Complex Variables

Samuel Lindskog

February 20, 2025

Contents

1 Complex numbers

1.1 Fundamental definitions and identities

Definition 1.1 (Complex number)**.** A complex number is an expression with of the form $z = x + iy$, where x and y are real numbers.

Definition 1.2. Ever complex number $z \neq 0$ has a multiplicative inverse given by

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

Definition 1.3 (Modulus)**.** The modulus of a complex number $z = x + iy$ is the length of the vector (x, y), and is denoted $|z|$.

$$|z| = \sqrt{x^2 + y^2}.$$

Proposition 1.4. For $z, w \in \mathbb{C}$, it follows from the triangle inequality that

$$|z + w| \leq |z| + |w|$$
$$|z - w| \geq |z| - |w|$$

Definition 1.5 (Multiplication)**.** $(x + iy)(u + iv) = xu - yv + i(xv + yu)$.

Definition 1.6 (Complex conjugate)**.** The complex conjugate of a complex number $z = x + iy$ is defined to be $\overline{z} = x - iy$.

Proposition 1.7. For $z, w \in \mathbb{C}$, the following identities hold:

$$\overline{\overline{z}} = z$$
$$\overline{z + w} = \overline{z} + \overline{w}$$
$$\overline{zw} = \overline{z}\overline{w}$$
$$\overline{z\overline{w}} = \overline{z}w$$
$$|z| = |\overline{z}|$$
$$|z|^2 = z\overline{z}$$
$$|zw| = |z||w|$$

Proposition 1.8. The real and imaginary parts of z can recovered from z by

$$\operatorname{Re} z = (z + \overline{z})/2$$
$$\operatorname{Im} z = (z - \overline{z})/2i$$

Lemma 1.9 (Triangle inequality in $\mathbb{R}^n$)**.** Suppose $a, b \in \mathbb{R}^n$, with $|a|$ the distance from a to 0 under the euclidean metric. Then

$$|a + b| \leq |a| + |b|.$$

Proof: If dot product of two vectors is zero, they are LI. Prove basis exists such that each vector dotted with all vectors in basis is zero (use nullity potentially). if a, b vectors such that $b \cdot a = 0$, then $a \cdot (a + b) = a \cdot a$. If $|a + b| < a$ then $a \cdot (a + b) < a \cdot a$, so $|a + b| \geq |a|$. $|a|, |b|$ are both geq than magnitude of their sides made of a scalar multiple of $a + b$. □

Proposition 1.10. Let $a, b \in \mathbb{C}$. Then

$$|a + b|^2 = |a|^2 + |b|^2 + a\overline{b} + b\overline{a} = |a|^2 + |b|^2 + 2\operatorname{Re} a\overline{b}.$$

Lemma 1.11 (Triangle inequality in $\mathbb{C}$)**.** For $x, y \in \mathbb{C}$, $|x + y| \leq |x| + |y|$.
Proof: Suppose $u, v \in \mathbb{R}$. Then

$$|u + iv| = \sqrt{u^2 + v^2} \geq \sqrt{u^2} = |u| \geq u.$$

Therefore $\operatorname{Re} x + y \leq |x + y|$ and

$$2\operatorname{Re} x\overline{y} \leq 2|x\overline{y}| = 2|xy| = 2|x||y|$$

Because $(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x||y|$, it follows from proposition 1.10 that $(|x| + |y|)^2 \leq (|x| + |y|)^2$, and therefore $|x + y| \leq |x| + |y|$. □

Definition 1.12 (Cauchy's inequality)**.**

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 + \sum_{i=1}^{n} |b_i|^2$$

1.2 Polar representation

Definition 1.13 (Polar representation)**.** The polar representation of a complex number $z = x + iy$ is

$$re^{i\theta} = r(\cos\theta + i\sin\theta).$$

Here $r = |z|$. The *argument* of z is a multivalued function of θ, with

$$\arg z \in \{\theta + 2\pi k \mid k \in \mathbb{Z}\}.$$

The principle value of $\arg z$ denoted $\operatorname{Arg} z$ is the unique member of $\arg z$ such that $-\pi < \operatorname{Arg} z \leq \pi$.

