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FINITE DIMENSIONAL RANKED VECTOR SPACES

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After the introduction of the method of ranked spaces by K. Kunugi in 1954 ([1]), ranked vector spaces have been investigated by many researchers with various definitions ([2], [4], [6]). In this paper, we consider the finite dimensional ranked vector spaces (abbreviated as f. d. r. v. s.) over the field $\mathbb{K}$ which is $\mathbb{R}$ or $\mathbb{C}$ defined by the condition that the scalar multiplication and the addition are r-continuous as in S. Nakanishi ([4]). Defining r-isomorphisms between two f. d. r. v. s., we prove that every f. d. r. v. s. which satisfies the separation property (r-T_1) is r-isomorphic to the standard f. d. r. v. s. $(\mathbb{K}^d, \{B^d(0, \frac{1}{2^i})\})$, that is, we have essentially only one f. d. r. v. s. when the dimension d is given just as in the case of finite dimensional topological vector spaces.

Let us remark that we use essentially the r-continuity of scalar multiplication and addition in our proof. In fact, if we define f. d. r. v. s. by using continuity in the sense of r-convergent sequences, i.e., for any sequences $\{\alpha_i\}$ in $\mathbb{K}$, $\{x_i\}$ in E and $\{y_i\}$ in E which r-converge to $\alpha \in \mathbb{K}$, $x \in E$ and $y \in E$ respectively, the sequences $\{\alpha_i x_i\}$ and $\{x_i+y_i\}$ in E r-converge to $\alpha x \in E$ and $x+y \in E$ respectively, then the theorem is no longer true even for the one-dimensional case. (See the example in §3.)

Here is a brief outline of this paper: In §1, we start with some definitions and properties mainly to unify terminologies. Then we introduce the standard f. d. r. v. s. $(\mathbb{K}^d, \{B^d(0, \frac{1}{2^i})\})$ in §2. Finally, in §3, we prove the main theorem.

§1. Preliminaries.

1.1. Ranked spaces ([1], [3]).

A space E is called a ranked space (of indicator ω_0) if, for every point x of E, there is associated a non-empty family $\mathcal{V}(x)$ consisting of subsets of E and, for every non-negative integer n, there is associated a subfamily $\mathcal{V}_n$ of $\mathcal{V} = \{\mathcal{V}(x);\ x \in E\}$ satisfying following conditions:

(A) For every member $U(x)$ of $\mathcal{V}(x)$, $x \in U(x)$.

(a) For every point x of E, every member $U(x)$ of $\mathcal{V}(x)$ and every non-negative integer n, there are another integer m and a member $V(x)$ of $\mathcal{V}_m$ such that $m > n$ and $V(x) \subset U(x)$.

Members of $\mathcal{V}(x)$ are called preneighborhoods of x and written $U(x)$, $V(x)$, etc. Preneighborhoods of x belonging to $\mathcal{V}_n$ are of rank n and written $U(x, n)$, $V(x,n)$, etc. These preneighborhoods are also written simply U, V, etc.

A sequence $\{U_i\} = \{U(x, n_i)\}$ of preneighborhoods of center x is called fundamental if it satisfies the following conditions:

(f.1) $U_1 \supset U_2 \supset \ldots \supset U_i \supset \ldots$

(f.2) $n_1 < n_2 < \ldots < n_i < \ldots$

Sequences of preneighborhoods are sometimes written u, v, etc. For two sequences $u = \{U_i\}$ and $v = \{V_i\}$, $u > v$ means that, for every U_i, there is a V_j such that $U_i \supset V_j$ and the equivalence $u \sim v$ means that $u > v$ and $v > u$. Any subseuqnce v of a fundamental sequence u is also a fundamental sequence equivalent to u.

A sequence $\{x_i\}$ in a ranked space E is said to be r-convergent or to r-converge to x if there is a fundamental sequence $u = \{U_j\}$ of center x satisfying that, for every j, there is an i_0 such that $x_i \in U_j$ where $i \geqq i_0$. Then we write $x = \text{r-lim}\, x_i$.

