Point Set Topology, the Brouwer Fixed Point Theorem and Applications

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1 Introduction

These notes form a foundation for understanding Point-set Topology, which deals with identifying and exploring the properties of underlying structures in mathematics. We initially discuss Metric Spaces and properties within them, before generalizing to Topological Spaces. Two essential properties of Topological Spaces, Compactness and Connectedness, are then explored, with specific focus on an important equivalence between Compactness, Sequential Compactness, and Completeness and Precompactness, as well as the notion of fixed points.

Once a sufficient foundation has been established, we explore the main part of the project: the Brouwer Fixed Point Theorem. These notes offer an in-depth proof of the theorem, as well as an exploration of its applications in a variety of areas and concepts in mathematics, such as Linear Algebra, the Brouwer-Lebesgue Tiling Theorem, the n-dimensional Intermediate Value Theorem, and Topological Invariance of Domain and Dimension.

2 Metric Spaces

2.1 Basic Definitions

We begin by first defining what a metric is:

Definition 2.1. (Simmons) Let X be a **non-empty set**. Then a metric on X is a real function d of ordered pairs of elements in X, $d : X \times X \to \mathbb{R}_+$, such that the following conditions hold:

1. $d(x, y) \geq 0$ (the codomain of d is non-negative)
2. $d(x, y) = d(y, x)$ (d is symmetric)
3. $d(x, y) = 0 \iff x = y$
4. $d(x, y) \leq d(x, z) + d(z, y)$ (the triangle inequality holds for d), for all $x, y, z \in X$.

We define a metric space to be a set X with a metric d on X. We denote the metric space by (X, d).

2.2 Examples

Example 2.2. A common metric used in $\mathbb{R}$ is the following:

$$d(x, y) = |x - y| \tag{1}$$

where, of course, $|x - y|$ is the absolute value of $x - y$. That is,

$$|x - y| = \max\{(x - y), -(x - y)\}. \tag{2}$$

We will prove that this is a metric in $\mathbb{R}$:

Proof. 1. Of course, $|x - y| \geq 0 \;\; \forall \; x, y \;\in \mathbb{R}$, this is trivial.

2. Note that $|x - y| = \max\{(x - y), -(x - y)\} = \max\{-(y - x), (y - x)\} = |y - x|$.
3. ($\Rightarrow$) If $|x - y| = 0$, then either $x - y = 0$, or $y - x = 0$. In either case, $x = y$. ($\Leftarrow$) If $x = y$, then $x - y = 0$ so $|x - y| = 0$.
4. We wish to show that $|x - y| \leq |x - z| + |z - y|$:
Consider that

$$\begin{aligned}
(|x + y|)^2 &= |x + y||x + y| = |(x + y)(x + y)| \\
\Rightarrow (|x + y|)^2 &= |x^2 + 2xy + y^2| \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2 \\
\Rightarrow (|x + y|)^2 &\leq (|x| + |y|)^2
\end{aligned}$$

and as $|x + y| \geq 0$ and $|x| + |y| \geq 0 \; \forall x, y, \in \mathbb{R}$,

$$\Rightarrow |x + y| \leq |x| + |y|.$$

This is a version of the triangle inequality. We have that $|x - y| = |x - z + z - y|$, so, applying the above, $|x - y| \leq |x - z| + |z - y|$, as required. Hence, $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$.

□

Example 2.3. Let X be any arbitrary set. We define the discrete metric on X to be:

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

To prove that this is a metric, consider the following:

Proof.

1. Clearly, $d(x,y) \geq 0$ for all x, y in X.
2. If $x = y$ then $y = x$, and similarly, if $x \neq y$ then $y \neq x$. Hence, $d(x,y) = d(y,x)$. This is trivial.
3. By definition $x = y \iff d(x,y) = 0$.
4. Consider that $d(x,y)$, $d(x,z)$, and $d(z,y)$ are all either 0 or 1. So, it is easy to see that regardless, the triangle inequality will hold. The only possible place where the triangle inequality may not hold is when $d(x,y) = 1$ and $d(x,z) = d(z,y) = 0$. Suppose that it does not hold. By definition $x = z$ and $z = y$ so $x = y$, which is a contradiction. The rest of the cases are trivial, and so the triangle inequality holds for all values x, y, z.

$\square$

Example 2.4. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean vector space with elements $x = (x_1, ..., x_n)$, $(x_i \in \mathbb{R})$, and let

$$|x| = \sqrt{\sum_{i=1}^{n} {x_i}^2} \geq 0$$

The usual or Euclidean metric defined on the space $\mathbb{R}^n$ is given by:

$$d(x,y) = |x - y|.$$

To prove that this is a metric, consider the following:

Proof.

1. By definition, $d(x,y) \geq 0$.
2. $|x-y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} = |y - x|$.
3. Suppose $x = y$. Then $|x-y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (x_i - x_i)^2} = 0$. Conversely, suppose $\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0$. As squares are always non-negative, $x_i = y_i$ for all $i = 1, ..., n$ and so $x = y$.
4. Take any $x, y, z \in \mathbb{R}^n$. Let $x_i - y_i = r_i$, $y_i - z_i = s_i$, and all summations be over $i = 1, ..., n$. Then we wish to prove that:

   $$\left(\sum (r_i + s_i)^2\right)^{1/2} \leq \left(\sum r_i^2\right)^{1/2} + \left(\sum s_i^2\right)^{1/2}.$$

   As both sides are non-negative, this is equivalent to proving the square of both sides, that is:

   $$\sum r_i^2 + \sum s_i^2 + 2\sum r_i s_i \leq \sum r_i^2 + \sum s_i^2 + 2\left(\sum r_i^2\right)^{1/2}\left(\sum s_i^2\right)^{1/2}.$$

   Simplifying, this is just the Cauchy-Schwarz Inequality:

   $$\sum_{i=1}^{n} (r_i s_i) \leq \sqrt{\left(\sum_{i=1}^{n} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right)} \tag{3}$$

Cauchy-Schwarz Inequality Proof:
We will prove 3 by induction. Let $P(n)$ be the above statement. $P(1)$ obvious, as

$$\sqrt{\left(\sum_{i=1}^{1} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{1} s_i^2\right)} = r_i s_i = \sum_{i=1}^{1}(r_i s_i).$$

To prove $P(2)$, observe that

$$\begin{aligned} & r_1 s_1 + r_2 s_2 \leq \sqrt{r_1^2 + r_2^2}\sqrt{s_1^2 + s_2^2} \\ \iff & (r_1 s_1 + r_2 s_2)^2 \leq (r_1^2 + r_2^2)(s_1^2 + s_2^2) \\ \iff & r_1^2 s_1^2 + 2 r_1 s_1 r_2 s_2 + r_2^2 s_2^2 \leq r_1^2 s_1^2 + r_1^2 s_2^2 + r_2^2 s_1^2 + r_2^2 s_2^2 \\ \iff & 0 \leq r_1^2 s_2^2 - 2 r_1 s_1 r_2 s_2 + r_2^2 s_1^2 \\ \iff & 0 \leq (r_1 s_2 - r_2 s_1)^2 \end{aligned}$$

Which is true. Hence $P(2)$ holds.

Assume that $P(n)$ is true. Then,

$$\begin{aligned} \sum_{i=1}^{n}(r_i s_i) &\leq \sqrt{\left(\sum_{i=1}^{n} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right)} \\ \Rightarrow \sum_{i=1}^{n}(r_i s_i) + (r_{n+1} s_{n+1}) = \sum_{i=1}^{n+1}(r_i s_i) &\leq \sqrt{\left(\sum_{i=1}^{n} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right)} + (r_{n+1} s_{n+1}) \end{aligned}$$

But by $P(2)$,

$$\begin{aligned} & \sqrt{\left(\sum_{i=1}^{n} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right)} + (r_{n+1} s_{n+1}) \leq \sqrt{\left(\sum_{i=1}^{n} r_i^2\right) + r_{n+1}^2}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right) + s_{n+1}^2} \\ \Rightarrow & \sqrt{\left(\sum_{i=1}^{n} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n} s_i^2\right)} + (r_{n+1} s_{n+1}) \leq \sqrt{\left(\sum_{i=1}^{n+1} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n+1} s_i^2\right)} \end{aligned}$$

Hence, we have that

$$\sum_{i=1}^{n+1}(r_i s_i) \leq \sqrt{\left(\sum_{i=1}^{n+1} r_i^2\right)}\sqrt{\left(\sum_{i=1}^{n+1} s_i^2\right)}$$

And so $P(n+1)$ holds whenever $P(n)$ holds [12].

Thus, the triangle inequality holds for all $x, y, z \in \mathbb{R}^n$.

□

2.3 Open and Closed Sets in Metric Spaces

2.3.1 Open Sets

We first define an open ball, and hence an open set:

Definition 2.5. Let (X, d) be a metric space. If x_0 is a point of (X, d) and r is a positive real number, then the open ball $B_r(x_0)$ with centre x_0 and radius r is the subset of X defined by

$$B_r(x_0) := \{x \in X \mid d(x, x_0) < r\} \tag{4}$$

A subset G of the metric space (X, d) is called an open set if and only if, given any point x in G, there exists a positive real number r such that $B_r(x) \subseteq G$.

Proposition 2.6. In any metric space X, each open ball is an open set.

Proof. Let $B_r(x_0)$ be an open ball in X. Take any x in $B_r(x_0)$. We wish to prove that there exists an r_1 such that $B_{r_1}(x) \subseteq B_r(x_0)$. Consider that, for any $x \in B_r(x_0)$, $d(x, x_0) < r$. So, take $r_1 = r - d(x, x_0)$, a positive real number. Then, for any $y \in B_{r_1}(x)$, $d(y, x) < r_1$, and by the triangle inequality $d(y, x_0) \leq d(y, x) + d(x, x_0) < r_1 + d(x, x_0) = (r - d(x, x_0)) + d(x, x_0) = r$. That is, $d(y, x_0) < r$, and so $y \in B_r(x_0)$. Hence, as y is arbitrary, $B_{r_1}(x) \subseteq B_r(x_0)$, and thus the open ball $B_r(x_0)$ is an open set. $\square$

There are several important properties of open sets with respect to metric spaces:

Proposition 2.7. Let (X, d) be a metric space. Then the empty set $\emptyset$ and the full space X are open sets.

Proof. Let us first prove that the empty set $\emptyset$ is open. Note that $\emptyset \subset X$. We can call $\emptyset$ open if we can show that for any point in $\emptyset$, there exists a positive real number r such that $B_r(x) \subseteq \emptyset$. But of course, there are no points in the empty set, so this condition automatically holds. To prove that X, the full space, is open, consider that the full space contains every possible open ball in X. Hence, for all x in X, there exists an open ball $B_r(x) \subseteq X$. Hence X is open. $\square$

Proposition 2.8. The intersection of any finite collection of open sets in a metric space X is open in X.

Proof. Let U and V be open sets in a metric space X. Then by definition, there exist positive real numbers r_1 and r_2 such that $B_{r_1}(x) \subseteq U$ and $B_{r_2}(x) \subseteq V$ for all $u \in U$, $v \in V$. Now take any $x \in U \cap V$. Then we have that there exist positive real numbers r_1 and r_2 such that $B_{r_1}(x) \subseteq U \cap V$ and $S_{r_2}(x) \subseteq U \cap V$. Take $r = \min\{r_1, r_2\}$. Then we have that $B_r(x) \subseteq U \cap V$ (It may be useful to remember that both of these open balls are centred on x). Hence, the intersection $U \cap V$ of two open sets U,V, is also open. It follows by induction that the intersection of a finite number of open sets is also open. $\square$

Proposition 2.9. The union of any arbitrary collection of open sets in a metric space X is also open in X.

Proof. Let $x \in \cup_{i \in I} U_i$, with $\{U_i\}$ a (possibly infinite) collection of open sets. Then x is an interior point (refer to Definition 2.13) of some U_k and there is an open ball centred on x contained in U_k by definition of open sets. This ball is therefore contained in $\cup_{i \in I} U_i$, and as x is arbitrary the union is open. Note that this proof does not rely on the assumption that the union is finite. $\square$

Remark 2.10. Note that Proposition 2.8 only applies to **finite** intersections, not **infinite** intersections. The following counterexample demonstrates this.

Example 2.11. Take the metric space $\mathbb{R}$ under the usual metric. Consider the open sets given by $(-\frac{1}{n}, \frac{1}{n})$. The infinite intersection of these intervals is the singleton $\{0\}$, which is not open. If it were true that singleton sets were open, then as every set can be written as an arbitrary union of singleton sets, by Proposition 2.9 the set $[0, 1)$ would be open. It is, of course, not open, as there is no $r > 0$ such that $B_r(0) \subseteq [0, 1)$.

The following is quite useful to know:

Theorem 2.12. Let X be a metric space. A subset G of X is open $\iff$ it is a union of open balls.

Proof. Let X be a metric space. We will first prove the forwards implication and then the backwards implication:
$\Rightarrow$: Suppose we have a subset G of X, and that G is open. Then, by definition, for every point x in G, there exists an open ball centred on x of radius r such that $x \in B_r(x) \subseteq G$. Then, as for each x in G there is an open ball containing x, then we can write G as the union of all the open balls $B_r(x)$ for each x. That is, if G is open, then G is a union of open balls.
$\Leftarrow$: We already know by Proposition 2.6 that an open ball is an open set, and so by Proposition 2.9, the union of open balls is open. So, if G is a union of open balls, then G is open. $\square$

A fairly useful concept is the **interior** of a subset U:

Definition 2.13. Let X be a metric space and $U \subseteq X$. A point $x \in U$ is said to be an **interior** point of U if there exists a positive real number $r > 0$ such that the ball centred at x with radius r is a subset of U, that is, $B_r(x) \subseteq U$. The interior of U refers to the set of all interior points of U, and is denoted $\text{int}(U)$.

Remark 2.14. The interior of a subset, $\text{int}(U)$, is always open.

Proof. We have that for each $x \in \text{int}(U)$, there exists an $r > 0$ such that $B_r(x) \subseteq \text{int}(U)$. So we can write the interior of U as the union of open balls. By Theorem 2.12, it follows that $\text{int}(U)$ is open. $\square$

Example 2.15. Equipped with the usual metric, the following examples help to visualise open balls in certain metric spaces:

1. In $\mathbb{R}$, $B_r(x_0) = (x_0 - r, x_0 + r)$.
2. In $\mathbb{R}^2$, $B_r(x_0)$ is the interior (refer to Definition 2.13) of a disc of radius r.
3. In $\mathbb{R}^3$, $B_r(x_0)$ is the interior (refer to Definition 2.13) of a sphere of radius r.

Note that $B_r(x_0)$ depends in general on the metric, d, as well as the underlying set.

Definition 2.16. Let X be a metric space and $U \subseteq X$. A point $x \in U$ is said to be a boundary point of U if for every positive real number $r > 0$ we have that there exists points $a, b \in B_r(x)$ such that $a \in U$ and $b \in U'$. The boundary of U refers to the set of all boundary points of U, and is denoted ∂U.

2.3.2 Closed Sets

Definition 2.17. Let X be a metric space. A subset A of X is closed $\iff$ its complement A' is open.

This is equivalent to the following definition:

Definition 2.18. A subset A of the metric space X is called a closed set if it contains each of its limit points.

where limit points are defined to be the following:

Definition 2.19. If A is a subset of X, a point x in X is called a limit point of A if each open centred on x contains at least one point of A different from x.

To prove that these two definitions of closed sets are equivalent, consider the following:

Proof. Suppose that A is closed. Then A' is open. Also suppose x is a limit point of A. Then if $x \notin A$, then there is some open set U such that $x \in U \subset A'$, which contradicts x being a limit point. So A contains all of its limit points.

Now suppose that A contains all of its limit points, and take some $x \notin A$. Since x cannot be a limit point, there is some open set $U \in A'$ such that $x \in U$. So, A' is open and hence A is closed. $\square$

The following are several important properties of closed sets:

Proposition 2.20. In any metric space X, the empty set $\emptyset$ and the full space X are closed sets.

Proof. The empty set has no elements, and so really contains all of its limit points, and is therefore closed. To prove that the full space X is closed, consider that it contains all points, and so automatically contains all of its limit points and hence is closed. $\square$

Definition 2.21. Let (X, d) be a metric space. If x_0 is a point of (X, d) and r is a positive real number, then the closed ball $D_r(x_0)$ with centre x_0 and radius r is the subset of X defined by

$$D_r(x_0) := \{x \in X \mid d(x, x_0) \leq r\} \tag{5}$$

Proposition 2.22. In any metric space X, each closed ball is a closed set.

Proof. Let X be a metric space with metric d. Consider an arbitrary closed ball $D = D_r(x_0)$ centred on x_0 with radius r in X. The claim is equivalent to showing that D' is open, by Definition 2.17. That is, we need to show that for every $y \in D'$, there exists an open ball centred on y contained in D'. Since $y \notin D$, then $d(x_0, y) > r$. So, $d(x_0, y) - r > 0$. Define $r_1 = d(x_0, y) - r$. We claim that the open ball $B_{r_1}(y)$ is contained in D'. Consider any z in $B_{r_1}(y)$. Then by the triangle inequality,

$$\begin{aligned}
&d(x_0, y) \leq d(x_0, z) + d(z, y) \\
\Rightarrow &d(x_0, z) \geq d(x_0, y) - d(z, y) > d(x_0, y) - r_1 \\
\Rightarrow &d(x_0, z) > d(x_0, y) - (d(x_0, y) - r) = r \\
\Rightarrow &d(x_0, z) > r.
\end{aligned}$$

Hence, z is not contained in D. As z and y are arbitrary points in D', it follows that D' is open, and so D is closed. That is, the closed ball $D_r(x_0)$ is a closed set. $\square$

Proposition 2.23. Let X be a metric space. Then

1. Any arbitrary intersection of any collection of closed sets in X is closed.
2. The finite union of any collection of closed sets in X is closed.

Proof. To prove the above, we can use De Morgan's Law:

Lemma 2.24. De Morgan's Law: Let S and T be sets, and let $\{T_i\}_{i \in I}$ be a collection of subsets of T. Then

$$S \setminus \bigcap_{i \in I} T_i = \bigcup_{i \in I} (S \setminus T_i)$$

Both of these follow directly from De Morgan's Law, Proposition 2.8, and Proposition 2.9. $\square$

Example 2.25. Every subset of the discrete space X is both open and closed.

Proof. Take any arbitrary $U \subseteq \mathbb{R}$. If $U = \emptyset$ or $U = \mathbb{R}$, then we are done as we have already proven that the empty set and the full space are both open and closed for any metric space. Take any $x \in U$. Consider that at the most extreme, $U = \{x\}$ and so consider that the ball $B_1(x) = \{x\}$, and hence we have found a ball contained in U centred at x. By definition, U is open. By Definition 2.17 the complement U' is closed. And as U is arbitrary, we have that all subsets of the discrete space $\mathbb{R}$ are both open and closed. $\square$

2.4 Continuity in Metric Spaces

We will begin with the most fundamental and intuitive definition of continuity:

Definition 2.26. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$ continuous iff for every $a \in X$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d(x, a) < \delta \Rightarrow \rho\left(f(x), f(a)\right) < \epsilon. \tag{6}$$

This is equivalent to the following definition:

Definition 2.27. A map $f : M_1 \to M_2$ between metric spaces is continuous at x in M_1 if given any $S_\epsilon(f(x))$ there exists a $B_\delta(x)$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$.

Proposition 2.28. Suppose that $f : M_1 \to M_2$ is a map between metric spaces. Then f is continuous $\iff$ for every set G open in M_2, $f^{-1}(G)$ is open in M_1.

Proof. We will proceed by proving the forwards implication, followed by the backwards implication of the above statement:

$\Longrightarrow$: Suppose $f : M_1 \to M_2$ is continuous, and suppose we have an open set G in M_2. We wish to prove that $f^{-1}(G)$ is open in M_1. Take any $x \in f^{-1}(G)$. Then $f(x) \in G$. Hence, by definition of an open set, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset G$. And so, by definition of continuity, there exists a $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$. So, as $B_\epsilon(f(x)) \subset G$, then $f(B_\delta(x)) \subset G$ and $B_\delta(x) \subset f^{-1}(G)$. Hence, $f^{-1}(G)$ is open in M_1.

