Complex Variables

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Contents

1 Complex numbers

1.1 Fundamental definitions and identities

Definition 1.1 (Complex number)**.** A complex number is an expression of the form $z = x + iy$, where x and y are real numbers.

Definition 1.2. Ever complex number $z \neq 0$ has a multiplicative inverse given by

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

Definition 1.3 (Modulus)**.** The modulus of a complex number $z = x + iy$ is the length of the vector (x, y), and is denoted $|z|$.

$$|z| = \sqrt{x^2 + y^2}.$$

Proposition 1.4. For $z, w \in \mathbb{C}$, it follows from the triangle inequality that

$$\begin{aligned} &|z + w| \leq |z| + |w| \\ &|z - w| \geq |z| - |w| \end{aligned}$$

Definition 1.5 (Multiplication)**.** $(x + iy)(u + iv) = xu - yv + i(xv + yu)$.

Definition 1.6 (Complex conjugate)**.** The complex conjugate of a complex number $z = x + iy$ is defined to be $\overline{z} = x - iy$.

Proposition 1.7. For $z, w \in \mathbb{C}$, the following identities hold:

$$\begin{aligned} &\overline{\overline{z}} = z \\ &\overline{z + w} = \overline{z} + \overline{w} \\ &\overline{zw} = \overline{z}\overline{w} \\ &\overline{z\overline{w}} = \overline{z}w \\ &|z| = |\overline{z}| \\ &|z|^2 = z\overline{z} \\ &|zw| = |z||w| \\ &\operatorname{Re} z + \operatorname{Re} w = \operatorname{Re} z + w \\ &\operatorname{Im} z + \operatorname{Im} w = \operatorname{Im} z + w \end{aligned}$$

Proposition 1.8. The real and imaginary parts of z can recovered from z by

$$\begin{aligned} &\operatorname{Re} z = (z + \overline{z})/2 \\ &\operatorname{Im} z = (z - \overline{z})/2i \end{aligned}$$

Lemma 1.9 (Triangle inequality in $\mathbb{R}^n$)**.** Suppose $a, b \in \mathbb{R}^n$, with $|a|$ the distance from a to 0 under the euclidean metric. Then

$$|a + b| \leq |a| + |b|.$$

Proof: If dot product of two vectors is zero, they are LI. Prove basis exists such that each vector dotted with all vectors in basis is zero (use nullity potentially). if a, b vectors such that $b \cdot a = 0$, then $a \cdot (a + b) = a \cdot a$. If $|a + b| < a$ then $a \cdot (a + b) < a \cdot a$, so $|a + b| \geq |a|$. $|a|, |b|$ are both geq than magnitude of their sides made of a scalar multiple of $a + b$. □

Proposition 1.10. Let $a, b \in \mathbb{C}$. Then

$$|a + b|^2 = |a|^2 + |b|^2 + a\overline{b} + b\overline{a} = |a|^2 + |b|^2 + 2\operatorname{Re} a\overline{b}.$$

Lemma 1.11 (Triangle inequality in $\mathbb{C}$)**.** For $x, y \in \mathbb{C}$, $|x + y| \leq |x| + |y|$.

Proof: Suppose $u, v \in \mathbb{R}$. Then

$$|u + iv| = \sqrt{u^2 + v^2} \geq \sqrt{u^2} = |u| \geq u.$$

Therefore $\operatorname{Re} x + y \leq |x + y|$ and

$$2\operatorname{Re} x\overline{y} \leq 2|x\overline{y}| = 2|xy| = 2|x||y|$$

Because $(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x||y|$, it follows from proposition 1.10 that $(|x| + |y|)^2 \leq (|x| + |y|)^2$, and therefore $|x + y| \leq |x| + |y|$. □

Definition 1.12 (Cauchy's inequality)**.**

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 + \sum_{i=1}^{n} |b_i|^2$$

Proposition 1.13. If $z = x + iy$ then

$$\begin{aligned} |x| &\leq |z|, \\ |y| &\leq |z|. \end{aligned}$$

Proof: From the definition of modulus,

$$|z|^2 = x^2 + y^2. \tag{1}$$

Thus wlog

$$\begin{aligned} x &= \pm\sqrt{|z|^2 - y^2}, \\ |x| &= \sqrt{|z|^2 - y^2}. \end{aligned}$$

