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COMPLEX ANALYSIS: THE GEOMETRIC VIEWPOINT

SECOND EDITION

STEVEN G. KRANTZ

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Second Edition

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Second Edition

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To my parents

Acknowledgments

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—S.G.K.

Preface to the Second Edition

The warm reception with which the first edition of this book has been received has been a source of both pride and pleasure. It is a special privilege to have created a "for the record" version of Ahlfors's seminal ideas in the subject. And the geometric viewpoint continues to develop.

In the intervening decade, this author has learned a great deal more about geometric analysis, and his view of the subject has developed and broadened. It seems appropriate, therefore, to bring some new life to these pages, and to set forth a fresh enunciation of the role of curvature in basic complex function theory.

In this new edition, we explain how, in a natural and elementary manner, the hyperbolic disc is a model for the non-Euclidean geometry of Bolyai and Lobachevsky. Later on, we explain the Bergman kernel and provide an introduction to the Bergman metric.

I have many friends and colleagues to thank for their incisive remarks and suggestions about the first edition of this book. I hope that I do them justice in my efforts to implement a second edition. As always, the Mathematical Association of America has been an exemplary publisher and has provided all possible support in the publication process. I offer my humble thanks.

Preface to the First Edition

The modern geometric point of view in complex function theory began with Ahlfors's classic paper [AHL]. In that work it was demonstrated that the Schwarz lemma can be viewed as an inequality of certain differential geometric quantities on the disc (we will later learn that they are curvatures). This point of view—that substantive analytic facts can be interpreted in the language of Riemannian geometry—has developed considerably in the last fifty years. It provides new proofs of many classical results in complex analysis, and has led to new insights as well.

In this monograph we intend to introduce the reader with a standard one semester background in complex analysis to the geometric method. All geometric ideas will be developed from first principles, and only to the extent needed here. No background in geometry is assumed or required.

Chapter 0 gives a bird's eye view of classical function theory of one complex variable. We pay special attention to topics which are developed later in the book from a more advanced point of view. In this chapter we also sketch proofs of the main results, with the hope that the reader can thereby get a feeling for classical methodology before embarking on a study of the geometric method.

Chapter 1 begins a systematic treatment of the techniques of Riemannian geometry, specially tailored to the setting of one complex variable. In order that the principal ideas may be brought out most clearly, we shall concentrate on only a few themes: the Schwarz lemma, the Riemann mapping theorem, normal families, and Picard's theorems. For many readers this will be a first contact with the latter two results. The geometric method provides a particularly cogent explanation of these theorems, and can be contrasted with the more classical proofs which are discussed in Chapter 0. We shall also touch on Fatou-Julia theory, a topic which is rather technical from the analytic standpoint but completely natural from the point of view of geometry.

In Chapter 3 we introduce the Carathéodory and Kobayashi metrics, a device which is virtually unknown in the world of one complex variable. This decision allows us to introduce invariant metrics on arbitrary planar domains without resort to the uniformization theorem. We are then able to give a "differential geometric" interpretation of the Riemann mapping theorem.

The last chapter gives a brief glimpse of several complex variables. Some of the themes which were developed earlier in the book are carried over to two dimensions. Biholomorphic mappings are discussed, and the inequivalence of the ball and bidisc is proved using a geometric argument.

The language of differential geometry is not generally encountered in a first course in complex analysis. It is hoped that this volume will be used as a supplement to such a course, and that it may lead to greater familiarity with the fruitful methodology of geometry.

Contents

Epilogue

In complex analysis, geometric methods provide both a natural language for analyzing and recasting classical problems and also a rubric for posing new problems. The interaction between the classical and the modern techniques is both rich and rewarding.

Many facets of this symbiosis have yet to be explored. In particular, very little is known about explicitly calculating and estimating the differential invariants described in the present monograph. It is hoped that this book will spark some new interest in these matters.

Appendix on the Structure Equations and Curvature

1. Introduction

Here we give a brief presentation of the connection between the calculus notion of curvature (see [THO]) and the more abstract notion of curvature which leads to the definition of κ in Chapter 2. It is a pleasure to acknowledge our debt to the clear and compelling exposition in [ONE].

First, a word about notation. We use the language of differential forms consistently in this appendix. On the one hand, classically trained analysts are often uncomfortable with this language. On the other hand, the best way to learn the language is to use it. And the context of curvature calculations on plane domains may in fact be the simplest nontrivial context in which differential forms can be profitably used. In any event, this appendix would be terribly clumsy if we did not use forms, so the decision is essentially automatic. All necessary background on differential forms may be found in [RU1] or in [ONE].