A ranked space E is said to satisfy the separation property (r-T_1) if, for every x of E and for every fundamental sequence $u = \{U(x, n_i)\}$ of center x, the intersection $\cap u = \bigcap_i U(x, n_i) = \{x\}$ (a set consisting of the element x alone).

1.2. Cartesian products of ranked spaces ([5]).

The Cartesian product of two ranked spaces E and F, written $E \times F$, is the ranked space associated with $\mathcal{W}(x,y)$ ($(x,y) \in E \times F$)) and $\mathcal{W}_l$ defined as follows:

Let E and F be associated with $\mathcal{U}(x), \mathcal{U}_n$ and $\mathcal{V}(y), \mathcal{V}_m$ respectively. The set $E \times F$ is the Cartesian product of sets E and F,

$$\mathcal{W}(x, y) = \{U \times V;\ U \in \mathcal{U}(x),\ V \in \mathcal{V}(y)\}\ ((x,y) \in E \times F),$$

$$\mathcal{W}_l = \{U \times V;\ U \in \mathcal{U}_n,\ V \in \mathcal{V}_m,\ \min(n,m) = l\}.$$

For two sequences $u = \{U_i\}$ in E and $v = \{V_i\}$ in F, $u \times v$ means the sequence $\{U_i \times V_i\}$ in the Cartesian product of ranked spaces E and F. It is easily verified ([5],1.4.) that we may take fundamental sequences of center (x, y) in $E \times F$ only of the form $u \times v$ where u is of center x and v is of center y respectively.

1.3. Mappings of ranked spaces.

Let E and F be ranked spaces. A mapping f of ranked space E into ranked space F is called r-continuous at $x \in E$ if, for every fundamental sequence $u = \{U_i(x)\}$ of center x in E, there is a fundamental sequence $v = \{V_i(f(x))\}$ of center $f(x)$ in F satisfying $v > f(u)$, where $f(u) = \{f(U_i(x))\}$. f is called r-continuous on E if f is r-continuous at each point x in E.

Ranked spaces E and F are called r-isomorphic if there is a one-to-one mapping of E onto F which is r-continuous with r-continuous inverse.

1.4. Finite dimensional ranked vector spaces.

Let E be a vector space over the scalar field $\mathbb{K}$. E is called a ranked vector space if E is a ranked space with the property that the operations of addition σ: $E \times E \to E$ and multiplication by scalars μ: $\mathbb{K} \times E \to E$ are r-continuous ([4]). A ranked vector space is called a finite dimensional ranked vector space if E is of finite dimension over the scalar field $\mathbb{K}$.

§2. The standard d-dimensional ranked vector space $(\mathbb{K}^d, \{B^d(0, \frac{1}{2^i})\})$.

In this paper, we understand that $\mathbb{K}$ means at the same time the one-dimensional vector space over the scalar field $\mathbb{K}$ and $\mathbb{K}^d$ means the d-dimensional vector space over the same scalar field $\mathbb{K}$ which is the Cartesian product of d copies of one-dimensional vector space $\mathbb{K}$.

2.1. $(\mathbb{K}, \{B(0, \frac{1}{2^i})\})$.

If, for every point λ of the vector space $\mathbb{K}$ and every non-negative integer n, we associate families $\mathcal{V}(\lambda)$ and $\mathcal{V}_n$ such that $\mathcal{V}(\lambda) = \cup\{\lambda + B(0, \frac{1}{2^n}); n \geq 0\}$ where $B(0,\frac{1}{2^n}) = \{\alpha \in \mathbb{K}; |\alpha| \leq \frac{1}{2^n}\}$ and $\mathcal{V}_n = \cup\{\lambda + B(0, \frac{1}{2^n}); \lambda \in \mathbb{K}\}$, it is easily shown that the vector space $\mathbb{K}$ becomes one-dimensional ranked vector space, which is written as $(\mathbb{K}, \{B(0, \frac{1}{2^i})\})$. In this case, $\{B(0, \frac{1}{2^i})\}$ is essentially only one, therefore called the fundamental sequence of center $0 \in \mathbb{K}$.