Suppose to the contrary $|x| > |z|$. Then $|x|^2 = x^2 > |z|^2$, and because y^2 is nonnegative, equation 1 is not true. □

1.2 Polar representation

Definition 1.14 (Polar representation)**.** The polar representation of a complex number $z = x + iy$ is

$$re^{i\theta} = r(\cos\theta + i\sin\theta).$$

Here $r = |z|$. The *argument* of z is a multivalued function of θ, with

$$\arg z \in \{\theta + 2\pi k \mid k \in \mathbb{Z}\}.$$

The principle value of $\arg z$ denoted $\operatorname{Arg} z$ is the unique member of $\arg z$ such that $-\pi < \operatorname{Arg} z \leq \pi$.

Remark. The value of certain functions depend on the argument of elements in it's domain. Thus in order to prevent the function from being multi-valued, a 'branch' of the domain must be chosen with only one argument. In order to maintain continuity, 'branch cuts' must be made to the functions domain, where the argument of elements on a branch coincide.

Definition 1.15 (de Moiver's formulae)**.** The identies obtained by equating the imaginary and real parts of the expansions of $e^{in\theta}$ and $(e^{in})^\theta$ are known as de Moivre's formulae, e.g.

$$\begin{aligned} &e^{2i\theta} = (e^{i\theta})^2 \\ &\cos 2\theta + i\sin 2\theta = \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta \\ &\cos 2\theta = \cos^2\theta - \sin^2\theta \\ &\sin 2\theta = 2\cos\theta\sin\theta \end{aligned}$$

Definition 1.16 (nth root)**.** A number $z \in \mathbb{C}$ is the nth root of $w \in \mathbb{C}$ if $z^n = w$. If $w = \rho e^{i\varphi} \neq 0$, then the nth roots of w are

$$\rho^{1/n} e^{i\varphi/n + 2\pi k/n}, \quad k = 0, 1, \ldots, n-1.$$

This is equivalent to multiplying $\rho^{1/n} e^{i\varphi/n}$ by the nth roots of unity, i.e. all nth roots of 1.

1.3 Exp, log, and power functions

Definition 1.17 (Extended complex plane)**.** The extended complex plane is the complex plane together with the point at infinity, denoted $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

Proposition 1.18. If $z \in \mathbb{C}$ with $z = x + iy$ then

$$e^z = e^x e^{iy} = e^x(\cos y + i\sin y).$$

2 Analytic Functions

Remark. Going forward, if a function's domain and codomain are not specified, the function is from $\mathbb{C}$ to $\mathbb{C}$.

2.1 Limits

Definition 2.1 (Limits)**.** If the limit of $f(z)$ as z approaches z_0 is w_0, this means that for all $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

This is written as $\lim_{z \to z_0} f(z) = w_0$. If the domain or range of the function we are taking the limit of is $\mathbb{R}^n$, the definition remains the same and uses the euclidean metric.

Lemma 2.2. The limit of a function is unique.

Proof: Suppose $f : \mathbb{C} \to \mathbb{C}$ a function and that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} f(z) = w_1$ with $w_0 \neq w_1$. Because $w_0 \neq w_1$, $|w_0 - w_1| = L > 0$. If we take $\epsilon = L/2$, there exists $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - w_0| < L/2 \wedge |f(z) - w_1| < L/2$. Because $(z - w_1) + (w_0 - z) = w_0 - w_1$, by the triangle inequality $|w_0 - w_1| \leq |f(z) - w_0| + |f(z) - w_1| = L$, a contradiction. □

Theorem 2.3. *Suppose* $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$, *and that* $z_0 = x_0 + iy_0$ *and* $w_0 = u_0 + iv_0$. *Then* $\lim_{z \to z_0} f(z) = w_0$ *iff*

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0. \tag{2}$$

Proof: Suppose $\lim_{z \to z_0} f(z) = w_0$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned}
&|(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow \\
&|u(x, y) - u_0 + i(v(x, y) - v_0)| = \sqrt{(u(x, y) - u_0)^2 + (v(x, y) - v_0)^2} < \epsilon.
\end{aligned}$$

Because $\sqrt{(u(x, y) - u_0)^2} \leq \sqrt{(u(x, y) - u_0)^2 + (v(x, y) + v_0)^2}$,

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow \sqrt{(u(x, y) - u_0)^2} < \epsilon.$$

Therefore wlog $\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$.

