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Midterm Exam

Problem 1: Let x_0 be deterministic and $x_1, \ldots, x_N$ denote random variables satisfying (an autoregressive model of order 1)

$$x_n = \alpha x_{n-1} + w_n, \quad n = 1, ..., N,$$

where $w_1, ..., w_N$ are independent and identically distributed Gaussian random variables with mean 0 and variance 1 while α denotes an unknown parameter.

a) Find the joint density of $x_1, ..., x_N$ given α (remember x_0 is deterministic).

b) Compute the maximum likelihood estimate of α when you are given x_0 and a realization of $x_1, ..., x_N$.

Problem 2: Let $x_n, n = 1, ..., N$ be random variables and consider the two scenarios:

$$\mathsf{H}_0 : x_n = -s\alpha_n + w_n,$$
$$\mathsf{H}_1 : x_n = s\alpha_n + w_n,$$

where w_n are independent and identically distributed Gaussian random variables with mean 0 and variance σ_2 where σ_2 is unknown, $\alpha_1, ..., \alpha_N$ are deterministic and known and, finally $s > 0$ is a deterministic and unknown parameter. If the prior probabilities are $P(\mathsf{H}_0) = P(\mathsf{H}_1) = 0.5$

a) Find the optimum decision mechanism that decides between the two scenarios and minimizes the probability of making an error. Start by assuming that all unknown parameters are magically known.

   Let D be the actual outcome, C be the cost, then we have:

$$\{D_0, H_0\} \text{ with cost } C_{00},$$
$$\{D_0, H_1\} \text{ with cost } C_{01},$$
$$\{D_1, H_0\} \text{ with cost } C_{10},$$
$$\{D_1, H_1\} \text{ with cost } C_{11}.$$

   Then the average error(cost) will be:

$$C(\delta_0, \delta_1) = C_{00}\mathbb{P}(D_0\&_0) + C_{01}\mathbb{P}(D_0\&_1) + C_{10}\mathbb{P}(D_1\&_0) + C_{11}\mathbb{P}(D_1\&_1).$$

   Let $C_{00} = C_{11} = 0$, $C_{01} = C_{10} = 1$, then

$$\begin{aligned} C(\delta_0, \delta_1) &= \mathbb{P}(D_0\&H_1) + \mathbb{P}(D_1\&H_0) \\ &= \int \delta_0(X) f_1(X) dX \cdot \mathbb{P}(H_1) + \int \delta_1(X) f_0(X) dX \cdot \mathbb{P}(H_0). \end{aligned}$$

By definition, we can get:

$$F_0(x) = \prod_{i=1}^{N} P(X_i < x) = \prod_{i=1}^{N} P(-s\alpha_i + \omega_i < x) = \prod_{i=1}^{N} P(\omega_i < x + s\alpha_i) = \prod_{i=1}^{N} F_{w_i}(x + s\alpha_i),$$

$$f_0(x) = \prod_{i=1}^{N} f_{w_i}(x + s\alpha_i).$$

$$F_1(x) = \prod_{i=1}^{N} P(X_i < x) = \prod_{i=1}^{N} P(s\alpha_i + \omega_i < x) = \prod_{i=1}^{N} P(\omega_i < x - s\alpha_i) = \prod_{i=1}^{N} F_{w_i}(x - s\alpha_i),$$

$$f_1(x) = \prod_{i=1}^{N} f_{w_i}(x - s\alpha_i).$$

We want to minimize the cost,

$$\arg\min_{\delta_0,\delta_1} C(\delta_0, \delta_1) = \arg\min_{\delta_0,\delta_1} \int (\delta_0(X) f_1(X) dX \cdot \mathbb{P}(H_1) + \delta_1(X) f_0(X) dX \cdot \mathbb{P}(H_0)).$$

If $\frac{f_1(x)}{f_0(x)} \geq \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}$, we take H_1; If $\frac{f_1(x)}{f_0(x)} < \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}$, we choose H_0.

That is, if $\frac{f_w(x-s\alpha_i)}{f_w(x+s\alpha_i)} \geq 1$, we choose H_1.

$$\frac{f_w(x - s\alpha_i)}{f_w(x + s\alpha_i)} = \prod_{i=1}^{N} \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-s\alpha_i)^2}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x+s\alpha_i)^2}} = \prod_{i=1}^{N} e^{\frac{4xs\alpha_i}{2\sigma^2}}$$

The optimum decision mechanism is, if $\prod_{i=1}^{N} e^{\frac{4xs\alpha_i}{2\sigma^2}} \geq 1$, then choose H_1, otherwise choose H_0

b) The decision mechanism you found in a) depends on the unknown parameters s and σ_2 . Apply suitable transformations to find an equivalent mechanism (by taking for example the logarithm and removing unnecessary terms) which does not depend on these two unknown parameters.

c) Explain what are the optimality properties of the mechanism you ended up with.

Problem 3: Consider a random vector X for which we have three possible scenarios

$$\mathsf{H}_0 : X \sim f_0(X),$$
$$\mathsf{H}_1 : X \sim f_1(X),$$
$$\mathsf{H}_2 : X \sim f_2(X),$$

with all the prior probabilities assumed equal. Find the optimum decision mechanism that minimizes the probability of making an error. Consider now the two likelihood ratios $\mathsf{L}_1 = \frac{f_1(X)}{f_0(X)}$ and $\mathsf{L}_2 = \frac{f_2(X)}{f_0(X)}$. For every realization X you can compute the two likelihood ratios which are in fact all you need to make your decision.

a) In the 2D space with axes $\mathsf{L}_1, \mathsf{L}_2$ identify the regions for which you decide in favor of each of the three scenarios $\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2$.

b) What happens at the boundaries between two regions? What happens at the single point which belongs to the boundary of all three regions?

Problem 4: As discussed in the class the space of all random variables constitutes a vector space. We can also define an inner product (also mentioned in class) between two random vectors x, y

$$< x, y >= E[xy].$$

Consider now the following random variables x, z, w. We are interested in linear combinations of the form $\hat{x} = az + bw$ where a, b are real deterministic quantities.

a) By using the orthogonality principle find the $\hat{x}_*$ (equivalently the optimum coefficients a_*, b_*) that is closest to x in the sense of the norm induced by the inner product.

b) Compute the optimum (minimum) distance and its optimum approximation $\hat{x}_*$ in terms of $E[xz], E[xw], E[z^2], E[zw], E[w^2]$.

c) Explain what is the physical meaning of this approximation.