

Homework 2

Problem 1: Let $\mathcal{X}_1, \mathcal{X}_2$ be two jointly Gaussian vectors with means μ_1, μ_2 covariance matrices Σ_{11}, Σ_{22} and cross covariance matrix $\Sigma_{12} = \mathbb{E}[(\mathcal{X}_1 - \mu_1)(\mathcal{X}_2 - \mu_2)^t]$. By computing the conditional probability density prove that \mathcal{X}_1 given \mathcal{X}_2 continuous to be Gaussian with mean that depends on \mathcal{X}_2 but with a covariance matrix which is independent of \mathcal{X}_2 .

Problem 2: Consider a Bernoulli random variable χ that takes the value a_1 with probability p and the value $a_2 (a_2 \neq a_1)$ with probability $1p$.

- a) Compute the the average and the variance of χ .
- b) Suppose now that you generate N independent realizations of χ . Propose a way to estimate $p = \mathbb{P}(\chi = a_1)$.
- c) Compute the mean and variance of your estimate. What can you conclude from this computation when you consider $N \rightarrow \infty$?

Problem 3: \mathcal{X} is a random vector and there are K different possibilities that can generate realizations of this vector. Let $f_1(X), \dots, f_K(X)$ the corresponding pdfs and p_1, \dots, p_K the corresponding prior probabilities that each case can occur of each possibility (with $p_1 + \dots + p_K = 1$). Using total probability and the trick that relates a pdf to the probability of a differential event, show that the pdf $f(X)$ of \mathcal{X} satisfies

$$f(X) = p_1 f_1(X) + \dots + p_K f_K(X).$$

Let now χ_1, χ_2 be two random variables which 99% of the time are independent and Normally (Gaussian) distributed, both with mean 0 and variance 1 and 1% of the time they are independent and Normally distributed both with mean 0 and variance $\sigma_2 \neq 1$.

- a) Compute the joint pdf of the two random variables.
- b) Examine if the two random variables are *independent*.
- c) Give an example of two random variables that are *uncorrelated* but not independent.

Problem 4: Let χ, ζ be random variables that are related through the equality

$$\zeta = |\chi + s|.$$

- a) If the pdf of χ is $f_\chi(x)$ compute the pdf of ζ when s is a deterministic quantity.
- b) Repeat the previous question when s is a random variable independent from χ and takes only the two values 0 and 1 with probabilities 0.2 and 0.8 respectively.
- c) Under the assumptions of question b) compute the posterior probability $\mathbb{P}(s = 0 | \zeta = z)$. *Hint: For the computation of the pdf of a random variable the simplest way is to start with the computation of the cdf and then take the derivative. For b) use total probability.*

Problem 5: Consider the space of all scalar random variables.

- a) Show that this is a vector space by defining properly the operation of addition and multiplication.
- b) For any two random variables χ, ψ we define the mapping $\langle \chi, \psi \rangle = \mathbb{E}[\chi\psi]$. Show that this mapping is an inner product in our vector space.
- c) What particular form do you obtain when you apply the general Schwarz inequality?
- d) How would you extend the previous definitions if you want a vector space comprised of *random vectors* of length d ? Define properly the inner product and find the new form of the Schwartz inequality.