2. Expressing Curvature Intrinsically

First we recall the concept of curvature for a smooth, two-dimensional surface $M \subseteq \mathbb{R}^3$. All of our calculations are local, so it is convenient to think of M as parametrized by two coordinate functions over a connected open set $U \subseteq \mathbb{R}^2$:

$$U \ni (u, v) \xrightarrow{p} (x_1(u, v), x_2(u, v), x_3(u, v)) \in M.$$

We require that the matrix

$$\begin{pmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \\ \dfrac{\partial x_1}{\partial v} & \dfrac{\partial x_2}{\partial v} & \dfrac{\partial x_3}{\partial v} \end{pmatrix}$$

have rank 2 at each point of U. The vectors given by the rows of this matrix span the tangent plane to M at each point. Applying the Gram-Schmidt orthonormalization procedure to these row vectors, and shrinking U, M if necessary, we may create vector fields

$$E_1 : M \longrightarrow \mathbb{R}^3$$

$$E_2 : M \longrightarrow \mathbb{R}^3$$

such that $E_1(x_1, x_2, x_3)$ and $E_2(x_1, x_2, x_3)$ are orthonormal and tangent to M at each $P = (x_1, x_2, x_3) \in M$. Denote by $T_P(M)$ the collection of linear combinations

$$aE_1(x_1, x_2, x_3) + bE_2(x_1, x_2, x_3), \qquad a, b \in \mathbb{R}.$$

We call $T_P(M)$ the *tangent space* to M at P.

Let $E_3(P)$ be the unit normal to M at P given by the $E_1(P) \times E_2(P)$. The functions E_1, E_2, E_3 are smooth *vector fields* on M: they assign to each $P \in M$ a triple of orthonormal vectors.

Let $\delta_1, \delta_2, \delta_3$ denote the standard basis for vectors in $\mathbb{R}^3$:

$$\delta_1 = (1, 0, 0),$$

$$\delta_2 = (0, 1, 0),$$

$$\delta_3 = (0, 0, 1).$$

(Many calculus books call these vectors **i**, **j**, and **k**.) Then we may write

$$E_i = \sum_j a_{i,j}(x_1, x_2, x_3)\delta_j, \qquad i = 1, 2, 3.$$

The matrix

$$\mathcal{A} \equiv \left(a_{i,j}\right)_{i,j=1}^{3},$$

where the $a_{i,j}$ are *functions* of the space variables, is called the *attitude matrix* of the frame (or basis) E_1, E_2, E_3. Since $\mathcal{A}$ transforms one orthonormal frame to another, $\mathcal{A}$ is an orthogonal matrix. Hence

$$\mathcal{A}^{-1} = {}^t\mathcal{A}.$$

Definition 1. If $P \in M$, $v \in T_P(M)$, and f is a smooth function on M, then define

$$D_v f(P) = \frac{d}{dt} f \circ \phi(t)\bigg|_{t=0},$$

where ϕ is any smooth curve in M such that $\phi(0) = P$ and $\phi'(0) = v$. One checks that this definition is independent of the choice of ϕ.

Definition 2. If

$$\alpha : M \longrightarrow \mathbb{R}^3$$

is a vector field on M,

$$\alpha(P) = \alpha_1(P)\delta_1 + \alpha_2(P)\delta_2 + \alpha_3(P)\delta_3,$$

and $v \in T_P(M)$ then define

$$\nabla_v\alpha(P) = (D_v\alpha_1)(P)\delta_1 + (D_v\alpha_2)(P)\delta_2 + (D_v\alpha_3)(P)\delta_3.$$

The operation ∇_v is called *covariant differentiation* of the vector field α.

Definition 3. If $P \in M$, $v \in T_P(M)$, we define the *shape operator* (or *Weingarten map*) for M at P to be

$$S_P(v) = -\nabla_v E_3(P).$$

Lemma 4. *We have that $S_P(v) \in T_P(M)$.*

Proof. Now $E_3 \cdot E_3 \equiv 1$, hence

$$\begin{aligned} 0 &= D_v(E_3 \cdot E_3)\Big|_P \\ &= (2 \nabla_v E_3) \cdot E_3\Big|_P \\ &= -2S_P(v) \cdot E_3(P). \end{aligned}$$

Hence $S_P(v) \perp E_3(P)$ so $S_P(v) \in T_P(M)$. ■

Notice that the shape operator assigns to each $P \in M$ a linear operator S_P on the 2-dimensional tangent space $T_P(M)$.

We can express this linear operator as a matrix with respect to the basis $E_1(P)$, $E_2(P)$. So S assigns to each $P \in M$ a 2×2 matrix $\mathcal{M}_P$.

