Real Analysis

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February 13, 2025

Contents

1 The natural numbers

1.1 Peano axioms

Definition 1.1 (Peano axioms)**.** Using ++ as the successor operation, the natural numbers are defined as follows:

(a) 0 is a natural number.

(b) If n is a natural number, then $n++$ is also a natural number.

(c) For all natural numbers n, $n++ \neq 0$.

Definition 1.2 (Addition of natural numbers)**.** Let m be a natural number. $0 + m := m$ and $(n++) + m := (n + m)++$.

Proposition 1.3. $m + 0 = m$.
Proof: Let $n \in \mathbb{N}$. $0 + 0 := 0$, so by inductive hypothesis $n + 0 = n$. $(n++) + 0 := (n + 0)++$, and from the inductive hypothesis equals $n++$. □

Lemma 1.4. For any natural numbers n and m, $n + (m++) = (n + m)++$.
Proof: Suppose $n, m \in \mathbb{N}$. $0+(m++) := m++ = (0+m)++$. By inductive hypothesis $n+(m++) = (n+m)++$. From the definition of addition $(n++)+(m++) = (n+(m++))++$ and from the inductive hypothesis $n + (m++) = (n + m)++$ so we have

$$\begin{aligned}(n++) + (m++) &= (n + (m++))++ \\ &= ((n + m)++)++ \\ &= ((n++) + m)++\end{aligned}$$

□

Proposition 1.5 (Commutivity of addition)**.** For $n, m \in \mathbb{N}$, $n + m = m + n$.
Proof: Let $n, m \in \mathbb{N}$. From proposition 1.3, $0+m = m+0$, so by inductive hypothesis $n + m = m + n$. $(n++) + m = (n + m)++$ and from inductive hypothesis this equals $(m + n)++$. From lemma 1.4, this equals $m + (n++)$. □

Proposition 1.6 (Associativity of addition)**.** Let $a, b, c \in \mathbb{N}$. Then $(a+b)+c = a+(b+c)$.
Proof: exercise □

Proposition 1.7 (Cancellation law)**.** Let $a, b, c \in \mathbb{N}$. Iff $a + b = a + c$, then $b = c$.
Proof: If $0 + b = 0 + c$ then from the definition of addition $b = c$. By inductive hypothesis for any $n \in \mathbb{N}$, $n+b = n+c$. $(n++)+b = (n+b)++$ and $(n++)+c = (n+c)++$, so from the inductive hypothesis and the axioms of natural numbers, $(n++)+b = (n++)+c$. □

Definition 1.8 (Positive natural number)**.** A natural number n is said to be positive iff it is not 0.

Definition 1.9 (Ordering of natural numbers)**.** Let $n, m \in \mathbb{N}$. We write $n \geq m$ or $m \geq n$ iff $n = m + a$ for some $a \in \mathbb{N}$.

Proposition 1.10. Let $m_0, m, m' \in \mathbb{N}$, and let $P(x)$ be a property of arbitrary $x \in \mathbb{N}$. Suppose that for each $m \geq m_0$ the following implication holds:

$$\Big(\forall m' \in [m_0, m),\ P(m')\Big) \Rightarrow P(m).$$

Then we can conclude $P(m)$ is true for all natural numbers $m \geq m_0$.

1.2 Multiplication

Definition 1.11 (Multiplication of natural numbers)**.** Let m be a natural number. $0 \times m := 0$ and $(n++) \times m := (n \times m) + m$.

Lemma 1.12 (Commutivity of multiplication)**.** Let $n, m \in \mathbb{N}$. Then $n \times m = m \times n$.
Proof: exercise □

Lemma 1.13. Let $n, m \in \mathbb{N}$. Then $n \times m = 0$ iff n or m is zero.

Proof: exercise $\square$

Proposition 1.14 (Distributive law)**.** For any natural numbers a, b, c, we have $a(b + c) = ab + ac$.

Proposition 1.15 (Associativity of multiplication)**.** If $a, b, c \in \mathbb{N}$ then $(a \times b) \times c = a \times (b \times c)$.

Proposition 1.16. If a, b are natural numbers such that $a < b$, and c is positive, then $ac < bc$.

Corollary 1.17. Let $a, b, c \in \mathbb{N}$ such that $ac = bc$ and c is non-zero. Then $a = b$.

Proposition 1.18 (Euclid's division lemma)**.** Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.

Definition 1.19 (Exponentiation for natural numbers)**.** Let $m \in \mathbb{N}$. $m^0 := 1$, and $m^{n++} = m^n \times m$.

2 Set theory

2.1 Fundamentals

Definition 2.1 (Axioms of sets)**.**

(a) (Sets are objects) If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.

(b) (Equality of sets) Two sets A and B are equal iff every element of A is an element of B and vice versa.

(c) (Empty set) There exists a set known as the empty set, denoted $\emptyset$, which contains no elements. In other words, for all objects x we have $x \notin \emptyset$.

(d) (Singleton sets) If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e. for every object y we have $y \in \{a\}$ iff $y = a$. $\{a\}$ is referred to as a singleton set.

(e) (Pairwise union) Given any two sets A and B, there exists a set $A \cup B$, called the union of A and B, which consists of all the elements which belong to A or B. In other words,
$$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B).$$

(f) (Axiom of specification) Let A be a set, and for each $x \in A$ let $P(x)$ be a property pertaining to x. Then there exists a set $\{x \in A \mid P(x)\}$ whose elements are precisely the elements x in A for which $P(x)$ is true.

(g) (Replacement) Let A be a set. For any object $x \in A$ and any object y, suppose we have a property $P(x, y)$ that is true for at most one y for each $x \in A$. Then
$$z \in \{y \mid P(x, y),\, x \in A\} \Leftrightarrow P(x, z).$$

(h) (Infinity) There exists a set $\mathbb{N}$, whose elements are called natural numbers, as well as an object $0 \in \mathbb{N}$, and an object $N++$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms hold.

Lemma 2.2. Let A be a non-empty set. Then there exists an object x such that $x \in A$.

Definition 2.3 (Subset)**.** Let A, B be sets. We say that A is a subset of B, denoted $A \subseteq B$, iff every element of A is also an element of B. We say that A is a proper subset of B, denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Definition 2.4 (Intersection)**.** The intersection $S_1 \cap S_2$ of two sets is defined to be the set
$$S_1 \cap S_2 := \{x \in S_1 \mid x \in S_2\}.$$

Definition 2.5 (Disjoint)**.** Two sets are disjoint if $A \cap B = \emptyset$.

Definition 2.6 (Difference set)**.** If A and B are sets, the set $A \setminus B$ is the set A with any elements of B removed, i.e.

$$A \setminus B := \{x \in A \,|\, x \notin B\}.$$

Proposition 2.7. Let A, B, C be subsets of set X.

(a) (Minimal element) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

(b) (Maximal element) $A \cup X = X$ and $A \cap X = A$.

(c) (Identity) $A \cap A = A$ and $A \cup A = A$.

(d) (Commutativity) $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

(e) (Associativity) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

(f) (Distributivity) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(g) (Partition) $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$.

(h) (De Morgan Laws) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.