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Section - I

DIFFERENTIAL GEOMETRY

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THE EUCLIDEN SPACES

Unit Structure :

Differential geometry makes use of a lot of linear algebra and multi-variable calculus. We utilize this unit consisting of Chapters 1,2,3, of the study material to recollect basic concepts and elementary results of both, linear algebra and multi-variable calculus.

To begin with, in this chapter, we will recapitulate elementary algebra and geometry of the Euclidean Spaces $\mathbb{R}^n\left(n=2,3,4......\right)$. We discuss their basic features ab initio in three parts; (i) the real vector space structure of $\mathbb{R}^n$, (ii) the inner product and the resulting metric topology of $\mathbb{R}^n$ and (iii) its standard orientation.

In **Chapter 2** we recall the algebra of linear endomorphisms of $\mathbb{R}^n$, reaching finally the group $SO\left(\mathbb{R}^n\right)$ of its orientation preserving linear automorphisms and discuss some of its properties. Actually we introduce the whole group $GL\left(\mathbb{R}^n\right)$ and them concentrate more on its sub-group $O\left(\mathbb{R}^n\right)$ consisting of all orthogonal automorphisms of $\mathbb{R}^n$ and their matrix representations. We explain here, the total derivative $Df\left(p\right)$ of a vector valued function $f\left(x\right)$ of a multi-variable $x=\left(x_1,x_2,....,x_n\right)$ as a linear transformation elaborating its role as a local linear approximation to f in neighborhoods of the point p (in the domain of)

Chapter 3 is a mix-bag of some more linear algebra and a rather long recap of basic concepts and elementary and yet fundamental results of differential calculus (such as the inverse function theorem, implicit function theorem the rank theorem….). Throughout we are emphasizing the role of $Df\left(p\right)$ as a linear transformation approximating f around p.

In what is to follow, we make use of both-the linear algebra apparatus and the multi-variable calculus machinery in a crucial way. For example, we differentiate a curve at a point to get the tangent line - a linear (and hence a more amicable) curve approximating the bending and twisting the curve.

Similarly we approximate a (continuously bending) surface by the tangent plane to the surface at a point of it.

Approximating the non-linear real world by linear objects is indeed a fruitful, common practice. Differential geometry emphasizes this practice.

Actually smooth curves in $\mathbb{R}^2/\mathbb{R}^3$ and smooth surfaces in $\mathbb{R}^3$ are the main geometric objects of our interest but the analysis of their geometry often leades us to higher dimensional Euclidean geometry. Therefore we are treating their generality, emphasizing particular cases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

For further details regarding the portion of this unit, the reader should consult (1) Linear Algen, (2) Undergraduate Analysis, both books authored by Serg Lang; and of course, the text books recommended by the University.

1.1 THE VECTOR SPACE $\mathbb{R}^n$

Throughout this set of notes, $\mathbb{R}$ denotes the real number system (aka the "real line"). Following subsets of it appear here and there in the text :

- $\mathbb{N} = \{1, 2, 3, ...\}$
- $\mathbb{Z}^+ = \mathbb{N} U \{0\}$
  $= \{0, 1, 2, ..., n, ...\}$
- $\mathbb{Z} = \{...., -2, -1, 0, 1, 2, ...\}$

Let n be any integer ≥ 2.

$\mathbb{R}^n$ stands for the set of all **ordered** **n-tuples** $(x_1, x_2, ..., x,)$ of real numbers. For the sake of notational economy, we denote it by x; thus; $x := (x_1, x_2, ...x_k, ...x_n)$ the real number x_k occupying the k^{th} place in the $n-tuple\ x$ above is the k^{th} coordinate of x.

For any **n-tuples** $x=(x_1,x_2,....,x_n)$ and $y=(y_1,y_2,....,y_n)$ and for any $a\in\mathbb{R}$, we put :

- $x+y:=(x_1+y_1,x_2+y_2,....x_n+y_n)$

and • $\bullet\, a\cdot x:=(ax_1,ax_2,.......,ax_n)$

(again, for the notational simplicity, we will often write ax in place of $a\cdot x\cdot$)

The declaratives $\bullet$ and $\bullet$ give rise to the algebraic operations:

a) <u>addition of vectors</u> :

$$+:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$$
$$(x,y)\mapsto x+y$$

and b) <u>multiplication of vectors by real numbers</u> :

$$\bullet:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$$
$$(a,x)\mapsto a\cdot x$$

The resulting algebraic system $(\mathbb{R}^n,+,\bullet)$ is a <u>real vector space</u> (and therefore we call its elements <u>vectors</u>. Instead of the complete triple $(\mathbb{R}^n,+,\bullet)$ we will indicate only $\mathbb{R}^n$, the underlying vector space operations $+,\bullet$ being understood.

The <u>dimension</u> of this vector space is n. For, the elements $e_1,e_{2,},....,e_n$ of $\mathbb{R}^n$ given by

$$\begin{aligned}
e_1&=(1,0,....,0)\\
e_2&=(0,1,....,0)\\
&\ \vdots\\
e_k&=(0,....,\underset{\substack{\uparrow\\ \left(k^{th}\ place\right)}}{1},....,0)\\
&\ \vdots\\
e_n&=(0,.......,0)
\end{aligned}$$

enable us to write every $x=(x_1,x_2....,x_n)\in\mathbb{R}^n$ uniquely in the form : $x=x_1e_1+x_2e_2+.....+x_ne_n$ and therefore, the set $\{e_1,e_2,...,e_n\}$ consisting of the n vectors is a vector basis of $\mathbb{R}^n$.

We call $\{e_1,e_2,...,e_n\}$ the <u>standard vector basis</u> of $\mathbb{R}^n$.

It turns out that any n dimensional real vector space can be identified with $\mathbb{R}^n$ (the identification being by meanse of an isomorphism of vector spaces). Thus, the Euclidean spaces $\mathbb{R}^n\,(n = 2,3....)$ are prototypes of all finite dimensional real vector spaces.

Let us note at this stage a slight deviation from the classical vector notations in case of $\mathbb{R}^2$ and $\mathbb{R}^3$:

In the 2-dimensional coordinate geometry we identified a plane with $\mathbb{R}^2$ by meanse of a Cartesian coordinate frame XOY and then we dealt with the points of the plane in terms of the coordinate pairs (x, y) w.r.t. our choice frame XOY. Similarly we used to identify the physical space with $\mathbb{R}^3$ by meanse of an orthogonal coordinate frome $O - XYZ$ and the resulting Cartesian coordinates of a point were (x, y, z). In the present context, we use the notations (x_1, x_2) in place of (x, y) of the planar coordinate geometry and the triples (x_1, x_2, x_3) in place of (x, y, z). Also instead of the unit vectors $\vec{i}, \bar{j}, \vec{k}$ (along the axes of the $O - XYZ$ frame) we will bring e_1, e_2, e_3 of the standard basis.

Also, the arrows $\vec{u}, \bar{v}, \vec{w}$ over the vectors u, v, w are banished, we simply write u, v, w even though they are vectors.

One more point : We often consider a lower dimensional $\mathbb{R}^m$ imbedded in a higher dimensional $\mathbb{R}^n$ by meanse of the natural imbedding map :

$$\mathbb{R}^m \hookrightarrow \mathbb{R}^n$$

taking a point $(x_1, x_2, ..., x_m)$ of $\mathbb{R}^m$ to the point $\left(x_1, x_2, ..., x_m, \underbrace{0.......0}_{n-m}\right)$ of $\mathbb{R}^n$. Thus occasionally we consider the vector space $\mathbb{R}^m$ as a subspace of a higher dimensional $\mathbb{R}^n$.

1.2 THE INNER PRODUCT OF $\mathbb{R}^n$:

For any $x = (x_1, x_2...., x_n)$ and $y = (y_1, y_2...., y_n)$ in $\mathbb{R}^n$ we consider the sum : $x_1y_1 + x_2y_2 + ... + x_ny_n$. Denoting it by $\langle x, y\rangle$ we get the map :

$$\langle -,- \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
$$(x,y) \mapsto \langle x,y \rangle$$

Note the following properties of the map $\langle -,- \rangle$:

a) The map $\langle -,- \rangle$ is <u>bilinear</u>ie i.e.for any x, y, z in $\mathbb{R}^n$ and for any a, b, c in $\mathbb{R}$, we have

i) $\langle ax+by, z \rangle = a\langle x,z \rangle + b\langle y,z \rangle$

ii) $\langle x, by+cz \rangle = b\langle x,y \rangle + c\langle x,z \rangle$

b) $\langle \, , \, \rangle$ is <u>symmetric,</u>

$\langle x,y \rangle = \langle y,x \rangle$ for all x, y in $\mathbb{R}^n$ and

c) $\langle \, , \, \rangle$ is <u>positive definite</u> i.e.

$\langle x,x \rangle \geq 0$ for all $x \in \mathbb{R}^n$ and move over

$\langle \; x \, x \; \rangle = 0$ when and only when $x = 0\big(=(0\,0....0)\big)$

The map $\langle -,- \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called the standard inner product of $\mathbb{R}^n$.

In what is to follow, we consider the vector space $\mathbb{R}^n$ equipped with the inner product $\langle -,- \rangle$ i.e we consider the quadruple $\left(\mathbb{R}^n, +, \cdot, <, >\right)$; it is the <u>$n-$dimensional Euclidean space</u>. For the usual reason, we adopt and use the shorter notation $\mathbb{R}^n$ for the quadruple.

Thus, the Euclidean space $\mathbb{R}^n$ is not just a barren set, it is a mathematical space carrying two distinct structures, namely its $n-$dimensional real vector space structure together with the standard inner product of it. Of course, these two structures are compatible with each other. One manifestation of this compatibility is the bilineanity of the inner product : the inner product respects the vector space operations of $\mathbb{R}^n$. Several other forms of the compatibility between the algebraic and geometric features of $\mathbb{R}^n$ will be witnessed while studying these notes.

We proceed to explain that the inner product of $\mathbb{R}^n$ is geometric in nature; it gives rise to a metric i.e. a distance function on $\mathbb{R}^n$:

For each $x \in \mathbb{R}^n$ we write $\|x\|$ for $+\sqrt{\langle x, x\rangle}$.

This gives rise to the function :

$$\|\cdot\| : \mathbb{R}^n \to [0, \infty)$$
$$x \mapsto \|x\|$$

We interpret $\|x\|$ as the <u>length</u> of the vector x and call the map $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ the <u>Euclidean norm</u> on $\mathbb{R}^n$.

The norm $\| \ \|$ and the inner product < , > are related by the following inequality :

Proposition 1 : For every x, y in $\mathbb{R}^n$ we have $|\langle x, y\rangle| \le \|x\|\|y\|$ and the equality holds when and only when $y = ax$ for some $a \in \mathbb{R}$ (i.e. when x and y are <u>parallel</u> vectors).

The above inequality is variously called the <u>Schwarz inequality</u>, the <u>Cauchy</u> - <u>Schwarz</u> <u>inequality</u> or the <u>CBS inequality</u> (CBS being the acronym for Cauchy - Buniyakowski - Schwarz, the mathematicians who invented this inequality independently.)

Proof : The inequality is a trivial equality in case when either of x, y is a zero vector, say y = 0. For, in that case, we have

$$\begin{aligned}\langle x, 0\rangle &= \langle x, 0+0\rangle \\ &= \langle x, 0\rangle + \langle x, 0\rangle\end{aligned}$$

Thus $\langle x, 0\rangle = 2\langle x, 0\rangle$ which implies $\langle x, 0\rangle = 0$. We therefore proceed to consider $y \neq 0$ (and therefore $\|y\| \neq 0$.) Now, for any $a \in \mathbb{R}$, we have $\langle x - ay, x - ay\rangle \ge 0$ that is, $\langle x, x\rangle - 2\,a\langle x \cdot y\rangle + a^2\langle y, y\rangle \ge 0$.

i.e. $\|x\|^2 - 2\,a\langle x, y\rangle + a^2\|y\|^2 \ge 0$ for any $a \in \mathbb{R}$.

In particular, for $a = \dfrac{\langle x, y\rangle}{\|y\|^2}$ the above inequality reduces to

$$\|x\|^2 - 2\frac{\langle x, y\rangle^2}{\|y\|^2} + \frac{\langle x, y\rangle^2 \|y\|^2}{\|y\|^4} \ge 0 .$$

Thus, $\|x\|^2 - \frac{\langle x, y\rangle^2}{\|y\|^2} \geq 0$ or equivalenty put, we get $\|x\|^2 \|y\|^2 \geq \langle x, y\rangle^2$ which gives the desired inequality.

Next if x, y are parallel, say $y = ax$ for some $a \in \mathbb{R}$ then we get $\|y\| = |a|\|x\|$ and then

$$\begin{aligned}|\langle x\ y\rangle| = |\langle x, ax\rangle| &= |a|\|y\|^2 \\ &= \|x\|\|y\|\end{aligned}$$

Thus when x and y are parallel vector, the Schwarz inequality becomes equality.

Finally suppose $|\langle x\ y\rangle| = \|x\|\|y\|$ with $y \neq o$ and therefore, $\|y\| \neq 0$. Consider $a = \frac{\langle x, y\rangle}{\|y\|^2}$ and then we have

$$\begin{aligned}\langle x - ay, x - ay\rangle &= \|x\|^2 - 2\,a\langle x, y\rangle + a^2\|y\|^2 \\ &= \|x\|^2 - 2\frac{\langle x, y\rangle^2}{\|y\|^2} + \frac{\langle x, y\rangle^2\|y\|^2}{\|y\|^4} \\ &= \|x\|^2 - \frac{\langle x, y\rangle^2}{\|y\|^2} \\ &= 0 \quad \text{by the assumed equality.}\end{aligned}$$

Thus, we have $\langle x - ay\ x - ay\rangle = 0$ and therefore $x = ay$ (with $a = \frac{\langle x, y\rangle}{\|y\|^2}$) $\square$.

The CBS inequality leades us to a geometric interpretation of the inner product : Already we have treated $\|x\| := +\sqrt{\langle x, x\rangle}$ as the length of the vector $x \in \mathbb{R}^n$. Note that this interpretation is consistant with the usual length (Pythagorean) of a vector in $\mathbb{R}^3/\mathbb{R}^2$.

Secondly consider any pair x, y of non zero vector in $\mathbb{R}^n$. We rewrite the CBS inequality in the form $-1 \leq \frac{\langle x, y\rangle}{\|x\|\|y\|} \leq 1$.

This suggests that we interprete the quantity $\frac{\langle x, y\rangle}{\|x\|\|y\|}$ as the cosine - $\cos\theta$ -of the angle-between the vectors x, y.

This consideration inspires us to declare the perpendicularity relation between vectors in $\mathbb{R}^n$; $x \perp y$ if $\langle x, y\rangle = 0$.

Also note that the classical Pythagoream property (about the lengths of sides of a right angled triangle) continues to hold in the present (higher dimensional) context : If x, y are any elements of $\mathbb{R}^n$ with $x \perp y$ then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

To see this, consider,

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x\,x\rangle + 2\langle x\,y\rangle + \langle y\,y\rangle \\ &= \|x\|^2 + 2\cdot 0 + \|y\|^2\end{aligned}$$

Thus $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ holds for all x, y in $\mathbb{R}^n$ with $x \perp y$. Note that the vectors $e_1, e_2 ...., e_n$ in the standard basis $E = \{e_1, e_2, ..., e_n\}$ are pairwise orthogonal and each of them has unit length. We express this property by saying that the standard basis of $\mathbb{R}^n$ is <u>orthonormal</u>. More generally a subset A of $\mathbb{R}^n$ is <u>orthonormal</u> if its elements satisfy the following two conditions :

i) $\|x\| = 1$ for each $x \in \mathrm{A}$

ii) If x, y are any two distinct elements of A then $x \perp y$ (i.e. $\langle x, y\rangle = 0$).

Note that an orthonormal subset $\mathrm{A} = \{v_1\, v_2 .....v_n\}$ of $\mathbb{R}^n$ is linearly independent. For if

$$a_1 v_1 + a_2 v_2 + ... + a_m v_m = 0 \qquad \text{.................................} \qquad (*)$$

holds for some real numbers $a_1, a_2, ..., a_m,$ then we deduce that $a_1 = a_2 \quad = a_m = 0$. To get this take the inner product of the equality(*) with each v_1 to get.

$$a_1 \langle v_1 v_i\rangle + a_2 \langle v_2 v_i\rangle + ... + a_i \langle v_i v_i\rangle + ... + a_m \langle v_m v_i\rangle = 0$$

i.e. $a_i 0 + a_2 0 + \quad a_i \cdot 1 + \quad + a_m \cdot 0 = 0$ which gives $a_i = 0$ for each $i(1 \le i \le m)$. This justifies our claim that the orthonormal set $A = \{v_1, ...v_m\}$ is linearly independent. On the other hand any linearly independent subset of $\mathbb{R}^n$ gives rise to an orthonormal subset having as many elements as those of the linearly independent subset. We prove this fact in the following proposition :

Proposition 2 : Any linearly independent subset $A = \{v_1, v_2, ..., v_m\}$ of $\mathbb{R}^n$ $\mathbb{R}^n$ gives rise to an orthonormal subset $B = \{f_1\ f_2 ... f_m\}$ of $\mathbb{R}^n$ in which each $f_1^{'}$ is a linear combination of $v_1, v_2, ..., v_i\ (|\le| \le m)$.

Proof : v_1 being an element of linearly independent set is non zero. Therefore $\|v_1\| \ne 0$ and therefore $f_1 := \dfrac{v_1}{\|v_1\|}$ is a well defined unit vector.

Next, we consider $v_2 - \langle v_2\ f_1 \rangle f_1 = v_2 - \dfrac{\langle v_2\ v_1 \rangle v_1}{\|v_1\|^2}$. This vector also is non-zero. (For, otherwise we would get $v_2 = \dfrac{\langle v_2\ v_1 \rangle v_1}{\|v_1\|^2}$ which contradicts the linear independence of the elements of the set A. We put

$$f_2 = \frac{v_2 - \langle v_2\ v_1 \rangle f_1}{\|v_2 - \langle v_2, v_1 \rangle f_1\|}$$

Clearly $\|f_1\| = \|f_2\| = 1$ and $f_1 \perp f_2$.

In the neat step, we consider v_3 and obtain the vector $v_3 - \langle v_3\ f_1 \rangle f_1 - \langle v_3\ f_2 \rangle f_2$ from it. Invoking the linear independence of the set A, we again get that this vector is non-zero. Using this last observation, we construct :

$$f_3 = \frac{v_3 - \langle v_3\ f_1 \rangle f_1 - \langle v_3\ f_2 \rangle f_2}{\|v_3 - \langle v_3 f_1 \rangle f_1 - \langle v_3\ f_2 \rangle f_2\|}$$

Imitating this procedure successively, we obtain the desired ortho-normal set $\{f_1, f_2 .... f_m\}$ where the f_k for $2 \le k \le m$ is given inductively by

$$f_k = \frac{v_k - \langle v_k, f_1 \rangle f_1 - \langle v_k \ f_2 \rangle f_2 - \ldots - \langle v_k, f_{k-1} \rangle f_{k-1}}{\left\| v_k - \langle v_k, f_1 \rangle f_1 - \langle v_k \ f_2 \rangle f_2 - \ldots - \langle v_k, f_{k-1} \rangle f_{k-1} \right\|}$$

This method of obtaining an orthonormal set $\{ f_1, f_2 .... f_m \}$ from a linearly independent set $\{ v_1, v_2 .... v_m \}$ of vectors is called the **<u>Gram - Schmidt orthonormalization</u>** process. Application of this process to an arbitrary basis of $\mathbb{R}^n$ enables us to get a new vector basis which is orthonormal.

1.3 THE METRIC TOPOLOGY OF $\mathbb{R}^n$:

The inner product $\langle ; \rangle$ of $\mathbb{R}^n$ gives rise to a complete separable metric topology on it in the following way :

For any x, y in $\mathbb{R}^n$, we put :

$$d(x, y) = +\sqrt{x - y, x - y} = \|x - y\|$$

or equivalently $d(x, y) = +\sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$. This assignment gives rise to the map :

$$d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$$
$$(x, y) \mapsto d(x, y) = \|x - y\|$$

This map is in fact a metric on $\mathbb{R}^n$:

We readily have :

i) $d(x, y) \geq 0$ for all x, y in $\mathbb{R}^n$ and $d(x, y) = 0$ if and only if $x = y$.

ii) $d(x, y) = d(y, x)$ for x, y in $\mathbb{R}^n$.

More over, for any x, y, z in $\mathbb{R}^n$, we have

$$\begin{aligned} d(x, z)^2 &= \|x - z\|^2 \\ &= \|x - y + y - z\|^2 \\ &= \langle x - y + y - z, x - y + y - z \rangle \\ &= \langle x - y, x - y \rangle + 2\langle x - y, y - z \rangle + \langle y - z, y - z \rangle \\ &= \|x - y\|^2 + 2\langle x - y, y - z \rangle + \|y - z\|^2 \\ &\leq \|x - y\|^2 + 2\|x - y\|\|y - z\| + \|y - z\|^2 \end{aligned}$$

By the CBS inequality.

$$= \left(\|x - y\| + \|y - z\| \right)^2$$
$$= \left(d\left(x; y\right) + d\left(y, z\right) \right)^2$$

Thus $d\left(x\,z\right)^2 \leq \left(d\left(x,y\right)+d\left(y,z\right)\right)^2$ for all x, y, z in $\mathbb{R}^n$.
Thereby we get the triangle inequality,
$d\left(x\,z\right) \leq d\left(x,y\right)+d\left(y,z\right)$ for all x, y, z in $\mathbb{R}^n$.

Thus, the Euclidean space $\mathbb{R}^n$ is actually a metric base but we will not indicate its metric. All the topological considerations will be in reference to this Euclidean metric topology. Among all the properties of the metric space $\mathbb{R}^n$, we mention only the following two :

i) $\mathbb{R}^n$ is a complete metric space;
ii) $\mathbb{R}^n$ is separable.

Property (ii) can be seen here itself : Let $\mathbb{Q}^n$ be the set of all ordered n-types $\left(a_1, a_2, ...a_n\right)$ of rational numbers. Then the set $\mathbb{Q}^n$ is a countable, and dense subject $\mathbb{R}^n$ and hence $\mathbb{R}^n$ is separable.

We prove property in the following proposition :

Proposition 3 : The metric space $\mathbb{R}^n$ is complete.

Proof : We consider a Cauchy sequence $\left(v_k : k \in \mathbb{N}\right)$ in $\mathbb{R}^n$. Writing each term v_k in terms of its coordinates $v_k = \left(v_k^1, v_k^2; ...v_k^n\right)$.

We split the sequence $\left(v_k : k \in \mathbb{N}\right)$ into n sequences of real numbers: $\left(v_k^1 : k \in \mathbb{N}\right), \left(v_k^2 : k \in \mathbb{N}\right)......\left(v_k^n : k \in \mathbb{N}\right)$.

Note that for each $k.\ell$ in $\mathbb{N}$ and for each i $\left(1 \leq i \leq n\right)$ we have :
$\left|v_{k+e}^j - v_k^j\right| \leq \left\|v_{k+e} - v_k\right\|$,……………………….. (*)

The inequalities (*) imply that the Cauchy property of $\left(v_k : k \in \mathbb{N}\right)$ induces Cauchy property in each of the coordinate sequences : $\left(v_k^1 : k \in \mathbb{N}\right), \left(v_k^2 : k \in \mathbb{N}\right)......\left(v_k^n : k \in \mathbb{N}\right)$.

By the completeness of the real line $\mathbb{R}$, we get real numbers $w_1, w_2, ...w_n$ which are limits of the (Cauchy) coordinate sequences :

$w_1 = \lim_{k\to\infty} v_k^1, w_2 = \lim_{k\to\infty} v_k^2, ,....w_n = \lim_{k\to\infty} v_k^n$.

We form the vector $w = (w_1, w_2, ..., w_n) \in \mathbb{R}^n$.

Finally note that $d(v_k, w) \le \sum_{\ell=1}^{n} \left| v_k^\ell - w_\ell \right|$ which (together with the above deduction that $v_k^\ell \to w_\ell$ as $k \to \infty$ for $1 \le \ell \le n$) implies that $v_k \to \omega$.

Thus, each Cauchy sequence $(v_k : k \in \mathbb{N})$ in $\mathbb{R}^n$ converges to a $w \in \mathbb{R}^n$ and therefore $\mathbb{R}^n$ is complete. $\square$

We observe one more property of the metric topology of $\mathbb{R}^n$.

Let A be any subset of a $\mathbb{R}^m$ and let a be any point of A. Let $f_1, f_2, ... f_n : A \to \mathbb{R}$ be functions, all being continuous at a.

Let $f : A \to \mathbb{R}^n$ be the map given by $f(y) = (f_1(y), f_2(y), ... f_n(y)) \in \mathbb{R}^n$ for each $y \in \mathrm{A}$.

Proposition 4 : The map $f : A \to \mathbb{R}^n$ is continuous at *a*.

Proof : Let $\in > o$ be given. Then for $\frac{\in}{\sqrt{n}} > o$, continuity of each $f_i (1 \le i \le n)$ at a implies that there exist $\delta_i < o$ such that $\left| f^i(y) - f^i(a) \right| < \frac{\in}{\sqrt{n}}$ for all $y \in \mathrm{B}(a, \delta i)$.

Consider $\delta = \min\{\delta_1, \delta_2 ....\delta_n\}$. Then $\delta > o$ and $y \in \mathrm{B}(a, \delta)$ implies $y \in \mathrm{B}(a, \delta i)(1 \le i \le n)$ and therefore $\left| f^i(y) - f^i(a) \right| < \frac{\in}{\sqrt{n}}$ for $1 \le i \le n$. This set of inequalities implies :

$\left\| f(y) - f(a) \right\| < \in$ for all $y \in \mathrm{B}(a, \delta)$ proving continuity of f at a. $\square$

1.4 ORIENTING $\mathbb{R}^n$:

Orientation of $\mathbb{R}^n$ and its orientability in two different ways is yet another aspect of its geometry. Here, we give a brief, heuristic introduction to the main ideas related to the orientations of $\mathbb{R}^n$. We use only elementary geometric concepts. A precise algebraic formulation of it (in terms of orthonormal transformations of $\mathbb{R}^n$**)** will be given in the next chapter.

The term <u>orientation</u> applies primarily to orthogonal frames in $\mathbb{R}^n$. W try to reach the vast expense of $\mathbb{R}^n$by means of an orthogonal frame $\mathscr{F}_s = \mathscr{F}_s(f_1\ f_2....f_n)$associated with an orthonormal vector basis $(f_1\ f_2....f_n)$.

Recall, an orthogonal frame $\mathscr{F}_s(f_1\ f_2....f_n)$is obtained by laying its axes $OX_1\ OX_2.....OX_n$ along the vectors $f_1, f_2,....f_n$respectively.

Schematic depition of an orthogonal frame $\mathscr{F}_s(f_1\ f_2....f_n)$

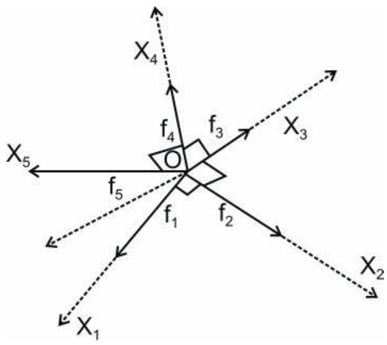


Fig. 1

Clearly the frame $\mathscr{F}_s(f_1\ f_2....f_n)$ and the ordered orthonormal vector basis $(f_1\ f_2....f_n)$specify each other and therefore, we often talk of them interchangeably.

We consider various $\mathscr{F}_s(f_1\ f_2....f_n)$

Of course, there is the standard orthornormal frame $\mathscr{F} = (e_1\ e_2....e_n)$ introduced earlier. But this particular frame may not be the best choice to study a specific geometric / physical problem. For example in studying the rotational motion of a spinning top (with its nail tip remaining stationary on the ground) we need consider besides the stationary frame the rotating **<u>body frame</u>** $\widehat{\mathscr{F}}_s$which is an orthogonal frame fixed in the top and therefore it is a moving orthogonal frame. And we study the rotational motion of the top by studying how the body frame $\widehat{\mathscr{F}}_s$changes its orientation with respect to the stationary frame $\mathscr{F}_s$. Thus, we need two distinct orthogonal frames to study the dynamics of a spinning top.

We therefore consider **all** orthogonal frames $\mathscr{F}_s(f_1\ f_2 .... f_n)$ and compare them with the standard frame $\mathscr{F}(e_1\ e_2 .... e_n) = \mathscr{F}_s$

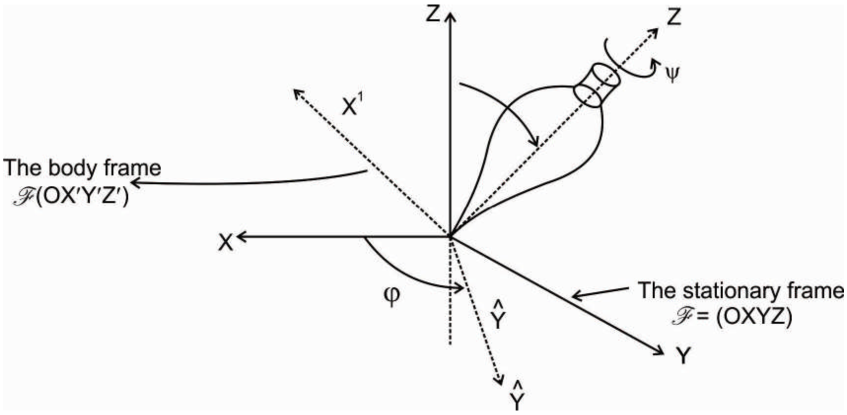


Fig. 2 : The stationary frame $\mathscr{F}_s$ and the body frame $\widehat{\mathscr{F}}_s$

How do we compare two frames?

It is intuitively clear that we can rotate $\mathscr{F}_s$ about the common origin and make it coincide with $\mathscr{F}_s$. This corresponds to a change $(f_1\ f_2 .... f_3) \to (e_1\ e_2 .... e_3)$ of the orthogonal bases associated with the two frames $\mathscr{F}_s$ and $\mathscr{F}_s 0$.

Recall now the elementary facts of linear algebra. (We will discuss more about there in the next chapter.)

- Each change of orthonormal basis (and therefore that of the associated orthonormal frames) $(e_1\ e_2 .... e_n) \to (f_1\ f_2 .... f_n)$ gives rise to a unique orthogonal linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(e_i) = fi$ for $1 \le i \le n$.
- Each such T has the property det (T)=+1 or -1. We use these properties to compare two frames. We say that two orthogonal frames $\mathscr{F}_s$ **and** $\mathscr{F}_s 0$ **have the same orientation** if det (T) = +1 and they have the **opposite orientation** if det T = -1.

We regard this classification of orthogonal frames into two disjoint families as two orientations of the space $\mathbb{R}^n$; we call them the "standard orientation" and the "opposite orientation" of $\mathbb{R}^n$. Thus we have the following :

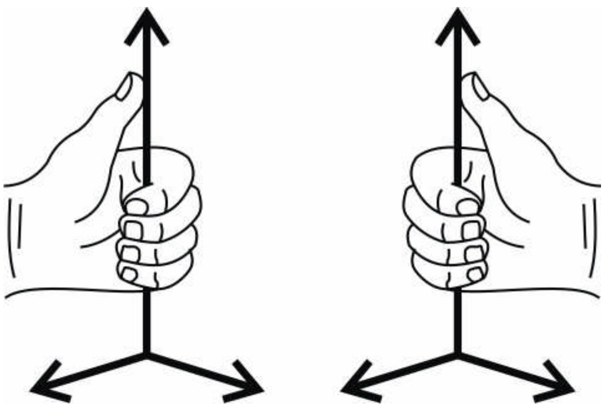
- The **standard orientation** of $\mathbb{R}^n$ pertains to the orthogonal frames $\mathscr{F}_s(f_1, f_2 \ldots. f_n)$ with the property that the associated $T : \mathbb{R}^n \to \mathbb{R}^n$ (with $T(e_i) = f_i \;\; 1 \leq i \leq n$) has det T = +1.

•• The **opposite orientation** of $\mathbb{R}^n$ pertains to any orthogonal frame $\mathscr{F}_s(f_1 \; f_2 \ldots. f_n)$ with det T = -1.

Thus each Euclidean space $\mathbb{R}^n$ carries two distinct orientations, namely (a) the standard orientation as described in • and (b) the opposite orientation described in ••.

Applying all this consideration to $\mathbb{R}^3$, our physical space; we have an equivalent, but rather tangible description in the popular language : Orthogonal frames being **left handed** and **right handed** :

- $\mathscr{F}_s(f_1 \; f_2 \ldots. f_n)$ is right handed if the frame can be grabed by right hand so that the thumb points in the direction of f_3.
- On the other hand $\mathscr{F}_s(f_1 \; f_2 \ldots. f_n)$ is left handed if it can be grabed by the left hand so that the thumb (again) points in the direction of f_3.



Left and Right Handedness of Orthogonal Frames
Figure 3

Of course the right handed frames determine the standard orientation of $\mathbb{R}^3$ while the left handed frames determine the opposite orientation of it.

Exercises :

1) Prove that $\mathbb{Q}^n = \underbrace{\mathbb{Q}\times\mathbb{Q}\times\ldots\times\mathbb{Q}}_{n}$ is a dense subset of $\mathbb{R}^n$.

2) Apply Grahm - Schmidt process to obtain orthonormal sets from the given (linearly independent) subsets :

a) $\{(1,3);(2,4)\}\subset\mathbb{R}^2$

b) $\{(1,3,1)(1,4,1)\ \ (0,2,1)\}\subset\mathbb{R}^3$

c) $\{(1,2,3),(2,3,1),(3,1,2)\}\subset\mathbb{R}^3$

3) Prove that any n-dimensional real vector space is isomorphic with $\mathbb{R}^n$.

4) Prove that any two vector bases in $\mathbb{R}^n$ have equal number of elements.

5) Give all the details regarding the proof that $d:\mathbb{R}^n\times\mathbb{R}^n\to[0,\infty)$ given by $d(x,y)=\|x-y\|\ \ x,y$ in $\mathbb{R}^n$, is a metric.

6) Describe a real vector space which is not isomorphic with any $\mathbb{R}^n$. (Justify your claims)

7) Let $f:\mathbb{R}^n\to\mathbb{R}^m$, $g:\mathbb{R}^m\to\mathbb{R}^\ell$ be maps such that *f* is continuous at a $p\in\mathbb{R}^n$ and g is continuous of $f(p)=q$. Prove continuity of $gof:\mathbb{R}^n\to\mathbb{R}^\ell$ at p.

8) Recall : (i) a multi-index is an ordered n-tuple. $\alpha=(\alpha_1,\alpha_2,....,\alpha_n)$ where each $\alpha_i\in\mathbb{Z}^+$.

ii) $|\alpha|=\sum_{i=1}^{n}\alpha^i$

iii) $x^\alpha=\prod_{i=1}^{n}(x_i)^{\alpha i}\ (\in\mathbb{R}),\ x=(x_1,x_2,...,x_n)\in\mathbb{R}^n$

iv) A polynomial in the multi-variable $x=(x_1,x_2,...,x_n)\in\mathbb{R}^n$ is finite linear combination. $p(x):=\sum_{|\alpha|\le m}a_\alpha x^\alpha$

Prove (a) each monomial $x^\alpha:\mathbb{R}^n\to\mathbb{R}$ is continuous on $\mathbb{R}^n$.

b) and therefore each $p:\mathbb{R}^n\to\mathbb{R}$ is continuous on $\mathbb{R}^n$.

❖❖❖❖

2

ORTHOGONAL TRANSFORMATIONS

Unit Structure :

In the preceding chapter, an Euclidean space was introduced as a mathematical system consisting of the set $\mathbb{R}^n$ carrying three mutually compatible structures, namely (i) the n-dimensional real vector space structure, (ii) the inner product giving rise to the metric topology of $\mathbb{R}^n$ and (iii) the standard orientation of it.

In this chapter we will discuss linear transformations between Euclidean spaces and their properties. In particular, we will come across the group $GL(\mathbb{R}^n)$ ($GL(n), GL(n,\mathbb{R})$ are other notations for the same) consisting of bijective linear self maps of the vector space $\mathbb{R}^n$ (a <u>self-map</u> is a map of the type $f : \mathrm{X} \to \mathrm{X}$ i.e. a map of a set X to the same set.) Actually we are moving towards a sub-group $SO(\mathbb{R}^n)$ (or $SO(n,\mathbb{R}), SO(n)$ etc.) of $GL(\mathbb{R}^n)$; it is the group of symmetries or the automorphisms of the Eudidean space $\mathbb{R}^n$. These transformations - being symmetries of $\mathbb{R}^n$ - help us understand the shapes of geometric objects residing in $\mathbb{R}^n$: smooth curves, smooth surfaces, higher dimensional smooth manifolds…. Also, being automorphisms of the vector space, they play an important role in the derivation of many results of differential geometry.

We begin with a recall of basic concepts of linear algebra. (: linear transformations between Euclidean spaces, their matrix representation, the algebra of linear transformations and its reflectin in the albegra of matrices and so on, reaching finally the groups $SO(n)$). We will say a little more about the forms of the matrices in $SO(2)$ and $SO(3)$.

We will also recall a bit of differential calculus of vector valued functions $f:\Omega\left(\subseteq\mathbb{R}^n\right)\to\mathbb{R}^m$ of a multi-variable $x=\left(x_1,x_2,....,x_n\right)\in\Omega$. Recalling the definition of the total derivative $Df\left(p\right)$ of such a f at a $p\in\Omega$ as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ we take the view-point that differentiation of a function at a point is a process employed to approximate a general (differentiable) map locally by a linear transformation. This is an important interpretation, because we can now use all the machinery of linear algebra to get information about the local behaviour of such a f around a point p of its domain of definition.

Basic results of differential calculus mentioned in this chapter and the next one are : the inverse mapping theorem, the implicit mapping and the rank theorems Picard's existence / uniqueness theorem about the solution of an ODE and so on. We state these results (they go without proof) here in this set of notes because they are used here and there in differential geometry and therefore, a student should know at least the precise enunciations of these results. Detailed proofs of them are equally important and the reader can consult a suitable analysis book (e.g. one of the text -books by Serg Lang)

2.1 LINAR TRANSFORMATIONS

Definition 1 :

a) <u>A linear transformation</u> from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a map :

$$T:\mathbb{R}^n\to\mathbb{R}^n$$

which satisfies the identity :

$$T\left(ax+by\right)=aT\left(x\right)+bT\left(y\right)$$

for all x,y in $\mathbb{R}^n$ and for all a, b in $\mathbb{R}$.

Occasionally we speak of a <u>linear map</u> instead of a linear transformation.

b) A linear self map $T:\mathbb{R}^n\to\mathbb{R}^n$ is called a <u>linear endomorphism</u> (or often merely an endomorphism) of $\mathbb{R}^n$. It is also said to be an operator on $\mathbb{R}^n$.

c) A bijective linear endomorphism $T:\mathbb{R}^n\to\mathbb{R}^n$ is said to be a <u>linear automorphism</u> (or only an <u>automorphism</u>) of $\mathbb{R}^n$

d) A linear map $T:\mathbb{R}^n\to\mathbb{R}$ is called a <u>linear form</u> on $\mathbb{R}^n$.

We adopt the following notations :

- $L(\mathbb{R}^n, \mathbb{R}^m)$ denotes the set of all linear maps : $T:\mathbb{R}^n \to \mathbb{R}^m$.
- End $(\mathbb{R}^n)$ is the set of all linear endomorphisions $T:\mathbb{R}^n \to \mathbb{R}^n$
- Aut $(\mathbb{R}^n)$ is the set of all linear automorphisons of $\mathbb{R}^n$.
- $(\mathbb{R}^n)^*$ denotes the set of all linear forms on $\mathbb{R}^n$

We note here a few basic properties of linear linear transformations and their spaces listed above. Most of these properties are stated here without proof, because they are discussed routinely in any linear algebra courses. The reason why these properties are listed here is only to refresh readers memory about the precise statements and the full import of these properties :

I) If S, T are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ and if α, β are an real numbers then they combine to give a map :

$$\alpha S + \beta T : \mathbb{R}^n \to \mathbb{R}^m \text{ .... } (*)$$

which is given by :

$$(\alpha S + \beta T)(x) = \alpha \cdot S(x) + \beta \cdot T(\ell) \text{ for all } x \in \mathbb{R}^n.$$

This map $\alpha S + \beta T$ is also a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

II) Let $L(\mathbb{R}^n, \mathbb{R}^m)$ be the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then the operation (*) (described above) combining two linear transformations S, T and two real number α, β producing the linear transformation $\alpha S + \beta T$ is an algebraic operation giving the set $L(\mathbb{R}^n, \mathbb{R}^m)$ the structure of a real vector space. Thus, the set $L(\mathbb{R}^n, \mathbb{R}^m)$ together with the operation (*) is a real vector space. We will prove that the dimension of this vector space is m.n.

III) In particular the set $(\mathbb{R}^n)^*$ is a vector space and its dimension is $n.1 = n.$

We justify this claim by describing a bijective linear map $\Theta : (\mathbb{R}^n)^* \to \mathbb{R}^n$ as follows.

Let $T \in \left(\mathbb{R}^n\right)^*$ be arbitrary.

For each $i\left(1 \le i \le n\right)$ we put $y_i = T\left(e_i\right)$. We form the vector $y = \left(y_1, y_2, ...y_n\right)$. Now for each $x \in \mathbb{R}^n$ we have :

$$
\begin{aligned}
T\left(x\right) &= T\left(x_1, e_1 + x_2 e_2 + ... + x_n e_n\right) \\
&= x_1 T\left(e_1\right) + x_2 T\left(e_2\right) + ... x_n T\left(e_n\right) \\
&= x_1 y_1 + x_2 y_2 + ... x_n y_n \\
&= \left\langle x\ y \right\rangle
\end{aligned}
$$

Thus, with each $T \in \left(\mathbb{R}^n\right)^*$ is associated a $y \in \mathbb{R}^n$ satisfying.

$$T\left(x\right) = \left\langle x, y\right\rangle, \forall x \in \mathbb{R}^n.$$

Clearly this y (associated with the $T \in \left(\mathbb{R}^n\right)^*$) is unique. We put $O\left(T\right) = y$. Now we have the map $O : \left(\mathbb{R}^n\right)^* \to \mathbb{R}^n$. It is easy to prove that this map (H) is bijective and linear.

IV) a) If $S : \mathbb{R}^n \to \mathbb{R}^m, T : \mathbb{R}^m \to \mathbb{R}^k$ then $T_0 S : \mathbb{R}^n \to \mathbb{R}^k$ is also linear.

b) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is bijective linear then its inverse $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ also is linear.

