Real Analysis

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Contents

1 The natural numbers

1.1 Peano axioms

Definition 1.1 (Peano axioms)**.** Using $++$ as the successor operation, the natural numbers are defined as follows:

(a) 0 is a natural number.

(b) If n is a natural number, then $n++$ is also a natural number.

(c) For all natural numbers n, $n++ \neq 0$.

Definition 1.2 (Addition of natural numbers)**.** Let m be a natural number. $0 + m := m$ and $(n++) + m := (n + m)++$.

Proposition 1.3. There is only one zero, i.e. for $a \in \mathbb{N}$ if $0 + a = 0' + a = a$, then $0 = 0'$.

Proof: Suppose $0 \neq 0'$. Then 0 is a successor of $0'$ or $0'$ is a successor of 0. Because no successor of a natural number is 0, this is impossible. $\square$

Proposition 1.4. $m + 0 = m$.

Proof: Let $n \in \mathbb{N}$. $0+0 := 0$, so by inductive hypothesis $n+0 = n$. $(n++)+0 := (n+0)++$, and from the inductive hypothesis equals $n++$. $\square$

Lemma 1.5. For any natural numbers n and m, $n + (m++) = (n + m)++$.

Proof: Suppose $n, m \in \mathbb{N}$. $0 + (m++) := m++ = (0 + m)++$. By inductive hypothesis $n+(m++) = (n+m)++$. From the definition of addition $(n++)+(m++) = (n+(m++))++$ and from the inductive hypothesis $n + (m++) = (n + m)++$ so we have

$$\begin{aligned}(n++) + (m++) &= (n + (m++))++ \\ &= ((n + m)++)++ \\ &= ((n++) + m)++\end{aligned}$$

$\square$

Proposition 1.6 (Commutivity of addition)**.** For $n, m \in \mathbb{N}$, $n + m = m + n$.

Proof: Let $n, m \in \mathbb{N}$. From proposition 1.4, $0 + m = m + 0$, so by inductive hypothesis $n + m = m + n$. $(n++) + m = (n + m)++$ and from inductive hypothesis this equals $(m + n)++$. From lemma 1.5, this equals $m + (n++)$. $\square$

Proposition 1.7. If $a, b \in \mathbb{N}$ and $a + b = a$, then $b = 0$.

Proof: Suppose $a, b \in \mathbb{N}$ with $a + b = a$. $\square$

Proposition 1.8 (Associativity of addition)**.** Let $a, b, c \in \mathbb{N}$. Then $(a + b) + c = a + (b + c)$.

Proof: Suppose $a, b \in \mathbb{N}$. From here we utilize the definition of addition, and commutivity of addition for the rest of the proof. It follows that $(a + b) + 0 = a + b = a + (b + 0)$. By inductive hypothesis suppose $(a + b) + c = a + (b + c)$ for $c \in \mathbb{N}$. Then

$$\begin{aligned}(a + b) + c++ &= [(a + b) + c]++ \\ &= [a + (b + c)]++ \\ &= a + (c + b)++ \\ &= a + [(c++) + b] \\ &= a + (b + c++)\end{aligned}$$

$\square$

Proposition 1.9 (Cancellation law)**.** Let $a, b, c \in \mathbb{N}$. Iff $a + b = a + c$, then $b = c$.

Proof: If $0 + b = 0 + c$ then from the definition of addition $b = c$. By inductive hypothesis for any $n \in \mathbb{N}$, $n+b = n+c$. $(n++)+b = (n+b)++$ and $(n++)+c = (n+c)++$, so from the inductive hypothesis and the axioms of natural numbers, $(n++) + b = (n++) + c$. $\square$

Definition 1.10 (Positive natural number)**.** A natural number n is said to be positive iff it is not 0.

Definition 1.11 (Ordering of natural numbers)**.** Let $n, m \in \mathbb{N}$. We write $n \geq m$ or $m \geq n$ iff $n = m + a$ for some $a \in \mathbb{N}$.

Proposition 1.12. If a or b are not zero, then $a + b \neq 0$.

