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CS 536 : Decision Trees

1. For a given value of k, m, (number of features, number of data points), write a function to generate a training data set based on the above scheme.

```
def get_data(k, m, save = False):
    # input args:
    # k: number of features
    # m: number of data points
    X = np.zeros((m, k))
    Y = np.zeros((m, 1))
    w = np.zeros(k)
    denom = 0.9 * 10 * (1 - (0.9 ** (k - 1)))
    for j in range(m):
        s = 0
        if np.random.rand() >= 0.5:
            X[j][0] = 1
        for i in range(1, k):
            if np.random.rand() >= 0.75:
                X[j][i] = X[j][i - 1]
            else:
                X[j][i] = 1 - X[j][i - 1]
            w[i] = (0.9 ** i) / denom
            s += w[i] * X[j][i]
        if s >= 0.5:
            Y[j] = X[j][0]
        else:
            Y[j] = 1 - X[j][0]
    if save == True:
        save_arr(np.concatenate((X, Y), axis = 1))
    return X, Y
```

2. Given a data set, write a function to fit a decision tree to that data based on splitting the variables by maximizing the information gain. Additionally, return the training error of this tree on the data set, $err_{train}(\hat{f})$. *It may be useful to have a function that takes a data set and a variable, and returns the data set partitioned based on the values of that variable.*

```
import numpy as np

eps = 1e-5

class Node:
    x_id = None
    y = None
```

```
        equals_zero = None
        equals_one = None
        def __init__(self, x_id):
            self.x_id = x_id

    class DecisionTree:
        root = None
        def __init__(self):
            self.root = Node(None)

    def entropy(P, P_y0):
        if P < eps or P > 1 - eps or P_y0 < eps or P_y0 > 1 - eps:
            return 0
        return -P * np.log(P) - P_y0 * np.log(P_y0)

    def IC(X, Y):
        m = X.shape[0]
        x_1 = np.count_nonzero(X)
        if x_1 == 0 or x_1 == m:
            return 0
        c_00 = 0
        c_01 = 0
        c_10 = 0
        c_11 = 0
        for x, y in zip(X, Y):
            if x < eps:
                if y < eps:
                    c_00 += 1
                else:
                    c_01 += 1
            else:
                if y < eps:
                    c_10 += 1
                else:
                    c_11 += 1
        P_x = x_1 / m
        P_00 = c_00 / (c_00 + c_01)
        P_01 = c_01 / (c_00 + c_01)
        P_10 = c_10 / (c_10 + c_11)
        P_11 = c_11 / (c_10 + c_11)
        return P_x * entropy(P_11, P_10) + (1 - P_x) * entropy(P_01, P_00)

    def fit_decision_tree(X, Y, node, p_vis):
```

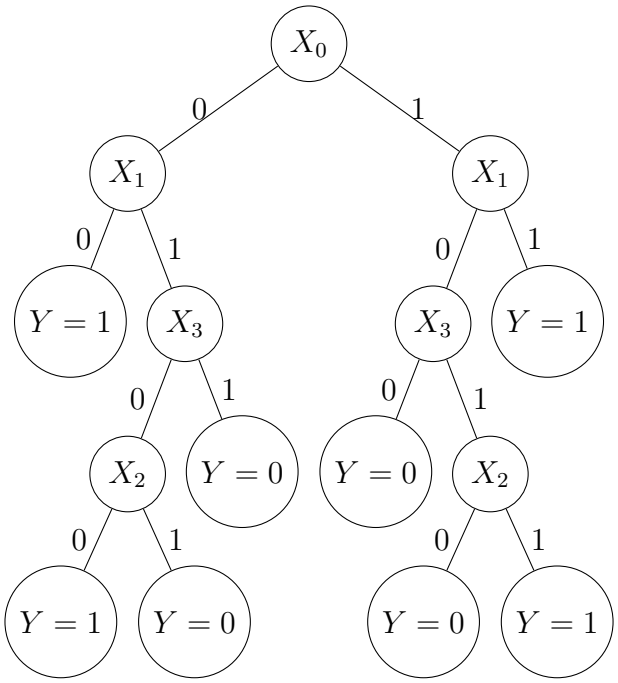
```
    m, k = X.shape
    P_y = np.count_nonzero(Y) / m
    H_y = entropy(P_y, 1 - P_y)
    maxx = 0
    max_id = -1
    vis = np.copy(p_vis)
    for i in range(k):
        if vis[i] == 1:
            continue
        IG = H_y - IC(X[:, i], Y)
        if IG > maxx:
            maxx = IG
            max_id = i
    if max_id != -1:
        vis[max_id] = 1
        new_X_0 = np.copy(X)
        new_X_1 = np.copy(X)
        new_Y_0 = np.copy(Y)
        new_Y_1 = np.copy(Y)
        for i in range(m - 1, -1, -1):
            if new_X_0[i][max_id] > 1 - eps:
                new_X_0 = np.delete(new_X_0, i, axis = 0)
                new_Y_0 = np.delete(new_Y_0, i, axis = 0)
            if new_X_1[i][max_id] < eps:
                new_X_1 = np.delete(new_X_1, i, axis = 0)
                new_Y_1 = np.delete(new_Y_1, i, axis = 0)
        node.x_id = max_id
        if new_X_0.shape[0] > 0:
            node.equals_zero = Node(None)
            fit_decision_tree(new_X_0, new_Y_0, node.equals_zero, vis)
        if new_X_1.shape[0] > 0:
            node.equals_one = Node(None)
            fit_decision_tree(new_X_1, new_Y_1, node.equals_one, vis)
    else:
        if P_y >= 0.5:
            node.y = 1
        else:
            node.y = 0
```