V) Let $\left\{f_1, f_2, ... f_n\right\}$ be any vector basis of $\mathbb{R}^n$ and let $v_1, v_2, ...v_n$ be any vectors in a $\mathbb{R}^m$. Then there exists a unique linear $T : \mathbb{R}^n \to \mathbb{R}^m$ having the property : $T\left(f_i\right) = v_i \quad 1 \le i \le n$.

The unique linear T is given as follows :

Let $x = x_1 f_1 + x_2 f_2 + ... + x_n f_n$ be any vector in $\mathbb{R}^n$. Then by linearity of T, we have

$$
\begin{aligned}
T\left(x\right) &= T\left(x_1 f_1 + x_2 f_2 + \quad + x_n f_n\right) \\
&= x T\left(f_1\right) + x_2 T\left(f_2\right) + x_n T\left(f_n\right) \\
&= x_1 v_1 + x_2 v_2 + \qquad + x_n v_n
\end{aligned}
$$

VI) In particular we consider m = n and in place of $\left\{f_1, f_2, ..., f_n\right\}$ we take the standard basis $\left\{e_1, e_2, ..., e_n\right\}$ of $\mathbb{R}^n$. Next for any pair $i, j \quad \left(1 \le i, j \le n\right)$, we consider the set $\left(v_1, v_2, ..., v_n\right)$ where

$v_i = e_j$ and $v_i = 0$ for all other $\ell(1 \le \ell \le n)$. By property (V) above, we get a unique linear $T_{ij} : \mathbb{R}^n \to \mathbb{R}^n$ satisfying.

$$T_{ij}(e_k) = o \ \ if \ k \ne i$$
$$= e_j \ if \ k = i$$

Thus, $T_{ij}(x) = x_i e_j$ for all $x \in \mathbb{R}^n$.

We consider the set $\{T_{ij} : 1 \le i, j \le n\}$. It is easy to prove that this set is linearly independent $\sum_{1 \le i\, j \le n} \alpha_{ij} T_{ij} \equiv o$ only when all $\alpha_{ij} = o$.

On the other hand we prove that any T is a linear combination of $\{T_{ij} : 1 \le i, j \le n\}$: In fact let $T(ei)\sum_{j \le n} \alpha ij\, ej \ for\, 1 \le i \le n$. Then $T = \sum_{1 \le i\, j \le n} \alpha_{ij} T_{ij}$.

This shows that $\{T_{ij} : 1 \le i, j \le n\}$ is a a vector basis of $\text{End}(\mathbb{R}^n)$ and thus it is a vector space having dimension $= n^2$.

VII) Thus, the set $\text{End}(\mathbb{R}^n)$ carries two kinds of algebraic operations namely. (a) the vector space operations and (b) the composition $o : End(\mathbb{R}^n) \times End(\mathbb{R}^n) \to End(\mathbb{R}^n)$ taking a pair S, T to SoT. Note that 'o' distributes over the vector space operations

$$S \circ (\alpha T + \beta R) = \alpha\, S \circ T + \beta \cdot S \circ R.$$

Thus End $(\mathbb{R}^n)$ is a real n-dimensional algebra.

VIII) Let $GL(\mathbb{R}^n)$ be the set of all linear automorphisms of $\mathbb{R}^n$ (We often denote it by $GL(n,\mathbb{R})$ or by $GL(n)$.) The set $GL(\mathbb{R}^n)$ has the following properties :

i) $I \in GL(\mathbb{R}^n)$, (I being the identity transformation on $(\mathbb{R}^n)$.

ii) If $T \in GL(\mathbb{R}^n)$, then $T^{-1}\, also \in GL(\mathbb{R}^n)$

iii) If S, T are both in $GL(\mathbb{R}^n)$ then $S \circ T \in GL(\mathbb{R}^n)$

In other words, the system $\{GL(\mathbb{R}^n), o\}$ is a group. We call it the <u>n-dimensional general linear group</u>.

2.2 ALGEBRA OF MATRICES :

Matrices are computational counterparts of linear transformations. With each operator T of $\mathbb{R}^n$, we associate square matrix and the neumerical calculations done on matrices given information about their predessors.

Recall for any m, n in $\mathbb{N}$, a matrix of size $m \times n$ is an array $A = \left[a_{ij}\right]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ of real numbers arranged in m rows and n columns (the numbers a_{ij} being placed at the cross-roads of ith row and jth column:

$$A = \left[a_{ij}\right]_{\substack{1 \le i \le m \\ 1 \le j \le n}} = \begin{bmatrix} a_{11} \quad a_{1z} \ldots\ldots a_{1j} \ldots\ldots a_{1n} \\ \ldots\ldots\ldots\ldots\vdots\ldots\ldots\ldots \\ a_{i1} \quad a_{1z} \ldots\ldots a_{1j} \ldots\ldots a_{in} \\ \ldots\ldots\ldots\ldots\vdots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\vdots\ldots\ldots\ldots \\ a_{m1}, a_{m2} \ldots\vdots\ldots a_{mj} \ldots\ldots a_{mn} \end{bmatrix}$$

We often write only $\left[a_{ij}\right]$ instead of $\left[a_{ij}\right]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ whenever the size of the matrix is understood.

We represent a linear $T : \mathbb{R}^n \to \mathbb{R}^m$ by a $m \times n$ matrix (T) and use the latter as a computational device to get information of the linear transformation T.

Recall two matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ of the same size are equal A =B if and only if $a_{ij} = b_{ij}$ holds for all pairs (i, j) -

$M\left(m, n, \ \mathbb{R}\right)$ denotes the set of all real matrices of size $m \times n$.

When m = n, we write $M\left(n, \mathbb{R}\right)$ for the set $M\left(n, n, \mathbb{R}\right)$ and the matrices in it are said to be square matrices (of size $n \times n$.)

The set $M\left(m, n, \mathbb{R}\right)$ has the structure of a real vector space : If $A = \left[a_{ij}\right] \quad B = \left[b_{ij}\right]$ are any two matrices and if α, β are any two real numbers, then we define $\alpha A + \beta B$ as the $m \times n$ matrix $c = \left[C_{ij}\right]$ given by :

$$c_{ij} = \alpha a_{ij} + \beta b_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$$

Thus $\alpha A + \beta B$ is the matrix $\left[\alpha a_{ij} + \beta b_{ij}\right]$.

For any pair (i, j) $1 \le i \le m$ $1 \le j \le n$, let A_{ij} be the matrix :

$$A_{ij} = \begin{bmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{bmatrix} \leftarrow ith\ row$$

$$\uparrow$$

$$jth\ column$$

there being zeros at all places in A_{ij} except at the (ij)th-place where we have 1.

It now follows that any matrix $A = \left[a_{ij}\right] \in M(m, n, \mathbb{R})$ can be expressed uniquely as the linear combination.

$$A = \sum a_{ij}\ A_{ij}$$

the sum above extending over all pairs (ij) with $1 \le i \le m$ and $1 \le j \le n$ and consequently, the vector space $M(m, m, \mathbb{R})$ has dimension m.n.

We now recall the multiplication of matrices : for any m,n and p in $\mathbb{N}$, let $A = \left[a_{ij}\right] \in M(m, n\,\mathbb{R})$, $B = \left[b_{jk}\right] \in M(n, p, \mathbb{R})$. Then the matrix $D = \left[d_{ik}\right] \in M(m, p, \mathbb{R})$ given by $d_{ik} = \sum\limits_{1 \le j \le m} a_{ij} b_{jk}$ is defined as the product $D = A \cdot B$ (the factors A, B of D in the indicated order).

Note that both, the products $A \cdot B$ and $B \cdot A$ are defined only when m = n = p i.e. when both A, B are square matrices of the same size. We pursue this case (i.e. of square matrices) by the following hands-on account :

The set $M(n, \mathbb{R})$ carries the following algebraic operations (all explained above in the more general context) :

- Addition of matrices :

$$+ : M(n, \mathbb{R}) \times M(n, \mathbb{R}) \to M(n, \mathbb{R})$$

$$(A, B) \mapsto A + B$$

- Multiplication of a matrix by a real number :

  $\cdot : \mathbb{R} \times M(n, \mathbb{R}) \to M(n, \mathbb{R})$

  $(\alpha, A) \mapsto \alpha \cdot \mathrm{A}$
- Multiplication of matrices

  $\cdot : M(n, \mathbb{R}) \times M(n, \mathbb{R}) \to M(n, \mathbb{R})$

  $(A, B) \mapsto A \bullet B,$

The set $M(n, \mathbb{R})$ together with the above three algebraic operations is an n^2 dimensional associative (real) algebra with identity (i.e. it is a combination of a n^2-dimensional real vector space and a ring with identity.)

Thus, on one hand we have the operator algebra $\mathrm{Aut}(\mathbb{R}^n)$ and on the other hand, we have the algebra $M(n, \mathbb{R})$ of $n \times n$ real matrices. We proceed to explain below that an orthonomal vector basis of $\mathbb{R}^n$ establishes an isomorphism (of algebras) between the two.

Thus let $\mathscr{F} = \{f_1, f_2, ..., f_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Now for a $T \in End(\mathbb{R}^n)$ and for each $f_j\,(1 \le j \le n)$ we get the vector $T(f_j)$ expressing it as the linear combination :

$$T(f_j) = \sum_{i=1} a_{ij}\ f_i \quad 1 \le j \le n \ \ .... (*)$$

We collect the coefficients a_{ij} in (*) above and form the matrix $[a_{ij}]$ which we denote by [T] or more accurately by $[T]_{\mathscr{F}}$.

Thus the orthonormal basis of $\mathbb{R}^n$ gives rise to the map

$$\mathrm{End}\ (\mathbb{R}^n) \xrightarrow{\Im_s} M(n, \mathbb{R})$$

$$A \mapsto [A] = [A]_{\mathscr{F}}$$

We note the following properties of this map -

i) The map is a bijection between $End(\mathbb{R}^n)$ and $M(n, \mathbb{R})$

ii) The map preserves the algebraic operations on the two sets, that is, the following equalities hold :

- $[I]_{F_s} = \left[\delta_{ij}\right]$, I being the identity operator on $\mathbb{R}^n$ and $\left[\delta_{ij}\right]$ is the identity $n \times n$ matrix :

$$\left[\delta_{ij}\right] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

- $[S+T]=[S]+[T]$
- $[\alpha S]=\alpha \cdot [S]$
- $[S \circ T]=[S] \bullet [T]$

These properties-described in (I) and (II) above-taken together imply that the map (*) is an isomorphism between the two algebraic systems.

The third proper-property (III) stated below-is about the dependence of the matrix representation $[T]_{\mathscr{F}}$ of an operator T on the orthonormal basis $\mathscr{F}$:

Let $\mathscr{F} = \{ f_1, f_2, ..., f_n \}$ and $\mathscr{G} = \{ g_1, g_2, ...g_n \}$ be two orthonormal bases of $\mathbb{R}^n$. If T is any operator on $\mathbb{R}^n$, then the two bases associate the matrices $[T]_{\mathscr{F}}$ and $[T]_{\mathscr{G}}$. We seek a relation between the two matrix representation. Towards this aim, we consider the matrix $C = \left[C_{ij}\right]$ describing the change of the vector bases $\mathscr{F} \to \mathscr{G}$ thus for each $j, 1 \le j \le n$ we have :

$$g_j = \sum_{i=1}^{n} c_{ij} f_i$$

Applying T to this equality, we get

$$T\left(g_j\right) = \sum_{i=1}^{n} c_{ij} T\left(f_i\right)$$

Now if $[T]_{\mathscr{F}} = \left[a_{ij}\right]$ and $[T]_{\mathscr{G}} = \left[b_{ij}\right]$ then we have $T\left(f_i\right) = \sum_{k=1}^{n} a_{ki} f_k \quad T\left(g_i\right) = \sum_{k=1}^{n} b_{ki} g_k$ and therefore, we get

$$\sum_{\ell=1}^{n} b_{ej} g_e = \sum_{i=1}^{n} c_{ij} T\left(f_{ij}\right)$$

$$= \sum_{i=1}^{n} c_{ij} \sum_{k=1}^{n} a_{ki} f_k$$

Therefore $\sum_{e=1}^{n} b_{ej} \sum_{k=1}^{n} c_{ke} f_k = \sum_{\ell=1}^{n} c_{\ell j} \sum_{k=1}^{n} a_{ke} f_k$ that is,

$$\sum_{k}\left(\sum_{\ell} c_{kl} b_{ej}\right) f_k = \sum_{k}\left(\sum_{\ell} a_{k\ell} c_{ej}\right) f_k .$$

Equating coefficients of each f_k, we get $\sum_{\ell=1}^{n} c_{ke} b_{ej} = \sum_{\ell=1}^{n} a_{ke} c_{ej}$.

The above equalities are obtained for each pair (k, j) with $1 \le k, j \le n$ and therefore, we get the equality of the matrices :

$$\left[\sum_{e} c_{ke} b_{ej}\right] = \left[\sum_{e} a_{ke} c_{ej}\right]$$ that is, we have $CB = AC$.

Now note that C is invertible (it being the matrix onnecting two vector bases, $\Im$ and g) and therefore the last equality implies

$$B = C^{-1}.AC$$

that is : $[T]_{\mathscr{G}} = C^{-1}[T]_{\mathscr{F}} C$.

We summarize it in the third property of matrices :

III) For any two orthonormal bases $\mathscr{F}, \mathscr{G}$ of $\mathbb{R}^n$ and for any operator T on $\mathbb{R}^n$, we have :

$$[T]_{\mathscr{G}} = C^{-1}[T]_{\mathscr{F}} C$$

C in above being the matrix of the change of vector bases from $\mathscr{F}$ to $\mathscr{G}$.

We use this property crucially in defining the determinant of an operator T on $\mathbb{R}^n$.

2.3 DETERMINANT OF A LINEAR EUDOMORPHISM OF $\mathbb{R}^n$:

First we define the determinant of a square matrix and then extend it to linear endomorphisms.

Recall, first, the permutation group S(n) of set {1,2,….,n}. Also recall that each $\sigma \in S(n)$ has its signature $\in(\sigma) \in \{+1,-1\}$.

Definition 3 : For a square matrix $A=\left[a_{ij}\right]$ of size $n \times n$, the determinant det(A) is the number

$$\det(A) = \sum_{\sigma \in S(n)} \in(\sigma)\, a_{1,\sigma(i)} \cdot a_{2,\sigma(i)} \ldots\ldots a_{n,\sigma(n)}$$

$$= \sum_{\sigma \in S(n)} \in(\sigma) \prod_{1 \le i \le n} a_{i,\sigma(i)} \ldots\ldots\ldots\ldots\ldots\ldots \text{(*)}$$

Now we have a function :

$$\det : M(n,\mathbb{R}) \to \mathbb{R}$$
$$A \mapsto \det(A)$$

We mention (without proof) following three properties of this function.

1) det (I) =1, I being the identity $n \times n$ matrix : $I=\left[\delta_{ij}\right]$
2) $\det(A \cdot B) = \det(A) \cdot \det B$ for any A, B in $M(n,\mathbb{R})$ and
3) a matrix $A=\left[a_{ij}\right]$ is invertible if an only if $\det(A) \neq o$.

Note that property (2) above has the following important consequence : If C is any invertible $n \times n$ matrix, then for any $n \times n$ matrix A we have the equality :

$$\det(A) = \det\left(C^{-1}AC\right)$$

In fact we have

$$\det\left(C^{-1}AC\right) = \det\left(C^{-1}\right)\det(AC)$$
$$= \det\left(C^{-1}\right) \cdot \det(A) \cdot \det(C)$$
$$= \det A$$

We also have $1 = \det(I) = \det\left(C^{-1}C\right) = \det\left(C^{-1}\right)\frac{1}{\det C}$ and therefore $\det\left(C^{-1}\right) = \frac{1}{\det C}$. Applying this result we get :

$$\det\left(C^{-1}AC\right) = \det\left(C^{-1}\right)\det(A)\det(C)$$
$$= \det(A)$$

We use the property of determinant of square matrices to define $\det(T)$ of an operator T on $\mathbb{R}^n$.

For an orthonormal basis $\mathscr{F}$ we consider the matrix $[T]_{\mathscr{F}}$ of T w.r.t. $\mathscr{F}$. Involving the formula $(*)$ above, we consider det $[T]_{\mathscr{F}}$ and then we observe that this number, thus arrived at, is actually independent of the vector basis $\mathscr{F}$ used (and therefore, it is actually an attribute of the operator T itself and not that of its matrix representation.) For, if g is any other orthonormal basis of $\mathbb{R}^n$, then we have :

$$[T]_{\mathscr{G}} = C^{-1} \cdot [T]_{\mathscr{F}} \bullet C \text{ and therefore,}$$

$$\begin{aligned} \det[T]_{\mathscr{G}} &= \det\left(C^{-1}[T]_{\mathscr{F}} \bullet C\right) \\ &= \det[T]_{\mathscr{F}} \end{aligned}$$

Thus, $\det[T]_{\mathscr{F}} = \det[T]_{\mathscr{G}}$ for any orthonormal bases $\mathscr{F}$ and $\mathscr{G}$ of $\mathbb{R}^n$. We define $\det(T)$ to be this common value :

$$\det T = \det[T]_{\mathscr{F}} = \det[T]_{\mathscr{G}}.$$

Now we have the function : $\det : End\left(\mathbb{R}^n\right) \to \mathbb{R}$.

This map has the following properties

- $\det(I) = 1$, I being the identity operator on $\mathbb{R}^n$
- $\det(S \bullet T) = \det(S) \bullet \det(T)$ for all S T in $End\left(\mathbb{R}^n\right)$.
- An operator T is invertible if and only if $\det(T) \neq O$.
- If T is invertible, then $\det\left(T^{-1}\right) = \dfrac{1}{\det(T)}$.

2.4 TRACE OF AN OPERATOR

There is yet another invariant associated with an operator T on $\mathbb{R}^n$, namely its <u>trace</u>. Like the determinant of an operator, we define it first for a square matrix and then extend it to an operator.

Definition 4 : The trace of a matrix $A = \left[a_{ij}\right]$-denoted by $tr(A)$- is given by $tr(A) = \sum_{i=1}^{n} a_{ii}$.

Note that for any $A=\left[a_{ij}\right]$, $B=\left[b_{ij}\right]$ we have $AB=\left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]$ and therefore,

$$\begin{aligned} tr(AB) &:= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} \\ &= tr(BA) \end{aligned}$$

Thus $tr(AB)=tr(BA)$ for any A, B in $M(n,\mathbb{R})$. We use this property to define $tr(T)$ of an operator T : Choose any orthonormal basis $\mathscr{F}$ and consider $Tr\left([T]_{\mathscr{F}}\right)$ as defined above. We claim that this number does not depend on the orthonormal basis $\mathscr{F}$. For, let $\mathscr{F}$ and $\mathscr{G}$ be two orthnormal bases with C as the matrix describing the change $\mathscr{F}$ to $\mathscr{G}$. Then for any operator T on $\mathbb{R}^n$, we have

$$[T]_{\mathscr{G}} = C^{-1}[T]_{\mathscr{F}}\, C$$

and therefore,

$$\begin{aligned} tr[T]_{\mathscr{G}} &= tr\left\{C^{-1}[T]_{\mathscr{F}}\, C\right\} \\ &= tr\left([T]_{\mathscr{F}}\, C, C^{-1}\right) \\ &= tr\left([T]_{\mathscr{F}}\right) \end{aligned}$$

This leades us to the definition

$tr(T)=tr[T]_{\mathscr{F}}=Tr[T]_{\mathscr{G}}$.

Now, we have the function

$tr : End(\mathbb{R}_n) \to \mathbb{R}$

Two of the, properties of this map are

1) $tr(I)=n$
2) $tr(S\cdot T)=tr(T\,S)$ for any, S, T in $End\left(\mathbb{R}^n\right)$.

In what is to follow, we will be using only the standard orthonormal basis $\sum=\{e_1,e_2,.....,e_n\}$ as a convenient choice and therefore the matrix representation $[T]$ of T will be understood to be with respect to the standard orthornormal basis $\sum : [T]=[T]_{\Sigma}$.

2.5 ORTHOGONAL LINEAR TRANSFORMATIONS :

We single out a sub-group of the group $GL(\mathbb{R}^n)$.

Definition : A linear $T:\mathbb{R}^n \to \mathbb{R}^n$ is <u>orthogonal</u> if it preserves the inner product.

$\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all x, y in $\mathbb{R}^n$.

$O(\mathbb{R}^n)$ denotes the set of all orthorgonal $T:\mathbb{R}^n \to \mathbb{R}^n$.

Note the following elementary properties of orthogonal transformations.

- A linear $T:\mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if it preserves the Euclidean norm of the vectors.
- As an immediate consequence of the above we get that an orthogonal T is bijective.
- T is orthogonal if and only if $\langle T(e_i), Te_j\rangle = \delta_{ij}$ for all $i, j,\quad 1 \le i, j \le n$.
- i) The identity map $I:\mathbb{R}^n \to \mathbb{R}$ is orthogonal
- ii) If T is orthogonal then so is T^{-1}
- iii) If S, T are orthogonal, then so is SoT.

Thus, the composition operation : $S, T \to SoT$ becomes a binary operation on the set $O(\mathbb{R}^n)$ in such a way that $(O(\mathbb{R}^n), o)$ is a sub-group of $GL(\mathbb{R}^n)$. We denote this sub-group by the underlying set $O(\mathbb{R}^n)$ only and call it the <u>n-dimensional orthogonal group</u>.

We characterize an orthogonal $T:\mathbb{R}^n \to \mathbb{R}^n$ in terms of a property of its matrix.

Proposition 1 : A linear $T:\mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if its matrix $A(=[T])$ satisfies $A^t A = I$.

Proof : Suppose, T is orthogonal. Then for any pair (i, j) with $1 \le i, j \le n$ we have

$$\langle T(e_i), T(e_j)\rangle = \delta_{ij}\left(= \langle e_i, e_j\rangle\right)$$

Now $T(e_i) = \sum_{k=1}^{n} a_{ki} e_k, T(e_j) = \sum_{k=1}^{n} a_{ej} e_\ell$ and therefore :

$$\begin{aligned}\left\langle T(e_i), T(e_j) \right\rangle &= \left(\sum_{k=1}^{n} a_{ki} e_k, \sum_{\ell=1}^{n} a_{\ell j} e_\ell, \right\rangle \\ &= \sum_{k,\ell}{}^{1} a_{ki} a_{\ell j} \left\langle e_k, e_\ell \right\rangle \\ &= \sum_{k,\ell} a_{ki} a_{\ell j} \delta_{k\ell} \\ &= \sum_{k} a_{ki} a_{kj}\end{aligned}$$

Note that $\sum_{k=1}^{n} a_{ki} a_{kj}$ is the $(ij)^{th}$ entry in the matrix $A^t.A$. Now the equalities.

$$\sum_{k=1}^{n} a_{ki} a_{kj} = \delta_{ij} \quad 1 \le i, j \le n$$

$$\left[\sum_{k=1}^{n} a_{ki} a_{kj} \right] = \left[\delta_{ij} \right] \text{ that is } A^t \cdot A = I .$$

The proof of the converse is left as an exercise.

We consider the determinant of an orthogonal T. On one hand$\left(A^t A\right) = 1$ for $A = [T]$and therefore, we get $\det\left(A^t A\right) = 1$. But $\det\left(A^t A\right) = \det\left(A^t\right) \mathit{product} \det(A) = \det(A)\det(A) = \det(A)^2$.Thus $\det(A)^2 = 1$ holds for an orthogonal T with A = det (T). We consider all orthogonal T with $\det[T] = +1$.

Let $SO\left(\mathbb{R}^n\right) = \left\{ T \in O\left(\mathbb{R}^n\right) : \det[T] = +1 \right\}$.

Note that because the map $\det : O\left(\mathbb{R}^n\right) \to \mathbb{R}$ is multiplicative, the set $SO\left(\mathbb{R}^n\right)$is a sub-group of the group $O\left(\mathbb{R}^n\right)$.

Definition : The group $SO\left(\mathbb{R}^n\right)$ is the <u>n-dimensional special linear group</u>.

In the next chapter we will define orientations of $\mathbb{R}^n$ using the group $SO\left(\mathbb{R}^n\right)$.

2.6 THE TOTAL DERIVATIVE :

Let Ω be an open subset of $\mathbb{R}^n$, p a point of Ω and let $f:\Omega\to\mathbb{R}^m$ be any map.

We explain in few words the concept of total derivative of such a vector valued function of a multi-variable $x=(x_1\, x_2 ... x_n)\in\Omega$ as a linear transformation : $Df(p):\mathbb{R}^n\to\mathbb{R}^m$.

Definition 7 : f is differentiable at p if there exists a linear map $T:\mathbb{R}^n\to\mathbb{R}^m$ such that

$$\lim_{h\to o}\frac{f(p+h)-f(p)-T(h)}{\|h\|}=O$$

Note that the quantity $\frac{f(p+h)-f(p)-T(h)}{\|h\|}$ is defined for non-zero but small $h\in\mathbb{R}^n$ and the limit being zero indicates that $f(p+h)-f(p)-T(h)$ is a quantity of second order smallness in comparison with the "increment" h. Thus differentiability of f at p is about approximating the variation $f(p+h)-f(p)$ of f around p by the linear map $T:\mathbb{R}^n\to\mathbb{R}^m$.

Recall from analysis that any linear $T:\mathbb{R}^n\to\mathbb{R}^m$ is continuous at O (actually at every point of $\mathbb{R}^n$). Consequently for a f; differentiable at $p\in\Omega$; $f(p+h)-f(p)-T(h)\to O$ and $T(h)\to O$ as $h\to O$ implies that $f(p+h)-f(p)\to O$ as $h\to O$ i.e. f is continuous at p. Thus, the classical result : differentiability of a function at a point implies continuity of it at the same point - continues to hold in the present context also.

Next, note that if f is differentiable at a $p\in\Omega$ then the linear T appearing in the definition must be unique. To see this consider two linear maps $T_1,T_2:\mathbb{R}^n\to\mathbb{R}^m$ satisfying :

$$\frac{f(p+h)-f(p)-T_1(h)}{\|h\|}\to O$$

$$\frac{f(p+h)-f(p)-T_2(h)}{\|h\|}\to O$$

as $h \to O$. Then we get $\dfrac{T_1(h)-T_2(h)}{\|h\|} \to O$ as $h \to O$. But this implies $T_1 = T_2$. To see this, consider any hon-zero $x \in \mathbb{R}^2$. Then for $k \in \mathbb{N}$, large enough use consider $h = \dfrac{x}{k}$ so that $h \to O$ as $k \to \infty$. Thus

$$\frac{T_1\left(\frac{x}{k}\right)-T_2\left(\frac{x}{k}\right)}{\left\|\frac{x}{k}\right\|} \to O \text{ as } k \to \infty$$

But $$\frac{T_1\left(\frac{x}{k}\right)-T_2\left(\frac{x}{k}\right)}{\left\|\frac{x}{k}\right\|} = \frac{T_1(x)-T_2(x)}{\|x\|}$$

Therefore, $$\frac{T_1(x)-T_2(x)}{\|x\|} = \lim_{k\to\infty} \frac{T_1\left(\frac{x}{k}\right)-T_2\left(\frac{x}{k}\right)}{\left\|\frac{x}{k}\right\|} = O.$$

This gives $T_1(x)-T_2(x)$ whenever $x \neq o$. But $T_1(o)=T_2(o)=o$ by the linearity of T_1 and T_2. Therefore $T_1(x)=T_2(x)$ holds for all $x \in \mathbb{R}^n$ i.e. $T_1 = T_2$.

We call the unique T the total derivative of f at p and denote it by $Df(p)$. Thus, when f is differentiable at a point p of its domain, its total derivative $Df(p)$ is a linear transformation $Df(p) \because \mathbb{R}^n \to \mathbb{R}^m$.

We consider the matrix $[Df(p)]$ of the total derivative. Suppose $Df(p) = \left[a_{ij}\right]_{\substack{1\le i\le m \\ 1\le j\le n}}$. We ask what are a_{ij}. To answer this we have

$$Df(p)(e_j) = \lim_{t\to o} \frac{f(p+tej)-f(p)}{t}.$$

The i^th component of this vector equation is

$$a_{ij} = \lim_{t\to o} \frac{f_i(p+tej)-f_i(p)}{t}$$

that is $a_{ij} = \frac{\partial f_i}{\partial x_j}(p)$. Thus we get $Df(p) = \left[\frac{\partial f_i}{\partial x_j}(p)\right]$. In classical literature this matrix $\left[\frac{\partial f_i}{\partial x_j}(p)\right]$ is called the Jacobean matrix of the total derivative $Df(p)$.

In particular if the map f is differentiable at every $p \in \Omega$ then we get the map $Df : \Omega \to L(\mathbb{R}^n\ \mathbb{R}^m)$ with the partial derivatives $\frac{\partial f_i}{\partial x_j} : \Omega \to \mathbb{R}$.

We say that the map $f : \Omega \to \mathbb{R}^m$ is <u>continuously differentiable</u> on Ω if (i) f is differentiable at every $p \in \Omega$ and if (ii) all the partial derivatives $\frac{\partial f_i}{\partial x_j} : \Omega \to \mathbb{R}\ 1 \le i \le m, 1 \le j \le n$ are continuous on Ω.

We will discuss more differential calculus in Chapter 3.

Exercises :

1) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear. Prove that there exists a constant $C < \infty$ such that $\|T(x)\| \le C\|x\|$ holds for all $x \in \mathbb{R}^n$.

   Hence or otherwise deduce that any linear $T : \mathbb{R}^n \to \mathbb{R}^n$ is continuous at every point of $\mathbb{R}^n$.

2) Prove that any linear $T : \mathbb{R}^n \to \mathbb{R}$ satisfies $\langle T(s), T(y)\rangle = \langle x, y\rangle$ for every x, y in $\mathbb{R}^n$ if and only if $\|T(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$.

3) Let $A = [T]$ for a $T \in End(\mathbb{R}^n)$. Suppose A satisfies $A^t A = I$. Prove that T is ortrogonal.

4) Prove that $O(\mathbb{R}^n)$ is a group and $SO(\mathbb{R}^n)$ is a normal sub-group of it.

5) Prove that $SO(\mathbb{R}^n)$ has exactly two cosets in $O(\mathbb{R}^n)$.

6) Prove $tr(ST) = tr(TS)$ holds for all linear S, $T : \mathbb{R}^n \to \mathbb{R}^n$.

7) Prove : If $T:\mathbb{R}^n \to \mathbb{R}^n$ is linear, then $DT(p) \equiv T$ for every $p \in \mathbb{R}^n$.

8) Let $\Omega \subset \mathbb{R}^n, \tilde{\Omega} \subset \mathbb{R}^m$ be open sets and lot $P \in \Omega$ be arbitrary with $a = f(p)$.

   Let $f:\Omega \to \Omega, g:\tilde{\Omega} \to \mathbb{R}^\ell$ be maps such that (i) f is differentiable at p, (ii) g is differentiable at $q = f(p)$. Prove that gof is differentiable at p and derive : $D(gof)(p) = Dg(q) \circ Df(p)$.

9) Prove that $GL(\mathbb{R}^n)$ is an open subset of $End(\mathbb{R}^n)$.

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3

ISOMETRIES OF $\mathbb{R}^n$, SMOOTH FUNCTIONS ON $\mathbb{R}^n$

Unit Structure :

Having introduced the group $O(\mathbb{R}^n)$ of orthogonal linear transformation of $\mathbb{R}^n$, we discuss a larger group of transformations of a $\mathbb{R}^n$, namely the group of isometries of $\mathbb{R}^n$ where an isometry of $\mathbb{R}^n$ is a bijective self map of $\mathbb{R}^n$ which preserves the distance between its points. First, we derive the basic result describing an isometry as a rigid motion of $\mathbb{R}^n$ ie. a map which is a composition of a rotation and a translation in $\mathbb{R}^n$. We verify that such rigid motions in $\mathbb{R}^n$ form a group.

In the remaining part, we discuss some basic theorems of differential calculus. We introduce the function space $C^\infty(\Omega)$ of smooth real valued functions of a multivariable ranging in an open subset Ω of $\mathbb{R}^n$.

3.1 ISOMETRIES OF $\mathbb{R}^n$

Definition 1 : An isometry of $\mathbb{R}^n$ is a bijective map $f:\mathbb{R}^n \to \mathbb{R}^n$ which preserves distance between any two points of $\mathbb{R}^n$:

$$\|f(x)-f(y)\| = \|x-y\| \text{ for all } x, y \text{ in } \mathbb{R}^n.$$

Here are some simple facts about the isometries of $\mathbb{R}^n$:

- Every orthogonal linear $T:\mathbb{R}^n \to \mathbb{R}^n$ is an isometry of $\mathbb{R}^n$; for, let x, y in $\mathbb{R}^n$ be arbitrary. Then we have :
  $\langle T(x-y), T(x-y)\rangle = \langle x-y, x-y\rangle$

for all x, y in $\mathbb{R}^n$ i.e.
$\langle T(x)-T(y), T(x)-T(y)\rangle = \langle x-y, x-y\rangle$
and therefore :
$\|T(x)-T(y)\| = \|x-y\|$
for every x, y in $\mathbb{R}^n$

- For each $a \in \mathbb{R}^n$ let $T_a : \mathbb{R}^n \to \mathbb{R}^n$ be the map given by $T_a(x) = x + a$ for every $x \in \mathbb{R}^n$. The bijective map T_a is called the translation map in $\mathbb{R}^n$ determined by its element a. Clearly, each T_a is an isometry of $\mathbb{R}^n$.
- If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of $\mathbb{R}^n$ them its inverse $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ also is an isometry of $\mathbb{R}^n$.
- If $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are isometries of $\mathbb{R}^n$, then so is their composition $g \circ f : \mathbb{R}^n \to \mathbb{R}^n$

Let $Iso(\mathbb{R}^n)$ be the set of all isometries of $\mathbb{R}^n$. It then follows that the composition of self maps of $\mathbb{R}^n$ when restricted to $Iso(\mathbb{R}^n)$ becomes a binary operation on $Iso(\mathbb{R}^n)$ and the resulting algebraio system :

$$\left(Iso(\mathbb{R}^n), 0\right)$$

is a group. It is the group of isometies of $\mathbb{R}^n$. It is easy to see that orthogonal transformations of $\mathbb{R}^n$ constitute a sub group of $\left(Iso(\mathbb{R}^n), 0\right)$. Also the set of all translational maps i.e. $\{T_a : a \in \mathbb{R}^n\}$ is also another subgroup of the isometry group.

Now, we obtain a result regarding the structure of an isometry of $\mathbb{R}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry.

Let $a = f(0)$. Define $R : \mathbb{R}^n \to \mathbb{R}^n$ by $R(x) = f(x) - f(0)$ for each $x \in \mathbb{R}^n$. Thus, we have :

$$\begin{aligned} f(x) &= f(x) - f(0) + f(0) \\ &= f(x) - f(0) + a \\ &= T_a 0 R(x) \quad \forall x \in \mathbb{R}^n \end{aligned}$$

We prove below that $R:\mathbb{R}^n \to \mathbb{R}^n$ is linear and preserves inner product of $\mathbb{R}^n$ (and therefore is an orthogonal transformation of $\mathbb{R}^n$).

Now, for any x, y in $\mathbb{R}^n$ we have
$f(x) - f(y) = R(x) - R(y)$
and therefore $\|R(x) - R(y)\| = \|f(x) - f(y)\| = \|x - y\|$ (since f is isometry. Thus we have

$\langle R(x) - R(y), R(x) - R(y)\rangle = \langle x - y, x - y\rangle$ and therefore :

$\langle R(x), R(x)\rangle - 2\langle R(x), R(x)\rangle + \langle R(y), R(y)\rangle$
$= \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle$
i.e. $\|R(x)\|^2 - 2\langle R(x), R(y)\rangle + \|R(y)\|^2$
$= \|x\|^2 - 2\langle x\ y\rangle + \|y\|^2$ for all x, y in $\mathbb{R}^n$ ………………..(*)

Recall $R(0) = 0$ and therefore
$\|R(x)\| = \|R(x) - 0\| = \|R(x) - R(0)\| = \|x - 0\| = \|x\|$ and similarly $\|R(y)\| = \|y\|$. Using these results, above yields. ………………….. (*)

$\langle R(x), R(y)\rangle = \langle x, y\rangle$ for all x, y in $\mathbb{R}^n$.

We use the identity (**) to deduce linearly of R as follows.

First, note that (**) implies that $\{T(e_i) : 1 \le i \le n\}$ is orthonormal. Therefore, for any $x \in \mathbb{R}^n$, we have

$$R(x) = \sum_{i=1}^{n} \langle R(x), R(e_i)\rangle R(e_i)$$

But again by (**) we have $\langle R(x), R(e_i)\rangle = (x, e_i) = x_i$ for each $i,\ 1 \le i \le n$. Therefore

$$R(x) = \sum_{z=1}^{n} x_i R(e_i) \qquad \ldots (***)$$

for each $x \in \mathbb{R}^n$.

Now for any x, y in $\mathbb{R}^n$ and for any a, b in $\mathbb{R}$ we have

$$R(ax+by) = \sum_{1=i}^{n}(ax_i + by_i)R(e_i)$$

$$= a\sum_{1=i}^{n} x_i R(e_i) + b\sum_{i=1}^{n} y_i R(e_i)$$

$$= a\,R(x) + b\,R(y) \text{ using } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ (***)}$$

This prove linearly of R. Thus we have prove above both : linearly in (***) and inner product preserving property (**) and therefore T is orthogonal.

Uniqueness of the decomposition $f = T_a 0\, R$ is left as an exercise for the reader.

We summarize this result in the following :

Proposition 1 : Every isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is expressible uniquely in the form $f = T_a 0 R$ where R is an orthogonal transformation of $\mathbb{R}^n$ and T_a is the translation with $a = f(0)$.

3.2 ORIENTATIONS OF $\mathbb{R}^n$:

The concept of orientation of $\mathbb{R}^n$ was introduced in Chapter 1 in terms of families of orthogonal frames of $\mathbb{R}^n$. It was shown that $\mathbb{R}^n$ has exactly two orientations. In this chapter we reformulate it shightly differently so as to involve the group $SO(n)$. We bring orthonormal bases in place of the orthogonal frames and decompose the set of all orthonormal bases into two classes, they are equivalence classes of a certain equivalence relation, the later being introduced in terms of the group $SO(\mathbb{R}^n)$.

To begin with, note that each orthogonal frame $\mathscr{F}$ in $\mathbb{R}^n$ determines and is determined by an ordered orthornormal basis $(f_1, f_2 ..., f_n)$, the i[th] unit vector f_i pointing along the i[th] axis of $\mathscr{F}$. Thus there is a 1-1 correspondance between orthogonal frames $\mathscr{F}$ in $\mathbb{R}^n$ and ordered orthonormal vector bases $(f_1, f_2 ..., f_n)$. Now we consider ordered orthonormal bases instead of orthogonal frames to specify orientations of $\mathbb{R}^n$. We make this choice because now we are acquainted with the group $SO(\mathbb{R}^n)$, the elements of the group enabling us to change from one frame to another similarly oriented frame.

Let Σ denote the set of all ordered orthonormal vector bases of $\mathbb{R}^n$. We introduce a relation $\simeq$ on the set Σ as follows : Let $(f_1, f_2..., f_n)$ and $(g_1, g_2..., g_n)$ be any two ordered orthonormal bases. Then there exists a unique orthogonal linear $T:\mathbb{R}^n \to \mathbb{R}^n$ satisfying $T(f_i) = g_i$ for $1 \le i \le n$. Moreover, we have : $\det(T) = +1$ or -1.

We set : $(f_1, f_2,...., f_n) \simeq (g_1, g_2,...., g_n)$ if an only if $\det T = +1$.

Clearly the relation $\simeq$ thus defined is an equivalence relation on Σ. Therefore it decomposes the set Σ into disjoint subsets of if namely the equivalence classes of the relation $\simeq$:

Each equivalence class is said to determine an orientation of $\mathbb{R}^n$.

Finally because $\det T = +1$ or -1 for each $T \in \Sigma$ we see that there are two distinct equivalence classes and hence two distinct orientations of $\mathbb{R}^n$.

- The equivalence class containing the standard basis $(e_1, e_2, ..., e_n)$.

To describe the other class consider the vector basis $(-e_1, e_2, ..., e_n)$ of $\mathbb{R}^n$. Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation of $\mathbb{R}^n$ associated with the change $(e_1, e_2, ..., e_n)$ to $(-e_1, e_2, ..., e_n)$. Clearly

$$[T]=\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

and therefore $\det T = -1$. Thus $(-e_1, e_2, ..., e_n)$ belongs to the other equivalence class ie the other orientation of $\mathbb{R}^n$. Therefore, we have

- The equivalence class containing $(-e_1, e_2, ..., e_n)$ is the opposite orientation of $\mathbb{R}^n$.

The Groups $SO(\mathbb{R}^2)$ and $SO(\mathbb{R}^3)$:

We describe these groups in terms of their matrices.

First the matrices in $SO(2)$:

Let (f_1, f_2) be any orthonomal basis of $\mathbb{R}^2$ belonging to the same orientation class of the standard basis (e_1, e_2) of $\mathbb{R}^2$.

Let θ be the angle between e_1 and f_1 which is measured counter-clockwise. Then the matrix of T is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

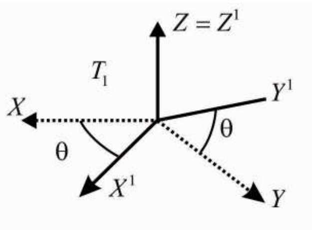
This shows that $SO(\mathbb{R}^2)$ consists of all $T:\mathbb{R}^2 \to \mathbb{R}^2$ having matrix representations : (with respect to the standard basis (e_1, e_2):

$$SO(\mathbb{R}^2) = \left\{ T \in O(\mathbb{R}^2) : [T] = \begin{bmatrix} \cos\theta, -\sin\theta \\ \sin\theta, \cos\theta \end{bmatrix}, o \le \theta < 2\pi \right\}$$

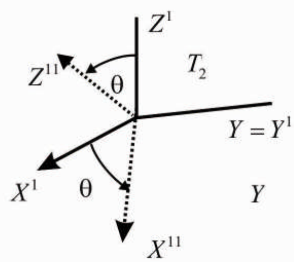
Next we describe the group $SO(\mathbb{R}^3)$ by means of the matrix representations $[T]$ of its elements T with respect to the standard basis (e_1, e_2, e_3): We consider T obtained as the resultant $T = T_2 o T_1$ of two rotations where (i) T_1 is the rotation of the XOY-plane about the Z-axis through an angle θ measured counter clockwise, $o \le \theta \le 2\pi$ and (ii) T_2 is the rotation of the frame about the Y-axis through an angle o :

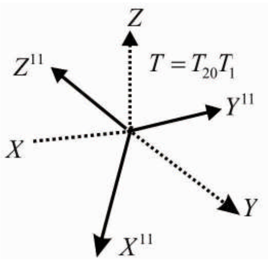
The matrices are :

$$[T_1] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[T_2] = \begin{bmatrix} \cos\varnothing & 0 & -\sin\varnothing \\ 0 & 1 & 0 \\ \sin\varnothing & 0 & \cos\varnothing \end{bmatrix}$$





$$[T] = [T_2] \bullet [T_1] = \begin{bmatrix} \cos\varnothing & 0 & -\sin\varnothing \\ 0 & 1 & 0 \\ \sin\varnothing & 0 & \cos\varnothing \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\varnothing & -\cos\varnothing\sin\theta & -\sin\varnothing \\ \sin\theta & \cos\theta & 0 \\ \cos\varnothing\sin\theta & -\sin\theta\sin\theta & \cos\theta \end{bmatrix}$$

$$SO\left(\mathbb{R}^3\right) = \left\{ T \in O\left(\mathbb{R}^3\right) : [T] = \begin{bmatrix} \cos\varnothing\cos\varnothing & -\cos\varnothing\sin\varnothing & -\sin\varnothing \\ \sin & \cos\varnothing & 0 \\ \cos\theta\sin\varnothing & -\sin\theta\sin\varnothing & \cos\varnothing \end{bmatrix} : o \le \theta \le \pi \right\}$$

3.3 SMOOTH FUNCTIONS :

We return to differential calculus and recall some more terminology.