Proof: Suppose $a, b \in \mathbb{N}$ with $b \neq 0$. If $a = 0$ then $a + b = 0 + b = b \neq 0$. If $a \neq 0$, because no natural number has zero as a successor it follows from the definition of addition that $a + b \neq 0$. $\square$

Proposition 1.13 (Trichotomy of order for natural numbers)**.** Let $a, b \in \mathbb{N}$. Then exactly one of the following statements is true: $a < b$, $a = b$, $a > b$.

Proof: Suppose $a, b \in \mathbb{N}$ and $a < b$. Then for some $c \in \mathbb{N}$, $a = b + c$ with $b \neq a$. If $c = 0$ then $a = b$, a contradiction. If $b < a$, then for some $d \in \mathbb{N}$, $b = a + d$ with $a \neq b$. If $d = 0$ then $a = b$, a contradiction. Because $b = b + d + c$ and $c, d \neq 0$, it follows from commutivity and propositions 1.12 and 1.3 that this is impossible. Therefore wlog if $a < b$ then a is not greater than or equal to b. Suppose $a = b$. If $a < b$ then $a = b + c$ for some $c \in \mathbb{N}$ with $b \neq c$, a contradiction. Therefore wlog if $a = b$ then a is not less than or greater than b. $\square$

Proposition 1.14 (Strong principle of induction)**.** Let $m_0, m, m' \in \mathbb{N}$, and let $P(x)$ be a property of arbitrary $x \in \mathbb{N}$. Suppose that for each $m \geq m_0$ the following implication holds:

$$\left(\forall m' \in [m_0, m),\ P(m')\right) \Rightarrow P(m).$$

Then we can conclude $P(m)$ is true for all natural numbers $m \geq m_0$.

1.2 Multiplication

Definition 1.15 (Multiplication of natural numbers)**.** Let m be a natural number. $0 \times m :=$ 0 and $(n++) \times m := (n \times m) + m$.

Proposition 1.16. $m \times 0 = 0$.

Proof: From the definition of multiplication, $0 \times 0 = 0$. By inductive hypothesis suppose $m \times 0 = 0$. Then $(m++) \times 0 = (m \times 0) + 0 = 0$. $\square$

Proposition 1.17. For $n, m \in \mathbb{N}$, $n \times (m++) = (n \times m) + n$.

Proof: Let $n, m \in \mathbb{N}$. $0 \times (m++) = 0 = (0 \times m) + 0$. By inductive hypothesis, $(n \times (m++)) = (n \times m) + n$. It follows that

$$\begin{aligned}
(n++) \times (m++) &= (n \times (m++)) + (m++) \\
&= (n \times m) + n + (m++) \\
&= (n \times m) + m + (n++) \\
&= ((n++) \times m) + (n++)
\end{aligned}$$

$\square$

Proposition 1.18. For $m \in \mathbb{N}$, $1m = m$.

Proof: If $m \in \mathbb{N}$ $0 \times m = 0$. Then $(0++) \times m = 1 \times m = 0 + m = m$. $\square$

Lemma 1.19 (Commutivity of multiplication)**.** Let $n, m \in \mathbb{N}$. Then $n \times m = m \times n$.

Proof: Let $n, m \in \mathbb{N}$. $0 \times m = m \times 0 = 0$. By inductive hypothesis, $n \times m = m \times n$. It follows from proposition 1.17 that

$$\begin{aligned}
(n++) \times m &= (n \times m) + m \\
&= (m \times n) + m \\
&= m \times (n++)
\end{aligned}$$

$\square$

Proposition 1.20 (Distributive law)**.** For any natural numbers a, b, c, we have $a(b+c) = ab + ac$.

Proof: TODO $\square$

Proposition 1.21 (Associativity of multiplication)**.** If $a, b, c \in \mathbb{N}$ then $(a\times b)\times c = a\times(b\times c)$.

Proof: TODO $\square$

Proposition 1.22. If $a, b \in \mathbb{N}^+$, then $ab \neq 0$.

Proof: Let $a \in \mathbb{N}^+$. By proposition 1.18 $1a = a$ and a is positive. By inductive hypothesis if $n \in \mathbb{N}^+$ then na is positive. $n++$ is a successor to n, and no successor of a natural number is zero, so $n++$ is positive. $(n++)a = na + a$. Both na and a are positive and by proposition 1.12, $na + a$ is positive and thus not zero. $\square$

Proposition 1.23. If a, b are natural numbers such that $a < b$, and c is positive, then $ac < bc$.

