. $J(u) = 18u_1 + 20u_2 - u_1^2 - 2u_2^2 + 2u_1u_2 \to \max,$

$u_1 + u_2 \le 5, \quad u_1 \ge 2,\ u_2 \ge 0.$

36. $J(u) = 12u_1 - 2u_2 - \frac{3}{2}u_1^2 - \frac{1}{2}u_2^2 + u_1u_2 \to \max,$

$u_1 \le 4,\ u_2 \le 3, \qquad u_1 + 3u_2 \le 6,\ u_2 \ge 0.$

37. $J(u) = 26u_1 + 20u_2 - 3u_1^2 - 2u_2^2 - 4u_1u_2 \to \max,$

$2u_1 + u_2 \le 4,\ u_1 \ge 0, \qquad u_2 \le 2,\ u_2 \ge 0.$

38. $J(u) = 10u_1 - 8u_2 - u_1^2 - 2u_2^2 + 2u_1u_2 \to \max,$

$-u_1 + u_2 \le -2,\ u_1 \ge 0, \qquad u_1 \le 5,\ u_2 \ge 0.$

39. $J(u) = \frac{13}{2}u_1 + 5u_2 - 2u_1^2 - u_2^2 + \frac{1}{2}u_1u_2 \to \max,$

$u_1 + u_2 \le 3,\ u_1 \ge 0, \qquad u_1 - u_2 \le 1,\ u_2 \ge 0.$

40. $J(u) = \frac{9}{2}u_1 + \frac{27}{2}u_2 - \frac{1}{2}u_1^2 - 2u_2^2 - \frac{1}{2}u_1u_2 \to \max,$

$u_1 + u_2 \ge 2,\ u_1 \ge 0, \qquad u_1 + u_2 \le 4,\ u_2 \ge 0.$

41. $J(u) = -2u_1 + 8u_2 - u_1^2 - 5u_2^2 + 4u_1u_2 \to \max,$

$u_1 + u_2 \le 3,\ u_1 \ge 0, \qquad -2u_1 + 3u_2 \le 6,\ u_2 \ge 0.$

42. $J(u) = -8u_1 + 18u_2 - u_1^2 - 2u_2^2 + 2u_1u_2 \to \max,$

$-u_1 + u_2 \le 3,\ u_1 \ge 0, \qquad u_1 \le 2,\ u_2 \ge 0.$

43. $J(u) = \dfrac{23}{2}u_1 - u_1^2 - 2u_2^2 + \dfrac{3}{2}u_1u_2 \to \max,$

$5u_1 + 4u_2 \le 20,\ u_1 \ge 0, \qquad u_1 - u_2 \le 2,\ u_2 \ge 0.$

44. $J(u) = 48u_1 + 28u_2 - 4u_1^2 - 2u_2^2 - 4u_1u_2 \to \max,$

$2u_1 + u_2 \le 6,\ u_1 \ge 0, \qquad -2u_1 + u_2 \le 4,\ u_2 \ge 0.$

45. $J(u) = u_1 + 10u_2 - u_1^2 - 2u_2^2 + u_1u_2 \to \max,$

$3u_1 + 5u_2 \le 15,\ u_1 \ge 0, \qquad u_1 - 2u_2 \le 4,\ u_2 \ge 0.$

46. $J(u) = -6u_1 + 18u_2 - 2u_1^2 - 2u_2^2 + 2u_1u_2 \to \max,$

$2u_1 - u_2 \le 2,\ u_1 \ge 0, \qquad 5u_1 + 3u_2 \le 15,\ u_2 \ge 0.$

47. $J(u) = -u_1 + 5u_2 - u_1^2 - u_2^2 + u_1u_2 \to \max,$

$u_1 + u_2 \le 3,\ u_1 \ge 0, \qquad u_1 \le 2,\ u_2 \ge 0.$

48. $J(u) = 14u_1 + 10u_2 - u_1^2 - u_2^2 - u_1u_2 \to \max,$

$-3u_1 + 2u_2 \le 6,\ u_1 \ge 0, \qquad u_1 + u_2 \le 4,\ u_2 \ge 0.$

49. $J(u) = -u_1^2 - u_2^2 + 6 \to \max, \quad u_1 + u_2 \le 5, \quad u_2 \ge 0.$

50. $J(u) = 5u_1^2 - u_2^2 + 4u_1u_2 \to \max,\ u_1 \ge 0,\ u_2 \ge 0.$

$u_1^2 + u_2^2 + 2u_1 - 4u_2 \le 4,$

51. $J(u) = u_1^2 + u_1u_2 + u_2u_3 + 6 \to \max,$

$u_1 + 2u_2 + u_3 \le 3, \quad u_j \ge 0,\ j \in \overline{1,3}.$

52. $J(u) = u_2u_3 + u_1 + u_2 - 10 \to \min,$

$2u_1 + 5u_2 + u_3 \le 10, \quad u_2 \ge 0, \quad u_3 \ge 0.$

53. $J(u) = -u_1u_2 + u_1 - u_2 + 5 \to \min,$

$2u_1 + u_2 = 3,\ u_1 \ge 0, \quad 5u_1 - u_2 = 4,\ u_2 \ge 0.$

54. $J(u) = 10u_1^2 + 5u_2^2 - u_1 + 2u_2 - 10 \to \min,$

$2u_1^2 + u_2 \le 4,\ u_2 \ge 0,\ \ u_1 + u_2 \le 8.$

55. $J(u) = u_3^2 + u_1 u_2 - 6 \to \min,$

$2u_2 + u_3 \le 3,\ u_j \ge 0,\ \ u_1 + u_2 + u_3 \le 2,\ j = 1,2,3\,.$

56. $J(u) = -2u_1^2 - 3u_2^2 + u_1 - 6 \to \max,$

$u_1^2 + u_2 \le 3,\ u_2 \ge 0,\ \ \ 2u_1 + u_2 \le 5.$

57. $J(u) = u_1^2 + 3u_1 + 5 \to \min,$

$u_1^2 + u_2^2 - 2u_1 + 8u_2 \le -16,\ \ -u_1^2 + 6u_1 - u_2 \le 7.$

58. $J(u) = u_1^2 + \frac{5}{2}u_2^2 - u_1 u_2 - 7 \to \min,$

$u_1^2 - 4u_1 - u_2 \le -5,\ \ -u_1^2 + 6u_1 - u_2 \le 7.$

59. $J(u) = -u_1^5 + 7 \to \max,$

$u_1^2 + u_2^2 - 4u_2 \le 0,\ \ u_1 u_2 - 1 \ge 0,\ u_1 \ge 0.$

60. $J(u) = 4u_1^2 + \frac{7}{2}u_2^2 - u_1 u_2 - u_2 + 5 \to \min,$

$2u_1^2 + 9u_2^2 \le 18,\ u_2 \ge 0,\ \ \ \ -u_1 - u_2 \le 1.$

61. $J(u) = 2u_1^5 + 3u_2^3 - 11 \to \min,$

$u_1^2 + u_2^2 - 6u_1 + 16u_2 \le -72,\ \ u_2 + 8 \le 0.$

62. $J(u) = 3u_1^2 + u_2^2 - 2u_1 u_2 + 5 \to \min,$

$25u_1^2 + 4u_2^2 \le 100,\ u_1 \ge 0,\ \ \ \ \ \ u_1^2 + 1 \ge 0.$

63. $J(u) = -5u_1^2 - 2u_2^5 + 3u_1 - 4u_2 - 18 \to \max,$

$3u_1^2 - 6 - u_2 \le 0,\ \ u_1^2 + u_2^1 \le 9,\ \ \ \ u_1 \ge 0,\ \ \ \ \ \ u_2 \ge 0.$