$\Longrightarrow$: Now suppose that we have that, for every open set G in M_2, $f^{-1}(G)$ is open in M_1. Consider an arbitrary point x in M_1, then $f(x) \in M_2$. By definition of an open set, there exists an $\epsilon > 0$ such that $B_\epsilon(f(x))$ is open in M_2. And so by assumption, $f^{-1}(B_\epsilon(f(x)))$ is open in M_1. But we know that $f(x) \in B_\epsilon(f(x))$ and so $x \in \ f^{-1}(B_\epsilon(f(x)))$. By definition of an open set, there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. But this means that $f(B_\delta(x)) \subset B_\epsilon(f(x))$, which is the definition of continuity. That is, f is a continuous function.

Hence, f is continuous $\iff$ for every set G open in M_2, $f^{-1}(G)$ is open in M_1.

□

2.5 Equivalent Metrics

Definition 2.29. We say that two metrics d_1 and d_2 on a set X are equivalent if the identity map $i : (X, d_1) \to (X, d_2)$ is continuous, and if the map $i^{-1} : (X, d_2) \to (X, d_1)$ is continuous.

2.6 Convergence In a Metric Space

Definition 2.30. A sequence (x_n) of points in a metric space X with metric d **converges** to a point x in X if given any (real number) $\epsilon > 0$, there exists (an integer) N such that $x_n \in B_\epsilon(x)$ for all $n \geq N$.

This can equivalently be written as:

Definition 2.31. A sequence (x_n) of points in a metric space X with metric d converges to a point x (that is, $\lim_{n\to\infty}(x_n) = x$) if, given any $\epsilon > 0$, there exists an integer N such that

$$n \geq N \Rightarrow d(x, x_n) < \epsilon.$$

Theorem 2.32. Let X be a metric space, and let $U \subseteq X$. If $x \in X$ is a limit point of U, then there exists a sequence $(x_n) \in U$ such that $\lim_{n\to\infty} x_n = x$.

Proof. Take any $x_0 \in U, x_0 \neq x$. Then take some $x_1 \in U \cap B_{\frac{d(x,x_0)}{2}}, x_1 \neq x$. We continue this process and define x_n to be some point in $U \cap B_{\frac{d(x,x_n)}{2}}, x_n \neq x$. The existence of this point is guaranteed by the definition of a limit point. Clearly (x_n) is a convergent sequence tending to x. $\square$

Theorem 2.33. Let X be a metric space and $U \subseteq X$. Then U is closed iff the limit of every convergent sequence $(x_n) \in U$ satisfies

$$\lim_{n\to\infty} x_n \in U.$$

Proof. We will first prove the forwards implication and then the backwards implication:
$\Rightarrow$: Suppose that $U \neq \emptyset$ is closed, and suppose there exists a convergent sequence (x_n) in U whose limit converges to a point x not in U (i.e. $x \in U'$). By definition of convergence, given any $\epsilon > 0$, there exists some N such that $x_n \in B_\epsilon(x)$ for all $n \geq N$. We know that as U is closed, by Definition 2.17 U' is open. By definition of openness, there is some $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq U'$. But then there is some N such that for all $n \geq N$, $x_n \in B_{\epsilon_0}(x)$, which would imply that some terms of the sequence (x_n) are outside of U, which is a contradiction.
$\Leftarrow$: Now suppose that the limit of every convergent sequence $(x_n) \in U$ satisfies $\lim_{n\to\infty} x_n \in U$. Then by Theorem 2.32, all limit points of U are contained within U. So, by Definition 2.18, U is closed. $\square$

Definition 2.34. Cauchy: A sequence (x_n) in a metric space X is Cauchy if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $n, m \geq N$ implies that $d(x_n, x_m) < \epsilon$.

Definition 2.35. Bounded: A sequence (x_n) in a metric space X is bounded if there is an $M > 0$ and a $b \in X$ such that $d(x_n, b) \leq M$ for all $n \in \mathbb{N}$.

Proposition 2.36. Let X be a metric space.

1. A sequence in X can have at most one limit.
2. If $x_n \in X$ converges to A and (x_{n_k}) is any subsequence of (x_n), then x_{n_k} converges to a as $k \to \infty$.
3. Every convergent sequence in X is bounded.
4. Every convergent sequence in X is Cauchy.

Proposition 2.37. Let X be a metric space and let (x_n) be a Cauchy sequence. Then (x_n) converges to x iff (x_n) has a subsequence that converges to x.

Proof. $\Rightarrow$: This trivially follows as (x_n) is a subsequence of itself that converges to x.
$\Leftarrow$: Suppose (x_n) is a Cauchy sequence with a subsequence (x_{n_k}) that converges to x. By definition of a Cauchy sequence, given any $\epsilon > 0$, there is an $N_1 \in \mathbb{N}$ such that $n, m \geq N_1$ implies that $d(x_n, x_m) < \epsilon/2$. And as (x_{n_k}) is a convergent subsequence, then by definition of convergence, there exists an $N_2 \in \mathbb{N}$ such that $d(x_{n_k}, x) < \epsilon/2$ for all $n \geq N_2$. Take $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, by the triangle inequality, $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon/2 + \epsilon/2 = \epsilon$. So, for all $n \geq N$, $d(x_n, x) < \epsilon$ and so (x_n) converges to x. $\square$

2.6.1 Uniform Convergence in Metric Spaces

The following propositions will be of use later when discussing Function Spaces.

Definition 2.38. We say that a subset $U \subset X$ of a metric space X is bounded if $U \subset B_r(x)$ for some $r > 0$ and $x \in X$.

Definition 2.39. We say that a **function** $f : X_1 \to X_2$ is **bounded** if $f(X_1) \subset X_2$ is bounded.

Definition 2.40. We say that a sequence (f_n) of functions $f : X_1 \to X_2$ is **uniformly convergent** to a function $f : X_1 \to X_2$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $m, n > N$ implies that $d_2(f_m(x), f_n(x)) < \epsilon$ for all $x \in X_1$.

It is important to note the difference between pointwise convergence and uniform convergence. Uniform convergence is a stronger notion. If a sequence converges uniformly, it is guaranteed to converge under the given metric. It is possible for a sequence to converge pointwise to a point, but not converge with respect to the particular metric. The following example demonstrates this fact.

Example 2.41. Consider the sequence (f_n), $f_n(x) = \frac{x}{n}$ in the metric space $\mathbb{R}$ with the usual metric. The sequence is pointwise convergent to 0:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = x \lim_{n\to\infty} \frac{1}{n} = x \cdot 0 = 0$$

But (f_n) is not uniformly convergent. Take $\epsilon = 1$ for example. Supposing (f_n) is uniformly convergent, then there is some integer N such that $m, n > N$ implies that $|\frac{x}{m} - \frac{x}{n}| < 1$. But we can take x to be arbitrarily large and $m \neq n$, so this is a contradiction.

Proposition 2.42. Let (f_n) be a sequence of functions $f_n : X_1 \to X_2$. If each f_n is bounded and $f_n \to f$ uniformly, then $f : X_1 \to X_2$ is bounded. [11]

Proof. Denote the metrics associated with X_1 and X_2 by d_1 and d_2 respectively. Supposing f_n is uniformly convergent, by Definition 2.40 there exists an $N \in \mathbb{N}$ such that $m, n > N$ implies that $d_2(f_m(x), f_n(x)) < 1$ for all $x \in X_1$. And supposing f_n is bounded, by Definition 2.39, there exists some $f(x_0) \in X_2$ and some $r > 0$ such that $d_2(f_n(x), f(x_0)) < r$ for all $x \in X_1$. By the triangle inequality,

$$d_2(f(x), f(x_0)) \leq d_2(f(x), f_n(x)) + d_2(f_n(x), f(x_0)) < 1 + r.$$

Hence, by definition f is bounded. □

Definition 2.43. We say that a sequence of functions (f_n), $f_n : X_1 \to X_2$, is uniformly Cauchy if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $m, n > N$ implies that $d(f_m(x), f_n(x)) < \epsilon$ for all $x \in X_1$.

2.6.2 Completeness

Definition 2.44. We say that a metric space X is **complete** iff every Cauchy sequence $(x_n) \in X$ converges to some point in X.

It will be useful to recall the definition of a Cauchy sequence in a metric space, Definition 2.34.

Remark 2.45. $\mathbb{R}$ equipped with the usual metric is a complete metric space.

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$. Then $\{x_n\}$ is bounded, and so all are all subsequences of $\{x_n\}$. There exists a subsequence of $\{x_n\}$ that is monotone and bounded, and hence convergent. But if a Cauchy sequence has a subsequence that converges to some x in $\mathbb{R}$, then it too converges to x. Hence, $\{x_n\}$ is convergent to some x in $\mathbb{R}$ [20] . □

Theorem 2.46. Let X be a complete metric space and U a subset of X. Then U (as a subspace) is complete iff U (as a subset) is closed.

Proof. $\Rightarrow$: Suppose that U is complete and take some sequence $(x_n) \in U$ that converges. Any convergent sequence is a Cauchy sequence, and so (x_n) is Cauchy. By assumption, if x is the limit of (x_n), then $x \in U$. By Theorem 2.33, it follows that U is closed.
$\Leftarrow$: Now suppose that U is a closed subset and that (x_n) is Cauchy in U. Then (x_n) is also Cauchy in X as U is a subspace of X. So (x_n) converges to some x in X. But by assumption, U is closed and so by Theorem 2.33, it follows that x must be in U, and so U is complete. □

3 Topological Spaces

3.1 Basic Definitions

First, we need to establish what a topological space is:

Definition 3.1. A topological space (X, τ) consists of a non-empty set X together with a fixed collection τ of subsets of X satisfying:

1. $\emptyset, X \in \tau$.
2. If U_1 and U_2 are in τ, then $U_1 \cap U_2 \in \tau$. By induction, the finite intersection of $U_1, ...U_n \in \tau$ is also in τ.
3. The arbitrary union of sets in τ is again in τ. That is, if $U_\lambda \in \tau$ for each $\lambda \in \Lambda$, where Λ is some indexing set, then $\bigcup\limits_{\lambda \in \Lambda} U_\lambda \in \tau$.

The collection τ is called a topology for X, and the members of τ are called the open sets of X. That means that '$U \in \tau$' and 'U is open in X' are equivalent statements.

3.2 Examples

We can already easily construct topological spaces from what we already know:

Definition 3.2. Discrete topology on X: Take any non-empty set X, and let τ be the set of all subsets of X. Then a topological space on X is given by (X, τ).

Proof. We will check each of the axioms required to define a topological space:

1. $\emptyset$ and X are subsets of X, and so the first axiom is fulfilled.
2. Suppose we have two subsets $U_1, U_2 \subseteq X$. Then $U_1 \cap U_2$ is also a subset of X. As τ contains every possible subset of X, it follows that $U_1 \cap U_2$ is also in τ. It follows by induction that the finite intersection of $U_1, ..., U_n \in \tau$ is also in τ.
3. Let U_λ be in X (and so in τ by definition of **discrete** topology) for $\lambda \in \Lambda$, where Λ is some indexing set. Then all the elements of the arbitrary union $\bigcup\limits_{\lambda \in \Lambda} U_\lambda$ are in X, so $\bigcup\limits_{\lambda \in \Lambda} U_\lambda \subseteq X$. By definition of the discrete topology, $\bigcup\limits_{\lambda \in \Lambda} U_\lambda$ is in τ.

□

Definition 3.3. Indiscrete or Trivial topology on X: Take any non-empty set X, and let $\tau = \{\emptyset, X\}$. Then a topological space on X is given by (X, τ), that is, $(X, \{\emptyset, X\})$. X is known as the trivial or indiscrete topological space.

Proof. We will check each of the axioms required to define a topological space:

1. Automatically, the first axiom is fulfilled, as $\tau = \{\emptyset, X\}$.
2. We have that: $\emptyset \cap \emptyset = \emptyset$, $\emptyset \cap X = X \cap \emptyset = \emptyset$, and $X \cap X = X$. Hence, all possible intersections of two elements in τ are also contained within τ.
3. The arbitrary union of sets in τ can only give either X or $\emptyset$, and both possibilities each is contained within τ.

□

Example 3.4. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean vector space with elements $x = (x_1, ..., x_n)$, $(x_i \in \mathbb{R})$, and let

$$|x| = \sqrt{\sum_{i=1}^{n} {x_i}^2} \geq 0$$

be the length of x. We declare a subset U of $\mathbb{R}^n$ to be *open* iff for each $a \in U$, there exists an $r > 0$ such that

$$|x - a| < r \Rightarrow x \in U.$$

The collection of open sets thus defined is called the **usual topology** on $\mathbb{R}^n$.

Proof. Let τ denote the usual topology on $\mathbb{R}^n$.

The full space $\emptyset$ and $\mathbb{R}^n$ are both open. Consider that $\emptyset$ does not have any elements for which there must exist an $r > 0$ such that the above holds. $\mathbb{R}^n$, the full space, is also open, as $|x - a|$ is always contained within $\mathbb{R}$ and so there will always exist an $r > 0$ such that the above holds. So $\emptyset$ and $\mathbb{R}^n$ are both in τ.

2. Let U_1 and U_2 be in τ. Let $a \in U_1 \cap U_2$. Then we know that as $a \in U_1$ and $a \in U_2$, by definition, there exist $r_1 > 0$ and $r_2 > 0$ such that $|x - a| < r_1 \Rightarrow x \in U_1$ and $|x - a| < r_2 \Rightarrow x \in U_2$. If we take $r = \min\{r_1, r_2\}$, then $|x - a| < r \Rightarrow x \in U_1 \cap U_2$. As $r_1, r_2 > 0$, $\min\{r_1, r_2\} > 0$ and so it follows that $U_1 \cap U_2$ is also open.

3. Let $U_\lambda \in \tau$, for all $\lambda \in \Lambda$ where Λ is some indexing set. Then Let $a \in \bigcup_{\lambda \in \Lambda} U_\lambda$. Then for at least one $\lambda_0 \in \Lambda$ we have $a \in U_{\lambda_0}$. So, there exists an $r > 0$ such that $B_r(a) \subseteq U_{\lambda_0}$. Thus, considering each $a \in \bigcup_{\lambda \in \Lambda} U_\lambda$, then $B_r(a) \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$ and so $\bigcup_{\lambda \in \Lambda} U_\lambda$ is also open.

□

Example 3.5. Let X be a metric space with metric d. The collection of open sets of X defines a topology. The result is called a metric topology on X arising from the metric d.

Proof. The Topological space axioms are satisfied:

1. By Proposition 2.20, $\emptyset$ and X are open sets, and hence are contained in τ.
2. By Proposition 2.8, the intersection of any finite collection of open sets is also open. So, for any (open) sets in τ, their intersection is also contained in τ.
3. By Proposition 2.9, the union of any arbitrary collection of open sets is also open. So, for any (open) sets in τ, their union is also contained in τ.

□

Further, if (X, τ) is a topological space, and X admits a metric whose metric topology is precisely τ, then we say that (X, τ) is metrisable.

3.3 Subspaces

We may form subspaces of topological spaces.

Definition 3.6 (Subspace topology on X)**.** Let (X, τ) be a topological space on X, and let $A \subseteq X$ be any subset. Then the subspace topology on A consists of all sets of the form $U \cap A$, for all $U \in \tau$. That is, the subspace topology on A, τ_A, is given by:

$$\tau_A = \{A \cap U \mid U \in \tau\}.$$

The proof of why τ_A forms a topology on A is trivial:

Proof.

1. Of course $\emptyset$ is contained within τ_A, as $\emptyset \in \tau$ and so $A \cap \emptyset = \emptyset$. The full space A is also in τ_A, as by definition of a topology $X \in \tau$ so $A \cap X = A \in \tau_A$.

2. Let $V_\lambda \in \tau_A$, for $\lambda \in \Lambda$, where Λ is a finite indexing set.. Then there exists, for each $\lambda \in \Lambda$ a $U_\lambda \in \tau$ such that $V_\lambda = A \cap U_\lambda$. Then $\bigcap\limits_{\lambda \in \Lambda} (V_\lambda) = \bigcap\limits_{\lambda \in \Lambda} (A \cap U_\lambda) = A \cap \bigcap\limits_{\lambda \in \Lambda} U_\lambda$. But by definition of a topology, $\bigcap\limits_{\lambda \in \Lambda} U_\lambda \in \tau$ and so by definition of a subspace topology $A \cap \bigcap\limits_{\lambda \in \Lambda} U_\lambda \in \tau_A$.

3. The third condition for τ_A to be a topology is satisfied similarly. With V_λ and U_λ defined as above for $\lambda \in \Lambda$ where Λ is an arbitrary indexing set, we have that $\bigcup\limits_{\lambda \in \Lambda} (V_\lambda) = \bigcup\limits_{\lambda \in \Lambda} (A \cap U_\lambda) = A \cap \bigcup\limits_{\lambda \in \Lambda} U_\lambda$. As $\bigcup\limits_{\lambda \in \Lambda} U_\lambda \in \tau$, it follows that $A \cap \bigcup\limits_{\lambda \in \Lambda} U_\lambda \in \tau_A$.

$\square$

3.4 Open and Closed Sets

The notion of open and closed sets with respect to topological spaces is key in understanding how they work. Their definitions are fairly simple. We define the following notions:

Definition 3.7. Let (X, τ) be a topological space. We say that a set U, $U \subset X$, is open if $U \in \tau$.

Definition 3.8. Let (X, τ) be a topological space. We say that a set U, $U \subset X$, is closed if $U' \in \tau$, where U' is the complement of U.

Remark 3.9. It is important to note that not open $\not\Rightarrow$ closed, and vice versa. Some sets can be both open and closed. The following example confirms this fact.

Example 3.10. Let (X, τ) be a topological space. Then the empty set $\emptyset$ and the full space X are both open and closed.

Proof. Define an arbitrary topological space (X, τ). Then by definition of a topological space, $\emptyset$ and $X \in \tau$. So, by Definition 3.7, both $\emptyset$ and X are open sets. But also consider that $X' = \emptyset$, and as $\emptyset$ is open, Definition 3.8 implies that X is closed. Similarly, as $\emptyset' = X$ and X is open, by Definition 3.8 $\emptyset$ is closed. Hence, X and $\emptyset$ are both open and closed. $\square$

Remark 3.11. In order to show that a set $U \subseteq X$ is open, it is enough to show that for every $x \in U$ there is an open set V with $x \in V \subseteq U$.

3.5 Continuity in Topological Spaces

Definition 3.12. Given two topological spaces (X_1, τ_1) and (X_2, τ_2), and a map $f : X_1 \to X_2$, we say that f is continuous (with respect to the topologies of τ_1 and τ_2) if $U \in \tau_2 \Rightarrow f^{-1}(U) \in \tau_1$. Specifically, we say that f is (τ_1, τ_2)-continuous. Note that the definition concerns **inverse** images and not **direct** images.

This is equivalent to the following definition:

Definition 3.13. The function f is continuous at $x \in X_1$ if given any U_2 in τ_2 such that $f(x) \in U_2$, there exists U_1 in τ_1 such that $x \in U_1$ and $f(U_1) \subset U_2$.

Proposition 3.14. A map $f : X_1 \to X_2$ is continuous iff for every closed V in X_2, $f^{-1}(V)$ is closed in X_1.

Proof. $\Rightarrow$: Suppose f is continuous. Let V be a closed subset of X_2. Then $X_2 \setminus V$ is open in X_2 and $f^{-1}(X_2 \setminus V)$ is open in X_1. But, $f^{-1}(X_2 \setminus V) = f^{-1}(X_2) \setminus f^{-1}(V) = X_1 \setminus f^{-1}(V)$, and so $X_1 \setminus f^{-1}(V)$ is open in X_1. It follows that$f^{-1}(V)$ is closed in X_1.
$\Leftarrow$: Let U be open in X_2. Then $X_2 \setminus U$ is closed in X_2, and by assumption $f^{-1}(X_2 \setminus U)$ is closed in X_1. But this means that $f^{-1}(X_2 \setminus U) = f^{-1}(X_2) \setminus f^{-1}(U) = X_1 \setminus (f^{-1}(U))$ is closed in X_1. Hence, $f^{-1}(U)$ is open in X_1, and by definition f is continuous. □

Proposition 3.15. If X_1, X_2, and X_3 are topological spaces and $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are continuous maps, then $g \circ f : X_1 \to X_3$ is also continuous [18, pp.48].