The scalar field $\mathbb{K}$ also becomes a ranked space with the families $\mathcal{V}(\lambda)$, $\mathcal{V}_n$ defined as above. Then, the fundamental sequence of 0 means the sequence $\{B(0, \frac{1}{2^i})\}$. Let us remark, in this case, a sequence $\{x_n\}$ in $\mathbb{K}$ r-converges to an $x \in \mathbb{K}$ if and only if it converges topologically in $\mathbb{K}$.

2.2. $(\mathbb{K}^d, \{B^d(0, \frac{1}{2^i})\})$.

The vector space $\mathbb{K}^d$ becomes a ranked vector space if the families $\mathcal{W}(\Lambda)$ and $\mathcal{W}_n$ are associated with each $\Lambda \in \mathbb{K}^d$; $\Lambda = (\lambda_1, \ldots, \lambda_d)$ and non-negative integer n in the following manner: Let $B^d(0, \frac{1}{2^n}) = \{\Lambda \in \mathbb{K}^d; \|\Lambda\| = \sqrt{|\lambda_1|^2+\ldots+|\lambda_d|^2} \leq \frac{1}{2^n}\}$, $\mathcal{W}(\Lambda) = \cup\{\Lambda + B^d(0, \frac{1}{2^n}); n \geq 0\}$ and

$\mathcal{W}_n = \bigcup\{\Lambda + B^d(0, \frac{1}{2^n});\ \Lambda \in \mathbb{K}^d\}$. This is called the standard d-dimensional ranked vector space and is written as $(\mathbb{K}^d, \{B^d(0, \frac{1}{2^i})\})$. As in the one-dimensional case, the fundamental sequence of center $0 \in \mathbb{K}^d$ may be taken as $\{B^d(0, \frac{1}{2^i})\}$ and the same remark applies for the convergence of sequences in $\mathbb{K}^d$.

2.3. $\mathbb{K} \times E$.

Let $\mathbb{K}$ be the ranked space of scalars and let E be a ranked space associated with the families $\mathcal{V}(x)$ and $\mathcal{V}_n$ for each x in E and each non-negative integer n. The Cartesian product $\mathbb{K} \times E$ defined in 1.2. becomes a ranked vector space with the families $\mathcal{U}(\lambda, x)$ and $\mathcal{U}_l$ associated with each $(\lambda, x) \in \mathbb{K} \times E$ and each non-negative integer l respectively such that $\mathcal{U}(\lambda, x) = \{(\lambda + B(0, \frac{1}{2^n})) \times V;\ n \geq 0,\ V \in \mathcal{V}(x)\}$ and $\mathcal{U}_l = \{(\lambda + B(0, \frac{1}{2^n})) \times V;\ V \in \mathcal{V}_m, \min(n,m)=l\}$. By the standard nature of $\mathbb{K}$, a fundamental sequence of center $(0, 0) \in \mathbb{K} \times E$ may be taken in the form $\{B(0, \frac{1}{2^i})\} \times u = \{B(0, \frac{1}{2^i}) \times U_i\}$ where $u = \{U_i\}$ is a fundamental sequence of center 0 in E.

§3. Proof of the main theorem.

Theorem. Let E be a d-dimensional ranked vector space satisfying the separation property $(r\text{-}T_1)$. Then, E and the standard d-dimensional ranked vector space $(\mathbb{K}^d, \{B^d(0, \frac{1}{2^i})\})$ are r-isomorphic.

For the proof, we will need the following lemma which is proved in [2]p.270, [4]p.182 and [5]p.362:

Lemma. Let E be a ranked vector space satisfying the separation property $(r\text{-}T_1)$. If $x \in E$ and $y \in E$ are r-limits of a sequence $\{x_j\}$ in E, then x=y.

Proof of the theorem. For each $x \in E$, let us denote by Φ_x the family of all fundamental sequences of center x. The set $\{x_1, \dots, x_d\}$ is chosen as a $\mathbb{K}$- basis of E. Let f be the mapping of E onto $\mathbb{K}^d$ defined by $f(x) = (\lambda_1, \dots, \lambda_d)$ where $x = \lambda_1 x_1 + \dots + \lambda_d x_d$. f is an algebraic isomorphism of E onto $\mathbb{K}^d$.