Let Ω be an open subset of $\mathbb{R}^n$.

For a $f : \Omega \to \mathbb{R}$ and for a $p \in \Omega$, recall that the limits :

$\lim\limits_{t \to o} \dfrac{f(p + te_i) - f(p)}{t}$ are the <u>partial derivatives</u> $\dfrac{\partial f}{\partial x_i}(p)$ $(1 \le i \le n)$ of f at p.

Suppose the function f is such that $(i) \dfrac{\partial f}{\partial x_i}(p)$ for $1 \le i \le n$ and for all $p \in \Omega$ exist. Then we get the functions :

$$\frac{\partial f}{\partial x_i} : \Omega \to \mathbb{R} \;\; 1 \le i \le n$$

from the function f.

We say that the function $f : \Omega \to \mathbb{R}$ is <u>continuously differentiable</u> on Ω if (i) $\dfrac{\partial f}{\partial x_i}(p)$ exists for each $p \in \Omega$, each $i(1 \le i \le n)$ and (ii) all the function $\dfrac{\partial f}{\partial x_i} : \Omega \to \mathbb{R}$ $(1 \le i \le n)$ are continuous on Ω.

$C^1(\Omega)$ denotes the set of all continuously differentiable function on Ω.

Next, we say that $f:\Omega\to\mathbb{R}$ is twice continuously differentiable on Ω if (i) $f\in C^1(\Omega)$ and (ii) for each i $1\le i\le n$, $\frac{\partial f}{\partial x_i}\in C^1(\Omega)$. If f is twice continuously differentiable, then it follows that for each $i,j\,(1\le i\ \ j\le n)$ and for each $p\in\Omega$, $\frac{\partial^2 f}{\partial x_i \partial x_j}(p)=\frac{\partial^2 f}{\partial x_j \partial x_i}(p)$; $1\le i,j\le n$.

$C^2(\Omega)$ denotes the space of all twice continuously differentiable $f:\Omega\to\mathbb{R}$.

Higher order continuously differentiability of $f:\Omega\to\mathbb{R}$ is defined inductively : Suppose k times continuous differentiability of f on Ω is defined. Then we say that f is $k+1$ times continuously differentiable on Ω if (i) $\frac{\partial f}{\partial x_i}(x)$ exists for each $x\in\Omega$ and (ii) the functions $\frac{\partial f}{x_i}:\Omega\to\mathbb{R}$ are k times continuously differentiable on Ω.

If f is k times continuously differentiable on Ω then it follows that for any multi-index $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$ with $\alpha_1+\alpha_2+......+\alpha_n\le k$ the mixed partial derivative $D^\alpha f(p):=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f(p)$ exists for all $p\in\Omega$ and the resulting function :

$$D^\alpha f:\Omega\to\mathbb{R};\quad p\to D^\alpha f(p) \text{ is continuous on } \Omega.$$

$C^k(\Omega)$ is the functions space of all k times continuous differentiable functions $f:\Omega\to\mathbb{R}$. The functions space $C^k(\Omega)$ has the structure of a commutative ring with identity; the ring operations being addition and multiplication of function on Ω.

Now, we have a decreasing sequence of functions spaces :

$$C^1(\Omega)\supset C^2(\Omega)\supset\cdots\cdots\supset C^k(\Omega)\supset\cdots\cdots$$

We consider the intersection :

$$C^{\infty}(\Omega) := \bigcap \left\{ C^k(\Omega) : k \in \mathbb{N} \right\}.$$

This space contains non-trivial (= non constant) function : In fact, we have the following :

a) If $\alpha = (\alpha_1, \alpha_2, \cdots\cdot, \alpha_n)$ is any multi-ndex, then the monomial

$x^{\alpha} : \Omega \to \mathbb{R}; x \to (x_1)^{\alpha_1} (x_2)^{\alpha_2} \cdots\cdot (x_n)^{\alpha_n}$ is in $C^{\infty}(\Omega)$.

Consequently any polynomial : $p(x) = \sum_{1\alpha 1 \le m} a_{\alpha} x^{\alpha}$ is in $C^{\infty}(\Omega)$.

b) Given any compact k, open $\bigcup$ with $K \subset U \subset \Omega$, there exists a $f \in C^{\infty}(\Omega)$ satisfying :

i) $f \equiv 1 \ on\, k$,

ii) $f \equiv 0 \ on\, \Omega - U$.

Above, we mentioned smooth functions defined on open subsets of $\mathbb{R}^n$. In Chapter 8 we will extend the property of smoothness to functions defined on open subsets of smooth surfaces.

Also, recall the smoothness of vector valued functions defined on open subsets of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}^m$ be any vector valued functions. Let its components be $f_1, f_2, \ldots.. f_m : \Omega \to \mathbb{R}$ thus $f(x) = (f_1(x), f_2(x), \ldots.. f_m(x))$ for all $x \in \Omega$. Now we declare that f is smooth if each of $f_1 \ldots f_m$ is in $C^{\infty}(\Omega)$ in the above sease. Moreover, for any multi-index $\alpha = (\alpha_1, \alpha_2, \cdots\cdot, \alpha_n)$ we define :

$D^{\alpha} f(x) = (D^{\alpha} f_1(x), D^{\alpha} f_2(x), \ldots\ldots, D^{\alpha} f_m(x))$ for each $x \in \Omega$. $D^{\alpha} f(x)$ is the mixed partial derivative of f at x.

3.4 BASIC THEOREMS OF DIFFERENTIAL CALCULUS :

We recall here three of the basic theorems of differential calculus, namely :

- The inverse function theorem,
- The implicit function theorem,
- The rank theorem.

Of the three of them the first is an independent result of fundamental nature and the other theorems are deduced from the first. Other basic theorems of differential calculus such as Picard's theorem (regarding the existence and uniqueness of solution of an ODE), the Frobaneous theorem and so on will be explained in the latter chapters.

We begin with the first theorem of the above list :

Theorem 1 (The Inverse Function Theorem) :

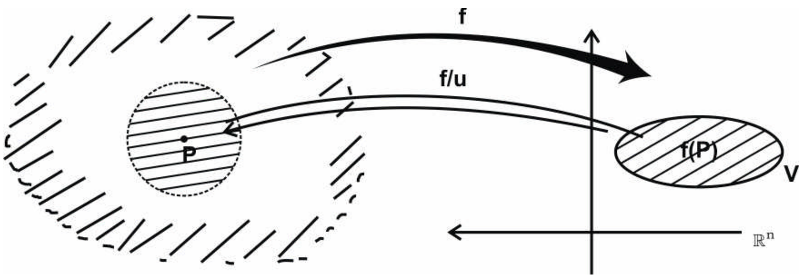
Let Ω be an open subset of $\mathbb{R}^n$, p a point of Ω and let $f : \Omega \to \mathbb{R}^n$ be a smooth map.

Suppose, the derivative $Df\left(p\right) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism. Then there exist open subsets U of Ω, V of $\mathbb{R}^n$ having the following properties :

i) $p \in U$, $f\left(p\right) \in V$

ii) $f\left(U\right) = V$ and

iii) ${}^{f}\!/_{U} : U \to V$ is bijective with the inverse $\left({}^{f}\!/_{U}\right)^{-1} : V \to U$ also being smooth. (In other words, (iii) means ${}^{f}\!/_{U}$ is a <u>diffcomorphism</u> between U and V).



$\Omega\left(c\mathbb{R}^n\right)$

Fig. 1 (Inverse Function Theorem)

As explained earlier, the total derivative $Df\left(p\right)$ is a linear map approximating the given f in a neighborhood of p and therefore, some of the properties of the approximating map $Df\left(p\right)$ should reflect back on the local behaviour of f around the point p.

The theorem above asserts that indeed, the invertibility of the approximating linear map $Df\left(p\right)$ ensures local invertibility of the function f, the local inverse of f also being smooth.

A smooth bijective map $f: U \to V$ with $f^{-1}: V \to U$ also being smooth is said to be a smooth diffeomorphism between the sets U and V. Thus, the inverse function theorem asserts that a smooth map with its derivative at a point being invertible is a (local) diffeomorphim in a neighborhood of that point.

Next, we discuss the implicit function theorem.

Theorem 2 (The Implicit Function Theorem) :

Let, a) $U \, C \, \mathbb{R}^m, V \, C \, \mathbb{R}^n$ be open subsets. (Here we denote a point of U by $x = (x_1, x_2, ..., x_m)$ and a point of V by $y = (y_1, y_2, ..., y_n)$.)

b) $f = f(x, y): U \times V \to \mathbb{R}^m$ be a smooth map.

c) Suppose, for a point $p = (a, b) \in U \times V$ the matrix $\left[\frac{\partial f_i}{\partial y_i}(a\, b) \right] 1 \le i, j \le n$ is invertible.

Then there exists an open $\widetilde{U}$ with $a \in \widetilde{U} \in U$ and a smooth map $g: \widetilde{U} \to V$ which satisfies

i) $g(a) = b$ and

ii) $f(x, g(x)) = f(a, b)$ for all $x \in \widetilde{U}$.

Thus, the theorem asserts that when the condition (c) is satisfied, the equation $f(x\, y) = C(= f(a\, b))$ can be **solved** to get the variable y as a functions (smooth) $y = g(x)$ satisfying the additional proviso : $b = g(a)$.

This result has applications everywhere in differential geometry; we explain here only a small aspect of it :

We are given a smooth function

$$f: \Omega \to \mathbb{R},$$

Ω being an open subset of $\mathbb{R}^3$. For a $d \in \mathbb{R}$ we consider the set $M = \{(x, y, z) \in \Omega : f(x, y, z) = d\}$. If not empty, then such a $M = M(d)$ is often called a level set of the function through the value d.

We consider a non-empty level set $M(d)$ of the function f satisfying the additional condition :

$$\frac{\partial f}{\partial z}(x,y,z) \neq 0$$

for all $(x,y,z) \in M(d)$. Then for any $(a,b,c) \in M(d)$ the implicit function theorem asserts that there exists an open $U c \mathbb{R}^2$ with $(a,b) \in U$ and smooth map $g : U \to \mathbb{R}$ with $g(a,b) = c$ satisfying $f(x,y,g(x,y)) \equiv d$ on U.

In other words a part of the level set $M(d)$ containing a given $p = (a,b,c)$ is the graph of a smooth function g and therefore it looks like a surface. This observation is used very often in studying local properties of smooth surfaces.

Finally, we describe the rank theorem.

First recall that a matrix (of size $m \times n$) has rank k if the matrix contains an invertible sub-matrix of size $k \times k$ and has no invertible sub-matrix of size larger than $k \times k$.

Now, the theorem :

Theorem : (The Rank Theorem) :

Let $f : \Omega \to \mathbb{R}^n$ be a smooth map- Ω being an open subset of $\mathbb{R}^m$, the map f having the property that $rk(Df(p)) = k$ for every $p \in \Omega$. Then for every $p \in \Omega$ there exist :

i) an open $U \, c \, \mathbb{R}^m$ with $0 \in U$

ii) an open $V \, c \, \mathbb{R}^n$ with $f(p) \in V$

iii) a diffeomorphism $g : U \to g(U) \subset \Omega$ with $g(0) = p$

iv) a diffeomorphism $h : V \to h(V) \subset \mathbb{R}^n$

Such that the map

$$h \circ f \circ g : U \to \mathbb{R}^n$$

is given by

$$h \circ f \circ g(x_1 x_2 ...., x_m) = \left(x_1 x_2 ...., x_k, \underbrace{o.....o}_{n-k} \right) \text{for all } (x_1 x_2 ...., x_m) \in U .$$

Exercises :

1) Check if each of the following $T:\mathbb{R}^3 \to \mathbb{R}^3$ is an isometry and then express each of the T in the form $T_a \circ R$ in case T is an isometry :

   i) $T(x,y,z) = \left(x+2, \dfrac{2y}{\sqrt{13}} - \dfrac{3z}{\sqrt{13}} + 4, \dfrac{3y}{\sqrt{13}} + \dfrac{2z}{\sqrt{13}} + 3 \right)$

   ii) $T(x,y,z) = \left(\dfrac{x}{2} + \dfrac{\sqrt{3}}{2} z + 5, y+2, \dfrac{-\sqrt{3}}{2} x + \dfrac{z}{2} + 4 \right)$

   iii) $T(x,y,z) = (4x+5y+z, 4x+3y, 5z)$

   iv) $T(x,y,z) = \left(\dfrac{\sqrt{2}x}{3}, y \dfrac{\sqrt{7}z}{3} \right)$

2) Let $f \in C^1(\Omega)$, $p \in \Omega$ (Ω being an open subset of $\mathbb{R}^n$)

Prove :

a) $Df(p)(u) = \sum_{2=1}^{n} u_i \dfrac{\partial f}{\partial xi}(p)$ for all $u \in \mathbb{R}^n$

b) $Df(p)(\alpha v + \beta w) = \alpha Df(p)(v) + \beta Df(p)(w)$

3) If $f \in C^2(\Omega)$ then prove that for every $p \in \Omega$, $D^2 f(p)$ is a symmetric bilinear form.

4) i) Construct a smooth map $f:\mathbb{R} \to [0,1]$ such that

   $f(x) = 1$ if $|x| \le 1$

   $\quad = 0$ if $|x| > 2$

   ii) Using the map f of (i), construct a smooth

   $g:\mathbb{R}^2 \to [0,1]$

   such that $g \equiv 1 \; on \; B(01)$

   $\qquad \equiv 0 \; on \; \mathbb{R}^2 - B(0,4)$

❖❖❖❖

4

SMOOTH CURVES

Unit Structure :

In this unit (consisting of this chapter and the next, two) we study the geometry of smoothly parametrized space curves. (After discussing such curves, we will give indications of the geometry of curves in higher dimensional Euclidean spaces also). In this chapter we begin with basic geometric features of j a smoothly prametrized space curve, its reparametrization, its unit speed version, a moving orthonormal frame along it and so on.

Actually, we will consider a smaller class of curves consisting of Frenet curves and explain how differentiation leades us to geometric features of such curves. In particular. We introduce the concepts of **curvature** and **torsion** of a curve which are smooth functions defined along a Frenet curve. Explaining their geometric significance, we proceed to derive the basic equations - the Serret - Frenet equations - associated with such curves. It is the central result of the theory of Frenet curves that the two functions curvature and torsion functions - of a curve determine the curve uniquely to within an isometry of $\mathbb{R}^3$. We derive this important result - the fundamental theorem of curves - using Picard's existence / uniqueness theorem of solutions of ODE.

Throughout, we are considering curves which are smooth (= infinitely differentiable) This assumption (infinite differentaibility of curves) is superfluous, for, we are using only thrice continuous differentiability of the parametrized curves. We have adopted here infinite differentiability as only a convenient set - up to derive the basic theory. But on the other hand if a curve is not as much as thrice continuously differentiable, then some of the tools of differential calculus may not be applicable.

4.1 SMOOTH CURVES.

In this chapter, *I, J, K* denote intervals

Definition: A <u>smooth curve</u> is a smooth map

$$c : I \to \mathbb{R}^3$$

The curve is said to be parametrized by the independent variables of the map. s is the parameter of the curve and for a $s_0 \in I$ the point $c(s_0)$ of the curve is said to have the parametric value s_0.

The set $\{ c(s) : s \in I \}$ is called the **trace** of the curve c.

For each $s \in I$, writing the point $c(s)$ in terms of its Cartesian coordinates: $c(s) = ((x_1(s), x_2(s), x_3(s)))$ we get the real valued function:

$$x_1 : I \to \mathbb{R}, \quad x_2 : I \to \mathbb{R}, \; x_3 : I \to \mathbb{R}$$

Note that the curve $c : I \to \mathbb{R}^3$ is smooth if and only if the function $x_{1,}, x_2, x_3 \; : \; I \to R$ are smooth.

Let now $\Theta : J \to I$ be a smooth, strictly monotonic increasing, bijective function. The curve c and the function Θ combine to get yet another curve:

$$\tilde{c} = c0\,\Theta : J \to \mathbb{R}^3$$

Definition 2: The curve $\tilde{c} : J \to \mathbb{R}^3$ is said to be a reparamentrization of the curve $c : I \to \mathbb{R}^3$

If $r \in J$ is the variable raging in J, then we speak of r as the new parameter and Θ the reparametrizing map.

For the curve $c : I \to \mathbb{R}^3$, we write $\dot{c}(s)$ for $\dfrac{dc(s)}{dt}, \ddot{c}(s) \; for \; \dfrac{d^2c(s)}{ds^2} \, etc.$

Note that $\dot{c}(s) = \left(\dfrac{dx_1(s)}{ds}, \dfrac{dx_2(s)}{ds}, \dfrac{dx_3(s)}{ds} \right)$

$$\ddot{c}(S)=\left(\frac{d^2x_1(s)}{ds^2},\frac{d^2x_2(s)}{ds^2},\frac{d^2x_3(s)}{ds^2}\right) etc.$$

Definition 3: $c:I\to\mathbb{R}^3$ is said to be (i) regular if $\dot{c}(s)\neq 0$ for all $s\in I$ and (ii) a <u>Frenet curve</u> if $\dot{c}(s)$ and $\ddot{c}(s)$ are linearly independent.

We will consider only smooth Frenet curves.

Definition 4: A Frenet curve is said to be a unit speed curve if $\|\dot{c}(s)\|\equiv 1.$

Below, we show that a Frenet curve can be reparmetrized so as to make it a unit speed curve.

Let $c:I\to\mathbb{R}^3$ be a Frenet curve.

For an arbitrary chosen $s_0\in I$ we consider the integral :

$$\ell(s)=\int_{s_0}^{s}\|\dot{c}(x)\|dr \quad s\in I$$

$\ell(s)$ is the (signed) length of the segment of the curve c lying between the points $c(s_0)$ *and* $c(s)$ of it. Note the following

- $\ell(s)<0$ if $s<s_0$ and $\ell(s)>0$ if $s>s_0$
- $\dfrac{d\ell(s)}{ds}=|\dot{c}(s)|>0$ for all $s\in I$ (by the regularity assumption on c) and therefore the function $s\mapsto\ell(s)$ is a strictly monotonic increasing function on its domain interval I.
- The map $s\mapsto\ell(s)$ being continuous, its range-we denote it by J is an interval. Now we have the function :

  $$\ell:I\to J$$

  which is strictly monotonic increasing and bijcetive function between I and J.

We consider the inverse function $\ell^{-1}:J\to I;$ We denote it by Θ. Thus we get the strictly monotonic increasing, smooth map $\Theta:J\to I$ which is bijection between the indicted intervals.

We use Θ to reparametrize the given curve :
$\tilde{c} = c \circ \Theta : J \to \mathbb{R}^3$

Finally, we have: For any $r \in J$

$$\dot{\tilde{c}}(r) = \frac{d\tilde{c}(r)}{dr}$$

$$= \frac{dc(s)}{ds}\frac{ds}{dr}$$

$$= \frac{\dot{c}(s)}{\frac{dr}{ds}}$$

$$= \frac{\dot{c}(s)}{\|\dot{c}(s)\|}$$

and therefore $\left\|\frac{d\tilde{c}(r)}{dr}\right\| = \left\|\frac{\dot{c}(s)}{\dot{c}(s)}\right\| = 1$ for all $r \in J$, that is, the reparametrized curve $\tilde{c} : J \to \mathbb{R}^3$ is a unit speed curve. Thus, a regular curve when re- parametrized by its are-length becomes a unit speed curve.

Note that we can regain the original curve c from its unit speed version $\tilde{c}$:

$$c = \widetilde{C}_0\, \Theta^{-1}.$$

Therefore, we introduce many of the geometric aspects of the given curve c in terms of those of its unit speed version. Also. note that c and its (unit speed) reparametrization $\tilde{c}$, both have the same trace.

Let us discuss a few simple examples of smooth curves, some of which are Frenet curves while some of them are not.

- The curve $c : \mathbb{R} \to \mathbb{R}^3$ given by $C(s) = (s^2, s^3, s^4)$ for $s \in \mathbb{R}$ is smooth but fails to be regular at $C(0) = (0,0,0)$. It is regular when restricted to any interval I not containing 0. In fact c / I is a Frenet curve for such a I.
- Let $c : (0, \infty) \to \mathbb{R}^3$ be the curve given by
  $c(s) = \left(s, \frac{s^2}{2}, \frac{2\sqrt{2}}{3} s^{3/2} \right) \quad s > 0$

Then we have $\dot{c}(s)=\left(1,s,\sqrt{2}\,s^{1/2}\right)$,

$$\ddot{c}(s)=\left(0,1,\frac{1}{\sqrt{2s}}\right)$$

Clearly $\dot{c}(s)$ and $\ddot{c}(s)$ are linearly independent (i.e. non-parallel) vectors for every $c(s)$ and consequently c is a Frenet curve.

Measuring are length from the $C(0)$ end, we get

$$\ell(s)=\int_0^{\delta}\left\|\dot{C}(u)\right\|du$$
$$=\int_0^{\delta}(1+u)du$$
$$=\frac{s^2}{2}+2s \qquad s>0$$

Putting $\ell(s)=r$ $(=\Theta^{-1}(s)$ in the above notation) we get :
$$s=\sqrt{(2r+1)}-1 \qquad (r>0)$$

Therefore, the re-parametrization using r as the new parameter gives the curve

$$\tilde{c}:(0,\infty)\to\mathbb{R}^3$$
$$\tilde{c}(r)=\left(\sqrt{(2r+1)}-1,r+1-\sqrt{2r+1},\frac{2\sqrt{2}}{3}\left(\sqrt{(2r+1)}-1\right)\right)^{3/2}$$ for $r>0$.

We consider a planar curve called the exponential spiral. Its the curve

$$c:\mathbb{R}\to\mathbb{R}^2$$
given by $c(s)=\left(e^s\cos s,e^s\sin s\right),s\in\mathbb{R}$

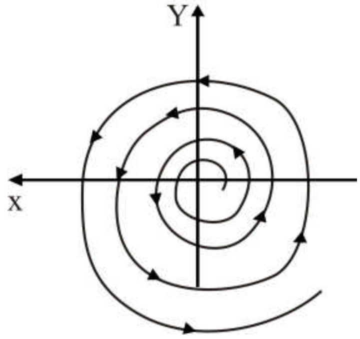


Fig. 1. The Exponential Spiral

Note that $\dot{c}(s) = \left(\left(e^s \cos s - \sin s\right), \left(e^s \cos s + \sin s\right)\right)$ giving $\left\|\dot{c}(s)\right\| = 2 \cdot e^s$. Therefore c is a regular, but not a unit speed curve. Moreover, we have

$$\ddot{c}(s) = \left(-2\overset{s}{e}\sin s, 2\overset{s}{e}\cos s\right)$$

Thus, in fact c is a Frenet curve.

To reparametrize it with respect to its are length we consider its signal arc length function. Taking $c(0) = (1,0)$ as the reference point, we obtain the (signed) length function given by

$$\begin{aligned} r = \ell(s) &= \int_0^{\delta} \left\|\dot{c}(\gamma)\right\| dr \\ &= \int_0^{s} 2e^r dr \\ &= 2e^s - 2 \end{aligned}$$

This gives $s = \log\left(\frac{r}{2} + 1\right)$ and therefore, the reparametrization of the exponential spiral :

$$\tilde{c}(r) = \left(\left(\frac{r}{2} + 1\right)\cos\log\left(\frac{s}{2} + 1\right), \left(\frac{s}{2} + 1\right)\sin\log\left(\frac{s}{2} + 1\right)\right)$$

The Cycloide :

A wheel of radius a is rolling on the (horizontal $X - axis$ of a vertical $XOY - plane$, moving with constant velocity ω. Then a point P held fix on the wheel rim traces a curve. This curve is called a cycloide. Its parametric representation (Parametrized by the time t) is

$$c(t) = \left(wt + a\cos wt, a + a\sin\frac{wt}{a}\right), t \in \mathbb{R}$$

It is not a unit speed curve It is left as an exercise for the reader to reparametrize it so as to get a unit speed curve.

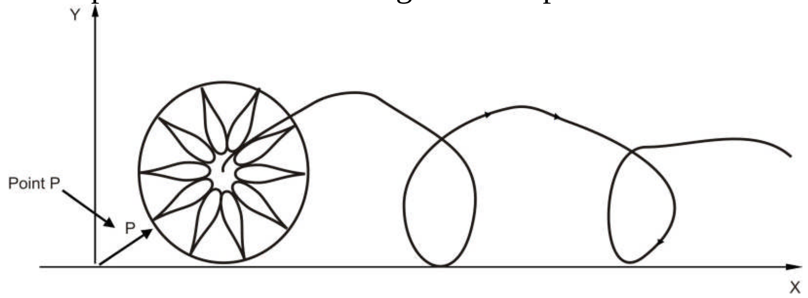


Fig. Cycloide

The Elliptical Helix :

It is a curve which climbs up an elliptical cylinder with cross section

$$\frac{x^2}{a^2}+\frac{x_2{}^2}{b^2}=1 \qquad a \neq b$$

Choosing yet another constant $C>0$
We get the curve.

$$c:\mathbb{R}\to\mathbb{R}^3$$
$$c(t)=(a\cos t, b\sin t, c.t),\ \ t\in\mathbb{R}.$$

The resulting curve is a Frenet curve The reader is invited to verify this fact and to reparametrize it so as to get a unit speed curve.

4.2 CURVATURE AND TORSION OF FRENET CURVES

Let $c: I\to R^3$ be a Frenet curve, its parameter being denoted by $s\in I$. As explained in the preceding section, we assume without loss of generally that it a unit speed curve.
We use the notations:

$$\dot{c}(s) \text{ for } \frac{dc(s)}{ds}$$

$$\ddot{c}(s)\text{for } \frac{d^2c(s)}{ds^2}$$

$$\dddot{c}(s) \text{ for } \frac{d^3c(s)}{ds^3} \text{ and so on.}$$

Putting $t(s)=\hat{c}(s)$, we get the tangential vector having unit length. Moreover $\dot{t}(s)=\ddot{c}(s)\neq 0$ and is not parallel to $t(s)$In fact, $i^0(s)\perp t(s);$ for, differentiating the identity.

$$<t(s),t(s)>=1$$

We get 2 $<\dot{t}(s),t(s)>=0$ and therefore, indeed $\dot{t}(s)\perp t(s)$.

Again using $\dot{t}(s)=\ddot{c}(s)\neq 0$ we introduce

i) $n(s)=\dfrac{\ddot{c}(s)}{\|\ddot{c}(s)\|}$ and

ii) $b(s)=t(s)\times n(s)$.

Now we get an orthonormal triade $(t(s),n(s),b(s))$ of vectors located at the point $c(s)$ of c. We call

- $t(s)$ the <u>unit tangent</u> to c at $c(s)$
- $n(s)$ the <u>principal normal</u> to c at $c(s)$
- $b(s)$ the <u>binormal</u> to c at $c(s)$
- the ordered triple $(t(s),n(s),b(s))$ is called the Serret - Frenet frame or the principal triade to c at $c(s)$ (Often the Serret - Frenet frame is referred to as the Frenet Frame. and the scalar $k(s)=\|\ddot{c}(s)\|$ is the curvature of the curve at the point $c(s)$.

At a later stage we will associate one more scalar called the torsion of c at $c(s)$ and denote it by $\tau(s)$; it quantifies the twisting of the curve c at the point $c(s)$.

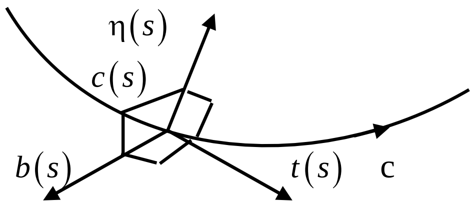


Fig. 3 : The Principle Triade $(t(s),n(s),b(s))$

We proceed to explain how the scalar $k(s)\geq 0$ describes the bending of the curve C at its point $c(s)$.

Fix arbitrarily a point say $c(s_0)=p$ of c. Take two more points $c(s_1)$ and $c(s_2)$ on the curve (without loss of generality, assume that $s_1 < s_0 < s_2$.) We now prove the following

Proposition 2: suppose $k(s_0)>0$. If s_1,s_2 are near enough to s_0 then $c(s_0),c(s_1),c(s_2)$ are non- collinear (and therefore there is a unique circle passing thought them.)

Proof: (By contradiction)

Under contrary assumption, suppose we can choose parametric values s_1, s_2 arbitrary near to s_0 such that the points $c(s_1), c(s_0), c(s_2)$ are collinear. Now, because the (smooth) curve c is bending continuously, there exist parametric values r_1, r_2 with $s_1 < r_1 < s_0 < r_2 < s_2$ such that the tangent vectors $\dot{c}(r_1)$ $\dot{c}(r_2)$ are both parallel to the line L. (The geometric situation is as in Fig.4 below.)

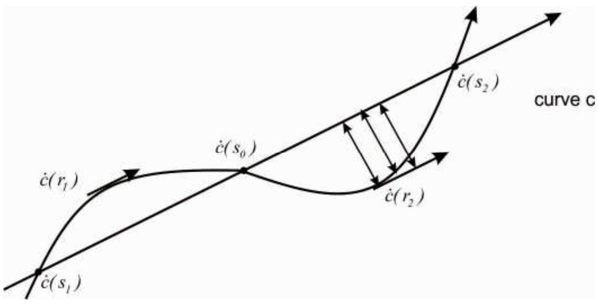


Fig. 4

Recall, $\dot{c}(r_1)$ and $\dot{c}(r_2)$ are both unit victors and therefore, their being parallel to the line L implies their equality: $\dot{c}(r_1) = \dot{c}(r_2)$ or equivalently put:

$$\frac{\dot{c}(r_2) - \dot{c}(r_1)}{r_2 - r_1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (*)$$

Recall, s_1 s_2 are arbitrarily near to s_0 ; we make $s_1; s_2$ both approach s_0 indefinitely. Then $r_1 \to s_0, r_2 \to s_0$ and therefore the equation $(*)$ in the limit becomes

$$\lim_{\substack{r_1 \to s_0 \\ r_2 \to s_0}} \frac{\dot{c}(r_1)(r_2) - \dot{c}(r_1)(r_1)}{r_2 - r_1} = 0$$

But, the above limit is $\ddot{c}(s_0)$, Thus $\ddot{c}(s_0) = 0$ Thus, we have arrived at a contradiction to the assumption $\ddot{c}(s_0) \neq 0$

Therefore, indeed, when s_1, s_2 are near enough to s_0 the three points $c(s_0), c(s_1), c(s_2)$ are non - collinear □

We consider the circle determined by the (Non - collinear) points $c(s_0), c(s_1), c(s_2)$; let it be denoted by $S(s_0, s_1, s_2)$ and its centre by $D(s_0, s_1, s_2)$.

We prove below that the circle $S(s_{0,}s_1,s_2)$takes a limiting position in the plane through $c(s_0)$ containing $t(s_0)$and $n(s_0)$. Clearly the limiting circle is the best curve reflecting the bending of the curve $c$at its point $c(s_0)$ (The circles $S(s_{0,}s_1,s_2)$ approximate c around the point $c(s_0)$ and the approximation improves as $s_1, s_2 \to s_0$.

It turns out that radius of this limiting circle is $\frac{1}{k(s_0)}$ .We prove this result in the following proposition

Proposition 3:

The circle $S(s_{0,}s_1,s_2)=S$ takes a limiting position in the plane through $c(s_0)$containing $t(s_0)$, $n(s_0)$and its radius is $\frac{1}{k(s_0)}$.

Proof: Let $D=D(s_0,s_1,s_2)$be the centre of the circle S. For a fixed pair s_1,s_2in I (near enough to s_0) we consider the function:

$$f : I \to \mathbb{R}$$

given by $f(s)=<c(s)-D,c(s)-D>$

Because the circle S passes through $C(s_0)$, $C(s_1)$ and $C(s_2)$, we get

$$f(s_o)=f(s_1)=f(s_2)$$

Applying mean value theorem of differential calculate to $f(s)-f(s_i)$, $i=0,1,2,$ we get r_1 r_2 in I with $s_1<r_1<s_0<r_2<s_2$ such that

$$\dot{f}(r_1)=\dot{f}(r_2) \text{ .... (*)}$$

Application of the same theorem to $\dot{f}$ with $(*)$ gives $r_3 \in I$ with $r_1<r_3<r_2$ such that

$$\ddot{f}(r_3)=0 \text{ .... (**)}$$

We take limit of $(*)$ and $(**)$ as $s_1, s_2 \to s_0$ and consequently $r_1, r_2, r_3 \to s_0$ This gives

$$\dot{f}(s_0) = 0 = \ddot{f}(s_0)$$

But, we have:

$$\begin{aligned}\dot{f}(s_0) &= \lim_{s \to s_0} \dot{f}(s) \\ &= \lim_{s \to s_0} 2 < \dot{c}(s); c(s) - D > \\ &= 2 < \dot{c}(s_0); c(s_0) - \lim_{x \to s} D >\end{aligned}$$

This gives: $< \dot{c}(s_0); c(s_0) - \lim_{x \to 0} D >= 0$ ……………………… (***)

Next differentiation of $f(s)$ twice gives:

$$\ddot{f}(s) = 2 < \ddot{c}(s), c(s) - D > +2 < \dot{c}(s), \dot{c}(s)$$

$$= 2 < \ddot{c}(s), c(s) - D > +2$$

Therefore:

$$\begin{aligned}0 &= \ddot{f}(s_0) \\ &= \lim_{s \to s_0} \{ 2 < \ddot{c}(s), c(s) - D > +2 \}\end{aligned}$$

This gives $< n(s_0), c(s_0) - \lim D > = -\dfrac{1}{k(s_0)}$ …………………. (****)

Above, we have been writing $\lim D$ for the limit

$$\lim_{\substack{s_1 \to s_0 \\ s_2 \to s_0}} D(s_0, s_1, s_2)$$

Thus, from $(***)$ we get:

(a) the point $\lim D$ lies on the line through $c(s_0)$ and going perpendicular to the vector $t(s_0)$ (equivalently put $\lim D$ is a point lying on the line through $c(s_0)$ and extending in the direction of $n(s_0)$

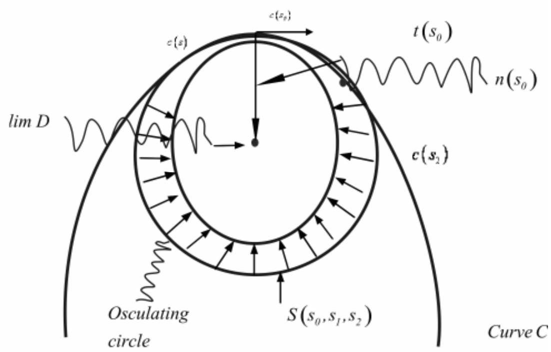
From $(****)$ we get:

(b) $\left\| c(s_0) - lim\, D \right\| = \frac{1}{k(s_0)}$.

The observations (a), and (b) above give:

$$lim\, D = c(s_0) + \frac{n(s_0)}{k(s_0)}.$$

Therefore the circle $S(s_0, s_1, s_2)$ indeed takes a limiting position, lying in the plane through $c(s_0)$ parallel to $t(s_0)$ *and* $n(s_0)$ in such a way that its centre is $c(s_o) + \frac{n(s_0)}{k(s_0)}$. See Fig. 5 below:



We call the limiting circle the <u>osculating circle</u> of the given curve c at its point $c(s_0)$.

Here is some more terminology.

- The plane through $c(s)$ spanned by $t(s)$ and $n(s)$ is the <u>osculating plane</u> of c at its point $c(s)$.
- The plane through $c(s)$ spanned by $n(s)$ and $b(s)$ is the normal plane to c at its point $c(s)$.
- the plane through $c(s)$ spanned by $t(s)$ and $b(s)$ is the plane rectifying plane of c at $C(s)$.

Thus, we have obtained above that to within second order of approximation, the curve seems to live within its osculating plane at $c(s)$ and is approximately a circle - the osculating circle at $c(s)$ - and having radius $\frac{1}{k(s)}$.

Note one more geometric fact: the binormal maintains its perpendicularity to the osculating plane as we move along the curve. Therefore the map $s \to b(s)$ describes the movement of the binormal as its foot traces the curve c while the foot moves forward, the vector $b(s)$ rotates about the tangent line as its axis of rotation; in other words it describes the twist in the curve We are interested in the rate of twist- the rotation of the vector $b(s)$. we denote the rate of rotation of $b(s)$ by $\Im(s)$ and call it the torsion of the curve c at the point $c(s)$.

4.3 THE SERRET - FRENET FORMULAE :

In the last section, we singled out a class of regular curves which we called the Frenet curves and associated with such a $c : I \to \mathbb{R}^3$ the geometric objects namely

(i) a moving orthonormal frame

$$s \to (t(s), n(s), b(s)), \quad s \in I$$

and

ii) The two functions:

$k : I \to [0, \infty)$

$c : I \to \mathbb{R}$

describing the bending and twisting of the curves. We now derive differential equations in the vector fields $s \to t(s), s \to n(s)$, $s \to b(s)$, which rolate all the quantities described in (i) and (ii) above.

We already have:

$$\dot{t}(s) = k(s)n(s), \quad s \in I$$

Next, we have $\|n(s)\| \equiv 1$ i.e. $< \eta(s), n(s) > \equiv 1$. Differentiating this identity we get.

$$< \dot{n}(s), n(s) > \equiv 0$$

Therefore the vector $\dot{n}(s)$ is expressible as a linear combination of $(t(s), b(s))$ (Here we are using the facts that $(t(s), n(s), b(s))$ is an orthonormal vector basis and the vector $\dot{\eta}(s)$ has no component along $n(s)$ as derived above) We get.

$n(s) = \alpha(s)t(s)+\beta(s)b(s)$ ……………………… (*)

for some smooth functions $\alpha, \beta : I \to \mathbb{R};$ we find these functions. Taking inner product of the identity $(*)$ with $t(s)$, we get

$$<n(s),t(s)=>\alpha(s)<t(s),t(s)+\beta(s)<t(s),b(s)>$$
$$= \alpha(s).I+\beta(s).0$$
$$i.e<n(s),t(s)\equiv\alpha(s) \text{ ……………………………. (**)}$$

On the other hand, differentiating the identity $<n(s),t(s)>\equiv 0$ gives $<\dot{\eta}(s),t(s)>+<\eta(s),\dot{t}(s)>\equiv 0$

Therefore $<\dot{\eta}(s),t(s)>+<\eta(s),k(s)n(s)>\equiv 0$ *i.e*
$<n(s),t(s)>+<k(s)n(s),\ n(s)>\equiv 0$
and therefore

$$<n(s),t(s)>=-k(s) \text{ ……………………….. (***)}$$

Now,
$(**)$ and $(***)$ give $<n(s),t(s)>=-k(s)$. Finally combining this identity with $(*)$ yields :

$$n(s)=-k(s)t(s)+\beta(s)b(s)$$

Recall, we have introduced the function $t(s)$ as the function describing the rotation of the unit vector $b(s)$ about the vector $t(s)$ as its axis of rotation. Consequently we have the equation :
$\dot{b}(s)=-t(s)n(s)$
(the negative sign being introduced as a rotational convenience). On the other hand, differentiation of the identity $<n(s),b(s)>\equiv 0$ gives
$<\dot{n}(s)b(s)>=+<\dot{n}(s),\dot{b}(s)>=0$

i.e. $<\dot{n}(s)b(s)>=+<n(s),\alpha(s)n(s)>$ ……………………… (****)
i.e $<\dot{n}(s)b(s)>=t(s)$ ……………………………… (****)

Again taking inner product of the equation
$\dot{n}(s)=-k(s)t(s)+t(s)b(s)$

with $b(s)$ gives $\langle n(s), b(s)\rangle = p(s)$; thus by (****) above we get $\beta(s) = \tau(s)$. This gives :

$$\dot{n}(s) = -k(s)t(s) + t(s)b(s)$$

Thus, we have obtained the triple of ODE

$$\dot{t}(s) = k(s)n(s)$$

$$n(s) = -k(s)t(s) + t(s)b(s)$$

$$\dot{b}(s) = -t(s)n(s)$$

These equations are often written in the matrix form

$$\frac{d}{ds}\begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix} = \begin{bmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & \alpha(s) \\ 0 & t(s) & 0 \end{bmatrix}\begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}$$

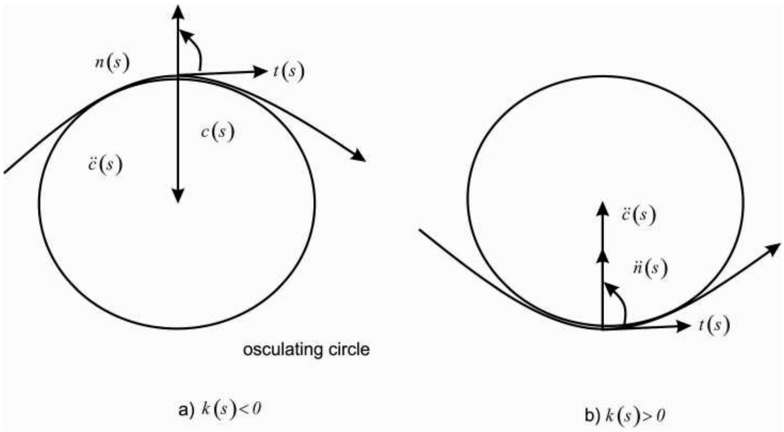
These equations are called the <u>Serret</u> - <u>Frenet equations</u> of a (Frenet) curve.

Thus associated with a Frenet curve is a pair of scalar valued functions, defined along the curve namely the curvature k and the torsion t. In Chapter 6: we will prove that these two functions together determine the curve uniquely to within a rigid motion of the curve.