Proof. Suppose that X_1, X_2, and X_3 are topological spaces and $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are continuous maps. Let U be any open subset of X_3. Then $g^{-1}(U)$ is open in X_2 by Definition 3.12, and further $f^{-1}(g^{-1}(U))$ is continuous in X_1 by Definition 3.12. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$, hence, $g \circ f$ is continuous. □

Proposition 3.16. For any topological spaces X_1, X_2 and $y_0 \in X_2$, the constant function

$$f_0 : X_1 \to X_2 \; ; \; x \to y_0$$

is a continuous function.

Proof. Let $f(x) = y_0$ for all $x \in X$. Then let U be open in Y. Then $f^{-1}(U)$ is either equal to $\emptyset$ if y_0 is not in U, or equal to X if y_0 is in U. In either case, $f^{-1}(U)$ is open and so by definition f is continuous. □

Remark 3.17. *Continuity in Metric Topological Spaces:* We have already defined continuity in Metric Spaces in Definition 2.26. The same definition of continuity can be used to define continuity in Metric Topological Spaces. That is, a map f is topologically continuous (under Definition 3.12) $\iff$ f is analytically continuous (under Definition 2.26). This follows from Proposition 2.28.

3.6 Bases for Topologies

We will now define the notion of a basis of a topology.

Definition 3.18. A Basis for a topology on a set X is a collection $\mathcal{B}$ of subsets $B \subseteq X$ such that:

1. $X = \bigcup\limits_{B \in \mathcal{B}} B$.
2. The intersection of two sets B_1, B_2 in $\mathcal{B}$ is a set $B_1 \cap B_2 \in \mathcal{B}$. Specifically, there exists a set $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$.

Definition 3.19. The topological space (X, τ) generated by a basis $\mathcal{B}$ has open sets that are the arbitrary unions of basis elements $B_\lambda \in \mathcal{B}$. That is, any open set U in X is given by:

$$U = \bigcup_{\lambda \in \Lambda} B_\lambda$$

Proof. We will prove each of the axioms required to define a topology. Let τ be the topology generated by the basis $\mathcal{B}$ for the arbitrary set X in the manner above.

1. Consider that $\emptyset \in \tau$ as it is the union of no sets, which is possible under the above definition, and $X \in \tau$, as by Definition 3.18, $X = \bigcup\limits_{B \in \mathcal{B}} B$. It is the union of all the sets in $\mathcal{B}$.

2. If $U_\lambda \in \tau$ for each $\lambda \in \Lambda$, then $\bigcup\limits_{\lambda \in \Lambda} U_\lambda \in \tau$. This is trivial.

3. Let $U_1, U_2 \in \tau$. Then we can write $U_1 = \bigcap\limits_{\lambda \in \Lambda} B_\lambda$ and $U_2 = \bigcap\limits_{\omega \in \Omega}$., where Λ and Ω are finite indexing sets. But
$$U_1 \cap U_2 = \bigcap_{\lambda \in \Lambda} B_\lambda \cap \bigcap_{\omega \in \Omega} B_\omega = \bigcap_{\lambda \in \Lambda, \omega \in \Omega} B_\lambda \cap B_\omega.$$
And any intersection of finite sets is contained within the basis, and so $U_1 \cap U_2 \in \tau$.

□

Proposition 3.20. If τ is the topology on a set X arising from a basis $\mathcal{B}$ for X, then $\mathcal{B}$ is a basis for τ.

3.7 Products

Definition 3.21. Given Topological spaces $(X_1, \tau_1)$and (X_2, τ_2), the product topology τ for $X_1 \times X_2$ is the topology with basis
$$\mathcal{B} = \{U_1 \times U_2 \mid U_1 \in \tau_1, U_2 \in \tau_2\}$$
The space $(X_1 \times X_2, \tau)$ is called the topological product of X_1 and X_2.

Proof. The following proves that this does, in fact, form a topology on $X_1 \times X_2$:

1. $\emptyset \in \tau_1, \tau_2$, then $\emptyset \times \emptyset = \emptyset \in \tau$. Also, $X_1 \in \tau_1$ and $X_2 \in \tau_2$, so $X_1 \times X_2 \in \tau$.

2. Let $U_\lambda \in \tau_1$ for all $\lambda \in \Lambda$, and let $V_\omega \in \tau_2$ for all $\omega \in \Omega$. Then by definition of a topology, $\bigcup\limits_{\lambda \in \Lambda} U_\lambda \in \tau_1$ and $\bigcup\limits_{\omega \in \Omega} V_\omega \in \tau_2$. But $\bigcup\limits_{\lambda \in \Lambda, \omega \in \Omega} (U_\lambda \times V_\omega) = (\bigcup\limits_{\lambda \in \Lambda} U_\lambda, \bigcup\limits_{\omega \in \Omega} V_\omega) \in \tau_2$.

3. Let $U_1, U_2 \in \tau_1$ and $V_1, V_2 \in \tau_2$. Then $U_1 \cap U_2 \in \tau_1$ and $V_1 \times V_2 \in \tau_2$. But $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \tau$. So the intersection of two subsets of $X_1 \times X_2$ is again in $X_1 \times X_2$. It follows by induction that the finite intersection of a collection of open subsets of $X_1 \times X_2$ is again open in $X_1 \times X_2$.

□

It is important to note that it is incorrect to assume that every open set in $X_1 \times X_2$ is of the form $U_1 \times U_2$, where $U_1 \in \tau_1$ and $U_2 \in \tau_2$.

Proposition 3.22. Given a point (x, y) in $X_1 \times X_2$ and an open set W in $X_1 \times X_2$ with $(x, y) \in W$, there exist sets U_1 and U_2 open in $X_1 \times X_2$ respectively such that $(x, y) \in U_1 \times U_2 \subset W$.

Proposition 3.23. The arbitrary (and possibly infinite) product of topological spaces is again a topological space. That is, if X_λ is a topological space for all $\lambda \in \Lambda$, then $T = \Pi_{\lambda\in\Lambda}X_\lambda$ is a topological space. We define the open sets in the topology to be unions of sets of the form $\Pi_{\lambda\in\Lambda}U\lambda$, where each U_λ is open in X_λ and $U_\lambda \neq X_\lambda$ for only finitely many λ.

Proof. Let $T = \Pi_{\lambda\in\Lambda}X_\lambda$ and let τ denote the arbitrary product topology on this set.

1. Consider that for all $X_\lambda, \lambda \in \Lambda$, then $\emptyset$ and $X_\lambda \in \tau_\lambda$, so $\emptyset$ and $T \in \tau$.

2. Let $x_\omega \in \tau$ for $\omega \in \Omega$. Then we can write $x_\omega = \Pi_{\lambda\in\Lambda}U_{\omega_\lambda}$, with $U_{\omega_\lambda} \in \tau_\lambda$ and $\omega \in \Omega$. Then

$$\begin{aligned}\bigcup_{\omega\in\Omega} x_\omega = &\bigcup_{\omega\in\Omega} \Pi_{\lambda\in\Lambda}U_{\omega_\lambda}\\ \equiv &\Pi_{\lambda\in\Lambda}\bigcup_{\omega\in\Omega} U_{\omega_\lambda}\end{aligned}$$

   But as $U_{\omega_\lambda} \in \tau$, by definition of a topology $\bigcup_{\omega\in\Omega} U_{\omega_\lambda} \in \tau_\lambda$. So then $\Pi_{\lambda\in\Lambda}\bigcup_{\omega\in\Omega} U_{\omega_\lambda} \in \tau$.

3. Suppose that we have V_1 and $V_2 \in \tau$. We wish to show that $V_1 \cap V_2 \in \tau$. We can write $V_1 = \Pi_{\lambda\in\lambda}U_{\alpha_\lambda}$ and $V_2 = \Pi_{\lambda\in\Lambda}U_{\beta_\lambda}$. Then $V_1 \cap V_2 = \Pi_{\lambda\in\lambda}U_{\alpha_\lambda} \cap \Pi_{\lambda\in\Lambda}U_{\beta_\lambda} = \Pi_{\lambda\in\lambda}(U_{\alpha_\lambda} \cap U_{\beta_\lambda})$. But $U_{\alpha_\lambda} \cap U_{\beta_\lambda} \in \tau_\lambda$ for all $\lambda \in \Lambda$. So, it follows that $V_1 \cap V_2 \in \tau$.

□

Definition 3.24. The map p_λ between a product topological space $\Pi_{\lambda\in\Lambda}X_\lambda$ and one of its factors X_λ is called the projective map.

Proposition 3.25. The projections $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$ of a topological product onto its factors are continuous.

Proof. Take any open set U in X_1. Then U is open in X_1, and so $(p_1)^{-1} : U \times X_2$ is an open set in the basis of $X_1 \times X_2$. This proves that p_1 is continuous, the proof for p_2 is similar. □

Proposition 3.26. Let $f' : X' \to X_1 \times X_2$ be a map from a topological space X' to the topological product $X_1 \times X_2$. Then f is continuous iff $p_1 \circ f$ and $p_2 \circ f$ are continuous.

Proof. If f is continuous then so is $p_1 \cdot f$ and $p_2 \cdot f$ by Proposition 3.15 and Proposition 3.25. Suppose that $p_1 \cdot f$ and $p_2 \cdot f$ are continuous. then for any subsets $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ we have that:

$$\begin{aligned} f^{-1}(U_1 \times U_2) &= f^{-1}((U_1 \times X_2) \cap (X_1 \times U_2)) \\ &= f^{-1}(U_1 \times X_2) \cap f^{-1}(X_1 \times U_2) \\ &= f^{-1}(p^{-1}(U_1)) \cap f^{-2}(p^{-2}(U_2)) \\ &= (p_1 \cdot f)^{-1}(U_1) \cap (p_2 \cdot f)^{-1}(U_2) \end{aligned}$$

So, if U_1 and U_2 are open in X_1 and X_2 respectively, then $(p_1 \cdot f)^{-1}(U_1)$ and $(p_2 \cdot f)^{-1}(U_2)$ are open in X' too. So, by definition f is continuous. □

3.8 Homeomorphisms

Definition 3.27. A homeomorphism $f : X_1 \to X_2$ of topological spaces is a bijection such that f and f^{-1} are both continuous. Equivalently, a homeomorphism f is a bijection such that U is open in X_1 iff $f(U)$ is open in X_2.

A homeomorphism preserves all the structure that topological spaces possess. If there exists a homeomorphism $f : X_1 \to X_2$, we say that X_1 and X_2 are homeomorphic. f also defines an equivalence relation.

Proposition 3.28. If X_1 and X_2 are topological spaces and $f : X_1 \to X_2$ is continuous, then the graph G_f of f is homeomorphic to X_1.

If a property of a topological space is preserved by a homeomorphism, it is classified as a topological invariant.

Proposition 3.29. Consider the identity map

$$i : (X, \tau_1) \to (X, \tau_2) \ ; \ x \to x$$

. We have that,

1. The map i is continuous under the condition $\tau_2 \subseteq \tau_1$.

2. The map i is a homeomorphism iff the topologies $\tau_1 = \tau_2$.

Proof. The first statement is trivial. If we have a set $U \in \tau_2$, then $i^{-1}(U) = i(U) = U$, and therefore is also open in X as $\tau_2 \subseteq \tau_1$. The second statement then follows from the first. □

Proposition 3.30. Consider the n-dimensional sphere

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \ : \ |x| = 1\}$$

with metric topology inherited from $\mathbb{R}^{n+1}$. Let $x \in \mathbb{S}^n$. Then $\mathbb{S}^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$.

Proof. Without loss of generality, assume that the point x is the north pole, that is, $x = (0, 0, 0, ...1) \in \mathbb{S}^n$. The hyperplane T given by $\{x \in \mathbb{R}^n \mid x_{n+1} = -1\}$ is tangent to $\mathbb{S}^n$ at the south pole $-x$ and is homeomorphic to $\mathbb{R}^n$. This is trivial.

For any $y \neq x \in \mathbb{S}^n$, let $f(y)$ be the unique point where the line L_1 joining y and x given by

$$L_1 = \{\lambda x + (1 - \lambda)y \mid \lambda \in \mathbb{R}\} \subseteq \mathbb{R}^{n+1}$$

intersects T. At this point, λ is given by:

$$\begin{aligned} f(y) =& \lambda x + (1 - \lambda)y \\ \Rightarrow f(y)_{n+1} =& \lambda x_{n+1} + (1 - \lambda)y_{n+1} \end{aligned}$$

But $x_{n+1} = 1$, and $f(y)_{n+1} = -1$. So,

$$\begin{aligned} -1 =& \lambda + (1 - \lambda)y_{n+1} \\ \Rightarrow \lambda + 1 =& (\lambda - 1)y_{n+1} \\ \Rightarrow \lambda + 1 =& \lambda y_{n+1} - y_{n+1} \\ \Rightarrow \lambda(1 - y_{n+1}) =& -1 - y_{n+1} \\ \Rightarrow \lambda(y) =& \frac{y_{n+1} + 1}{y_{n+1} - 1} \end{aligned}$$

$\lambda(y)$ is a continuous function of $y \neq x \in \mathbb{S}^n$. So the function

$$f(y) = \lambda(y)x + (1 - \lambda(y))y$$

is a continuous, and clearly bijective map between the $\mathbb{S}^n \setminus \{x\}$ and T. The inverse function is also continuous. Considering the same L as above, we wish to find

Let $y' \in T$, and x still be the north pole of the n-sphere. Then the line L_2 joining x and y' is given by

$$L_2 = \{\alpha x + (1-\alpha)y' \mid \alpha \in \mathbb{R}\} \subseteq \mathbb{R}^{n+1}$$

The line intersects the n-sphere at some point, $g(y')$, that is,

$$g(y') = \alpha x + (1-\alpha)y'$$

But $|g(y')| = 1$, so $|\alpha x + (1-\alpha)y'| = 1$

$$\begin{aligned}
&\Rightarrow \alpha^2(y'y') + (1-\alpha)^2(x,x) + 2\alpha(1-\alpha)(x,y') = 1\\
&\Rightarrow \alpha^2(y'y') + (1-2\alpha+\alpha^2)(x,x) + 2\alpha(x,y') - 2\alpha^2(x,y') = 1\\
&\Rightarrow \alpha^2((y',y') + 1 - 2(x,y')) + \alpha(-2+2(x,y')) + 1 = 1\\
&\Rightarrow \alpha^2((y',y') + 1 - 2(x,y')) + \alpha(-2+2(x,y')) = 0
\end{aligned}$$

Hence, either $\alpha = 0$, which could not be the case as then $g(y') = y'$, or $\alpha = \frac{2-2(x,y')}{(y',y')+1-2(x,y')}$. This α is a continuous function of y'. Hence, g is also a continuous function of y', but it is also the inverse function of f. Thus, f is a homeomorphism $f : \mathbb{S}^n \setminus \{x\} \to \mathbb{R}^n$. □

Remark 3.31. It is important to note that not all bijective functions are continuous in topological spaces. The following example demonstrates this:

Example 3.32. Consider the function $f : \mathbb{R} \to \mathbb{R}$ with the usual topology given by:

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Z} \\ x+1 & \text{if } x \notin \mathbb{Z} \end{cases}$$

This is a bijective function, however it is not continuous.

Remark 3.33. More so, not all continuous bijections are homeomorphisms. The following example demonstrates this:

Example 3.34. Consider the identity mapping between $\mathbb{R}$ with the discrete topology and $\mathbb{R}$ with the usual topology. We know that (1) the mapping is bijective as it sends $x \to x$, and has an inverse, and (2) is continuous, as we have already proven that the identity map is continuous in Proposition 3.29 when $\tau_2 \subseteq \tau_1$ (this holds as the discrete topology contains all possible subsets of $\mathbb{R}$, and so also contains those within the usual topology on $\mathbb{R}$). However, the topologies are not equal, and so we have that i is not a homeomorphism.

3.9 Closures

If A is a subset of a topological space, then its *closure*, denoted by $\overline{A}$ is the intersection of all closed supersets of A. Formally,

Definition 3.35. The closure of a subset $A \subseteq X$ is

$$\overline{A} = \bigcap_{C \subseteq X \text{ closed } ; A \subseteq C} C.$$

Remark 3.36. We see many different properties arise from closures:

1. $\overline{A}$ is a closed superset of A.
2. $\overline{A}$ is the intersection of all closed sets containing A.

3. $\overline{A}$ is the smallest closed set containing A.

These are all trivial concepts that follow from the definition. We can see that, from these properties, any closed set equals its closure.

Some more complicated properties are as follows:

Proposition 3.37. Let X be a topological space. If A and B are arbitrary subsets of X, then the operation of forming closures has the following properties [17]:

1. $\overline{\emptyset} = \emptyset$,
2. $A \subseteq \overline{A}$,
3. $\overline{\overline{A}} = \overline{A}$,
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
5. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Proof. 1. Any closed set equals its closure. As $\emptyset$ is a closed set, it follows that $\overline{\emptyset} = \emptyset$.

2. By definition, $\overline{A}$ is the smallest closed set containing A. So $A \subseteq \overline{A}$.
3. By definition, $\overline{A}$ is a closed set, and any closed set equals its closure so $\overline{\overline{A}} = \overline{A}$.
4. Note that $A \subseteq A \cup B$ and $B \subseteq A \cup B$ implies $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$. So, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Also, the subset $\overline{A} \cup \overline{B}$ is closed and contains both A and B, so $A \cup B \subseteq \overline{A} \cup \overline{B}$. But by Remark 3.36, $\overline{A \cup B}$ is defined to be the **smallest** closed set containing $A \cup B$, so any closed set that also contains $A \cup B$ also contains $\overline{A \cup B}$ and so $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.
5. Since $\overline{A} \cap \overline{B}$ is the intersection of two closed sets, it is closed. Also note that $A \cap B$, and so it also contains $\overline{A \cap B}$. Hence, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

□

Note: The reverse inclusion for 5 only holds for all $A, B \in X$ when X is discrete. That is, $\overline{A \cap B} = \overline{A} \cap \overline{B}$ for all $A, B \in X$ iff X is discrete. To prove this, consider a topological space X is discrete iff every subset in X is closed. So for contradictions sake, suppose there exists a non-closed subset $A \subseteq X$. Then take any $x \in \overline{A}'$. Then $x \in \overline{\{x\}} \cap \overline{A} = \overline{\{x\} \cap A} = \overline{\emptyset} = \emptyset$. Which is a contradiction. So for $\overline{A \cap B} = \overline{A} \cap \overline{B}$ to hold for all $A, B \in X$, every subset of X must be closed and so X must be discrete.

However, this is not to say that for there to be two subsets $A, B \in X$ such that $\overline{A \cap B} = \overline{A} \cap \overline{B}$ X must be discrete. Take for example the space $\mathbb{R}$ equipped with the usual topology. Let $A = (1, 5)$ and $B = (4, 10)$. Then $A \cap B = (4, 5)$ so $\overline{A \cap B} = [4, 5]$. But $\overline{A} = [1, 5]$ and $\overline{B} = [4, 10]$ so $\overline{A} \cap \overline{B} = [4, 5]$. So for this specific example, $\overline{A \cap B} = \overline{A} \cap \overline{B}$, even though we know that $\mathbb{R}$ with the usual topology is not discrete.

Example 3.38. Regarding metric Topological Spaces, one common error is the assumption that the closure of an open ball of radius r centred on x is a closed ball of radius r centred on x. The following example shows that this is not necessarily the case. Let X be any topological space with $|X| \geq 2$ and define a metric d on X to be:

$$d(x, y) = \begin{cases} 0 & \text{iff } x = y \\ 1 & \text{otherwise} \end{cases}$$

Then the open unit ball around any point x is the singleton set $\{x\}$, and its closure is also $\{x\}$. However, the closed unit ball around any x is X. So, if $|X| \geq 2$ in this example, then the closure of an open ball does not equal the corresponding closed ball. [Is there anything more to be said?]