Step I. We will show that the inverse mapping $f^{-1}: \mathbb{K}^d \to E$ is r-continuous. It is enough to show it at the origin $0 = (0, \dots, 0) \in \mathbb{K}^d$.

For each k where $1 \leqq k \leqq d$, the k-th projection mapping $p_k: \mathbb{K}^d \to \mathbb{K}$ is defined by $p_k(\Lambda) = \lambda_k$ for $\Lambda = (\lambda_1, \dots, \lambda_d)$. Then, we can express the inverse mapping f^{-1} as

$$f^{-1}(\Lambda) = p_1(\Lambda)x_1 + \dots + p_d(\Lambda)x_d.$$

From the definition of $B(0, \frac{1}{2^i})$ and $B^d(0, \frac{1}{2^i})$, we have

$$p_k(B^d(0, \frac{1}{2^i})) \subset B(0, \frac{1}{2^i}),$$

hence

$f^{-1}(B^d(0, \frac{1}{2^i})) \subset B(0, \frac{1}{2^i})x_1 + \ldots + B(0, \frac{1}{2^i})x_d$ for every i. Next, since the mapping μ: $\mathbb{K} \times E \to E$ defined by the scalar multiplication is r-continuous, for each k where $1 \leqq k \leqq d$, the mapping $\mu(\lambda, x_k)$ is r-continuous at $(0, x_k)$. That is, for any $u_k \in \Phi_{x_k}$, there exists a $v_k \in \Phi_0$ satisfying $\{B(0, \frac{1}{2^i})\}\cdot u_k < v_k$ for each k where $1 \leqq k \leqq d$. Moreover, the mapping σ: $E \times E \to E$ defined by the addition is r-continuous, consequently, by repeated applications of σ, there exists a $w \in \Phi_0$; $w = \{W_i\}$ satisfying $v_1 + \ldots + v_d < w$. That is, for every $W_i \in w$, there is an m such that $V_{1m} + \ldots + V_{dm} \subset W_i$ where $V_{km} \in v_k$ for each k where $1 \leqq k \leqq d$.

Combining these considerations, we conclude that, for the fundamental system $\{B^d(0, \frac{1}{2^i})\}$ in $\mathbb{K}^d$ there exists a $w \in \Phi_0$ such that for every $W_i \in w$, there is an l satisfying $f^{-1}(B^d(0, \frac{1}{2^l})) \subset B(0, \frac{1}{2^l})x_1 + \ldots + B(0, \frac{1}{2^l})x_d \subset V_{1m} + \ldots + V_{dm} \subset W_i$. This shows that $f^{-1}(\{B^d(0, \frac{1}{2^i})\}) < w$.

<u>Step II</u>. We will show that the mapping f: $E \to \mathbb{K}^d$ is r-continuous. It is enough to show it at the origin $0 \in E$.

We suppose that the mapping f: $E \to \mathbb{K}^d$ were not r-continous at the origin $0 \in E$. Then, for some $u \in \Phi_0$, $f(u) \not< \{B^d(0, \frac{1}{2^i})\}$. That is, there would exist j such that $f(U_i) \not\subset B^d(0, \frac{1}{2^j})$ for each i. This means that, for each i, there would exist $y_i \in U_i$ such that $f(y_i) \notin B^d(0, \frac{1}{2^j})$, i.e., $\| f(y_i) \| \geqq \frac{1}{2^j}$.

(Case I) where the sequence $\{f(y_i)\}$ in $\mathbb{K}^d$ would have an accumulation point A in the topological space $\mathbb{K}^d$. Then, $\| A \| \geqq \frac{1}{2^j}$, so that $A \neq 0$. We choose a suitable subsequence $\{y_{i_k}\} \subset \{y_i\}$ so that the sequence $\{f(y_{i_k})\}$ would r-converge to A, noted $f(y_\infty)$, when $k \to \infty$ in $(\mathbb{K}^d, \{B^d(0, \frac{1}{2^i})\})$. By the step I, the sequence $\{y_{i_k}\}$ would r-converge to y_∞ when $k \to \infty$. We know that $y_\infty \neq 0$ since $A \neq 0$. On the other hand, the sequence $\{y_{i_k}\}$ would r-converge to 0 when $k \to \infty$ since $y_{i_k} \in U_{i_k}$ and $u = \{U_i\} \in \Phi_0$. By the lemma, this is a contradiction.