Suppose $\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$. Then for all $\epsilon > 0$ there exists $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}
\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1 &\Rightarrow \sqrt{(u(x, y) - u_0)^2} < \epsilon/2, \\
\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1 &\Rightarrow \sqrt{(v(x, y) - v_0)^2} < \epsilon/2.
\end{aligned}$$

If $0 < \delta < \delta_1, \delta_2$, it follows from the triangle inequality, the definition of f, and the definition of modulus that

$$|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

Thus $\lim_{z \to z_0} f(z) = w_0$. □

Remark. Going forward, for $x, y \in \mathbb{R}^n$, $d(x, y)$ refers to the euclidean metric.

Theorem 2.4. *Suppose that*

$$\lim_{z \to z_0} f(x) = w_0 \text{ and } \lim_{z \to z_0} F(z) = W_0.$$

Then the following is true:

$$\begin{aligned}
&\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0, \\
&\lim_{z \to z_0} [f(z)F(z)] = w_0 W_0, \\
&\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}, \quad W_0 \neq 0.
\end{aligned}$$

Proof: prove □

Theorem 2.5. *If z_0 and w_0 are points in the z and w planes respectively, then the following properties hold:*

$$\lim_{z \to z_0} f(z) = \infty \Leftrightarrow \lim_{z \to z_0} \frac{1}{f(z)} = 0$$
$$\lim_{z \to \infty} f(z) = w_0 \Leftrightarrow \lim_{z \to 0} f(\frac{1}{z}) = w_0$$
$$\lim_{z \to \infty} f(z) = \infty \Leftrightarrow \lim_{z \to 0} \frac{1}{f(1/z)} = 0$$

Proof: prove □

2.2 Derivatives

Definition 2.6 (Continuity)**.** A function f is continuous at a point z_0 if all three of the following conditions are satisfied:

(a) $\lim_{z \to z_0} f(z)$ exists.

(b) $f(z_0)$ exists.

(c) $\lim_{z \to z_0} f(z) = f(z_0)$.

Theorem 2.7. *The composition of continuous functions is continuous.*

Proof: prove □

Theorem 2.8. *If a function $f(z)$ is continuous and nonzero at a point z_0, then $f(z) \neq 0$ throughout some neighborhood of that point*

Proof: prove □

Theorem 2.9. *If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M and $z' \in R$ such that*

$$\forall z \in R,\ |f(z)| \leq M$$

and

$$f(z') = M.$$

Proof: prove □

Definition 2.10 (Derivative)**.** Let f be a function whose domain of definition contains an ϵ-neighborhood of z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

The function f is said to be differentiable at z_0 if $f'(z_0)$ exists. If we set $\Delta z = z - z_0$, we can write the definition as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

When using this form of the derivative, the subscript on z is often dropped and we introduce the number $\Delta w = f(z + \Delta z) - f(z)$ so that the derivative becomes

$$f'(z_0) = \frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

Proposition 2.11. Because the derivative is a limit, if it exists it must be unique.

Proposition 2.12. If a function $f : \mathbb{C} \to \mathbb{C}$ is differentiable at a point $z_0 \in \mathbb{C}$, then f is continuous at z_0.

