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Homework 1

Problem 1: Consider the matrix

$$A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1+\epsilon \end{bmatrix}.$$

(a) Find the eigenvalues/eigenvectors of A assuming $\epsilon \neq 0$. Force your eigenvectors to have unit norm.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0.5 \\ 0 & 1+\epsilon-\lambda \end{bmatrix}$$
$$\det(A - \lambda I) = (1-\lambda) * (1+\epsilon-\lambda) - 0.5 * 0$$
$$= (1-\lambda) * (1+\epsilon-\lambda)$$

Let $\det(A - \lambda I) = 0$ we can get eigenvalues $\lambda_1 = 1, \lambda_2 = 1+\epsilon$.
For $\lambda = 1$, solve $A\vec{x} = \lambda\vec{x}$:

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1+\epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\downarrow$$
$$x_1 + 0.5x_2 = x_1$$
$$(1+\epsilon)x_2 = x_2$$
$$\downarrow$$
$$x_1 = x_1 (x_1 \neq 0)$$
$$x_2 = 0$$

Therefore, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a eigenvector of A associated with the eigenvalue $\lambda = 1$.

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1+\epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (1+\epsilon) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\downarrow$$
$$x_1 + 0.5x_2 = (1+\epsilon)x_1$$
$$(1+\epsilon)x_2 = (1+\epsilon)x_2$$
$$\downarrow$$
$$x_1 = \frac{x_2}{2\epsilon}$$
$$x_2 = x_2$$

Therefore, $\begin{bmatrix} \frac{1}{\sqrt{1+4\epsilon^2}} \\ \frac{2\epsilon}{\sqrt{1+4\epsilon^2}} \end{bmatrix}$ is a eigenvector of A associated with the eigenvalue $\lambda = 1+\epsilon$.

(b) Diagonalize A using the eigenvalues/eigenvectors you computed.

Let T be the matrix with eigenvectors as its columns.

$$T = \begin{bmatrix} 1 & \frac{1}{\sqrt{1+4\epsilon^2}} \\ 0 & \frac{2\epsilon}{\sqrt{1+4\epsilon^2}} \end{bmatrix}$$

$$T^{-1} = \frac{1}{\frac{2\epsilon}{\sqrt{1+4\epsilon^2}}} \begin{bmatrix} \frac{2\epsilon}{\sqrt{1+4\epsilon^2}} & -\frac{1}{\sqrt{1+4\epsilon^2}} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2\epsilon} \\ 0 & \frac{\sqrt{1+4\epsilon^2}}{2\epsilon} \end{bmatrix}$$

$$T^{-1}AT = \begin{bmatrix} 1 & -\frac{1}{2\epsilon} \\ 0 & \frac{\sqrt{1+4\epsilon^2}}{2\epsilon} \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 1+\epsilon \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{\sqrt{1+4\epsilon^2}} \\ 0 & \frac{2\epsilon}{\sqrt{1+4\epsilon^2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2\epsilon} \\ 0 & \frac{(1+\epsilon)\sqrt{1+4\epsilon^2}}{2\epsilon} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{\sqrt{1+4\epsilon^2}} \\ 0 & \frac{2\epsilon}{\sqrt{1+4\epsilon^2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1+\epsilon \end{bmatrix}$$

(c) Start now sending $\epsilon \to 0$. What do you observe is happening to the matrices you use for diagonalization as ϵ becomes smaller and smaller? So what do you conclude when $\epsilon = 0$?

When ϵ is becoming closer to 0, the determinant of matrix T is becoming closer to 0.
Thus, when $\epsilon = 0$, matrix T will become non-invertable. Matrix A will become non-diagonalizable.