Midterm Exam

Foundations of Computer and Data Science CS-596

Problem 1: Let x_0 be deterministic and $x_1, \ldots, x_N$ denote random variables satisfying (an autoregressive model of order 1)

$$x_n = \alpha x_{n-1} + w_n, \quad n = 1, \ldots, N,$$

where $w_1, \ldots, w_N$ are independent and identically distributed Gaussian random variables with mean 0 and variance 1 while α denotes an unknown parameter. a) Find the joint density of $x_1, \ldots, x_N$ given α (remember x_0 is deterministic). b) Compute the maximum likelihood estimate of α when you are given x_0 and a realization of $x_1, \ldots, x_N$.

Problem 2: Let $x_n, n = 1, \ldots, N$ be random variables and consider the two scenarios:

$\mathsf{H}_0 : x_n = -s\alpha_i + w_n,$
$\mathsf{H}_1 : x = s\alpha_i + w_n,$

where w_n are independent and identically distributed Gaussian random variables with mean 0 and variance σ^2 where σ^2 is unknown, $\alpha_1, \ldots, \alpha_N$ are deterministic and known and, finally $s > 0$ is a deterministic and unknown parameter. If the prior probabilities are $\mathbb{P}(\mathsf{H}_0) = \mathbb{P}(\mathsf{H}_1) = 0.5$ a) Find the optimum decision mechanism that decides between the two scenarios and minimizes the probability of making an error. Start by assuming that all unknown parameters are magically known. b) The decision mechanism you found in a) depends on the unknown parameters s and σ^2. Apply suitable transformations to find an **equivalent** mechanism (by taking for example the logarithm and removing unnecessary terms) which **does not depend on these two unknown parameters**. c) Explain what are the optimality properties of the mechanism you ended up with.

Problem 3: Consider a random vector X for which we have three possible scenarios

$\mathsf{H}_0 : X \sim f_0(X),$
$\mathsf{H}_1 : X \sim f_1(X),$
$\mathsf{H}_0 : X \sim f_2(X),$

with all the prior probabilities assumed equal. Find the optimum decision mechanism that minimizes the probability of making an error. Consider now the two likelihood ratios $\mathsf{L}_1 = \frac{f_1(X)}{f_0(X)}$ and $\mathsf{L}_2 = \frac{f_2(X)}{f_0(X)}$. For every realization X you can compute the two likelihood ratios which are in fact all you need to make your decision. a) In the 2D space with axes $\mathsf{L}_1, \mathsf{L}_2$ identify the regions for which you decide in favor of each of the three scenarios $\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2$. b) What happens at the boundaries between two regions? What happens at the single point which belongs to the boundary of all three regions? *Hint: We know what the optimum decision mechanism is. Find its specific form for the given priors and propose an equivalent decision mechanism that involves comparisons between the pdfs and therefore needs only the two likelihood ratios.*

Problem 4: As discussed in the class the space of all random variables constitutes a vector space. We can also define an inner product (also mentioned in class) between two random vectors x, y

$$<x, y> = \mathbb{E}[xy].$$

Consider now the following random variables x, z, w. We are interested in linear combinations of the form $\hat{x} = az + bw$ where a, b are real deterministic quantities. a) By using the orthogonality principle find the $\hat{x}_*$ (equivalently the optimum coefficients a_*, b_*) that is closest to x in the sense of the norm induced by the inner product. b) Compute the optimum (minimum) distance and its optimum approximation $\hat{x}_*$ in terms of $\mathbb{E}[xz], \mathbb{E}[xw], \mathbb{E}[z^2], \mathbb{E}[zw], \mathbb{E}[w^2]$. c) Explain what is the physical meaning of this approximation.

Your reports, in *hard copy*, must be submitted to Mr. Kumar tomorrow, Friday, November 9, between 5:00-6:00PM, Room: CBIM-05. (NOT SOONER and NOT LATER!!!)

There will be no meeting in Hill 101 and you are not allowed to ask me or the TA for any help.