Real Analysis test 2

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1 Series

1.1 Useful properties

Proposition 1.1 (Triangle inequality)**.**

$$\left|\sum_{i=m}^{n} a_i\right| \leq \sum_{i=m}^{n} |a_i|.$$

Proposition 1.2. The series below converges iff for every real number $\epsilon > 0$, there exists an integer $N \geq m$ such that

$$\forall p, q \geq N, \left|\sum_{n=p}^{q} a_n\right| \leq \epsilon$$

Proposition 1.3. If $\sum_{n=m}^{\infty} a_n = C_1$ and $\sum_{n=m}^{\infty} b_n = C_2$ then

$$\sum_{n=m}^{\infty} (a_n + b_n) = C_1 + C_2.$$

Proposition 1.4. If $\sum_{n=m}^{\infty} a_n = L$, then

$$\sum_{n=m}^{\infty} ca_n = cL.$$

Definition 1.5 (Geometric series)**.** Let x be a real number. If $|x| \geq 1$, then the series $\sum_{n=0}^{\infty} x^n$ is divergent. If $|x| < 1$, then the series is absolutely convergent and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

1.2 Series convergence tests

Proposition 1.6 (Comparison test for finite series)**.** Let $m \leq n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \leq i \leq n$. Suppose that $a_i \leq b_i$ for all $m \leq i \leq n$. Then we have

$$\sum_{i=m}^{n} a_i \leq \sum_{i=m}^{n} b_i.$$

Proposition 1.7 (Zero test)**.** If a series converges, the limit of the same sequence of its elements converges to zero.

Proposition 1.8 (Absolute convergence test)**.** If a series is absolutely convergent, it is convergent.

Proposition 1.9 (Alternating series test)**.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which are non-negative and decreasing. Then the series $\sum_{n=m}^{\infty}(-1)^n a_n$ is convergent iff $(a_n)_{n=m}^{\infty}$ converges to zero.

Proposition 1.10 (Comparison test)**.** Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two series, and $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ converges, then

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Proposition 1.11 (Divergence test)**.** The contrapositive of the comparison test.

Proposition 1.12 (Cauchy criterion)**.** Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of non-negative real numbers. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent iff the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k}$$

is convergent.

Proposition 1.13 (P-test)**.** Let $q > 0$ be a real number. Then the series $\sum_{n=1}^{\infty} \frac{1}{n^q}$ is convergent when $q > 1$ and divergent when $q \leq 1$.

Proposition 1.14 (Root test)**.** Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$.

(a) If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent.

(b) If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is not convergent.

(c) If $\alpha = 1$, we cannot assert any conclusion.

Proposition 1.15 (Ratio test)**.** Let $\sum_{n=m}^{\infty} a_n$ be a series of non-zero numbers.

(a) If $\limsup \frac{|a_{n+1}|}{|a_n|} < 1$, the above series is absolutely convergent

(b) If $\liminf \frac{|a_{n+1}|}{|a_n|} > 1$, the above series is not convergent.

(c) In the remaining cases, we cannot assert any conclusion.