HW 10

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Exercise 10.2.5

Let $f : [a, b] \to \mathbb{R}$ with $a < b$ be continuous and differentiable on (a, b). Let $g : [a, b] \to \mathbb{R}$ be defined by

$$g(x) = \frac{f(b) - f(a)}{b - a}(x - a).$$

It follows that $(f - g)(a) = (f - g)(b) = f(a)$ because

$$f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = f(a) - 0 = f(a),$$

and

$$f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - (f(b) - f(a)) = f(a).$$

Thus from Rolle's theorem, there exists $d \in (a, b)$ such that $(f - g)'(d) = 0$. $g'(d)$ can be found from the limit definition of the derivative:

$$\begin{aligned} g'(x) &= \lim_{x \to d} \frac{\frac{f(b)-f(a)}{d-a}(d - a) - \frac{f(b)-f(a)}{b-a}(x - a)}{d - x} \\ &= \lim_{x \to d} \frac{\frac{f(b)-f(a)}{d-a}(d - x)}{d - x} \\ &= \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

It follows from the difference rule that

$$(f - g)'(x) = f'(d) - g'(d) = 0,$$

so

$$\begin{aligned} f'(d) &= g'(d) \\ &= \frac{f(b) - f(a)}{b - a}, \end{aligned}$$

proving the mean value theorem.

Exercise 10.2.6

Suppose to the contrary that $|f(x) - f(y)| > M|x - y|$ for some $x, y \in [a, b]$. Wlog assuming $y > x$, it follows from MVT for derivatives that for some $c \in (x, y)$,

$$f|'_{[x,y]}(c) = \frac{f(x) - f(y)}{x - y}.$$

Thus

$$\begin{aligned} \left| f|'_{[x,y]}(c) \right| &= \left| \frac{f(x) - f(y)}{x - y} \right| \\ &= \frac{|f(x) - f(y)|}{|x - y|} \\ &> \frac{M|x - y|}{|x - y|} \\ &= M. \end{aligned}$$

The limit definition of the derivative yields the same result as $f|'_{[x,y]}$ for f' at d because $[x, y]$ is connected. Thus f' is not bounded by M, a contradiction.

Exercise 10.2.7

It follows from the previous exercise that f is Lipschitz continuous. Therefore

$$|f(x) - f(y)| \leq M|x - y|.$$

If $\epsilon > 0$ and $0 < \delta < \epsilon/M$, it follows that

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in \mathbb{R}, \big(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon\big),$$

so f is uniformly continuous.

Exercise 10.3.2

If $f : [-1, 1] \to \mathbb{R}$ is defined

$$f(x) = \begin{cases} x & x < 0 \\ x + 1 & x \geq 0 \end{cases}$$

Then

$$\lim_{x \to 0^-} \frac{1 - f(x)}{x} = \lim_{x \to 0^-} \frac{1}{x}$$

diverges. Thus

$$\lim_{x \to 0} \frac{1 - f(x)}{x}$$

diverges or does not exist and the function is not differentiable at 0.

Exercise 10.4.1

Let $f : (0, \infty) \to (0, \infty)$ be defined $x^{1/n}$ for $x \in (0, \infty)$ and $n \in \mathbb{Z}^+$. For part (a), because $1/n \in \mathbb{R}$, it follows from Proposition 9.9.10 that $x^{1/n}$ continuous on $(0, \infty)$. For part (b), we must first establish that f is bijective, in order to establish that its unique inverse f^{-1} exists.

Let $a, b \in (0, \infty)$ with $a \neq b$, and assume to the contrary that $a^{1/n} = b^{1/n}$. Then $(a^{1/n})^n = (b^{1/n})^n$ so $a = b$, a contradiction. Thus $a \neq b \Rightarrow f(a) \neq f(b)$ so f is injective. Also, for $y \in (0, \infty)$, $y^n > 0$ so $y^n \in (0, \infty)$ and $f(y^n) = y$. Therefore $\forall y \in (0, \infty)$, $\exists x \in (0, \infty)$ such that $f(x) = y$ and f is surjective. Thus f is bijective and has a unique inverse.

If $f^{-1} : (0, \infty) \to (0, \infty)$ is defined by y^n for $y \in (0, \infty)$, then $(f^{-1} \circ f) : (0, \infty) \to (0, \infty)$ is defined by $(x^{1/n})^n = x$ for $x \in (0, \infty)$, i.e. the indentity function, so f^{-1} is the inverse function of f.

Next, we find the derivative of $f^{-1} : (0, \infty) \to (0, \infty)$ defined by y^n for $y \in (0, \infty)$. Note the sum on lines 4 & 5 disappears for $n < 2$.

$$\begin{aligned}
(f^{-1})'(y) &= \lim_{h \to 0} \frac{f^{-1}(y+h) - f(y)}{h} \\
&= \lim_{h \to 0} \frac{(y+h)^n - y^n}{h} \\
&= \lim_{h \to 0} \frac{-y^n + \sum_{k=0}^{n} \binom{n}{k} y^{n-k} h^k}{h} \\
&= \lim_{h \to 0} \frac{y^n - y^n + \frac{n!}{(n-1)!} y^{n-1} h + \sum_{k=2}^{n} \binom{n}{k} y^{n-k} h^k}{h} \\
&= \lim_{h \to 0} n y^{n-1} + \sum_{k=2}^{n} \binom{n}{k} y^{n-k} h^{k-1} \\
&= n y^{n-1}.
\end{aligned}$$

Therefore f^{-1} is differentiable, with $(f^{-1})'(y)$ is defined ny^{n-1} for $y \in (0, \infty)$, and by proposition 6.7.3 $(f^{-1})'$ is greater than zero on its domain. $(f^{-1})^{-1} = f$, and from part (a) f is continuous, so by the inverse function theorem f is differentiable, and for $x \in (0, \infty)$,

$$\begin{aligned}
f'(x) &= \frac{1}{(f^{-1})'(f(x))} \\
&= \frac{1}{n(x^{1/n})^{n-1}} \\
&= \frac{1}{n}(x^{1-1/n})^{-1} \\
&= \frac{1}{n} x^{1/n-1}.
\end{aligned}$$

Exercise 11.1.4

Let $x \in \bigcup P\#P'$. Then $x \in p$ for some $p \in P\#P'$ and $p = a \cap a'$ for some $a \in P$ and $a' \in P'$. From the definition of a partition, $\bigcup P$ and $\bigcup P'$ are equal to I, so $a, a' \subseteq I$ and thus $p \subseteq I$ so $x \in I$. Because $\bigcup P = I$ and $\bigcup P' = I$, for any $x \in I$ there exists $a \in P$ and $a' \in P'$ with $x \in a$ and $x \in a'$. It follows $x \in a \cap a'$, and $a' \cap a \in P\#P'$, so $x \in \bigcup P\#P'$ and $\bigcup P\#P' = I$.

If $x, y \in P\#P'$, then for some $a, b \in P$ and $a', b' \in P'$ we have $x = a \cap a'$ and $y = b \cap b'$. Thus if $s \in x$ and $s \in y$, then $s \in a, b$. Because the elements of each partition are pairwise disjoint, this directly implies $a = b$. By the same logic $a' = b'$, so $x = y$. By contrapositive for $x, y \in P\#P'$ with $x \neq y$ implies $x \cap y = \emptyset$ so $\bigcap P\#P' = \emptyset$. Therefore $P\#P'$ is a partition of I. Trivially $a \cap a' \subseteq a$, $a \cap a' \subseteq a'$, thus all elements of $P\#P'$ are subsets of some element of P and some element of P'. Therefore $P\#P'$ is finer then both P and P'.