Corollary 1.24. Let $a, b, c \in \mathbb{N}$ such that $ac = bc$ and c is non-zero. Then $a = b$.

Proposition 1.25 (Euclid's division lemma)**.** Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq+r$.

Definition 1.26 (Exponentiation for natural numbers)**.** Let $m \in \mathbb{N}$. $m^0 := 1$, and $m^{n++} = m^n \times m$.

2 Set theory

2.1 Fundamentals

Definition 2.1 (Axioms of sets)**.**

(a) (Sets are objects) If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.

(b) (Equality of sets) Two sets A and B are equal iff every element of A is an element of B and vice versa.

(c) (Empty set) There exists a set known as the empty set, denoted $\emptyset$, which contains no elements. In other words, for all objects x we have $x \notin \emptyset$.

(d) (Singleton sets) If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e. for every object y we have $y \in \{a\}$ iff $y = a$. $\{a\}$ is referred to as a singleton set.

(e) (Pairwise union) Given any two sets A and B, there exists a set $A \cup B$, called the union of A and B, which consists of all the elements which belong to A or B. In other words,
$$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B).$$

(f) (Axiom of specification) Let A be a set, and for each $x \in A$ let $P(x)$ be a property pertaining to x. Then there exists a set $\{x \in A \,|\, P(x)\}$ whose elements are precisely the elements x in A for which $P(x)$ is true.

(g) (Replacement) Let A be a set. For any object $x \in A$ and any object y, suppose we have a property $P(x, y)$ that is true for at most one y for each $x \in A$. Then
$$z \in \{y \,|\, P(x,y),\, x \in A\} \Leftrightarrow P(x, z).$$

(h) (Infinity) There exists a set $\mathbb{N}$, whose elements are called natural numbers, as well as an object $0 \in \mathbb{N}$, and an object $N++$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms hold.

(i) (Universal specification) DANGER - Suppose for every object x we have a property $P(x)$. Then there exists a set $\{x \,|\, P(x)\}$.

(j) (Regularity). If A is a non-empty set, then there is at least one element x of A which is either not a set, or is disjoint from A.

(k) (Power set) Let X and Y be sets. Then there exists a set, denoted Y^X, which consists of all the functions from X to Y, thus

$$f \in Y^X \Leftrightarrow f \text{ is a function from } X \text{ to } Y.$$

(l) (Union) Let A be a set whose elements are all sets. Then there exists a set $\bigcup A$ defined

$$x \in \bigcup A = \{x \mid \exists S \in A,\, x \in S\}.$$

Remark. The axioms of set theory introduced, excluding universal specification, are known as the Zermelo-Fraenkel axioms of set theory.

Lemma 2.2 (Single choice)**.** Let A be a non-empty set. Then there exists an object x such that $x \in A$.
Proof: Suppose there does not exist any object x such that $x \in A$. Simultaneously $x \notin \emptyset$, so $x \in A \Leftrightarrow x \in \emptyset$ and $A = \emptyset$, a contradiction. □

Definition 2.3 (Subset)**.** Let A, B be sets. We say that A is a subset of B, denoted $A \subseteq B$, iff every element of A is also an element of B. We say that A is a proper subset of B, denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Theorem 2.4. *Let A be a set. Then $\emptyset \subseteq A$.*
Proof: If $\emptyset \subseteq A$ then for all objects x,

$$x \in \emptyset \Rightarrow x \in A.$$

This is vacuously true because there does not exist x such that $x \in \emptyset$. □

Definition 2.5 (Intersection)**.** The intersection $S_1 \cap S_2$ of two sets is the set

$$S_1 \cap S_2 = \{x \mid x \in S_1 \wedge x \in S_2\}.$$

Definition 2.6 (Union)**.** The union $S_1 \cup S_2$ of two sets is the set

$$S_1 \cup S_2 = \{x \mid x \in S_1 \vee x \in S_2\}.$$

Definition 2.7 (Disjoint)**.** Two sets are disjoint if $A \cap B = \emptyset$.

