Shengjie Li
RUID: 188008047

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CS 536 : Estimation Problems

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Uniform Estimators

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables, uniformly distributed on $[0, L]$ (i.e., with density $1/L$ on this interval). In the posted notes on estimation, it is shown that the method of moments and maximum likelihood estimators for L are given by

$$\hat{L}_{\text{MOM}} = 2\overline{X}_n$$
$$\hat{L}_{\text{MLE}} = \max_{i=1,\ldots,n} X_i. \tag{1}$$

We want to consider the question of which estimator is better. Recall the definition of the mean squared error of an estimator as

$$\text{MSE}(\hat{L}) = \mathbb{E}[(\hat{L} - L)^2]. \tag{2}$$

Note: the answers to homework zero may also be useful here.

1) Show that in general, $\text{MSE}(\hat{\theta}) = \text{bias}(\hat{\theta})^2 + \text{var}(\hat{\theta})$, where var is the variance, and bias is given by

$$\text{bias}(\hat{\theta}) = \theta - \mathbb{E}[\hat{\theta}]. \tag{3}$$

Solution:

$$
\begin{aligned}
\text{MSE}(\hat{\theta}) &= \mathbb{E}[(\hat{\theta} - \theta)^2] \\
&= \mathbb{E}[\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2] \\
&= \mathbb{E}[\hat{\theta}^2] - 2\theta\mathbb{E}[\hat{\theta}] + \theta^2 \\
&= \mathbb{E}[\hat{\theta}^2] - 2\theta\mathbb{E}[\hat{\theta}] + \theta^2 + \mathbb{E}[\hat{\theta}]^2 - \mathbb{E}[\hat{\theta}]^2 \\
&= (\theta - \mathbb{E}[\hat{\theta}])^2 + (\mathbb{E}[\hat{\theta}^2] - \mathbb{E}[\hat{\theta}]^2) \\
&= \text{bias}(\hat{\theta})^2 + \text{var}(\hat{\theta})
\end{aligned}
$$

2) Show that $\hat{L}_{\text{MOM}}$ is *unbiased*, but that $\hat{L}_{\text{MLE}}$ has bias. In general, $\hat{L}_{\text{MLE}}$ consistently underestimates L - why?

Solution:

$$\begin{aligned}
\mathrm{bias}(\hat{L}_{\mathrm{MOM}}) &= L - \mathbb{E}[\hat{L}_{\mathrm{MOM}}]\\
&= L - \mathbb{E}[2\overline{X}_n]\\
&= L - 2\mathbb{E}[\overline{X}_n]\\
&= L - 2\mathbb{E}[\frac{1}{n}\sum_{i=1}^{n}X_i]\\
&= L - \frac{2}{n}\sum_{i=1}^{n}\mathbb{E}[X_i]\\
&= L - \frac{2}{n}n\mathbb{E}[X]\\
&= L - 2\mathbb{E}[X]\\
&= 0
\end{aligned}$$

Thus $\hat{L}_{\mathrm{MOM}}$ is unbiased.

The cdf of MLE:

$$F(x) = \mathbb{P}(\max_{i=1,\ldots,n} X_i \le x) = \mathbb{P}(X_1 \le x, \ldots, X_n \le x) = \mathbb{P}(X_1 \le x)\ldots\mathbb{P}(X_n \le x) = (\frac{x}{L})^n.$$

Thus, the pdf of MLE is:

$$f(x) = F'(x) = \frac{n \cdot x^{n-1}}{L^n}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}[\hat{L}_{\mathrm{MLE}}] &= \int_{-\infty}^{\infty} x f(x)dx\\
&= \int_{0}^{L} x\frac{n \cdot x^{n-1}}{L^n}dx\\
&= \int_{0}^{L} n(\frac{x}{L})^n dx\\
&= \frac{n}{n+1}\cdot\frac{x^{n+1}}{L^n}\Big|_0^L\\
&= \frac{n}{n+1}\cdot\frac{L^{n+1}}{L^n}\\
&= \frac{n}{n+1}L.
\end{aligned}$$

We have

$$\mathrm{bias}(\hat{L}_{\mathrm{MLE}}) = L - \mathbb{E}[\hat{L}_{\mathrm{MLE}}] = L - \frac{n}{n+1}L = \frac{1}{n+1}L \neq 0.$$

Thus, $\hat{L}_{\text{MLE}}$ has bias.

$$\mathbb{P}(\hat{L}_{\text{MLE}} \geq L) = \mathbb{P}(\max_{i=1,\ldots,n} X_i \geq L) = 0.$$

Thus, $\hat{L}_{\text{MLE}}$ consistently underestimates L.

