"Calculus: Early Transcendantals" Notes

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February 12, 2025

Contents

TODO: binomial theorem, cosh sinh, complete the square, difference of squares, partial fractions, 2.3 epsilon delta proofs

1 Chapter 2: Limits and continuity

1.1 Rates of change and tangent lines to curves

Definition 1.1 (Average rate of change)**.** The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$

Definition 1.2 (Secant line)**.** A line joining two points of a curve is called a secant line.

Remark. The average rate of change of f from x_1 to x_2 is the slope of the secant line between these points.

1.2 Limit of a function and limit laws

Definition 1.3 (Limit)**.** Let $f(x)$ be defined on an open interval about c. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x \to c} f(x) = L,$$

if for every number $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Definition 1.4 (Limit laws)**.** If L, M, c, k are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

$$\begin{aligned}
&\text{Sum Rule:} && \lim_{x\to c}(f(x) + g(x)) = L + M \\
&\text{Difference Rule:} && \lim_{x\to c}(f(x) - g(x)) = L - M \\
&\text{Constant Multiple Rule:} && \lim_{x\to c}(k \cdot f(x)) = k \cdot L \\
&\text{Product Rule:} && \lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M \\
&\text{Quotient Rule:} && \lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0 \\
&\text{Power Rule:} && \lim_{x\to c}[f(x)]^n = L^n, \quad n \in \mathbb{Q}^+
\end{aligned}$$

If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.

Theorem 1.5. *If $P(x)$ is some polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, then*

$$\lim_{x\to c} P(x) = P(c).$$

Theorem 1.6. *If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then*

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Theorem 1.7 (Sandwich theorem)**.** *Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that*

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

1.3 One-sided limits

Definition 1.8 (Right limit)**.** Assume the domain of f contains an interval (c, d) to the right of c. We say that $f(x)$ has a right-handed limit L at c and write

$$\lim_{x \to c^+} f(x) = L$$

if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$c < x < c + \delta |f(x - L)| < \epsilon.$$

1.4 Continuity

Definition 1.9 (Continuity)**.** Let c be a real number that is in the interval of the domain of a function f. f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

f is right-continuous at c if

$$\lim_{x \to c^+} f(x) = f(c.)$$

f is left-continuous at c if

$$\lim_{x \to c^-} f(x) = f(c).$$

Remark. If a function is not continuous at a point c of its domain, we say that f is discontinuous at c, and that f has a discontinuity at c.

Proposition 1.10 (Continuity test)**.** A function $f(x)$ is continuous at a point $x = c$ iff it meets the following three conditions:

(a) $f(c)$ exists.

(b) $\lim_{x \to c} f(x)$ exists.

(c) $\lim_{x \to c} f(x) = f(c)$.

Definition 1.11 (Continuous function)**.** A function is continous if it is continous at every point in its domain.

Theorem 1.12. *If the function f and g are continous at $x = c$, then the following algebraic combination are continous at $x = c$.*

$$\begin{aligned}
&f + g \\
&f - g \\
&k \cdot f,\ k \in \mathbb{R} \\
&f \cdot g \\
&f/g,\ g(c) \neq 0 \\
&f^n,\ n \in \mathbb{N}^+ \\
&f^{1/n},\ \ \textit{if defined on an interval containing } c.
\end{aligned}$$

Theorem 1.13. *If $\lim_{x \to c} f(x) = b$ and g is continous at the point b, then*

$$\lim_{x \to c} g(f(x)) = g(b).$$

Theorem 1.14 (Intermediate value theorem for continuous functions)**.** *If f is a continous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c \in [a, b]$.*

1.5 Limits involving infinity

Definition 1.15 (Limits approaching infinity)**.** We say that $f(x)$ has the limit L as x approaches infinity and write

$$\lim_{x\to\infty} f(x) = L$$

if, for all $\epsilon > 0$ there exists M such that for all x in the domain of f

$$x > M \Rightarrow |f(x) - L| < \epsilon$$

Theorem 1.16. *Theorem 1.4 applies to limits that appraoch infinity.*

Definition 1.17 (Horizontal asymptote)**.** A line $y = b$ is a horizontal asymptote of a graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b.$$

Definition 1.18 (Infinite limits)**.** We say that $f(x)$ approaches infinity as x approaches c and write

$$\lim_{x\to c} = \infty,$$

if for every number $B > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow f(x) > B.$$

Definition 1.19. A line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty.$$

2 Chapter 3: Derivatives

2.1 Tangent lines and the derivative

Definition 2.1 (Tangent line)**.** The slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists. The tangent line to the curve at P is the line through P with this slope.

Definition 2.2 (Derivative)**.** The derivative of a function f at a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists.

Definition 2.3 (Differentiability)**.** A function $y = f(x)$ is differentiable on an open interval if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the right/left hand limits of the derivative exist on the left/right side of the interval respectively.

Theorem 2.4. *If f has a derivative at $x = c$, then f is continuous at $x = c$.*

3 Chapter 10: Parametric equations

3.1 Parametrizations of plane curves

Definition 3.1 (Parametric curve)**.** If x and y are given as functions of t

$$x = f(t), \quad y = g(t),$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ is a parametric curve. These equations are called parametric equations for the curve.

Remark. The variable t is a parameter for the curve, and its domain I is the parameter interval. When we give parametric equations for a curve, we say that we have parameterized the curve. The equations and interval together constitute a parametrization of the curve.

3.2 Calculus with parametric curves

Remark. A parametrized curve $x = f(t)$ and $y = g(t)$ is differentiable at t if f and g are differentiable at t.

4 Multiple integrals

4.1 Double integrals over rectangles

Theorem 4.1 (Fubini's theorem)**.** *Let $f(x, y)$ be continuous on a region R.*

(a) If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx.$$

Proposition 4.2. The area of a closed, bounded plane region R is

$$A = \iint_R dA.$$

5 Line integrals

5.1 Line integrals

Definition 5.1 (Path independence)**.** Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent in D and the field $\mathbf{F}$ is conservative in D.

Theorem 5.2 (Fundamental theorem of line integrals)**.** *Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\boldsymbol{F} = \nabla f$ on a domain D containing C. Then*

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(B) - f(A).$$