Two other important notions are interiors and boundaries (we have already referred to boundaries in Remark 3.36):

Definition 3.39. The interior of a subset $A \subseteq X$ is given by:

$$A^0 = \bigcup_{U \subseteq X \text{ open}; U \subseteq A} U.$$

Definition 3.40. The boundary of a subset $A \subseteq X$ is

$$\partial A = \overline{A} \setminus A^0.$$

Remark 3.41. $\overline{A}$ is the union of A and its boundary ∂A

Definition 3.42. We say that a subset A of X is **dense** in X iff $\overline{A} = X$.

3.10 Convergence in Topological Spaces

Definition 3.43. A sequence (x_n) of points in a topological space X converges to a point X in X if, given any open set U containing x, there exists an N such that $x_n \in U$ for all $n \geq N$.

Example 3.44. Let X be an indiscrete space. Then any sequence (x_n) in X converges to any point x of X. For given any open set U containing x, we must have $U = T$, so $x_n \in U$ for all $n \geq 1$.

3.11 Hausdorff Spaces

Definition 3.45. A topological space (X, τ) is called Hausdorff if, for each $x, y \in X$, with $x \neq y$, there exist *disjoint* open sets U and V such that $x \in U$ and $y \in V$.

Proposition 3.46. Every metric space is Hausdorff, and in particular $\mathbb{R}^n$ is Hausdorff.

Proof. Let (X, d) be a metric space with $x, y \in X$, $x \neq y$. Let $r = d(x, y)$. Define $U = B_{r/2}(x)$ and $V = B_{r/2}(y)$. Then $U \cap V = \emptyset$. To prove that this is true, suppose that there exists $z \in U \cap V$. Then $d(x, z) < r/2$ and $d(z, y) < r/2$. So, by the triangle inequality,

$$r = d(x, y) \leq d(x, z) + d(z, y) < r/2 + r/2 = r$$

$$\Rightarrow r < r$$

which is a contradiction. Hence, $U \cap V = \emptyset$, and therefore, as x, y are arbitrary, X is Hausdorff. □

Remark 3.47. By the above, the topological space $\mathbb{R}^n$ with its usual topology is Hausdorff.

Some important properties are the following:

Proposition 3.48.

1. Any subspace of a Hausdorff Space is Hausdorff.
2. The topological product of two Hausdorff spaces is Hausdorff.
3. If $f : X_1 \to X_2$ is injective and continuous, and if X_2 is Hausdorff then so is X_1.
4. Hausdorffness is a topological invariant.

Proof. We will prove these all individually:

1. Let X be a topological space and A be some subspace of X. If x and y are two distinct points of A, then there exist two disjoint subsets U and V of X such that $x \in U$ and $y \in V$. But then $U \cap A$ and $V \cap A$ are open in A, and are still disjoint, so there exist two disjoint subsets in A such that $x \in U \cap A$, and $y \in V \cap A$. As the choice of x and y is arbitrary, A is Hausdorff.

2. We will specifically prove that the product of any (i.e. even infinitely many) Hausdorff spaces is Hausdorff. Let X_α be Hausdorff spaces for $\alpha \in A$ (A is some indexing set). Let $T = \Pi_{\alpha \in A} X_\alpha$ be the topological product of these spaces. If x and y are distinct elements of T, there is at least one $\alpha_0 \in A$ where they differ. X_{α_0} is Hausdorff, so there exist subsets U_{α_0} and V_{α_0} in X_{α_0} such that $x_{\alpha_0} \in U_{\alpha_0}$ and $y_{\alpha_0} \in V_{\alpha_0}$, and $U_{\alpha_0} \cap V_{\alpha_0} = \emptyset$. Let $U_\alpha = V_\alpha = X_\alpha$ for all $\alpha \in A \setminus \{\alpha_0\}$, and let $U = \Pi_{\alpha \in A} U_\alpha$ and $V = \Pi_{\alpha \in A} V_\alpha$. Then U and V are basic open sets in T, $x \in U$, $y \in V$. Most importantly, $U \cap V = \emptyset$. To prove this, suppose there exists a $z \in U \cap V$. Then $z_{\alpha_0} \in U_{\alpha_0} \cap V_{\alpha_0} = \emptyset$, which is a contradiction. So, there exist disjoint subsets U and V of T such that $x \in U$ and $y \in V$. As x and y are arbitrary, it follows that T is Hausdorff.

3. Suppose that $f : X_1 \to X_2$ is continuous and injective, and let X_2 be Hausdorff. Then by injectivity, for $x, y \in X_1, x \neq y$, then $f(x) \neq f(y)$. As X_2 is Hausdorff, we can choose some neighbourhood of $f(x)$ and $f(y)$, U and V respectively, such that $U \cap V = \emptyset$. Note that $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$. These two subsets of X_1, $f^{-1}(U)$ and $f^{-1}(V)$, are neighbourhoods of x and y respectively. As the choice of x and y are arbitrary, it follows that X_1 is Hausdorff.

4. It is easy to see why this follows from the previous statement. If $f : X_1 \to X_2$ is a homeomorphism and X_2 is Hausdorff, then X_1 is also Hausdorff by definition of a homeomorphism. Also note that if f is a homeomorphism from $X_1 \to X_2$, then f^{-1} is a homeomorphism from X_2 to X_1 and so if X_1 is Hausdorff then so is X_2. Combining these, if two topological spaces are homeomorphic and one is Hausdorff, then the other is necessarily Hausdorff too.

□

4 Compactness

4.1 Basic Definition

A cover for a set X is a collection $\mathcal{C}$ of subsets of X such that $X \subset \bigcup_{\lambda \in \Lambda} C_\lambda$. In the context of topological spaces, we can write this definition formally as:

Definition 4.1. If (X, τ) is a topological space and $A \subseteq X$, then a collection of subsets $\mathcal{C}$ of X is said to be a cover of A if

$$A \subseteq \bigcup_{\lambda \in \Lambda} C_\lambda$$

Definition 4.2. A subcover $\mathcal{V}$ of a given cover $\mathcal{C}$ for a set X is a subcollection $\mathcal{V} \subset \mathcal{C}$ which still forms a cover for X.

Definition 4.3. An **open** cover of a topological space $\{X, \tau\}$ is a collection $\{C_\lambda \mid \lambda \in \Lambda\}$ of **open** subsets C_λ of X such that

$$\bigcup_{\lambda \in \Lambda} C_\lambda = X.$$

Example 4.4. Consider the set $X = \{1, 2, 3\}$ and the topology $\tau = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$. Then $\mathcal{T}$ is a topological space on X with topology τ. Let $\mathcal{C} = \{\{1\}, \{3, 2\}, \{1, 3\}, \{2, 3\}\}$. Then $\mathcal{C}$ is a subspace of $\mathcal{T}$. The check for both of these statements is trivial. We claim that $\mathcal{C}$ is an open cover for X.

Proof. Proving that a set $\mathcal{C}$ is a cover of X is a simple task, we simply need to take the union of all of the elements in $\mathcal{C}$ and show that it equals X:

$$\bigcup_{C_i \in \mathcal{C}} C_i = \{1\} \cup \{3, 2\} \cup \{1, 3\} \cup \{2, 3\} = \{1, 2, 3\} = X.$$

□

Note that $\mathcal{C}$ is also an open cover, as all of the elements in $\mathcal{C}$ are open (recall that being "open" is equivalent to being an element of the topology, τ). Hence, we have satisfied Definition 4.3.

Remark 4.5. If a cover $\mathcal{C}$ is open, then a subcover $\mathcal{V} \subset \mathcal{C}$ is also open. This is trivial.

We say that a cover is finite if $\mathcal{C}$ is finite.

Definition 4.6. A compact space is a topological space in which every open cover has a finite subcover. Symbolically, whenever $\mathcal{C} = \{C_\lambda \mid \lambda \in \Lambda\}$ is an open cover, there exist $\lambda_1, \lambda_2, ..., \lambda_n$ such that

$$X = \bigcup_{j=1}^{n} C_{\lambda_j}.$$

It is important to understand this definition well. In layman's terms, given any open cover $\mathcal{C}$ of X, there exists a finite number of the open sets in $\mathcal{C}$ which are enough to cover X [18, pp. 77]. The key concept is that **every** open cover is finite, not just **(at least) one** open cover is finite. The latter is trivially true, taking the singleton collection $\{X\}$ as a finite open cover.

Definition 4.7. Let X be a topological space. Then a subspace $A \subseteq X$ is said to be compact in X iff A itself is a compact topological space.

Proposition 4.8. Let (X, τ) be a topological space and $A \subseteq X$. Then A is compact iff every open cover of A has a finite subcover.

Proof. We will first prove the forwards implication and then the backwards implication.
($\Rightarrow$) Suppose that X is a topological space, A is a compact subset of X, and that $\{C_\lambda\}$ is an open cover for X. By definition of compactness, there exists a finite subset $\Lambda' \subset \Lambda$ such that $\{C_\lambda\}_{\lambda \in \Lambda'}$ is a finite open cover for X. Then $\{C_\lambda \cap A\}_{\lambda \in \Lambda'}$ is a finite collection of open sets whose union covers A. Thus, as the original choice of open cover is arbitrary, we can construct finite subcovers for any open cover of a subset A and so the forwards implication holds.
($\Leftarrow$) Suppose we have a topological space X and that every open cover of a subset A of X has a finite subcover. Take an arbitrary collection of open sets in A, $\{U_\lambda\}$, such that the union of the open sets equals A. Each U_λ can be written in the form $U_\lambda = V_\lambda \cap A$, where V_λ is an open set in X. It follows that $U_\lambda \subset V_\lambda$ and so $\{V_\lambda\}$ forms an open cover for A in X. By assumption each open cover of A has a finite subcover, and so $\{V_\lambda\}$ has a finite subcover. Thus A is compact and the backwards implication holds.

□

4.2 Properties of and Theorems Relating to Compact Spaces

Several important theorems and propositions follow from these definitions:

Theorem 4.9. Let X be a topological space, and $A \subseteq X$ be a finite subspace. Then A is compact in X.

Proof. Suppose A is a finite subset of X. Then we can write $A = \{x_1, x_2, ..., x_n\}$. Let $\mathcal{C} = \{C_i \mid i \in I\}$ be an open cover of A. Then by Definition 4.1,

$$A \subseteq \bigcup_{i \in I} C_i.$$

At most, A can be partitioned into n groups (as there are n elements), where all elements are contained in $\mathcal{C}$. So there exists a subcollection $I^* \subseteq I$ such that

$$A \subseteq \bigcup_{i \in I^*} C_i$$

(where $|I^*| \leq |I|$). By definition, $\mathcal{C}^*$, the subset relating to I^*, defines a subcover of A. The choice of $\mathcal{C}$ is arbitrary. So A is compact in X. $\square$

Remark 4.10. From the above theorem, it follows that if (X, τ) is a topological space, and X is finite, then (X, τ) is a compact topological space.

Proposition 4.11. Let $f : X \to Y$ be a continuous map between topological spaces. If X is compact, so is $f(X)$.

Proof. Let U_λ be open subsets of Y which cover $f(X)$. Then $f^{-1}(U_\lambda)$ are open sets in X which cover X. Hence, as X is compact, there exists a finite subcover $\{f^{-1}(U_{\lambda_1}), ...f^{-1}(U_{\lambda_n})\}$ in this cover of X and so $\{U_{\lambda_1}, ..., U_{\lambda_n}\}$ covers $f(X)$. So $f(X)$ is compact. $\square$

Proposition 4.12. Let X_1 and X_2 be topological spaces, $A \subseteq X_1$, and $f : A \to X_2$ be a continuous map. If A is compact in X_1 then $f(A)$ is compact in X_2.

Proof. The proof of this is similar to Proposition 4.11. $\square$

Remark 4.13 (Extreme Value Theorem and Compact Spaces)**.** One incredible result that follows from these propositions is the Extreme Value Theorem: if $f : X \to \mathbb{R}$ is a real valued, continuous function from a compact space X to the real numbers $\mathbb{R}$, then there is an $x \in X$ such that $f(x) \geq f(y)$ for all $y \in X$. The proof is as follows:

Proof. The proof of this refers to the Heine-Borel Theorem (Theorem 4.23), which is proven later in this section. Since X is compact, by Proposition 4.11, it follows that the image $f(X)$ is compact in $\mathbb{R}$. But by the Heine-Borel Theorem (Theorem 4.23), it follows that $f(X)$ is closed and bounded. By the Completeness axiom there exists an upper bound of $f(X)$. Denote this upper bound $M \in f(X)$. By definition of the supremum, M is a limit point of $f(X)$, and as $f(X)$ is closed and therefore must contain all of its limit points, $M \in f(X)$. So, there must exist an element $x \in X$ such that $f(x) = M$. $\square$

Remark 4.14. Compactness is a topological invariant.

Proof. Let X_1 and X_2 be topological spaces and let $f : X_1 \to X_2$ be a homeomorphism. Suppose that X_2 is compact and that $\{U_\lambda \mid \lambda \in \Lambda\}$ is an open cover for X_1. Consider the collection $\{f(U_\lambda) \mid \lambda \in \Lambda\}$. As f is by definition continuous and surjective, this collection forms an open cover for X_2. As X_2 is compact, by definition there exists a finite subcover $\{f(U_\lambda) \mid \lambda \in \Lambda'\}$ from this open cover for X_2. Again, as f^{-1} is by definition continuous and surjective, $\{f^{-1}(f(U_\lambda)) \mid \lambda \in \Lambda'\} = \{U_\lambda \mid \lambda \in \Lambda'\}$ forms an open cover for X_1. Note that $\{U_\lambda \mid \lambda \in \Lambda'\}$ is a finite subcover in $\{U_\lambda \mid \lambda \in \Lambda\}$ of X_1, and so by definition X_1 is also compact. Showing the reverse (i.e. that X_1 is compact $\Rightarrow$ X_2 is compact) can be done in the same exact manner considering that if f is a homeomorphism from X_1 to X_2, then so is f^{-1} from X_2 to X_1. $\square$

Proposition 4.15. Let (X, τ) be a topological space.

1. If X is compact and $A \subseteq X$ is closed, then A is compact in X.

2. If X is Hausdorff and $A \subseteq X$ is compact, then A is closed.

Proof. We will prove each point individually:

1. Let A be a closed subspace of a compact space X. Let $\mathcal{C} = \{C_i \mid i \in I\}$ be an open cover of A. Trivially, A' is an open cover of A', as A is closed so A' is open. Hence, $\mathcal{C} \cup A'$ forms a cover for X. But X is compact, and so there exists a finite subcover of $\mathcal{C} \cup A'$ for X, which is in turn a finite subcover for A and so A is compact.

2. Recall the definition of Hausdorff 3.45. Let X be Hausdorff and $A \subseteq X$ be compact. Then fix $x \in A'$. Since X is Hausdorff, for each $y \in A$, there are disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. The collection $\{V_y \mid y \in A\}$ is an open cover of A, and also has a finite subcover, $\{V_y \mid y \in F\}$, where F is some **finite** subset of A. Let

$$U = \bigcap_{x \in F} U_x.$$

By the finiteness of F, U is an open neighbourhood of x disjoint from A and so U is open. Further A' is open. By Definition 3.8, A is closed. [1]

□

Proposition 4.16. A topological product $X_1 \times X_2$ is compact iff X_1 and X_2 are both compact. $X_1 \times X_2$ is compact with respect to the product topology.

Proof. We will first prove the forwards implication, and then the backwards implication.
$\Rightarrow$: Suppose $X_1 \times X_2$ is compact. Then since X_1 and X_2 are continuous images of $X_1 \times X_2$ by the projections p_1 and p_2, X_1 and X_2 are also compact by Proposition 4.11.
$\Leftarrow$: Now suppose X_1 and X_2 are compact. We need the following lemma:

Lemma 4.17. Let X_1 be a topological space, X_2 a compact space, $x \in X_1$, N an open set in $X_1 \times X_2$ such that $\{x\} \times X_2 \subseteq N$. Then there is an open set $W \subseteq X_1$ such that $x \in W$ and $W \times X_2 \subseteq N$.

Proof. N is a union of sets $U_\lambda \times V_\lambda$ with U_λ open in X_1 and V_λ open in X_2. Now $\{x\} \times X_2$ is compact (as it is homeomorphic to X_2) and so there is a finite collection $U_{\lambda_j} \times V_{\lambda_j}$ with $1 \leq j \leq n$, with x in each U_{λ_j}, which covers it. Then $W := \bigcup_{j=1}^{n} U_{\lambda_j}$ does the job. □

Let $\{T_\lambda\}$ be an open cover of $X_1 \times X_2$. For each $x \in X_1$, $\{x\} \times X_2$ is homeomorphic to X_2, and so may be covered by finitely many sets in the collection $\{T_\lambda\}$. Let $N(x)$ be the union of these finitely many open sets. By the lemma, there is an open set $W(x)$ such that $x \in W(x)$ and $W(x) \times X_2 \subseteq N(x)$. So, for all $x \in X$, there is an open set $W(x)$ which contains x and is such that $W(x) \times X_2$ is covered by finitely many of the T_λ's. Now the collection $\{W(x) \mid x \in X\}$ is an open cover for X_1, and so there is a finite subcover $\{W(x_1), W(x_2), W(x_3), ..., W(x_n)\}$. So, the collection of sets $W(x_1) \times X_2, ..., W(x_n) \times X_2$ covers $X \times Y$, and since each is covered by only finitely many T_λ's, we conclude that we only need finitely many T_λ's to cover all of $X_1 \times X_2$ [9]. □

Remark 4.18. By Induction, the product of finitely many compact spaces is also compact.

Theorem 4.19. Suppose X_1 is compact, X_2 is Hausdorff and that $f : X_1 \to X_2$ is a continuous bijection. Then f is a homeomorphism.

Proof. To see that this is true, we simply need to show that f^{-1} is also continuous. For any $V \subset X_1$, $(f^{-1})^{-1}(V) = f(V)$. For $y \in f(V) \iff y = f(x)$ for some $x \in V \iff x = f^{-1}(y_0)$ in $V \iff y \in (f^{-1})^{-1}(V)$.We shall prove that if V is closed in X_1 then $(f^{-1})^{-1}(V)$ is closed in X_2. So, suppose that V is closed in X_1. Since X_1 is compact, as the continuous image of a compact space and as X_2 is Hausdorff, $f(V)$ is closed by Proposition 4.15. But $(f^{-1})^{-1}(V) = f(V)$, so $(f^{-1})^{-1}(V)$ is closed in X_2. It follows by Proposition 3.14 that f^{-1} is also continuous and so f is a homeomorphism. □

4.2.1 Compactness of $[a, b]$

Theorem 4.20. The real line $\mathbb{R}$ is not compact.

Proof. Let $\mathbb{R}$ be a topological space equipped with the usual topology. Consider that, for a topological space to be compact, every open cover must admit a finite subcover. To prove that $\mathbb{R}$ is not compact, we will use proof by contradiction. Consider the open cover $\mathcal{C} = \{(-n, n) \mid n \in \mathbb{N}\}$. Seeing that this is an open cover is trivial. Now suppose that there is a finite subcover of $\mathcal{C}$. That is, there exists $N \in \mathbb{N}$ such that $(-N, N)$ contains every other element of $\mathcal{C}$. But take any $x > N$. Clearly $x \in \mathcal{C}$, as $\mathcal{C}$ is a cover of $\mathbb{R}$. However, $x \notin (-N, N)$, and hence we have a contradiction. So, $\mathbb{R}$ is not compact. $\square$

Proposition 4.21. Any closed, bounded interval $[a, b]$ in $\mathbb{R}$ is compact.

Proof. Suppose that $[a, b]$ is not compact, then there exists an open cover $\mathcal{C}_\Lambda$ such that $[a, b] \subset \bigcup_{\lambda \in \Lambda} U\lambda$ with no finite subcover. We can write $[a, b] = [a, m_1] \cup [m_1, b]$, where $m_1 = (a + b)/2$ (the midpoint of a and b). Consider that the union of two intervals with finite subcovers will itself have a finite subcover. Because there is no finite subcover for $[a, b]$, then for at least one of $[a, m_1]$ or $[m_1, b]$, there is no finite subcover for the interval.