3) Compute the variance of $\hat{L}_{\text{MOM}}$ and $\hat{L}_{\text{MLE}}$.

Solution:

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}[(X - \frac{X}{2})^2] = \frac{1}{4}\mathbb{E}[X^2] = \frac{1}{4}\int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{4}\int_0^L x^2 \frac{1}{L}dx$$

$$= \frac{1}{4}\frac{1}{3}\frac{1}{L}x^3 \Big|_0^L = \frac{1}{12L}L^3 = \frac{L^2}{12}$$

$$\text{Var}(\hat{L}_{\text{MOM}}) = \text{Var}(2\overline{X}_n) = 4\text{Var}(\overline{X}_n) = 4\text{Var}(\frac{1}{n}\sum_{i=1}^n X_i) = \frac{4}{n^2}\sum_{i=1}^n \text{Var}(X_i) = \frac{4}{n^2}n\text{Var}(X) = \frac{L^2}{3n}$$

$$\text{Var}(\hat{L}_{\text{MLE}}) = \mathbb{E}[\hat{L}_{\text{MLE}}^2] - \mathbb{E}[\hat{L}_{\text{MLE}}]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - (\frac{n}{n+1}L)^2$$

$$= \int_0^L x^2 \frac{n \cdot x^{n-1}}{L^n}dx - \frac{n^2 L^2}{(n+1)^2}$$

$$= n\int_0^L \frac{x^{n+1}}{L^n}dx - \frac{n^2 L^2}{(n+1)^2}$$

$$= \frac{n}{n+2}\frac{x^{n+2}}{L^n}\Big|_0^L - \frac{n^2 L^2}{(n+1)^2}$$

$$= \frac{nL^2}{n+2} - \frac{n^2 L^2}{(n+1)^2}$$

$$= \frac{nL^2((n+1)^2 - n(n+2))}{(n+2)(n+1)^2}$$

$$= \frac{nL^2}{(n+2)(n+1)^2}$$

4) Which one is the better estimator, i.e., which one has the smaller mean squared error?

$$\text{MSE}(\hat{L}_{\text{MOM}}) = \text{bias}(\hat{L}_{\text{MOM}})^2 + \text{var}(\hat{L}_{\text{MOM}}) = \frac{L^2}{3n}$$

$$\text{MSE}(\hat{L}_{\text{MLE}}) = \text{bias}(\hat{L}_{\text{MLE}})^2 + \text{var}(\hat{L}_{\text{MLE}}) = (\frac{1}{n+1}L)^2 + \frac{nL^2}{(n+2)(n+1)^2} = \frac{2L^2}{(n+2)(n+1)}$$

5) Experimentally verify your computations in the following way: Taking $n = 100$ and $L = 10$,

- For $j = 1, \ldots, 1000$:

- Simulate $X_1^j, \ldots, X_n^j$ and compute values for $\hat{L}_{\mathrm{MOM}}^j$ and $\hat{L}_{\mathrm{MLE}}^j$

- For $n = 100, L = 10$, simulate $X_1, \ldots, X_n$, and compute values for $\hat{L}_{\mathrm{MOM}}$ and $\hat{L}_{\mathrm{MLE}}$.

- Estimate the mean squared error for each population of estimator values.

- How do these estimated MSEs compare to your theoretical MSEs?

6) You should have shown that $\hat{L}_{\mathrm{MLE}}$, while biased, has a smaller error over all. Why? The mathematical justification for it is above, but is there an explanation for this?

7) Find $\mathbb{P}(\hat{L}_{\mathrm{MLE}} < L - \epsilon)$ as a function of L, ϵ, n. Estimate how many samples I would need to be sure that my estimate was within ϵ with probability at least δ.

8) Show that

$$\hat{L} = \left(\frac{n+1}{n}\right) \max_{i=1,\ldots,n} X_i, \tag{4}$$

is an unbiased estimator, and has a smaller MSE still.