Proposition 2.13 (Differentiation formulas)**.** Let $c \in \mathbb{C}$ be a constant, $z \in \mathbb{C}$ an independent variable, $n \in \mathbb{Z}$, and f a function from $\mathbb{C} \to \mathbb{C}$ which is differentiable at z. These differentiation formulas can be derived from the definition of the derivative:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z} c = 0 \\
&\frac{\mathrm{d}}{\mathrm{d}z} z = 1 \\
&\frac{\mathrm{d}}{\mathrm{d}z} [cf(z)] = cf'(z) \\
&\frac{\mathrm{d}}{\mathrm{d}z} z^n = nz^{n-1} \\
&\frac{\mathrm{d}}{\mathrm{d}z} [f(z) + g(z)] = f'(z) + g'(z) \\
&\frac{\mathrm{d}}{\mathrm{d}z} f(z)g(z) = f(z)g'(z) + f'(z)g(z) \\
&\frac{\mathrm{d}}{\mathrm{d}z} \frac{f(z)}{g(z)} = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}
\end{aligned}$$

Theorem 2.14 (Chain rule)**.** *If f, g functions from $\mathbb{C} \to \mathbb{C}$ differentiable at $z \in \mathbb{C}$, then*

$$\frac{\mathrm{d}}{\mathrm{d}z} g \circ f(z) = g' \circ f(z) \cdot f'(z).$$

2.3 Cauchy-Riemann equations

Theorem 2.15. *Suppose that $f(z) = u(x, y) + iv(x, y)$ with $z = x + iy$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Additionally, let u_x, v_x, u_y, v_y be the partial derivatives of the component functions of f with respect to x and y at x_0, y_0. Then the first order partial derivatives of u and v exist at x_0, y_0, and they must satisfy the Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

at that point. In polar form, these equations are

$$ru_r = v_\theta, \quad u_\theta = -rv_r.$$

Proof: The proof follows from the equality of the limit definition of the derivative approaching a point z_0 from the x-axis, and approaching z_0 from the y-axis. We utilize the fact that the limit of a sum of component functions of z is the sum of the limits of the individual functions. These individual limits then become the partial derivatives of f with respect to x and y. Approaching z_0 along the x-axis, the derivative of $f = u(x, 0) + v(x, 0)$ at z_0 is

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} + i\frac{v(x + \Delta x) - v(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \to 0} i\frac{v(x + \Delta x) - v(x)}{\Delta x}
\end{aligned}$$

Approaching z_0 along the y-axis, the derivative of $f(u(0, y) + v(0, y))$ at z_0 is

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta y \to 0} \frac{u(y + \Delta y) - u(y)}{i\Delta y} + i\frac{v(y + \Delta y) - v(y)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(y + \Delta y) - u(y)}{i\Delta y} + \lim_{\Delta y \to 0} i\frac{v(y + \Delta y) - v(y)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} -i\frac{u(y + \Delta y) - u(y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(y + \Delta y) - v(y)}{\Delta y}
\end{aligned}$$

Thus $u_x = v_y$ and $u_y = -v_x$. □

Remark. $1/i = -i$.

Corollary 2.16. $f'(z_0) = u_x + iv_x = v_y - iu_y$.

Theorem 2.17. *Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ-neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that:*

(a) The first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood.

(b) Those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations at (x_0, y_0).

Then $f'(z_0)$ exists.

Proof: prove □

Theorem 2.18. *Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be defined throughout some ϵ-neighborhood of a nonzero point $z_0 = r_0 \exp(i\theta_0)$, and suppose that*

(a) The first order partial derivatives of the function u and v with respect to r and θ exists everywhere in the neighborhood;

(b) Those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form of the Cauchy Reimann equations.

Then $f'(z_0)$ exists, its value being $f'(z_0) = \exp(i\theta_0)(u_r + iv_r)$.

Proof: prove □

2.4 Analytic and harmonic functions

Definition 2.19 (Analytic function)**.** A function f of the complex variable z is analytic at a point z_0 if it has a derivative at each point in some neighborhood z_0. A function is analytic in an open set if it has a derivative everywhere in that set.

Definition 2.20 (Entire function)**.** An entire function from $\mathbb{C} \to \mathbb{C}$ is a function that is analytic at each point in its domain.

Remark. Every polynomial is an entire function.