The linear operator S_P measures the rate of change of E_3 in any tangent direction v. It can be shown that $\mathcal{M}_P$ is diagonalizable. The (real) eigenvectors of $\mathcal{M}_P$ correspond to the *principal curvatures* of M at P (these are the directions of greatest and least curvature) and the corresponding eigenvalues measure the amount of curvature in those directions.

The preceding observations about $\mathcal{M}_P$ motivate the following definition.

Definition 5. The *Gaussian curvature* $\kappa(P)$ of M at a point $P \in M$ is the determinant of $\mathcal{M}_P$—the product of the two eigenvalues.

Our aim is to express $\kappa(P)$ in terms of the intrinsic geometry of M, without reference to E_3—that is, without reference to the way that M is situated in space.

To this end, we define *covector fields* θ_i which are dual to the vector fields E_i:

$$\theta_i E_j(P) = \delta_{ij}.$$

Then the θ_i may be expressed as linear combinations of the standard basis covectors dx_1, dx_2, dx_3. Indeed, if $\mathcal{A} = (a_{i,j})$ is the attitude matrix then

$$\theta_i = \sum a_{i,j}\, dx_j$$

(remember that $\mathcal{A}^{-1} = {}^t\mathcal{A}$). Thus, for each i, θ_i is a differential form; and the standard calculus of differential forms—including exterior differentiation—applies.

From now on, we restrict attention to 1- and 2-forms acting on tangent vectors to M. Thus any 1-form α may be expressed as

$$\alpha = \alpha(E_1)\theta_1 + \alpha(E_2)\theta_2$$

and any 2-form β may be expressed as

$$\beta = \beta(E_1, E_2)\theta_1 \wedge \theta_2.$$

Now we define covector fields $\omega_{i,j},\ i, j \in \{1, 2, 3\}$, by the formula

$$\omega_{i,j}(v) = (\nabla_v E_i) \cdot E_j(P).$$

Here "$\cdot$" is the Euclidean dot product. We think of $\omega_{i,j}$ as a differential 1-form. Notice that, since $E_i \cdot E_j \equiv \delta_{i,j}$, we have for $v \in T_P(M)$ that

$$\begin{aligned} 0 &= D_v(E_i \cdot E_j) \\ &= (\nabla_v E_i) \cdot E_j + E_i \cdot (\nabla_v E_j) \\ &= \omega_{i,j}(v) + \omega_{j,i}(v). \end{aligned}$$

Thus

$$\omega_{i,j} = -\omega_{j,i}.$$

In particular,

$$\omega_{i,i} = 0.$$

If $v \in T_P(M)$ then it is easy to check, just using linear algebra, that

$$\nabla_v E_i = \sum_j \omega_{i,j}(v) E_j, \qquad 1 \leq i \leq 3.$$

We call the $\omega_{i,j}$ the *connection forms* for M. We can now express the shape operator in terms of these connection forms.

Proposition 6. *Let $P \in M$ and $v \in T_P(M)$. Then*

$$S_P(v) = \omega_{1,3}(v) E_1(P) + \omega_{2,3}(v) E_2(P).$$

Proof. We have that

$$\begin{aligned} S_P(v) &= -\nabla_v E_3 \\ &= -\sum_{j=1}^{3} (\nabla_v E_3 \cdot E_j) E_j \\ &= -\sum_{j=1}^{3} \omega_{3,j}(v) E_j \\ &= \omega_{1,3}(v) E_1 + \omega_{2,3}(v) E_2, \end{aligned}$$

since $\omega_{3,3} = 0$. ■

Now we can express Gaussian curvature in terms of the $\omega_{i,j}$.

Proposition 7. *We have that*

$$\omega_{1,3} \wedge \omega_{2,3} = \kappa\, \theta_1 \wedge \theta_2.$$

Proof. We need to calculate $\mathcal{M}_P$ in terms of the $\omega_{i,j}$. Proposition 6 gives that

$$S_P(E_1) = \omega_{1,3}(E_1) E_1 + \omega_{2,3}(E_1) E_2$$

and

$$S_P(E_2) = \omega_{1,3}(E_2) E_1 + \omega_{2,3}(E_2) E_2$$

so the matrix of S_P, in terms of the basis E_1, E_2, is

$$\mathcal{M}_P = \begin{pmatrix} \omega_{1,3}(E_1) & \omega_{2,3}(E_1) \\ \omega_{1,3}(E_2) & \omega_{2,3}(E_2) \end{pmatrix}.$$

We know that $\omega_{1,3} \wedge \omega_{2,3}$, being a 2-form, can be written as $\lambda \cdot \theta_1 \wedge \theta_2$. On the other hand,

$$\begin{aligned} \kappa &= \det \mathcal{M}_P \\ &= \omega_{1,3}(E_1)\omega_{2,3}(E_2) - \omega_{1,3}(E_2)\omega_{2,3}(E_1) \\ &= (\omega_{1,3} \wedge \omega_{2,3})(E_1, E_2) \\ &= \lambda. \end{aligned}$$

Therefore

$$\omega_{1,3} \wedge \omega_{2,3} = \lambda\, \theta_1 \wedge \theta_2 = \kappa\, \theta_1 \wedge \theta_2. \qquad \blacksquare$$

Our goal now is to express κ using only those $\omega_{i,j}$ with $i \neq 3$, $j \neq 3$. For this we require a technical lemma about the attitude matrix.