64. $J(u) = 3u_1^2 + \frac{5}{2}u_2^2 - 3u_1 u_2 + 7 \to \min,$

$3u_1 - u_2 \le -1,\ \ \ u_2^2 \le 2.$

65. $J(u) = u_1^6 + 2u_1u_2 - u_1 + 6 \to \max,$

$3u_1^2 \le 15,\ u_1 \ge 0\ \ -u_1 - 5u_2 \ge -10,\ \ u_2 \ge 0.$

66. $J(u) = 4u_1^2 + 3u_2^2 + 4u_1u_2 - u_1 + 6u_2 - 5 \to \max,$

$-u_1^2 - u_2^2 \ge -3,\ \ -3u_1^2 - u_2 \ge -4.$

67. $J(u) = -u_1^2 - 2u_2^2 + u_1u_2 + u_1 - 26 \to \max,$

$u_1^2 \le 25,\ u_1 + 2u_2 \le 5, u_2 \ge 0.$

68. $J(u) = -u_1^2 - u_2^2 \to \max,$

$2u_1 + 3u_2 \le 6,\ \ -u_1 - 5u_2 \ge -10, u_1 \ge 0.$

69. $J(u) = 2u_1^2 + u_2^2 + 4u_1u_2 - u_1 + 6 \to \max,$

$-u_1 + u_2 \ge -1,\ u_1 \ge 0,\ \ 2u_1 + u_2 \le 5, u_2 \ge 0.$

70. $J(u) = -2u_1^2 - 3u_2^2 - u_1u_2 + 6 \to \max,$

$u_1 + u_2 \le 3.\ \ -u_1^2 + u_2 \ge -5, u_1 \ge 0$

71. $J(u) = -u_1^2 - u_2^2 + u_1 + 5u_2 - 5 \to \max,,\ u_1 + u_2 \le 5.$

72. $J(u) = -u_1^2 - u_2^2 - u_1u_3 \to \min,$

$3u_1 + u_2 + u_3 \le 4,\ u_j \ge 0,\ \ u_1 + 2u_2 + 2u_3 = 3,\ j = 1,2,3.$

73. $J(u) = -5u_1^2 - 6u_2^2 - u_3^2 + 8u_1u_2 + u_1 \to \max,$

$u_1^2 - u_2 + u_3 \le 5,\ \ u_1 + 5u_2 \le 8,\ \ u_1 \ge 0,\ \ u_2 \ge 0.$

74. $J(u) = u_1^2 + u_2^2 + u_3^2 \to \min,$

$u_1 + u_2 + u_3 = 3,\ \ 2u_1 - u_2 + u_3 \le 5.$

75. $J(u) = -3u_1^2 - u_2^2 - u_3^2 - u_1u_2 - 6 \to \min,$

$2u_1 + u_2 + u_3 \le 5,\ \ u_2 + 3u_3 \le 8, u_2 \ge 0.$

76. $J(u) = -u_1u_2 + u_1u_3 - 2u_2u_3 + u_1 - 5 \to \max,$

$u_1 + u_2 + u_3 = 3,\ \ 2u_1 + u_2 \le 5,\ \ u_1 \ge 0.$

77. $J(u) = u_1^2 + \frac{1}{2}u_2^2 + \frac{3}{2}u_3^2 - 12u_1 + 13u_2 - 5u_3 \to \min,$

$u_1 - 5u_2 + 4u_3 = 16,\ \ 2u_1 + 7u_2 - 3u_3 \le -2,\ \ u_1 \ge 0.$

78. $J(u) = u_1^2 + 2u_2^2 + 30u_1 - 16u_3 + 10 \to \min,$

$5u_1 + 3u_2 - 4u_3 = -20, \;\; u_1 - 6u_2 + 3u_3 \le 0, u_3 \ge 0.$

79. $J(u) = \frac{1}{2}u_1^2 + u_2^2 - 5u_1 + u_3 - 16 \to \min,$

$u_1 + u_2 - 2u_3 \le 3, \;\; 2u_1 - u_2 - 3u_3 \ge -11, \;\; u_j \ge 0, \;\; j = 1,2,3.$

80. $J(u) = -2u_1^2 - 2u_1 - 4u_2 + 3u_3 + 8 \to \max,$

$8u_1 - 3u_2 + 3u_3 \le 40, \; -2u_1 + u_2 - u_3 = -3, \;\; u_2 \ge 0.$

81. $J(u) = u_1u_3 - u_1 + 10 \to \min, \quad u_1^2 + u_3 \le 3, u_j \ge 0$

$u_2^2 + u_3 \le 3, j = 1,2,3.$

82. $J(u) = -3u_1^2 - u_2^2 + u_2 + 7u_3 \to \max,$

$4u_1 + u_2 - 2u_3 = 5 - 1, \;\; u_1 \ge 0, \;\; 2u_2 + u_3 \le 4.$

83. $J(u) = u_1^2 + u_2^2 - u_1u_2 - 6u_1u_3 + u_2u_3 + u_1 - 25 \to \min,$

$u_1^2 + u_2^2 + u_1u_2 + u_3 \le 10,$

$u_1u_2 + u_1 + 2u_2 - u_3 = 4, \;\; u_j \ge 0, \; j = 1,2,3.$

84. $J(u) = -e^{(u_1 + u_2 + u_3)} \to \max,$

$-u_1^2 - u_2^2 - u_3^2 \le 10, \;\; u_2 \ge 0 \;\; u_1^3 \le 5, u_3 \ge 0.$

85. $J(u) = -3u_2^2 + 11u_1 + 3u_2 + u_3 + 27 \to \max,$

$u_1 - 7u_2 + 3u_3 \le -7, \; 5u_1 + 2u_2 - u_3 \le 2, \; u_3 \ge 0.$

86. $J(u) = 4u_1^2 - 8u_1 + u_2 + 4u_3 + 12 \to \min,$

$3u_1 - u_2 + u_3 \le 5$, $u_1 + 2u_2 - u_3 \ge 0, \;\; u_j \ge 0, \;\; j = 1,2,3.$

87. $J(u) = \frac{1}{2}u_2^2 + \frac{3}{2}u_3^2 - 2u_2 - 9u_3 \to \min,$

$3u_1 - 5u_2 + u_3 \ge -19, \quad 2u_2 - u_3 \ge 0, \;\; u_j \ge 0, \;\; j = 1,2,3.$

Task 3

To evaluate the task of the linear programming (1 – 89 variants):

$$J(u) = c'u \to \min; \ U = \{u \in E^n \,/\, Au = b, \, u \ge 0\}.$$

1. $c' = (5,-1,1,0,0), \ b' = (5,4,11), \ A = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 & 0 \\ 0 & 5 & 6 & 1 & 0 \end{pmatrix}.$

2. $c' = (6,1,-1,-2,0), \ b' = (4,1,9), \ A = \begin{pmatrix} 1 & 2 & 1 & 6 & 1 \\ 3 & -1 & -1 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 \end{pmatrix}.$

3. $c' = (0,6,1,-1,0), \ b' = (6,6,6), \ A = \begin{pmatrix} 3 & -1 & 1 & 6 & 1 \\ 1 & 0 & 5 & 1 & -7 \\ 1 & 2 & 3 & 1 & 1 \end{pmatrix}.$

4. $c' = (7,1,1,-1,0), \ b' = (5,3,2), \ A = \begin{pmatrix} 5 & 1 & 1 & 3 & 1 \\ 0 & -2 & 4 & 1 & 1 \\ 1 & -3 & 5 & 0 & 0 \end{pmatrix}.$

5. $c' = (8,1,-3,0,0), \ b' = (4,3,6), \ A = \begin{pmatrix} -1 & 1 & 1 & 2 & 1 \\ 2 & 0 & 1 & -3 & 5 \\ 3 & 0 & -1 & 6 & 1 \end{pmatrix}.$

6. $c' = (0,1,-3,-1,-1), \ b' = (2,8,5), \ A = \begin{pmatrix} -2 & -1 & 2 & 0 & 0 \\ 1 & 1 & 4 & 1 & 3 \\ 3 & 1 & -1 & 0 & 6 \end{pmatrix}.$