4.4 SIGNED CURVATURE FOR A PLAIN CURVE:

For a curve living in a plain the binormal $b(s)$ remains a constant unit vector, namely one of the two unit vectors which are perpendicular to the plain in which the curve is situated Consequently the third of the Serret - Frenet equations gives $t \equiv 0$. On the other hand we can make use of the standard (counter clockwise) orientation of the plane to refine the (blunt) non- negative curvature function and make it a function taking both non - negative/ negative values. We ascribe a signature to $k(s)$ as follows We replace the principal normal $n(s)$ by the vector $\hat{n}(s)$ (say) which is obtained by rotating $t(s)$(about its foot $c(s)$ through $\frac{\pi}{2}$ the rotation being anticlockwise (It is here that we are using the standard orientation of $\mathbb{R}^2$) Now, we obtain the signed curvature $k(s)$ of c at $c(s)$ by using the defining equation:

$$\ddot{c}(s) = k(s)n(s)$$



For example, the curve $c(s)=(s,s^2)\quad s\in\mathbb{R}$ has positive curvature while the curve $\hat{c}(s)=(s,-s^2),\ s\in\mathbb{R}$ has negative curvature.

In passing, note the following simple fact : Identify the plane with the complex plane $\mathbb{C}$. Then rotation of vectors anti - clockwise through the angle $\frac{\pi}{2}$ corresponds to multiplication of the vector (as a complex number) by the imaginary unit i. This consideration leads to the definition of the signed curvature:

$$\frac{d}{dx}t(s)=k(s)\,i\,t(s).$$

Exercises :

1) Obtain principle triade of the curves given below at the indicated points.

a) $c(s)=\left(s,s^2,s^3\right)$ at $(1,1,1)$

b) $c(s)=(4\cos s,4\sin s,3s)$ at $(4,0,0)$

c) $c(s)=\left(e^{\delta},4s,e^{2s}\right)$ at $(1,0,1)$

2) Reparametrize the following curves so as to get unit speed curves.

a) $c(s)=\left(s^2,2s^2,3s^3\right)s>0$ at $(1,1,1)$

b) $c(s)=\left(e^s c\,\mathrm{os}\,2s,e^s\sin 2s,0\right)\ s\in\mathbb{R}$

c) $c(s)=(4,2s,3s),\ s\in\mathbb{R}$

3) Verify that the curve $c:\mathbb{R}\to\mathbb{R}^3$ given by $c(s)=\left(\frac{3}{10}\cos 2s, \frac{2}{5}\cos 2s, \frac{1}{2}\sin 2s-1\right), s\in\mathbb{R}$ is a unit speed curve and obtain the curvature and torsion function of it.
4) Let $L:\mathbb{R}^3\to\mathbb{R}^3$ be an isometry of $\mathbb{R}^3$ and let $c:I\to\mathbb{R}^3$ be a Frenet curve.

Prove :

i) Loc is also a Frenet curve
ii) both, *c*, Loc have the same curvature and torsion functions.

5) Suppose the curve $c:\mathbb{R}\to\mathbb{R}^3$ has non-vanishing curvature. Prove that if all osculating planes of c pass through a fixed point, then *c* is a plane curve.

6) Calculate the signed curvature function the curves :

i) $c(s)=(s,s^2), s>O$
ii) $c(s)=(s,-s^2), s>O$

❖❖❖❖

5

CURVATURE AND TORSION

Unit Structure :

In the last chapter we considered smooth Frenet curves and defined the curvature and torsion functions of such curves. In defining these terms, we used the unit speed kind of paramentrization of the curves in an essential way (for example, we were using the unit length property of the tangent vector $\dot{c}(s)$ in getting the perpendicularity $t(s) \perp n(s)$.) However, curves are seldom in the unit speed parametrized form. We therefore need develop equations to calculate these quantities applicable even when the curves of our interest are arbitrarily parametrized (regular Frenet) curves.

In this chapter, we develop the desired formulae for the Frenet curves and then we proceed to study the geometry of such curves in terms of the curvature and tersion functions.

5.1 CURVATURE AND TORSION FUNCTIONS :

Let $c : I \to \mathbb{R}^3$ be an arbitrarily parametrized regular Frenet curve; its parameter being denoted by $r \in I$. Besides r, we need consider the natural are-length (and hence unit speed) parametrization for a while. Thus, we consider $\tilde{c} : J \to \mathbb{R}^3$ the are-length parameter manifestation of c, the are-length parameter, as usual, being denoted by s and the reparametrization map being $\alpha : I \to J : \alpha(r) = s$ where α is strictly monotonic increasing bijetive map between the indicated intervals. For the sake of convenience we will denote differentiation w.r.t. r by a dot "." e.g. $\dot{c}(r)\left(= \dfrac{d\,c(r)}{dr}\right)$ while we will use the derivative notation $\dfrac{d\,c(s)}{ds}$ for the are length parametrization. Now the relations between the two parametrizations are :

$$c(r) = \tilde{c}(\alpha(r))$$
$$= \tilde{c}(s(r))$$

Where we are writing $S(r)$ for $\alpha(r)$

In terms of these notations, we have :

$$c(r) = \tilde{c}(s(r))$$
$$\dot{c}(r) = \frac{d}{ds}\tilde{c}(s(r))s(r)$$
$$= t(r)\cdot s(r) \text{ .... (*)}$$
$$\ddot{c}(r) = \frac{d}{ds}t(s)\dot{s}(r)^2 + \ddot{s}(r)t(s)$$
$$= k(s)\dot{s}(r)^2 n(s) + \ddot{s}(r)t(s) \text{...................... (**)}$$
$$\dddot{c}(r) = \frac{d}{ds}k(s)\dot{s}(r)^3 n(s) + k(s)\dot{s}(r)^3 \dot{n}(s)$$
$$+2\dot{s}(r)\ddot{s}(r)k(s)n(s) + \dddot{s}(r)t(s) + \ddot{s}(r)\dot{s}(r)k(s)n(s)$$
$$= \frac{d}{ds}k(s)\dot{s}(r)^3 n(s) + k(s)\dot{s}(r)\left[k(s)n(s) + 7(s)b(s)\right]$$
$$+2\dot{S}(r)\ddot{s}(r)k(s)n(s) + \dddot{s}(r)t(s) + \ddot{s}(r)\dot{s}(r)k(s)n(s)$$

This $\dddot{c}(r) = \dddot{s}(r)^3 t(s) + \left[\dfrac{dk(s)}{ds}\dot{s}(r)^3 - k(s)\cdot\dot{s}(r)^3 3\dot{s}(r)\ddot{s}k(s)\right]$

$n(s) + k(s)\cdot t(s)\dot{S}(r)b(s)$ .... (***)

Farming cross product of (*) and (**) we get :

$$\dot{c}(r)\times\ddot{c}(r) = \dot{s}(r)^3 k(s)t(s)\times n(s) + \ddot{s}(r)^2 t(s)\times t(s)$$
$$= \dot{s}(r)^3 k(s)b(s) + 0 \text{ .... (4*)}$$

And therefore $\left\|\dot{c}(r)\times\ddot{c}(r)\right\| = \left|\dot{s}(r)\right|^3 k(s(r))\left\|b(s)\right\|$

$$= \left|\dot{s}(r)\right|^3 k(s)\cdot 1$$

Which gives $k(r)\left(=k(s(r))\right) = \dfrac{\left\|\dot{c}(r)\times\ddot{c}(r)\right\|}{\left\|\dot{c}(r)\right\|^3}$

Next, taking inner product of (***) with (4*) we get :

$$\left\langle \dot{c}(r)\times\ddot{c}(r),\dddot{c}(r)\right\rangle = k^2(r)t(s)s(r)^6$$

$$= \frac{\left\|\dot{c}(r)\times\ddot{c}(r)\right\|}{\left\|\dot{c}(r)\right\|^6} t(s)\left\|\dot{c}(r)\right\|^6$$

Using the above obtained expression for $k(r)\left(=k(s(r))\right)$. Thus we get : $t(r) = \dfrac{\left\langle \dot{c}(r)\times\ddot{c}(r),\dddot{c}(r)\right\rangle}{\left\|\dot{c}(r)\times\ddot{c}(r)\right\|^2}$

(Above we have adapted the notation $k(r)$ for $\left(=k(s(r))\right)$.

Thus we have proved the following :

Proposition 1 : The curvature and torsion functions.

$k : I \to [0,\infty), t : I \to \mathbb{R}$ for a regular Frenet curve $c : I \to \mathbb{R}^3$ are given by :

$$k(r) = \frac{\left\|\dot{c}(r)\times\ddot{c}(r)\right\|}{\left\|\dot{c}(r)\right\|^3}$$

And $\quad t(r) = \dfrac{\left\langle \dot{c}(r)\times\ddot{c}(r),\dddot{c}(r)\right\rangle}{\left\|\dot{c}(r)\times\ddot{c}(r)\right\|^2}$

$$= \frac{det\left\langle \dot{c}(r),\ddot{c}(r),\dddot{c}(r)\right\rangle}{\left\|\dot{c}(r)\times\ddot{c}(r)\right\|^2}$$

(In the above determinant notation : $det\left\langle \dot{c}(t),\ddot{c}(t),\dddot{c}(t)\right\rangle$ the vectors $\dot{c}(r),\ddot{c}(r),\dddot{c}(r)$ are in the columns of the matrix $\left[\dot{c}(r),\ddot{c}(r),\dddot{c}(r)\right]$.

An Illustrative Example : Obtain the curvature and torsion functions for the circular helix :

$c(r)=(a\cos r, a\sin r, br); r\in\mathbb{R}$ a and b being both non-zero constants.

Solution : We have :

$$\dot{c}(r)=(-a\sin r, a\cos r, b)$$
$$\ddot{c}(r)=(-a\cos r, -a\sin r, o)$$

And $\dddot{c}(r)=(a\sin r, -a\cos r, o)$

Therefore $\dot{c}(r)\times\ddot{c}(r)=(ab\sin r, -ab\cos r, a^2), \langle\dot{c}(r), \ddot{c}(r), \dddot{c}(r)\rangle$

$$= a^2b\sin^2 r + a^2b\cos^2 r + 0.a^2$$
$$= a^2b$$

$$\left\|\dot{c}(r)\times\ddot{c}(r)\right\| = \sqrt{(a^2+1)b^2}$$
$$= b\sqrt{(a^2+1)}$$

And $\left\|\dot{c}(r)\right\| = \sqrt{(a^2+b^2)}$. This gives :

$$k(r)=\frac{b\sqrt{(a^2+1)}}{(a^2+b^2)^{3/2}} \text{ and } t(r)=\frac{a^2b\sqrt{(+1)}}{b^2(a^2+1)}=\frac{a^2}{b^2(a^2+1)}\square$$

Here is another Illustrate Example :

Calculate k, t of the space curve :
$c(r)=(e^r, r, r^2), r\in\mathbb{R}$.

Solution : We have :

$$c(r)=(e^r, r, r^2)$$

And therefore $\dot{c}(r)=(e^r, 1, 2r)$

$$\ddot{c}(r)=(e^r, 0, 2)$$

And $\dddot{c}(r)=(e^r, 0, 0)$

This gives : $\dot{c}(r)\times\ddot{c}(r) = det\begin{bmatrix} e_1 & e_2 & e_3 \\ e^r & 1 & 2r \\ e^r & 0 & 2 \end{bmatrix}$

$$= \left(2, e^r(2r-1), -2e^r\right)$$

And therefore $\left\|\dot{c}(r)\right\| = \sqrt{\left(e^{2r}+4r^2+1\right)}$

$$\left\|\dot{c}(r)\times\ddot{c}(r)\right\| = \sqrt{\left[e^{2r}\left(4r^2-8r+5\right)+4\right]}$$

And $\left\langle \dot{c}(r), \ddot{c}(r), \dddot{c}(r)\right\rangle = e^{2r}$

These equalities give :

$$k(r) = \frac{\left\|\dot{c}(r)\times\ddot{c}(r)\right\|}{\left\|\dot{c}(r)\right\|^3}$$

$$= \frac{\sqrt{\left[e^{2r}\left(4r^2-8r+5\right)+4\right]}}{\left(e^{2r}+4r^2+1\right)^{3/2}}$$

$$t(r) = \frac{\left\langle \dot{c}(r)\times\ddot{c}(r), \dddot{c}(r)\right\rangle}{\left\|\dot{c}(r)\times\ddot{c}(r)\right\|^3}$$

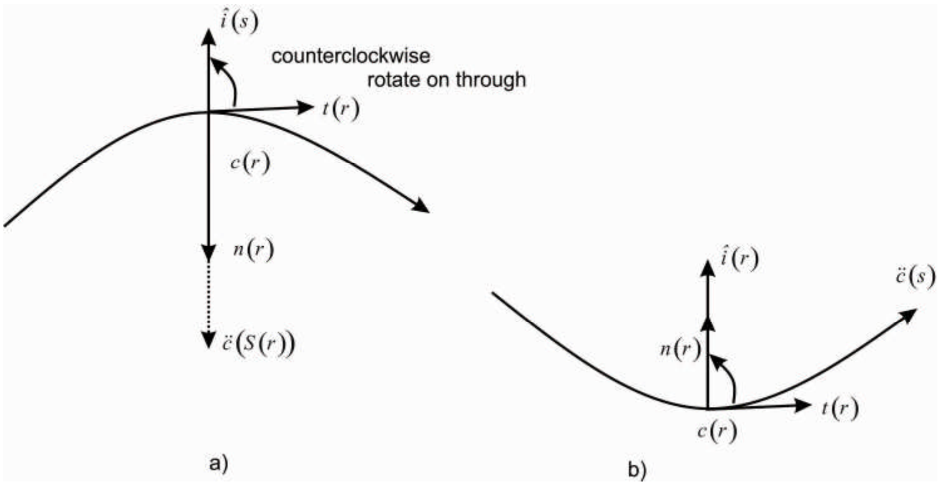
$$= \frac{e^{2r}}{\left(e^{2r}+4r^2+1\right)}$$

5.2 SIGNED CURVATURE OF A PLANE CURVE :

The concept of signed curvature of a plane curve was introduced in Chapter 4. Here we tarry a while to explain a little more about the underlying heuristics formula for the same of a planar, regular but arbitrarily paramethrized Frenet curve.

Thus, let $c : I \to \mathbb{R}^2$ be a Frenet curve, its parameter being denoted by r. we consider its unit speed parametrization also, the associated unit speed parameter being (as usual) $S = S(r)$. In order to employ as few notations as possible, we write both the parametrizations of the curve by the same symbol $c : c(r) = c(r)$ in the sense $c(r)=\tilde{c}(S(r))=(\tilde{c}(S))$ being the unit speed version of $c(r)$ and $S \mapsto S(r)$ being the unit speed parametrization map. With this notational understanding in mind, we write $t(r)$ for $t(s(r))$ $n(r)$ for $n(s(r))$ and so on.

Now returning to the signed curvature we recall that we were considering rotation of the unit tangent $t(r)$ about the point $c(r)$ through the angle $\frac{\pi}{2}$ and thus getting $\hat{t}(r)$. See the figure below:



Thus, at the point $c(r)$ of the curve c, we have the two unit vectors $n(r)$ and $\hat{t}(r)$. Clearly we have either $\hat{t}(r)=n(r)$ as indicated in part (a) of the figure or $\hat{t}(r)=n(r)$ as shown in part (b).

Now, let us note the difference between the earlier (rather blunt) case of the non-negative curvature $k(s)$ and that of the present signed curvature $\hat{k}(s)$.

In defining $k(s)$ we compared $\ddot{c}(s)$ with the principal normal $n(s)$:

$\ddot{c}(s)=k(s)n(s)$ .... (*)

While introducing the signed curvature $\hat{k}(s)$ we are comparing $\ddot{c}(s)$ with $\hat{t}(s)$:

$\ddot{c}(s)=\hat{k}(s)\,\hat{t}(s)$ .... (**)

thus arriving at the definition of signed curvature $\hat{k}(s)$ of c at the point $c(s)$. Consequently in view of the above observation $n(s) = \pm\hat{t}(s)$ (as illustrated in cases (a), (b) above) the equations (*) and (**) give two possibilities : $\hat{k}(s) = \pm k(s)$.

Now a few words about the notations : As mentioned above, we are desirous of using as few notations as possible. Above, we introduced the notation $\hat{k}(s)$ for the signed curvature besides the earlier $k(s)$. However, in a plane we will be considering the signed curvature only and as such the two notations : $k(s)$ and $\hat{k}(s)$ are superflows. We therefore abandon $\hat{k}(s)$ and revert to the old notation $k(s)$ through we are dealing with the signed curvature. Thus from now-onwards $k(s)$ stands for the signed curvature of a planar curve while in $\mathbb{R}^3$ it is the old non-negative curvature. (Also, we continue with the practice of denoting by a dot : "." differentiation with respect to the given parameter, while $\frac{d}{ds}$ is the differentiation with respect to the natural are length s of the curve). Now we write the vector equation $\frac{d^2c(s)}{ds^w} = k(s)\hat{t}(s)$ in terms of its components.

$$\begin{bmatrix} \frac{d^2c(s)}{ds^2} \\ \frac{d^2c_2(s)}{ds^2} \end{bmatrix} = k(s)\begin{bmatrix} -t_2(s) \\ \\ t_1(s) \end{bmatrix}$$

$$= k(s)\begin{bmatrix} \frac{d_2c_2}{ds}(s) \\ \frac{d}{ds}c_1(s) \end{bmatrix}$$

Equivalently, put, we have the pair

$$\left.\begin{aligned} \frac{d^2c_1(s)}{ds_2} &= -k(s)\frac{dc_2(s)}{ds} \\ \frac{d^2c_2(s)}{ds_2} &= k(s)\frac{dc_1(s)}{ds} \end{aligned}\right\} \ldots\ldots\odot$$

The pair $\odot$ expressed in terms of the given parameters takes the form :

a) $$\ddot{c}_1(r)\left(\frac{dr}{ds}\right)^2+\dot{c}_1(r)\frac{d^2r}{ds^2}=-k(r)\dot{c}_2(r)\frac{dr}{ds},$$

b) $$\ddot{c}_2(r)\left(\frac{dr}{ds}\right)^2+\dot{c}_2(r)\frac{d^2r}{ds^2}=k(r)\dot{c}_2(r)\frac{dr}{ds}.$$

Multiplying (a) by $\dot{c}_2(r)$ and (b) by $\dot{c}_1(r)$ gives :

c) $$\ddot{c}_1(r)\left(\frac{dr}{ds}\right)^2+\dot{c}_2(r)+\dot{c}_1(r)\dot{c}_2(r)\frac{d^2r}{ds^2}=-k(r)\dot{c}_2(r)^2\frac{dr}{ds}$$

d) $$\ddot{c}_2(r)\left(\frac{dr}{ds}\right)^2+\dot{c}_1(r)+\dot{c}_1(r)\dot{c}_2(r)\frac{d^2r}{ds^2}=k(r)\dot{c}_1(r)^2\frac{dr}{ds}.$$

Subtract of (c) from (d) gives :

$$\left[\dot{c}_1(r)\ddot{c}_2(r)-\dot{c}_2(r)\ddot{c}_1(r)\right]\left(\frac{dr}{ds}\right)^2=k(r)\left[\dot{c}_1(r)^2+\dot{c}_2(r)^2\right]\frac{dr}{ds}$$ and therefore :

$$\left[\dot{c}_1(r)\ddot{c}_2(r)-\dot{c}_2(r)\ddot{c}_1(r)\right]\left(\frac{dr}{ds}\right)=k(r)\left[\dot{c}_1(r)^2+\dot{c}_2(r)^2\right];$$ which in turn gives :

$$\frac{det\left[\dot{c}_1(r),\ddot{c}_2(r)\right]}{\left(\frac{dr}{ds}\right)^3}=k(r)$$

This is the desired formula for the signed curvature $k(r)$ of the planer curve.

Note one more aspect of the signed curvature namely the general non-negative curvature given by $k(r)=\dfrac{\left\|\dot{c}(r)\times\ddot{c}(r)\right\|}{\left\|\dot{c}(r)\right\|^3}$ involves differentiation of the curve only but the singed curvature

$k(r)=\frac{det\left[\dot{c}(r),\ddot{c}(r)\right]}{\left\|\dot{c}(r)\right\|^3}$ is not only terms of the derivatives $\dot{c}(r),\ddot{c}(r)$ of c, but it takes into consideration the anti-clockwise orientation of the ambiant space $\mathbb{R}^2$ in a crucial way! For defining the signed curvature, we were considering the anticlockwise rotation of the unit tangent $t(r)$. Had we chosen to rotate it in the clockwise manner, the curvature could have changed its sign!

Let us consider two simple examples of curves and calculate their signed curvature functions.

(I) $c:\mathbb{R}\to\mathbb{R}^2$ is given by the graph of the cosine curve : $c(r)=(r,cos\,r)$.

Then we have (i) $\dot{c}(r)=(1,sin\,r)$

(ii) $\left\|\dot{c}(r)\right\|=\sqrt{1+sin^2 r}$

(iii) $\ddot{c}(r)=(0,-cos\,r)$

Now $det\left[\dot{c}(r),\ddot{c}(r)\right]=det\begin{bmatrix}1 & 0\\ -sin\,r & -cos\,r\end{bmatrix}=-cos\,r$ and therefore the (signed) curvature of this curve is $k(r)=\frac{-cos\,r}{\left(1+sin^2 r\right)^{3/2}}$.

II) $c:\mathbb{R}\to\mathbb{R}^2$ is given by $c(r)=\left(e^r cos\,r,e^r sin\,r\right)$.

Then we have (i) $\dot{c}(r)=\left(e^r(cos\,r-sin\,r),e^r(sin\,r+cos\,r)\right)$

(ii) $\left\|\dot{c}(r)\right\|=\sqrt{2e^r}$

(iii) $\ddot{c}(r)=\left(2e^r sin\,r,2e^r cos\,r\right)$

Therefore

$$det\left[\dot{c}(r),\ddot{c}(r)\right]=det\begin{bmatrix}e^r(cos\,r-sin\,r) & -2e^r sin\,r\\ e^r(cos\,r+sin\,r) & 2e^r cos\,r\end{bmatrix}=2e^{2r}.$$

Therefore $k(r)=\frac{2e^{2r}}{2\sqrt{2}e^{3r}}=\frac{1}{\sqrt{2}.e^r}$ .... $\square$

5.3 ELEMENTARY PROPERTIES OF CURVATURE AND TORSION

Here is another description of the curvature of a space curve: The curvature of a curve at a point of it measures angle variation of the tangent vector per unit length of the are. To be more precise, we have the following :

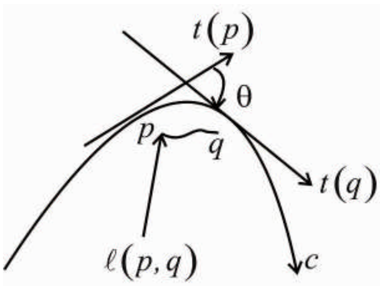
Proposition 2 : The curvature $k(p)$ of *c* at a point *p* of it is given by

$k(p) = \lim\limits_{q \to p} \dfrac{\theta}{\ell(p,a)}$ where *q* is a point on *c* with $q \neq p, \ell(p,q)$ is the length of the are of c between its points *p*, *q* and θ is the angle between the tangents at *p* and *q*.

Proof : The angle θ is obtained by using the formula for the angle in an isoscelese triangle :

$$sin\left(\frac{\theta}{2}\right) = \frac{\left\| t(q) - t(p) \right\|}{2}$$

Therefore $\lim\limits_{q \to p} \dfrac{\theta}{\ell(p,a)} = \lim\limits_{\ell \to 0} \dfrac{\frac{\theta}{2}}{\sin Q/2} \lim\limits_{\ell \to 0} \dfrac{2 \sin \frac{\theta}{2}}{\ell(p,q)}$



$$= 1 \cdot \lim_{\ell \to 0} \frac{\left\| t(p+\ell) - t(p) \right\|}{\ell}$$

$$= \lim_{\ell \to 0} \frac{\left\| \dot{c}(p+\ell) - \dot{c}(p) \right\|}{\ell}$$

$$= \left\| \ddot{c}(p) \right\|$$

$$= k(p)$$

Definition 5.1: A regular curve having the property that the tangent lines at all points of which make a constant angle with a fixed direction is called a <u>slope line</u>.

Proposition 3 : (Lancert's Theorem) a Frennet Curve is a slope line if and only if the quotient $\frac{t(s)}{k(s)}$ is constant.

Proof : First, suppose that there exists a constant unit vector *e*(the direction) such that $\langle t(s),e\rangle$ is the same for all s. Then we have

$$\frac{d}{ds}\langle \bar{t}(s),e\rangle = 0 .$$

i.e. $\langle k(s),n(s),e\rangle = 0$ and $k(s)\neq 0$ for all s implies $\langle n(s),e\rangle \equiv 0$. Differentiating this equation we get $\langle -k(s),t(s)+t(s)b(s),e\rangle = 0$; which gives : $\frac{t(s)}{k(s)} = \frac{\langle b(s),e\rangle}{\langle k(s),e\rangle}$ .... (*)

Now, the above observation that $n(s)\perp e$ for all s implies that the vector e remains in the rectifying planes. Combining this observation with the assumption that $\langle t(s),e\rangle =$constant implies that $\langle b(s),e\rangle$ also is constant. Now (*) above gives constancy of the function $r \rightarrow \frac{t(s)}{k(s)}$.

Conversely suppose, $\frac{t(s)}{k(s)}$ is independent of s and consider the vector $a(s) = b(s) + \frac{t(s)}{k(s)}t(s)$. Differentiation of the functions $s \mapsto a(s)$ gives.

$$\frac{d\dot{v}(s)}{ds} = -t(s)n(s) + \frac{t(s)}{k(s)}k(s)n(s)$$

$$= 0$$

and thus $a(s)\equiv a$ for a constant vector. The constancy of a and that of $\frac{t(s)}{k(s)}$ now implies that the tangent vectors make constant angle with the vector a and therefore, the curve is a slope line.

Proposition 4 : A Frennet curve $c : J \rightarrow \mathbb{R}^3$ lies on a sphere of radius $R > 0$ if and only if its curvature and torsion functions k,t, satisfy the identity

$$\frac{1}{k^2(s)}+\left(\frac{k(s)}{k(s)^2\,t(s)}\right)^2\equiv R^2$$

Proof : First, suppose that the curve *c* lies on the sphere of radius $\mathbb{R}$ centred at 0. Then we have : $\langle c(s)-0,c(s)-0\rangle\equiv R^2$.

Differentiating this identity w.r.t. s, we get $2\langle t(s),c(s)\rangle\equiv 0$.

The above identity implies that $c(s)$ lies in the normal plane :

$c(s)=\alpha(s)\eta(s)+\beta(s)t(s)$ and $\|c(s)\|^2=R^2$ gives $\alpha(s)^2+\beta(s)^2=R^2$.

Differentiating the identity $\langle t(s),c(s)\rangle\equiv 0$ gives

$k(s)\langle n(s),c(s)\rangle+\langle t(s),t(s)\rangle\equiv 0$

i.e. $k(s)\langle n(s),\alpha(s)\eta(s)+\beta(s)b(s)\rangle+1\equiv 0$ which gives $\alpha(s)=-\dfrac{1}{k(s)}$ .... (*)

Next, differentiating the identity $k(s)\langle n(s),c(s)\rangle+1\equiv 0$ gives $k^1(s)\langle n(s),c(s)\rangle+k(s)\langle -k(s)t(s)+t(s)b(s),c(s)\rangle\equiv 0$. Which gives $k^1(s)\alpha(s)+k(s)t(s),\beta(s)=0$ and therefore we get $\beta(s)=\dfrac{k^1(s)}{k(s)^2\,t(s)}$ .... (**)

These values of $\alpha(s),\beta(s)$ substituted in the equation $\alpha(s)^2+\beta(s)^2=\mathbb{R}^2$ gives :

$$\frac{1}{k(s)^2}+\left(\frac{k^1(s)}{k^2(s)t(s)}\right)^2=R^2.$$

Conversely, suppose the above equation is satisfied. Differentiating it, we get :

$$\frac{-xk^1(s)}{k(s)^3}+\frac{zk^1(s)}{k^2(s)\cdot\Im(s)}\cdot\frac{d}{ds}\left(\frac{k^1(s)}{k^2(s)\Im(s)}\right)\equiv 0.$$

This gives $\dfrac{t(s)}{k(s)}=\dfrac{d}{ds}\left(\dfrac{k^1(s)}{k^2(s)t(s)}\right)$.

Next, consider the vector $a(s)=c(s)+\dfrac{\eta(s)}{k(s)}-\dfrac{k^1(s)}{k^2(s)t(s)}b(s)$.

Differenting it, we get :

$$\dot{a}(s)=t(s)-\frac{\dot{k}(s)n(s)}{k^2(s)}+\frac{(-k(s)t(s)+t(s)b(s))}{k(s)}$$

$$-\frac{d}{ds}\left(\frac{\dot{k}(s)}{k(s)^2\,t(s)}\right)b(s)+\frac{\dot{k}(s)t(s)n(s)}{k(s)^2\,t(s)}$$

$$=0$$

Thus, $\dfrac{d}{ds}c(s)=0$ and therefore $a(s)$ is a constant vector, say $a(s)\equiv a$ and then we get :

$$a-c(s)=\frac{\eta(s)}{k(s)}-\frac{k(s)}{k(s)^2\,\Im(s)}\cdot b(s).$$

This gives : $\|c(s)-a\|^2=\dfrac{1}{k(s)^2}+\left(\dfrac{\dot{k}(s)}{k^2(s)\Im(s)}\right)^2$.

But, by assumption, we have $\dfrac{1}{k(s)^2}+\left(\dfrac{\dot{k}(s)}{k(s)^2\,\Im(s)}\right)^2=R^2$

therefore, we get $\|c(s)-a\|^2=R^2$ i.e. the curve lies on the sphere centred at a and having radius R.

Proposition 5 : Let c be a closed plane curve.

Then the integral $\dfrac{1}{2\pi}\int_C k(s)\,ds$ is an integer.

Proof : We identify $\mathbb{R}^2$ with $\mathbb{C}$.

Also, we recall an elementary result of complex analysis. For any $x\in\mathbb{C}, e^z=1$ if and only if $z=2\pi im$ for some $m\in\mathbb{Z}$.

Define $f:[0,L]\to\mathbb{C}\setminus\{0\}$ by putting

$$f(s) = exp\left(\int_0^s k(r)\,dr\right); s \in [0\ L]$$

(Here L is the length of the curve c)

Also, we consider the map $g : [0,L] \to \mathbb{C}$ given by $g(s) = \dfrac{t(s)}{f(s)}, s \in [0,L]$. Then we have,

$$\dot{g}(s) = \left[\dot{t}(s) f(s) - f(s) t(s)\right] \Big/ \left(f(s)\right)^2$$

Then we get

$$\dot{g}(s) = \frac{\dot{t}(s) f(s) - \dot{f}(s) t(s)}{f(s)^2}$$

$$= \frac{ik(s)t(s) f(s) - ik(s) f(s) t(s)}{f(s)^2}$$

(Above we are dealing with the signed curvature of c and therefore $f(s) = k(s)\hat{t}(s) = k(s) it(s) = ik(s)t(s) = 0$.

Therefore g is a constant function. In particular, $g(0) = g(L)$ that is : $\dfrac{t(0)}{f(0)} = \dfrac{t(L)}{f(L)}$. Now because c is a closed curve, we have $t(0) = t(L)$ which in view of the last equality gives $f(0) = f(L)$. But $f(0) = 1$ and therefore we get $f(L) = 1$ i.e. $exp\left(i\int_0^2 k(r)\,dr\right) = 1$. Therefore, by the above quoted result, we get $i\int_O^L k(r)\,dr = 2\pi im$ for some integer m. This gives

$$\frac{1}{2\pi}\int_0^L k(r)\,dr = m \in \mathbb{Z}.$$

Lastly, we prove the following result (which is of considerable technical importance in geometry / analysis).

Lemma : Let $Q : [a\,b] \to \mathbb{R}$ be a continuously differentiable function. Suppose the function $f : [a,b] \to \mathbb{C}$ given by $f(s) = exp(iQ(s)) s \in [a,b]$ satisfies : $f(a) = 1$ and $f(b) = -1$. Then $\int_a^b \left|\dot{f}(r)\right| dr \geq \pi$ and the inequality becomes equality if and only if it is monotonic and $\left|Q(b) - Q(a) = \pi\right|$.

Proof : We have : $\dot{f}(s) = iQ(s)\,f(s)$ and therefore :

$$\begin{aligned}\int_a^b \left|\dot{f}(s)\right| ds &= \int_a^b \left|\dot{Q}(s)\right| \left\| f(s) \right| ds \right| \\ &\geq \left| \int_a^b \dot{Q}(s)\, ds \right| \\ &= \left| Q(b) - Q(a) \right|\end{aligned}$$

Because $f(a) = +1, f(b) = -1$, there exist integers ℓ, m such that $Q(a) = 2\pi\ell$ and $Q(b) = (2\pi m + \pi) = (2m+1)\pi$.

Therefore $\int_a^b \left|\dot{f}(r)\right| dr \geq \pi \left| 1 + 2(l - m) \right|$

$$\geq \pi.$$

The statement regarding the equality follows directly.

Exercises :

1) Compute the curvature and torsion functions of the following curves.
   a) $c(t) = \left(t, \frac{a}{2}\left(e^{t/a} + e^{-t/a} \right), 0 \right)$ $t \in \mathbb{R}$
   b) $c(t) = \left(a(t - \sin t), a(1 - \cos t), bt \right)$ $b \in \mathbb{R}$
   c) $c(t) = \left(t, t^2, t^3 \right)$ $t > 0$

2) Obtain the principal triade $r \mapsto \left(t(r), n(r), b(r) \right)$ for the following curves :
   i) $c(r) = \left(r, r^2, 2r \right)$ $r > 0$
   ii) $c(r) = \left(4e^r, r, e^{-r} \right)$
   iii) $c(r) = \left(2, 10\cos r, 5\sin r \right)$
   iv) $c(r) = \left(a\cos r, a\sin r, br \right)$ $r \in \mathbb{R}, a, b$ being constants.

3) Prove : If all tangent vector (unit length) are drawn from the origin of the curve $c(t) = \left(3t, 3t^2, 2t^3 \right)$ then their end points are on the surface of a circular cone having axis $x - z - y = 0.$

4) Let a plane curve be given in polar coordinates (r,Θ) by $r = r/\Theta$. Using the notation $\dot{r} = \dfrac{dr}{d\Theta 1}$ prove that the are length of the curve segment corresponding to Θ verying in $[a,b]$ is given by $\int_a^b \sqrt{\left(\dot{r}^2 + r^2\right)}\, d\Theta$ and the curvature function $k(\Theta)$ is :

$$k(\Theta) = \frac{2\dot{r}^2 - r\ddot{r} + r^2}{\left(\dot{r}^2 + r^2\right)^{3/2}}.$$

5) Obtain the curvature function $k(\Theta)$ of the curve (called Archimedean spiral) : $r(\Theta) = a\Theta$, a being a constant.

6) If a circle is rolled along a line (without slipping) then a fixed point on the circle describes a curve called the "cycloide".

i) Obtain a parameterization of the cycloide generated by a circle of radius.
ii) Obtain a unit speed parameterization of the same curve.
iii) Obtain expressions for the functions $s \mapsto t(s), s \mapsto n(s), s \mapsto b(s)$ for the (unit speed) cycloide of rad a>0.
iv) Obtain its curvature function.

❖❖❖❖

6

FUNDAMENTAL THEOREM OF CURVES

Unit Structure :

In the preceding chapter we studied that with each Frenet curve $c : I \to \mathbb{R}^3$ are associated two scalar functions, namely its curvature function. $k : I \to [0,\infty)$ and the torsion function $t : I \to \mathbb{R}$. The fundamental theorem of curves, which we will study in this chapter, deals with the converse : it asserts that the two functions $k : I \to [0,\infty), t : I \to \mathbb{R}$ determine the curve uniquely to within an isometry of $\mathbb{R}^3$.

The proof of this important theorem is based on a basic existence / uniqueness theorem for the theory of ODE, namely the Picard's existence / uniqueness theorem on the solution of a first order ODE. We therefore recall Picard's theorem (statement only) and then proceed to prove the fundamental theorem of curves.

After proving the main theorem, we discussed a few exercises which illustrate various concepts related to space curves we have come across.

A point regarding our differentiability assumptions need be explained here : We are assuming throughout that all curves $c : I \to \mathbb{R}^3$ are infinitely differentiable on I, we are also imposing regularity conditions on the derivatives : $\dot{c}(t) \neq o, \ddot{c}(t) \neq o$ for all $t \in I$ and so on. Actually, we seldom differentiate curves more than thrice in deriving any result or in calculating any quantity associated with a curve. Infinite differentiability of curves is indeed superfluous but it is used as a general set-up, it can be relaxed to just three times continuous differentiability (but not any further because we are using differentiation as a tool involving $\dot{c}(t), \ddot{c}(t), \dddot{c}(t)$ and their linear independence and so on.)

For a long time, a curve was considered as a thin line in $\mathbb{R}^2$ or in $\mathbb{R}^3$ which was mere a continuous image of an internal. Apart from the fact that the tools of differential calculus are not applicable to such curves, there are space filling curves, which shatter the classical expectation of a curve as a thin line. In 6.6 we discuss (rather concisely) an example of a fat continuous curve filling a square.

6.1 THE FUNDAMENTAL THEOREM OF CURVES

We begin here with the recall of some of the concepts associated with a Frenet curve and then (only) state the enunciation of the fundamental theorem. The proof of the theorem (as explained above) makes use of Picard's theorem in ODE and therefore we discuss Picard's theorem in the next section (again only the statement, no proof!) and then develop the proof of the main theorem in 6.3. It is hoped that this approach will help the reader develop the context to study the proof of the main theorem.

Recall that a smooth curve : $c : I \to \mathbb{R}^3$ with $\left\| \dot{c}(s) \right\| \equiv 1, \ddot{c}(s) \neq 0$ for all $s \in I$ gives rise to the two functions :

- *curvature* $k : I \to [0,\infty)$ and
- • *torsion* $t : I \to \mathbb{R}$.

These functions and the principal triade $\left(t(s), n(s), b(s)\right)$ for each $s \in I$ associated with the curve satisfy the ODE called the Serret - Frenet formulae :

$$\frac{d}{ds}\begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix} = \begin{bmatrix} o & k(s) & o \\ -k(s) & o & t(s) \\ o & -t(s) & o \end{bmatrix}\begin{bmatrix} b(s) \\ n(s) \\ b(s) \end{bmatrix}$$

Now we ask : Conversely given the following data :

- *two smooth functions*

$k : I \to [o,\infty)$

$t : I \to \mathbb{R}$

•• $a \ point \ p \in \mathbb{R}^3 \ with \ a \ parametric \ value \ s \in I$, and
••• $an \ orthonormal \ triade \ of \ vectors \left(t_0, n_0, b_0\right)$,

is there a smooth curve $c : I \to \mathbb{R}^3$ having the given smooth functions $k, \Im$ as its curvature, and torsion; passing through the point p i.e. $c\left(s_\circ\right) = p$ and having the principal triade $\left(t_0, n_0, b_0\right)$ at its point $p = c\left(s_\circ\right)$?

The fundamental theorem gives an affirmative answer.

Theorem 1 (The Fundamental Theorem of Space Curves) : Given :
i) smooth functions : $k : I \to \left[0, \infty\right), t : I \to \mathbb{R}$
ii) $p_0 \in \mathbb{R}^3, s_0 \in I$ and
iii) orthonormal vectors $\left(t_0, n_0, b_0\right)$

there exists a unique Frenet curve $c : I \to \mathbb{R}^3$ which has the properties :

• $c\left(s_0\right) = p$

•• $c \ has \ the \ principal \ triade \left(t_0, n_0, s_0\right) at \ c\left(s_\circ\right)$ and
••• $c \ has, k, t \ as \ its \ curvature \ and \ torsion \ functions.$

We prove this theorem in 6.3.

6.2 THE INITIAL VALUE PROBLEM OF ODE

We introduce here the initial value problem of ODE and state without proof the existence / uniqueness theorem regarding the solution of the initial value problem. The precise statement of the theorem is to be used in proving the fundamental theorem (of curves) in the next section.

Let I denote an open interval and let so be an arbitrary point of it. Let $A : I \to M_n\left(\mathbb{R}\right)$ be any smooth matrix valued map and let x_0 be any point of $\mathbb{R}^n$. We consider the ODE.

$\dfrac{dx}{ds} = A(s)\,X$, (X being a variable ranging in $\mathbb{R}^n$) and a solution $t \mapsto X\left(t\right)$ of the ODE is required to satisfy $X\left(s_0\right) = x_0$. This constitutes the initial value problem (I.V.P.) :

$$\frac{dx}{ds} = A(s)\,X\,, \qquad X\left(s_0\right) = x_0 \text{ .... (*)}$$

Now, the theorem regarding the existence and uniqueness of the solution of the I.V.P. (initial value problem) is the following :

Theorem 2 : The initial value problem (*) has a unique solution $X : I \to \mathbb{R}^n$ defined on the whole of the internal I.

Consult Chapter 2 of this series of study material.

6.3 PROOF OF THE FUNDAMENTAL THEOREM

To begin with, we consider the principal triade map $s \mapsto \left(t(s), n(s), b(s)\right)$ through $\left(t_0, n_0, b_0\right)$ of a prospective curve $c(s)$ passing through the given point P_0. Putting

$$X\left(s\right) = \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}, \qquad s \in I$$

We treat $X\left(s\right)$ in two different but equivalent ways, namely :

- Being an ordered triple of vectors in $\mathbb{R}^3$ it is a vector in $\mathbb{R}^9$.
- It is also a 3×3 matrix of which the top row consists of the three components of $t(s)$, the middle row consists of those of $n(s)$ and the bottom row consisting of the components of $b(s)$,

$$X\left(s\right) = \begin{bmatrix} t_1(s) & t_2(s) & t_2(s) \\ n_1(s) & n_2(s) & n_3(s) \\ b_1(s) & b_2(s) & b_3(s) \end{bmatrix}$$

We now consider the initial value problem :

$$\frac{dx}{ds} = A(s)\,X\left(s\right), X\left(s_0\right) = \begin{bmatrix} t_0 \\ n_0 \\ b_0 \end{bmatrix} \text{ .... (*)}$$

Where $A(s)=\begin{bmatrix} o & k(s) & o \\ -k(s) & o & t(s) \\ o & -t(s) & o \end{bmatrix}$.

Note that the ODE in (*) is nothing but the system of the Secret - Frenet equations.

By Theorem 2 above, we get a unique solution $X : I \to M(3,\mathbb{R}) \simeq \mathbb{R}^9$ of the initial value problem (*). Thus we get the functions.

$$t : I \to \mathbb{R}^3$$
$$n : I \to \mathbb{R}^3$$
$$b : I \to \mathbb{R}^3$$

With $t(s_0)=t_0, n(s_0)=n_0, b(s_0)=b_0$.