Definition 2.8 (Difference set)**.** If A and B are sets, the set $A \setminus B$ is the set A with any elements of B removed, i.e.

$$A \setminus B \coloneqq \{x \mid x \in A \wedge x \notin B\}.$$

Proposition 2.9. Let A, B, C be subsets of set X.

(a) (Minimal element) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

(b) (Maximal element) $A \cup X = X$ and $A \cap X = A$.

(c) (Identity) $A \cap A = A$ and $A \cup A = A$.

(d) (Commutativity) $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

(e) (Associativity) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

(f) (Distributivity) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(g) (Partition) $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$.

(h) (De Morgan Laws) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Definition 2.10 (Ordered pair)**.** If x and y are any objects, we define the ordered pair (x, y) to be a new object which consists of x as its "first component" and y as its "second component". Two ordered pairs x, y and x', y' are equal if

$$x = x', \quad y = y'.$$

Definition 2.11 (Cartesian product)**.** Let A, B be sets. Then the cartesian product of A and B, written $A \times B$, is

$$A \times B = \{(a, b) \,|, a \in A,\, b \in B\}.$$

Definition 2.12 (Ordered n-tuple)**.** Let n be a natural number. An ordered n-tuple $(x_i)_{1\leq i\leq n}$ is a collection of objects x_i, one for every natural number i between 1 and n. Two ordered n-tuples $(x_i)_{1\leq i\leq n}$ and $(y_i)_{1\leq i\leq n}$ are said to be equal iff $x_i = y_i$ for all $1 \leq i \leq n$.

Definition 2.13 (n-fold Cartesian product)**.** If $(X_i)_{1\leq i\leq n}$ is an ordered n-tuple of sets, their Cartisian product $\prod_{i=1}^{n} X_i$ is defined

$$\prod_{i=1}^{n} X_i = \{(x_i)_{1\leq i\leq n} \,|\, x_i \in X_i\}.$$

Definition 2.14 (Indexed family)**.** If for each element $j \in J$ with $J \neq \emptyset$, there corresponds a set A_j, then

$$\mathscr{A} = \{A_j \,|\, j \in J\}.$$

Is called an indexed family of sets with J as the index set. If $J = \{1, 2, \ldots, n\}$ we may index the set similarly to sum notation.

Definition 2.15 (Union and intersection of indexed family)**.** The union of all sets in an indexed family $\mathscr{A}$ with index set J is

$$\bigcup_{j\in J} A_j = \{x \,|\, \exists A_j \in \mathscr{A},\, x \in A_j\}.$$

The intersection of all sets in $\mathscr{A}$ is

$$\bigcap_{j\in J} A_j = \{x \,|\, \forall A_j \in \mathscr{A},\, x \in A_j\}.$$

Lemma 2.16 (Finite choice)**.** Let $n \geq 1$ be a natural number, and for each natural number $1 \leq i \leq n$, let X_i be a non-empty set. Then there exists an n-tuple $(x_i)_{1\leq i\leq n}$ such that $x_i \in X_i$ for all $1 \leq i \leq n$. In other words if each X_i is non-empty, then its n-fold cartesian product is nonempty.
Proof: Let $\mathscr{A} = \{A_i \,|\, 1 \leq i \leq n\}$ with $n \in \mathbb{N}$ be an indexed family of nonempty sets. It follows from lemma 2.2 that for each A_i, $1 \leq i \leq n$, there exists $a_i \in A_i$. Using this fact, define an ordered n-tuple $(a_i)_{1\leq i\leq n}$. □

2.2 Functions

Definition 2.17 (Relation)**.** Let A, B be sets. A relation between A and B is an subset of $A \times B$.

Definition 2.18 (Equivalence relation)**.** An equivalence relation on a set S is a relation such that for all $x, y, z \in S$, the relation satisfies the following properties:

(a) (Reflexive property) xRx.

(b) (Symmetric property) $xRy \Rightarrow yRx$.

(c) (Transitive property) $xRy \wedge yRx \Rightarrow xRz$.

Definition 2.19 (Partition)**.** A partition of a set S is a collection $\mathscr{P}$ of nonempty subsets of S that are pairwise disjoint, and whose union is S, i.e.

(a) $A = \bigcup \mathscr{P}$.