Lemma 8. *We have that*

$$\omega_{i,j} = \sum_k a_{j,k}\, da_{i,k}, \qquad 1 \leq i,\ j \leq 3.$$

Proof. If $v \in T_P(M)$ then

$$\omega_{i,j}(v) = \nabla_v E_i \cdot E_j(P).$$

But

$$E_i = \sum_k a_{i,k} \delta_k.$$

Therefore

$$\nabla_v E_i = \sum_k (D_v a_{i,k}) \delta_k.$$

Then

$$\begin{aligned}\omega_{i,j}(v) &\equiv \nabla_v E_i \cdot E_j\\ &= \left(\sum_k (D_v a_{i,k})\delta_k\right) \cdot \left(\sum_k a_{j,k}\delta_k\right)\\ &= \sum_k (D_v a_{i,k}) a_{j,k}\\ &= \sum_k \, da_{i,k}(v) a_{j,k}.\end{aligned}$$

As a result,

$$\omega_{i,j} = \sum_k a_{j,k}\, da_{i,k}. \qquad \blacksquare$$

Now we have reached a milestone. We can derive the important *Cartan structural equations*, which are the key to our intrinsic formulas for curvature.

Theorem 9. *We have*

$$d\theta_i = \sum_j \omega_{i,j} \wedge \theta_j, \tag{1}$$

$$d\omega_{i,j} = \sum_k \omega_{i,k} \wedge \omega_{k,j}. \tag{2}$$

Proof. We have that

$$\theta_i = \sum_j a_{i,j}\, dx_j$$

hence

$$d\theta_i = \sum_j \, da_{i,j} \wedge dx_j.$$

Since the attitude matrix $\mathcal{A}$ is orthogonal, we may solve the equations in Lemma 8 for $da_{i,k}$ in terms of $\omega_{i,j}$. Thus

$$da_{i,j} = \sum_k \omega_{i,k} a_{k,j}.$$

Substituting this into our last formula gives

$$\begin{aligned} d\theta_i &= \sum_j \left[\left(\sum_k \omega_{i,k} a_{k,j} \right) \wedge dx_j \right] \\ &= \sum_k \left[\omega_{i,k} \wedge \sum_j a_{k,j}\, dx_j \right] \\ &= \sum_k \omega_{i,k} \wedge \theta_k. \end{aligned}$$

This is the first structural equation.

For the second equation, notice that the formula

$$\omega_{i,j} = \sum_k a_{j,k}\, da_{i,k}$$

implies

$$d\omega_{i,j} = -\sum_k da_{i,k} \wedge da_{j,k}.$$

On the other hand,

$$\begin{aligned} \sum_k \omega_{i,k} \wedge \omega_{k,j} &= \sum_k \left(\sum_\ell a_{k,\ell}\, da_{i,\ell} \right) \wedge \left(\sum_m a_{j,m}\, da_{k,m} \right) \\ &= \sum_k \left(\sum_\ell a_{k,\ell}\, da_{i,\ell} \right) \wedge \left(-\sum_m a_{k,m}\, da_{j,m} \right) \\ &= -\left(\sum_{\ell,m} \left\{ \sum_k a_{k,\ell} a_{k,m} \right\} \cdot da_{i,\ell} \wedge da_{j,m} \right) \end{aligned}$$

$$= -\sum_m da_{i,m} \wedge da_{j,m}$$
$$= d\omega_{i,j}.$$

In the antepenultimate line we have used the fact that $\mathcal{A}^{-1} = {}^t\mathcal{A}$. The result now follows. ■

The following corollary will prove critical.

Corollary. *We have*

$$d\omega_{1,2} = -\kappa\theta_1 \wedge \theta_2.$$

Proof. The second structural equation gives

$$d\omega_{1,2} = \sum_k \omega_{1,k} \wedge \omega_{k,2}$$
$$= \omega_{1,1} \wedge \omega_{1,2} + \omega_{1,2} \wedge \omega_{2,2} + \omega_{1,3} \wedge \omega_{3,2}.$$

Only the third summand doesn't vanish. But we calculated in Proposition 7 that it equals $-\kappa\theta_1 \wedge \theta_2$. ■

The Corollary has been our main goal in this subsection. It gives an intrinsic way to calculate Gaussian curvature in the classical setting, hence a way to *define* Gaussian curvature in more abstract settings. We now proceed to develop this more abstract point of view.