(Case II) where the sequence $\{f(y_i)\}$ in $\mathbb{K}^d$ would have no accumulation points. We can choose a suitable subsequence $\{y_{i_k}\} \subset \{y_i\}$ such that $\| f(y_{i_k}) \| \geqq 2^k$ $(k = 1,2,\ldots)$. Putting $A_k = f(y_{i_k}) / \| f(y_{i_k}) \|$ $(k = 1,2,\ldots)$, then $\| A_k \| = 1$ $(k = 1,2,\ldots)$. As the set $S = \{A \in \mathbb{K}^d;\ \| A \| = 1\}$ is compact in the topological space $\mathbb{K}^d$, we can take a suitable subsequence $\{A_{k_j}\} \subset \{A_k\}$ which would converge to some element $A_\infty \in S$ when $j \to \infty$.

By the step I, the sequence $\left\{ y_{k_j} \middle/ \| f(y_{k_j}) \| \right\}$ would r-converge to y_∞; $y_\infty = f^{-1}(A_\infty) \neq 0$. On the other hand, the sequence $\{y_{k_j}\}$, which is a subsequence of the sequence $\{y_i\}$ which r-converges to 0, would r-converge to 0 too. Hence, if the r-convergence to 0 of the sequence $\left\{ y_{k_j} \middle/ \| f(y_{k_j}) \| \right\}$ $(k \to \infty)$ would be shown, there arises a contradiction to the above lemma.

To complete the proof, we have to show that for a sequence $\{z_i\}$ in E which r-converges to 0 and a sequence $\{\alpha_i\}$ in $\mathbb{K}$; $|\alpha_i| \geqq 2^i$, the sequence $\{z_i / \alpha_i\}$ in E is r-convergent to 0.

From the hypothesis, there is a $u = \{U_i\} \varepsilon \Phi_0$ satisfying that, for every i, there exists a k, such that $z_n \varepsilon U_i$ where $n \geqq k$. By the condition of r-continuity of the scalar multiplication μ: $\mathbb{K} \times E \to E$, there is a $v = \{V_i\} \varepsilon \Phi_0$ satisfying $\{B(0, \frac{1}{2^i})\} \cdot u < v$. That is, for each $V_i \varepsilon v$, there is a j such that $B(0, \frac{1}{2^j}) \cdot U_j \subset V_i$. Accordingly, for each V_i, we choose a j and for this j, we choose a k. Putting $l = \max(k,j)$, we have, for any $n \geqq l$,

$\frac{1}{\alpha_n} \cdot z_n \varepsilon B(0, \frac{1}{2^j}) \cdot U_j \subset V_i$. This shows r-convergence to 0 of the sequence $\{z_i / \alpha_i\}$.

Example. We take the underlying set of E as $\mathbb{C}$ itself. Let $\mathcal{A}$ be the family of all subsets A of $[0, 2\pi)$ which are non-empty and at most countable, and define $U(A, \eta) = \{re^{\sqrt{-1}a};\ a \varepsilon A,\ 0 \leqq r \leqq \eta\}$ $(A \varepsilon \mathcal{A}, \eta > 0)$. Then, the preneighborhoods and the ranks in E are defined by

$$\mathcal{U}(x) = \{x + U(A, \eta);\ A \varepsilon \mathcal{A},\ \eta > 0\} \quad (x \varepsilon E)$$
$$\mathcal{U}_n = \{x + U(A, \tfrac{1}{2^n});\ A \varepsilon \mathcal{A},\ x \varepsilon E\} \quad (n = 1,2,\dots).$$

Now, one can prove easily that, in the ranked space E, a sequence $\{x_n\}$ is r-convergent to an x if and only if $|x_n - x| \to 0$, i.e., x_n converges to x topologically in $\mathbb{C}$. Hence, E becomes a ranked vector space defined by using continuity in the sense of r-convergent sequences. Nevertheless, the ranked vector space E is not r-isomorphic to the standard one-dimensional ranked vector space $(\mathbb{C}, \{B(0, \frac{1}{2^i})\})$.

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