3. For $k = 4$ and $m = 30$, generate data and fit a decision tree to it. Does the ordering of the variables in the decision tree make sense, based on the function that defines Y ? Why or why not? Draw the tree.

   Solution:

The generated data looks like this:

X =					Y =
	[1	0	0	1]	0
	[1	0	1	1]	1
	[1	0	0	1]	0
	[0	1	0	1]	0
	[0	1	0	0]	1
	[1	0	1	0]	0
	[0	1	1	0]	0
	[0	1	0	0]	1
	[0	0	1	0]	1
	[0	0	1	0]	1
	[1	0	1	0]	0
	[0	1	1	0]	0
	[1	1	0	1]	1
	[1	0	1	1]	1
	[1	0	0	1]	0
	[0	1	0	1]	0
	[0	1	0	0]	1
	[0	1	0	1]	0
	[0	0	0	1]	1
	[1	0	1	0]	0
	[0	1	1	0]	0
	[0	1	0	1]	0
	[1	1	0	1]	1
	[1	0	1	0]	0
	[0	1	1	0]	0
	[0	1	1	0]	0
	[1	1	1	0]	1
	[1	1	0	1]	1
	[0	1	0	1]	0
	[0	1	0	1]	0



I think the ordering of variables of this decision tree kind of makes sense. Because the generation of X_i heavily depends on the previous value X_{i-1}, it's intuitive to split the data in the order of X_0, X_1, X_2, X_3. Note that if the data point have more $1s$, the sum of $w_2X_2 + w_3X_3 + \cdots + w_kX_k$ would be relatively large. Thus, the leaves nodes with a path of more $1s$ are likely to have the different value than X_0, and vice versa.

4. Write a function that takes a decision tree and estimates its typical error on this data $err(\hat{f})$; i.e., generate a lot of data according to the above scheme, and find the average error rate of this tree over that data.

```
def predict(node, x):
    if node == None:
        return 1 # No data captured
    if node.y != None:
        return node.y
    if x[node.x_id] < eps:
        return predict(node.equals_zero, x)
    else:
        return predict(node.equals_one, x)

def get_err(tree, X, Y):
    s = 0
    m, k = X.shape
    for i in range(m):
        prediction = predict(tree.root, X[i])
```

```
        if prediction != Y[i]:
            s += 1
    print('Training error is: %f' % (1.0 * s / m))
    return 1.0 * s / m
```

I generated data of $m = 10,000$ and used the decision tree I got in the previous question to predict Y and get the error. I did this for 50 times and then got the average error: 0.046790.

5. For $k = 10$, estimate the value of $|err_{train}(\hat{f}) - err(\hat{f})|$ for a given m by repeatedly generating data sets, fitting trees to those data sets, and estimating the true and training error. Do this for multiple m, and graph this difference as a function of m. What can you say about the marginal value of additional training data?