7. $c' = (1,-2,-1,-1,0), \ b' = (2,7,2), \ A = \begin{pmatrix} 2 & 0 & 1 & -1 & 1 \\ 4 & 1 & 3 & 1 & 2 \\ -1 & 0 & 1 & 2 & 1 \end{pmatrix}.$

8. $c' = (0,1,-6,1,-3),\ b' = (9,14,3),\ A = \begin{pmatrix} 6 & 1 & 1 & 2 & 1 \\ -1 & 0 & -1 & 7 & 8 \\ 1 & 0 & 2 & 1 & 1 \end{pmatrix}.$

9. $c' = (-8,-1,-1,1,0),\ b' = (5,9,3),\ A = \begin{pmatrix} -2 & 0 & 3 & 1 & 1 \\ 3 & 1 & 1 & 6 & 2 \\ -1 & 0 & 2 & -1 & 2 \end{pmatrix}.$

10. $c' = (-1,3,-1,1,0),\ b' = (4,4,15),\ A = \begin{pmatrix} 2 & 0 & 3 & 1 & 0 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & 3 & 6 & 3 & 6 \end{pmatrix}.$

11. $c' = (0,2,0,1,-3),\ b' = (6,1,24),\ A = \begin{pmatrix} 4 & 1 & 1 & 0 & 1 \\ -1 & 3 & -1 & 0 & 3 \\ 8 & 4 & 12 & 4 & 12 \end{pmatrix}.$

12. $c' = (10,5,-25,5,0),\ b' = (32,1,15), A = \begin{pmatrix} 8 & 16 & 8 & 8 & 24 \\ 0 & 2 & -1 & 1 & 1 \\ 0 & 3 & 2 & -1 & 1 \end{pmatrix}.$

13. $c' = (6,0,-1,1,2),\ b' = (8,2,2),\ A = \begin{pmatrix} 4 & 1 & 1 & 2 & 1 \\ 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$

14. $c' = (-5,-1,3,-1,0),\ b' = (7,7,12),\ A = \begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 3 & -1 & 4 & 0 \\ 0 & 4 & 0 & 8 & 1 \end{pmatrix}.$

15. $c' = (5,3,2,-1,1),\ b' = (12,16,3),\ A = \begin{pmatrix} 3 & 4 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 \\ 1 & -3 & 0 & 0 & 1 \end{pmatrix}.$

16. $c' = (7,0,1,-1,1),\ b' = (1,12,4),\ A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}.$

17. $c' = (6,-1,2,-1,1),\ b' = (2,11,6),\ A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 1 & 1 & 1 \\ 3 & 2 & 0 & 0 & 1 \end{pmatrix}.$

18. $c' = (0,0,3,-2,-1),\ b' = (5,7,2),\ A = \begin{pmatrix} 2 & 1 & 1 & 1 & 3 \\ 3 & 0 & 2 & -1 & 6 \\ 1 & 0 & -1 & 2 & 1 \end{pmatrix}.$

19. $c' = (1,7,2,1,-1),\ b' = (20,12,6),\ A = \begin{pmatrix} 6 & 3 & 1 & 1 & 1 \\ 4 & 3 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 1 \end{pmatrix}.$

20. $c' = (2,0,1,-1,1),\ b' = (2,14,1),\ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 3 & 5 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}.$

21. $c' = (6,1,0,1,2),\ b' = (2,18,2),\ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 2 & 6 & 2 & 1 & 1 \\ 1 & -2 & 0 & 0 & 1 \end{pmatrix}.$

22. $c' = (0,3,1,-1,1),\ b' = (2,2,6),\ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \end{pmatrix}.$

23. $c' = (3,0,1,-2,1),\ b' = (6,2,2),\ A = \begin{pmatrix} 2 & 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$

24. $c' = (0,5,1,-1,1),\ b' = (2,2,10),\ A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \end{pmatrix}.$

25. $c' = (1,5,2,-1,1),\ b' = (12,1,3),\ A = \begin{pmatrix} 3 & 4 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 1 & 1 \end{pmatrix}.$

26. $c' = (5,0,1,-1,1),\ b' = (1,3,12),\ A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 2 \end{pmatrix}.$

27. $c' = (7,0,2,-1,1),\ b' = (2,3,11),\ A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 5 & 2 & 1 & 1 & 1 \end{pmatrix}.$

28. $c' = (1,-4,1,1,1),\ b' = (28,2,12),\ A = \begin{pmatrix} 5 & 5 & 1 & 2 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 0 & 0 & 1 \end{pmatrix}.$

29. $c' = (0,8,2,1,-1),\ b' = (2,20,6),\ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 6 & 3 & 1 & 1 & 1 \\ 3 & -2 & 0 & 0 & 1 \end{pmatrix}.$

30. $c' = (0,-2,1,-1,1),\ b' = (14,10,1),\ A = \begin{pmatrix} 3 & 5 & 1 & 1 & 2 \\ 2 & 5 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}.$

31. $c' = (7,2,0,1,2),\ b' = (2,12,18),\ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ -2 & 6 & 2 & 1 & 1 \end{pmatrix}.$

32. $c' = (1,3,1,-1,1),\ b' = (2,6,1),\ \ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}.$

33. $c' = (5,1,-1,1,2),\ b' = (2,8,2),\ \ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 4 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$

34. $c' = (1,2,1,-1,1),\ b' = (11,2,3),\ \ A = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$

35. $c' = (10,5,2,-1,1),\ b' = (17,1,3),\ \ A = \begin{pmatrix} 2 & 3 & 1 & 2 & 1 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 & 1 \end{pmatrix}.$

36. $c' = (2,-1,-3,1,1),\ b' = (6,16,7),\ \ A = \begin{pmatrix} 1 & 0 & 2 & 1 & 3 \\ 2 & 2 & 4 & 8 & 4 \\ 1 & 0 & -1 & 7 & 1 \end{pmatrix}.$

37. $c' = (4,-1,1,2,-1),\ b' = (2,13,16),\ \ A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{pmatrix}.$

38. $c' = (2,2,1,2,-1),\ b' = (12,2,26),\ \ A = \begin{pmatrix} 4 & -3 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 6 & 3 & 1 & 1 & 1 \end{pmatrix}.$

39. $c' = (5,2,-1,1,1),\ b' = (26,12,6),\ \ A = \begin{pmatrix} 9 & 1 & 1 & 1 & 2 \\ 4 & 3 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 1 \end{pmatrix}.$

40. $c' = (1,11,1,2,-1)$, $b' = (13,10,1)$, $A = \begin{pmatrix} 2 & 6 & 1 & 1 & 1 \\ 2 & 5 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}$.

41. $c' = (-5,1,1,-2,0)$, $b' = (1,6,2)$, $A = \begin{pmatrix} 3 & 1 & -3 & 1 & 0 \\ 2 & 3 & 1 & 2 & 1 \\ 3 & 1 & -2 & -1 & 0 \end{pmatrix}$.

42. $c' = (0,3,1,-1,1)$, $b' = (6,2,1)$, $A = \begin{pmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}$.

43. $c' = (8,1,-3,0,0)$, $b' = (4,3,6)$, $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 2 & 0 & 1 & -3 & 5 \\ 3 & 0 & -1 & 6 & 1 \end{pmatrix}$.

44. $c' = (2,1,1,-1,1)$, $b' = (2,11,3)$, $A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$.

45. $c' = (5,0,1,2,-1)$, $b' = (13,3,6)$, $A = \begin{pmatrix} 4 & 3 & 2 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{pmatrix}$.

46. $c' = (1,3,1,1,1)$, $b' = (1,17,4)$, $A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 5 & 2 & 2 & 1 & 3 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$.

47. $c' = (9,5,2,-1,1)$, $b' = (12,17,3)$, $A = \begin{pmatrix} 3 & 4 & 1 & 0 & 0 \\ 2 & 3 & 1 & 2 & 1 \\ 1 & -3 & 0 & 0 & 1 \end{pmatrix}$.