Definition 1.14 (de Moiver's formulae)**.** The identies obtained by equating the imaginary and real parts of the expansions of $e^{in\theta}$ and $(e^{in})^{\theta}$ are known as de Moivre's formulae, e.g.

$$\begin{aligned}
&e^{2i\theta} = (e^{i\theta})^2 \\
&\cos 2\theta + i\sin 2\theta = \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta \\
&\cos 2\theta = \cos^2\theta - \sin^2\theta \\
&\sin 2\theta = 2\cos\theta\sin\theta
\end{aligned}$$

Definition 1.15 (nth root)**.** A number $z \in \mathbb{C}$ is the nth root of $w \in \mathbb{C}$ if $z^n = w$. If $w = \rho e^{i\varphi} \neq 0$, then the nth roots of w are

$$\rho^{1/n} e^{i\varphi/n + 2\pi k/n}, \quad k = 0, 1, \ldots, n-1.$$

This is equivalent to multiplying $\rho^{1/n} e^{i\varphi/n}$ by the nth roots of unity, i.e. all nth roots of 1.

1.3 Exp, log, and power functions

Definition 1.16 (Extended complex plane)**.** The extended complex plane is the complex plane together with the point at infinity, denoted $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

Proposition 1.17. If $z \in \mathbb{C}$ with $z = x + iy$ then

$$e^z = e^x e^{iy} = e^x(\cos y + i\sin y).$$

2 Analytic Functions

2.1 Limits

Definition 2.1 (Limits)**.** If the limit of $f(z)$ as z approaches z_0 is w_0, this means that for all $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

This is written as $\lim_{z\to z_0} f(z) = w_0$. If the domain or range of the function we are taking the limit of is $\mathbb{R}^n$, the definition remains the same and uses the euclidean metric.

Lemma 2.2. The limit of a function is unique.
Proof: Suppose $f : \mathbb{C} \to \mathbb{C}$ a function and that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} f(z) = w_1$ with $w_0 \neq w_1$. Because $w_0 \neq w_1$, $|w_0 - w_1| = L > 0$. If we take $\epsilon = L/2$, there exists $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - w_0| < L/2 \wedge |f(z) - w_1| < L/2$. Because $(z - w_1) + (w_0 - z) = w_0 - w_1$, by the triangle inequality $|w_0 - w_1| \leq |f(z) - w_0| + |f(z) - w_1| = L$, a contradiction. □

Theorem 2.3. *Suppose $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, and that $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Then $\lim_{z \to z_0} f(z) = w_0$ iff*

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0. \tag{1}$$

Proof: Suppose $\lim_{z\to z_0} f(z) = w_0$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned}
&|(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow \\
&|u(x,y) - u_0 + i(v(x,y) - v_0)| = \sqrt{(u(x,y) - u_0)^2 + (v(x,y) - v_0)^2} < \epsilon.
\end{aligned}$$

Because $\sqrt{(u(x,y) - u_0)^2} \leq \sqrt{(u(x,y) - u_0)^2 + (v(x,y) + v_0)^2}$,

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow \sqrt{(u(x,y) - u_0)^2} < \epsilon.$$

Therefore wlog $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

Suppose $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$. Then for all $\epsilon > 0$ there exists $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}
\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1 \Rightarrow \sqrt{(u(x,y) - u_0)^2} < \epsilon/2, \\
\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1 \Rightarrow \sqrt{(v(x,y) - v_0)^2} < \epsilon/2.
\end{aligned}$$

If $0 < \delta < \delta_1, \delta_2$, it follows from the triangle inequality, the definition of f, and the definition of modulus that

$$|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

Thus $\lim_{z\to z_0} f(z) = w_0$. □

Remark. Going forward, for $x, y \in \mathbb{R}^n$, $d(x,y)$ refers to the euclidean metric.

Theorem 2.4. *Suppose that*

$$\lim_{z\to z_0} f(x) = w_0 \text{ and } \lim_{z\to z_0} F(z) = W_0.$$

Then the following is true:

$$\begin{aligned}
&\lim_{z\to z_0} [f(z) + F(z)] = w_0 + W_0, \\
&\lim_{z\to z_0} [f(z)F(z)] = w_0 W_0, \\
&\lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}, \quad W_0 \neq 0.
\end{aligned}$$

Proof: prove dis □