3. Curvature Calculations on Planar Domains

Let $\Omega \subseteq \mathbb{C}$ be a domain which is equipped with a metric ρ. Assume for simplicity that $\rho(z) > 0$ at all points of Ω. Define functions

$$E_1 \equiv \frac{(1,0)}{\rho} \qquad \text{and} \qquad E_2 \equiv \frac{(0,1)}{\rho}.$$

Then

$$\theta_1 = \rho\, dx \qquad \text{and} \qquad \theta_2 = \rho\, dy$$

are the dual covector fields. We define $\omega_{i,j}$ according to the first structure equation:

$$\begin{aligned} d\theta_1 &= \omega_{1,2} \wedge \theta_2, \\ d\theta_2 &= \omega_{2,1} \wedge \theta_1. \end{aligned}$$

We define Gaussian curvature according to the Corollary to Theorem 9 in the previous section:

$$d\omega_{1,2} = -\kappa\theta_1 \wedge \theta_2.$$

One can check that these definitions are independent of the choice of frame (or basis) E_1, E_2, but this is irrelevant for our purposes.

We conclude this Appendix by proving that the definition of curvature which we just elicited from the structural equations coincides with the one given in Section 2.1. First, by the way that we've defined θ_1 and θ_2, we have

$$\begin{aligned} d\theta_1 &= d\rho \wedge dx \\ &= (\rho_x\, dx + \rho_y\, dy) \wedge dx \\ &= \rho_y\, dy \wedge dx \\ &= -\frac{\rho_y}{\rho}\, dx \wedge \rho\, dy \\ &= -\frac{\rho_y}{\rho}\, dx \wedge \theta_2. \end{aligned}$$

Similarly,

$$\begin{aligned} d\theta_2 &= d\rho \wedge dy \\ &= (\rho_x\, dx + \rho_y\, dy) \wedge dy \\ &= \rho_x\, dx \wedge dy \end{aligned}$$

$$= -\frac{\rho_x}{\rho}\,dy \wedge \rho\,dx$$
$$= -\frac{\rho_x}{\rho}\,dy \wedge \theta_1.$$

Comparison with the first structural equation gives

$$\omega_{1,2} = -\frac{\rho_y}{\rho}\,dx + \tau\,dy$$

and

$$\omega_{1,2} = -\omega_{2,1} = -\left(-\frac{\rho_x}{\rho}\,dy\right) + \sigma\,dx,$$

for some unknown functions τ and σ.

The only way these equations can be consistent is if

$$\omega_{1,2} = -\frac{\rho_y}{\rho}\,dx + \frac{\rho_x}{\rho}\,dy.$$

Thus

$$\begin{aligned}
d\omega_{1,2} &= -\frac{\partial}{\partial y}\left(\frac{\rho_y}{\rho}\right) dy \wedge dx + \frac{\partial}{\partial x}\left(\frac{\rho_x}{\rho}\right) dx \wedge dy \\
&= \left(-\frac{\rho_{yy}}{\rho} + \frac{\rho_y \rho_y}{\rho^2}\right) dy \wedge dx + \left(\frac{\rho_{xx}}{\rho} - \frac{\rho_x \rho_x}{\rho^2}\right) dx \wedge dy \\
&= \frac{1}{\rho^2}\left(\rho\Delta\rho - (\rho_y)^2 - (\rho_x)^2\right) dx \wedge dy \\
&= \frac{1}{\rho^4}\left(\rho\Delta\rho - (\rho_y)^2 - (\rho_x)^2\right) \theta_1 \wedge \theta_2.
\end{aligned}$$

The second structural equation now implies that

$$\kappa = -\frac{1}{\rho^4}(\rho\Delta\rho - |\bigtriangledown\rho|^2).$$

On the other hand, in Section 2.1 we defined

$$\kappa = -\frac{\Delta \log \rho}{\rho^2}.$$

We have

$$\frac{\partial}{\partial \bar{z}} \log \rho = \frac{1}{\rho} \frac{\partial \rho}{\partial \bar{z}}$$

and

$$\begin{aligned}
\Delta \log \rho &= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \log \rho \\
&= 4 \left(-\frac{1}{\rho^2} \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial \bar{z}} + \frac{1}{\rho} \frac{\partial^2 \rho}{\partial z \partial \bar{z}} \right) \\
&= -\frac{1}{\rho^2} \cdot | \bigtriangledown \rho|^2 + \frac{1}{\rho} \cdot \Delta \rho.
\end{aligned}$$

It follows that, according to the definition in Section 2.1,

$$\kappa = -\frac{1}{\rho^4} (\rho \Delta \rho - | \bigtriangledown \rho|^2).$$

Thus, as claimed, the definition of curvature from Chapter 2 is equal to that which arises from the structural equations.