48. $c' = (1,1,1,2,-1),\ b' = (12,26,12),\ \ A = \begin{pmatrix} 4 & -3 & 1 & 0 & 0 \\ 6 & 3 & 1 & 1 & 1 \\ 3 & 4 & 0 & 0 & 1 \end{pmatrix}.$

49. $c' = (0,-7,-1,1,1),\ b' = (2,26,6),\ \ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 9 & 1 & 1 & 1 & 2 \\ 3 & -2 & 0 & 0 & 1 \end{pmatrix}.$

50. $c' = (4,8,1,2,-1),\ b' = (2,13,1),\ \ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 2 & 6 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}.$

51. $c' = (3,-1,-1,1,0),\ b' = (3,6,5),\ \ A = \begin{pmatrix} 1 & -1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 2 & 3 \\ 3 & 2 & 0 & -3 & 8 \end{pmatrix}.$

52. $c' = (1,-3,1,2,-1),\ b' = (2,2,5),\ \ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$

53. $c' = (0,1,1,-2,1),\ b' = (2,2,6),\ \ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 1 \end{pmatrix}.$

54. $c' = (0,5,1,-1,1),\ b' = (2,2,11),\ \ A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 & 1 \end{pmatrix}.$

55. $c' = (9,2,-1,0,1),\ b' = (12,1,17),\ \ A = \begin{pmatrix} 3 & 4 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 2 & 1 \end{pmatrix}.$

56. $c' = (1,0,1,1,1),\ b' = (1,3,17),\ A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 5 & 2 & 2 & 1 & 3 \end{pmatrix}.$

57. $c' = (3,-2,1,2,-1),\ b' = (2,3,13),\ A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 & 1 \end{pmatrix}.$

58. $c' = (9,0,-1,1,1),\ b' = (2,12,26),\ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 4 & 3 & 0 & 1 & 0 \\ 9 & 1 & 1 & 1 & 2 \end{pmatrix}.$

59. $c' = (5,5,1,2,-1),\ b' = (26,2,12),\ A = \begin{pmatrix} 6 & 3 & 1 & 1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 0 & 0 & 1 \end{pmatrix}.$

60. $c' = (0,10,1,2,-1),\ b' = (2,10,1),\ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 2 & 5 & 0 & 1 & 0 \\ 2 & 6 & 1 & 1 & 1 \end{pmatrix}.$

61. $c' = (3,2,1,-1,0),\ b' = (5,5,5),\ A = \begin{pmatrix} 3 & 1 & 3 & 1 & 2 \\ 3 & 2 & 1 & 1 & 1 \\ 7 & -2 & 2 & 0 & -1 \end{pmatrix}.$

62. $c' = (1,2,-1,-1,0),\ b' = (25,3,5),\ A = \begin{pmatrix} 5 & 10 & 5 & 15 & 10 \\ 0 & 1 & -1 & 6 & 2 \\ 0 & 6 & 1 & -1 & -1 \end{pmatrix}.$

63. $c' = (1,1,-2,-1,0),\ b' = (4,7,9),\ A = \begin{pmatrix} 2 & -1 & 0 & 1 & 1 \\ 3 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 & 6 \end{pmatrix}.$

64. $c' = (1,-3,1,0,0),\ b' = (4,3,6),\ A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & -1 & -3 \\ 3 & 0 & 3 & 1 & 2 \end{pmatrix}.$

65. $c' = (-2,-1,1,-5,0),\ b' = (7,1,9),\ A = \begin{pmatrix} 3 & 1 & 1 & 2 & 3 \\ 2 & 0 & 3 & 2 & -1 \\ 3 & 0 & -1 & 1 & 6 \end{pmatrix}.$

66. $c' = (6,0,-1,1,2),\ b' = (8,2,2),\ A = \begin{pmatrix} 4 & 1 & 1 & 2 & 1 \\ 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$

67. $c' = (-5,-1,3,-1,0),\ b' = (7,7,12),\ A = \begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 3 & -1 & 4 & 0 \\ 0 & 4 & 0 & 8 & 1 \end{pmatrix}.$

68. $c' = (5,3,2,-1,1),\ b' = (12,16,3),\ A = \begin{pmatrix} 3 & 4 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 \\ 1 & -3 & 0 & 0 & 1 \end{pmatrix}.$

69. $c' = (7,0,1,-1,1),\ b' = (1,12,4),\ A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}.$

70. $c' = (6,-1,2,-1,1),\ b' = (2,11,6),\ A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 1 & 1 & 1 \\ 3 & 2 & 0 & 0 & 1 \end{pmatrix}.$

71. $c' = (0,0,3,-2,-1),\ b' = (5,7,2),\ A = \begin{pmatrix} 2 & 1 & 1 & 1 & 3 \\ 3 & 0 & 2 & -1 & 6 \\ 1 & 0 & -1 & 2 & 1 \end{pmatrix}.$

72. $c' = (1,7,2,1,-1),\ b' = (20,12,6),\ A = \begin{pmatrix} 6 & 3 & 1 & 1 & 1 \\ 4 & 3 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 1 \end{pmatrix}.$

73. $c' = (2,0,1,-1,1),\ b' = (2,14,1),\ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 3 & 5 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}.$

74. $c' = (6,1,0,1,2),\ b' = (2,18,2),\ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 2 & 6 & 2 & 1 & 1 \\ 1 & -2 & 0 & 0 & 1 \end{pmatrix}.$

75. $c' = (0,3,1,-1,1),\ b' = (2,2,6),\ A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \end{pmatrix}.$

76. $c' = (3,0,1,-2,1),\ b' = (6,2,2),\ A = \begin{pmatrix} 2 & 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$

77. $c' = (0,5,1,-1,1),\ b' = (2,2,10),\ A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 2 \end{pmatrix}.$

78. $c' = (1,5,2,-1,1),\ b' = (12,1,3),\ A = \begin{pmatrix} 3 & 4 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 1 & 1 \end{pmatrix}.$

79. $c' = (5,0,1,-1,1),\ b' = (1,3,12),\ A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 2 \end{pmatrix}.$

80. $c' = (7,0,2,-1,1)$, $b' = (2,3,11)$, $A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 5 & 2 & 1 & 1 & 1 \end{pmatrix}$.

81. $c' = (1,-4,1,1,1)$, $b' = (28,2,12)$, $A = \begin{pmatrix} 5 & 5 & 1 & 2 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 0 & 0 & 1 \end{pmatrix}$.

82. $c' = (0,8,2,1,-1)$, $b' = (2,20,6)$, $A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 6 & 3 & 1 & 1 & 1 \\ 3 & -2 & 0 & 0 & 1 \end{pmatrix}$.

83. $c' = (7,2,0,1,2)$, $b' = (2,12,18)$, $A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ 2 & 6 & 2 & 1 & 1 \end{pmatrix}$.

84. $c' = (1,3,1,-1,1)$, $b' = (2,6,1)$, $A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}$.

85. $c' = (1,2,1,-1,1)$, $b' = (11,2,3)$, $A = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$.

86. $c' = (10,5,2,-1,1)$, $b' = (17,1,3)$, $A = \begin{pmatrix} 2 & 3 & 1 & 2 & 1 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 & 1 \end{pmatrix}$.

87. $c' = (2,-1,-3,1,1)$, $b' = (6,16,7)$, $A = \begin{pmatrix} 1 & 0 & 2 & 1 & 3 \\ 2 & 2 & 4 & 8 & 4 \\ 1 & 0 & -1 & 7 & 1 \end{pmatrix}$.

88. $c' = (1,11,1,2,-1),\ b' = \left(13,10,1\right),\quad A = \begin{pmatrix} 2 & 6 & 1 & 1 & 1 \\ 2 & 5 & 0 & 1 & 0 \\ 7 & -4 & 0 & 0 & 1 \end{pmatrix}.$

89. $c' = (2,1,1,-1,1),\ b' = \left(2,11,3\right),\quad A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$

KEYS

To the 1-st task

It is possible to construct the matrixes $J''(u)$ and check its determination on U according to the theorem 3 (lecture 4). Similarly tasks 2-89 are executed.

