Homework #1

Foundations of Computer and Data Science CS-596

Problem 1: Consider the matrix

$$A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1+\epsilon \end{bmatrix}.$$

a) Find the eigenvalues/eigenvectors of A assuming $\epsilon \neq 0$. Force your eigenvectors to have unit norm. b) Diagonalize A using the eigenvalues/eigenvectors you computed. c) Start now sending $\epsilon \to 0$. What do you observe is happening to the matrices you use for diagonalization as ϵ becomes smaller and smaller? So what do you conclude when $\epsilon = 0$?

Problem 2: Let A, B be two matrices of the same dimensions $k \times m$. a) With direct computation show that $\text{trace}(AB^T) = \text{trace}(B^T A) = \text{trace}(BA^T) = \text{trace}(A^T B)$. b) Use question a) to compute $\mathsf{E}[\mathbf{x}^T A \mathbf{x}]$ where $\mathsf{E}[\cdot]$ denotes expectation, A is a constant matrix and $\mathbf{x}$ is a random vector for which we know that $\mathsf{E}[\mathbf{x}\mathbf{x}^T] = Q$. *Hint: The trace of a scalar is the scalar itself.* c) Using the previous properties show that for any matrix A of dimensions $k \times k$ we have $\text{trace}(A) = \text{trace}(UAU^{-1})$ for any nonsingular matrix U of dimensions $k \times k$. In other words that the trace does not change if we apply a similarity transformation. d) Use question c) to prove that if matrix A of dimensions $k \times k$ is diagonalizable then its trace is equal to the sum of its eigenvalues (actually this is true even if the matrix is not diagonalizable). e) Regarding question d) how do you explain this equality given that when A is real the trace is also real whereas the eigenvalues can be complex? f) Using again question d) what can you say about the coefficient c_{k-1} of the characteristic polynomial $\lambda^k + c_{k-1}\lambda^{k-1} + \cdots + c_0$ of A. We recall that we already know that $c_0 = (-1)^k \lambda_1 \cdots \lambda_k = (-1)^k \det(A)$.

Problem 3: A matrix A is called *nilpotent* if $A^r = 0$ for some integer $r > 1$. a) Show that all eigenvalues of A must be equal to 0. b) Is such a matrix diagonalizable? c) Give an example of a 2×2 matrix which is nilpotent. c) If we apply a similarity transformation to a nilpotent is the resulting matrix nilpotent? Apply the previous observation to your example in c) to obtain a second example of a nilpotent matrix.

Problem 4: Let Q be symmetric and positive definite with dimensions $k \times k$. Denote with Q_i the matrix we obtain from Q by eliminating its ith column and ith row. a) Show that Q_i is also symmetric and positive definite. Extend this to when we eliminate more than one columns and rows. b) Show that the largest eigenvalue of Q is larger than the largest eigenvalue of Q_i and, that the smallest eigenvalue of Q is smaller than the smallest eigenvalue of Q_i. c) If A is a square matrix (not necessarily symmetric and positive definite) of dimensions $k \times k$ and $\lambda_1, \ldots, \lambda_k$ are its eigenvalues while $\sigma_1, \ldots, \sigma_k$ its singular values (obtained by applying SVD) then show

$$\min_i \sigma_i \leq \min_i |\lambda_i| \leq \max_i |\lambda_i| \leq \max_i \sigma_i.$$

In other words the eigenvalues are located on the complex plane inside an annulus with the two circle radii defined by the largest and smallest singular value. *Use the result we proved in class regarding the bounds of the ratio* $\frac{\mathbf{x}^T C \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$ *for* C *symmetric. Consider that this is true even if* $\mathbf{x}$ *is a vector with complex elements.*

We will meet on Friday, October 12, at 6:00PM, in Core 101, to discuss the problems
(after our regular class!!!)

Your answers, in *hard copy*, must be submitted: Monday, October 15, in CBIM-05, 5:00PM - 6:00PM, to Mr Neelesh Kumar. Our TA, will be happy to collect them.

Please respect the indicated time because Neelesh has many other tasks to occupy him, besides TAing.