Table of Symbols

Symbol	Page Number	Meaning
$\mathcal{A}$	193	the attitude matrix
$A^2(\Omega)$	138	the Bergman space
$A_{r,R}$	122	an annulus
$\mathrm{Aut}(U)$	121, 179	the automorphism group of U
$[az+b]/[cz+d]$	14	linear fractional transformation
$\mathbf{B}(0,r)$	48	Poincaré metric ball
$B(P,R)$	128	metric ball
$\mathcal{B}(P,r)$	165	a ball in $\mathbb{C}^2$
$\mathbb{C}$	1	complex numbers
$\widehat{\mathbb{C}}$	82	the Riemann sphere
$\mathbf{C}$	62	arc of a circle
$\mathcal{C}$	168	domain of convergence
$\mathbb{C}_{0,1}$	76	the domain $\mathbb{C} \setminus \{0, 1\}$
C^1	3	continuously differentiable
C^2	105	twice continuously differentiable
C^k	105	k times continuously differentiable
C^∞	106	infinitely differentiable
$\mathcal{C}_\Omega(P,Q)$	37	all piecewise continuously differentiable curves connecting P to Q
$c(P)$	111	center of curvature
C_R	53	points in D having Poincaré distance R from 0
$C(z,\zeta)$	140	the Cauchy kernel
D	2	unit disc
$\mathbf{d}$	111	internally tangent disc
$\triangle$	40	the Laplacian
d_ρ	47	the Poincaré distance
D_v	193	directional derivative
$\delta_1, \delta_2, \delta_3$	192	standard basis vectors
dA	138	area measure
$\det\ \mathrm{Jac}\,\phi$	141	the Jacobian determinant
$D(P,r)$	2	open disc
$\overline{D}(P,r)$	2	closed disc

Symbol	Page Number	Meaning
$\partial D(P, r)$	2	boundary of disc
$D'(0, \epsilon)$	87	a punctured disc
$D(C(P), r_0)$	110	externally tangent disc
$D(C'(P), r_0)$	110	internally tangent disc
$D^2(P, r)$	165	a bidisc in $\mathbb{C}^2$
$\overline{D}^2(P, r)$	166	closure of a bidisc
$(D, U)_P$	90, 180	holomorphic functions f from U to D such that $f(P) = 0$
$\mathrm{dist}_C(z, \tau_p)$	120	Carathéodory distance of z to τ_p
$d_\rho(P, Q)$	37	ρ-metric distance of P to Q
$d_\rho(P, Q)$	47	Poincaré distance
$d_\sigma(z, w)$	83	spherical distance
$d\omega_{i,j} = \sum_k \omega_{i,k} \wedge \omega_{k,j}$ $d\theta_i = \sum_j \omega_{i,j} \wedge \theta_j$	198	Cartan structural equations
dV	139	volume measure
E_1, E_2	192	a frame (basis) of vector fields
E_3	192	the unit normal vector field
F	2	a holomorphic function
F'	2	complex derivative of F
$\mathcal{F}$	16	a normal family
$F_C^U(P)$	90, 180	Carathéodory metric on U at P
$F_K^U(P)$	94, 181	Kobayashi metric on U at P
$f_*\gamma$	43, 181	the push-forward of γ
f^*	42	the pullback mapping
$f^\#(z)$	85	the spherical derivative
f^n	58, 101	nth iterate of f
$\\|f\\|_{A^2(\Omega)}$	138	the Bergman norm
$\langle f, g\rangle_{A^2(\Omega)}$	139	the Bergman inner product
$\mathcal{G}$	81	a family of holomorphic functions
γ	3, 30, 31	a curve
$\dot{\gamma}$	30	derivative of γ
$\gamma_{P,Q}(t)$	50	curve of least Poincaré length connecting P to Q
$\Gamma_\alpha(P)$	118	Stolz region, non-tangential approach region