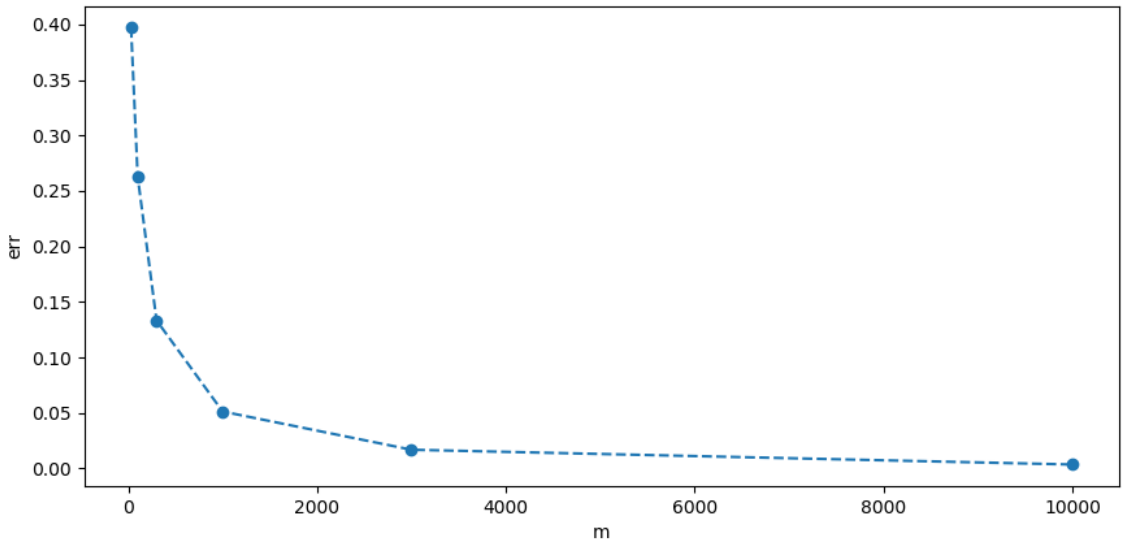


Figure 1: Differnt errors versus different ms

I did this experiment for $m = 30, 100, 300, 1000, 3000, 10000$. For every m, I generated 10 different training sets and fitted decision trees on these training sets, tested these decision trees on 10 different test sets of $m = 10,000$ and got the average errors. (100 tests for every m)

The average errors are: $0.396951, 0.262908, 0.133341, 0.051315, 0.016886, 0.003579$, respectively.

Note that $2^k = 2^{10} = 1024$, the larger the m is, the more likely we have got all possible combinations of X.

From the 5 we can see, the line tend to level out after $m \geq 3000$, thus I guess the marginal value of m is around 3000.

6. Design an alternative metric for splitting the data, not based on information content / information gain. Repeat the computation from (5) above for your metric, and compare the performance of your trees vs the ID3 trees.

   Solution:

   I designed a pretty greedy yet simple metric: find the X that agrees on Y the most, i.e.

$$\arg\max_i \max(\sum_{j=1}^{m} \mathbb{1}X_i^j = Y^j, \sum_{j=1}^{m} \mathbb{1}X_i^j \neq Y^j)$$

   The results are actually pretty good as 6 shows.

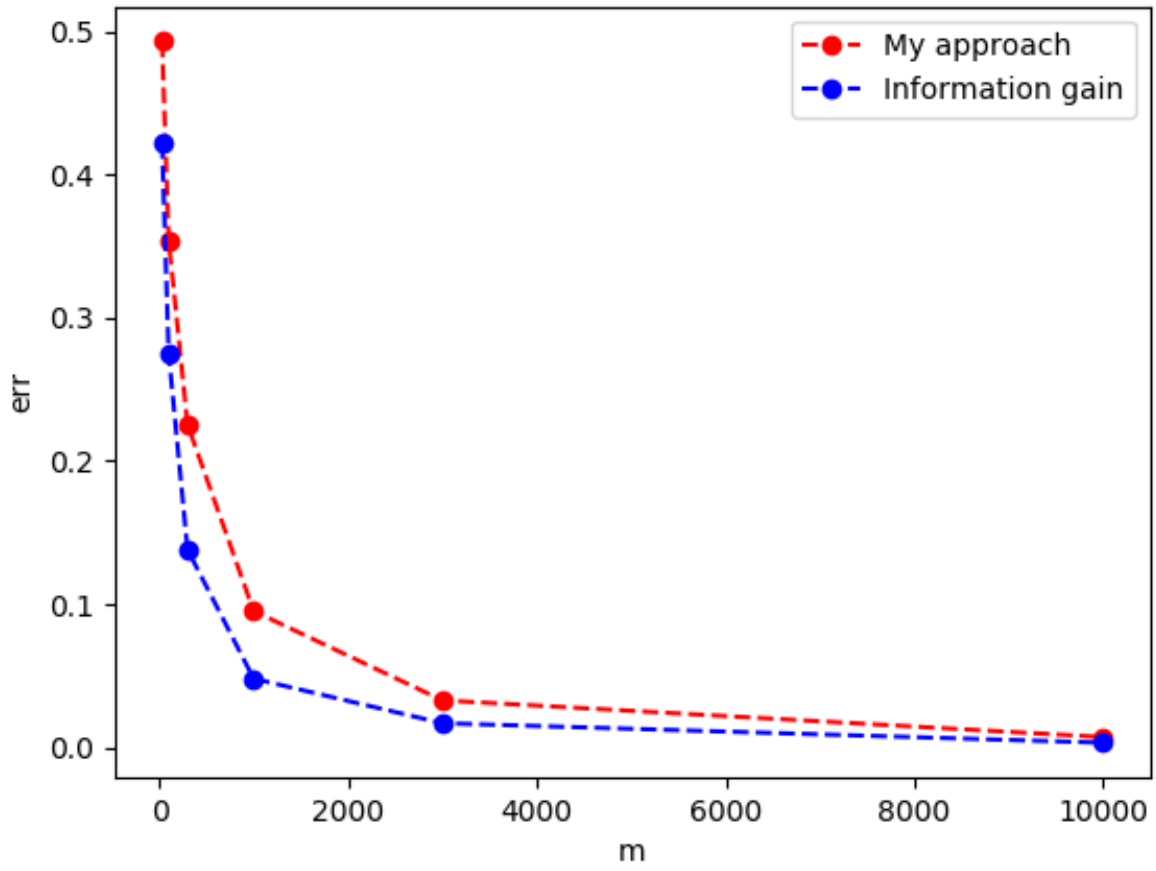


Figure 2: Comparison