To the 2-nd task

1. $J(u_*) = 2{,}50$;
2. $J(u_*) = 4{,}75$;
3. $J(u_*) = 11{,}00$;
4. $J(u_*) = 11{,}00$;
5. $J(u_*) = 11{,}12$;
6. $J(u_*) = 2{,}50$;
7. $J(u_*) = 5{,}67$;
8. $J(u_*) = 5{,}29$;
9. $J(u_*) = 6{,}25$;
10. $J(u_*) = 9{,}59$;
11. $J(u_*) = 12{,}50$;
12. $J(u_*) = 9{,}59$;
13. $J(u_*) = 24{,}34$;
14. $J(u_*) = 38{,}91$;
15. $J(u_*) = 27{,}99$;
16. $J(u_*) = 4{,}57$;
17. $J(u_*) = 8{,}00$;
18. $J(u_*) = 5{,}75$;
19. $J(u_*) = 8{,}40$;
20. $J(u_*) = 12{,}76$;
21. $J(u_*) = 17{,}00$;
22. $J(u_*) = 15{,}84$;
23. $J(u_*) = 4{,}45$;
24. $J(u_*) = 3{,}77$;
25. $J(u_*) = 4{,}20$;
26. $J(u_*) = 11{,}02$;
27. $J(u_*) = 10{,}12$;
28. $J(u_*) = 9{,}53$;
29. $J(u_*) = 13{,}00$;
30. $J(u_*) = 6{,}10$;
31. $J(u_*) = 41{,}00$;
32. $J(u_*) = 25{,}81$;
33. $J(u_*) = 20{,}82$;
34. $J(u_*) = 15{,}00$;
35. $J(u_*) = 66{,}0$;
36. $J(u_*) = 25{,}12$;
37. $J(u_*) = 47{,}0$;
38. $J(u_*) = 23{,}50$;
39. $J(u_*) = 12{,}07$;
40. $J(u_*) = 25{,}13$;
41. $J(u_*) = 4{,}60$;
42. $J(u_*) = 40{,}0$;
43. $J(u_*) = 23{,}45$;
44. $J(u_*) = 112{,}0$;
45. $J(u_*) = 103{,}5$;
46. $J(u_*) = 41{,}23$;
47. $J(u_*) = 6{,}75$;
48. $J(u_*) = 40{,}0$.

To the 3-rd task

1. $J(u_*) = 5$;

2. $J(u_*) = 6$;

3. $J(u_*) = 6$;

4. $J(u_*) = \frac{14}{3}$;

5. $J(u_*) = \frac{134}{7}$;

6. $J(u_*) = -\frac{137}{33}$;

7. $J(u_*) = 0$;

8. $J(u_*) = \frac{11}{3}$;

9. $J(u_*) = 0$;

10. $J(u_*) = 7$;

11. $J(u_*) = \frac{54}{13}$;

12. $J(u_*) = -10$;

13. $J(u_*) = 6$;

14. $J(u_*) = -\frac{7}{6}$;

15. $J(u_*) = 26$;

16. $J(u_*) = \frac{29}{5}$;

17. $J(u_*) = 15$;

18. $J(u_*) = \frac{4}{3}$;

19. $J(u_*) = 14$;

20. $J(u_*) = \frac{43}{7}$;

21. $J(u_*) = \frac{99}{5}$;

22. $J(u_*) = \frac{17}{3}$;

23. $J(u_*) = 7$;

24. $J(u_*) = 6$;

25. $U = \varnothing$;

26. $J(u_*) = \frac{27}{5}$;

27. $J(u_*) = 16$;

28. $J(u_*) = 26$;

29. $J(u_*) = 12$;

30. $J(u_*) = -\frac{3}{7}$;

31. $J(u_*) = \frac{118}{5}$;

32. $J(u_*) = \frac{19}{3}$;

33. $J(u_*) = \frac{20}{3}$;

34. $J(u_*) = \frac{22}{3}$;

35. $J(u_*) = \frac{437}{13}$;

36. $J(u_*) = \frac{31}{3}$;

37. $U = \varnothing$;

38. $J(u_*) = \frac{89}{5}$;

39. $J(u_*) = 19$;

40. $J(u_*) = 21$;

41. $J(u_*) = -3$;

42. $J(u_*) = \frac{17}{3}$;

43. $J(u_*) = \frac{110}{7}$;

44. $J(u_*) = \frac{19}{2}$;

45. $J(u_*) = 5$;

46. $J(u_*) = \frac{77}{5}$;

47. $J(u_*) = \frac{389}{13}$;

48. $J(u_*) = \frac{72}{5}$;

49. $J(u_*) = 16$;

55. $J(u_*) = \frac{438}{13}$;

61. $J(u_*) = 5$;

50. $J(u_*) = 23$;

51. $J(u_*) = \frac{23}{3}$;

52. $J(u_*) = 6$;

53. $J(u_*) = 4$;

54. $J(u_*) = \frac{26}{5}$;

56. $J(u_*) = 8$;

57. $J(u_*) = \frac{24}{5}$;

58. $J(u_*) = 22$;

59. $J(u_*) = 28$;

60. $U = \varnothing$;

62. $J(u_*) = \frac{133}{37}$;

63. $J(u_*) = \frac{21}{10}$;

64. $J(u_*) = -1$;

65. $J(u_*) = -\frac{5}{17}$.

TESTS

1) Weierstrass' theorems define

A) Sufficient conditions that ensemble

$U_* = \{u_* \in U \ / \ I(u_*) = \min_{u \in U} I(u)\} \neq \varnothing$

B) Necessary and sufficient conditions that ensemble

$U_* = \{u_* \in U / I(u_*) = \min_{u \in U} I(u)\} \neq \varnothing$

C) Necessary conditions that ensemble

$U_* = \{u_* \in U \ / \ I(u_*) = \min_{u \in U} I(u)\} \neq \varnothing$

D) Necessary and sufficient conditions to ensemble convexity

$U_* = \{u_* \in U / I(u_*) = \min_{u \in U} I(u)\}$

E) Necessary conditions that the point $u_* \in U$ is a point of the minimum to function $I(u)$ on ensemble U

2) Sufficient conditions that the ensemble $U_* = \{u_* \in U / I(u_*) = \min_{u \in U} I(u)\} \neq \varnothing$ **give**

A) Weierstrass' theorems

B) Farkas's theorems

C) Kuhn-Tucker's theorems

D) Lagrange theorems

E) There isn't any correct answer amongst A)-D).

3) Ensemble $U \subseteq E^n$ is identified convex if

A) $\forall u \in U, \forall v \in U, \ \forall \alpha \in [0,1]$ the point $u_\alpha = \alpha u + (1-\alpha)v \in U$

B) $\forall u \in U, \forall v \in U, \forall \alpha \in [0,1]$, $u_\alpha \leq \alpha u + (1-\alpha)v$

C) $\forall u \in U, \forall v \in U, \ \forall \alpha \in [0,1]$, $u_\alpha \geq \alpha u + (1-\alpha)v$

D) $\forall u \in U, \forall v \in U$, $\exists$ number $\alpha \in [0,1]$ for which

$u_\alpha = \alpha u + (1-\alpha)v \in U$

E) $\forall u \in U, \forall v \in U, \forall \alpha \in E^1$ the point

$u_\alpha = \alpha u + (1-\alpha)v \in U$

4) Ensemble $U \subseteq E^n$ is convex iff

A) it contains all convex linear combinations of any final number own points

B) $\forall u \in U, \ \forall v \in U$ possible construct the point $u_\alpha = \alpha u + (1-\alpha)v \in U$ which is a convex linear combination of the points u, v

C) It is limited and closed

D) $\forall v \in U, \ \exists \varepsilon > 0$ vicinity $\overline{S}_\varepsilon(v) \subset U$

E) From any sequence $\{u_k\} \subset U$ it is possible to select converging subsequence

5) Function $I(u)$ determined on convex ensemble $U \subseteq E^n$ is identified convex if:

A) $\forall u \in U, \forall v \in U, \forall \alpha \in [0,1]$,

$I(\alpha u + (1-\alpha)v) \le \alpha I(u) + (1-\alpha)I(v)$

B) $\forall \{u_k\} \in U : \lim\limits_{k\to\infty} u_k = u, \ I(u) \le \lim\limits_{k\to\infty} I(u_k)$

C) $\forall u \in U, \forall v \in U, \forall \alpha \in [0,1]$,

$I(\alpha u + (1-\alpha)v) \ge \alpha I(u) + (1-\alpha)I(v)$

D) $\forall \{u_k\} \in U : \lim\limits_{k\to\infty} u_k = u, \quad I(u) \ge \underline{\lim}_{k\to\infty} I(u_k)$

E) There isn't any correct answer amongst A)-D).