Symbol	Page Number	Meaning
$G(z, \zeta)$	148	the Green's function
id	122	the identity mapping
$\mathbf{i}(z)$	127	the identity map
$\mathbf{i}_P^C(U)$	182	Carathéodory indicatrix
$\mathbf{i}_P^K(U)$	182	Kobayashi indicatrix
$I(z)$	83	inversion mapping
$\oint_\gamma F(z)\,dz$	4	complex line integral
∞	86	point at infinity on the Riemann sphere
$\mathrm{Jac}\,\phi$	141, 172	the Jacobian matrix
K	17, 18	a compact set
$\kappa = \kappa(P)$	67	curvature of the metric ρ
$\kappa_{U,\rho}(z) = \kappa_\rho(z)$	67	curvature of the metric ρ
K_Ω	139, 140	the Bergman kernel for Ω
λ	30	a metric
ℓ	61, 62	a line in Euclidean geometry
$\ell(\gamma)$	30	length of γ
$\ell_\rho(\gamma)$	34	length of curve γ in metric ρ
$\ell_K(\gamma)$	181	Kobayashi length of the curve γ
$\mathcal{M}_\beta(P)$	119	metric approach region
$\mathcal{M}_P$	194	matrix of the shape operator
$\mu(z)$	76	metric of negative curvature on $\mathbb{C} \setminus \{0, 1\}$
$\nabla_v \alpha(P)$	193	covariant derivative
ν_P	108	unit outward normal at P
ν'_P	108	unit inward normal at P
$\omega_{i,j}$	195	connection forms
P	82	the north pole of the Riemann sphere
P_k	25	kth degree Taylor polynomial
$p(x, y)$	82	the stereographic projection map
$p(z)$	9	a polynomial
Ω	8	a domain
(Ω_1, ρ_1)	43	metric pair
$\partial/\partial z,\ \partial/\partial\overline{z}$	38	complex differential operators
$\phi_a(z_1, z_2)$	177	biholomorphic mapping of B
$\phi_a(\zeta)$	13	a Möbius transformation
$\{\phi_j\}$	145	an orthonormal system for $A^2(\Omega)$
$\psi(z, w)$	155	the pseudohyperbolic metric

Symbol	Page Number	Meaning
Ψ	129	a homotopy
$\mathbb{R}$	14	the real numbers
$r(P)$	111	radius of curvature
ρ	31	a metric (weight)
ρ	70	the Poincaré metric
$\rho(z)$	106	defining function
$\rho_\alpha^A(z)$	72	dilated, scaled Poincaré distance metric
ρ_C^Ω	103, 104	Carathéodory metric
ρ_E^Ω	103, 104	Euclidean metric
ρ_K^Ω	103, 104	Kobayashi metric
ρ_Ω	152	Bergman metric
$\rho_r(z)$	70	the dilated Poincaré metric
ρ_τ	14	a rotation
$\rho(z)\|dz\|$	34	a conformal metric
$S_P(v)$	193	shape operator or Weingarten map
σ	122	the reflection map
$\sigma(z)$	69	the spherical metric
σ_0	20	extremal function
$\sum_j a_j(z-P)^j$	2	power series expansion
$\sum a_{jk}(z_1-P_1)^j(z_2-P_2)^k$	166	power series expansion
T	109	tubular coordinate mapping
$T_P(M)$	192	the tangent space to M at P
τ	84	induced Euclidean metric on sphere
τ_p	120	inward normal segment at p
θ_i	194	covector fields
U	2, 8	a domain
$(U, D)^P$	93, 180	holomorphic functions f from D to U such that $f(0) = P$
U_0	81	the slit plane
U_w	164	slice of a domain in $\mathbb{C}^2$
U^z	165	slice of a domain in $\mathbb{C}^2$
W	108	tubular neighborhood
$\\|\xi\\|_{\rho,z}$	31	metric length of ξ
$\|\xi\|$	31	Euclidean length of ξ

References

[AHL1] L. Ahlfors, An extension of Schwarz's lemma, *Trans. Amer. Math. Soc.* 43 (1938), 359–364.

[AHL2] ——, *Complex Analysis,* 3rd. ed. McGraw-Hill, New York, 1979.

[APF] L. S. Apfel, thesis, Washington University, 2003.

[BEK] S. R. Bell and S. G. Krantz, Smoothness to the boundary of conformal mappings, *Rocky Mountain J. Math.* 17(1987), 23–40.

[BEL] S. R. Bell, Biholomorphic mappings and the $\bar{\partial}$-problem, *Ann. Math.*, 114 (1981), 103–112.

[BER] S. Bergman, *The Kernel Function and Conformal Mapping*, 2nd ed., American Mathematical Society, Providence, 1970.

[BOAS] R. P. Boas, *Invitation to Complex Analysis,* Random House, New York, 1987.

[CHM] S. S. Chern and J. Moser, Real hypersurfaces in complex manifolds, *Acta Math.* 133(1974), 219–271.

[COH] R. Courant and D. Hilbert, *Methods of Mathematica Physics*, Interscience Publishers, New York, 1953–1962.

[DFN] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry—Methods and Applications*, Spring-er-Verlag, New York and Berlin, 1984.

