Homework 2

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Problem 1

(a) $A = (1,2) \cup (2,\infty)$.

Proof: Let $(n, 2n) = A_n$ with $n \in \mathbb{N}$, then $A_1 = (1,2)$. Iff $x \in \bigcup_{n=2}^{3} A_n$, then

$$x \in (2,4) \wedge x \in (3,6) \Leftrightarrow 2 < x < 6 \Leftrightarrow x \in (2,6).$$

So $\bigcup_{n=2}^{3} A_n = (2,6)$. By inductive hypothesis, if $m \in \mathbb{N}$ with $m \geq 2$ then $\bigcup_{n=2}^{m} A_n = (2, 2m)$. If $x \in \bigcup_{n=2}^{m+1} A_n$ then

$$x \in \bigcup_{n=2}^{m} A_n \cup (m+1, 2(m+1)) \Leftrightarrow (2 < x < 2m) \vee (m+1 < x < 2(m+1)) \Leftrightarrow x \in (2, 2(m+1))$$

Therefore $\bigcup_{n=1}^{m+1} A_n = (2, 2(m+1))$. Because for all $\epsilon > 0$ there exists $N = \lceil \epsilon \rceil$ such that $2N > \epsilon$, We see that $\bigcup_{n=2}^{\infty} A_n = (2,\infty)$, so $A = (1,2) \cup (2,\infty)$. □

(b) $C = (-\infty, \infty)$.

Proof: Let $C_n = (n, n+2)$. Then $C_0 = (0,2)$, $C_1 = (1,3)$, and $C_2 = (2,4)$, and thus $\bigcup_{n=0}^{2} C_n = (0,4)$. By inductive hypothesis, if $m \in \mathbb{N}$ then $\bigcup_{n=1-m}^{1+m} C_m = (1-m, 3+m)$. Then

$$\bigcup_{n=1-(m+1)}^{1+(m+1)} C_n \Leftrightarrow (-m < x < 2-m) \vee (2+m < x < 4+m) \vee (1-m < x < 3+m) \Leftrightarrow x \in (1-(m+1), 3+(m+1)).$$

Because for all $\epsilon > 0$ there exists $N = 3\lceil \epsilon \rceil$ such that $|3+N|, |1-N| > \epsilon$, so $C = (-\infty, \infty)$. □

Problem 2

Define $f : (1,4) \to \mathbb{R}$

(a) Is f one-to-one? No. $f^{-1}(\{-\frac{1}{2}\}) = \{2,3\}$.

(b) What is $f\big((1,2)\big)$? $(-\infty, -1/2)$. $f(x)$ has no critical points on the interval $(1,2)$, so $f\big((1,2)\big)$ is the interval $(\lim_{x\to 1^+} f(x), f(2))$.

(c) What is $f\big((1,3)\big)$? $(-\infty, -\frac{4}{9})$. $f(x)$ has one critical point at $x = 2.5$, and because $lim_{x\to 1^+} f(x) = -\infty$ and $f(3 = -1/2)$, $f(2.5) = \frac{4}{9}$ is the relative maximum value on $(1,3)$.

Problem 3

(a) Is f one-to-one? No. $f(\frac{4}{3}) = f(-\frac{2}{3})$.

(b) Is f onto? No. The codomain of f is always positive so there exists x in the codomain $\mathbb{Z}$ such that $x \notin f(\mathbb{Z})$.

Problem 4

(a) The pre-image of $(0,1)$ in $\mathbb{R}$, $f^{-1}\big((0,1)\big) = (-\infty, \frac{1}{2})$.

(b) $f^{-1}\big((1,2)\big) = (\frac{1}{2}, 1)$.

(c) $f^{-1}\big((-\infty,2)\big) = (-\infty, 1)$.

Problem 5

(a) The image of $\{1,2\}$ in $\mathbb{R}$, $g\big(\{1,2\}\big) = \{1,2\}$.

(b) $g\big((1,2)\big) = \{1, \frac{3}{2}\}$.

(c) $g\big([1,2]\big) = \{1, \frac{3}{2}, 2\}$.