At this stage, we claim that the assumed orthonormility of (t_0,n_0,b_0) implies orthronomality of $(t(s),n(s),b(s))$ for each $s \in I$. To get this result, we use the antisymmetry of the matrix $A(s)$, that is $A(s)^t=-A(s)$; (where $A(s)^t$ is the transpose of $A(s)$

We have :

$$\begin{aligned}
\frac{d}{ds}\left(X(s)^t \cdot X(s)\right) &= \left(\frac{d}{ds}X^t(s)\right)\cdot X(s)+X^t(s)\cdot\frac{d}{ds}X(s) \\
&=\left(\frac{d}{ds}X(s)\right)^t X(s)+X(s)^t\cdot\left(A(s)X(s)\right) \\
&=\left(A(s)X(s)\right)^t\cdot X(s)+X(s)^t\cdot A(s)X(s) \\
&=\left(X(s)^t A(s)^t\right)\cdot X(s)+X(s)^t\cdot A(s)\cdot X(s) \\
&=-\left(X(s)^t A(s)\right)\cdot X(s)+X(s)^t\cdot A(s)\cdot X(s) \\
&=-X(s)^t\cdot A(s)\cdot X(s)+X(s)^t\cdot A(s)\cdot X(s) \\
&=0
\end{aligned}$$

This proves constancy of the matrix valued function $s \mapsto X(s)^t \cdot X(s)$:

$$X(s)^t \cdot X(S) = X(s_0)^t \bullet X(S_0)$$

$$= [t_0, n_0, b_0] \cdot \begin{bmatrix} t_0 \\ n_0 \\ b_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus $X^t(s) \cdot X(s) = I$, that is, each $X(s)$ is an orthogonal matrix; in other words for each $s \in I, (t(s), n(s), b(s))$ is an orthonormal triade of vectors.

Finally we get the desired curve $c : I \to \mathbb{R}^3$ by putting :

$$c(s) = p + \int_{s_0}^{s} t(r)\, dr$$

Clearly the curve $c : I \to \mathbb{R}^3$ is well-defined and satisfies; (a) $c(s_0) = p$ and (b) $\dot{c}(s) = t(s)$. Moreover, we have :

$\dot{t}(s) = k(s)n(s)$, $\dot{n} = -k(s)t(s) + t(s)\, b(s)$, $\dot{b} = -t(s)n(s)$ that is, the curve $c : I \to \mathbb{R}^3$ satisfied the Serret-Frenet equations having k, t as its curvature and torsion functions . And then, the initial conditions - that is, $c(s_0) = p, (t(s_0), n(s_0), b(s_0)) = (t_0, n_0, b_0)$ impose uniqueness on the solution curve c.

Thus, given smooth functions $k : I \to [0, \infty)$ and $t : I \to \mathbb{R}$ the theorem guarantees that there exist curves $c : I \to \mathbb{R}^3$ having k, t as their curvature and torsion functions. Next we claim that any two such curves are related by an isometry of $\mathbb{R}^3$ i.e. one curve is the isometric image of the other. To see this, consider any two such curves say $c : I \to \mathbb{R}^3$ and $\tilde{c} : I \to \mathbb{R}^3$ choose $s_0 \in I$ arbitrarily.

Let $p = c(s_0)$ and $\tilde{p} = \tilde{c}(s_0)$.

We put $d = \tilde{p} - p$ that is, $\tilde{p} = p + d$.

Also, let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the unique orthogonal transformation having the property :

$A(t_0) = \tilde{t_0}$ $A(n_0) = \tilde{n_0}$ and $A(b_0) = \tilde{b_0}$.

The vector d and the orthogonal transformation A combine to give the isometry $L : \mathbb{R}^3 \to \mathbb{R}^3$ where $L(x) = A(x) + d$ for every $x \in \mathbb{R}^3$.

We claim $Loc = \tilde{c}$. To justify this claim, first denote the curve Loc by $c^* : I \to \mathbb{R}^3$. We have to verify $\tilde{c}(s) \equiv c^*(s)$. To verify this identity, it is enough to verify that both the curves $\tilde{c}, \overset{*}{c}$ have the following properties :

i) they both satisfy the Serret-Frenet equations with the same k, t,
ii) they pass through the point $\tilde{p}$ and
iii) at $\tilde{p}$, both of them have the same principal triade.

We leave the verification of these facts as an exercise for the reader.

6.4 ILLUSTRATIVE EXAMPLES

I) Determine all plane curves $c : I \to \mathbb{R}^2$ satisfying

i) $k(s) = a$, a being a constant.

ii) $k(s) = \dfrac{1}{\sqrt{s}}$ for $s > O$

iii) $k(s) = \dfrac{1}{\sqrt{(1-s^2)}}$ $-1 < s < 1$

Solution : Clearly because all curves are plain curves, we have $t = O$ and consequently there is only one Serret-Frenet equation :

$$\frac{dt(s)}{ds} = k(s)n(s).$$

Now $t(s)$ being a unit vector, we can write it in the form :

$t(s) = (cos\,\theta(s), sin\,\theta(s))$

$\theta(s)$ being the angle between the vector $t(s)$ and the X-axis. Then we have :

$$n(s) = (-sin\,\theta(s), cos\,\theta(s))$$

Consequently, the equation $\frac{dt(s)}{ds} = k(s)n(s)$ takes the form $(-sin\theta(s),\cdot cos\theta(s))\frac{d\theta(s)}{ds} = k(s)(-sin\theta(s), cos\theta(s))$ which gives $\frac{d\theta(s)}{ds} = k(s)$.

Now in case of (i) above we have :

$$\frac{d\theta(s)}{ds} = a$$

and therefore $\theta(s) = as + b$, for some constant b.

This gives $t(s) = (cos(as+b), sin(as+b))$

Integrating this expression for $t(s)$, we get

$c(s) = p + \int_0^s t(r)\,dr$ (because $\frac{dc(s)}{ds} = t(s)$) being a fixed point of $\mathbb{R}^3$.

$$\begin{aligned} P &= (p_1\ p_2) \\ &= (p_1, p_2) + \int_o^s (cos(ar+b), sin(ar+b))\,dr \\ &= (p_1, p_2) + \left(\frac{sin(as+b)}{a} - \frac{cos(as+b)}{a}\right) \end{aligned}$$

Therefore $c(s) = \left(p_1 + \frac{sin(as+b)}{a}, p_2 - \frac{cos(as+b)}{a}\right), s \in \mathbb{R}$.

ii) Now we have $\frac{d\theta(s)}{ds} = \frac{1}{\sqrt{s}}, s > o$ which gives $\theta(s) = 2\sqrt{s} + a$, for some constant and therefore, $t(s) = \left(cos\left(2\sqrt{s}+a\right), sin\left(2\sqrt{s}+a\right)\right)$.

Integrating this equation, we get

$$\begin{aligned} c(s) &= p + \int_o^s t(r)\,dr,\ p = (p_1, p_2) \in \mathbb{R}^2 \\ &= (p_1, p_2) + \left[\int_o^s cos\left(2\sqrt{r}+a\right)dr, \int_o^s sin\left(2\sqrt{r}+a\right)dr\right] \\ &= \left(p_1 + \int_o^s cos\left(2\sqrt{r}+a\right)dr, p_2 + \int_o^s sin\left(2\sqrt{r}+a\right)dr,\right) \end{aligned}$$

The two definite integrals are left for the reader to evaluate which he / she can, using methods of calculus (e.g. integration by parts.)

iii) Now, we have :

$$\frac{d\theta}{ds}(s) = \frac{1}{\sqrt{1-s^2}}$$

and therefore, $\theta(s) = sin^{-1}(s) + \text{constant}$.

We choose a frame of reference such that the constant of integration in above is zero :

$\theta(s) = sin^{-1}(s)$ i.e. $s = sin\,\theta(s)$ and therefore $cos\,\theta(s) = \sqrt{1-s^2}$

Now $t(s) = (cos\,\theta(s), sin\,\theta(s))$

$$= \left(\sqrt{1-s^2}, s\right)$$

This gives

$$c(s) = (p_1 p_2) + \left(\int_o^s \sqrt{(1-r^2)}dr, \int_o^s rdr\right)$$

$$= \left(p_1 + \int_o^s \sqrt{(1-r^2)}dr, p_2 + \frac{r^2}{2}\right)$$

(Again we leave the evaluation of the above definite integral be completed by the reader.)

ii) For a plane unit speed curve $c : I \to \mathbb{R}^2$ having curvature function $k : I \to \mathbb{R}$ and the Serret-Frenet frame $(t(s), n(s))$ at a point $c(s)$ of it, prove :

$$\begin{bmatrix} t(s) \\ n(s) \end{bmatrix} = \sum_{\ell \geq 0} \frac{1}{\ell!} \begin{bmatrix} o, \int_{s_0}^s k(r)\,dr \\ -\int_{s_0}^s k(r)\,dr, o \end{bmatrix} \begin{bmatrix} t(s_0) \\ n(s_0) \end{bmatrix}.$$

Proof : For $s \in I$, let $A(s)$ be the 2×2 matrix :

$$A(s) = \begin{bmatrix} o & , & \int_{s_0}^s k(r)\,dr \\ \int_{s_0}^s k(r)\,dr & , & o \end{bmatrix}$$

Also, let $X : I \to \mathbb{R}^2, Y : I \to \mathbb{R}^2$ be the functions given by :

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \sum_{\ell \geq 0} \frac{A(s)^{\ell}}{\ell!} \begin{bmatrix} t(s_0) \\ n(s_0) \end{bmatrix}.$$

Now we have :

- $$\frac{dA(s)}{ds} = \begin{bmatrix} 0, & k(s) \\ -k(s), & 0 \end{bmatrix}, \text{and}$$
- $$\frac{d}{ds}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \sum_{\ell \geq 0} \frac{A^{\ell}(s)}{\ell!} \begin{bmatrix} t(s_0) \\ n(s_0) \end{bmatrix}$$
$$= \sum_{\ell \geq 0} \frac{A^{\ell}(s)}{\ell!} \begin{bmatrix} 0, & k(s) \\ -k(s), & 0 \end{bmatrix} \begin{bmatrix} t(s_0) \\ n(s_0) \end{bmatrix}$$
$$= \begin{bmatrix} 0, & k(s) \\ -k(s), & 0 \end{bmatrix} \sum_{\ell \geq 0} \frac{A^{\ell}(s)}{\ell!} \begin{bmatrix} t(s_0) \\ n(s_0) \end{bmatrix}$$
$$= \begin{bmatrix} 0, & k(s) \\ -k(s), & 0 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix}$$

Now we have :

1) The function $s \to \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix}$ satisfies the ODE

$$\frac{d}{ds}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 0, & k(s) \\ -k(s), & 0 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} \text{ and}$$

2) $\begin{bmatrix} X(s_0) \\ Y(s_0) \end{bmatrix} = \begin{bmatrix} t(s_0) \\ n(s_0) \end{bmatrix}$

In other words the function on $s \to \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix}$ satisfies the ODE (1) and the initial conditions (2). Therefore, the equation has the solution $\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = e^{A(s)} \begin{bmatrix} t(s_0) \\ n(s_0) \end{bmatrix} = \sum_{\ell \geq 0} \frac{A^{\ell}(s)}{\ell!} \begin{bmatrix} t(s_0) \\ n(s_0) \end{bmatrix}$

$$= \sum_{\ell \geq 0} \frac{1}{\ell!} \begin{bmatrix} 0 & \int_{s_0}^{s} k(r)\,dr \\ -\int_{s_0}^{s} k(r)\,dr, & 0 \end{bmatrix} \begin{bmatrix} t(s_0) \\ n(s_0) \end{bmatrix}$$

6.5 SMOOTH CURVES IN HIGHER DIMENSIONAL EUCLIDEAN SPACES

We describe, in very few words, some of the geometry of smooth curves $c : I \to \mathbb{R}^n$ for $n > 3$. Our main intention is only to indicate generalization to higher dimensions of the geometry of the space curves which we have studied above. We only introduce concepts and state some of the elementary results, but every thing going without proof! Interested reader can consult a standard graduate level book such as : A course in Differential Geometry by Withelm Klingenberg. (A Springer - Verlag publication).

Now, a smooth curve in $\mathbb{R}^n$ is a smooth map $c : I \to \mathbb{R}^n$.

For a $r \in I$ writing $c(r)$ in terms of the Cartesian coordinates : $c(r) = \left(c^1(r), c^2(r) \ldots\ldots c^n(r)\right)$ we consider the derivatives ;

$$\dot{c}(r) = \left(\dot{c}^1(r), \dot{c}^2(r), \ldots\ldots, \dot{c}^n(r)\right)$$

$$\ddot{c}(r) = \left(\ddot{c}^1(r), \ddot{c}^2(r), \ldots\ldots, \ddot{c}^n(r)\right)$$

$$\vdots$$

$$c^{(k)}_{(r)} = \left(c^{1(k)}_{(r)}, c^{2(k)}_{(r)}, \ldots\ldots, c^{m(k)}_{(r)}\right)$$

the first one, namely $\dot{c}(r)$ is the tangent vector to the curve at its point $c(r)$.

We have the straight-forward generalization of the notion of reparametrization of c : Let $\theta : J \to I$ be a smooth, strictly monotonic increasing and bijective map. Then the (smooth) curve $c.\theta : J \to \mathbb{R}^n$ is said to be obtained from $c : I \to \mathbb{R}^n$ by reparametrization, the map $\theta : r(\in J) \mapsto \theta(r) = s$ being the parametrization map.

Again for a fixed $r_0 \in I$ (and thereby for a fixed point $p_0 = c(r_0)$ of c) and for a variable $c(r)(r \in I)$, the integral

$\ell(r)=\int_{r_0}^{r}\left\| \dot{c}(s) \right\| ds=\int_{r_0}^{r}\sqrt{\left\{\sum_{j=1}^{n}\dot{c}^{j}(s)^2\right\}}ds;$ $\ell(r)$ is the signed length of the segment of the curve with $c(r_0)$ and $c(r)$ as its end points.

Note that ℓ is a strictly monotonic increasing function whenever $\dot{c}(r)\neq o$ for all $r\in I$. Also, note that the set $J=\{\ell(r): r\in I\}$ is an interval.

In the following we consider those $c: I\to\mathbb{R}^n$ for which $\dot{c}(r)\neq o$ for all $r\in I$ holds. (which implies that the function $\ell: I\to J$ is bijective). We use the strictly monotonic $\ell: I\to J$ to reparametrize c:

The reparametrized curve $\tilde{c}$ has the property that $\left\|\dot{\tilde{c}}(s)\right\|=1$ for all $r\in J$, that is c is a unit speed curve.

Next, to get the n-dimensional analogue of the principal triade $(t(p),n(p),b(p))$ of a space curve at a point p of it, we assume the following property :

For each $r\in I$, the set $\left\{\dot{c}(r),\ddot{c}(r)........c^{(n)}(r)\right\}$ is linearly independent. Applying the Gram-Schmidt orthogonalization to each $\left\{\dot{c}(r),\ddot{c}(r)........c^{(n)}(r)\right\}$ we get the orthonormal set $\{e_1(r),e_2(r)........e_n(r)\}$ with the property that for each $k\leq n, c^{(k)}(r)$ is a linear combination of $e_1(r),e_2(r)........e_k(r)$. Now $\{e_1(r)........e_n(r)\}$ thus obtained, is the desired analogue of the principal triade of a space curve. We call the set $\{e_1,e_2........e_n\}$ of unit vector fields along c, the Frenet frame of the curve. Now, we have the following two results :

Theorem 3 : Let $c: I\to\mathbb{R}^n$ be a smooth curve having its Frenet frame $\{e_1........e_n\}$. Then there are smooth functions $k_1,k_2........k_n: I\to\mathbb{R}$ satisfying the equations :

$$\begin{bmatrix} e_1(r) \\ e_2(r) \\ e_3(r) \\ \vdots \\ \vdots \\ e_n(r) \end{bmatrix} = \begin{bmatrix} 0 & k_1(r) & & & & \\ -k_1(r) & 0 & k_2(r) & & & \\ 0 & -k_1(r) & 0 & k_3(r) & & \\ & & -k_3(r) & & & \\ & & & & 0 & k_n(r) \\ & & & & -k_{n-1}(r) & \end{bmatrix} \begin{bmatrix} e_1(r) \\ e_2(r) \\ e_3(r) \\ \vdots \\ \vdots \\ e_n(r) \end{bmatrix}$$

The functions $k_i : I \to \mathbb{R}$ are called the ith curvatures of c and the above set of equations are the Frenet equations.

Theorem 4 : (Fundamental Theorem of Curves.) Let $k_1, k_2 .... k_n : I \to \mathbb{R}$ be smooth cures $k_1, k_2 .... k_{n-2} > 0$ on I. For a fixed $r_0 \in I, p_0 \in \mathbb{R}^n$ and for any orthonormal set $\{e_{10}, e_{20} .... e_{n0}\}$ there exists a unique curve $c : I \to \mathbb{R}^n$ parametrized by its are-length r having the properties :

1) $c(r_0) = p_0$
2) $\{e_{10}, e_{20} .... e_{n0}\}$ is the Frenet frame of c at p. and
3) $k_1, k_2 .... k_{n-1} : I \to \mathbb{R}$ are the curvature functions of c.

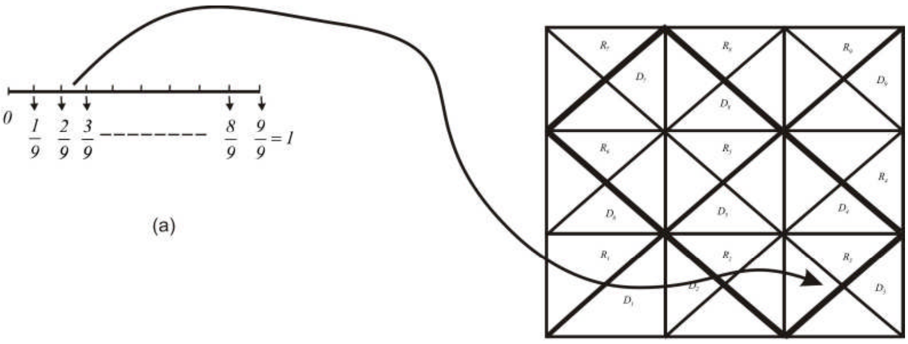
6.6 A SPACE-FILLING CONTINUOUS CURVE

We conclude this chapter by discussing an example of a continuous curve which is not a thin line but an area filling map because it is continuous, lacking any differentiability properties. This should convince the reader that a curve as a reasonable geometric object it should be more than a merely continuous map, it should have, differentiability properties and the successive derivatives having linear independence.

Theorem (Peano) : There exists a continuous surjective map (= a curve) $C : [0,1] \to [0,1] \times [0\,1] = R$

Proof : We obtain the desired C as the uniform limit of a sequence $\left(C_k : [0,1] \to \mathbb{R} : k \in \mathbb{N}\right)$ of continuous maps.

To construct C_1 we sub-divide (a) $[0,1]$ into $3^2=9$ sub-intervals of equal length : $\left[\frac{i-1}{9}\quad\frac{i}{9}\right]1\le i\le 9$ and (b) the rectangle R into nine Sub-rectangles of equal area as shown in the figure.



(a) (b)

We construct C_1 by mapping $\left[0,\frac{1}{9}\right]$ linearly onto the diagonal D_1 of the sub-rectangle $\mathbb{R}_1$, then mapping $\left[\frac{1}{9}\quad\frac{2}{9}\right]$ linearly onto the diagonal D_2 of the rectangle $\mathbb{R}_2$ and so on.

Next we construct C_2 in a similar manner : Sub-divide each $\left[\frac{i-1}{9}\quad\frac{i}{9}\right]$ into nine equal parts, the rectangle R, into nine sub rectangles of it having equal areas, and mapping the interval $\left[\frac{i-1}{9}\quad\frac{i}{9}\right]$ onto the nine diagonals of R_i in a similar manner.

Using the above procedure we get the sequence $\left(C_k:[0,1]\to\mathbb{R}:k\in\mathbb{N}\right)$.

Note the following properties of the sequence $\left(C_k:k\in\mathbb{N}\right)$ of curves :

•$\left\|C_k(t)-C_{k+1}(t)\right\|\le\frac{1}{\sqrt{2.3}}k-1$ for all $t\in[0,1]$

••$C_k\left(\frac{i}{3^{k+1}}\right)=C_{k+1}\left(\frac{i}{3^{k+1}}\right)=....$ for all $k\in\mathbb{N}$ and $i\le 3^{k+1}$ and

•••$\left\|C_k(t)-C_\ell(t)\right\|\le\frac{1}{\sqrt{2\cdot 3}}k-1$ for all $k\in[0,1]$ and for all k, ℓ in $\mathbb{N}$ with $k<\ell$.

The last of these properties implies that the sequence $(C_k : k \in \mathbb{N})$ is uniformly Cauchy on $[0,1]$ and as such it converges to a continuous map $C:[0,1] \to \mathbb{R}$.

Now, C is a continuous curve implies that its trace i.e. the set $\{C(r) : s \in [0,1]\}$ S a compact subset of $\mathbb{R}$. Moreover, this set contains all the point $\left(\frac{k}{3^m}, \frac{\ell}{3^m}\right)$ for $o \le k, \ell \le 3^m, m \in \mathbb{N}$ and therefore, the set is a dense subset of R. It then follows that this set is the whole of R i.e. the continuous curve C maps $[0,1]$ onto the rectangle R thus C is a fat set and not a thin line.

Exercise :

1) Let $c : I \to \mathbb{R}^3$ be a Frenet curve and let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry. Prove that both the curves C, Loc have the same curvature and torsion functions.

2) Is it true that all curves $C:[a\,b] \to \mathbb{R}^3$ having common curvature and torsion functions are isometric?

3) Let $\gamma = \gamma(s)$ be a curve in its natural parametriz ation (=Unit speed parametrization = are length parametrizational and let $u(t)$ be the same curve but with different parametrization the relation between then being $u(t) = \gamma(s(t))$. Prove :

$$\frac{d^2 u(t)}{dt^2} = k(t)\left(\frac{ds}{dt}\right)^{2\to} n(t) + \frac{d^2 s}{dt^2}(t)\vec{t}(s)$$

4) The Darbour vector of a curve with non-vanishing curvature is the vector $d = tt + kb$. Prove that the Serret-Frenet formulae can be written in the form :

$$\frac{dt}{ds} = d \times t,\ \frac{dn}{ds} = d \times n, \frac{db}{ds} = d \times b$$

5) Consider the curves $\tilde{c}:[0\,L] \to \mathbb{R}^3$ determined by the unit tangent of a regular curve $\tilde{c}:[0,L] \to \mathbb{R}^3$ i.e. $\tilde{c}(s) = t(s)$ ($\leftarrow$unit tangent of c at the point $c(s)$). Assume that $k(s)$ of c does not vanish anywhere and prove that $\tilde{c}$ is a regular curve and obtain expressions for its

curvature and torsion functions $\tilde{k}(s), \tilde{t}(s)$. Investigate $\tilde{k}(s), \tilde{t}(s)$ in case of $c(s)$, the helix :

$$c(s) = (a\cos s, a\sin s, bs)$$

6) Let a plane curve be given in polar coordinates (r, θ) by r=f(θ), $f:(0:2\pi) \to \mathbb{R}$ being a smooth function. Prove that the are-length s between two points $(\theta_1, f(\theta_1)), (\theta_2, f(\theta_2))$ on the curve $(\theta_1 < \theta_2)$ is given by $s = \int_{\theta_1}^{\theta_2} \sqrt{\left[f(\theta)^2 + f^2(\theta) \right]}\, d\theta$ and the curvature $k(\theta)$ of the curves give by $k(\theta) = \dfrac{2f^1(\theta)^2 - 2f(\theta)\, f(\theta) + f^2(\theta)}{\left[f(\theta)^2 + f^2(\theta) \right]^{3/2}}$

7) Calculate the curvature of the curve given by $r = a\theta$ where a is a positive constant.

8) Let $c: I \to \mathbb{R}^n$ be a Frenet curve in $\mathbb{R}^n$, Prove :

$$det \begin{bmatrix} \dot{c} \\ \ddot{c} \\ \vdots \\ c^{n(t)} \end{bmatrix} = \prod_{i=1}^{n-1} \left(k_i(t) \right)^{n-i}$$

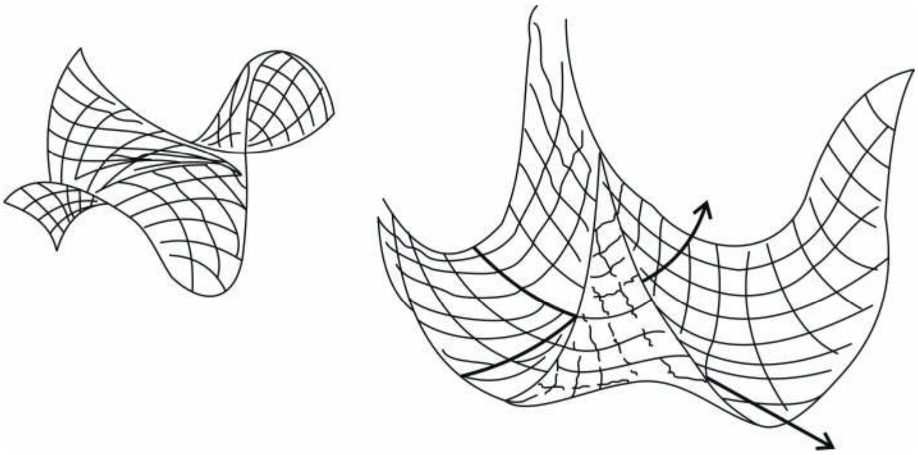
❖❖❖❖

7

REGULAR SURFACES

Unit Structure :

We think of a surface as a thin, smoothly bending sheet having no creases, no corners.......; a sheet spreading across a certain region in the physical space $\mathbb{R}^3$. Clearly we need two parameters - its coordinates - to specify the points of such a thin sheet. Moreover we need the coordinate systems which are adapted to the geometry of such smooth surfaces.



Observing common surfaces such as a sphere, a two dimensional torus, a cylinder, the Möbius band, a circular cone, etc we find that indeed such coordinate systems are available a plenty but only locally on a general surface, that is, each point of a surface has a small enough neighborhood carrying a reasonable coordinate system.

The above observation namely surfaces admitting coordinate systems only locally - each point has a small enough neighborhood carrying coordinates-leades as to the concept variously called a local coordinate system a coordinate chart or a local parametrization. Thus, mathematically, a smooth surface is a subset M of $\mathbb{R}^3$ admitting a nice set of coordinates in a neighborhood of each of its points. These coordinate systems, being local, are not unique but they are required to be smoothly related on the overlap of their domains : one set of coordinates should be smooth functions of the other coordinates! (This property will be explained in detail at the right stage.)

Using the local coordinates, we can differentiate functions defined on a surface and this gives rise to a full-fledged differential calculus on a surface. The resulting differential calculus is used as a tool to study the highly sophisticated geometry of a surface-a smoothly bending, thin portion of $\mathbb{R}^3$. In particular, we study the curvature properties of such a surface using the techniques of differential calculus.

In this chapter we introduce the notion of a differential structure of a surface and then proceed to explain differentiability of functions, smooth (tangential and normal) vector fields, smooth linear and bilinear forms on such smooth surfaces and so on. The chapters next to this will explain the geometric features of smooth surfaces.

Our discussion involves both the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$: we use coordinates of $\mathbb{R}^2$ to (loally) parametize the surface and $\mathbb{R}^3$ accommodates the surface. Although $\mathbb{R}^2$ is imbedded in $\mathbb{R}^3$, we will treat them as separate spaces, this is to avoide any notational confusion (Higher dimensional Euclidean spaces also crop-up here and there!)

The usual Cartesian coordinates in $\mathbb{R}^2$ will be denoted by $(u_1,u_2),(v_1,v_2)$ etc. In $\mathbb{R}^3$ we will use the triples such as $(x_1,x_2,x_3),(y_1,y_2,y_3)$ etc. for the Cartesian coordinates.

(r,θ) will be the usual polar coordinates in $\mathbb{R}^2\setminus\{0\}$ while $(r,\theta,\varnothing)$ are the familiar spherical polar coordinates in $\mathbb{R}^3\setminus\{0\}$.

7.1 LOCAL PARAMETRIZATION

Let M be a non-empty subset of $\mathbb{R}^3$. We will consider M equipped with the subspace topology of $\mathbb{R}^3$. Thus for each $p \in M$, the sets of the type $M \cap B(p,\delta)$ for $\delta > 0$ form a fundamental neighborhood system of p in the subspace topology of M. (Here, of course, $B(p,\delta)$ is the open ball in $\mathbb{R}^3$, centred at p and having radius $\delta > 0$).

Definition 1 : A local parametrization of M around a point p is a triple (U,Q,V) consisting of :

- An open subset U of $\mathbb{R}^2$
- An open neighborhood. V of p (V being open with respect to the subspace topology of M : V = $M \cap W$, W being an open subset of $\mathbb{R}^3$) and
- A homeomorphism $Q : U \to V$, the triple (U,Q,V) having the properties :

i) $Q : U \to \mathbb{R}^3$ is smooth and

ii) for each $q \in U$, the Jacobean matrix of Q at q:

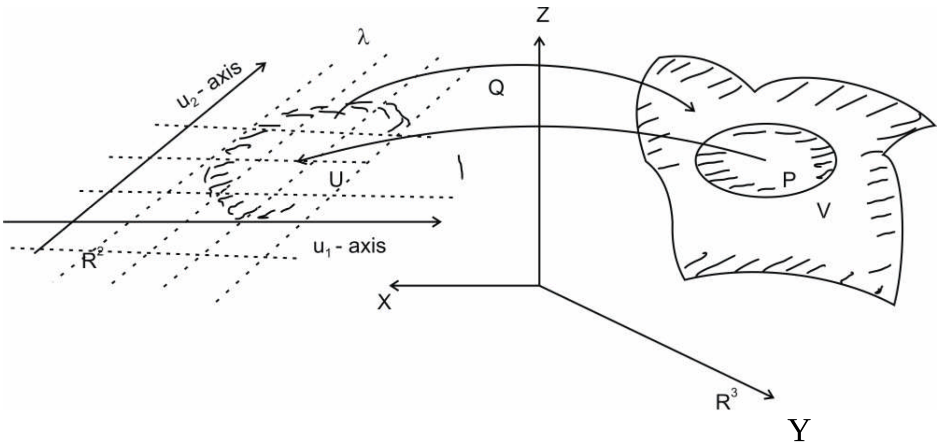
$$J_Q(q) = \begin{bmatrix} \frac{\partial Q_1}{\partial \mu_1}(q) & \frac{\partial Q_1}{\partial \mu_2}(q) \\ \frac{\partial Q_2}{\partial \mu_1}(q) & \frac{\partial Q_2}{\partial \mu_2}(q) \\ \frac{\partial Q_3}{\partial \mu_1}(q) & \frac{\partial Q_3}{\partial \mu_2}(q) \end{bmatrix}$$

has ran = 2.

Because $Q^{-1} : V \to U$ is well defined, for each $p \in V$. We write $Q^{-1}(p) = (u_1(p), u_2(p))$ and regard $(u_1(p), u_2(p))$ as the coordinates of p with respect to the local parametrization (U,Q,V). This consideration leads (us) to the functions :

$$(u_1, u_2) : V \to \mathbb{R}$$

and the resulting triple $(V,(u_1,u_2))$ is called a local coordinate chart on M around the point p; the functions $u_1, u_2 : V \to \mathbb{R}$ being called the coordinate functions of the coordinate chart.



Here are some more explanations regarding the notion of a loal parametrization :

- Recall, in a local parametrization (U,Q,V) the map $Q:U \to V = Q^{-1}(V)$. As such we may mention either (U,Q) or (V,Q^{-1}) instead of the whole triple. Using yet another symbol, say Ψ for Q^{-1}, it is found that the pair $(V,\Psi)\left(=\left(V,Q^{-1}\right)\right)$ is very useful. The map Ψ associates with each $p \in V$ the point (say) $q = \Psi(p)$ and then we identify the point p of M with $q = \Psi(p)$ of U and reard the coordinates $(u_1(q),u_2(q))$ as the coordinates of the point $p \in M$. Thus we are parametrizing the patch V on M by the coordinates on its image $\Psi(V) = U$.
- M, being a subset of $\mathbb{R}^3$, a point p of M has its natural Cartesian coordinates $(x_1(p),x_2(p),x_3(p))$. But it being a thin sheet (a 2 dimensional geometric object so to speak) the coordinates-three of them-are not independent, one of them is a function of the other two. Thus on the northern hemi-sphere M give by $M = \left\{(x,y,z) \in \mathbb{R}^3 x^2 + y^2 + z^2 = 1, z > 0\right\}$ we have $z = +\sqrt{\left(1 - x^2 - y^2\right)}$. Cartesian coordinates indeed are not independent and therefore not very useful in calculations. Secondly they do not reflect the spherical character of M. (Indeed navigators do not mention the Cartesian coordinates, the spherical polar coordinates $(\theta,\varnothing) =$ the (latitude, longitude) are their favourite choice!

  All in all, the Cartesian coordinates of the ambient space $\mathbb{R}^3$ are not used to describe the geometry of M.

- The main idea behind the new concept of a local parametrization (U,Q,V) is to put the points p of the part V of M in 1-1 correspondance with the points q of $UC\mathbb{R}^2$ by meanse of the homeomorphism Q so as to use the independent coordinates (u_1,u_2) of the associated point $q=Q^{-1}(p)$ as the coordinates $(u_1(p),u_2(p))$ of the point p of our interest. And a careful choice of the coordinates (u_1,u_2) may reflect better on the geometry of the portion V of M. Thus, for example, on the northern hemi-sphere, we prefer the independent coordinates $(\theta,\varnothing)-$the latitude and longitude-because they are better suited to the spherical geometry of the hemi-sphere.
- However, often a single parametrization fails to cover the whole of M. and we need find a system $\{(U_\lambda,Q_\lambda,V\lambda):\lambda\in\Lambda\}$ of local parametrizations which together cover the surface M, that is, $M=U\{V_\lambda:\lambda\in\Lambda\}$. Such a collection gives rise to the notion of a differential structure of M; this notion is explain below.

We first define the simpler concept a surface covered by a single coordinate chart.

Definition 2 : A <u>parametrized surface is</u> a subset M of $\mathbb{R}^3$ which is covered by a single parametrization i.e. there is a pair (U,Q) consisting of (i) an open set U of $\mathbb{R}^2$, (ii) a smooth map $Q:U\to\mathbb{R}^3$ such that the following conditions are satisfied.

a) $Q(U)=M$

b) $Q:U\to M$ is a homeomorphism and

c) $J_Q(q)$ has rank 2 at every $q\in U$

Here is an example of a parametrized surface; we consider the graph of a smooth function of two real variables :

Let U be an open cubset of $\mathbb{R}^2$ and let $f:U\to\mathbb{R}$ be any smooth function. We consider the graph of f i.e. the set $M\subset\mathbb{R}^3$ given by : $M=\{(u_1\,u_2\,f(u_1\,u_2)):(u_1\,u_2)\in U\}$.

Now, let $Q:U\to M$ be the smooth map given by

$Q(u_1\,u_2)=(u_1\,u_2,f(u_1\,u_2))$

for all $(u_1\, u_2) \in U$. Then indeed, Q is a homermorphism between U and M, moreover the Jacobean of Q at a $u = (u_1\, u_2) \in U$ is the matrix :

$$J_Q(u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u_1}(u_1, u_2) & \frac{\partial f}{\partial u_2}(u_1, u_2) \end{bmatrix}$$

Clearly, this matrix has rank =2. Therefore, the graph of such a smooth $f : U \to \mathbb{R}$ is a parametrized surface. For example, take $U = \left((u_1\, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 < 1\right)$ and let $f : U \to \mathbb{R}$ be the smooth map given by $f(u_1, u_2) = +\sqrt{\left(1 - u_1^2 + u_2^2\right)}, \quad (u_1, u_2) \in U$.

Clearly, the graph of this f is the northern hemisphere of unit radius. Note that the parametrization of the hemi-sphere using this f cannot be extended to any larger portion of the sphere. Thus on the whole sphere, we need more than one local parametrizations to cover it. This observation motivates the following definition.

We are consisdering a subset M of $\mathbb{R}^3$; it carrying the subspace topology of $\mathbb{R}^3$.

Definition 3 : A regular surface is a subset M of $\mathbb{R}^3$ having the following property :

For each $p \in M$, there exists a local parametrization (U, Q, V) on M with $p \in V$.

A regular surface is often called a smooth surface.

As observed, we have $Q(U) = V$ and therefore we often write only (U, Q) in place of the triple (U, Q, V).

A collection $D = \{(U_\lambda\, Q_\lambda) : \lambda \in \Lambda\}$ with the property $M = U\{U_\lambda : \lambda \in \Lambda\}$ is called a (smooth) coordinate atlas on M.

Thus a parametrized surface is a special case of a regular surface where a single coordinate chart is covering the underlying set. Of course we come across plenty of surfaces which are more general than parametrized surfaces. We discuss some examples of them by describing the set M and then specifying a coordinate atlas on it.

(I) A Sphere :

For a constant $a > 0$ let $M = \left\{ \left(x_1, x_2, x_3 \right) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = a^2 \right\}$

We consider the following open cover of the sphere M consisting of the six open hemi-spheres $H_1, H_2, H_3, H_4, H_5, H_6$, given by :

$$H_1 = \left\{ \left(x_1, x_2, x_3 \right) \in M, x_1 > 0 \right\}, H_2 = \left\{ \left(x_1, x_2, x_3 \right) \in M : x_1 < 0 \right\}$$
$$H_3 = \left\{ \left(x_1, x_2, x_3 \right) \in M, x_2 > 0 \right\}, H_4 = \left\{ \left(x_1, x_2, x_3 \right) \in M : x_2 < 0 \right\}$$
$$H_5 = \left\{ \left(x_1, x_2, x_3 \right) \in M, x_3 > 0 \right\}, H_6 = \left\{ \left(x_1, x_2, x_3 \right) \in M : x_3 < 0 \right\}$$

Also let $U = \left\{ \left(u_1\, u_2 \right) \in \mathbb{R}^2 : u_1^2 + u_2^2 < a^2 \right\}; U$ is an open subset of $\mathbb{R}^2$. We consider the following homeomorphism $\theta i : U \to Hi, 1 \le i \le 6 :$

$$\theta_1 \left(u_1\, u_2 \right) = \left(\sqrt{\left(a^2 - u_1^2 - u_2^2 \right)}, u_1\, u_2 \right); \left(u_1\, u_2 \right) \in U$$
$$\theta_2 \left(u_1\, u_2 \right) = \left(-\sqrt{\left(a^2 - u_1^2 - u_2^2 \right)}, u_1\, u_2 \right); \left(u_1\, u_2 \right) \in U$$
$$\theta_3 \left(u_1\, u_2 \right) = \left(u_1, \sqrt{\left(a^2 - u_1^2 - u_2^2 \right)}, u_2 \right); \left(u_1\, u_2 \right) \in U$$
$$\theta_4 \left(u_1\, u_2 \right) = \left(u_1, -\sqrt{\left(a^2 - u_1^2 - u_2^2 \right)}, u_2 \right); \left(u_1\, u_2 \right) \in U$$
$$\theta_5 \left(u_1\, u_2 \right) = \left(u_1\, u_2, \sqrt{\left(a^2 - u_1^2 - u_2^2 \right)} \right); \left(u_1\, u_2 \right) \in U$$
$$Q_6 \left(u_1\, u_2 \right) = \left(u_1\, u_2, -\sqrt{\left(a^2 - u_1^2 - u_2^2 \right)} \right); \left(u_1\, u_2 \right) \in U$$

Then $D = \left\{ \left(U, \theta_1 \right), \left(U, \theta_2 \right), \left(U, \theta_3 \right), \left(U, \theta_4 \right), \left(U, \theta_5 \right), \left(U, \theta_6 \right) \right\}$ is a coordinate atlas of the sphere M.

(II) The Möbius Band :

Let $Z = \mathbb{R} \times \left(-1, 1 \right) = \left\{ \left(x, y \right) \in \mathbb{R}^2 : y \in \left(-1, 1 \right) \right\}$.

Define an equivalence relation $\sim$ on Z by declaring $\left(x, y \right) \sim \left(x + 2, -y \right)$ for all $\left(x, y \right) \in \mathbb{R} \times \left(-1, 1 \right)$.

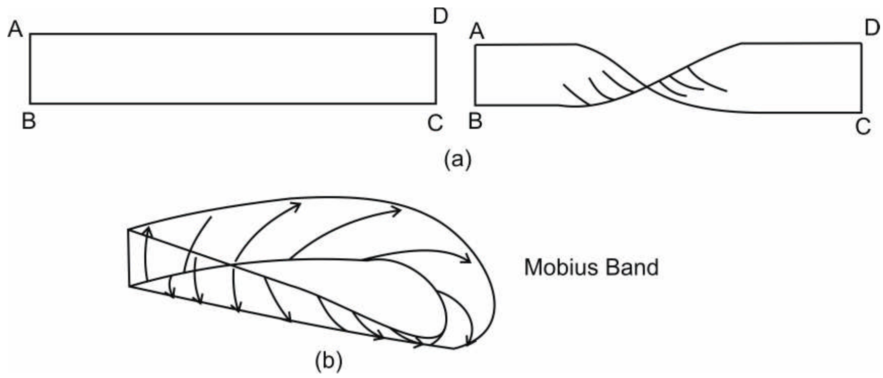
Let $M = Z/\sim$ and $\Pi : Z \to M$, the natural projection. M is given quotient topology of Z by the equivalence relation. Part (b) of the figure below depicts the Möbius band as a subset of $\mathbb{R}^3$.

Now, let $V_1 = \{\Pi(x,y) : -1 < x < 1, -1, < y < 1\}$ and
$V_2 = \{\Pi(x,y) : 0 < x < 2; -1, < y < 1\}$.

Also, let $U_1(-1,1) \times (-1,1)$ and $U_2(0,2) \times (-1,1)$. And finally let $\theta_1 = \Pi/U_1, \theta_2 = \Pi/U_2$. Then it can be seen that $D = \{(U_1\theta_1), (U_2\theta_2)\}$ is a coordinate atlas on the set M.

The set M equipped with D is called the Möbius band.

Here is a geometric description of the Möbius band : We consider the strip $R = [-1,1] \times (-1,1)$. Twisting the strip through $180°$ we bring the ends $\{-1\} \times (-1,1)$ and $\{+1\} \times (-1,1)$ together and glue them in such a way that the end $\{-1\} \times (-1,1)$ comes upside down and is glued to the other end.



An important property of regular surfaces is their orientability. Orientability property of regular surfaces is explained in the next chapter. Möbius band is a simple example of an unoriented surface.