6) Necessary and sufficient condition that continuously differentiated function $I(u)$ is convex on convex ensemble $U \subseteq E^n$:

A) $I(u) - I(v) \ge < I'(v), u - v >, \quad \forall u \in U, \ \forall v \in U$

B) $I(u) - I(v) \le I(\alpha u + (1-\alpha)v)$,

$\forall u \in U, \ \forall v \in U, \ \forall \alpha \in [0,1]$

C) $< I'(u_*), u - u_* > \geq 0, \quad \forall u \in U$

D) $< I'(u_*), u - u_* > \leq 0, \quad \forall u \in U$

E) $I(u) - I(v) \geq I(\alpha u + (1-\alpha)v),$
$\forall u \in U, \forall v \in U, \forall \alpha \in [0,1]$

7) Necessary and sufficient condition that continuously differentiated function $I(u)$ is convex on convex ensemble $U \subseteq E^n$

A) $< I'(u) - I'(v), u - v > \geq 0, \quad \forall u \in U, \forall v \in U$

B) $< I'(u) - I'(v), u - v > \leq 0, \quad \forall u \in U, \forall v \in U$

C) $I'(u) - I'(v) \geq I(u) - I(v), \quad \forall u \in U, \forall v \in U$

D) $I(\alpha u + (1-\alpha)v) \geq I'(u) - I'(v), \quad \forall u \in U, \forall v \in U$

E) $I(\alpha u + (1-\alpha)v) \leq I'(u) - I'(v), \quad \forall u \in U, \forall v \in U$

8) Function $I(u)$ determined on convex ensemble $U \subseteq E^n$ is identified strongly convex on U, if

A) $\exists \kappa > 0, I(\alpha u + (1-\alpha)v) \leq \alpha I(u) + (1-\alpha)I(v) - \alpha(1-\alpha)\kappa|u-v|^2,$
$\forall u \in U, \forall v \in U, \forall \alpha \in [0,1]$

B) $I(\alpha u + (1-\alpha)v) \leq \alpha I(u) + (1-\alpha)I(v),$
$\forall u \in U, \forall v \in U, \forall \alpha \in [0,1]$

C) $I(\alpha u + (1-\alpha)v) \geq \alpha I(u) + (1-\alpha)I(v),$
$\forall u \in U, \forall v \in U, \forall \alpha \in [0,1]$

D) $\exists \kappa > 0, I(\alpha u + (1-\alpha)v) \geq \alpha I(u) + (1-\alpha)I(v) - \alpha(1-\alpha)\kappa|u-v|^2,$
$\forall u \in U, \forall v \in U, \forall \alpha \in [0,1]$

E) There isn’t any correct answer amongst A)-D).

9) In order to twice continuously differentiated on convex ensemble U function $I(u)$ is convex on U necessary and sufficiently performing the condition

A) $< I''(u)\xi, \xi > \geq 0, \quad \forall u \in U, \forall \xi \in E^n$

B) $\det I''(u) \neq 0 \quad \forall u \in U$

C) $< I''(u)\xi, \xi > \le 0, \quad \forall u \in U, \forall \xi \in E^n$

D) $< I''(u)u, v > \ge 0, \quad \forall u \in U, \forall v \in U$

E) $< I''(u)u, v > \le 0, \quad \forall u \in U, \forall v \in U$

10) Choose correct statement

A) If $I(u), G(u)$ are convex functions determined on convex ensemble U, $\alpha \ge 0, \beta \ge 0$ that function $\alpha I(u) + \beta G(u)$ is also convex on ensemble U

B) If $I(u), G(u)$ are convex functions determined on convex ensemble $U \subseteq E^n$, that function $I(u) \cdot G(u)$ is also convex on U

C) If $I(u), G(u)$ are convex functions determined on convex ensemble $U \subseteq E^n$, that function $I(u) - G(u)$ is also convex on U

D) If $I(u), G(u)$ are convex functions determined on convex ensemble $U \subseteq E^n$, $G(u) \ne 0, \forall u \in U$, that $I(u)/G(u)$ is convex function on U

E) There isn't any correct answer amongst A)-D).

11) Choose correct statement

A) Intersection of two convex ensembles is convex ensemble

B) Union of two convex ensembles is convex ensemble

C) If $U \subseteq E^n$ is convex ensemble, that ensemble $E^n \setminus U$ is also convex

D) If $U_1 \subseteq E^n, U_2 \subseteq E^n$ is convex ensemble, that ensemble $U_1 \setminus U_2$ is also convex

E) There isn't any correct answer amongst A)-D).

12) Define type of the following problem $I(u) = < c, u > \to \inf$

$$u \in U = \{u \in E^n / u_j \ge 0, j \in I,$$
$$g_i(u) = < a^i, u > - b_i \le 0, i = \overline{1, m},$$
$$g_i(u) = < a^i, u > - b_i = 0, i = \overline{m+1, s}\}$$

A) General problem of linear programming

B) Canonical problem of linear programming

C) Nondegenerate problem of linear programming
D) Problem of nonlinear programming
E) The simplest variational problem

13) Define type of the following problem $I(u) = < c, u > \rightarrow \inf$

$$u \in U = \{u \in E^n / u_j = 0, i = \overline{1,n},\ Au = b\}$$

A) General problem of linear programming
B) Canonical problem of linear programming
C) Nondegenerate problem of linear programming
D) Problem of nonlinear programming
E) The simplest variational problem

14) Function $I(u) = u^2 - 2u_1u_2 + u_2^2$ on ensemble $U = E^n$ is
A) convex
B) concave
C) neither convex, nor concave
D) convex under $u_1 \geq 0$, $u_2 \geq 0$ and concave under $u_1 \leq 0$, $u_2 \leq 0$
E) convex under $u_1 \leq 0$, $u_2 \leq 0$ and concave under $u_1 \geq 0$, $u_2 \geq 0$

15) Define type of the problem $I(u) = 2u_1^2 - u_2^2 \rightarrow \inf$
$u \in U = \{u \in E^n / 2u_1 - u_2 \leq 3,\ u_1 + 4u_2 = 5\}$
A) Problem of nonlinear programming
B) Canonical problem of linear programming
C) Convex programming problem
D) General problem of linear programming
E) The simplest variational problem

16) Define type of the problem $I(u) = 2u_1 - u_2 \rightarrow \inf$
$u \in U = \{u \in E^n / u_2 \geq 0, u_1 + u_2^2 \leq 4, 2u_1 + u_2 \leq 2$
A) Convex programming problem
B) Canonical problem of linear programming
C) General problem of linear programming
D) Problem of nonlinear programming
E) The simplest variational problem

17) Define type of the problem
$I(u) = 2u_1 - u_2 \rightarrow \inf$
$u \in U = \{u \in E^n / u_1 \geq 0, -u_1 + 4u_2 \leq 2, u_1 - 3u_2 = 4\}$

A) General problem of linear programming
B) Canonical problem of linear programming
C) Convex programming problem
D) Problem of nonlinear programming
E) The simplest variational problem

18) Sequence $\{u_k\} \subset U$ is identified minimizing to function $I(u)$ determined on ensemble U, if

A) $\lim_{k\to\infty} I(u_k) = I_*,$ where $I_* = \inf_{u\in U} I(u)$

B) $I(u_{k+1}) \le I(u_k), \ \forall k = 1,2,...$

C) $\exists \lim_{k\to\infty} u_k = u,$ moreover $u \in U$

D) $I(u_{k+1}) \ge I(u_k), \ \forall k = 1,2,...$

E) There isn't any correct answer amongst A)-D).