[EAH] C. Earle and R. Hamilton, A fixed point theorem for holomorphic mappings, *Proc. Symp. Pure Math.*, Vol. XVI, 1968, 61–65.

[EPS] B. Epstein, *Orthogonal Families of Analytic Functions*, MacMillan, New York, 1965.

[FK] H. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, New York and Berlin, 1979.

[FEF1] C. Fefferman, The Bergman kernel and biholomorphic mappings, *Invent. Math.,* 26 (1974), 1–65.

[FEF2] C. Fefferman, Parabolic invariant theory in complex analysis, *Adv. Math.* 31(1979), 131–262.

[FUK] B. A. Fuks, *Introduction to the Theory of Analytic Functions of Several Complex Variables*, Translations of Mathematical Monographs, American Mathematical Society, Providence, R.I., 1963.

[GAR] J. Garnett, *Analytic Capacity and Measure,* Springer Lecture Notes #297 Springer-Verlag, 1972.

[GRK] R. E. Greene and S. G. Krantz, *Function Theory of One Complex Variable*, 2nd ed., American Mathematical Society, Providence, R.I., 2002.

[HAH] K. T. Hahn, Inequality between the Bergman metric and Carathéodory differential metrics, *Proc. Am. Math. Soc.* 68(1978), 193–194.

[HAL] P. R. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, 2nd ed., AMS Chelsea Publishing, Providence, R.I., 1998.

[HEI] M. Heins, On the number of 1-1 directly conformal maps which a multiply-connected plane region of finite connectivity $p(> 2)$ admits onto itself, *Bull. AMS* 52(1946), 454–457.

[HIL] E. Hille, *Analytic Function Theory*, 2nd ed., Chelsea Publishing, New York, 1982.

[KOB1] S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, 1970.

[KOB2] S. Kobayashi, *Hyperbolic Complex Spaces*, Springer-Verlag, Berlin, 1998.

[KOK] S. Kobayashi and K.Nomizu, *Foundations of Differential Geometry*, Vols. I and II, Interscience, New York, 1963, 1969.

[KR1] S. Krantz, *Function Theory of Several Complex Variables*, 2nd ed., American Mathematical Society, Providence, R.I., 2000.

[KR2] ——, What is several complex variables?, *Amer. Math. Monthly,* 94 (1987), 236–256.

[KR3] ——, Functions of one complex variable, *The Encyclopedia of Physical Science and Technology,* Academic Press, New York, 1987.

[KR4] ——, Functions of several complex variables, *The Encyclopedia of Physical Science and Technology,* Academic Press, New York, 1987.

[KR5] ——, *The Elements of Advanced Mathematics*, 2nd ed., CRC Press, 2002.

[KRP1] S. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, 2nd ed., Birkhäuser, Boston, in press.

[KRP2] S. Krantz and H. R. Parks, *The Geometry of Domains in Space*, Birkhäuser, Boston, 1999.

[KRP3] S. Krantz and H. R. Parks, *The Implicit Function Theorem*, Birkhäuser, Boston, 2002.

[KRP4] S. Krantz and H. R. Parks, Distance to C^k manifolds, *Jour. Diff. Equations* 40(1981), 116-120.

[LS] L. Loomis and S. Sternberg, *Advanced Calculus,* Addison-Wesley, Reading, 1969.

[MIS] D. Minda and G. Schober, Another elementary approach to the theorems of Landau, Montel, Picard and Schottky, *Complex Variables,* 2 (1983), 157–164.

[MUN] J. Munkres, *Elementary Differential Topology,* Princeton University Press, Princeton, 1963.

[ONE] B. O'Neill, *Elementary Differential Geometry,* Academic Press, New York, 1966.

[PAI] P. Painlevé, Sur les lignes singulières des fonctions analytiques, *Annales de la Faculté des Sciences de Toulouse* (1) 2 (1888), 1–130.

[PRI] I. Privalov, *Randeigenschaften Analytischer Functionen,* Deutsch Verlag der Wissenschaften, Berlin, 1956.

[RU1] W. Rudin, *Principles of Mathematical Analysis,* 3rd ed., McGraw-Hill, New York, 1976.

[RU2] ——, *Function Theory in the Unit Ball of* $\mathbf{C}^n$, Springer-Verlag, Berlin, 1981.

[RU3] ——, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.

[TAN] N. Tanaka, On generalized graded Lie algebras and geometric structures, *J. Math. Soc. Japan* 19(1967), 215–254.

[THO] G. Thomas, and R. Finney, *Calculus and Analytic Geometry,* Addison-Wesley, Reading, 1984.

[WOL] J. Wolf, *Spaces of Constant Curvature,* 4th ed., Publish or Perish Press, Berkeley, 1977.

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