Advanced Calculus

Samuel Lindskog

October 9, 2024

Test 1

Definition 1.1 (Relation)**.** A relation between A and B is any subset R of $A \times B$. If $(a, b) \in R$, then we say aRb.

Definition 1.2 (Equivalence Relation)**.** A relation R on a set S is an equivalence relation if it has the following properties for all x, y, z in S:

1. $xRx \qquad$ (reflexive property)
2. $xRy \Rightarrow yRx \qquad$ (Symmetric property)
3. $xRy \wedge yRz \Rightarrow xRz \qquad$ (Transitive property)

A partition of a set S is a collection $\mathscr{P}$ of nonempty subsets such that

1. $x \in S \Rightarrow x \in \bigcup_{A \in \mathscr{P}} A$
2. $\forall A, B \in \mathscr{P}, A \neq B \Rightarrow A \cap B = \emptyset$

Definition 1.3 (Function)**.** Let A and B be sets. A function from A to B is a nonempty relation $f \subseteq A \times B$ that satisfies the following two conditions:

1. $\forall a \in A, \exists b \in B, (a, b) \in f$
2. $(a, b) \in f \wedge (a, c) \in f \Rightarrow b = c$

Definition 1.4 (Upper bound)**.** Let $S \subseteq \mathbb{R}$. If there exists a real number m such that $m \geq s$ for all $s \in S$, then m is an upper bound of S

Definition 1.5 (Maximum)**.** If an upper bound m of S is a member of S, then m is called the maximum of S.

Definition 1.6 (Supremum)**.** Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then the least upper bound of S is called the supremum. Thus $m = \sup S$ iff

1. $\forall s \in S, m \geq s$
2. $m' < m \Rightarrow \exists s' \in S, s' > m'$

Axiom 1.1 (Completeness of $\mathbb{R}$)**.** Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound, i.e. $\sup S$ exists.

Test 2

Definition 1.7 (Convergence)**.** A sequence (s_n) is said to converge to the real number s provided that

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N \Rightarrow |s_n - s| < \epsilon.$$

If (s_n) converges to s, then s is called the limit of the sequence (s_n), and we write $\lim_{n \to \infty} s_n = s$. If a sequence does not converge it diverges.

Theorem 1.1. Let (s_n) and (a_n) be sequences of real numbers and let $s \in \mathbb{R}$. If for some $k > 0$ and some $m \in \mathbb{N}$ we have

$$|s_n - s| \leq k|a_n|, \quad \text{for all } n \geq m,$$

and if $\lim a_n = 0$, then it follows that $\lim s_n = s$.

Theorem 1.2. Every convergent sequence is bounded.

Theorem 1.3. If a sequence converges, its limit is unique.