Now pick whichever interval does not have a finite subcover (or either one if both do not). Suppose this interval is $[m_1, b]$. Again, dividing the interval in half, by the same logic above at least one of the subintervals will not have a finite subcover. We continue this process to obtain a sequence of **closed, bounded, and nested** intervals:

$$[a, b] \supset I_1 \supset I_2 \supset I_3 ...$$

Lemma 4.22. Consider that the intersection of closed, bounded nested intervals is non-empty

Proof. Each closed, bounded interval has a minimum and a maximum. Let (m_n) be the sequence of minima for the nested intervals, and (M_n) the sequence of maxima for the nested intervals. Observe that $m_n < M_n$ for all $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} m_n \leq \lim_{n\to\infty} M_n$. So either $\lim_{n\to\infty} m_n \neq \lim_{n\to\infty} M_n$, in which case there exists an interval $[m, M]$ contained in every interval I_n for all $n \in \mathbb{N}$. Or, $\lim_{n\to\infty} m_n = \lim_{n\to\infty} M_n = x$, in which case there exists an x contained in every interval I_n for all $n \in \mathbb{N}$. In either case, the intersection of all closed, bounded nested intervals is non-empty. [7] $\square$

But, there must exist an open set in our open cover containing x, U_x, by assumption. In $\mathbb{R}$, open sets are the unions of open intervals. Therefore U_x must have some open interval (μ_1, μ_2) containing x within it.

Consider this open interval (μ_1, μ_2). We wish to show that some I_k is contained within (μ_1, μ_2), which would be a contradiction as this would imply that the singleton set U_x is a finite subcover for the intervals $I_k, I_{k+1}, ...$ Consider that by Lemma 4.22, there exists some x in all I_n, $n \in \mathbb{N}$. Take any interval I_k, $k \in \mathbb{N}$ such that the length of the interval $|I_k| < q/2$, where $q = \max\{x - \mu_1, \mu_2 - x\}$. That is,

$$\frac{b-a}{2^k} < q$$

$$\Rightarrow \frac{b-a}{q} < 2^k$$

$$\Rightarrow k > \log_2 \left(\frac{b-a}{q} \right)$$

Then we are guaranteed to have an interval I_k contained within $[\mu_1, \mu_2]$. Then U_x is a finite subcover for all I_j, $k \leq j \in \mathbb{N}$. Hence, we have a contradiction. $\square$

One fundamental result related to compactness is the **Heine-Borel Theorem**. It is as follows:

Theorem 4.23. Heine-Borel: Any subset of $X \subseteq \mathbb{R}^n$ is compact iff X is a closed and bounded subset.

Proof. We will first prove the forwards implication and then the backwards implication.
$\Rightarrow$: Suppose that $X \subseteq \mathbb{R}^n$ is compact. To prove that X is closed, consider that $\mathbb{R}^n$ is Hausdorff (see section on Hausdorff spaces), and so by Proposition 4.15, X is closed. To prove that X is bounded, consider the open balls $B_k(o)$ centred around the origin o for $k = 1, 2, 3....$ These open balls form an open cover of $\mathbb{R}^n$. Hence, $\mathcal{C} = \{B_k(o) \cap X \mid k = 1, 2, 3...\}$ forms an open cover of X. Since, X is compact, there is a finite subcover of $\mathcal{C}$. That is, there is a finite subcover $\{B_{k_1}(o) \cap X, ..., B_{k_n}(o) \cap X\}$ of X. Let $m = \max\{k_1, ...k_n\}$. Then $B_m(o) \cap X$ covers X also. So, by definition, X is bounded. Hence, $X \subseteq \mathbb{R}^n$ is compact implies that X is closed and bounded.
$\Leftarrow$: Suppose that X is a closed and bounded subset of $\mathbb{R}$. Consider that if $X \subseteq \mathbb{R}^n$ is bounded, then we can enclose X in an n-box $T_0 = [-a, a]^n$, where $n > 0$. Hence, we know that $X \subseteq T_0$ (i.e. X is contained in T_0), and so it is enough to show that T_0 is compact by Proposition 4.15. We have proven that any closed, bounded interval $[a.b]$ in $\mathbb{R}$ is compact in Proposition 4.21, and also that the product of two compact spaces is compact in Proposition 4.16. Hence, as we can write T_0 as the cartesian product of an interval $[-a, a]$ n times, we can then conclude that T_0 is compact and so X is compact. □

4.2.2 Bolzano-Weierstrass Property and Compactness

We first recall the Bolzano-Weierstrass Theorem:

Theorem 4.24. Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

We define the Bolzano-Weierstrass Property to be the following:

Definition 4.25. A set U in a metric space has the **Bolzano-Weierstrass property** if every sequence in U has a convergent subsequence.

4.3 Compactness for Metric Spaces

Theorem 4.26. Lebesgue Number Let X be a compact metric space and let $\{U_\lambda \mid \lambda \in \Lambda\}$ be an open cover of X. Then there exists a positive number $\delta > 0$ known as the Lebesgue Number such that for all x in X, $B_\delta(x)$ lies entirely inside some U_λ.

4.3.1 Uniform Continuity

Definition 4.27. A map $f : X_1 \to X_2$ of metric spaces with metrics d_1 and d_2 is uniformly continuous on X_1 if given any $\epsilon > 0$, there exists a $\delta > 0$ such that $d_2(f(x), f(y)) < \epsilon$ for any x, y in X_1 satisfying $d_1(x, y) < \delta$.

Note that uniform continuity is a stronger notion than continuity in metric spaces, which is defined in Definition 2.26. This side by side comparison of these two definitions, adapted from [19], makes this clear. Note the order of quantifiers:

$$\begin{aligned}
\textbf{Continuity: } & (\forall \epsilon > 0)\ (\forall x \in X_1)\ (\exists \delta > 0)\ (\forall x_0 \in X_1), \\
& d_{X_1}(x, x_0) < \delta \Rightarrow d_{X_2}(f(x), f(x_0)) < \epsilon \\
\textbf{Uniform Continuity: } & (\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall x \in X_1)\ (\forall x_0 \in X_1), \\
& d_{X_1}(x, x_0) < \delta \Rightarrow d_{X_2}(f(x), f(x_0)) < \epsilon
\end{aligned}$$

The following example demonstrates this.

Example 4.28. Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, where both $\mathbb{R}$ are under the usual metric. f is continuous but not uniformly continuous.

Proof. Let $\epsilon > 0$. We can take $\delta = \min\left\{1, \frac{\epsilon}{1+2|x_0|}\right\}$. Assuming $|x - x_0| < \delta$, then we have that

$$|x^2 - x_0^2| = |x - x_0||x + x_0|$$

But if $|x - x_0| \leq 1$, then $-1 \leq x - x_0 \leq 1 \Rightarrow -1 + 2x_0 \leq x + x_0 \leq 1 + 2x_0$, and so $|x + x_0| \leq 1 + 2|x_0|$. So,

$$\begin{aligned} |x^2 - x_0^2| \leq& |x - x_0|(1 + 2|x_0|) \\ <& \delta(1 + 2x_0) = \epsilon \end{aligned}$$

and so f is continuous. However, f is not uniformly continuous. To prove this, suppose that f is uniformly continuous. Let $\epsilon > 0$. Then there exists some $\delta > 0$ such that, for all $x, x_0 \in \mathbb{R}$,

$$|x - x_0| < \delta \Rightarrow |x^2 - x_0^2| < \epsilon.$$

Consider $\epsilon = 1$. If such a δ existed and $x_0 = x + \delta$, then we would have that

$$\begin{aligned} |x^2 - (x + \delta)^2| <& 1 \\ \Rightarrow |2x\delta + \delta^2| <& 1 \end{aligned}$$

Which is a contradiction, as we can choose x to be arbitrarily large. So f is not uniformly continuous. $\square$

Proposition 4.29. If $f : X_1 \to X_2$ is a continuous map of metric spaces and if X_1 is compact, then f is uniformly continuous on X_1.

Proof. Let $\epsilon > 0$. As f is continuous for each x in X_1, there exists $\delta(x) > 0$ such that $d_2(f(x), f(y)) < \frac{1}{2}\epsilon$ for all y satisfying $d_1(x, y) < 2\delta(x)$. The collection $\{B_{\delta(x)}(x) \mid x \in X_1\}$ is an open cover for X_1. By the compactness of X_1, there is a finite subcover of $\{B_{\delta(x)}(x) \mid x \in X_1\}$, $\{B_{\delta(x)}(x_1), B_{\delta(x_2)}(x), ..., B_{\delta(x)}(x_n)\}$. Let $\delta = \min\{\delta(x_1), \delta(x_2), ..., \delta(x_n)\}$. Given any $x, y \in X_1$ satisfying $d_1(x, y) < \delta$, (1) there is some i in $1, 2, ..., r$ such that $d(x, x_i) < \delta(x_i)$, and then (2) $d_1(y, x_i) \leq d_1(y, x) + d_1(x, x_i) < \delta + \delta(x_i) < 2\delta$.

Now by (1), $d_2(f(x), f(x_i)) < \frac{1}{2}\epsilon$, and by (2), $d_2(f(y), f(x_i) < \frac{1}{2}\epsilon$, and so

$$d_2(f(x), f(y)) \leq d_2(f(x), f(x_i)) + d_2(f(x_i), f(y)) < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

as required [2].

$\square$

Remark 4.30. Compactness is NOT a necessary condition for uniform continuity. Consider any topological space X_1 and the identity map $i : X_1 \to X_1$. This is a uniformly continuous map that does not depend on X_1 being compact.

4.3.2 Sequential Compactness

Definition 4.31. A topological space X is sequentially compact if every sequence in X has a convergent subsequence converging to a point in X.

As metric spaces are also topological spaces, the above definition also applies to them. They possess interesting properties, which we will now explore.It may be useful to recall the definition of convergence in metric spaces, Definition 2.30.

Lemma 4.32. Let U be a subset of a metric space X and let $x \in X$. Then $x \in \overline{U}$ iff there exists a sequence (x_n) in U converging to x.

Proof. We will prove the forwards implication and then the backwards implication:
($\Rightarrow$) From the definition of closure, $\overline{U}$ is the union of the set of interior points of U (U^i) and the set of limit points of U (U'). If x is an interior point of U, i.e. $x \in U^i$, then $x \in U$ and so the sequence $(x_n) = x$ converges to x. If x is a limit point of U, i.e. $x \in U'$, then by Theorem 2.32, there exists a sequence (x_n) in U that converges to x.
($\Leftarrow$) Suppose that there exists a sequence (x_n) in U converging to x. Let U' be an open set containing x. By definition of an open set there exists a ball $b_\epsilon(x) \subseteq U$. But we assume that there exists a sequence (x_n) that converges to x, so by definition of convergence, there exists an $n \in \mathbb{N}$ such that: $d(x_n, x) < \epsilon$. So, $U \cap b_\epsilon(x) \neq \emptyset \Rightarrow U \cap U' \neq \emptyset$. Hence, $x \in \overline{U}$ [18, pp. 108]. $\square$

Proposition 4.33. Any sequentially compact space is also a complete space.

Proof. Suppose that X is a sequentially compact space and (x_n) is a Cauchy sequence in X. Then (x_n) has a convergent subsequence by definition of sequential compactness. But if (x_n) is Cauchy, (x_n) is convergent to x iff (x_n) has a subsequence convergent to x. So (x_n) is a convergent sequence to some x in X, and so by definition X is complete. $\square$

Remark 4.34. $\mathbb{R}$ is a sequentially compact metric space. This follows from Remark 2.45.

4.3.3 Precompactness

Definition 4.35. We say that a metric space X is precompact iff for every real number $\epsilon > 0$ there exists a finite collection of open balls in X of radius ϵ whose union contains X. Equivalently, a metric space X is precompact iff there exists a finite cover such that the radius of each element is at most ϵ.

Remark 4.36. Every precompact space is bounded but not every bounded space is precompact.

Proof. Every precompact space is bounded as the union of finitely many bounded sets is bounded. To show that the converse is not true, consider the example of an infinite set equipped with the discrete metric. It is bounded, but it is not precompact. $\square$

4.3.4 Equivalence of Sequential Compactness and Compactness for Metric Spaces

Now that we have explored compactness, sequential compactness, completeness and precompactness, we can prove a fundamental equivalence for metric spaces:

Theorem 4.37. Let U be a subset of the metric space X. Then U is sequentially compact iff U is complete and precompact.

Proof. $\Rightarrow$: Suppose that U is sequentially compact. We have already shown that any sequential compact space is also a complete space. To prove that any sequential compact space is precompact, we will prove the contrapositive. Suppose that U is not precompact. Then there exists some $\epsilon > 0$ such that there is no finite collection of open balls in X of radius ϵ whose union covers X. Take some arbitrary set of points $x_1, x_2, ..., x_n$ in U. Now choose some $x_{n+1} \notin \cap_{i=1,...n} B_\epsilon(x_i)$. As x_{n+1} is not contained within any of these balls, it follows that if $m, n \in \mathbb{N}$ and $m \neq n$, then $d(x_m, x_n) \geq \epsilon$. This proves that the sequence (x_n) has no convergent subsequence, and so U is not sequentially compact. So, taking the contrapositive of this statement, this proves that if U is sequentially compact then U is precompact.
$\Leftarrow$: Now suppose that U is complete and precompact. Take any sequence (x_n) in U. We wish to construct a Cauchy subsequence in U of (x_n). By constructing one, and by the completeness of U, we will find a subsequence of (x_n) in U that converges to a point in U. This will guarantee that U is sequentially compact. Since U is precompact, we can cover U with finitely many balls of radius 1.

At least one ball must contain an infinite number of x_i. Call this ball B_1, and let S_1 be the set of integers i for which $x_i \in B_1$. By induction we can define for each positive integer $k > 1$ a ball B_k of radius $1/k$ containing an infinite number of terms of the sequence (x_n). Defining S_k similarly to as defined for S_1, note that each S_k is infinite, and so we can choose a sequence contained in S_k with $n_k < n_k + 1$. But the sets S_k are nested, and so we have that for $i, j \geq k$, $n_i, n_j \in S_k$. So, for all $i, j \geq k$, x_{n_i} and x_{n_j} are contained within a ball of radius $1/k$. This proves that (x_{n_k}) is a Cauchy subsequence of (x_n), and is convergent. It follows that U is sequentially compact [13], [19]. □

Theorem 4.38. Let X be a metric space, and $U \subseteq X$. Then U is compact iff U is sequentially compact.

Proof. $\Rightarrow$: We will prove the contrapositive statement, that is, if U is not sequentially compact, then U is not compact. Suppose U is not sequentially compact. Then there exists a sequence in U with no convergent subsequence. This sequence must contain an infinite number of distinct points. Take some $x \in U$. If for every $\epsilon > 0$, the ball $b_\epsilon(x)$ contains a point in (x_n) distinct from x, then x would be the limit of a subsequence of (x_n). So, as there exists a non-convergent subsequence, there exists $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains no points from (x_n), except possibly x itself. The collection of these open balls, $\{B_{\epsilon_x}(x) \mid x \in U\}$ form an open cover of U.The union of a finite number of these balls $\{B_{\epsilon_x}(x_1), ..., B_{\epsilon_x}(x_n)\}$ contains at most n terms of the sequence. But because there are an infinite number of distinct terms in the sequence, no finite subset of these balls could ever form a cover of U. Hence, U is not compact.
$\Leftarrow$: Suppose U is sequentially compact. Let $\mathcal{C} = \{C_\lambda \mid \lambda \in \Lambda\}$ be an open cover for U. By Theorem 4.26, there exists a positive number $\delta > 0$ such that for all $x \in X$, $B_\delta(x)$ lies entirely inside some U_λ. By Theorem 4.37, we know that U is precompact, and so U is covered by a finite collection of open balls $\{B_1, ...B_m\}$ of radius less than $\delta/2$. For each $k \in \{1, ...m\}$, the diameter of B_k is less than δ. Hence, $B_k \subseteq C_{\lambda_k}$ for some $\lambda_k \in \Lambda$. Thus, $\{C_{\lambda_k} \mid k = 1, ..., m\}$ is a finite subcover of U and so U is compact [16]. □

5 Connectedness

5.1 Basic Definitions

Definition 5.1. (i) We say that a topological space X is **connected** iff it **cannot** be written as a union

$$X = A \cup B$$

where A and B are non-empty, open, disjoint subsets of X.

(ii) We say that a topological space is **disconnected** if it **can** be written as a union

$$X = A \cup B$$

where A and B are non-empty, open, disjoint subsets of X.

The property of connectedness is preserved under continuous mappings, that is

Proposition 5.2. If f is a continuous mapping from X_1 to X_2, and X_1 is connected, then $f(X_2)$ is also connected.

Proof. Suppose for contradiction's sake that X_1 is connected and $f : X_1 \to X_2$ is a continuous surjective mapping, and that $f(X_1)$ is disconnected. Then there exist $A, B \subset f(X_1)$ such that $A \cap B = \emptyset$ and $A \cup B = f(X_1)$. Then A and B are open in $f(X_1)$ as they form a basis for $f(X_1)$. By the continuity of f, it follows that then $f^{-1}(A)$ and $f^{-1}(B)$ are open and disjoint in X_1. But $f(X_1) = A \cup B \Rightarrow X = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, which would mean that X_1 is disconnected. This is a contradiction. □

This leads to an equivalent definition of connectedness:

Definition 5.3. A topological space X is connected iff any continuous mapping $f : X \to \{0, 1\}$ is constant.

Proof. ($\Rightarrow$) We will prove the contrapositive. Suppose there exists a continuous, non-constant mapping $f : X \to \{0, 1\}$. Set $A = f^{-1}(0)$ and $B = f^{-1}(1)$. By the continuity of f these are both open in X. As $A \cup B = X$ and $A \cap B = \emptyset$, X is disconnected.
($\Leftarrow$) We will prove the contrapositive. Suppose that X is disconnected. Then there exist $A, B \subset X$ such that $A \cap B = \emptyset$ and $A \cup B = X$. Define a function $f : X \to \{0, 1\}$ to be:

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

This is a continuous function, as for any U open in $\{0, 1\}$, $f^{-1}(U)$ is open in X. But, f is not a constant function. Hence, we have shown that if f is a continuous, constant function from $f : X \to \{0, 1\}$, then X is connected. □

Proposition 5.4. The topological space X is connected iff $\emptyset$ and X are the only both open and closed sets in X.

Proof. ($\Rightarrow$) We will use proof by contradiction. Suppose that X is connected, and that there exists a non-empty subset $U \subset X$ such that U is also open and closed. Then we have that U' is also open and closed, and we can write $X = U \cup U'$ and $U \cap U' = \emptyset$. But as X is connected, by definition this is a contradiction.
($\Leftarrow$) We will prove the contrapositive. Suppose that X is disconnected. Then there exist $A, B \subset X$ such that $A \cap B = \emptyset$ and $A \cup B = X$. Then $\{A, B\}$ form a basis for X and hence are open. But then $A' = B$ and $B' = A$, which means that A and B are also closed. So, we have found two non-empty sets in X that are both open and closed.

□

5.2 Examples

Some intuitive examples can simply be imagined in $\mathbb{R}$:

Example 5.5. (i) The topological space $[0, 1] \cup [3, 4]$ with the usual topology is disconnected, as $[0, 1] \cap [3, 4] = \emptyset$.

(ii) For all $a \in \mathbb{R}$, $\mathbb{R} \setminus a$ is disconnected.

More general examples are:

Example 5.6. (i) For any set X, the trivial topological space X is connected.

Proof. The topology on X is $\tau = \{\emptyset, X\}$. So the only possible open sets are $\emptyset$ and X, and so by Proposition 5.4, X is connected. □

(ii) For any set X, the discrete topological space X with at least two elements is disconnected.

Proof. Consider that for any discrete topological space X with at least two elements, we can take any $A \subset X$, and A and A' will be non-empty and $A, A' \in \tau$, where τ is the discrete topology. But $A \cap A' = \emptyset$ and $A \cup A' = X$, so by definition X is disconnected. □

Example 5.7. $\mathbb{R}$ with usual topology is connected:
We will prove this using Proposition 5.4. Take any open $U \subset \mathbb{R}$. We can write U as the intersection of disjoint, non-empty, open intervals in X. By definition, these open intervals do not contain their end points. So, there must be some interval such that the union does not contain its end point. Hence, U cannot be closed. Similarly, $\mathbb{R} \setminus U$ is closed but cannot be open. So the only closed and open sets in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$.