(b) $\forall A, B \in \mathscr{P},\, A \neq B \Rightarrow A \cap B = \emptyset$.

Definition 2.20 (Function)**.** A function from A to B, denoted $f : A \to B$ is a nonempty relation $f \subseteq A \times B$ that satisfies the following properties:

(a) (Existence) $\forall a \in A,\, \exists b \in B,\, (a, b) \in f$.

(b) (Uniqueness) $(a, b) \in f \wedge (a, c) \in f \Rightarrow b = c$.

Set A is called the domain of f, and set B is called the codomain. The range of f is $f(A)$, i.e. $\{b \in B \mid (a, b) \in f\}$.

Definition 2.21 (Equality of functions)**.** Two functions $f : X \to Y$ and $g : X' \to Y'$ are equal if their domains and codomains are equal, and furthermore that $f(x) = g(x)$ for all $x \in X$.

Definition 2.22 (Composition)**.** Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that the codomain of f is the same set as the domain of g. Then the composition $g \circ f : X \to Z$ of the two functions g and f is the function defined by the formula

$$(g \circ f)(x) = g(f(x)).$$

Lemma 2.23. Let $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.
Proof: $g \circ h$ is a function from X to Z, and $f \circ g$ is a function from $Y \to W$, so $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are functions from X to W. It follows from the definition of function composition that

$$\begin{aligned}(f \circ (g \circ h))(x) &= f((g \circ h)(x)) \\ &= f(g(h(x))) \\ &= (f \circ g)(h(x)) \\ &= ((f \circ g) \circ h)(x)\end{aligned}$$

□

Definition 2.24 (Injective)**.** A function $f : X \to Y$ is injective (one-to-one) if for $x, x' \in X$,

$$x \neq x' \to f(x) \neq f(x')$$

Definition 2.25 (Surjective)**.** A function $f : X \to Y$ is surjective (onto) if

$$\forall y \in Y,\ \exists x \in X,\ f(x) = y.$$

Definition 2.26 (Bijective)**.** A function is bijective (invertible) if it is injective and surjective.

Proposition 2.27. Let $f : A \to B$ and $g : B \to C$. Then

(a) If f and g are surjective, then $g \circ f$ is surjective.

(b) If f and g are injective, then $g \circ f$ is injective.

(c) If f and g are bijective, then $g \circ f$ is bijective.

Lemma 2.28. If $f : X \to Y$ is bijective then f is invertible. In other words for all $y \in Y$ there exists a unique $x \in X$ denoted $f^{-1}(y)$ such that $f(x) = y$. Therefore the inverse of f, $f^{-1} : Y \to X$ exists and is defined

$$f^{-1}(y) = x.$$

Definition 2.29 (Identity function)**.** A function defined on a set A that maps each element in A onto itself is called the identity function on A, and is denoted i_A.

Proposition 2.30. Let $f : A \to B$ be bijective. Then

(a) $f^{-1} : B \to A$ is bijective.

(b) $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Theorem 2.31. *Let $f : A \to B$ and $g : A \to B$ be bijective. Then the composition $g \circ f : A \to C$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Definition 2.32 (Image)**.** If $f : X \to Y$ is a function from X to Y, and $S \subseteq X$, we define the image of S under f, $f(S)$ to be the set

$$f(S) = \{f(x) \mid x \in S\}.$$

Definition 2.33 (Inverse image)**.** If U is a subset of Y, we define the set $f^{-1}(U)$ to be the set

$$f^{-1}(U) = \{x \in X \mid f(x) \in U\}.$$

We call $f^{-1}(U)$ the inverse image of U.

Proposition 2.34. If X, Y are sets and $f : X \to Y$ then $f(X) \subseteq Y$.
Proof: $y \in f(X)$ implies $y \in \{y \mid (x, y) \in f\}$ and f is a subset of $X \times Y$, so it follows from the definition of the cartesian product that $y \in Y$. □

