

Midterm Exam

Problem 1: Let x_0 be deterministic and x_1, \dots, x_N denote random variables satisfying (an autoregressive model of order 1) $x_n = \alpha x_{n-1} + w_n$, $n = 1, \dots, N$, where w_1, \dots, w_N are independent and identically distributed Gaussian random variables with mean 0 and variance 1 while α denotes an unknown parameter.

- Find the joint density of x_1, \dots, x_N given α (remember x_0 is deterministic).
- Compute the maximum likelihood estimate of α when you are given x_0 and a realization of x_1, \dots, x_N .

Problem 2: Let $x_n, n = 1, \dots, N$ be random variables and consider the two scenarios:

$$\begin{aligned} H_0 : x_n &= s\alpha_i + w_n, \\ H_1 : x &= s\alpha_i + w_n, \end{aligned}$$

where w_n are independent and identically distributed Gaussian random variables with mean 0 and variance σ_2 where σ_2 is unknown, $\alpha_1, \dots, \alpha_N$ are deterministic and known and, finally $s > 0$ is a deterministic and unknown parameter. If the prior probabilities are $P(H_0) = P(H_1) = 0.5$

- Find the optimum decision mechanism that decides between the two scenarios and minimizes the probability of making an error. Start by assuming that all unknown parameters are magically known.
- The decision mechanism you found in a) depends on the unknown parameters s and σ_2 . Apply suitable transformations to find an equivalent mechanism (by taking for example the logarithm and removing unnecessary terms) which does not depend on these two unknown parameters.
- Explain what are the optimality properties of the mechanism you ended up with.

Problem 3: Consider a random vector X for which we have three possible scenarios

$$\begin{aligned} H_0 : X &\approx f_0(X), \\ H_1 : X &\approx f_1(X), \\ H_2 : X &\approx f_2(X), \end{aligned}$$

with all the prior probabilities assumed equal. Find the optimum decision mechanism that minimizes the probability of making an error. Consider now the two likelihood ratios $L_1 = \frac{f_1(X)}{f_0(X)}$ and $L_2 = \frac{f_2(X)}{f_0(X)}$. For every realization X you can compute the two likelihood ratios which are in fact all you need to make your decision.

- In the 2D space with axes L_1, L_2 identify the regions for which you decide in favor of each of the three scenarios H_0, H_1, H_2 .
- What happens at the boundaries between two regions? What happens at the single point which belongs to the boundary of all three regions?

Problem 4: As discussed in the class the space of all random variables constitutes a vector space. We can also define an inner product (also mentioned in class) between two random vectors x, y

$$\langle x, y \rangle = E[xy].$$

Consider now the following random variables x, z, w . We are interested in linear combinations of the form $\hat{x} = az + bw$ where a, b are real deterministic quantities.

- a) By using the orthogonality principle find the \hat{x}_* (equivalently the optimum coefficients a_*, b_*) that is closest to x in the sense of the norm induced by the inner product.
- b) Compute the optimum (minimum) distance and its optimum approximation \hat{x}_* in terms of $E[xz], E[xw], E[z^2], E[zw], E[w^2]$.
- c) Explain what is the physical meaning of this approximation.