Example 5.8. $\mathbb{Q}$ is not connected.

Proof. Take any irrational $r \in \mathbb{R}$. Then we can partition $\mathbb{Q}$ into two open, disjoint subsets, $\mathbb{Q}\cap(-\infty, r)$ and $\mathbb{Q} \cap (r, \infty)$. □

5.3 Properties of Connected Spaces

Proposition 5.9. Connectedness is a topological invariant.

Proof. We wish to show that if $f : X_1 \to X_2$ is a homeomorphism, and if X_1 is connected, then so is X_2. But this follows directly from Proposition 5.2 (as f is bijective and so $f(X_1) = X_2$). The reverse (i.e. if X_2 is connected and we wish to show X_1 is also connected), simply follows from the definition of a homeomorphism - both f and f^{-1} are continuous and so we follow the same argument using f^{-1}. □

Proving that two spaces are not homeomorphic is often easily done by considering connectedness. The following example demonstrates this:

Example 5.10. The circle S_r of radius r and the real line $\mathbb{R}$ are not homeomorphic.

Proof. Suppose for contradiction's sake that S_r and $\mathbb{R}$ are homeomorphic. Then there exists a homeomorphism $f : S_r \to \mathbb{R}$. These are both connected spaces. If you remove a point $x \in S_r$, then the remaining space $S_r \setminus \{x\}$ would be homeomorphic to $\mathbb{R} \setminus f(x)$. But, consider that the space $S_r \setminus \{x\}$ is still connected, whereas the space $\mathbb{R} \setminus f(x)$ is not. As connectedness is a topological invariant, this is a contradiction. □

Proposition 5.11. The arbitrary union of connected subspaces of a topological space is also connected if their intersection is nonempty. That is, if for each λ in some indexing set Λ, A_λ is a connected subspace of a topological space X and for all $i, j \in \Lambda$ $A_i \cap A_j \neq \emptyset$, then $\bigcup\limits_{\lambda\in\Lambda} A_\lambda$ is also connected [18, pp. 98].

Proof. Let $\{A_\lambda\}$ be a set of connected subspaces of a topological space, with $\lambda \in \Lambda$ as an arbitrary indexing set. Suppose for contradiction's sake that $\bigcup\limits_{\lambda\in\Lambda} A_\lambda$ is not connected. Then it can be partitioned into two disjoint subsets B, C such that $B \cup C = \bigcup\limits_{\lambda\in\Lambda}$ and $B \cap C = \emptyset$. Take any $x \in \bigcup\limits_{\lambda\in\Lambda}$. Suppose that $x \in B$. Then $x \notin C$. Take an arbitrary $y \in C$. There is some A_λ such that $y \in A_\lambda$. But $x \in A_\lambda$. So $x \in A_\lambda \cap B$ and $y \in A_\lambda \cap C$. But these are relatively open, nonempty, proper subsets of A_λs such that $(A_\lambda \cap B) \cup (A_\lambda \cap C) = A_\lambda$ and $(A_\lambda \cap B) \cap (A_\lambda \cap C) = \emptyset$. This implies that A_λ is disconnected, which is a contradiction.

□

Proposition 5.12. Let X_1 and X_2 be topological spaces. Then $X_1 \times X_2$ is connected iff X_1 and X_2 are connected.

Proof. ($\Rightarrow$) Suppose $X_1 \times X_2$ is connected. Consider the continuous projection maps $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$. By Proposition 5.2, X_1 and X_2 are also connected.
($\Leftarrow$) Suppose X_1 and X_2 are connected. To prove that $A \times B$ is also connected, we will use Proposition 5.3. That is, we wish to prove the map $f : A \times B \to \{0, 1\}$ is constant. It is simple to see that the

function f is constant for every set of the form $\{a\} \times B$ or $A \times \{b\}$ for $a \in A$ and $b \in B$. Take some $(a, b) \in A \times B$. Then for any other $(a^*, b^*) \in A \times B$, we have that $f(a, b) = f(a, b^*) = f(a^*, b^*)$. So, the map f is constant and hence we have that $X_1 \times X_2$ is also connected. [3] $\square$

Proposition 5.13. Let A be a connected subset of a topological space X and suppose $A \subseteq B \subseteq \overline{A}$. Then B is connected.

Proof. Suppose for contradiction's sake that B is disconnected. Then there exist open sets $U, V \subseteq X$ such that $B \cap U$ and $B \cap V$ are disjoint, non-empty and $B \subseteq U \cup V$. Then $A \cap U$ and $A \cap V$ are disjoint open subsets of A such that $A \cap U \cup A \cap V = A$. But this would be a contradiction as A is connected by assumption. So, it must then be that one is empty. Suppose $A \cap U$ is empty. But as $B \subseteq \overline{A}$, every open set intersecting B also intersects A. But $B \cap U \neq \emptyset$ by assumption. Hence, $A \cap U \neq \emptyset$. The same argument can also be applied for $A \cap V$, and so this is a contradiction, as one must be empty [8]. $\square$

5.4 Intervals

Definition 5.14. Recall that an interval can take the form of open intervals: (a, b), closed intervals: $[a, b]$, and half open half closed (mixed) intervals: $[a, b)$ or $(a, b]$.

Remark 5.15. Every non-empty interval $I \subseteq \mathbb{R}$ is connected.

Proof. This follows by Proposition 5.13. $\square$

Definition 5.16. We say that a subset $X \subseteq \mathbb{R}$ has the interval property if $x, y \in X$ implies $z \in X$ for all $x < z < y$

Proposition 5.17. Let $X \subseteq \mathbb{R}$ be non-empty. Then the following statements are equivalent:

1. X is connected.
2. X has the interval property.
3. X is a non-empty interval in $\mathbb{R}$.

Proof. (1) $\Rightarrow$ (2): To prove that (1) $\Rightarrow$ (2), suppose that X is connected, and that X does not have the interval property. Then for some $x < y \in X$, there exists $z \notin X$, such that $x < z < y$. Then we have that $U = (-\infty, z) \cap X$ and $V = X \cap (z, \infty)$ are two subsets such that $U \cap V = \emptyset$ and $U \cup V = X$. But this is a contradiction. So X must have the interval property.

(2) $\Rightarrow$ (3): Suppose that $X \subseteq \mathbb{R}$ is non-empty and has the interval property. Let

$$a = \inf(U) \ (= \infty \text{ if not bounded above})$$

$$b = \sup(U) \ (= -\infty \text{ if not bounded below})$$

If $a < b$, then for all $z \in (a, b)$, $a < z < b$ so by definition of infimum/supremum, there exist some $x, y \in X$ such that $a < x < z < y < b$. So by assumption that X has the interval property, $(a, b) \subseteq X$. By definition of infimum/supremum, $X \subseteq [a, b]$. But if $(a, b) \subseteq X \subseteq [a, b]$, then X can only be one of the various types of interval. Hence, X is an interval.

(3) $\Rightarrow$ (2): This is trivial, by definition of an interval.

(3) $\Rightarrow$ (1): By Remark 5.15, this is true.

Hence, as (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3), then (1) $\Rightarrow$ (3). As (3) $\Rightarrow$ (1), it follows that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). $\square$

Remark 5.18. If f is a map $f : X_1 \to \mathbb{R}$ and X_1 is connected, then $f(X_1)$ is an interval.

Proof. Of course if X_1 is connected, then by Proposition 5.2 $f(X_1)$ is connected. We have shown that this is equivalent to $f(X_1)$ being an interval, as $f(X_1) \subseteq \mathbb{R}$, in Proposition 5.17. $\square$

5.4.1 Intermediate Value Theorem

Proposition 5.19. If $f : [a, b] \to \mathbb{R}$ is continuous, then f has the Intermediate Value Property. That is, for all $y \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]$, there is some $c \in [a, b]$ such that $f(c) = y$.

Proof. Consider that $[a, b]$ is a closed, bounded (compact), and connected subspace of $\mathbb{R}$. As compactness and connectedness are both preserved by continuous functions, $f[a, b]$ is also closed, bounded (compact), and connected. It follows that $f[a, b]$ is of the form $[x, y]$, for some $x, y \in \mathbb{R}$. $\square$

5.5 Fixed Points

Definition 5.20. A fixed point of a map $f : X \to X$ is a point $x \in X$ such that $f(x) = x$.

Example 5.21. For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ has a fixed point, $x = 0$. But $f(x) = x + 1$ does not have a fixed point.

Theorem 5.22. The One Dimensional Brouwer Fixed Point Theorem:
Every continuous map $f : [0, 1] \to [0, 1]$ has a fixed point. I.e., there exists an $x \in [0, 1]$ such that $f(x) = x$.

Proof. Define a (continuous) map $g : [0, 1] \to \mathbb{R}$ where $g(x) = x - f(x)$. Then $g(0) = -f(0) \leq 0$ and $g(1) = 1 - f(1) \geq 0$. By the Intermediate Value Theorem, as g is a continuous function, then there exists an $x \in [0, 1]$ such that $g(x) = x - f(x) = 0, \Rightarrow f(x) = x$. $\square$

More generally,

Theorem 5.23. If $a \leq b$, then any continuous function $f : [a, b] \to [a, b]$ has a fixed point. That is, there exists $x \in [a, b]$ such that $f(x) = x$.

Proof. Define $g(x) = f(x) - x$. Then g is a continuous function. We also have that $g(a) \geq 0$ and $g(b) \leq 0$. So, as g is continuous it has the intermediate value property and so there must exist some $c \in [a, b]$ such that $g(c) = 0$. That is, where $f(c) - c = 0 \Rightarrow f(c) = c$. $\square$

Remark 5.24. The One Dimensional Brouwer Fixed Point Theorem and the Intermediate Value Theorem are equivalent. The above proof is verification of this.

5.6 Path-Connectedness

Definition 5.25. A path in a topological space X, a path from a to b in X is a continuous map $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$. If such a map exists for all $a, b \in X$, then we say that X is path-connected.

Definition 5.26. We say that a subset $U \subseteq \mathbb{R}^n$ is convex if for any $x_0, x_1 \in X$ the segment

$$[x_0, x_1] = \{(1 - t)x_0 + tx_1 \mid 0 \leq t \leq 1\}$$

is wholly contained in X.

Remark 5.27. Any convex subset X of $\mathbb{R}^n$ is path connected, as any two points can be connected by a straight line contained within X.

Remark 5.28. For all $n \in \mathbb{N}$, $\mathbb{R}^n$ is path-connected.

Proof. Consider that $\mathbb{R}^n$ is itself convex, as any line segment $[x_0, x_1]$ where $x_0, x_1 \in \mathbb{R}^n$ is itself contained within $\mathbb{R}^n$. So, this follows from Definition 5.26. $\square$

Theorem 5.29. If a topological space X is path-connected, then X is also connected.

Proof. We will prove this by contradiction. Suppose that X is path-connected, but not connected. Then there exist some open $U, V \subset X$ such that $U \cap V = \emptyset$ and $U \cup V = X$. Let $u \in U$ and $v \in V$. Let f be a continuous map/path $f : [0,1] \to X$ such that $f(u) = 0$ and $f(v) = 1$. Consider the subsets $f^{-1}(U)$ and $f^{-1}(V)$ in $[0,1]$. They are open by the continuity of f, and disjoint as $U \cap V = \emptyset$, and $f^{-1}(U) \cup f^{-1}(V) = [0,1]$. Hence, $[0,1]$ is disconnected. This is a contradiction. $\square$

Theorem 5.30. If $n \geq 2$, $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic.

Proof. Suppose $\mathbb{R}^n$ and $\mathbb{R}$ are homeomorphic for some $n \geq 2$. Then there exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}$. So, for all $x \in \mathbb{R}^n$, $\mathbb{R}^n \setminus \{x\}$ is homeomorphic to $\mathbb{R} \setminus \{f(x)\}$. But $\mathbb{R}^n \setminus \{x\}$ is path connected and so connected by Theorem 5.29, and $\mathbb{R} \setminus \{f(x)\}$ is disconnected. As connectedness is a topological invariant, this is a contradiction. $\square$

Remark 5.31. From Proposition 5.30, it follows that $(0,1)^2$ and $(0,1)$ are not homeomorphic.

6 Function Spaces

In this section we will consider important examples of metric spaces, function spaces. Of course it follows that as any function space equipped with one of the metrics described below is a metric space (which we will prove), any function space is also a topological space.

6.0.1 Function Space from Continuous Functions

Definition 6.1. Let X be a compact metric space, and let $\mathscr{C}(X) = \{f : X \to \mathbb{R} \,|\, f \text{ is bounded and continuous}\}$. Define a metric d_∞ on $\mathscr{C}(X)$ to be

$$d_\infty(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

Then $\mathscr{C}(X)$ is a metric space equipped with d_∞.

Proof. We will check that the metric space axioms hold for the above definition.

1. The metric is defined to be the supremum of non-negative numbers, so it must be that $d_\infty(f,g) \geq 0$.

2. If $f = g$, then they are identical functions from $X \to \mathbb{R}$, which means that $f(x) = g(x)$ for all $x \in X$. Hence,
$$d_\infty(f,g) = \sup_{x \in X} |f(x) - g(x)| = \sup_{x \in X} |0| = 0$$
Conversely, $\sup\limits_{x \in X} |f(x) - g(x)| = 0$ if and only $f(x) = g(x)$, as $|f(x) - g(x)| \geq 0$ for all $x \in X$. This implies that $f = g$.

3. Since $|f(x) - g(x)| = |g(x) - f(x)|$, it follows that $d_\infty(f,g) = d_\infty(g,f)$.

4. Let $f, g, h \in \mathscr{C}(X)$. Then for any $c \in X$, by the triangle inequality
$$\begin{aligned} |f(c) - h(c)| \leq &|f(c) - g(c)| + |g(c) - h(c)| \\ \leq &\sup_{x \in X} |f(x) - g(x)| + \sup_{x \in X} |g(x) - h(x)| \\ = &d_\infty(f,g) + d_\infty(g,h) \end{aligned}$$
But this implies that $d_\infty(f,g) + d_\infty(g,h)$ is an upper bound for $\{|f(x) - h(x)| \mid x \in X\}$. By definition of supremum, it follows that $\sup\limits_{x \in X} |f(x) - h(x)| \leq d_\infty(f,g) + d_\infty(g,h)$ and so $d_\infty(f,h) \leq d_\infty(f,g) + d_\infty(g,h)$ [18, pp. 28].

□

Proposition 6.2. If X is a compact metric space, then $\mathscr{C}(X)$ is a complete metric space.

Proof. Let $x \in X$. Then suppose that (f_n) is a Cauchy sequence with respect to d_∞. Then for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n, m \geq N$ implies $d_\infty(f_n, f_m) < \epsilon$. Note that $(f_n(x))$ is a sequence of real numbers. Take some $\epsilon > 0$. Then consider that

$$|f_n(x) - f_m(x)| \leq \sup_{x \in X} |f_n(x) - f_m(x)|$$

But by definition of d_∞, and the fact that $(f_n(x))$ is Cauchy with respect to d_∞,

$$\Rightarrow |f_n(x) - f_m(x)| \leq d_\infty(f_n, f_m) < \epsilon$$

This shows that (f_n) is also Cauchy with respect to the usual metric. But consider that by Remark 2.45, as $\mathbb{R}$ is complete, the limit $\lim_{n\to\infty} f_n(x)$ is in $\mathbb{R}$. We have now proven that $f = \lim\limits_{n\to\infty} f_n(x)$ exists and is real-valued.

Now we just need to show that $\lim\limits_{n\to\infty} f_n$ converges to f under d_∞. That is, $d_\infty(f_n, f) \to 0$ as $n \to \infty$. By Definition 2.30, we wish to show that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $n > N$ we have $d_\infty(f_n, f) \leq \epsilon$.

Take some $\epsilon > 0$ and fix $N \in \mathbb{N}$ such that for $n, m \leq N$, $d_\infty(f_n, f_m) < \frac{\epsilon}{2}$. Such an N exists under the assumption that the terms of $(f_n(x))$ are Cauchy. As d_∞ is a metric, the triangle inequality holds and we have that

$$d_\infty(f_n, f) \leq d_\infty(f_n, f_N) + d_\infty(f_N, f)$$

By choice of N we have that $d_\infty(f_n, f_N) < \frac{\epsilon}{2}$. But also consider that $d_\infty(f_m, f_N) < \frac{\epsilon}{2}$. So we can conclude that $\lim\limits_{m\to\infty} d_\infty(f_m, f_N) \leq \frac{\epsilon}{2}$. That is, $d_\infty(f, f_N) \leq \frac{\epsilon}{2}$. Hence, we have that

$$d_\infty(f_n, f) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

And so we have shown that $d_\infty(f_n, f) \to 0$ as $n \to \infty$. In essence, we have shown that the limit $(f_n(x))$ converges to pointwise is the limit $(f_n(x))$ converges to uniformly.

Finally, we wish to show that $f \in \mathscr{C}(X)$. That is, that it actually is a continuous function. By Definition 2.26, we wish to show that for any $\epsilon > 0$ and any $x \in X$, there exists a $\delta > 0$ such that for $a \in X$ with $d(a, x) < \delta$ we have that $|f(a) - f(x)| < \epsilon$, where d is the associated metric with the compact metric space X. By the triangle inequality,

$$|f(a) - f(x)| \leq |f(a) - f_n(a)| + |f_n(a) - f_n(x)| + |f_n(x) - f(x)|$$

For some n. We can pick an n such that $|f(a) - f_n(a)| < \frac{\epsilon}{3}$ and $|f_n(x) - f(x)| < \frac{\epsilon}{3}$. We can also choose some n such that $|f_n(a) - f_n(x)| < \frac{\epsilon}{3}$ because $f_n(x)$ is continuous as it is in $\mathscr{C}(X)$. Hence,

$$|f(a) - f(x)| < \epsilon$$

and so f is continuous and so $f \in \mathscr{C}(X)$. This concludes the proof. [14]

□

6.0.2 Function Spaces from Integrable Functions

Definition 6.3. Let $\mathcal{L}^1[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is integrable}\}$ be the set of continuous functions from $[a, b]$ to $\mathbb{R}$ where, for any $f \in \mathcal{L}^1[a, b]$, $\int_a^b |f(x)| < +\infty$. Define a metric $d_{\mathcal{L}^1}$ on $\mathcal{L}^1[a, b]$ to be

$$d_{\mathcal{L}^1}(f, g) = \int_a^b |f(x) - g(x)| dx$$

Then $\mathcal{L}^1[a, b]$ is a metric space equipped with $d_{\mathcal{L}^1}$ [10].

Proof. We will check that the metric space axioms hold for the above definition.

1. By definition $d_{\mathcal{L}^1}(f,g) \geq 0$ for all $f, g \in \mathcal{L}^1[a,b]$.
2. Suppose that $f = g$. Then $|f(x) - g(x)| = 0$ for all $x \in [a,b]$, and so $d_{\mathcal{L}^1}(f,g) = 0$. Conversely, if $d_{\mathcal{L}^1}(f,g) = 0$, then $\int_a^b |f(x) - g(x)|dx = 0$. But $|f(x) - g(x)| \geq 0$ for all $x \in [a,b]$, so it follows that for $\int_a^b |f(x) - g(x)|dx = 0$, then $|f(x) - g(x)| = 0$ for all $x \in [a,b]$. So $f = g$.
3. As $|f(x) - g(x)| = |g(x) - f(x)|$, it follows that $d_{\mathcal{L}^1}(f,g) = d_{\mathcal{L}^1}(g,f)$
4. We wish to check that the triangle inequality holds under $d_{\mathcal{L}^1}$. Let $f, g, h \in \mathcal{L}^1[a,b]$.

$$d_{\mathcal{L}^1}(f,h) = \int_a^b |f(x) - h(x)|dx$$

But by the triangle inequality, $|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$. So,

$$\begin{aligned} \int_a^b |f(x) - h(x)|dx \leq& \int_a^b |f(x) - g(x)| + |g(x) - h(x)|dx \\ \leq& \int_a^b |f(x) - g(x)|dx + \int_a^b |g(x) - h(x)|dx \\ =& d_{\mathcal{L}^1}(f,g) + d_{\mathcal{L}^1}(g,h) \\ \Rightarrow d_{\mathcal{L}^1}(f,h) \leq& d_{\mathcal{L}^1}(f,g) + d_{\mathcal{L}^1}(g,h). \end{aligned}$$

As required.