Definition 2.21 (Singular point)**.** If $f : \mathbb{C} \to \mathbb{C}$ is a function not analytic at a point z_0, but analytic at some point in every neighborhood of z_0, then z_0 is called a singular point.

Proposition 2.22. If two functions P and Q are analytic in a domain D, their sum and product are analyitic. Their quotient is analytic in D provided the denominator is nonzero in D.

Proposition 2.23. The composition of two analytic functions is analytic.

Theorem 2.24. *If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.*

Definition 2.25 (Harmonic)**.** A real-valued function H of two real variables x and y is said to be harmonic in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order, and satisfies

$$H_{xx} + H_{yy} = 0.$$

Theorem 2.26. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then it's component functions u and v are harmonic in D.*

Definition 2.27 (Harmonic conjugate)**.** If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy-Riemann equations (in the same order as u and v appear in the c.r. equations) throughout D, then v is said to be the harmonic conjugate of u.

Remark. It is possible that v is the harmonic conjugate of u with the reverse not being true.

Theorem 2.28. *A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D iff v is the harmonic conjugate of u.*

2.5 read 27/28

3 Elementary functions

Definition 3.1 (Branch). A branch of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f.

Definition 3.2 (Complex exponents). When $z \neq 0$ and the exponent c is any complex number, the function z^c is defined by means of the equation

$$z^c = e^{c \log z}.$$

Definition 3.3 (log and Log). Let $z = re^{i\theta}$. $\log z$ is a multiple-valued function defined by

$$\log z = \ln|z| + i \arg z.$$

$\operatorname{Log} z$ is the single valued function

$$\operatorname{Log} z = \ln r + i\Theta,$$

Where $\Theta = \operatorname{Arg} z$. $\operatorname{Log} z$ is not analytic because it is not continuous when $z \in \mathbb{R}^-$.

Proposition 3.4 (Branches and derivatives of logarithms). If $\theta = \Theta + 2n\pi$ for $n \in \mathbb{Z}$, and $z = re^{i\theta}$, then $\log z$ can be defined

$$\log z = \ln r + i\theta.$$

If we let α denoted any real number, and $\alpha < \theta < \alpha + 2\pi$ with $r > 0$, $\log z$ becomes a single-valued continuous function. $\log z$ restricted to this domain is a branch of $\log z$. If $\alpha = -\pi$, this branch is called the principal branch, and is equal to $\operatorname{Log} z$ with the additional restriction $\Theta < \pi$.

Definition 3.5 (Sin and Cos).

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Definition 3.6 (Sinh and Cosh).

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$
$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

Proposition 3.7.

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$
$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

4 Integrals

4.1 Derivatives and contours

Definition 4.1 (Derivative). Let $w(t) = u(t) + iv(t)$ be a complex-valued function of a real variable, where the functions $u(t)$ and $v(t)$ are real valued functions of a real variable. Then the derivative of $w(t)$ with respect to t is

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t) + i\frac{\mathrm{d}}{\mathrm{d}t}v(t).$$

Definition 4.2 (Definite integral for function of a real variable)**.** When $w(t)$ is a complex-valued function of a real variable t, written

$$w(t) = u(t) + iv(t),$$

where u and v are real-valued, the definite integral of $w(t)$ over an interval $a \leq t \leq b$ is

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt,$$

Provided the integrals on the left exist.

Remark. This is analagous to derivatives of vector functions in calculus, where the definite integral of a function $f : \mathbb{R} \to \mathbb{R}^n$ is a vector in $\mathbb{R}^n$.

Definition 4.3 (Arc)**.** A set of points $z = (x, y)$ in the complex plane is said to be an arc if

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b$$

where x and y are continuous functions of the real parameter t. An arc C is called a simple arc, or a Jordan arc, if it does not cross itself. When the arc is simple except for the fact that $z(b) = z(a)$, we say that C is a simple closed curve. Such a curve is positively oriented when it is in the counterclockwise direction.

Definition 4.4 (Smooth arc)**.** A smooth arc $z = z(t)$ defined on $a \leq t \leq b$ has continuous first derivatives on its domain $a \leq t \leq b$ which are nonzero on $a < t < b$.