Lemma 2.35. Let X be a set. Then the set

$$\{Y \mid Y \subseteq X\}$$

Is a set.
Proof: Let X be a set and $A \subseteq X$ with $A \neq \emptyset$. Then there exists $p \in A$, and we can define a function $f : X \to A$ with $x \in X$ by

$$f(x) = \begin{cases} x \in A & f(x) = x \\ x \notin A & f(x) = p \end{cases}$$

Thus for all $a \in f(X)$, $a \in A$ or $a = p \in A$, so $f(X) \subseteq A$. Next, for all $x \in X$, $(x, f(x)) \in f(X)$. Because for all $a \in A$ we have $a \in X$ then for all $a \in A$, $(a, f(a)) = (a, a) \in f(X)$ so from the definition of an image $A \subseteq f(X)$. Thus $A = F(X)$. From the power set axiom in definition 2.1,

$$\{f : X \to A \mid A \subseteq X \wedge A \neq \emptyset\} \subseteq X^X$$

From replacement, pairwise union, and singleton set axioms in definition 2.1, we can define a set $\mathrm{P}(X)$ that is the union of all images of functions in X^X, and $\{\emptyset\}$. As established above, all nonempty subsets of X are included in this set, and from proposition 2.34 all images of functions in X^X are subsets of X. □

Definition 2.36 (Power set)**.** For a set X, the set $\{Y \mid Y \subseteq X\}$ is called the power set of X, and is denoted $\mathrm{P}(X)$ or 2^X.

Definition 2.37 (Cardinality)**.** We say that two sets X and Y have equal cardinality iff there exists a bijection $f : X \to Y$ from X to Y.

Proposition 2.38. Let X, Y, Z be sets.

(a) X has equal cardinality with X.

(b) If X has equal cardinality with Y, then Y has equal cardinality with X.

(c) If X has equal cardinality Y and Y has equal cardinality with Z, then X has equal cardinality with Z

Proof: □

Definition 2.39 (Cardinality n)**.** Let n be a natural number. A set X is said to have cardinality n, if it has equal cardinality with $\{\in \mathbb{N} \mid 1 \leq i \leq n\}$. In this case we say that X has n elements.

Lemma 2.40. Suppose that $n \geq 1$, and set X has cardinality n. Then X is non-empty, and if x is any element of X, then the set $X - \{x\}$ has cardinality $n - 1$.

Proposition 2.41. Let X be a set with some cardinality n. Then X cannot have any other cardinality, i.e. X cannot have cardinality m for any $m \neq n$.

Definition 2.42 (Finite set)**.** A set is finite iff it has cardinality n for some natural number n; otherwise, the set is called infinte.

Theorem 2.43. *The set of natural numbers is infinite.*

3 Integers and rationals

3.1 The integers

Definition 3.1 (Integers)**.** An integer is an expression of the form $a - b$, where a and b are natural numbers. Two integers are considered to be equal, $a - b = c - d$, iff $a + d = c + b$. The set of all integers is denoted $\mathbb{Z}$.

Remark. The use of $-$ is purely notational (until subtraction is defined). $a - b$ can be interpreted as an ordered pair in $\mathbb{N} \times \mathbb{N}$.

Definition 3.2 (Integer addition)**.** The sum of two integers $(a - b) + (c - d)$ is defined by the formula

$$(a - b) + (c - d) = (a + c) - (c + d)$$

Definition 3.3 (Integer multiplication)**.** The product of two integers $(a - b) \times (c - d)$ is defined by the formula

$$(a - b) \times (c - d) = (ac + bd) - (ad + bc).$$

Remark. We may identify the integers with natural numbers by setting $n \equiv n - 0$. Definitions of equality and previously defined operations remain consistent with each other.

Proposition 3.4. If $a, b \in \mathbb{Z}$ and $a + b = b$ then $a = 0$.
Proof: prove □

Lemma 3.5. Addition and multiplication are well defined.

Definition 3.6 (Negation of integers)**.** If $(a-b)$ is an integer, we define the negation $-(a-b)$ to be the integer $b - a$.

Lemma 3.7 (Trichotomy of integers)**.** Let x be an integer. Then either x is zero, equal to a positive natural number, or x negated is a positive natural number.

Definition 3.8 (Positive integer)**.** If n is a positive natural number, we call n a positive integer, and $-n$ a negative integer.