□

7 The Brouwer Fixed Point Theorem

We have already discussed the one-dimensional Brouwer Fixed Point Theorem, and now will focus on it's general form:

Theorem 7.1 (Brouwer Fixed Point Theorem (BFPT))**.** Define $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$. Let $f : \mathbb{D}^n \to \mathbb{D}^n$ be a continuous function. Then there is an $x \in \mathbb{D}^n$ such that $f(x) = x$.

Proof. Let $n \geq 2$. Consider that we can approximate any continuous function f by C^1 functions. Specifically, take a continuous $f : \mathbb{D}^n \to \mathbb{D}^n$ and approximate it by C^1 functions f_m such that $\sup_{x \in \mathbb{D}^n} |f(x) - f_m(x)| \to 0$ as $m \to \infty$ (i.e., the sequence of f_m converges uniformly to f). We require the following lemma:

Lemma 7.2. All C^1 maps have a fixed point. That is, if $g : \mathbb{D}^n \to \mathbb{D}^n$ and $g \in C^1$, then $\exists\ x \in \mathbb{D}^n$ such that $g(x) = x$.

As $f_m \in C^1$ for all m, then by Lemma 7.2, there exists a fixed point x_m of f_m (i.e. $f_m(x_m) = x_m$). $\mathbb{D}^n$ is closed and bounded, and so by Theorem 4.23, $\mathbb{D}^n$ is compact. It is sequentially compact by Proposition 4.38, and by definition then there is a subsequence of x_m convergent to some $x \in \mathbb{D}^n$. Now we wish to prove that $f(x) = x$. Consider that $f_m(x_m) = (f_m(x_m) - f(x_m)) + f(x_m)$. But $f_m(x_m) - f(x_m) \to 0$ and $f(x_m) \to f(x)$ as $m \to \infty$, so $f_m(x_m) \to f(x)$ as $m \to \infty$ by the continuity of f and the uniform approximation of f by f_m. But also, $f_m(x_m) = x_m$ as x_m is a fixed point, and $x_m \to x$ as $m \to \infty$ so $f_m(x_m) \to x$ as $m \to \infty$. Hence, $f(x) = x$, and so x is a fixed point of f.

Now, all that remains is to prove Lemma 7.2. We will proceed by contradiction. If the lemma is false, then there is some $f \in C^1$ and for all $x \in \mathbb{D}^n$, we have $f(x) \neq x$. Take any $f \in C^1$. Then construct a new map f^* using f that sends x to the **unique** point on $\mathbb{S}^{n-1}$ on the line segment **starting** at $f(x)$ and **passing** through x. Note that f^* is a linear combination of f and x, and so it is also in C^1.

When the domain of the function is restricted to the boundary (i.e. x lies on $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$,), then clearly f^* is the identity function ($f^*_{\partial\mathbb{D}^n}(x) = x$). So to prove Lemma 7.2 true by contradiction, it suffices to prove that no such f^* exists. That is, there is no C^1 map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ that is the identity. [1] Now we have a statement that we can contradict. Suppose that such a map does exist.

Consider that:

$$\int_{\mathbb{D}^n} \det Dg = \int_{\mathbb{D}^n} \mathrm{d}g_1 \wedge \mathrm{d}g_2 \wedge ... \wedge \mathrm{d}g_n$$

Where Dg is the derivative matrix of g. We will prove that no g exists by proving two contradictory claims:

Claim 1. $\det Dg \equiv 0$ on $\mathbb{D}^n$, and so

$$\int_{\mathbb{D}^n} \det Dg = 0.$$

Consider that Dg is a matrix whose columns are $\nabla g_1, \nabla g_2, ..., \nabla g_n$. As $g_1^2 + ... + g_n^2 = 1$, taking the derivative both sides we get $2g_1\nabla g_1 + ... + 2g_n\nabla g_n = 0$. So, $\nabla g_1, ..., \nabla g_n$ are linearly dependent for all x. So, $\det Dg \equiv 0 \Rightarrow \int_{\mathbb{D}^n} \det Dg = 0$.

Claim 2. Consider that:

$$\int_{\mathbb{D}^n} \det Dg = \mathrm{vol}(\mathbb{D}^n) > 0$$

Let $\mathbf{F} = g_1 \mathrm{d}g_2 \wedge ... \wedge \mathrm{d}g_n$, where $\mathrm{d}g_2 \wedge ... \wedge \mathrm{d}g_n$ is the matrix whose first column consists of entries $\mathbf{e_1}, ..., \mathbf{e_n}$ and whose j'th column for $j \geq 2$ is ∇g_j. Then

$$\mathrm{div}(\mathbf{F}) = \det Dg$$

So, applying divergence theorem,

$$\int_{\mathbb{D}^n} \det Dg = \int_{\mathbb{D}^n} \mathrm{div}(\mathbf{F}) = \int_{\mathbb{S}^{n-1}} \mathbf{F} \cdot \mathbf{n} \mathrm{d}S$$

If we parametrise $\mathbb{S}^{n-1}$by $\mathbf{x}(s) \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ where $s \in \mathbb{R}^{n-1}$, then

$$\mathbf{n}\mathrm{d}S = \mathbf{x}_{s_1}(s) \wedge ... \wedge \mathbf{x}_{s_{n-1}}(s) \mathrm{d}s_1 ... \mathrm{d}s_{n-1}$$

where the subscripts denote the partial derivatives. Hence, we have that

$$\mathbf{F}(\mathbf{x}(s)) \cdot \mathbf{n}\mathrm{d}S = g_1(\mathrm{d}g_2 \wedge ... \wedge \mathrm{d}g_n) \cdot (\mathbf{x}_{s_1}(s) \wedge ... \wedge \mathbf{x}_{s_{n-1}}(s)) \mathrm{d}s_1 ... \mathrm{d}s_{n-1}$$

A useful result that will help to simplify the above is the **Cauchy-Binet Formula:**

Lemma 7.3. Let A and B be $n \times n$ square matrices with row vectors $a_1, ...a_n$ and $b_1, ...b_n$ respectively. Then

$$|a_1 \wedge ... \wedge a_n||b_1 \wedge ... \wedge b_n| = |\det(\mathbf{a_i} \cdot \mathbf{b_j})|$$

[1] This is a well known result known as the No Retraction Theorem. Though there are many variations of proofs of the theorem, the argument below is fairly easy to follow.

By Lemma 7.3, we can rewrite $\mathbf{F}(\mathbf{x}(s)) \cdot \mathbf{n}\mathrm{d}S$ as

$$\mathbf{F}(\mathbf{x}(s)) \cdot \mathbf{n}\mathrm{d}S = g_1(\mathbf{x}(s)) \det(\nabla g_i(\mathbf{x}_{s_i}(s)) \cdot \mathbf{x}_{s_j}(s))_{2\leq i\leq n, 1\leq j\leq n-1} \mathrm{d}s_1 ... \mathrm{d}s_{n-1}$$

By the chain rule $\nabla g_i(\mathbf{x}_{s_i}(s)) \cdot \mathbf{x}_{s_j}(s) = \frac{\partial}{\partial_{s_j}} g_i(\mathbf{x}_{s_i}(s)$, and so

$$\int_{\mathbb{S}^{n-1}} \mathbf{F} \cdot \mathbf{n}\mathrm{d}S = \int g_1(\mathbf{x}(s)) \det\left(\frac{\partial}{\partial_{s_j}} g_i(\mathbf{x}_{s_i}(s))\right) \mathrm{d}s_1 ... \mathrm{d}s_{n-1}$$

This integral clearly depends on the values of g only in $\mathbb{S}^{n-1}$. But on $\mathbb{S}^{n-1}$, we have defined g to be the identity (I). Hence, it follows that

$$\int_{\mathbb{D}^n} \det DI = \mathrm{vol}(\mathbb{D}^n)$$

This is, of course, non-zero and so claim 1 contradicts claim 2. Thus, There is no C^1 map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ that is the identity. This proves the BFPT. $\square$

Remark 7.4. Proving the BFPT requires some subtleties that make it rather difficult. The fact that one can approximate any continuous function by C^1 functions requires a proof in its own right, as goes for the Cauchy-Binet Formula, Lemma 7.3. These proofs, however, have not been included in these notes.

Remark 7.5. The theorem can be even further generalized to not only consider a continuous map from an n-ball $\mathbb{D}^n \subset \mathbb{R}^n$ to itself, but from a compact, convex set to itself [6].

7.1 Applications of the BFPT

There are many real-world applications of the BFPT. For example, the theorem is fundamental in proving the classic economic principle of a Nash Equilibrium (although this will not be covered further). A rather mindblowing consequence of the theorem is that if you take a map, scrunch it up and place it on the ground, there is at least one point on the map that is directly above the place on earth it corresponds to. We will rather focus on some pure mathematics applications, namely in Eigenvectors in Linear Algebra, the Brouwer-Lebesgue Tiling Theorem and Topological Invariance of Dimension.

7.1.1 Eigenvectors in Linear Algebra

Proposition 7.6. If A is an $n \times n$ matrix with non-negative entries, the there is an eigenvector $x \in \mathbb{R}^n$ with non-negative entries and corresponding eigenvalue non-negative.

Proof. Let $S = \{(x_1, ...x_n) \mid x_1 + .. + x_n = 1,\ x_j \geq 0\} \subset \mathbb{R}^n$. Then S is topologically a disc (i.e. homeomorphic to $\mathbb{D}^{n-1}$). If there is some $\mathbf{x} \in S$ such that $A\mathbf{x} = \mathbf{0}$, then A has 0 as a corresponding eigenvalue and we are done. But, if not, then let $f : S \to S$ be the map $f(\mathbf{x}) = \frac{A\mathbf{x}}{|A\mathbf{x}|}$. This is continuous, as we are not considering the case where $A\mathbf{x} = \mathbf{0}$. Then, by the BFPT, there is some $\mathbf{x_0} \in S$ such that $f(\mathbf{x_0}) = \mathbf{x_0}$. Then

$$\begin{aligned} \mathbf{x_0} =& \frac{A\mathbf{x_0}}{|A\mathbf{x_0}|} \\ \Rightarrow A\mathbf{x_0} =& |A\mathbf{x_0}|\mathbf{x_0} \end{aligned}$$

and so $\mathbf{x_0}$ is a non-negative eigenvector of A as $\mathbf{x_0} \in S$, with a non-negative eigenvalue $|A\mathbf{x_0}|$. $\square$

7.1.2 n-Dimensional Intermediate Value Theorem

A fundamental result that underpins a great deal of mathematics is the Intermediate Value Theorem (IVT). We have already encountered it in one-dimension. In the n-dimensional case, the IVT is the following:

Theorem 7.7 (Intermediate Value Theorem)**.** Suppose that $f : \mathbb{D}^n \to \mathbb{R}^n$ is continuous and suppose that when $|x| = 1$ ($x \in \partial\mathbb{D}^n$) we have

$$\langle f(x), x\rangle < 0$$

Then there exists an $x \in \mathbb{D}^n$ such that

$$f(x) = 0$$

Proof. Let $f : \mathbb{D}^n \to \mathbb{R}^n$ be a continuous map that satisfies the above criteria. To prove that there exists an $x \in \mathbb{D}^n$ such that $f(x) = 0$, we can construct a new map $g : \mathbb{D}^n \to \mathbb{D}^n$ using f such that, when we apply the BFPT to g, the result simplifies to $f(x) = 0$. Take $g(x) = \alpha f(x) + x$, for some $\alpha > 0$. Then by BFPT, there exists some $x \in \mathbb{D}^n$ such that

$$\begin{aligned} g(x) &= x \\ \Rightarrow \alpha f(x) + x &= x \\ \Rightarrow \alpha f(x) &= 0 \\ \Rightarrow f(x) &= 0 \end{aligned}$$

So, all that needs to be established is that for any continuous map $f : \mathbb{D}^n \to \mathbb{R}^n$ with the condition that $x \in \partial\mathbb{D}^n \Rightarrow \langle f(x), x\rangle < 0$, there exists an $\alpha > 0$ such that

$$g(x) = \alpha f(x) + x$$

is a continuous function $g : \mathbb{D}^n \to \mathbb{D}^n$.

To prove this, suppose that such a function g does not exist. We can write any $\alpha > 0$ as $\alpha = \frac{1}{m}$, $m \in \mathbb{R}$. Then for all m where $g(x) = \frac{1}{m}f(x) + x$, there is some $x_m \in \mathbb{D}^n$ such that $|g(x_m)| > 1$ ($x_m \notin \mathbb{D}^n$). As the choice of α is arbitrary, we have

$$\left|\frac{1}{m}f(x_m) + x\right| > 1.$$

We know that $\mathbb{D}^n$ is compact. By Heine-Borel, it is bounded and so by Bolzano-Weierstrass there is a subsequence of (x_m) convergent to some $x_0 \in \mathbb{D}^n$. As f is continuous, this implies $f(x_m) \to f(x_0)$. Hence, taking the limit of both sides of the above inequality,

$$\begin{aligned} &\Rightarrow |0f(x_0) + x_0| \geq 1 \\ &\Rightarrow |x_0| \geq 1 \end{aligned}$$

But we know that $x_0 \in \mathbb{D}^n$, so we deduce that $|x_0| = 1$. By assumption and the continuity of f, $\langle f(x), x\rangle < 0 \Rightarrow \langle f(x_0), x_0\rangle < 0$.

Again we deduce that there is some $\delta > 0$ and an $M \in \mathbb{N}$ such that for $m \geq M$, we have that

$$\langle f(x_m), x_m\rangle \leq -\delta < 0$$

But

$$\begin{aligned} \left|\frac{1}{m}f(x_m) + x_m\right|^2 &= \frac{1}{m^2}\left|f(x_m)\right|^2 + \frac{2}{m}\left|(x_m)f(x_m)\right|^2 + \left|x_m\right|^2 \\ &= \frac{1}{m^2}\left|f(x_m)\right|^2 + \frac{2}{m}\langle f(x_m), x_m\rangle^2 + \left|x_m\right|^2 \end{aligned}$$

By definition of supremum, and as $|x_m| \leq 1$,

$$\left|\frac{1}{m}f(x_m) + x_m\right|^2 \leq \frac{1}{m^2}\|f\|_\infty^2 + \frac{2}{m}\left|(x_m)f(x_m)\right|^2 + |x_m|^2$$
$$\leq \frac{1}{m^2}\|f\|_\infty^2 + \frac{2}{m}\left|(x_m)f(x_m)\right|^2 + 1$$

But $\langle f(x_m), x_m\rangle \leq -\delta$, so

$$\left|\frac{1}{m}f(x_m) + x_m\right|^2 \leq \frac{1}{m^2}\|f\|_\infty^2 + \frac{2}{m}(-\delta) + 1.$$

when $m \geq M$. We can now set up a contradiction. Suppose that $m > \frac{\|f\|_\infty^2}{\delta}$. Then $\frac{\delta}{m} > \frac{\|f\|_\infty^2}{m^2}$, and we would have that

$$\left|\frac{1}{m}f(x_m) + x_m\right|^2 < \frac{\delta}{m} - \frac{2\delta}{m} + 1$$
$$< 1 - \frac{\delta}{m}.$$

But as $m, \delta > 0$, this implies that $\left|\frac{1}{m}f(x_m) + x_m\right|^2 < 1$, which is a contradiction. So we can conclude that for any continuous $f : \mathbb{D}^n \to \mathbb{R}^n$, there exists an $\alpha > 0$ such that $g(x) = \alpha f(x) + x$ is a continuous function $g : \mathbb{D}^n \to \mathbb{D}^n$. This concludes the proof [9]. $\square$

Remark 7.8. The IVT and the BFPT are actually equivalent theorems. We have already seen that BFPT $\Rightarrow$ IVT. Now let us prove the converse.

Proposition 7.9. IVT $\Rightarrow$ BFPT.

Proof. Suppose $f : \mathbb{D}^n \to \mathbb{D}^n$ is continuous. Then by the BFPT, there is some $x_0 \in \mathbb{D}^n$ such that $f(x_0) = x_0$. Now set $g(x) = f(x) + x$. We know that $g : \mathbb{D}^n \to \mathbb{R}^n$. To check the two criteria of IVT hold, consider that

$$\langle g(x), x\rangle = \langle f(x) + x, x\rangle = |xf(x)| - |x|^2.$$

By the Cauchy-Schwarz Inequality,

$$\Rightarrow \langle g(x), x\rangle \leq |f(x)||x| - |x|^2.$$

But $|f(x)| \leq 1$, so

$$\Rightarrow \langle g(x), x\rangle \leq |x| - |x|^2.$$

Under the condition that $|x| = 1$, it holds that $\langle g(x), x\rangle \leq 0$. Note that

$$g(x_0) = f(x_0) - x_0 = x_0 - x_0 = 0.$$

Hence, the IVT holds for g. $\square$

The theorem works out nicely, however it is important to understand why the particular condition that when $|x| = 1 \Rightarrow \langle f(x), x\rangle < 0$ is stated. In fact, this proves to be sufficient in proving the more general case when $|x| = 1 \Rightarrow \langle f(x), x\rangle \leq 0$:

Suppose that f is a continuous function such that $\langle f(x), x\rangle \leq 0$ for $x \in \partial\mathbb{D}^n$. Define a new continuous function $f_m(x) = f(x) - \frac{x}{m}$. Then

$$\langle f_m(x), x\rangle = \langle f(x) - \frac{x}{m}, x\rangle = \langle f(x), x\rangle - \frac{1}{m}|x|^2.$$

By assumption, and under the condition $|x| = 1$, we have that

$$\langle f_m(x), x\rangle \leq 0 - \frac{1}{m} < 0,$$

and so we can apply the IVT to f_m. That is, f_m has a zero x_m. As $\mathbb{D}^n$ is compact, the sequence (x_m) has a subsequence that is convergent to some $x_0 \in \mathbb{D}^n$. But as $f_m(x_m) = 0$, and by the Cauchy-Schwarz Inequality, we can write

$$|f(x_0)| = |f(x_0) - f_m(x_m)| \leq |f(x_0) - f(x_m)| + |f(x_m) - f_m(x_m)|$$

$|f(x_m) - f_m(x_m)| \to 0$ and $|f(x_0) - f(x_m)| = \left|\frac{x}{m}\right| \leq \frac{1}{m} \to 0$ as $m \to \infty$. So, learly the right hand side of the above tends to 0 as $m \to \infty$. Hence, as $|f(x)| \geq 0$ for all $x \in \mathbb{D}^n$, this implies that $|f(x_0)| = 0$ and so the IVT holds for f.

Furthermore, the theorem works exactly the same for when $|x| = 1 \Rightarrow \langle f(x), x\rangle \geq 0$, simply you interchange f for $-f$ and proceed as normal. The condition highlights the fact that one cannot have a map f such that $\langle f(x), x\rangle \leq 0$ and $\langle f(x), x\rangle \geq 0$ when $|x| = 1$, and have the Intermediate Value Property. A simple counterexample proves this to be true: Take $f(x) = \mathbf{e_1}$. Then obviously $\langle f(x), x\rangle \leq 0$ and $\langle f(x), x\rangle \geq 0$ for different values of $x \in \partial\mathbb{D}^n$. Obviously there is no such $x_0 \in \mathbb{D}^n$ such that $g(x_0) = 0$.

7.1.3 The Brouwer-Lebesgue Tiling Theorem

Theorem 7.10 (Brouwer-Lebsegue Tiling Theorem)**.** Let $Q = Q^m$ be the unit cube $[-1/2, 1/2]^m \in \mathbb{R}^m$. Suppose that Q can be written as a finite union

$$Q = \bigcup_i D_i$$

where each D_i is a closed subset of Q which does not meet both of any pair of opposite faces of Q. Then there is some $x \in Q$ which belongs to at least $n + 1$ distinct D_i.

Remark 7.11. This theorem is very easy to visualise in one and two dimensions. Consider that Q is just the closed interval $[-1/2, 1/2]$ on the real line, and the closed subsets referred to are closed intervals. So, in order to form a covering of Q, two (i.e. $n + 1$) intervals must intersect at least once, otherwise there would be an element of Q not covered by the covering.A similar argument can be used to visualise the theorem in two dimensions, although it is important to note that the overlap of three closed rectangles (D_i) would be required.