A simplified description of orientability of a surface is that it admits a continuous (actually a smooth) unit normal field. One can see that the Möbius band does not admit such a unit normal field because of the twist applied to the rectangle $[-1,1] \times (-1,1)$ in getting the Möbius band out of it. Also note that the Möbius band has only one side.

(III) Surfaces of Revolution :

We consider a smooth curve $c:(a,b)\to\mathbb{R}^2$ in the vertical XOZ-plane $(=\mathbb{R}^2)$ given by $c(t)=(x_1(t),x_3(t))$, in terms of the two smooth functions $x_1,x_2:(a\,b)\to\mathbb{R}$.

Let $U=(a,b)\times(o,\pi)$.

We consider $\theta_1:U\to\mathbb{R}^3,\theta_2:U\to\mathbb{R}^3$ given by
$\theta_1(s,t)=(x_1(t)cos\,s,x_1(t)sin\,s,x_3(t))$ and
$\theta_2(s,t)=(x_1(t)cos(s-\pi),x_1(t)sin(s-\pi),x_3(t))$.

Let $M=\theta_1(U)u\theta_2(U)c\mathbb{R}^3$.

Then it can be seen that $(U_1,Q_1),(U_2,Q_2)$ are local parametrizations on M and $D=\{(U_1,Q_1),(U_2,Q_2)\}$ is a coordinate atlas on M; it being the surface of revolution of the curve C about the Z-axis of $\mathbb{R}^3$.

Before discussing more illustrative examples, let us prove a result. A variety of subsets of $\mathbb{R}^3$-called level sets of smooth functions - are regular surfaces. This claim is verified by applying the result proved in Proposition 1 given below.

Let Ω be an open subset of $\mathbb{R}^3$ and let $f:\Omega\to\mathbb{R}$ be a smooth function. For a constant $\alpha\in\mathbb{R}$, the set :
$M=\{x\in\Omega;\ f(x)=\alpha\}$
(if non-empty) is called a level set of the function.

Proposition 1 : Let f,M,α be as above. Suppose M is non-empty and has the following property :

For each $x\in M$, grad $(f)(x):=\left(\frac{\partial f}{\partial x_1}(x),\frac{\partial f}{\partial x_2}(x),\frac{\partial f}{\partial x_3}(x)\right)$ is a non-zero vector.

Then M is a regular surface.

Proof : Let $p=\left(p_1,p_2,p_3\right)$ be an arbitrary point of M. By assumption grad$(f)(p)\neq 0.$ Assume, withought loss of generality that $\frac{\partial f}{\partial x_3}(p)\neq 0$.

By the implicit function, there exists an open $U c \mathbb{R}^2$ and a smooth function $g:U\to\mathbb{R}$ having the following properties :

a) $\left(p_1,p_2\right)\in U$

b) $g\left(p_1,p_2\right)=p_3$

c) for any $\left(u_1,u_2\right)\in U,\left(u_1,u_2,g\left(u_1,u_2\right)\right)\in\Omega$ with $f\left(u_1 u_2,g\left(u_1 u_2\right)\right)=\alpha$.

(In other words, the function g solves the equation $f\left(x_1,x_2,x_3\right)=\alpha$ expressing x_3 as a function $\left(x_1,x_2\right)$.) The properties (a), (b), (c) imply that putting $\theta\left(u_1 u_2\right)=\left(u_1 u_2,g\left(u_1 u_2\right)\right)$ $\left(u_1 u_2\right)\in U$ the triple $\left(U,\theta,V\right)$ is a local parametrization on M.

Therefore, M is a regular surface.

As an application of this result, we discuss the following illustrative examples.

(IV) An ellipsoide $M=\left\{\left(x_1,x_2,x_3\right),\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1\right\}$ where $a>0,b>0,c>0$ are constants is a regular surface : Take $\Omega=\mathbb{R}^3, f:\mathbb{R}^3\to\mathbb{R}$ be the function $f=\left(x_1,x_2,x_3\right)=\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}$ and let $\alpha=1$. Clearly grad $(f)(p)\neq(0,0,0)$ for any $p\in M$ and therefore, M-the ellipsoide-is a regular surface.

V) The Parabolic Hyperboloid :

Let $M=\left\{\left(x_1,x_2,x_3\right)\in\mathbb{R}^3;x_3=x_1^2-x_2^2\right\}$

Take $\Omega=\mathbb{R}^3, f:\mathbb{R}^3\to\mathbb{R}$ given by

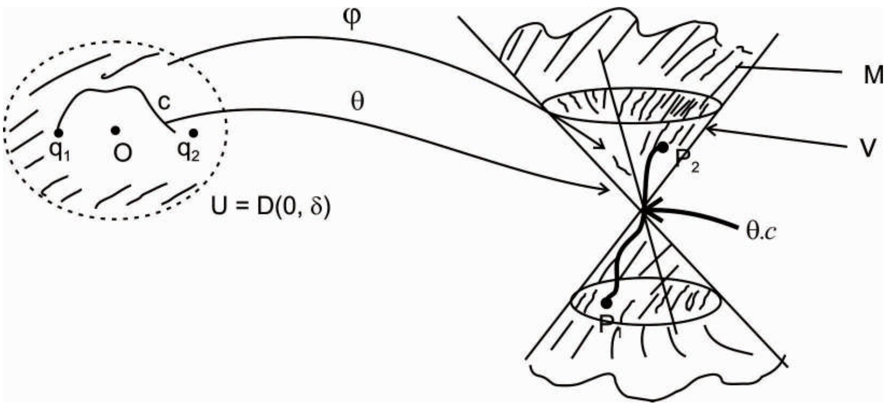
$f\left(x_1,x_2,x_3\right)=x_3-x_1^2+x_2^2,\left(x_1,x_2,x_3\right)\in\mathbb{R}^3$ and $\alpha=0$ we see that (grad f) $(p)=\left(1,-2p_2,2p_3\right)\neq\left(0,0\,0\right)$ and therefore, the set M is given by $M=\left\{\left(x_1,x_2,x_3\right)\in\mathbb{R}^3 x_3=x_1^2-x_2^2\right\}$ is a regular surface.

VI) Example of a set which is not a regular surface :

Let $N = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2\}$

We contend that this set is not a regular surface. Note that $O = (0,0,0)$ is a point of N. Now, if N were a regular surface, then every point of N would have a local parametrization about that point. We contend that the point $P = (0,0,0)$ of N has no local parametrization about it.

We justify this claim by contradiction. Assume the point P has a local parametrization (U, θ, V). Without loss of generally we assume that U is the disc $D(o, \delta)$ with $\varnothing(0,0) = (0,0,0) \in N$. Now consider any point p_1, p_2 as shown in the figure and let q_1, q_2 be the points in U= $D(0,\delta)$ with $\theta(q_1) = p_1, \theta(q_2) = p_2$



Now the contradiction is : the points, q_1, q_2 in $D(0,\delta)$ can be joined by a continuous curve c not passing through the point $q = (o\,o)$ but the curve $\theta(c)$. can not avoide $\theta(q) = (0,0,0)!$ consequently such a local parametrization around $p = (0,0,0)$ of M does not exist and therefore N is not a regular surface.

7.2 TRANSITION FUNCTIONS AND THEIR SMOOTHNESS

At this stage, we study on important aspect of local coordinates on a regular surface M : Let (U, θ) and (W, θ) be local parametrizations with $Q(U) \cap \theta(W)$ non-empty. Then any point p in $Q(U) \cap \theta(W)$=N (say) has two sets of coordinates :

$$Q^{-1}(p) = (u_1(p), u_2(p)) \text{ and } \theta^{-1}(p) = (w_1(p), w_2(p)).$$

This gives rise to coordinate functions $u_1 ; u_2 : N \to \mathbb{R}$ given by $\theta^{-1}(p) = (u_1(p), u_2(p))$ and $w_1 ; w_2 : N \to \mathbb{R}$ given by $\theta^{-1}(p) = (w_1(p), w_2(p))$ for all $p \in N$.

Now we can see that coordinates in one set are functions of the coordinates in other set. In fact we have :

$$(u_1, u_2) = (u_1(w_1, w_2), u_2(w_1, w_2)) = \theta^{-1} o\, \theta(w_1, w_2) \text{ and}$$
$$(w_1, w_2) = (w_1(u_1, u_2), w_2(u_1, u_2)) = \theta^{-1} \circ \theta(u_1, u_2).$$

It is an important (but tedius) result that these functions are smooth functions (of the indicated variables). Here we give a sketchy proof of this fact.

Proposition 1 : The following functions are smooth :

$u_1(w_1, w_2) : \theta^{-1}(N) \to \mathbb{R}$

$u_2(w_1, w_2) : \theta^{-1}(N) \to \mathbb{R}$

$w_1(u_1, u_2) : \theta^{-1}(N) \to \mathbb{R}$

$w_2(u_1, u_2) : \theta^{-1}(N) \to \mathbb{R}$

Proof : We prove smoothness of $(w_1(u_1, u_2),\ w_2(u_1, u_2))$ on the set $\theta^{-1}(N)$. (Smoothness of the other two functions is obtained in a similar proof.). We accomplish this by verifying smoothness of $\theta^{-1} o\, \theta$ in a neighborhood of each $q \in \theta^{-1}(N)$.

Thus choose arbitraity a $q \in \theta^{-1}(N)$. Let $\theta^{-1} o\, \theta(q) = p$.

Now recall $J_\theta(p)$ has rank = 2 and therefore some 2×2 sub-matrix of the matrix :

$$J_\theta(p) = \begin{bmatrix} \frac{\partial \theta^1}{\partial w_1}(p) & \frac{\partial \theta^1}{\partial w_2}(p) \\ \frac{\partial \theta^2}{\partial w_1}(p) & \frac{\partial \theta^2}{\partial w_2}(p) \\ \frac{\partial \theta^3}{\partial w_1}(p) & \frac{\partial \theta^3}{\partial w_2}(p) \end{bmatrix}$$

is non-singular. Assume without loss of generality that

$$\begin{bmatrix} \frac{\partial\theta^1}{\partial w_1}(p) & \frac{\partial\theta^1}{\partial w_2}(p) \\ \frac{\partial\theta^2}{\partial w_1}(p) & \frac{\partial\theta^2}{\partial w_2}(p) \end{bmatrix}$$

is non-singular. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map given by $\pi(x_1, x_2, x_3) = (x_1, x_2)$. Then the non-singularity of the above sub-matrix is the non-singularity of $J_{\pi o\theta}(p)$. Therefore, by the inverse mapping theorem, we get what $\pi o\theta$ is a local differmorphism in a neighbourhood of p. This implies the invertibility of π in a neighbourhood of $\theta(p)$. (Here, we are using local 1-1 ness of both $\pi o\theta$ and θ.) Now, we have :

$$\begin{aligned} \theta^{-1}o\theta = (q) &= \theta^{-1}o\pi^{-1}o\pi oQ \\ &= \left(Ho\theta^{-1}o(\pi oQ)\right) \end{aligned}$$

Thus, smoothness of both $-(\pi o\theta)^{-1}$ and $\pi o\theta$-implies smoothness of the map $\theta^{-1}o\theta$ which is the map which gives the change of coordinates $(u_1, u_2) \to (w_1, w_2)$.

For the two parametrizations (U, θ), (W, θ) of M with $Q(U) \cap \theta(W) N \neq \varnothing$ the maps

$$\begin{aligned} \theta^{-1}o\theta : \theta^{-1}(N) &\to \theta^{-1}(N) \\ (u_1, u_2) &\mapsto (w_1, w_2) \end{aligned} \quad \text{and} \quad \begin{aligned} \theta^{-1}o\,\theta : \theta^{-1}(N) &\to \varnothing^{-1}(N) \\ (w_1, w_2) &\mapsto (u_1, u_2) \end{aligned}$$

both describing the change of coordinates are alled the transition maps between the sets $\theta^{-1}(N)\, and\, Q^{-1}(N)$. Transition maps describe one set of coordinates as functions of the other set of coordinates. And we have proved above that the transition maps are smooth functions of the coordinates equivalently put: the two sets of coordinates -(u_1, u_2) and (w_1, w_2)- are smoothly related.

7.3 SMOOTH FUNCTIONS ON REGULAR SURFACES

Let M be a regular surface.

We will consider only two types of functions and define their being smooth :

- *curves* $c : I \to M$ and
- • *functions* $f : M \to \mathbb{R}$

Definition 4 : A curve $c : I \to M$ is smooth if $c : I \to \mathbb{R}^3$ is smooth.

It readily follows that if $c : I \to M$ is smooth in the sense of this definition than $c : I \to M$ is continuous.

Next, let (U, θ) be a local parametrization on M with associated coordinate functions $u_1 : u_2 : \theta(U)(\subseteq M) \to \mathbb{R}$. Then the curve $\theta^{-1} oC : I \to U$ can be written in terms of its coordinates :

$\theta^{-1} oC(t) = (u_1(t), u_2(t))$ for all $t \in I$ with $c(t) \in U$. Thus we get the functions $u_1(t), u_2(t)$ of the variable t. Now it can be seen that the curve c is differentiable (= smooth) if both the real valued functions $t \to u_1(t), t \to u_2(t)$ of the real variable $t \in I$ are smooth.

Finally we define smoothness of functions $f : M \to \mathbb{R}$.

Definitions 5 : $f : M \to \mathbb{R}$ is smooth if for every local parametrization (U, θ) of M, the function $f \, o \, \theta^{-1} : U \to \mathbb{R}$ is smooth.

Note that $f \, o \, Q^{-1} : U \to \mathbb{R}$ is a function of the two coordinate variables $u_1 \, u_2$ an U and therefore differentiability of $f \, o \, \theta^{-1}$ is a familiar concept.

We consider the set $C^\infty(M)$ of all smooth functions $f : M \to \mathbb{R}$. It is easy to see that the operations of addition and multiplication of functions $f : M \to \mathbb{R}$ give the set $C^\infty(M)$ the structure of a commutative and associative ring with identity.

Finally, let Ω be a non-empty open subset of a regular surface M. Then it is easy to see that Ω also is a regular surface. For if (U, θ) is a local paramentrization of M, then putting $\widetilde{U} = \theta^{-1}(\theta(U) \cap \Omega)$ and $\tilde{\theta} = \theta / \widetilde{U}$ we get a local parametrization $(\widetilde{U}, \tilde{\theta})$ on Ω. Such local parametrization $(\widetilde{U}, \tilde{\theta})$ on Ω obtained from (U, θ) of M give a coordinate atlas for Ω and thus, Ω becomes a regular surface in its own right. In particular, the function spaces $C^\infty(\Omega)$ for open $\Omega \subseteq M$ are well-defined.

In the next chapter, we will develop differential calculus on M using these function spaces $C^\infty(\Omega)$.

7.4 EXERCISES

1) Let M be the subset of $\mathbb{R}^3$ obtained by rotating the parabola $x_3 = 4x_1^2$ about the x_3 axis. Describe smooth function $f : M \to \mathbb{R}$ which generates M.

2) The 2-torus T_2 is the surface generated by revolving the circle $\left(x_1 - a\right)^2 + x_3^2 = b^2$ about the x_3-axis, a,b being constants with a<b. Exhibit a smooth coordinate atlas on T_2.

3) Although the set $M = \left\{\left(x_1, x_2, x_3\right) \in \mathbb{R}^3 : x_3^2 = x_1^2 + x_2^2\right\}$ is not a regular surface (as explained above) prove that its subset $\widetilde{M} = M - \{0\}$ is a regular surface.

4) Prove that a circular cylinder is a surface and describe a smooth atlas on it.

5) Let M be a regular surface and Ω an open subset of M. Let $f : M \to \mathbb{R}$ be a smooth map. Prove that ${}^{f}\!/\!_{\Omega}$ is smooth on Ω.

6) Let M_1 and M_2 be regular surfaces, with $M_1 \cap M_2$ open in both M_1 and M_2. Prove $M_1 \cup M_2$ is a regular surface.

7) Let M be a regular surface and let $f, g : M \to \mathbb{R}$ be smooth functions.
Prove :
a) $f + g$ is smooth on M.
b) $f \cdot g$ is smooth on M.

❖❖❖❖

8

CALCULUS ON REGULAR SURFACES

Unit Structure :

8.1 The Tangent Spaces $T_p\left(\mathbb{R}^3\right)$

8.2 The Tangent Space $T_p\left(M\right)$

8.3 Another Description of Tangent Vectors

8.4 Smooth Vector Fields

8.5 Smooth Forms on M

Having introduced regular surfaces M and the function spaces $C^{\infty}(\Omega)$ for various open $\Omega \subseteq M$ we consider some more concepts contributing to the calculus on a regular surface, namely : the tangent spaces $T_p(M)$ for $p \in M$, smooth vector fields on open subsets Ω of M smooth linear and bilinear forms and their properties and so on. The resulting calculus is then used as a tool to study the geometry of M. The primary geometri features of a regular surface M are two smooth symmetric bilinear forms the first fundamental form I and the second fundamental forms II - they will be introduced in the next chapter.

8.1 THE TANGENT SPACES $T_p\left(\mathbb{R}^3\right)$

In differential geometry, geometric object are highly localized. In particular, we need consider the classical vectors - the directed segments in $\mathbb{R}^3$ being located at various points of $\mathbb{R}^3$. Thus for a point $p \in \mathbb{R}^3$ and for a vector x in $\mathbb{R}^3$, we consider the ordered pair $(p,x);$ it represents the vector x not emanating from the origin of $\mathbb{R}^3$ but located at (or having its foot at) the point p.

$T_p\left(\mathbb{R}^3\right)$ denotes the set of all such ordered pairs

$$=\left\{(p\ x) : x \in \mathbb{R}^3\right.$$

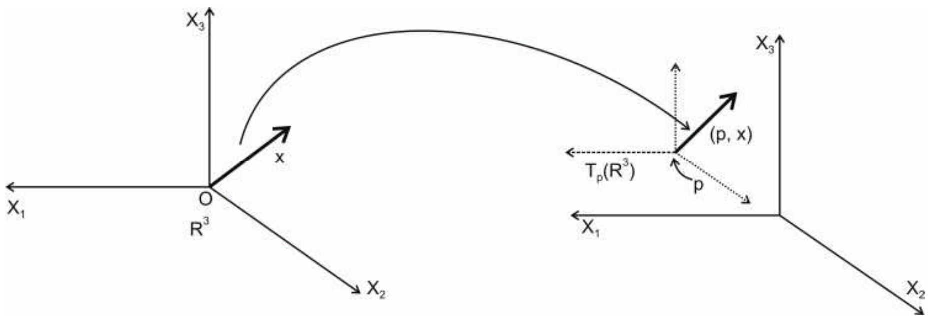
$$=\{p\} \times \mathbb{R}^3$$

Clearly for a fixed point p in $\mathbb{R}^3$, the set $T_p\left(\mathbb{R}^3\right)$ is in 1-1 correspondence with $\mathbb{R}^3$:

$$\mathbb{R}^3 \leftrightarrow T_p\left(\mathbb{R}^3\right)$$
$$x \leftrightarrow (p,x)$$

Therefore the familiar inner product space structure of $\mathbb{R}^3$ induces an inner produce space structure on $T_p(\mathbb{R})$:

- $(p,x)+(p,y)=(p,x+y)$
- $a(p,x)=(p,a\cdot x), a\in\mathbb{R}$
- $\langle (p,x),(p,y)\rangle = \langle x,y\rangle$



8.2 THE TANGENT SPACE $T_p(M)$

Let p be a point of the regular surface M.

Definition : A Vector $(p,x)\in T_p\left(\mathbb{R}^3\right)$ is tangential to M at the point p of M if there exists a smooth curve $c:(-\delta,\delta)\to M$ for some $\delta>0$ with the properties : $c(0)=P, \dot{c}(0)\left(=\frac{dc}{dt}(0)\right)=x$.

$T_p(M)$ denotes the set of all $(p,x)\in T_p\left(\mathbb{R}^3\right)$ which are tangential to M at p. we prove below that $T_p(M)$ is a two-dimensional subspace of the vector space $T_p\left(\mathbb{R}^3\right)$. Towards this aim, consider a local paramentrization (U,θ) of M with $p\in\theta(U):\theta(0)=p$. Recall that the derivative map $\theta_*(o):=D\theta(o)\ T_o(U)\to T_p\left(\mathbb{R}^3\right)$ is an injective linear map. We prove now that it maps $T_o\left(\mathbb{R}^2\right)=T_o(U)\simeq\mathbb{R}^2$ onto $T_p(M)$. To see this, consider a $(p,x)\in T_p(M)$. By definition of a tangential vector there exists a smooth curve $c:(-\delta,\delta)\to M$ with $c(0)=p$ and $\dot{c}(0)=x$. Assume without loss of generality that $c(t)\in\theta(U)$ for all t in $(-\delta,\delta)$. Now using bijective property of θ, consider $\tilde{c}:(-\delta,\delta)\to U$ such that $\theta\circ\tilde{c}\equiv c$. Let $\omega=\tilde{c}(o)$. This $\omega\in T_o(U)$ and $\theta_*(o)(\omega)=x$. Thus proves that $\theta_*:T_o(U)\to T_p(M)$ is surjective - Consequently $\theta_*(T_o(U))=T_p(M)$ is a linear subspace of $T_p(\mathbb{R}^3)$.

Clearly, the above result implies that the map $\theta_*(O):T_o(U)\to T_p(M)$ is an isomorphism and therefore $T_p(M)$ is a two dimensional subspace of $T_p(\mathbb{R}^3)$. We restate this fact in the following :

Proposition 1 : For each $p\in M, T_p(M)$ is a two dimensional subspace of $T_p(\mathbb{R}^3)$.

There is yet another noteworthy fact, namely the coordinate chart (U,θ) around a $p\in M$ gives rise to a vector basis of $T_p(M)$:

Consider the curves $\alpha_1:(-n,n)\to U, \alpha_2:(-n,n)\to U$ for small enough $n>0$; which are given by :

$\alpha_1(s)=(s,0)\alpha_2(s)=(0,s).\, s\in(-n,n)$. We have $\dot{\alpha}_1(0)=(1,0), \dot{\alpha}_2(0)=(0,1)$ which are vectors in $T_o(U)$ constituting a vector basis of $T_o(U)$ consequently the vectors.

$\theta_*(0)\left(\dot{\alpha}_1(0)\right)=\dfrac{\partial\theta}{\partial u_1}(0)$, $\theta_*(0)\left(\dot{\alpha}_2(0)\right)=\dfrac{\partial\theta}{\partial u_1}(0)$ form a vector basis of $T_p(M)$.

Note that the maps :
$(n,n)\to M; s\mapsto\theta(s,o)$ and $(-n,n)\to M; s\mapsto\theta(o,s)$

are two smooth curves passing through p and giving the basic tangent vectors $\dfrac{\partial\theta}{\partial u_1}(0,0), \dfrac{\partial\theta}{\partial u_2}(0,0)$ respectively and therefore they are vectors tangential to M at p.

Also, it is important to note that $p \in \theta(U)$ was arbitrary point of $\theta(U)$ and therefore the above discussion lead us to two vector fields on $\theta(U)$, both tangential to M at the points of $\theta(U)$: For each $u=(u_1,u_2)\in U$ (and not only $(0,0)$ as above) the vectors $\frac{\partial\theta}{\partial u_1}(u), \frac{\partial\theta}{\partial u_2}(u)$ are tangential to M at the point $\tilde{p}=\theta(u)$. Thus we get two tangential vector fields $\frac{\partial\theta}{\partial u_1}, \frac{\partial\theta}{\partial u_2}$ on $V=\theta(U)$ such that at each point $\tilde{p}=\theta(u)$ of V, $\frac{\partial\theta}{\partial u_1}(a), \frac{\partial\theta}{\partial u_2}(a)$ form a vector basis of $T_{\tilde{p}}(M)$.

There is one more point pertaining to the notation which we explain right here : We adapt the notations $\frac{\partial\theta}{\partial u_1}(\tilde{p}), \frac{\partial\theta}{\partial u_2}(\tilde{p})$ for the vectors $\frac{\partial\theta}{\partial u_1}(u), \frac{\partial\theta}{\partial u_2}(u)$ respectively at the point $\tilde{p}=\theta(u)\in V$.

These notations - the pair $\left\{\frac{\partial\theta}{\partial u_1}(\tilde{p}), \frac{\partial\theta}{\partial u_2}(\tilde{p})\right\}$ representing tangent vectors but partial differentiations in appearance are adapted everywhere in mathematical literature because vectors operate on functions by differentiation. We will explain more about this notational convention below, but at this stage but we note that because $\frac{\partial\theta}{\partial u_1}(p), \frac{\partial\theta}{\partial u_2}(p)$ is a vector basis of $T_p(M)$ for any point p of $\theta(U)=V$, any vector $(p,x)\in T_p(M)$ is expressible as a linear combination : $a_1\frac{\partial}{\partial u_1}(p)+a_2\frac{\partial}{\partial u_2}(p)$ for a unique pair a_1, a_2 of real number.

Now, about the action of a tangent vector on a smooth function : Let $(p,x)\in T_p(M)$ and let f be a smooth function defined on an open $W\subseteq M$ with $p\in W$. These two entities combine to produce the real number (denoted in differential calculus by) $D_x f(p)$ the derivative of f at p along x. It is obtained as follows. Choose a smooth curve $C:(-\delta,\delta)\to W$ with $c(o)=p$ and $\dot{c}(o)=x$. Then we lay :

$$D_x f(p)=\frac{d}{dt}f(c(t))_{t=0}$$

Now, let (U,θ) be any chart around p, its coordinate functions being u_1, u_2. Using these coordinates, we write :

$$c(t) = (c_1(t), c_2(t),)$$

Then we have :

$$\frac{d}{dt} f(c(t))_{t=o} = \frac{d}{dt} f(c_1(t), c_2(t),)_{t=o}$$

$$\dot{c}_1(0)\frac{\partial f}{\partial u_1}(p) + \dot{c}_2(0)\frac{\partial f}{\partial u_2}(p)$$

But, we also have $x = \frac{dc}{dt}(0)$

$$= \dot{c}_1(0)\frac{\partial}{\partial u_1}(p) + \dot{c}_2(0)\frac{\partial}{\partial u_2}(p)$$

and therefore, we get;

$$D_x f(p) = \left(\dot{C}_1(o)\frac{\partial}{\partial u_1}(p) + \dot{C}_2(o)\frac{\partial}{\partial u_2}(o)\right)(f)$$

$$= \dot{c}_1(0)\frac{\partial f}{\partial u_1}(p) + \dot{c}_2\frac{\partial f}{\partial u_2}(p).$$

To conclude, we have the following :

Given a point $p \in M$ and a pair of local coordinates (u_1, u_2) around p (determined by a local parametrization (U,θ), we have the following :

- Any $(p,x) \in T_p(M)$ can be expressed uniquely in the form $(p,x) = a_1\frac{\partial}{\partial u_1}(p) + a_2\frac{\partial}{\partial u_2}(p);\ \ a_1, a_2$ in $\mathbb{R}$.
- If $f : W \to \mathbb{R}$ is a smooth function, its domain of definition W being an open subset of M with $p \in W$ and if (U,θ) is a local parametrization around p, its coordinates being (u_1, u_2) then the real number $D_x f(p)$ - the derivative of f at p along x - is given by $(D_x f)(p) = a_1\frac{\partial f}{\partial u_1}(p) + a_2\frac{\partial f}{\partial u_2}(p)$ where $x = a_1\frac{\partial}{\partial u_1}(p) + a_2\frac{\partial f}{\partial u_2}(p)$.
- The resulting map $D_x(p) : C^\infty(W) \to \mathbb{R}$ has the following properties.

i) $D_x(af+bg)(p)=aD_x f(p)+bD_x(g)(p)$ for all f,g in $C^\infty(W)$ and a, b in $\mathbb{R}$

ii) $D_x(f\cdot g)(p)=D_x(f)(p)(g)(p)+(f)(p)\cdot D_x(g)(p)$ for all f,g in $C^\infty(W)$

iii) If $f:W\to\mathbb{R};\ g:\widetilde{W}\to\mathbb{R}$ are smooth functions, W and $\widetilde{W}$ being open neighborhoods of p, then $f\equiv g$ on $W\cap W$ implies : $D_x(f)(p)=D_x(g)(p)$

- For any $(p,x),(p,y)$ in $T_p(M)$, a, b in $\mathbb{R}$, $D_{(ax+by)}(f)(p)=a\cdot D_x(f)(p)+b\cdot D_y(f)(p)$ holds for all $f\in C^\infty(W)$ (W being an open neighborhood of p).

The last property implies that any smooth $f:W\to\mathbb{R}$ gives rise to a linear form on $T_p(M)$; we will denote it by $df(p)$. Thus the linear form $df(p):T_p(M)\to\mathbb{R}$ is given by :

$$
\begin{aligned}
df(p)(p,x) &= D_x(f)(p)\\
&= x_1\frac{\partial f}{\partial u_1}(p)+x_2\frac{\partial f}{\partial u_2}(p)
\end{aligned}
$$

for all $(p,x)=x_1\dfrac{\partial}{\partial u_1}(p)+x_2\dfrac{\partial}{\partial u_2}(p)\in T_p(M)$.

In particular, the coordinate functions u_1,u_2 of a local parametrization (U,θ) around p give rise to the linear forms $du_1(p),du_2(p)$ on $T_p(M)$. Note that $du_1(p),du_2(p)$ satisfy $du_1(p)\left(\dfrac{\partial}{\partial u_j}(p)\right)=\delta_{ij}$ and consequently we get :

$df(p)=\dfrac{\partial f}{\partial u_1}(p)+du_1(p)+\dfrac{\partial f}{\partial u_2}(p)\cdot du_2(p)$ for any smooth $f:W\to\mathbb{R}$.

Definition 2 : The linear form $df(p):T_p(M)\to\mathbb{R}$ is called the differential of f at the point p.

8.3 ANOTHER DESCRIPTION OF TANGENT VECTORS

Above we have defined a $(p,x)\in T_p(M)$ as a vector $x\in\mathbb{R}^3$ placed at p for which there corresponds a smooth curve

$c:(-\delta,\delta)\to M$ with $c(0)=p$ and $\dot{c}(0)=u$. The vector thus defined (tangential to M at p) operates on smooth functions $f:M\to\mathbb{R}$ producing real numbers $D_x(f)(p)$ given by.

$$D_x(f)(p)=\frac{d}{dt}f(C(t))_{t=o}$$

This action of (p,x) on smooth f has the following properties (as we have noted them above) :

i) If f, g in $C^\infty(M)$ are such that $f\equiv g$ in some neighborhood of p, then $D_x(f)(p)=D_x(g)(p)$.

ii) $D_x(af+bg)(p)=a\,D_x(f)(p)+b\,D_x(g)(p)$ for all f,g in $C^\infty(M)$ and for all a, b in $\mathbb{R}$.

iii) $D_x(f\cdot g)(p)=D_x(f)(p)(g)(p)+(f)(p)\cdot D_x(g)(p)$ for all f,g in $C^\infty(W)$.

We prove below that conversely, properties (1) , (2) and (3) above specify the vector $(p,x)\in T_p(M)$ completely. To be precise, we prove the following.

Proposition 2 : Let $L:C^\infty(M)\to\mathbb{R}$ be an operator satisfying the following conditions :

1) If f, g are such that $f\equiv g$ in some open neighborhood of p, then $L(f)=L(g)$.

ii) Let $L(af+bg)=a\,L(f)(p)+b\,L(g)$ for all f,g in $C^\infty(M)$ and for all a, b in $\mathbb{R}$

iii) Let $L(f\cdot g)=L(f)g(p)+f(p)L(g)$ for all f,g in $C^\infty(M)$

Then there exists a unique $x\in\mathbb{R}^3$, tangential to M at p such that $L(f)=D_x(f)(p)$ for all $f\in C^\infty(M)$

Next, to prove the existence of such of (p,x), note the following two properties :

- The result is a local result in the sense that by property (1) of L, the value $L(f)$ for any $f\in C^\infty(M)$ depends on the variation of f within (an arbitrarity chosen) neighborhood of p.

Accordingly, we can chose a local parametrization (U,θ) on M with (i) $U = B(0,\delta)$ for some $\delta > o$ and (ii) $Q(0) = p$ and then for $\theta(U)$; by property (1) of L, the behaviour of f outside $\theta(U)$ does not affect $L(f)$.

••*If* $f \equiv$ constant c (say), then $L(f) = 0$.

For, taking $f \equiv g \equiv 1$ we have :

$L(1) = L(1^2) = L(1)\cdot 1 + 1\cdot L(1) = 2L(1)$, thus $L(1) = 2L(1)$ and therefore $L(1) = 0$.

Now $L(C) = C \quad L(1) = 0$

Thus, $L(C) = 0$ for any constant function $f \equiv C$.

Now, for the above described choice of (U,θ) consider the finite Taylor expansion of a f around p

$$f(u_1 \ u_2) = f(p) + (u_1 - p_1)\frac{\partial f}{\partial u_1}(p) + (u_2 - p_2)\frac{\partial f}{\partial u_2}(p)$$

$+\sum_{i,j=1}^{2}(u_i - p_i)(u_j - p_j)g_{ij}(u)$ for some smooth functions $g_{ij} : U \to \mathbb{R}$.

Applying the operator L to this identity, we get :

$$\begin{aligned}
L(f) = L(f(p)) + L(u_1 - p_2)\frac{\partial f}{\partial u_1}(p) + L(u_2 - p_2)\frac{\partial f}{\partial u_2}(p) \\
+\sum_i (u_i - p_i)\sum_j (p_j - p_j)g_{ij}(p) \\
+\sum_j (u_j - p_j)\sum_i (p_i - p_i)g_{ij}(p) \\
= 0 + L(u_1 - p_2)\frac{\partial f}{\partial u_1}(p) + L(u_2 - p_2)\frac{\partial f}{\partial u_2}(p) + O \\
= x_1\frac{\partial f}{\partial u_1}(p) + x_2\frac{\partial f}{\partial u_2}(p)
\end{aligned}$$

where we are putting $x_1 = L(u_1 - p_1); x_2 = L(u_2 - p_2)$. We form the vector $x = x_1\frac{\partial}{\partial u_1}(p) + x_2\frac{\partial}{\partial u_2}(p) \in T_p(M)$ to get $L(f) = D_x(f)(p)$ for all $f \in C^\infty(M)$.

8.4 SMOOTH VECTOR FIELDS

A vector field on M is an assignment X, assigning a vector $X(p)$ to each $p \in M; X(p)$ being tangential to M at $p : X(p) \in T_p(M)$ for each $p \in M$.

A vector field X on M and a smooth function $f : M \to \mathbb{R}$ combine to produce a function on M, - we denote it by $X(f)$ where for each $p \in M$ the real number $X(f)(p)$ is given by :

$$X(f)(p) = D_x(f)(p)$$

where $X(p) \in T_p(M)$ is given by $X(p) = (p,x)$.

Now let (U,θ) be any local parametrization on M is; its coordinate functions being (u_1, u_2). Then for each $p \in \theta(U)$ we have: $X(p) = X_1(p)\frac{\partial}{\partial u_1}(p) + X_2(p)\frac{\partial}{\partial u_2}(p)$ with X_1, X_2 being smooth functions on $\theta(U) = V \subset M$. Therefore, for any smooth $f : M \to \mathbb{R}$, we get $X(f)(p) = X_1(p)\frac{\partial}{\partial u_1}(p) + X_2(p)\frac{\partial}{\partial u_2}(p)$ for every $p \in \theta(U)$. It now follows that $X(f)$ is smooth if the function f is smooth. We are interested in vector fields X on M which produce smooth functions.

Definition 3 : A vector field X on M is smooth if $X(f) : M \to \mathbb{R}$ is smooth whenever $f : M \to \mathbb{R}$ is smooth.

$X(M)$ denotes the set of all smooth vector field X on M.

It now follows that a vector field X on M smooth (i.e. $X \in X(M)$) if it satisfied the follows condition : For any local parametrization (U,θ), the representation : $X = X_1\frac{\partial}{\partial u_1} + X_2\frac{\partial}{\partial u_1}$ has the coefficient function $X_1, X_2 : U \to \mathbb{R}$ be smooth. It can be verified that the set $X(M)$ has the following algebric property :

If X, Y are smooth vector fields on M and if f, g are smooth functions on M, then the vector field $fX, +gY$ is also a smooth vector field on M.

Thus the set $X(M)$ of smooth vector fields on M is a module over the ring $C^{\infty}(M)$ of smooth functions.

Above, we considered smooth vector fields on a regular surface M. Since every non-empty open subset Ω of M because a regular surface, we have the well-defined concept of smooth vector fields on an open subset Ω of M. We denote the resulting $C^{\infty}(\Omega)$ module by $X(\Omega)$.

8.5 SMOOTH FORMS ON M

We consider now objects which are dual to the vector fields, they are called smooth one-forms on M. First, (an arbitrary) one form on M is an assignment of a liner form $w(p): T_p(M) \to \mathbb{R}$ to each $p \in M$. We denote the collection $\{w(p): p \in M\}$ by w.

Now, note that a vector field X on M and a one-form $w = \{w(p): p \in M\}$ on M combine to give a function $f: M \to \mathbb{R}$: For each $p \in M$, we evaluate the one form $w(p): T_p(M) \to \mathbb{R}$ on the vector $X(p) \in T_p(M)$ to get the real number $w(p)(X(p)) \in \mathbb{R}$; we put $f(p) = w(p)(X(p))$. This gives the function :

$$f: M \to \mathbb{R};\ p \mapsto w(p)(X(p))$$

We will be interested in those 1-forms to which differential calculus can be applied in a reasonable way. This motivates the following definition :

Definition 4 : A 1-form w on M is smooth if for every smooth vector field X on M, the function $w(X): M \to \mathbb{R}$ is smooth on M.

Now, we have the following list of simple facts related to smooth 1-forms and smooth vector fields on M :

1) If w, n are two smooth 1-forms and if $f, g: M \to \mathbb{R}$ are any two smooth functions then the combination $fw + gn$ given by $\left(f\dot{w} + gn\right)(p) = f(p)w(p) + g(p)n(p) \in T^*(p), p \in M$ is a smooth 1-form on M.

Therefore, the set of all smooth 1-forms on M is a module over the ring $C^{\infty}(M)$.

2) For any $p \in M$, the forms $du_1(p), du_2(p) : T_p(M) \to \mathbb{R}$ form a vector basis of the dual space $T_p^*(m)$, (of $T_p(M)$) and therefore, if w is a 1-form on M, then for each $p \in M$ we every $p \in M$, we seet that the 1-form w gives rise to two functions $f, g : M \to \mathbb{R}$ such that holds for every $p \in M$ and thus we have : $w = fdu_1 + gdu_2$.

3) Note further that $w\left(\frac{\partial}{\partial u_1}\right) = f$ and $w\left(\frac{\partial}{\partial u_1}\right) = g$ consequently, if w is a smooth 1-form on M then f,g (as above) must both be smooth

4) Now consider $w = fdu_1 + gdu_2$ and on arbitrary smooth vector field $X = h\frac{\partial}{\partial u_1} + k\frac{\partial}{\partial u_2}, h, k : M \to \mathbb{R}$ being both smooth functions.

Then we have $w(X) = fh + gk$.

Consequently, we have : w is smooth if and only if both, f, g are smooth functions. It also follows that the set of smooth 1 forms is a module over the ring $C^\infty(M)$ if w, n are smooth 1-forms and f, g are smooth functions on M, then $fw + gn$ is a smooth form on M.

5) If w is a smooth 1-form on M and if Ω is an open subset of M, then the restriction of w to Ω is a smooth 1-form on Ω.

6) And a smooth function $f : M \to \mathbb{R}$ gives rise to a smooth 1-form $\frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2$ on M, we denote it by df and call it the differential of f, thus, $df(X) = \frac{\partial f}{du_1} \cdot g + \frac{\partial f}{du_2} \cdot h$ with $X = g\frac{\partial}{du_1} + h\frac{\partial}{du_2}$.

We also consider smooth, symmetric 2 forms on M. first recall a few algebraic terms.

Let E be a finite dimensional real vector space.

A bilinear form on E is a map $\langle\ ,\ \rangle : \mathrm{E} \times \mathrm{E} \to \mathbb{R}; (x, y) \mapsto (x, y)$ which is linear in each of the two vector variables x, y ranging on E.

A bilinear form $\langle\ ,\ \rangle : \mathrm{E} \times \mathrm{E} \to \mathbb{R}$ is said to be

- Symmetric if $\langle x, y \rangle = \langle y, x \rangle$ holds for all x, y in E.
- Positive definite if $\langle x, x \rangle \geq 0$ for $x \in \mathrm{E}$ and $\langle x, x \rangle = 0$ only when $x = 0$.
- An <u>inner product</u> on E if $\langle\ ,\ \rangle$ is both symmetric and positive definite.

Let $\{e_1, e_2, ..., e_n\}$ be a vector basis of E, putting $a_{ij} = \langle e_i, e_j \rangle$ for $1 \le i \cdot j \le n$, we get the matrix $\left[a_{ij}\right]$ of $\langle\ ,\ \rangle$ with respect to the vector basis $\{e_1, e_2, ..., e_n\}$. Note that -

i) $\langle x, y \rangle = \sum_{1 \le i, j \le n} a_{ij} x_i y_j$ where $x = \sum_{i=1}^{n} x_i e_i$ and $y = \sum_{i=1}^{n} y_i e_i$.

ii) $\langle\ ,\ \rangle$ is symmetric if and only if $a_{ji} = a_{ji}$ holds for all $i, j, 1 \le i, j \le n$.

Now, we introduce the notion of a smooth, symmetric bilinear form on a regular surface.

Definition 5 : A bilinear form on a regular surface M is a rule-denoted by B - which associates with each $p \in M$, a bilinear form B(p) on the tangent space $T_p(M)$:

$$B(p): T_p(M) \times T_p(M) \to \mathbb{R}; (u, v) \mapsto B(p)(u, v)$$

A bilinear form B on M and two tangent fields X, Y on M combine to produce a function $B(X, Y): M \to \mathbb{R}$:

For each $p \in M$ the bilinear form $B(p): T_p(M) \times T_p(M) \to \mathbb{R}$ evaluated over $X(p), Y(p) \mapsto T_p(M)$ gives, the real number $B(p)(X(p), Y(p)) \in \mathbb{R}$. This specifies the function $B(X, Y): M \to \mathbb{R}: B(X, Y)(p) = B(p)(X(p), Y(p))$ for every $p \in M$. It now follows that the following identities hold :

- $B(fX, gY) = f \cdot gB(X, Y)$ for all functions $f, g: M \to \mathbb{R}$ and for all vector fields X, Y on M.
- i) $B(X_1 + X_2, Y) = B(X_1, Y) + B(X_2, Y)$

  ii) $B(X, Y_1 + Y_2) = B(X_1 Y_1) + B(X, Y_2)$ for all vector fields $X, X_1 X_2, Y, Y_1, Y_2$

Here is an example of an important bilinear form on M : Let, for each $p \in M$, $I(p): T_p(M) \times T_p(M) \to \mathbb{R}$ be given by :

$$I(p)(v, w) = \langle v, w \rangle_p\ v, w \in T_p(M).$$

This gives rise to the following map

I: $X(M) \times X(M) \to \mathbb{R}:$

$I(X,Y)(p) = I(p)(X(p),Y(p))$ for all $p \in M$ and for all smooth vector fields X,Y on M.