19) Choose correct statement for problem

$I(u) \to \inf, \quad u \in U \subset E^n$

A) For any function $I(u)$ and ensemble $U \subseteq E^n$ always exists a minimizing sequence $\{u_k\} \subset U$ for function $I(u)$

B) If function *I(u)* is continuously differentiable on ensemble U, that it reaches minimum value on U, i.e. $\exists u_* \in U$ such that $I(u_*) = \min_{u\in U} I(u)$

C) If minimizing sequence $\{u_k\} \subset U$ exists for function *I (u)*, that $\exists u_* \in U$ such that $I(u_*) \min_{u\in U} I(u)$

D) If the point $u_* \in U$ exists such that $I(u_*) \min_{u\in U} I(u)$, that minimizing sequence $\{u_k\} \subset U$ exists for function *I(u)*

E) There isn't any correct answer amongst A)-D).

20) Choose correct statement for problem

$I(u) \to \inf, \quad u \in U \subseteq E^n, \ U_* = \{u_* \in U / I(u_*) = \min_{u\in U} I(u)\}$

A) If function *I(u)* is semicontinuous from below on compact ensemble *U*, that ensemble $U_* \neq \varnothing$

B) If ensemble U is convex, but function $I(u)$ is continuous on *U*, that ensemble $U_* \neq \varnothing$

C) If ensemble U is limited, but function $I(u)$ is convex on ensemble *U*, that ensemble $U_* \neq \varnothing$

D) If function $I(u)$ is semicontinuous from below on ensemble *U*, but ensemble U is limited, that $U_* \neq \varnothing$

E) If function $I(u)$ is continuously differentiable on closed ensemble *U*, that $U_* \neq \varnothing$

21) Simplex method is used for solution

A) linear programming problem in canonical form
B) linear programming problem in general form
C) convex programming problem
D) nonlinear programming problem
E) optimal programming problem

22) Kuhn-Tucker theorems

A) define necessary and sufficient conditions that in the convex programming problem for each point $U_* = \{u_* \in U / I(u_*) = \min_{u \in U} I(u)\}$ exist Lagrange's multipliers $\lambda \in \Lambda_0$ such that pair (u_*, λ_*) forms saddle point to Lagrange' function

B) define necessary and sufficient existence conditions of function $I(u)$ minimizing sequences $\{u_k\} \subset U$

C) define necessary and sufficient convexity conditions to function $I(u)$ on convex ensemble $U \subseteq E^n$

D) define necessary conditions that in the convex programming problem ensemble $U_* = \{u_* \in U / I(u_*) = \min\limits_{u \in U} I(u)\}$ consists of single point

E) define necessary and sufficient conditions that ensemble $U_* = \{u_* \in U / I(u_*) = \min\limits_{u \in U} I(u)\} \neq \varnothing$

23) For the convex programming problem of the type $I(u) \to \inf, \quad u \in U = \{u \in E^n / u \in U_0, g_i(u) \le 0, \forall i = \overline{1,m}\}$

Lagrange's function has the form

A) $L(u,\lambda) = I(u) + \sum\limits_{i=1}^{m} \lambda_i g_i(u),\ u \in U_0,$

$\lambda \in \Lambda_0 = \{\lambda \in E^m / \lambda_1 \ge 0, \ldots, \lambda_m \ge 0\}$

B) $L(u,\lambda) = \sum\limits_{i=1}^{m} \lambda_i g_i(u),\ \lambda_i \ge 0,$

$\forall i = \overline{1,m},\ \overline{\sum\limits_{i=1}^{m} \lambda_i = 1,\ u \in U}_0$

C) $L(u,\lambda) = \lambda I(u),\ u \in U_0\ \ \lambda \in E^1$

D) $L(u,\lambda) = I(u + \sum\limits_{i=1}^{m} \lambda_i g_i(u)),\ u \in U_0,\ \lambda \in E^m$

E) $L(u,\lambda) = I(u) - \sum\limits_{i=1}^{m} \lambda_i g_i(u),\ u \in U_0,$

$\lambda \in \Lambda_0 = \{\lambda \in E^m / \lambda_1 \ge 0, \ldots, \lambda_m \ge 0\}$

24) For the convex programming problem of the type I(u) →inf
$u \in U = \{u \in E^n / u \in U_0, g_i(u) \le 0, i = \overline{1,m},$

$g_i(u) = \langle a^i, u \rangle - b_i = 0, i = \overline{m+1,s}\}$

Lagrange's function has the form

A) $L(u,\lambda)=I(u)+\sum_{i=1}^{s}\lambda_i g_i(u),\ u\in U_0,$

$\lambda\in\Lambda_0=\{u\in E^s / u_1\geq 0,\ u_2\geq 0,\dots,u_m\geq 0\}$

B) $L(u,\lambda)=\sum_{i=1}^{m}\lambda_i g_i(u),\ \lambda_i\geq 0,\ \forall i=\overline{1,m},\ \sum_{i=1}^{m}\lambda_i=1,\ u_0\in U$

C) $L(u,\lambda)=I(u)+\sum_{i=1}^{m}\lambda_i g_i(u),\ u\in U_0,\ \lambda\in E^m$

D) $L(u,\lambda)=I(u)+\sum_{i=1}^{s}\lambda_i g_i(u),\ u\in U_0,$

$\lambda\in\Lambda_0=\{u\in E^s / u_1\geq 0,\ u_2\geq 0,\dots,u_s\geq 0\}$

E) $L(u,\lambda)=\lambda I(u),\ u\in U,\ \lambda\in E^1$

25) For the convex programming problem necessary and sufficient conditions that for any point $u_*\in U_*=\{u\in U / I(u_*)=\min_{u\in U} I(u)$ **exist Lagrange's multipliers** $\lambda^*\in\Lambda_0$ **such that pair** (u_*,λ_*) **forms saddle point to Lagrange's function are defined**

A) Kuhn-Tucker's theorems
B) Lagrange's theorems
C) Weierstrass' theorem
D) Farkas' theorems
E) Bellman's theorems

26) Let $I(u)$ **be a convex function determined and continuously differentiated on convex ensemble *U*, ensemble** $U_*=\{u_*\in U / I(u_*)= =\min_{u\in U} I(u)\neq\varnothing$. **In order to point** $u_*\in U$ **be a point of the function minimum *I (u)* on ensemble *U* necessary and sufficient performing the condition**

A) $< I'(u_*), u-u_* >\geq 0,\quad \forall u\in U$

B) $< I'(u_*), u-u_* >\leq 0,\quad \forall u\in U$

C) $I(u_*) - I(u) \geq < I'(u), u_* - u >, \quad \forall u \in U$

D) $< I''(u_*)\xi, \xi > \geq 0, \quad \forall \xi \in E^n$

E) $I'(u_*) = 0$

27) Pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ is identified saddle point to Lagrange's function $L(u,\lambda) = I(u) + \sum_{i=1}^{s} \lambda_i g_i(u)$, if

A) $L(u_*, \lambda) \leq L(u_*, \lambda^*) \leq L(u, \lambda^*), \quad \forall u \in U_0, \forall \lambda \in \Lambda_0$