Proposition 3.9 (Integer laws for algebra)**.** Let x, y, z be integers. Then the following identities hold:

$$\begin{aligned}
&x + y = y + x \\
&(x + y) + z = x + (y + z) \\
&x + 0 = 0 + x = x \\
&x + (-x) = 0 \\
&xy = yx \\
&(xy)z = x(yz) \\
&1x = x \\
&x(y + z) = xy + xz
\end{aligned}$$

Proposition 3.10. If $a, b \in \mathbb{Z}$ with $a, b > 0$, then $ab > 0$.
Proof: If $a, b \in \mathbb{Z}$ with $a, b > 0$, then for some $x, y \in \mathbb{N}^+$, $a = x - 0$ and $b = y - 0$. Thus $ab = (xy + 0) = (0 + 0) = xy - 0$. Because $x, y \neq 0$, by proposition 1.22 $xy > 0$ so from the definition of a positive integer $ab > 0$. □

Proposition 3.11. If $a, b \in \mathbb{Z}$ with $a, b > 0$, then $a + b > 0$.
Proof: If $a, b \in \mathbb{Z}^+$, then for some $x, y \in \mathbb{N}^+$ we have $a = x - 0$ and $b = y - 0$, so $a + b = ((x + y) - 0)$. It follows from proposition 1.12 that $x + y > 0$ so from the definition of a positive integer, $a + b > 0$. □

Proposition 3.12. If $x \in \mathbb{Z}$ with $x = (a - b)$ then $-1 \cdot (a - b) = -(a - b)$.
Proof: $-1 \cdot (a - b) = (0 - 1) \cdot (a - b) = (0a + b) - (a + 0b) = -(a - b)$. □

Proposition 3.13 (Integers have no zero divisors)**.** If a, b are integers such that $ab = 0$, then $a = 0$ or $b = 0$.

Corollary 3.14 (Cancellation law)**.** If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.
Proof: Let $a, b, c \in \mathbb{Z}$ with $c =\neq 0$. If $a = 0$ it follows from proposition 3.13 that $ac = 0$ so $bc = 0$ and thus $b = 0$, so $a = b$. If $a \neq 0$, suppose to the contrary that $b \neq a$. It follows from proposition 3.4 that there exists $d \in \mathbb{Z}$ with $d \neq 0$ such that $a + d = b$. Using laws for algebra we see that $ac = ac + dc$. By proposition 3.13 $dc \neq 0$, a contradiction by proposition 3.4. Therefore $a = b$. □

Definition 3.15 (Ordering of integers)**.** Let $n, m \in \mathbb{Z}$. We say that n is greater than or equal to m and write $n \geq m$ or $m \leq n$ iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m and write $n > m$ or $m < n$ iff $n \geq m$ and $n \neq m$.

3.2 The rationals

Definition 3.16 (Rational number)**.** A rational number is an expression of the form $a//b$, where a and b are integers and $b \neq 0$. Two rational numbers are equal, $a//b = c//d$, iff $ad = bc$. The set of all rational numbers is denoted $\mathbb{Q}$.

Remark. We may indentify the rationals with natural numbers by setting $n//1 \equiv n$.

Definition 3.17 (Addition of rationals)**.** If $a//b$ and $c//d$ are rationals, their sum is

$$(a//b) + (c//d) = (ad + bc)//(bd).$$

Definition 3.18 (Product of rationals)**.** If $a//b$ and $c//d$ are rationals, their product is

$$(a//b) \cdot (c//d) = (ac)//(bd).$$

Definition 3.19 (Negation of rationals)**.** The negation of a rational $(a//b)$, denoted $=$ $(a//b)$ is

$$-(a//b) = (-a//b).$$

Definition 3.20 (Reciprocal of rationals)**.** If $x = a//b$ is a non-zero rational number, then the reciprocal of x^{-1} of x is defined

$$x^{-1} = b//a.$$

Lemma 3.21. The sum, product, negation, and reciprocal operations on rational numbers are well-defined.

Proposition 3.22. The negation of the negation of $x \in \mathbb{Q}$ is x.
Proof: The negation of the negation of an integer $x = (a - b)$ is $- - (a - b) = -(b - a) = (a - b)$ so $- - x = x$. The negation of the negation of a rational number $y = (c//d)$ is $- - (c//d) = -(-c//d) = (- - c//d) = c//d$. □

Definition 3.23 (Quotient)**.** The quotient of two rationals x and y with $y \neq 0$, denoted x/y, is

$$x/y = x \times y^{-1}.$$

Definition 3.24 (Subtraction)**.** The difference of two rationals x and y, denoted $x - y$, is defined

$$x - y = x + (-y).$$

Definition 3.25 (Positive rational number)**.** A rational number x is said to be positive iff we have $x = a/b$ for some positive integers a and b. It is said to be negative iff $x = -y$ for some positive rational y.