This biliner form is called the first fundamental form of the surface. Read more about it in Chapter 9.

Exercise :

1) For smooth vector fields X, Y on M and for smooth functions $f,g,h : M \to \mathbb{R}$, verify the following identies.

a) $X(f.g) = X(f)\cdot g + f \cdot X(g)$

b) If $X(f) = 0$ for all smooth $f : M \to \mathbb{R}$, then $X \equiv 0$

c) $X(af + bg) \equiv aX(f) + bX(g)$

2) Let X, Y be smooth vector fields on M giving the map $L : C^{\infty}(M) \to C^{\infty}(M)$:
$L(f) = X(Y(f)) - Y(X(f))$

Verify that L satisfies the properties (a), (b) (c) of exercise (1) above, using Proposition 2 deduce that L gives rise to a smooth vetor field on M. We denote this vector field by $[X,Y]$ and call it the Lie-prodct of X, Y in that order. It is also call the Lie-bracket of X, Y.

3) Prove that the operation of forming Lie-bracket $[X,Y]$ of two vector fields X,Y has the following properties :

i) $[X,Y] = -[Y\ X]$

ii) $[fX,Y] = f[X,Y] - Y(f)X$

iii) $[X[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = O$

4) Prove that combing a smooth 1-form w with a smooth vector field X on M produces the functions $w(X)$ which is smooth and the operation $(w,X) \to w(X)$ is bilinear.

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9

PARAMETRIZED SURFACES

Unit Structure :

9.1 An Oriented Parametrized Surface
9.2 The First Fundamental Form
9.3 The Shape Operator
9.4 Covariant Differentiation
9.5 Parallel Transport
9.6 Geodesics

In this chapter and the next, we will study some of the elementary aspects of the geometry of an oriented regular surface M. To begin with we wil discuss the geometry of such a M only at the local level, that is, the geometric structure of a small enough piece of a surface in the form of an open neighbourhood of a point of it. After getting familier with the local geometry, we will consider geometric properties of M as a whole and prove some basic results about them.

Accordingly we begin with a surface element in the form already introduced where it was termed a parametrized surface. Recalling the related concepts and explaining them again in the present context, we introduce two basic geometric ingradients of a parametrized surface namely the first and second fundamental forms I and II on the tangent bundle $T(M)$ of M. Both of them are symmetric two forms on $T(M)$. These forms will lead us to a number of geometric concepts on M : length of a smooth curve on M, covariant differentiation of vector fields, parallel transport of tangent vectors along smooth curves on M, geodesic curves on M, principal curvature of M at a point of it, the Gaussian and mean curvature tensor of M and so on. We introduce the intrinsic nature of some of the geometric properties and conclude the next chapter with the important theorem : Gauss' theorema egragium.

9.1 AN ORIENTED PARAMETRIZED SURFACE

Let M be a parametrized surface, its parametrization being (U,F); thus U is an open subset of $\mathbb{R}^2$, and $F:U\to\mathbb{R}^3$ is a smooth map having the properties :

- $F(U)=M$ and $F:U\to M$ is a homeomorphism, and
- The Jacobean $J_F(q)$ has rank = 2 at every $q\in U$

Now, using the homeomorphism $F:U\to M$, we identify each point p of M with the point $F^{-1}(p)=q$ of $U(c\mathbb{R}^2)$ and regard the native coordinates $(u_1(q),u_2(q))$ as the coordinates of p assigned by the parametrization (U,F).

Thus, there are two sets of coordinates on M :

i) The Cartesian co-ordinates $(x_1(p),x_2(p),x_3(p))$ given by the (Cartersian) Co-ordinate system of the ambinat space $\mathbb{R}^3$ and

ii) the co-ordinates $(u_1(p),u_2(p))$ determined by a parametrization (U,F) on M.

The co-ordinates (u_1,u_2) are independent and are often better adapted to the geometry of M while the Cartesian co-ordinates - being coordinates of the ambient space $\mathbb{R}^3$ - are often used as reference coordinates only. Thus, for a $q\in U$ we have :

$F(q)=(x_1(q),x_2(q),x_3(q))$, Cartesian coordinates of $Q(q)=p\in M$.

$$\frac{\partial F}{\partial u_1}(q)=\left(\frac{\partial x_1}{\partial u_1}(q),\frac{\partial x_2}{\partial u_2}(q),\frac{\partial x_3}{\partial u_1}(q)\right)$$

$$\frac{\partial F}{\partial u_2}(q)=\left(\frac{\partial x_1}{\partial u_2}(q),\frac{\partial x_2}{\partial u_2}(q),\frac{\partial x_3}{\partial u_2}(q)\right) \text{ and so on.}$$

Note that we have adapted the notations $\frac{\partial}{\partial u_1}(p)(or\ \partial_1(p))$ and $\frac{\partial}{\partial u_2}(p)(or\ \partial_2(p))$ for $\frac{\partial F}{\partial u_1}(q),\frac{\partial F}{\partial u_2}(q)$ respectively and in view of these notations, the rank condition - rank of $J_F(q)$ be 2 - is equivalent to the requirement that the vectors $\frac{\partial F}{\partial u_1}(q), \frac{\partial F}{\partial u_2}(q)$ be independent elements of the tangent space $T_p(M)$. Also, keep in mind that the pair $\left\{\frac{\partial}{\partial u_1}(q), \frac{\partial}{\partial u_2}(q)\right\}$ has to play a double role (i) as a vector basis of $T_p(M)$ and (ii) as differential operators operating on smooth functions $f : M \to \mathbb{R}$ giving real numbers $\frac{\partial f}{\partial u_1}(q), \frac{\partial f}{\partial u_2}(q)$. (In thse notations, the point $p \in Q$ appears but it is considerd to be identified with $q : p = F(q)$).

Let us now consider vector fields on M, first those vector fields which are tangential to M.

Recall, a vector field tangential to M (or a tangent field on M) is a rule X associating with each $p \in M$ a vector $X(p) \in T_p(M)$. Now since $\left\{\frac{\partial}{\partial u_1}(q), \frac{\partial}{\partial u_2}(q)\right\}$ is a basis of the vector space. $T_p(M)$, such a $X(p)$ can be expressed uniquely as a linear combination :

$$X(p) = X_1 \frac{\partial}{\partial u_1}(p) + X_2 \frac{\partial}{\partial u_2}(p)$$

$X_1(p), X_2(p)$ being real numbers. This way the vector field gives rise to the well-defined functions $X_1, X_2 : M \to \mathbb{R}$ the vector field then being expressible in the form :

$$X = X_1 \frac{\partial}{\partial u_1} + X_2 \frac{\partial}{\partial u_2}.$$

We regard the vector field X smooth if both the functions X_1, X_2 are smooth on M. Now, for any smooth function $f : M \to \mathbb{R}$, the vector field operates on f producing a smooth function $X(f) : M \to \mathbb{R}$ given by :

$$\begin{aligned} X(f)(p) &= X(p)(f) \\ &= X_1(p)\frac{\partial f}{\partial u_1}(p) + X_2(p)\frac{\partial f}{\partial u_2}(p); \ p \in M \end{aligned}.$$

On the other hand we have vector fields on M which are perpendicular to M : A smooth map $Y : M \to \mathbb{R}^3$ considered as a vector field on M (ie for each $p \in M$ the vector $Y(p)$ being considered located at p) is normal to M if $Y(p) \perp T_p(M)$ for each $p \in M$. For example the vector field Y given by $Y(p) = \frac{\partial F}{\partial u_1}(q) \times \frac{\partial F}{\partial u_2}(q)$ (with $F(q) = p$) for each $p \in M$ is such a normal vector field on M. In particular the vector field N on M given by $N(p) = \dfrac{\frac{\partial(p)}{\partial u_1} \times \frac{\partial(p)}{\partial u_2}}{\left\| \frac{\partial}{\partial u_1}(p) \times \frac{\partial}{\partial u_2}(p) \right\|}, p \in M$ has the unit normal property.

Consequently for each $p \in M$, the triple $\left\{ \frac{\partial}{\partial u_1}(p), \frac{\partial}{\partial u_2}(p), N(p) \right\}$ forms a vector basis of $T_p\left(\mathbb{R}^3\right)$ and the subset $\left\{ \frac{\partial}{\partial u_1}(p), \frac{\partial}{\partial u_2}(p) \right\}$ is a vector basis of the subspace $T_p(M)$ of $T_p\left(\mathbb{R}^3\right)$. On account of this property the unit normal field N on M orients the parametrized surface M. In what is follow, we will consider M to be oriented by this normal field N.

9.2 THE FIRST FUNDAMENTAL FORM

Now we consider the standard inner product $\langle .,\rangle$ of $\mathbb{R}^3$ which incudces the inner product $\langle ,\rangle_p$ on each $T_p\left(\mathbb{R}^3\right)$. We restrict $\langle ;\rangle_p$ to the subspace $T_p(M)$ of $T_p\left(\mathbb{R}^3\right)$ and denote it by $I(p)$. Thus, for each $p \in M$, we have the symmetric, positive definite bilinear form $I(p) : T_p(M) \times T_p(M) \to \mathbb{R}$ given by $I(p)\big((p\,v),(p\,w)\big) = \langle v,w\rangle$ for every pair $(p\,v),(p\,w)$ of vectors tangential to M at p.

Having introduced the inner product $I(p)$ on $T_p(M)$, we will write only $\langle v,w\rangle$ in place of the full form $I(p)\big((p\,v),(p\,w)\big)$. This is meant to simplify the notation whenever the point p of tangency of the vectors $(p,v),(p,w)$ is understood.

We consider the entire collection $\{I(p) : p(M)\}$ as a single entity and denote it by I.

Definition 1 : $I\left(=\left\{I(p): p\in M\right\}\right)$ is the first fundamental form of the surface.

For each $p\in I$, putting $g_{ij}(p)=\left\langle \frac{\partial}{\partial u_i}(p),\frac{\partial}{\partial u_j}(p)\right\rangle$ $=\left\langle \frac{\partial F}{\partial u_1}(p),\frac{\partial F}{\partial u_2}(p)\right\rangle$ for $p\in M, 1\le i,j\le 2$, we get the matrix valued function.

$$g: U\left\{T_p(M)\times T_p(M): p\in M\right\}\to M_2(\mathbb{R}).$$

It is the matrix of the first fundamental form.

Now if $X=\sum_i Xi\frac{\partial}{\partial u_i}, Y\sum Yi\frac{\partial}{\partial u_i}$ are two smooth vector fields (tangential to M) then we get the map $I(X,Y): M\to\mathbb{R}$ given by :

$$\begin{aligned}
I(X,Y)(p)&=\langle X(p),Y(p)\rangle\\
&=\left\langle \sum_i X_i(p)\frac{\partial}{\partial u_i}(p),\sum_j Y_j(p)\frac{\partial}{\partial u_j}(p)\right\rangle\\
&=\sum_{ij} Xi(p)Y_j(p)\left\langle \frac{\partial}{\partial u_i}(p),\frac{\partial}{\partial u_j}(p)\right\rangle\\
&=\sum_{ij} Xi(p)Y_j(p)g_{ij}(p)
\end{aligned}$$

We need consider the inverse of each $\left[g_{ij}(p)\right]$; we denote the resulting matrix by $\left[g^{ij}(p)\right]$, thus we have : $\sum_k g_{ik}(p)g^{kj}(p)\equiv\delta_{ij}$.

Let us consider following examples of surfaces and obtain the first fundamental forms for each of them :

(I) The (oriented) graph of a smooth functions : $f: U\to\mathbb{R}$;
U being an open subset of $\mathbb{R}^2$

Now, $M=\left\{\left(u_1,u_2,f(u_1,u_2)\right):(u_1,u_2)\in U\right\}$.

The parametrization map $F: U\to\mathbb{R}^3$ is :

$$F=(u_1,u_2)=\left(u_1,u_2,f\left(u_1,u_2\right)\right)=p,(u_1,u_2)\in U\,.$$

Therefore, $\dfrac{\partial}{\partial u_1}\times\dfrac{\partial}{\partial u_2}=\left(-\dfrac{\partial f}{\partial u_1},\dfrac{-\partial f}{\partial u_2},1\right)$ and then

$$N(p)=\frac{\left(-\dfrac{\partial f}{\partial u_1},\dfrac{-\partial f}{\partial u_2},1\right)}{\sqrt{\left(\dfrac{\partial f}{\partial u_1}\right)^2+\left(\dfrac{-\partial f}{\partial u_2}\right)^2+1}}$$

the right hand side of the above equally being evaluated at the point $p=\left(u_1,u_2,f\left(u_1,u_2\right)\right)$ for $(u_1,u_2)\in U$.

Also, we have :

$g_{11}=1+\left(\dfrac{\partial f}{\partial u_1}\right)^2, g_{22}=1+\left(\dfrac{\partial f}{\partial u_1}\right)^2$ and $g_{12}=g_{21}=\dfrac{\partial f}{\partial u_1},\dfrac{\partial f}{\partial u_2}$ and therefore the matrix of the first fundamental form of the surface :

$$\left[g_{ij}\right]=\begin{bmatrix} 1+\left(\dfrac{\partial f}{\partial u_1}\right)^2, & \dfrac{\partial f}{\partial u_1},\dfrac{\partial f}{\partial u_2} \\ \dfrac{\partial f}{\partial u_1},\dfrac{\partial f}{\partial u_2}, & 1+\left(\dfrac{\partial f}{\partial u_2}\right)^2 \end{bmatrix}$$

(II)A particular case of the above is the hemisphere of radius $a>o$:

$$M=\{x_1,x_2,x_3\}\in\mathbb{R}^3 : x_1^2+x_2^2<a^2; x_3=+\sqrt{\left(a^2-x_1^2+x_2^2\right)}$$

Now we have : $U=\left\{(u_1,u_2)\in\mathbb{R}^2; u_1^2+u_2^2<a^2\right\}$ and the map $f:U\to\mathbb{R}$ is $f\left(u_1,u_2\right)=+\sqrt{\left(a^2-u_1^2-u_2^2\right)}$. Finding expressions for the unit normal map $p\mapsto N(p)$, the matrix $\left[g_{ij}(p)\right]$ of the first fundamental form etc are left for the reader as an exercise.

(III) Let $U=\mathbb{R}^2$ and let $F:\mathbb{R}^2\to\mathbb{R}^3$ be the map given by $F\left(u_1,u_2\right)=\left(u_1,u_1^2+u_2,4u_1^2+u_2^3\right)$, $\left(u_1\ u_2\right)\in\mathbb{R}^2$. Now we have :

$J_F(u_1, u_2) = \begin{bmatrix} 1 & 0 \\ 2u_1 & 1 \\ 8u_1 & 3u_2^2 \end{bmatrix}$ for all $(u_1\, u_2) \in \mathbb{R}^2$; the matrix clearly has rank = 2 (because its submatrix $\begin{bmatrix} 1 & 0 \\ 2u_1 & 1 \end{bmatrix}$ is non singular) at every $(u_1\, u_2) \in \mathbb{R}^2$. Consequently the set M :

$$M = \left\{ \left(u_1, u_1^2 + u_2, 4u_1^2 + u_2^3\right) : \{u_1, u_2\} \in \mathbb{R}^2 \right\}$$

is a parametrized surface.

Now, we have

i) $F : U \to M \subset \mathbb{R}^3$ is given by $F(u_1, u_2) = \left(u_1, u_1^2 + u_2, 4u_1^2 + u_2^3\right)$,

ii) The vectors $\frac{\partial F}{\partial u_1}(u) = (1, 2u_1, 8u_1)$ and $\frac{\partial F}{\partial u_2} = \left(0, 1, 3u_2^2\right)$ span the tangent space $T_p(M)$ where $p = F(u_1, u_2)$;

iii) The unit normal field N on M is given by

$$N(p) = \frac{\left(2u_1\left(3u_2^2 - 4\right), -3u_2^2, 1\right)}{\sqrt{\left\{4u_1^2\left(3u_2^2 - 4\right)^2 + 9u_2^2 + 1\right\}}} \text{ and}$$

iv) The matrix of the first fundamental forms is :

$$\begin{bmatrix} 1 + 68u_1^2, & 2u_1\left(1 + 12u_2^2\right) \\ 2u_1\left(1 + 12u_2^2\right), & 1 + 9u_2^4 \end{bmatrix}$$

(IV) We consider a unit speed curve $C : I \to \mathbb{R}^3$ and the associated binormal field $b : I \to \mathbb{R}^3$ along it. Associated with the pair (a, b) is the parametrized surface M : $M = \{C(r) + sb(r) : r \in I, s \in \mathbb{R}\}$

Putting $U = I \times \mathbb{R}$, let $F : U \to \mathbb{R}^3$ be given by $F(r,s) = C(r) + sb(r), (r,s) \in I \times \mathbb{R}$.

Then we have : $\frac{\partial F}{\partial r}(r,s) = t(r) - s \cdot \mathfrak{I}(r)\, n(r)$

$$\frac{\partial F}{\partial r}(r,s) = b(r)$$

Clearly, $\frac{\partial F}{\partial r}(r,s), \frac{\partial F}{\partial s}(r,s)$ are livearly independent vectors and consequently (U,F) is a parametrization of the set M. Moreover, the unit normal field N on M is given by :

$$N(p) = \frac{-\left[s\Im(r)t(r)+\eta(r)\right]}{\sqrt{\left\{s^2\Im(r)^2+1\right\}}}$$

while the matrix of the first fundamental form is $\begin{bmatrix} 1+s^2\Im(r)^2, & 0 \\ 0, & 1 \end{bmatrix}$

we resume our study of a parametrized surface M having its parametrization (U, F) :

Let $S(2)$ be the unit sphere in $\mathbb{R}^3$ i.e.
$S(2) = \left\{(x_1\ x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$

Shifting the unit normal $N(p)$ from the point p of M and relocating it at $O \in \mathbb{R}^3$ we get the map (denoted by the same letters) :

$N : M \rightarrow S(2)$

We call this map the Gauss Map of the surface M. note that the Gauss map on the unit
$M = \left\{(x_1\ x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 > o\right\}$ is the identity map on M while that on the hemi sphere of radius $a > 0 : M = \left\{(x_1\ x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = a^2; x_3 > 0\right\}$ is : $N(p) = \frac{p}{a}$ for $p \in M$.

Illustrative examples (I) --- (IV) above describe the Gauss map of their surfaces.

9.3 THE SHAPE OPERATOR

We differentiate the Gauss map-defined above-at a point p of M with respect to the vectors $v \in T_p(M)$. The resulting linear map - the differential of the Gauss map at p-has important geometric prosperties; we describe them below.

Let p be a point of M and let $v \in T_p(M)$. We consider the derivative $(D_v N)(p)$. Thus, we choose a smooth $C : (-\delta, \delta) \rightarrow M$ with $C(0) = p$ and $\dot{C}(0) = v$. Then we have $(D,N)(p) = \frac{d}{dt} N(C(t))_{t=o}$

Note that $(D,N)(p) \in T_p(M)$. For, we have $\left\langle \dot{N}C(t), NC(t) \right\rangle \equiv 1$ for $t \in (-\delta, \delta)$ and therefore

$$\frac{d}{dt}\left\langle N(C(t))_{t=o}, N(C(p)) \right\rangle \equiv 0$$

i.e. $$2\left\langle \frac{d}{dt} N(C(t))_{t=o}, N(C(p)) \right\rangle \equiv 0$$

i.e. $$\left\langle (D_o N)(p), N(p) \right\rangle = O$$

the perpendicularity of $(D,N)(p)$ with $N(p)$ now implies $(D,N)(p)$ is in $T_p(M)$. Thus, the Gauss map, when differentiated at a $p \in M$ gives the linear map :

$$T_p(M) \to T_p(M)$$
$$v \mapsto (D,N)(p)$$

In what is to follow, we consider the map $v \to -(D,N)(p)$, the negative sign attached here is only to follow the standard practice in mathematics literature. We denote the resulting (linear) map by L_p :

$$L_p : T_p(M) \to T_p(M)$$

Definition 2 :

The linear map $L_p : T_p(M) \mapsto T_p(M)$ is called the <u>shape operator</u> of M at the point p.

The shape operator L_p is also called the <u>Weingarten map</u> of M at p.

Considering the Weignarten map L_p along with the linear product $I(p)$ of $T_p(M)$, we haave the important property of it :

Proposition 1 : L_p is a self-adjoint linear endomorphism of the inner product space $\left(T_p(M), I(p)\right)$.

Proof : Since $\left\{\frac{\partial}{\partial u_1}(p), \frac{\partial}{\partial u_2}(p)\right\}$ is a vector basis of $T_p(M)$ it is enough to verify the following equalities :

$$< L_p\left(\frac{\partial}{\partial u_i}(p), \frac{\partial}{\partial u_j}(p)\right) = \frac{\partial}{\partial u_i}(p), L_p\left(\frac{\partial}{\partial u_j}(p)\right) > \text{ for } 1 \leq i, j \leq 2$$

Proof : We have :

$$L_p \frac{\partial}{\partial u_i}(p) = D_{-\frac{\partial}{\partial u_i}(p)}(N)$$

$$= -\frac{\partial N}{\partial u_i}(p)$$

and therefore,

$$\left\langle L_p\left(\frac{\partial}{\partial u_i}(p)\right), \frac{\partial}{\partial u_j}(p)\right\rangle = -\left\langle \frac{\partial N}{\partial u_i}(p) \frac{\partial F}{\partial u_j}(p)\right\rangle$$

$$= \frac{-\partial}{\partial u_i}\left\langle N(p), \frac{\partial F}{\partial u_j}(p)\right\rangle + \left\langle N(p), \frac{\partial^2 F}{\partial u_i \partial u_j}(p)\right\rangle$$

$$= 0 + \left\langle N(p), \frac{\partial^2 F}{\partial u_i \partial u_j}(p)\right\rangle$$

The first summond above is O, because $\left\langle N(q), \frac{\partial F}{\partial u_i}(q)\right\rangle \equiv 0,$ ($\frac{\partial F}{\partial u_i}(q)$ being tangential to M at q while N(q) is perpendicular to the whole space $T_q(M)$). Thus

$$\left\langle L_p\left(\frac{\partial}{\partial u_i}(p)\right), \frac{\partial}{\partial u_j}(p)\right\rangle = \left\langle N(p), \frac{\partial^2 F}{\partial u_i \partial u_j}(p)\right\rangle.$$

Similarly we get :

$$\left\langle \frac{\partial}{\partial u_i}(p), L_p\left(\frac{\partial}{\partial u_j}(p)\right)\right\rangle = \left\langle N(p), \frac{\partial^2 F}{\partial u_i \partial u_j}(p)\right\rangle$$

$$= \left\langle N(p), \frac{\partial^2 F}{\partial u_i \partial u_j}(p)\right\rangle$$

Combining these two equalities, we get

$\left\langle L_p\left(\frac{\partial}{\partial u_i}(p)\right), \frac{\partial}{\partial u_j}(p)\right\rangle = \left\langle \frac{\partial}{\partial u_i}(p), L_p\left(\frac{\partial}{\partial u_j}(p)\right)\right\rangle$ which leades us to the self adjointness of L_p.

Next, we wish to find the matrix of L_p with respect to the vector basis $\left\{ \frac{\partial}{\partial u_1}(p), \frac{\partial}{\partial u_2}(p) \right\}$ of $T_p(M)$.

Suppose $L_p\left(\frac{\partial}{\partial u_i}(p)\right), = \sum_k \alpha ik \frac{\partial}{\partial u_k}(p)$ (of course, the summation being over $k = 1,2$.) Taking inner product of the above equality with $\frac{\partial}{\partial u_j}(p)$, we get $\left\langle L_p\left(\frac{\partial}{\partial u_i}(p)\right), \frac{\partial}{\partial u_j}(p)\right\rangle = \sum_k ai_k gk_j(p)$.

But we already have
$\left\langle L_p\left(\frac{\partial}{\partial u_i}(p)\right), \frac{\partial}{\partial u_j}(p)\right\rangle = \left\langle N(p), \frac{\partial^2 F}{\partial u_i \partial u_j}(p)\right\rangle$ and therefore the above equation gives $\left\langle N(p), \frac{\partial^2 F}{\partial u_i \partial u_j}(p)\right\rangle = \sum_k \alpha i_k gk_j(p)$.

Let $\left[g^{ij}(p)\right]$ be the inverse of the matrix $\left[g_{ij}(p)\right]$. Using this inverse matrix, we get

$$\sum_j \left\langle N(p), \frac{\partial^2 F}{\partial u_i \partial u_j}(p)\right\rangle g^{je}(p) = \sum_{jk} \alpha i_k gk_j(p)\, g^{jk}(p)$$
$$= \sum_k \alpha_{ik} \delta_k^e$$
$$= \alpha_{ie}$$

Thus $\alpha_{ij} = \sum_k \left\langle N(p), \frac{\partial^2 F}{\partial u_i \partial u_k}(p)\right\rangle g^{kj}(p)$ .... (*)

This gives the matrix $\left[\alpha_{ij}\right]$ of the Weingarten map L_p.

Let us consider the following illustrative examples : M being the graph of a smooth function $f: U \to \mathbb{R}$, (as usual U being an open subset of $\mathbb{R}^2$).

Now, we have the parametrisation map $F: U \to \mathbb{R}^3$ given by $F(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$ $(u_1, u_2) \in U$.

Writing $f_1 = \frac{\partial f}{\partial u_1}$, $f_2 = \frac{\partial f}{\partial u_2}$, $f_{ij} = \frac{\partial^2 f}{\partial u_i \partial u_j} | \le i, j \le x$ etc.

i) $\left[g_{ij}\right]=\begin{bmatrix}1+f_2^2, & f_1f_2\\ f_1f_2 \quad , & 1+f_2^2\end{bmatrix}\left[g^{ij}\right]=\dfrac{1}{\left(4f_1^2+f_2^2\right)}\begin{bmatrix}1+f_2^2, & -f_1f_2\\ -f_1f_2 \quad , & 1+f_1^2\end{bmatrix}$

ii) $N(p)=\left(\dfrac{-f_1(u_1,u_2)}{\sqrt{\left(1+f_1^2+f_2^2\right)}},\dfrac{-f_2(u_1,u_2)}{\sqrt{\left(1+f_1^2+f_2^2\right)}},\dfrac{1}{\sqrt{\left(1+f_1^2+f_2^2\right)}}\right)$ and

iii) $\dfrac{\partial^2 F}{\partial u_i \partial u_j}=\left(O,O,f_{ij}\right)$

Substituting these expressions in the formulae (*) we get :

$$\alpha_{11}=\frac{f_{11}\left(1+f_2^2\right)-f_{12}\cdot f_1\cdot f_2}{\left(1+f_1^2+f_2^2\right)^{3/2}}$$

$$\alpha_{12}=\frac{f_{12}\left(1+f_1^2\right)-f_{11}f_1f_2}{\left(1+f_1^2+f_2^2\right)^{3/2}}$$

$$\alpha_{21}=\frac{f_{21}\left(1+f_2^2\right)-f_{22}f_1f_2}{\left(1+f_1^2+f_2^2\right)^{3/2}}$$

and $$\alpha_{22}=\frac{f_{22}\left(1+f_2^2\right)-f_{22}f_1f_2}{\left(1+f_1^2+f_2^2\right)^{3/2}}$$

Taking $f:\mathbb{R}^2\to\mathbb{R}$ given by $f\left(u_1\,u_2\right)=u_1^2+u_1^2$ we get :

i) $f_1=2u_1, f_2=2u_2$

ii) $f_{11}=2=f_{22}, f_{12}=f_{21}=O$

iii) $\left(1+f_1^2+f_2^2\right)^{3/2}=\left(1+4u_1^2+4u_2^2\right)^{3/2}$ and therefore

$$\left[L_p\right]=\frac{1}{\left(1+4u_1^2+4u_2^2\right)^{3/2}},\begin{bmatrix}4\left(1+4u_2^2\right) & -16u_1u_2\\ -16u_1u_2 & 4\left(1+4u_2^2\right)\end{bmatrix}.$$

We combine the Weingartain maps $L_p:T_p(M)\to T_p(M)$ and the first fundamental form $I_p=\langle\quad\rangle_p=\langle\quad\rangle,:T_p(M)\times T_p(M)\to\mathbb{R}$ to get a bilinear map $II(p):T_p(m)\times T_p(m)\to\mathbb{R}$ for each $p\in M$ as follows : If v, w are vectors in $T_p(m)$, then

$$II(p)(v,w)=I(p)\left(L_p(v),w\right)=\left\langle L_p(v),w\right\rangle.$$

We denote the collection $\{II(p): p \in M\}$ by II and call it the <u>second fundamental</u> form on M.

Note that the second fundamental form II combines two smooth vector field X, Y on M and produces a smooth function $\pi(X,Y): M \to \mathbb{R}$ which is given by :

$$\pi(X,Y)(p) = \langle L_p(X(p)); Y(p)\rangle; \, p \in M \, .$$

Because I is bilinear and each L_p is self-adjoint, we get the following identities :

- $II(X,Y) = II(Y,X)$, X,Y being smooth vector fields on M.
- $\pi(fX + gY, Z) = f\,\pi(X,Z) + g\pi(Y,Z)$

Second fundamental form is used to express curvature properties of M, we will discuss this point in the next chapter.

9.4 COVARIANT DIFFERENTIATION

Given a smooth tangent field X on M and a $v \in T_p(M)$ covariant differentiation is a process producing a vector - denoted by $\nabla_v(X)$ – in $T_p(M)$.

Recall: To a $v \in T_p(M)$ there corresponds a smooth $C: (-\delta,\delta) \to M$ having the properties $C(0) = p, \dot{C}(0) = v$. The two - X, and C - Combine to give the smooth map

$$t \mapsto X(C(t)); t \in (-\delta,\delta).$$

Differentiation of it gives $D_v(X) = \frac{d}{dt} X(C(t)) \in T_p(\mathbb{R}^3)$. Note that though $D_v(X)$ is a vector located at p, it is not (in general) tangential to M at p.

To get a vector tangential to M at p, we project it down in the subspace $T_p(M)$ of $T_p(\mathbb{R}^3)$; that is let $\Pi(p): T_p(\mathbb{R}^3) \to T_p(M)$ be the desired projection thus, $\Pi(p)(w) = w - \langle w, N(p)\rangle N(p)$ for all $w \in T_p(\mathbb{R}^3)$. Now, we set

$$\nabla_v(X) = \Pi(p)\left(\frac{dX(C(t))}{dt}_{t=o}\right)$$

$$= \Pi(p)(D_v X) = D_v(X) - \langle D_v(X), N(p)\rangle N(p)$$

Definition 3 : $\nabla_v(X)$ is the <u>covariant derivative</u> of X with respect to $v \in T_p(M)$.

The covariant derivative has the following properties

- $\nabla_{(av+bw)}(X) = a\nabla_v(X) + b\nabla_w(X)$ for all v,w in $T_p(M)$ a, b in $\mathbb{R}$ and for all smooth vector fields X on M.
- $\nabla_v(X+Y) = \nabla_v(X) + \nabla_v(Y)$ for all $v \in T_p(M)$ and for all smooth vector fields X and Y.
- $\nabla_v(f \cdot X) = D_v(f)X(p) + f(p)\nabla_v(X)$ for all $v \in T_p(M)$ and for all smooth vector fields X (Recall $D_v(f)$ is the usual directional derivative of $f : D_v(f) = \frac{d}{dt} f(C(t))_{t=o}$.

All these properties follow from (i) the properties of $D_v(f) = \frac{d}{dt} f(C(t))_{t=o}$ and (ii) the linearity of the map $\Pi(p) : T_p(\mathbb{R}^3) \to T_p(M)$.

For a $v \in T_p(M)$ and for a tangential vector field X on M we intend to express $\nabla_v(X) \in T_p(M)$ using the vector basis $\left\{\frac{\partial}{\partial u_1}(p), \frac{\partial}{\partial u_2}(p)\right\}$.

We adapt the notations ∂_i for $\frac{\partial}{\partial u_1}$ and $\partial_i(p)$ for $\frac{\partial}{\partial u_i}(p), (i = 1,2)$ only for a short while.

Let $v = v_1\partial_1(p) + v_2\partial_2(p) \in T_p(M), v_1 v_2 \in \mathbb{R}$ and let $X = X_1\partial_1 + X_2\partial_2$ be a vector field on M with $X_1, X_2 : M \to \mathbb{R}$ being smooth functions.

Now, we have :

$$\begin{aligned}\nabla_v(X) &= v_1\nabla_{\partial_1(p)}(X_1\partial_1 + X_2\partial_2) + v_2\nabla_{\partial_2}(X_1\partial_1 + X_2\partial_2)\\ &= \left(v_1\frac{\partial X_1}{\partial u_1}(p) + v_2\frac{\partial X_1}{\partial u_2}(p)\right)\partial_1(p)\\ &+ \left(v_1\frac{\partial X_2}{\partial u_1}(p) + v_2\frac{\partial X_2}{\partial u_2}(p)\right)\partial_2(p)\\ &+ \Big(v_1X_1(p)\nabla_{\partial_1(p)}(\partial_1) + v_1X_2(p)\nabla_{\partial_1(p)}(\partial_2) +\\ &v_2X_1(p)\nabla_{\partial_2(p)} + v_1X_2(p)\nabla_{\partial_2(p)}(\partial_2)\Big)\end{aligned}$$

Therefore, we need express each $\nabla_{\partial_i(p)}(\partial_j)$ as a linear combination of $\partial_1(p)$ and $\partial_2(p)$. Suppose :

$\nabla_{\partial_i(p)}(\partial_j) = \Gamma_{ij}^1(p)\partial_1(p) + \Gamma_{ij}^2(p)\partial_2(p)$ where $\Gamma_{ij}^1(p)$, $\Gamma_{ij}^2(p)$ are real numbers. (Indeed they depend on i, j and p). Also, we write

$$\frac{\partial^2 F}{\partial u_i \partial u_j}(p) = \nabla_{\partial_i(p)}(\partial_j) + \alpha N(p).$$

Where α is some real number. Combining the above two equalities, we get $\frac{\partial^2 F}{\partial u_i \partial u_j}(p) = \Gamma_{ij}^1(p)\partial_1(p) + \Gamma_{ij}^2(p)\partial_2(p) + \alpha N(p)$........... (1)

Note right here that $\frac{\partial^2 F}{\partial u_i \partial u_j}(p) = \frac{\partial^2 F}{\partial u_j \partial u_i}(p)$ implies $\Gamma_{ij}^1(p) = \Gamma_{ji}^1(p)$ and $\Gamma_{ij}^2(p) = \Gamma_{ji}^2(p)$.

Taking inner product of the equation (1) with $\partial_k(p)$, we get :

$\left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}(p), \partial_k(p) \right\rangle = \Gamma_{ij}^1(p)g_{1k}(p) + \Gamma_{ij}^2(p)g_{2k}(p)$........... (2)

On the other hand, we have :

$$\left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}(p), \partial_k(p) \right\rangle = \frac{\partial}{\partial u_i}(p) \left\langle \frac{\partial F}{\partial u_j}(p), \partial_k(p) \right\rangle - \left\langle \frac{\partial F}{\partial u_j}(p), \frac{\partial^2 F}{\partial u_i \partial u_k}(p) \right\rangle$$

$$= \frac{\partial}{\partial u_i}(p) \langle \partial_j, \partial_k \rangle - \left\langle \partial_j(p), \frac{\partial^2 F}{\partial u_i \partial u_k}(p) \right\rangle$$

$$= \frac{\partial g_{ik}}{\partial u_i}(p) - \left\langle \partial_j(p), \Gamma_{ik}^1(p)\partial_1(p) + \Gamma_{ik}^2(p)\partial_2(p) \right\rangle$$

$$= \frac{\partial g_{ik}}{\partial u_i}(p) - \Gamma_{ik}^1(p) g_{1j}(p) - \Gamma_{ik}^2(p) g_{2j}(p)$$

Combining (2) and (3) above, we get :

$$0 = \frac{\partial g_{ik}}{\partial u_i}(p) - \Gamma_{ik}^1(p) g_{1k}(p) - \Gamma_{ij}^2(p) g_{2k}(p) + \Gamma_{ik}^1(p) g_{1j}(p)$$
$$+\Gamma_{ik}^2(p) g_{2j}(p)$$

Making cyclic permutations in $\{i, j, k\}$ we get two more equalities :

$$\bullet\bullet \; \frac{\partial g_{ki}}{\partial u_j}(p) = \Gamma_{jk}^1(p) g_{1i}(p) + \Gamma_{jk}^2(p) g_{2i}(p) + \Gamma_{ji}^1(p) g_{1j}(p) + \Gamma_{ji}^2(p) g_{2k}(p)$$

and

$$\bullet\bullet\bullet \; \frac{\partial g_{ij}}{\partial u_k}(p) = \Gamma_{ki}^1(p) g_{1j}(p) + \Gamma_{ki}^2(p) g = j(p) + \Gamma_{kj}^1(p) g_{1k}(p) + \Gamma_{kj}^2(p) g_2 i(p)$$

The operation $\bullet + \bullet\bullet - \bullet\bullet\bullet$ yields :

$$\frac{\partial g_{kj}(p)}{\partial u_i} + \frac{\partial g_{ik}(p)}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_k}(p) = 2\Gamma_{ij}^1(p) g_{1k}(p) + 2\Gamma_{ij}^2 g_{2k}(p)$$

$$= 2\sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} g_{\ell k}(p)$$

This $\frac{\partial g_{kj}(p)}{\partial u^i} + \frac{\partial g_{ik}(p)}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_k}(p) = 2\sum_{\ell} \Gamma_{ij}^{\ell} g_{\ell k}(p)$.

Multiplying the above equation by g^{km} and summing the resulting equations for $k = 1,2$ we get

$$\frac{1}{2}\sum_k\left[\frac{\partial g_{kj}(p)}{\partial u_i}+\frac{\partial g_{ik}(p)}{\partial u_j}-\frac{\partial g_{ij}}{\partial u_k}\right]g^{km}(p)=\sum_k \Gamma_{ij}^{\ell}(p)g_{\ell k}(p)g^{km}(p)$$

$$=\sum_k \Gamma_{ij}^{\ell}(p)g_{\ell k}\left(\sum_k g_{\ell k}(p)g^{km}(p)\right)$$

$$=\sum_{\ell}\Gamma_{ij}^{\ell}(p)\delta_{em}$$

$$=\Gamma_{ij}^{m}(p)$$

This gives us the desired formula :

$$\Gamma_{ij}^{m}(p)=\frac{1}{2}\sum_k g^{km}(p)\left[\frac{\partial g_{ki}(p)}{\partial u_i}+\frac{\partial g_{ik}(p)}{\partial u_j}-\frac{\partial g_{ij}}{\partial u_k}(p)\right]\text{............ (*)}$$

Definition 4 : $\left\{\Gamma_{ij}^{k}(p)1\le ij\ k\le 2\right\}$ are called the Christoffel Symbols of the surface M at the point p.

We thus get the function : $\Gamma_{ij}^{k}:M\to\mathbb{R}$.

Their defining property being : $\nabla_{\partial i}(\partial j)=\sum_{k=1}^{2}\Gamma_{ij}^{k}\partial k$.

Now fro any $v=\sum_{2=1}^{2}v_i\partial_i(p).\ X=\sum_{j=1}^{2}X_j\partial_i$, we have

$$\nabla_v(X)=\sum_i v_i\nabla_{\partial i(p)}\left(\sum_j X_j\partial_j\right)$$

$$=\sum_i v_i\sum_j\frac{\partial X_j}{\partial u_i}(p)\partial_j(p)+\sum_i v_iX_j(p)\nabla_{\partial_i}(p)\partial_j$$

$$=\sum_i\sum_j v_i\frac{\partial X_k}{\partial u_i}(p)\partial_k(p)+\sum_k\sum_{ij}v_i\frac{\partial X_j}{\partial u_i}(p)\Gamma_{ij}^{k}(p)\partial_k(p)$$

$$=\sum_k\left\{\sum_i v_i\frac{\partial X_k}{\partial u_i}(p)+\sum_{ij}v_iX_j(p)\Gamma_{ij}^{k}(p)\right\}\partial_k(p)$$

Thus $\nabla_v(X)=\sum_k\left\{\sum_i v_i\frac{\partial X_k}{\partial u_i}(p)+\sum_{ij}v_iX_j(p)\Gamma_{ij}^{k}(p)\right\}\partial_k(p)$.

The derivation (*) above gives a set of handy formulae to calculate the Christoffel symbols. In particular, applying them to the surfaces M which are graphs of functions $f:U\to\mathbb{R}$, we can obtain these functions Γ_{ij}^{k}, for example, the formulae can be applied to obtain Γ_{ij}^{k} on a hemisphere $x_1^2 + x_2^2 + x_3^2 = a^2$, or on a surface of revolution such as $x_3 = x_1^2 + x_2^2$ and so on. This is left as exercises for the reader.

Also, above we were considering the covariant different differentiatie $\nabla_v(X)$ of X at a point. The concept generatizes immediately : Given a pair of smooth vector fields say X, Y on M, differentiate one of them say X with respect to the other, obtaining a new vector field Z on M given by $Z(p) = \nabla_{Y(p)}(X)$. It can be verified that the resulting Z is a smooth vector field. We denote Z by $\nabla_Y(X)$ it is the covariant derivative of X with respect to Y.

9.5 PARALLEL TRANSPORT

We new use covariant differentiation (the Christoffel symbols Γ_{ij}^{k}) to move tangent vectors along smooth curves on M the movement preserving their tangentially, their length and the angle between two of them.

To be more specific, let $c : I \to M$ be a smooth curve, $p = c(t_o)$, and $v = v_1 \dfrac{\partial}{\partial u_i}(p) + v_2 \dfrac{\partial}{\partial u_2}(p)$ tangential to M at p. we want to transfer v from $p = c(t_o)$ to each point $c(t)$ of the curve in such a way that it is tangential to M at $c(t)$, its length remaining unaltered. This mode of transport of v then generates a vector field X along *c* i.e. a map : $t(\in I) \mapsto X(t) \in T_{c(t)}(M)$ with $\|X(t)\| = \|v\|$ and $X(t_o) = v$. We then say that the vector field X is obtained from v by parallel transporting v along *c*. Such a vector field is obtained by solving a pair of first order liner ODE (involving the Christoffe symbols Γ_{ij}^{k}.) and using the vector v (which is to be parallel transported) as the initial condition of the linear ODE.