B) $L(u_*, \lambda) \leq L(u, \lambda^*), \quad \forall u \in U_0, \forall \lambda \in \Lambda_0$

C) $L(u_*, \lambda^*) = 0$

D) $-L(u,\lambda) \leq L(u_*, \lambda^*) \leq L(u, \lambda), \quad \forall u \in U_0, \forall \lambda \in \Lambda_0$

E) $u_* \in U_* = \{u_* \in U / I(u_*) = \min_{u \in U} I(u)\}, \ \lambda_j^* \geq 0, \ \forall i = \overline{1,s}$

28) If pair $(u_*, \lambda^*) \in U_0 \times \Lambda_0$ is a saddle point to Lagrange's function $L(u,\lambda)$ in the convex programming problem, that

A) the point $u_* \in U_* = \{u \in U / I(u_*) = \min_{u \in U} I(u)\}$

B) Lebesgue ensemble $M(u_*) = \{u \in U / I(u) \leq I(u_*)\}$ is compact

C) there is a minimizing sequence $\{u_*\} \subset U$ for function $I(u)$, such that $\lim_{k \to \infty} |u_k| = \infty$

D) the convex programming problem is nondegenerate

E) ensemble $U_* = \{u_* \in U / I(u_*) = \min_{u \in U} I(u)\}$ contains single point

29) For the nonlinear programming problem

$$I(u)\to\inf,\ u\in U=\{u\in E^n / u\in U_0, g_i(u)\le 0, i=\overline{1,m}, g_i(u)=0, i=\overline{1,m}$$

generalized Lagrange's function has the form

A) $L(u,\lambda)=\lambda_0\ I(u)+\sum_{i=1}^{s}\lambda_i g_i(u),\ u\in U_0,\ \lambda\in\Lambda_0=$

$=\{\lambda\in E^{s+1} / \lambda_0\ge 0, \lambda_1\ge 0,\ldots,\lambda_m\ge 0\}$

B) $L(u,\lambda)=I(u)+\sum_{i=1}^{m}g_i(u),\ u\in U_0,\ \lambda\in E^m$

C) $L(u,\lambda)=\lambda_0 I(u)+\sum_{i=1}^{m}\lambda_i g_i(u),\ u\in U_0,\ \lambda\ge E^m$

D) $L(u,\lambda)=\lambda_0 I(u)+\sum_{i=1}^{s}\lambda_i g_i(u),\ u\in U_0,\ \lambda\ge E^{s+1}$

E) $L(u,\lambda)=\lambda I(u)+\sum_{i=1}^{s}g_i(u),\ u\in U_0,\ \lambda\ge 0$

30) Let $U\subseteq E^n$ **be convex ensemble, function** $I(u)\in C^1(U)$ **condition** $< I'(u_*), u-u_* >\ge 0,\ \forall u\in U$ **is**

A) necessary condition that the point $u_*\in U_*=\{u_*\in U / I(u_*)=$
$=\min_{u\in U} I(u)\}$

B) necessary and sufficient condition that the point $u_*\in U_*=\{u_*\in U / I(u_*)=\min_{u\in U} I(u)\}$

C) sufficient condition that the point $u_*\in U_*=\{u_*\in U / I(u_*)=\min_{u\in U} I(u)\}$

D) necessary and sufficient condition that the function *I(u)* is convex in the point $u_*\in U$

E) necessary and sufficient condition that ensemble $U_*=\{u_*\in U / I(u_*)=\min_{u\in U} I(u)\}\ne\varnothing$

31) Indicate faithful statement

A) convex programming problem is a partial case of the nonlinear programming problem

B) nondegenerate problem of nonlinear programming can be reduced to the convex programming problem

C) convex programming problem is a partial case of the linear programming problem

D) any nonlinear programming problem can be reduced to the convex programming problem

E) nonlinear programming problem is a partial case of the convex programming problem

32) For solving of the nonlinear programming problem is used

A) Lagrange multipliers method

B) Simplex-method

C) Method of the least squares

D) Pontryagin maximum principle

E) Bellman's dynamic programming method

33) What from enumerated methods can be used for solving of the convex programming problem

A) Lagrange multipliers method

B) Simplex-method

C) Method of the least squares

D) Pontryagin maximum principle

E) Bellman's dynamic programming method

34) What from enumerated methods can be used for solving of the linear programming problem

A) Simplex-method

B) Method of the least squares

C) Pontryagin maximum principle

D) Bellman's dynamic programming method

E) any method from A)- D)

35) In the convex programming problem the minimum to function *I(u)* on ensemble *U* can be reached

A) in internal or border points ensemble *U*

B) only in the border points ensemble *U*

C) only in the isolated points ensemble *U*

D) only in the internal points ensemble U
E) in internal, border, isolated points ensemble U

36) In the nonlinear programming problem minimum to function I(u) on ensemble *U* can be reached

A) in internal, border, isolated points ensemble U
B) only in the border points ensemble U
C) only in the isolated points ensemble U
D) in internal or border points ensemble U
E) only in the internal points ensemble U

37) If in the linear programming problem in canonical form $I(u) \to \inf,\ u \in U = \{u \in E^n / u_j \geq 0,\ j = \overline{1,n},\ Au = b\}$ **ensemble** $U_* = \{u_* \in U / I(u_*) = \min_{u \in U} I(u)\}$ **contains single point** u_***, that the point is**

A) an extreme point ensemble U
B) an isolated points ensemble U
C) an internal point ensemble U
D) an internal or extreme point ensemble U
E) extreme or isolated point ensemble U

38) The linear programming problem in canonical form has the form

A) $I(u) = \langle c, u \rangle \to \inf,$
$u \in U = \{u \in E^n / u_j \geq 0, j = \overline{1,n},\ Au = b\}$

B) $I(u) = \langle c, u \rangle \to \inf,$
$u \in U = \{u \in E^n / u_j \geq 0, j \in I, g_i(u) = \langle a^i,$
$u \rangle - b_i \leq 0, i = \overline{1,m},$
$g_i(u) = \langle a^i, u \rangle - b_i = 0, i = \overline{m+1,s}\}$

C) $I(u) = \langle c, u \rangle \to \inf,\ u \in U = \{u \in E^n / Au = b\}$

D) $I(u) = \langle c, u \rangle \to \inf,$

$u \in U = \{u \in E^n / u_j \geq 0, j \in I, g_i(u) = \langle a^i,$

$u \rangle - b_i \leq 0, i = 1, m\}$

E) $I(u) = \langle c, u \rangle \to \inf,$

$u \in U = \{u \in E^n / 0 \leq g_i(u) \leq b_i, i = \overline{1, m}\}$

39) The linear programming problem in canonical form $I(u) = \langle c, u \rangle \to \inf,$ $u \in U = \{u \in E^n / u_j \geq 0,\ j = \overline{1, n},\ Au = b\}$ **is identified nondegenerate, if:**

A) any point $u \in U$ has not less than *rang A* positive coordinates

B) *rangA = m,* where *A* is a constant matrix of dimensionality $m \times n, \quad m < n$

C) ensemble $U_* = \{u_* \in U \,/\, I(u_*) = \min_{u \in U} I(u)\} \neq \varnothing$

D) *rangA = m,* where *A* is a constant matrix of dimensionality $m \times n, \quad m < n$

E) any point $u \in U$ has no more than *rangA = m,* positive coordinates

40) By extreme point ensemble $U = \{u \in E^n / u_j \geq 0,\ j = \overline{1, n},\ Au = b\}$ **is called**

A) Point $u \in U$ which can not be presented in the manner of $u = \alpha v + (1 - \alpha) w$, where $\alpha \in (0,1),\ v \in U,\ w \in U$

B) Isolated point ensemble *U*

C) Border point ensemble *U*

D) Point $u \in U$ presented in the manner of $u = \alpha v + (1 - \alpha) w$, where $\alpha \in (0,1),\ v \in U,\ w \in U$

E) Internal point ensemble *U*

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