Definition 3.26 (Ordering of rationals)**.** Let $x, y \in \mathbb{Q}$. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - y$ is a positive negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$, and $x \leq y$ iff either $x < y$ or $x = y$.

Proposition 3.27. $x \in \mathbb{Q}$ is positive iff $x > 0$, and negative iff $x < 0$.
Proof: If $x = a//b$ is a positive rational number then $a, b > 0$. Because $0 = 0//d$ for some $d \in \mathbb{N} \setminus \{0\}$, $x - 0 = x + 0 = ad//bd = a//b$ and thus $x > 0$. If $x > 0$ then $x - 0$ is positive. Because $0 = 0//d$ for some $d \in \mathbb{N} \setminus \{0\}$ we have $x - 0 = x + 0 = ad//bd = a//b$, which is positive. □

Proposition 3.28 (Laws of algebra for rationals)**.** Let x, y, z be rationals. Then the following laws of algebra hold:

$$\begin{aligned}
&x + y = y + x \\
&(x + y) + z = x + (y + z) \\
&x + 0 = x \\
&x + (-x) = 0 \\
&xy = yx \\
&(xy)z = d(yz) \\
&1x = x \\
&x(y + z) = xy + xz
\end{aligned}$$

Proposition 3.29. $-1x = -x$.
Proof: If $x = a//b$ then $-1 \cdot x$ is $(-1//1) \cdot (a//b) = -1a//b$. From proposition 3.12, $-1a = -a$ so $-1a//b = -(a//b) = -x$. □

Proposition 3.30. If $a, b \in \mathbb{Q}$ with $a > 0$ and $b > 0$ then $a + b > 0$.
Proof: Suppose $a, b \in \mathbb{Q}$ with $a, b > 0$. It follows from the definition of positive rational number that for some positive $x, y, z, w \in \mathbb{Z}$, $a = x//y$ and $b = z//w$, so $ab = xw + zy/yw$. By proposition 3.10 $xw, zy, yw > 0$, so by 3.11, $a + b > 0$. □

Lemma 3.31 (Trichotomy of rationals)**.** Let x be a rational number. Then exactly one of the following three statements is true:

(a) $x = 0$.

(b) x is positive.

(c) x is negative.

3.3 Absolute value and exponentiation

Definition 3.32 (Absolute value)**.** If x is a rational number, the absolute value $|x|$ of x is defined as follows:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Definition 3.33 (Distance)**.** The distance between $x, y \in \mathbb{Q}$, sometimes denoted $d(x, y)$, is

$$d(x, y) = |x - y|.$$

Proposition 3.34. For all $x \in \mathbb{Q}$, $|x| \geq 0$.
Proof: If $x \geq 0$ then $|x| = x$ so $|x| \geq 0$. If $x < 0$ then $|x| = -x$. By proposition 3.27 x is negative. Therefore there exists $y \in \mathbb{Q}^+$ such that $x = -y$, so by proposition 3.22 $-x = --y = y$ and $-x$ is positive. By proposition 3.27, $-x > 0$. □

Proposition 3.35 (Triangle inequality)**.** For $x, y \in \mathbb{Q}$, $|x + y| \leq |x| + |y|$.
Proof: □

Definition 3.36 (ϵ-closeness)**.** Let $\epsilon > 0$ be a rational number, and let x, y be rational numbers. We say that y is ϵ-close to x iff $d(y, x) < \epsilon$.

Definition 3.37 (Exponentiation to a natural number)**.** Let x be a rational number. To raise x to the power 0, we define $x^0 = 1$ and for all $n \in \mathbb{N}$, $x^{n+1} = x^n \times x$.

Definition 3.38 (Exponentiation to a negative number)**.** Let x be a non-zero rational number. Then for any negative integer $-n$,

$$x^{-n} = 1/x^n.$$