Writing $X(t) = X_1(t)\dfrac{\partial}{\partial u_1}(c(t)) + X_2(t)\dfrac{\partial}{\partial u_2}(c(t))$, we get the (unknown function $X_1, X_2 : I \to \mathbb{R}$. Now, we consider the initial value problem :

$$\left.\begin{aligned}&\frac{dX_1(t)}{dt}+\sum_{ij}\Gamma^1_{ij}\dot{c}(t)X_j(t)=0, X_1(t_o)=v_1\\&\frac{dX_2(t)}{dt}+\sum_{ij}\Gamma^2_{ij}(c(t))\dot{c}(t)X_j(t)=0, X_2(t_o)=v_2\end{aligned}\right\}\text{............ (*)}$$

(In above $v=v_1\frac{\partial}{\partial u_1}(c(t_o))+v_2\frac{\partial}{\partial u_2}(c(t_o))$)

Note that this initial value problem (*) is equivalent to :
$\nabla_{\dot{C}(t)}X(t)=o, X(t_0)=v$ .... $(*^1)$

By Picards theorem, the above initial value problem (*) (or equivalent version $(*^1)$ of it) has a unique solution :

$X : I\to\mathbb{R}^3$

We say that the vector $X(t)\in T_{C(t)}(M)$ is obtained from the vector v by parallel transporting it to $c(t)$ along *c*.

At this stage, we improve our notation slightly : Taking into consideration the initial condtion $X(t_o)=v$, we write X_v for X. Thus each $v\in T_{c(t_o)}(M)$ gives rise to the vector field $X_v : I\to\mathbb{R}^3$ having the properties :

i) $X_v(t)\in T_{c(t)}(M)$ for each $t\in I$,

ii) $\nabla_{\dot{c}(t)}X_v(t)=0$

iii) If v, w are in $T_{c(t_o)}(M)$ a, b in $\mathbb{R}$, then $X_{av+bw}=aX_v+bX_w$

iv) For any v, $w\in T_{(c(t_o))}M$, the associated vector fields X_v, X_w satisfy the identity $\langle X_v(t), X_w(t)\rangle=\langle v,w\rangle$ that is, the parallel transport of any two vectors v, $w\in T_{(c(t_o))}M$, pressures the angle between them (throughout the transport along *c*.)

To justify this last property, we have :

$$\begin{aligned}\frac{d}{dt}\langle X_v(t), X_w(t)\rangle &= \left\langle\nabla_{\dot{c}(t)}X_v(t), X_w(t)\right\rangle\\&\quad+\left\langle X_v(t),\nabla_{\dot{c}(t)}X_w(t)\right\rangle\\&=\langle 0, X_w(t)\rangle+\langle X_v(t)0\rangle\\&=0\end{aligned}$$

Therefore $\langle X_v(t), X_w(t)\rangle = \langle X_v(t_o), X_w(t_o)\rangle = \langle v, w\rangle$

This completes the verification of the claim that the parallel transport preserves the inner product. In particular we have :

a) $\|X_v(t)\| = \|v\|$ i.e. parallel transport preserves the length of the vectors and

b) If $\theta(t)$ is the angle between $X_v(t)$ and $X_w(t)$ then

$$Cos\,\theta(t) = \frac{\langle X_v(t), X_w(t)\rangle}{\|X_v(t)\|\|X_w(t)\|}$$

$$= \frac{\langle X_v(t_o), X_w(t_o)\rangle}{\|X_v(t_o)\|\|X_w(t_o)\|}$$

$= Cos\,\theta(t_o)$ for all t and therefore $\theta(t) \equiv \theta(t_o)$ i.e. parallel transport of tangent vectors along a smooth curve preserves the angle between them.

9.6 GEODESICS

Geoddesics are smoth curves on a surface which have parallel tangent fields.

Definition 5 : A smooth curve $c : I \to M$ satisfying $\nabla_{\dot{C}(t)}\left(\dot{c}(t)\right) \equiv O$ is called a geodesic curve (or simply a geodesic)

Equivalently put, a smooth curve $c : I \to M$ the second derivative $\ddot{c}(t)$ of which is along the normal to the surface is a geodesic.

Writing $c(t) = (c_1(t), c_2(t))(= u_1(c_2(t))u_2(c_2(t)))$ we have

$$\dot{c}(t) = \left(\dot{c}_1(t), \dot{c}_2(t)\right)$$

$$= \dot{c}_1(t)\frac{\partial(c(t))}{\partial u_1} + \dot{c}_2(t)\frac{\partial(c(t))}{\partial u_2}$$

and therefore we have

$$\nabla_{\dot{C}(t)}\left(\dot{c}(t)\right) = \left[\ddot{c}_1(t) + \sum_{ij}\Gamma^1_{ij}(c(t))\dot{c}_i(t)\dot{c}_j(t)\right]\frac{\partial}{\partial_{u1}}(c(t))$$

$$+\left[\ddot{c}_2(t) + \sum_{ij}\Gamma^2_{ij}(c(t))\dot{c}_i(t)\dot{c}_j(t)\right]\frac{\partial}{\partial_{u2}}(c(t))$$

Now $\nabla_{\dot{c}(t)}\left(\dot{c}(t)\right)=O$ yields.

$$\ddot{c}_1(t)+\sum_{ij}\Gamma_{ij}^{1}\left(c(t)\right)\dot{c}_i(t)\dot{c}_j(t)\equiv O$$

$$\ddot{c}_2(t)+\sum_{ij}\Gamma_{ij}^{2}\left(c(t)\right)\dot{c}_i(t)\dot{c}_j(t)\equiv O$$

And then the existence and uniqueness theorem of solution of the second order ODE with a prescribed initial conditions gives the following result.

Theorem 1 : Given $p\in M$, and the $v\in T_p(M)$ there exists a unique geodesic curve $c=c_{(p,v)}:I\to M$ (I being an open interval containing 0) having the following properties :

1) c is defined on the largest open interval I.
2) $c(0)=p$ and $\dot{c}(0)=v$.

Exercises :

1) Let p, a, b be any vectors in $\mathbb{R}^3$ and let $F:\mathbb{R}^2\to\mathbb{R}^3$ be the map given by $F(u,v)=p+ua+vb$ for $(u,v)\in\mathbb{R}^2$.

Prove :

i) $\left[\mathbb{R}^2,F\right]$ give rise to a parametrized surface if and only if $a\times b\neq 0$.

ii) Putting $c=a\times b$, prove that a $w\in\mathbb{R}^3$ is a point of the surface $M=F\left(\mathbb{R}^2\right)$ if and only if $\langle c;w-p\rangle=0$.

2) For each of the following surfaces obtain the matrix $\left[g_{ij}\right]$, its determinant $g=det\left[g_{ij}\right]$ the inverse matrix $\left[g^{ij}\right]$ and the unit normal N :

a) $F(u,v)=\left(R\cos u\cdot\cos v,R\sin u,\cos v,R\sin v\right)$
b) $F(u,v)=\left(u\cos v;u\sin v,bv\right)$
c) $F(u,v)=\left(\left(R+r\cos u\right)\cos v,\left(R+r\cos u\right)\sin v,r\sin u\right)$
R, r being constants.

3) Calculate $\left\{\Gamma_{ij}^{k}\right\}$ for the surfaces M = Graph (f) $f:U\subseteq\left(\mathbb{R}^2\right)\to\mathbb{R}$ being given by

a) $f(u,v) = \frac{(u^2 - v^2)}{2}$; $(u,v) \in \mathbb{R}^2$

b) $f(u,v) = (u^3 - 2uv + 4uv^2 + v^3)$ $(u,v) \in \mathbb{R}2$

4) Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $F(u,v) = (u,v,u^2v)$ $(u,v) \in \mathbb{R}^2$.

Obtain

i) Expression for $\{\Gamma_{ij}^k\}$ for the surface $M = F(\mathbb{R}^2)$

ii) Derive equations for the geodesics on the above surface.

5) Obtain equations for the geodesics on the sphere (part of it) parametrized by the usual longitude-lattitude angles (u,v):

$$F(u,v) = (\cos v \cos u, \cos v \sin u, \sin v)$$

and prove that the great ciucles are the geodesic curves on the sphere.

❖❖❖❖

10

CURVATURE OF A REGULAR SURFACE

Unit Structure :

We study now the main geometric feature of a regular surface M, namely, its curvature. First, we introduce a number of scalar quantities defined at each point p of M, namely.

i) the normal curvature of M along a tangential direction at p;
ii) the principal curvatures of M at p and
iii) the Gaussian and mean curvatures of M at a p.

And then we intro the Riemann curvature tensor which is a biquadratic form on the tangent bundle of M. it is the carrier of complete information about the curvature properties of the surface M. Next, explaining the intrinsic / exterensic nature of geometric. properties of M, we conclude the chapter by proving the important result - the Theorema Egragium of C.F. Gauses - that the Gaussian curvature function is an intrinsic property of a regular surface.

Throughout this chapter, a regular surface is a subset M of $\mathbb{R}^3$ with $F : U \to M$ as its parametrization, its orientation being specified by a given unti normal field $N : M \to \mathbb{R}^3$.

10.1 THE NORMAL CURVATURE

Let p be a point of M and let v be a unit vector tangential to M at p; it is to be treated as a direction vector (tangential to M) at p.

We choose a smooth unit speed curve :

$c : (-\delta, \delta) \to M$ satisfying $c(o) = p$ and $\dot{c}(o) = v$.

Assuming $\ddot{c}(0) \neq o$ we get the curvature $k(p)$ of c at p, which given by : $\ddot{c}(0) = k(p)n(p)$ where $n(p)$ is the principal normal to c at $c(0) = p$.

Now we have two unit vectors located at the point p, namely :

i) the principal normal $n(p)$ of c at p and

ii) the unit normal $N(p)$ to M at p.

In general, the two vectors are distinct.

We consider the decomposition of $\ddot{c}(0)$ into its components : one along the normal $N(p)$ and the other in the tangent plane $T_p(M)$ of M :

$$\begin{aligned}
\ddot{c}(o) &= \ddot{c}(0)(tan) + \ddot{c}(0)(normal) \\
&= \ddot{c}(0)(tan) + \left\langle \ddot{c}(0), N(p) \right\rangle N(p) \\
&= \ddot{c}(0)(tan) + k(p)\langle n(p), N(p) \rangle N(p)
\end{aligned}$$

This equality gives :

$k(p)n(p) = k(p)n(p)$(tangential)+ $k(p)\langle n(p), N(p)\rangle N(p)$.

Now, note the following :

$$\begin{aligned}
\left\langle \ddot{c} \right\rangle, N(p) &= \left\langle \frac{d}{dt}\dot{c}(t), N(c(t)) \right\rangle_{t=O} \\
&= \frac{d}{dt}\left\langle \dot{c}(t), N(c(t)) \right\rangle_{t=O} - \left\langle \dot{c}(0), \frac{d}{dt} N(c(t))_{t=0} \right\rangle \\
&= 0 + \left\langle \dot{c}(0), -\frac{d}{dt} N(c(t))_{t=O} \right\rangle \\
&= \left\langle \dot{c}(0), L_p\left(\dot{c}(0)\right) \right\rangle \\
&= II(p)\left\langle \dot{c}(0), \dot{c}(0) \right\rangle
\end{aligned}$$

Where, of course, $L_p : T_p(M) \to T_p(M)$ is the shape operator and $II(p) : T_p(M) \times T_p(M) \to \mathbb{R}$ is the second fundamental form of M; both at the point p.

Thus, the normal part of the curavature $k(p)$ depends only on the direction $v = \dot{c}(0)$ of c at p and not on the (whole) curve c: If c and $\tilde{c}$ are two curves on M with $c(o) = p = \tilde{c}(o)$ and $\dot{c}(o) = v = \dot{\tilde{c}}(o)$, then $k(p)$(normal) = $\tilde{k}(p)$(normal). This is naturally so, because, while passing through p in the direction v, the curve can wiggle on the surface thus affecting the tangential component (in the surface M) of its curvature but its normal bending being forced by the bending of M in the direction v at p. As such it (the normal part $k(p)$(normal)) is attributed to the curvature property of M at p in the direction $v\left(=\dot{c}(0)\right)$; we call it the normal curvature of M at p in the direction v. We adapt the notation k_v for the normal curvature.

Above we have derived the equality $k_v = II(p)(v,v)$

This result is often called Musiner's Theorem.

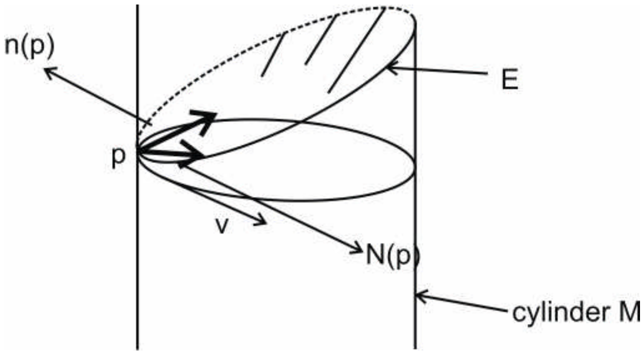
Consider the following simple cases :

- If M is a plane, then for any $c:(-\delta,\delta) \to M$ we have $\ddot{c}(o) = o$ $k(c(o)) = o$ and consequently the normal part of it is zero $k_0 = 0$ for any unit vector $v \in T_p(M)$.

•• Let M be a sphere of radius A>o and let p be a point of M. Then for any unit vector v tangential to the sphere M at p, we consider the great circle $c:\mathbb{R} \to M$ through p having tangent vector v at p. Now, we know that $n(p) \equiv N(p) = \frac{\overrightarrow{OP}}{a}$ and $k(p) = \frac{1}{a}$, consequently, $k_v = \frac{1}{a}$

••• Let M be the circular cylinder of radius a>o. We consider a point $p \in M$ and a unit vector v tangential to the cylinder at the point p. As usual, $N(p)$ is the unit normal to the cylinder at the point p. Thus, we have the two unit vectors, v and $N(p)$ determining a plane Σ through the point p. Note that the intersection $M \cap \Sigma$ is an ellipse E passing through p and the given vector v is tangential to the ellipse at the point p. Let θ be the angle between the plane Σ and $N(p)$. Clearly we can take the ellipse E for the curve $c:(-\delta,\delta)$. Now note that the curvature of E at the point p is $k(p)=\frac{cos\theta}{a}$ and the angle between $N(p)$ and v is θ and consequently, the desired normal curvature k_v of M at p in the direction v is given by :

$k_v=\frac{cos^2\theta}{a}$.



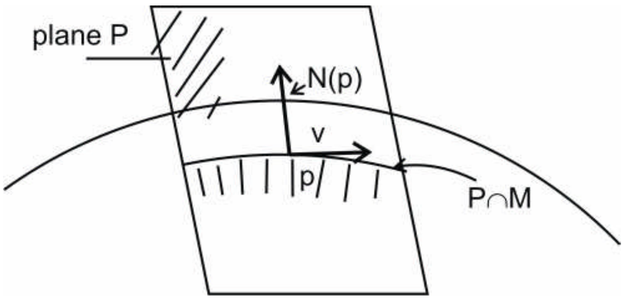
We summarise the above discussion and formulate the definition.

Let p be a point of a regular surface M and v, a unit vector tangential to M at p. Choosing a smooth curve $c:(-\delta,\delta)$ with $c(o)=p,\dot{c}(o)=v$ we consider its curvature $k(p)$ at p and the fraction $k(p)\langle v,N(p)\rangle$. We find that it depends only on the bending property of M at p in the direction v and not on the chosen curve : $k(p)\langle v,N(p)\rangle=II(p)(u,v)$. This leades us to the following definition :

Definition 1 : Given $v\in T_p(M)$ with $\|v\|=1$ the number $k(p)\langle v,N(p)\rangle=k_v$ is the normal curvature of M at p in the direction v.

Here is another realization of k_v : We consider the plane P through p containg the vector v and $N(p)$. It intersects the surface M along a smooth curve $c:(-\delta,\delta)\to M$. Obviously c passes through p and has unit tangent v at p.

We consider its curvature $k(p)$ and the associated quantity $k(p)\langle v,N(p)\rangle=II\left(\dot{c}(0),\dot{c}(0)\right)=\Pi(u,v)$ giving us the normal curvature k_v of M at p along v.



10.2 PRINCIPAL DIRECTIONS / PRINCIPAL CURVATURES

Above we have obtained the expression :

$$k_v = II(u,v)$$
$$= I(L_p u,v) = \langle L_p u,u \rangle$$

for the normal curvature of M at p along v; which involves the shape operator $L_p : T_p(M) \to T_p(M)$. We consider the eigenvalues and eigen-vectors of it. Recall L_p is self-adjoint and therefore its eigenvalues are real. We have the following two cases :

- L_p has a single (real) eigen-value say α and therefore, $L_p = \alpha I$, I being the identity operator of $T_p(M)$. In this case, every unit vector $v \in T_p(M)$ is an eigen-vector of $L_p : L_p(v) = \alpha v$
- L_p has two distinct (real) eigen-values say α, β with $\alpha < \beta$. Let u, v be the unit vectors in $T_p(M)$ corresponding to the eigen-values : $L_p(u) = \alpha v$ and $L_p(U) = \beta v$.

In the first case, that is when L_p has a single eigenvaue α, the point p is said to be an umbitic point of M. For such an umbilic point p of M, we have :

If $v \in T_p(M)$ with $\|v\| = 1$, then $L_p(u) = \alpha v$ and therefore

$$k_v = II(u,v)$$
$$= \langle L_p(v),U \rangle$$
$$= \langle \alpha v,U \rangle$$
$$= \alpha \langle U,U \rangle$$
$$= \alpha$$

This shows that the normal curvature k_v of M at an umbilic point is the same in all directions at p.

Here are simple examples of umbilic points.

i) On a plane P in $\mathbb{R}^3$, any point $p \in P$ is an umbilic point with $k_v = 0$ for every unit vector $v \in T_p(P)$.

ii) Any point P on a sphere S of radius a > 0 is an umbilic point with $k_v = \frac{1}{a}$ for every unit vector $v \in T_p(S)$.

iii) Let M be the surface of revolution generated by rotating the parabola $z = x^2, x \in \mathbb{R}$ about the Z-axis. Then the point $p = (0,0,0)$ is n umbilic point. (In fact it is the only umbilic point on the surface).

(Perhaps the above claim is clear to the reader, but we advise him / her to verify it mathematically in an exercise.)

In the other case, namely, when L_p has two distinct eigenvalues α, β with $\alpha < \beta$. let u, v be unit eigenvectors of α, β respectively (i.e. $L_p(u) = \alpha u, L_p(v) = \beta v$.) then as seen above we have $k_u = \alpha$ and $k_v = \beta$. Moreover $u \perp v$ and consequently, any unit vector $w \in T_p(P)$ can be expressed uniquenly in the form : $w = \cos\theta \cdot u + \sin\theta \cdot v$ where θ is the angle between u and w. Now, the normal curvature k_w of M at p in the direction w is given by

$$
\begin{aligned}
k_w &= (L_p w, w) \\
&= \langle L_p(\cos\theta \cdot u + \sin\theta \cdot u), (\cos\theta \cdot u + \sin\theta \cdot v) \rangle \\
&= \cos^2\theta \langle L_p(u), u \rangle + \sin\theta \cdot \cos\theta \langle L_p u, v \rangle \\
&\quad + \cos\theta \cdot \sin\theta \langle L_p(u), u \rangle + \sin^2\theta \langle L_p(u), v \rangle \\
&= \cos^2\theta \langle \alpha u, u \rangle + 0 + 0 + \sin^2\theta \langle \beta v, v \rangle \\
&= \alpha \cos^2\theta \langle u, u \rangle + \beta \sin^2\theta \langle v, v \rangle \\
&= \alpha \cos^2\theta + \beta \sin^2\theta
\end{aligned}
$$

(In above the middle terms are zero each because $\langle L_p(u), v \rangle = \beta \langle v, u \rangle = \beta \cdot 0$ because $u, \perp v$ and for the same reason $\langle L_p(u), v \rangle = 0$). Thus we get that the normal curvature k_w along such a $w (= \cos\theta u + \sin\theta v)$ is given by $k_w = \alpha \cos^2\theta + \beta \sin^2\theta$.

This formula for k_w which express k_w as a linear combination of the distinguished normal curvatures k_u, k_v involving the angle θ, is known as the Euler's formula.

Now in view of Euler's formula, it is clear that the eigen-values α, β of the shape operator L_p are respectively the minimum and maximum of the set $\left\{k_w : w \in T_p(M); \|w\| = 1\right\}$.

Definition 2 : Suppose, $p(\in M)$ is not an umbilic point. Then the unit eigen-vectors u, v belonging to the minimum and maximum of the normal curvatures α, β are called the principal curvature directions of the surface M at the point p.

Definition 3 : Let α, β be the minimum and maximum values of the normal curvature of M at p. Then the quantities;

$$K(p) = \alpha \cdot \beta$$
$$H(p) = \frac{1}{2}(\alpha + \beta)$$

are called respectively the Gaussian curvature and the mean curvature of M at the point p.

Note that when p is not an umbilical point of M, then the principal curvature directions u, v at p form an orthonormal basis of $\left(T_p(M), I(P)\right)$ and the matrix of the shape operator L_p with respect to this orthonormal basis $\{u, v\}$ is $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ and consequently we have :

i) $K(p) = \det L_p$ and

ii) $H(p) = \frac{1}{2} trace\left(L_p\right)$

We extend the above definition to an umbilic point also : Now we have $\alpha = \beta =$ the constant value k_w for all unit vectors $w \in T_p(M)$ and we them have : $K(p) = \alpha^2 = \beta^2$ and $H(p) = \frac{1}{2}(\alpha + \beta) = \alpha = \beta$.

Thus, we have :

1) In case of a plane P in $\mathbb{R}^3$, at any point p of P, we have : $\alpha = p = o$ and consequently $K(p) = 0 = H(p)$.

ii) Let M be the sphere of radius a > O. Then for any $p \in M$, have $\alpha = \frac{1}{a} = \beta$ and therefore $K(p) = \frac{1}{a^2}$ and $H(p) = \frac{1}{a}$.

iii) Let M be a circular cylinder of radius a > 0.

then at a point p of it, the principal directions are :

a) The line L_1 through p, parallel to the axis of the cylinder and

b) The line L_2 tangential to the cylinder at the point p and per pendicular to L_1.

The principal curvatures are $\alpha = 0$ (the curvature of the line L_1) and $\beta = \frac{1}{a}$, the curvature of the cylinder) and therefore, we get : $K(p) = 0$ and $H(p) = \frac{1}{2a}$.

iv) We consider upper half of the ellipsoide : $M = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, z > O \right\}$ a, b, c being constants with $a > b > c > 0$. Let p be the point $p = (0,0,c)$.

Note that $T_p(M)$ is the plane through p which is parallel to the XOY plane and the unit normal to M at p is the vector (0,0,1) located at the point p.

Now recall, for each unit vector $w \in T_p(M)$ we consider the plane $P(w)$ through p containing $N(p)$ and w. The intersection $P(w) \cap N$ is the half part of an ellipse through the point p and the curvature of this are (of the ellipse) at p is the normal curvature of the ellipsoide M at p in the direction w. in particular we consider the unit vectors $u = (1,0,0)$ and $v = (0,1,0)$ both located at the point (o,o,c) of M.

Clearly $P(u) \cap M$ is the ellipse : $\left\{ (x,0,z) \in \mathbb{R}^3 \, \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \right\}$ and its curvature at the point (o,o,c) is $-{}^{c}\!/_{a^2}$.

Similarly $P(u) \cap M$ is the ellipse : $\left\{ (o,y,z) \in \mathbb{R}^3 \, \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$ and its curvature at the point $(0,0,c)$ is $-{}^{c}\!/_{b^2}$.

Also note that the shape operator L_p has eigen-vectors (1,0,0) and (0,1,0) at $p=(0,0,c)$ and the respective eigen-values $-{}^{c}\!/_{a^2}, -{}^{c}\!/_{b^2}$. Therefore, the vectors $u=(1,0,0)$ $v=(0,1,0)$ are the principal directions of normal curvature and $-{}^{c}\!/_{a^2}, -{}^{c}\!/_{b^2}$ are the principal normal curvatures of the ellipsoide M at the point $p=(0,0,c)$. It now follows that the Gaussian and mean curvatures are given by : $K(p)=\dfrac{-c^2}{a^2\cdot b^2}$ and $H(p)=\dfrac{-c}{2}\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)$.

10.3 THE RIEMANNIAN CURVATURE TENSOR

We introduce now the sophishicated curvature tensor on a smooth, parametrized surface M. Being a smoothly verying field of biquadratic forms on all the tangent spaces $T_p(M)$ of M, it encodes all the curvature properties of the surface (and many more geometric properities of such a M. Naturaly it has very fine algetraic / geometric / analytical features. A comparesnsive study of it therefore leades one far beyond the scope of the syllabus; we cannot cover the topic completely here. Instead, we introduce it very briefly and mention some of its properties and relate the tensor to the Gaussian curvature of M. We then proceed to prove the grand "**theorema egregium**" of Gauss explaining the intrinsic nature of the geometry of M.

To begin with recall the equations (already explained) :

a) $\dfrac{\partial^2 F}{\partial u_i \partial u_j}=\sum_{\ell}\Gamma_{ij}^{\ell}\dfrac{\partial}{\partial u_e}+L_{ij}N$ and

b) $\dfrac{\partial N}{\partial u_i}=\sum_{j}L_i^j\dfrac{\partial F}{\partial u_j} \qquad 1\le i,j,\ell\le 2$

where the functions $L_{ij}:M\to\mathbb{R}$ have the properties :

i) for each $p\in M,\left[L_i^j(p)\right]$ is the matrix of the shape operator $L_p:T_p(M)\to T_p(M)$ with respect to the vector basis $\left\{\dfrac{\partial F}{\partial u_1}(p),\dfrac{\partial F}{\partial u_2}(p)\right\}$ of $T_p(M)$ and

ii) the functions L_i^j, L_{ij} are related as follows :

$$L_i^j = \sum_k g^{jk} L_{ik}, L_{ij} = \sum_k g_{ik} L_j^k .$$

Now, differentiating part (a) of (*) we get :

$$\frac{\partial^3 F}{\partial u_k \partial u_i \partial u_j} = \sum_\ell \frac{\partial \Gamma_{ij}^\ell}{\partial u_k} \frac{\partial F}{\partial u^\ell} + \sum_\ell L_{ij}^\ell \frac{\partial^2 F}{\partial u_k \partial u_\ell} + \frac{\partial L_{ij}^\ell}{\partial u_k} N + L_{ij} \frac{\partial N}{\partial u^k}$$

$$= \sum_m \frac{\partial \Gamma_{ij}^m}{\partial u_k} \frac{\partial F}{\partial u^m} + \sum_\ell L_{ij}^\ell \left\{ \sum_m \Gamma_{k\ell}^m \frac{\partial F}{\partial u_m} + L_{k\ell} N \right\}$$

$$= \frac{\partial L_{ij}}{\partial u_k} N + L_{ij}^\ell \sum_m L_k^m \frac{\partial F}{\partial u_m}$$

$$= \sum_m \left\{ \frac{\partial \Gamma_{ij}^m}{\partial u_k} + \sum_\ell \Gamma_{ij}^\ell \Gamma_{k\ell}^m + L_{ij}^\ell L_k^m \frac{\partial F}{\partial u_m} \right\}$$

$$+ \left\{ \frac{\partial L_{ij}}{\partial u_k} + \sum_\ell \Gamma_{ij}^\ell L_{k\ell} \right\}$$

Similarly, we have :

$$\frac{\partial^3 F}{\partial u_i \partial u_k \partial u_j} = \sum_m \left\{ \frac{\partial \Gamma_{kj}^m}{\partial u_i} + \sum_\ell \Gamma_{kj}^\ell \Gamma_{i\ell}^m + L_{kj} L_i^m \right\} F_m$$

$$+ \left\{ \frac{\partial L_{kj}}{\partial u_i} + \sum_\ell L_{kj}^\ell L_{i\ell} \right\} N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(***)}$$

The subtraction (**) - (***) gives

$$\frac{\partial^3 F}{\partial u_k \partial u_i \partial u_j} - \frac{\partial^3 F}{\partial u_i \partial u_k \partial u_j}$$

$$= \sum_m \left\{ \frac{\partial \Gamma_{kj}^m}{\partial u_k} - \frac{\partial \Gamma_{kj}^m}{\partial u_i} + \sum_\ell \left[\Gamma_{kj}^\ell \Gamma_{k\ell}^m - \Gamma_{kj}^\ell \Gamma_{i\ell}^m \right] \right.$$

$$\left. + \left[L_{ij} L_k^m - L_{kj} L_i^m \right] \right\} \frac{\partial F}{\partial u_m}$$

$$+ \left\{ \frac{\partial L_{ij}}{\partial u_k} - \frac{\partial L_{kj}}{\partial u_i} + \sum_\ell \left[L_{ij}^\ell L_{ke} - L_{kj}^\ell L_{ie} \right] \right\} N$$

Now we must have $\dfrac{\partial^3 F}{\partial u_k \partial u_i \partial u_j} - \dfrac{\partial^3 F}{\partial u_i \partial u_k \partial u_j} \equiv O$ and therefore

we get : $\dfrac{\partial \Gamma_{ij}^m}{\partial u_k} - \dfrac{\partial \Gamma_{kj}^m}{\partial u_i} + \sum_\ell \left[\Gamma_{ij}^\ell \Gamma_{k\ell}^m - \Gamma_{kj}^\ell \Gamma_{i\ell}^m \right] + L_{ij} L_k^m - L_{kj} L_i^m = O$

along with

$$\frac{\partial L_{ij}}{\partial u_k} - \frac{\partial L_{kj}}{\partial u_i} + \sum_{\ell} \left[\Gamma_{ij}^{\ell} \Gamma_{ke} - \Gamma_{kj}^{\ell} \Gamma_{ie} \right] = O .$$

We use the identity (***) written equivalently in the following way

$$\frac{\partial \Gamma_{ij}^{m}}{\partial u_k} - \frac{\partial \Gamma_{kj}^{m}}{\partial u_i} + \sum_{\ell} \left[\Gamma_{ij}^{\ell} \Gamma_{\ell k}^{m} - \Gamma_{kj}^{\ell} \Gamma_{\ell i}^{m} \right] = L_{ij} L_k^m - L_{kj} L_i^m .$$

Also reorganizing the indices $i\,k, j\;\ell, m$ wer write :

$R_{ijk}^{\ell} = \dfrac{\partial \Gamma_{ik}^{\ell}}{\partial u_j} - \dfrac{\partial \Gamma_{ij}^{\ell}}{\partial u_{k\ell k}} + \sum\limits_{m} \left[\Gamma_{ik}^{m} \Gamma_{mj}^{\ell} - \Gamma_{ij}^{m} \Gamma_{mk}^{\ell} \right]$ all the indices i, j, k, ℓ, m taking the values $1,2$.

Note that the functions $R_{ijk}^{\ell} : M \to \mathbb{R}$ satisfy : $R_{ijk}^{\ell} = -R_{ikj}^{\ell}$.

The collection $\left\{ R_{ijk}^{\ell} : 1 \le i, j, k, \ell \le 2 \right\}$ are components of a geometric object (related to M) called the curvature tensor of M.

We also introduce the functions $R_{\ell ijk} : M \to \mathbb{R} 1 \le i, j, k, \ell \le 2$ by $R_{\ell ijk} = \sum\limits_{m} g \ell m R_{ijk}^{m}$.

Note that we can retrieve R_{ijk}^{ℓ} from $R_{\ell ijk}$ by $R_{ijk}^{\ell} = \sum\limits_{m} g^{\ell m} R_{mijk}$. This is indeed so, because the matrices $\left[g_{ij} \right]$ and $\left[g^{ij} \right]$ are the inverses of each other.

Now the equality :

$$R_{ijk}^{m} = L_{ik} L_j^m - L_{ij} L_{km}$$

multiplied by $g_{\ell m}$ and then summed over $m = 1,2$ gives :

$$R_{\ell ijk} = L_{ik} L_{jm} - L_{ij} L_{km} .$$

In particular, we have

$$\begin{aligned} R_{1212} &= L_{22} L_{11} - \left(L_{21} \right)^2 \\ &= det \left[L_{ij} \right] \end{aligned}$$

Thus, for any $p \in M$, we have

$$R_{1212}(p) = det\left[L_{ij}(p)\right]$$
$$= det\left[\sum_k L_i^k(p)\, g_{kj}(p)\right]$$
$$= det\left[L_i^j(p)\right] det\left[g_{ij}(p)\right]$$
$$= det\left[L(p)\right] det\left[g_{ij}(p)\right]$$
$$= K(p)\, det\left[g_{ij}(p)\right]$$

$K(p)$ being, of course, the normal curvature of M at its point p. Thus, we have obtained $K(p) = \dfrac{R_{1212}(p)}{det\left[g_{ij}(p)\right]}$ ………………… (G)

This is offen called Gauss' formula for the normal curvature.

For the sake of convenience, we will refer to the Gauss formula by the symbol (G).

Now, looking at the right hand side of (G) we notice that it is a complex expression involving the entries g_{ij} of the first fundamental form and their partial derivatives $\dfrac{\partial g_{ij}}{\partial u_k}; \dfrac{\partial^2 g_{ij}}{\partial u_k \partial u_i}$. The functions g_{ij} are obtained by varying the parametrization maps F_1, F_2, F_3 on the surface and all the partial derivatives too are obtained by differentiating the F_i, g_{ij} etc.

Consequently we infer that the Gaussian curvature $K(p)$ of M at p is calculated by taking measurements on the surface and not referring to the ambient space $\mathbb{R}^3$.

On the other hand, there are geometric quantities pertaining to M which involve the ambian space also : For example the unit normal and its variation on the surface refer to the external space.

We call geometric quantities intrinsic to M if they are obtained by taking measurements taken strictly on the surface M. Thus, a geometric quantity is intrinsic if it is expressible in terms of the first fundamental form of the surface.

Above we have explained the proof (!) of the following :

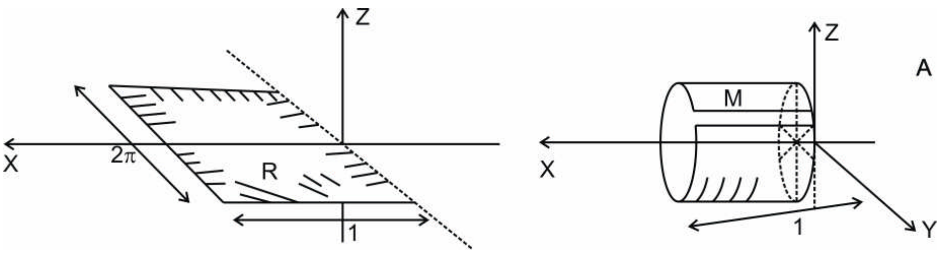
Theorema Egregium of Gauss : Gaussian curvature of a surface is an intrinsic property of a surface.

(Here "Egregium" means "extraordinary".)

And then let us note a property of surfaces which is not "intrinsic".

We consider the flat rectangle $R = \{(x, y, 0) : 0 < x < 1, 0 < y < 2\pi\}$ in the XOY plane $(= \mathbb{R}^2)$. We roll it up in the form of the circular cylinder :

$$M = \{(x, \cos y, \sin y) : 0 < x < 1, 0 < y < 2\Pi\}$$



Note that we obtained M from R without crumpling the paper (or without causing any kind of damage to the paper and consequently any measurements taken on the surface either in its rectangular form or in its cylindrical form are the same. Geometrically, both the suffaces , R M have the same first fundamental form.

But the mean curvature of $R \leq 0$ while that of the cylinder is $\frac{0+1}{2} = \frac{1}{2} \neq 0$.

The above example shows that the mean curvature of a surface is not an intrinsic property of surfaces. It dependes on the way in which it is imbedded in the ambient space (i.e. the space $\mathbb{R}^3$).

10.4 LOCALLY PARAMETRIZED SMOOTH SURFACES

In the preceeding part of this chapter, we considered smooth surfaces M which were covered by single parametrizations $(U, F) : M = F(U)$. But we come across surfaces which are parametrized only locally; such surfaces are overwheming in mathematics. We introduce the concept here formally.

Let M be a non-empty subset of $\mathbb{R}^3$. We consider M give the subspace topology of $\mathbb{R}^3$.

By a smooth, local parametrization on M, we mean a pair (U,F) consisting of an open subset U of $\mathbb{R}$ and a smooth map $F: U \to \mathbb{R}^3$, the pair having the following properties :

i) $F(U)$ is an open subset of M and the $F: U \to F(U)$ is a homeomorphism.
ii) For each $q \in U$, the Jacobean map $J_F(a): \mathbb{R}^2 \to \mathbb{R}^3$ is injective (equivalently put, it has rank 2)

A smooth atlas on M is a collection $D = \{(U_\lambda, F_\lambda) : \lambda \in \Lambda\}$ of smooth local parametrizations (U_λ, F_λ) on M with the property : $U\{F_\lambda(U_\lambda) : \lambda \in \Lambda\} = M$.

A smooth, locally parametrized surface is a set M on which is specified a smooth atlas D. We indicate it by the notation (M, D). The collection D is called a smooth atlas of the surface and an element $(U_\lambda, F_\lambda) \in D$ is often called a coordinate chart of (M, D).

Thus a (smooth) parametrized surface is a particular case of a (M, D) in which D has only one element (U, F). (We often speak of (M, D) being covered by a single coordinate chart.) But, of course, a set M may not be covered by a single coordinate chart. Moreover, there are subsets of $\mathbb{R}^3$ which are so scattered in $\mathbb{R}^3$ that they do not admit any smooth atlas.

We conclude this chapter by describing a smooth atlas on a sphere of radius a > o and then generalizing this in the form of a result which gives a large variety of locally parametrized surfaces :

Let $M = \{(x, y, z) : x^2 + y^2 + z^2 = a^2\}$

We consider the open cover $\{U^+,U^-,V^+,V^-,W^+,W^-\}$ of M where :

$$U^+=\{(x,y,z)\in M\,,z>o\},U^-=\{(x,y,z)\in M\,,z<o\}$$
$$V^+=\{(x,y,z)\in M\,,y>o\},V^-=\{(x,y,z)\in M\,,y<o\}$$
$$W^+=\{(x,y,z)\in M\,,x>o\}\,and\,W^-=\{(x,y,z)\in M\,,x<o\}$$

Also let $D=\{(u,v)\in\mathbb{R}^2,u^2+v^2<a\}$; it is an open subset of $\mathbb{R}^2$.

Now define $F^+:D\to U^+,F^-:D\to U$ by
$F^+(u,v)=\left(u,v,+\sqrt{(a^2-u^2-v^2)}\right)$ $F^-(u,v)=\left(u,v,-\sqrt{(a^2-u^2-v^2)}\right)$ and
$G^+:D\to V^+,G^-:D\to V^-$ by
$G^+(u,v)=\left(u,+\sqrt{(a^2-u^2-v^2)},u\right)$ $G^-(u,v)=\left(u,-\sqrt{(a^2-u^2-v^2)},u\right)$
and finally, $H^+:D\to W^+,H^-:D\to W^-$ by
$H^+(u,v)=\left(\sqrt{(a^2-u^2-v^2)},u,v\right)$, $H^-(u,v)=\left(-\sqrt{(a^2-u^2-v^2)},u,v\right)$.

Then $\{(U^\pm,F^\pm),(V^\pm,G^\pm),(W^\pm,H^\pm)\}$ is a smooth atlas on the sphere M.

Verification of this claim is left as an exercise for the reader.

Now, the following result.

Proposition 1 : Let W be an open subset of $\mathbb{R}^3$ and $f:W\to\mathbb{R}$ a smooth function. For $a\in\mathbb{R}$, let $M=\{(x,y,z)\in W:f(x,y,z)=a\}$. Suppose M satisfies : $\nabla f(p)\neq(o,o,o)$ for each $p\in M$.

Then M carries a smooth atlas D.

We give a sketchy proof below :

Proof : Let $p=(p_1,p_2,p_3)\in M$.

Then $\nabla f(p)\neq o$. Assume without loss of generality that $\frac{\partial f}{\partial x}(p)\neq O$. Then by implicit function theorem there exists an open

$U_p \subset \mathbb{R}^2$ with $(p_1, p_2) \in U$ and a smooth $g_p : U_p \to \mathbb{R}$ satisfying $f(x, y, g_p(x, y)) = a$ for all $(x, y) \in U_p$ and $g_p(p_1 p_2) = p_3$.

Define $G_p : U_p \to \mathbb{R}^3$ by putting $G_p(x, y) = (x, y, g_p(x, y))$ for all $(x, y) \in U_p$. Then (U_p, G_p) is a local parametrization of M around the point p. And then $\mathfrak{D} = \{(U_p, G_p) : p \in M\}$

is the desired smooth atlas on M.

Exercises :

1) Let M be the surface of revolution given by
$F(b, \theta) = (r(t)\cos\theta, r(t)\sin\theta t) : t \in I, O < \theta < 2\pi$ for a given $r : I \to \mathbb{R}$, Prove that the Gaussian curvature K and mean curvature H functions are given by

$$K(t, \theta) = \frac{\ddot{r}(t)}{r(t)\left(1 + \dot{r}(t)^2\right)^2}$$

$$H(t, \theta) = \frac{1}{2} \frac{r(t)\ddot{r}(t) - 1 - \dot{r}(t)^2}{r(t)\left(1 + \dot{r}(t)^2\right)^2}$$

2) Let S be the surface of reolution given by
$$F(t, \theta) = \left(\sin t \sin\theta, \sin t \cos t, \cos t + \log \tan\left(\frac{1}{2}\right)\right)$$
$$(t, \theta) \in \left(0, \frac{\pi}{2}\right) \times (0, 2\pi)$$
Show that the surface has constant Gaussian curvature $K \equiv -1$.

3) Let M be the ellipsoid :
$$M = \left\{(x, y, z) : x^2 + \frac{y^2}{2} + \frac{z^2}{3} = 1\right\}$$
Prove that none of the points $p_1 = (1, 0, 0), p_2 = (0, \sqrt{2}, 0)$, $p_3 = (0, 0, \sqrt{3})$ is an umbilic point.

4) Prove that (0,0,0) is the umbilic point of the surface $z = \sqrt{x^2 + y^2}$ and calculate the normal curvature of it at (0,0,0).

5) Find principal curvatures and principal directions of the following surfaces at a point of them

i) a circular cylinder

ii) the saddle surface $z = xy$

6) Let $\alpha : I \to M$ be a smooth curve. Show that the normal curvature of M at a point of α in the direction α (at that point) is given by : $K_{\eta}(\dot{\alpha}) = k \cos\theta$ where $k = \|\ddot{\alpha}\|$ is the curvature of α (as a curve in $\mathbb{R}^3$) and θ is the angle between the surface normal N and the principal normal vector of the curve α.

7) Find the normal, curvature of the surface $z = f(x, y)$ at a point p of it in the direction of the unit vector (a, b, c).

8) Let M be the hyperbolic paraboloid. $z = \frac{1}{2}(y^2)$. Show that the normal curvature of M at (0,0,0) along a unit vector $v = (\cos\theta, \sin\theta, 0)$ is :

$$k_n(v) = -\cos^2\theta + \sin^2\theta = -\cos 2\theta$$

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