Proof. This proof is based off of the version contained within the University of Edinburgh's Topology Course Lecture Notes (2012/2013) [9] as well as Adams et. al.'s article "When Soap Bubbles Collide", published in the *American Mathematical Monthly* (2007)[5]. Let $F_j^{\pm} := \{x \in Q \mid x_j = \pm 1/2\}$. That is, $F_j^{\pm}$ are the two faces of Q that are perpendicular to the x_j axis. The proof hinges on the following lemma being true:

Lemma 7.12. Suppose that B_j is a closed subset of Q which separates $F_j^{\pm}$ in Q. Then

$$\bigcap_{j=1}^{n} B_j \neq \emptyset.$$

A second lemma is also needed to construct these subsets, B_j:

Lemma 7.13. Let X be a metric space, and $F^{\pm}$ be dijoint, closed subsets of X. Let $A \subseteq X$ also be closed. Suppose $K \subseteq A$ is closed and separates $F^+ \cap A$ from $F^- \cap A$ in A. Then there exists a closed set $B \subseteq X$ which separates F^- from $F+$ such that $B \cap A \subseteq K$.

Under the assumption that these are true (which we will prove at the end), we can find such subsets B_j from the D_i and hence prove the theorem true. The proof constructs a chain of sets from D_i (where the latter is constructed from the former) until we eventually arrive at B_j:

Let

$$\begin{aligned}\mathscr{L}_1 &= \{D_i \mid D_i \cap F_1^- \neq \emptyset\},\\ \mathscr{L}_2 &= \{D_i \notin \mathscr{L}_1 \mid D_i \cap F_2^- \neq \emptyset\},\\ \mathscr{L}_k &= \{D_i \notin \mathscr{L}_1 \cup ... \cup \mathscr{L}_k \mid D_i \cap F_k^- \neq \emptyset\},\\ &...\\ \mathscr{L}_{n+1} &= \{D_i \notin \mathscr{L}_1 \cup ... \cup \mathscr{L}_n\}.\end{aligned}$$

[2]

Now, define

$$L_1 = \bigcup_{D_i \in \mathscr{L}_1} D_i, \quad ..., \quad L_{n+1} = \bigcup_{D_i \in \mathscr{L}_{n+1}} D_i.$$

By definition of $\mathscr{L}_k$ (for $k = 1, ..., n$) and the diameter restriction on D_i , L_k does not meet $f_1^-, ..., F_{k-1}^-$ and F_k^+. Similarly, L_{n+1} does not meet $F_1^-, ..., F_n^-$.

Now let

$$\begin{aligned}K_1 &= L_1 \cap (L_2 \cup ... \cup L_{n+1})\\ K_2 &= L_1 \cap L_2 \cap (L_3 \cup ... \cup L_{n+1})\\ &...\\ K_j &= (L_1 \cap ... \cap L_k) \cap (L_{k+1} \cup ... \cup L_{n+1})\end{aligned}$$

By De Morgan's Law,

$$\begin{aligned}K_1 &= L_1 \cap (L_2 \cup ... \cup L_{n+1})\\ \Rightarrow Q \setminus K_1 &= (Q \setminus L_1) \cup (Q \setminus (L_2 \cup ... \cup L_{n+1}))\end{aligned}$$

This is a disjoint union. Also, $F_1^- \subseteq Q \setminus L_1$ and $F_1^+ \subset Q \setminus (L_2 \cup ... \cup L_{n+1})$, so K_1 separates $F_1^\pm$ in Q.

Similarly, we have that

$$K_1 \setminus K_2 = (K_1 \setminus (L_1 \cap L_2)) \cup (K_1 \setminus (L_3 \cup ... \cup L_{n+1}))$$

Which is disjoint as if $x \in K_1 = L_1 \cap (L_2 \cup ... \cup L_{n+1})$, and $x \notin L_3 \cup ... L_{n+1}$, then $x \in L_1 \cap L_2$. Also,

$$F_2^+ \subseteq K_1 \setminus (L_1 \cap L_2) \quad \text{and} \quad F_2^- \subseteq K_1 \setminus (L_3 \cup ... \cup L_{n+1})).$$

Similarly, K_j separates $F_j^\pm \cap K_{j-1}$ in K_{j-1}. Consider that,

$$K_{j-1} \setminus K = K_{j-1} \cap (L_1 \cap ... \cap L_j) \cup K_{j-1} \setminus (L_{j+1} \cup ... \cup L_{n+1})$$

Which is a disjoint union of two sets, as if $x \in K_{j-1} = (L_1 \cap ... \cap L_{j-1}) \cap (L_j \cup ... \cup L_{n+1})$ and $L \notin L_{j+1} \cup ... \cup L_{n+1}$, then $x \in L_1 \cap ... \cap L_j$. Also,

[2] These sets $\mathscr{L}_k$ categorise the D_i by the faces they intersect. Should a set D_i intersect multiple faces, then the D_i gets categorised into the set for the lowest indexed face. So, for example, if D_i intersects faces F_1^- and F_2^-, the set D_i will be placed in the set containing sets intersecting F_1^-.

$$F_j^+ \subseteq K_{j-1} \setminus (L_1 \cap ... \cap L_j) \text{ and } F_j^- \subseteq K_{j-1} \setminus (L_{j+1} \cup ... \cup L_{n+1})).$$

Now by Lemma 7.13, with $X = Q$, $A = K_{j-1}$ (with $K_0 = Q$), $K = K_j$ and $F^\pm = F_j^\pm$, there exist B_j which separates F_j^- from F_j^+ in Q such that $B_j \cap K_j - 1 \subseteq K_j$. Then

$$\begin{aligned} B_1 \cap B_2 \cap ... \cap B_n &\subseteq K_1 \cap B_2 \cap ... \cap B_n \\ \subseteq K_2 \cap B_3 \cap ... \cap B_n &\subseteq ... \subseteq K_{n-1} \cap B_n \subseteq K_n \end{aligned}$$

But by Lemma 7.12, $B_1 \cap ... \cap B_n$ is non-empty, and therefore K_n is non-empty. As $K_n \neq \emptyset$, then there will be at least one common point in $L_1, ..., L_{n+1}$. This is equivalent to their being a common element in each of the **disjoint** collections, which is equivalent to the element belonging to at least $n+1$ D_i.

Now all we need to do is prove Lemmas 7.12 and 7.13.

Proof of Lemma 7.13. Suppose that K separates $F^\pm \cap A$ in A, then we can write

$$A \setminus K = U^+ \cup U^-$$

Where $U^\pm$ are disjoint, open sets in A (and hence open in $A \setminus K$) and $f^\pm \cap A \subseteq U^\pm$. Hence $U^\pm$ are also closed in $A \setminus K$, implying that any limit point of $U^\pm$ which lies in $A \setminus K$ also lies in $U^\pm$.

We need to prove that the sets $H^\pm := f^\pm cup U^\pm$ do not contain any of each other's limit points. Suppose for contradiction's sake that some $x_m \in H^-$ converges to some $x \in H^+$. Then some subsequence of x_m belongs either to F^- or U^-. If it belongs to F^-, then $x \in F^-$, and so $x \in F^+ \cup U^+$, which is a contradiction, as F^- and U^+ are disjoint. If x belongs to U^-, then as $U^- \subseteq A$ which is closed, $x \in A$. So either $x \in A \setminus K \Rightarrow x \in U^-$, or $x \in K$. But both U^- and K are disjoint from H^+, so we have a contradiction.

Define $f(x) = d(x, H^+) - d(x, H^-)$. f is clearly continuous. If $x \in H^+$, then $x \notin H^-$ and so $f(x) < 0$. Similarly, if $x \in H^-$, then $f(x) > 0$. We can define open sets $V^+ := \{x \in X \mid f(x) < 0\}$ and $V^- := \{x \in X \mid f(x) > 0\}$. These sets contain $H^\pm$ respectively. By definition, the set $B := \{x \in X \mid f(x) = 0\}$ is a closed set separating $H^\pm$ in X. □

Proof of Lemma 7.12. We now have B_i such that $Q \setminus B_i$ is the disjoint union of two relatively open sets $V_j^\pm$ in Q containing $F_j^\pm$ respectively.

Define a vector $v(\mathbf{x})$, whose j^{th} component is given by:

$$v_j(x) = \begin{cases} d(x, B_j) & \text{if } x \in V_j^- \\ -d(x, B_j) & \text{if } x \in V_j^+ \\ 0 & \text{if } x \in B_j \end{cases}$$

Define the continuous map $f : Q \to \mathbb{R}^n$ by $f(\mathbf{x}) = \mathbf{x} + v(\mathbf{x})$. This map takes Q to itself, as the coordinate changes by the distance to B_j, which is less than the distance to the opposite face [5].

We now get to use BFPT! As f is a continuous map from Q to itself, there exist $\mathbf{x_0} \in Q$ such that $f(\mathbf{x_0}) = \mathbf{x_0}$. But

$$\begin{aligned} f(\mathbf{x_0}) = v(\mathbf{x_0}) + \mathbf{x_0} &= \mathbf{x_0} \\ \Rightarrow v(\mathbf{x_0}) &= \mathbf{0} \end{aligned}$$

So, $v(\mathbf{x_0})$ is the zero vector, and so must be in the intersection $\bigcap_j B_j$. □

□

The Lebsesgue Tiling Theorem leads us to an interesting result:

Proposition 7.14. Suppose that

$$Q \subseteq \bigcup_i U_i$$

where each U_i is an open subset of $\mathbb{R}^n$ and has diameter at most 1. Then there is some $x \in Q$ which belongs to at least $n+1$ U_i.

Proof. We will construct some $D_i \subseteq U_i$ such that each D_i has a diameter strictly less than 1, and that $Q = \bigcup_i D_i$. Then we are satisfied to apply Theorem 7.10 and the conclusion will hold. Note that $\{U_i\}$ forms a cover of Q. By Theorem 4.26 (Lebesgue Number), there exists a positive number $\delta > 0$ such that $\forall x \in Q$, $B_\delta(x)$ lies entirely inside some U_i. Let

$$D_i = \{x \in Q \cap U_i \mid B_\delta(x) \subseteq U_i\}.$$

That is, the ball $B_\delta(x)$ is fully contained in a U_i. This can equivalently be written as $D_i = \{x \in Q \cap U_i \mid d(x, \partial U_i) \geq \delta\}$. Note that $D_i \subseteq U_i$. For all $x \in D_i$, These D_i are closed subsets of Q. To prove this, consider that if x_n is in D_i for all n and $x_n \to x$, we have that $x \in Q$ (as Q is closed), and $d(x, \partial U_i) \geq \delta$. To prove that $d(x, \partial U_i) \geq \delta$, consider that if $d(x, \partial U_i) < \delta$, then for all sufficiently large n, $d(x_n, \partial U_i) < \delta$ which is a contradiction. We also have that x is in U_i, and so x is certainly in the closure of U_i. But as $d(x, \partial U_i) \geq \delta$ and strictly $\delta > 0$, $d(x, \partial U_i) \neq 0$ and so x cannot lie on the boundary of U_i. So x is an interior point of U_i. As U_i is open, it follows that $x \in U_i$. Thus, as $x \in Q$, $x \in U_i$, and $d(x, \partial U_i) \geq \delta$, $x \in D_i$. Note that Certainly the diameter of D_i is strictly less than 1, as not all $x \in Q \cap U_i$ satisfy $B_\delta(x) \subseteq U_i$ (Take any x such that $d(x, \partial U_i) < \delta$). We wish to prove that $Q = \bigcup_i D_i$, and we are done.

All $D_i \subseteq Q$ by definition, so $\bigcup_i D_i \subseteq Q$. To prove the reverse, suppose the inclusion $\bigcup_i D_i \supseteq Q$ does not hold. Then there is an $x \in Q$ such that, for all i, $x \notin D_i$. So then either there is some $x \notin U_i$ for all i. But as $\{U_i\}$ forms a cover of Q, this is a contradiction. Or there is some $x \in U_i$ that satisfies $B_\delta(x) \nsubseteq U_i$. That is, there is some $B_\delta(x)$ not fully contained in one U_i. But this contradicts the definition of Lebesgue Number, δ. Hence, it must be that $\bigcup_i D_i \supseteq Q$ and so $\bigcup_i D_i = Q$. □

7.2 Topological Invariance of Domain and Dimension

Theorem 7.15 (Invariance of Domain)**.** Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be a continuous, injective function. Then if $f(U)$ is open.

Proof. This proof is based off of the one (Proposition 6.3) contained within the University of Ediburgh Mathematics 4 Topology course notes [9], and the lecture notes for Stony Brook University's MAT 530: Topology, Geometry 1 course [15]. We wish to show that every point $f(x) \in f(U)$ is an interior point of $f(U)$. Then, by definition $f(U)$ will be open. Consider that, for all $x \in U$, there is some $r > 0$ such that $\overline{B_r(x)} \subseteq U$. But note that $x \in \overline{B_r(x)} \subseteq U \Rightarrow f(x) \in f(\overline{B_r(x)}) \subseteq f(U)$ by the continuity of f. So, to show that $f(U)$ is an open subset of $\mathbb{R}^n$, it suffices to show that $f(x)$ is an interior point of $f(\overline{B_r(x)})$. Further, by rescaling and translation, it suffices to show that, if $f : \mathbb{D}^n \to \mathbb{R}^n$ is continuous and injective, then $f(0)$ is an interior point of $f(\mathbb{D}^n)$.

As $\mathbb{D}^n$ is compact, it follows that $f_{|\mathbb{D}^n} \to f(\mathbb{D}^n)$ is a homeomorphism. We define an inverse $g : f(\mathbb{D}^n) \to \mathbb{D}^n$. By the Tieze Extension Theorem, we extend g to $g : \mathbb{R}^n \to \mathbb{R}^n$. This function has a zero, that is of course, $f(0)$. We wish to show that any function sufficiently close to g has a zero. This will prove the above. More formally, let $\tilde{g} : f(\mathbb{D}^n) \to \mathbb{R}^n$ be continuous, with condition that

$|\tilde{g}(y) - g(y)| \leq 1$ for all $y \in f(\mathbb{D}^n)$. Then $\tilde{g}$ has a zero in $f(\mathbb{D}^n)$. We define a function $h(x) : \mathbb{D}^n \to \mathbb{D}^n$ such that

$$h(x) := x - \tilde{g}(f(x)).$$

Then by the BFPT, it follows that there exists $x \in \mathbb{D}^n$ such that

$$\begin{aligned} h(x) =& x \\ \Rightarrow x - \tilde{g}(f(x)) =& x \\ \Rightarrow \tilde{g}(f(x)) =& 0 \end{aligned}$$

Of course, $\tilde{g} : f(\mathbb{D}^n) \to \mathbb{R}^n$, and so it follows that $\tilde{g}$ has a zero, namely $f(x)$.

Suppose for contradiction's sake that $f(0)$ is not an interior point of $f(\mathbb{D}^n)$. Then $f(0)$ must lie on the boundary, and so a function has a zero on the boundary. We wish to construct some function $\tilde{g}$ from g which will contradict the fact that any function sufficiently close to g has a zero.

By the definition of continuity, there exists a $\delta > 0$ such that for all $y \in \mathbb{R}^n$, $|y - f(0)| < 2\delta \Rightarrow |g(y) - g(f(0))| < 1/4$. But of course we already have that $g(f(0)) = 0$, so $|y - f(0)| \Rightarrow |g(y)| < 1/4$. By assumption, there exists $\alpha \notin f(\mathbb{D}^n)$ such that $|\alpha - f(0)| < \delta$. Assume $\alpha = 0$. If not, we can translate it so that α gets moved to the origin.

So we have that $0 \notin f(\mathbb{D}^n)$, $|f(0)| < \delta$, and by the Triangle Inequality $|y| < \delta \Rightarrow |y - f(0)| < \delta$.

Define

$$L = L_1 \cup L_2 := (f(\mathbb{D}^n) \cap \{|y| \geq \delta\}) \cup \{y \in \mathbb{R}^n \setminus f(\mathbb{D}^n) \mid |y| = \delta\}$$

Notice that L_2 is the boundary of the ball of radius δ centred on the origin, and L_1 is the part of $f(\mathbb{D}^n)$ that lies outside that ball. By the compactness of $f(\mathbb{D}^n)$, L_1 and L_2 are also compact. Further, there are also no zeros on L_1.

We define a continuous function $\Phi : f(\mathbb{D}^n) \to L$ by

$$\Phi(y) = \max\left\{\frac{\delta}{|y|}, 1\right\} \cdot y$$

This is well-defined and continuous. When $y \in L_1$, then $\Phi(y) = y$. When $y \in f(\mathbb{D}^n)$ with $|y| < \delta$, $\Phi(y) = \frac{\delta y}{|y|} \in L_2$. Note that these are the boundary points of the ball.

Now take $\tilde{g} = g \circ \Phi : f(\mathbb{D}^n) \to \mathbb{R}^n$. When $y \in L_1$, $\tilde{g}(y) = g(y) \neq 0$. The only place for a zero then may be when $y \in L_2$. To prove that there is no zero when $y \in L_2$, if necessary we can perturb $\tilde{g}$ to be a better-suited function, by means of the Weierstrass Aproximation Theorem or perhaps by a more direct topological approach. This ensures that the function does not vanish in L_2.

After establishing this, we can conclude the proof by showing that $\tilde{g}$ is sufficiently close to g:

1. If $|y| \geq \delta$, then $\Phi(y) = y$, and $|g(y) - \tilde{g}(y)| = |g(y) - g \circ \Phi(y)| = |g(y) - g(y)| = 0 < 1$.
2. If $|y| \leq \delta$, then we have that $|g(y) < 1/4$ and $\tilde{g}(y) < 1/4$. So, $|g(y) - \tilde{g}(y)| < 1/4 + 1/4 = 1/2 < 1$.

In both cases, as $|g(y) - \tilde{g}| < 1$, $\tilde{g}$ should have a zero in $f(\mathbb{D}^n)$. But as we have proven it doesn't, this is a contradiction and our claim holds. □

Theorem 7.16 (Invariance of Dimension)**.** If U is an open subset of $\mathbb{R}^m$ and V an open subset of $\mathbb{R}^n$, and U is homeomorphic to V, then $m = n$.

Proof. To prove invariance of dimension, we will use invariance of domain. Suppose there exists a continuous, injective map $f : U \to \mathbb{R}^n$ where U is an open subset of $\mathbb{R}^m$. We will first prove that it must be that $m \leq n$.

We will proceed by contradiction. Suppose that $m > n$, then take some linear injection $p : \mathbb{R}^n \to \mathbb{R}^m$. The image, $p(\mathbb{R}^n)$ is a proper subspace of $\mathbb{R}^m$. Then $p \cdot f : U \to \mathbb{R}^m$ is a linear injection whose image is contained within a proper subspace of $\mathbb{R}^m$. However,

Lemma 7.17. Every proper subspace of $\mathbb{R}^m$ has an empty interior.

Proof. Let $S \subset \mathbb{R}^m$ be a proper subspace. Suppose S has a non-empty interior. Let $x \in S$. Then it contains some ball $B_r(x)$. Take some $y \in \mathbb{R}^m$. Let $z = x + \frac{r}{2\|y\|}y$. Then $z \in B_r(x) \subset S$. But then $y = (z-x)\frac{2\|y\|}{r}, \Rightarrow y \in S$. But as $y \in \mathbb{R}^m$ is arbitrary, it follows that $S = \mathbb{R}^m$. This is a contradiction, as we assumed S is a **proper** subspace of $\mathbb{R}^m$ [4]. $\square$

But by Theorem 7.15, this is a contradiction, as $p \cdot f(U)$ is assumed to be an open set. $\square$

8 Reflections

This project has set a solid foundation for any further studies in topology I may wish to undertake, perhaps in fourth year. Whilst researching I also touched on compactification and some topology using a more basis-oriented approach, however it soon became apparent that these areas were rather unnecessary (at least for the direction of the project) and so were for the most part not included within these notes. More so, the project has lead to an open question relating to theorems potentially equivalent to the Brouwer Fixed Point Theorem, and by extension the n-Dimensional Intermediate Value Theorem. This will perhaps be explored further at a later date.

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