Definition 4.5 (Contour)**.** A contour, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. When only the initial and final values of a contour C are the same, we say C is a simple closed contour.

Theorem 4.6 (Jordan curve theorem)**.** *The points on any simple closed contour C are boundary points of two distinct domains. One of these domains is the interior of C, and is bounded. The other is the exterior of C, and is unbounded.*

4.2 Contour integrals

Suppose a contour C is represented by the function $z : \mathbb{R} \to \mathbb{C}$ on the interval $(a \leq t \leq b)$. If $f : \mathbb{C} \to \mathbb{C}$ a function, and $f[z(t)]$ is piecewise continuous on the interval $a \leq t \leq b$, the function f is piecewise continuous on C. We then define the contour integral of f along C as

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt.$$

Proposition 4.7. It follows from the properties of complex-valued functions of a real variable that

$$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz,$$
$$\int_C \big[f(z) + g(z)\big]dz = \int_C f(z)dz + \int_C g(z)dz.$$

Remark. prove statements of section 40, 41, 42

Proposition 4.8.

4.3 Contour integral other shit

Lemma 4.9. If $w : \mathbb{R} \to \mathbb{C}$ is piecewise continuous on an interval $a \leq t \leq b$, then

$$\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt.$$

Theorem 4.10. *Let C denote a contour of length L, and suppose that a function $f : \mathbb{C} \to \mathbb{C}$ is continuous on C. If M is a nonnegative constant such that*

$$|f(z)| \leq M,$$

for all points z on C at which $f(z)$ is defined, then

$$\left| \int_C f(z)dz \right| \leq ML.$$

Theorem 4.11. *Suppose that a function $f : \mathbb{C} \to \mathbb{C}$ is continuous on domain D. The following statements are logically equivalent:*

(a) f has an antiderivative F throughout D.

(b) The integrals of f along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value.

(c) The integrals around closed contours lying entirely in D are equal to zero.

Theorem 4.12 (Cauchy-Goursat theorem)**.**
If a function $f : \mathbb{C} \to \mathbb{C}$ is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)dz = 0.$$

Theorem 4.13. *If a function $f : \mathbb{C} \to \mathbb{C}$ is analytic througout a simply connected domain D, then*

$$\int_C f(z)dz = 0$$

for every closed contour C lying in D.

Remark. Notice the lack of specificity that this contour is simple.

Theorem 4.14. *Suppose that*

(a) C is a simple closed contour, described in the counterclockwise direction.

(b) C_k, $k = 1, \ldots, n$ are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common.

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k, then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0.$$

Corollary 4.15 (Principle of deformation of paths)**.** Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If a function f is analytic in the closed region consisting of these contours and all points between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Theorem 4.16 (Cauchy integral formula)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0}.$$

This formula can be extended to find the derivative of functions analytic on and inside the interior of a simple closed contour C with repect to a point z on the interior of this curve:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)ds}{(s - z)^{n+1}},\ n \in \mathbb{Z}^+.$$

Theorem 4.17. *If a function f is analytic at a given point, then it's derivatives of all orders are analytic there too.*

Remark. read 51-53

Lemma 4.18. Suppose that $|f(z)| \leq |f_{z_0}|$ at each point z in some neighborhood $|z - z_0| < \epsilon$ in which f is analytic. Then $f(z)$ has constant value $f(z_0)$ throughout that neighborhood.

Theorem 4.19. *If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has non maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points z in it.*

Corollary 4.20. Suppose that a function f is continuous on a closed bounded region R, and that it is analytic and not constant in the interior of R. then the maximum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.

Lemma 4.21. Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \epsilon$ in which f is analytic. Then $f(z)$ has constant value $f(z_0)$ throughout that neighborhood.

Theorem 4.22. *If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points $z \in D$.*

5 Sequences

5.1 Convergence

Definition 5.1 (Convergence of sequences)**.** A sequence $(z_n)_{n=1}^{\infty}$ of complex numbers has a limit z if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \big(n > N \Rightarrow |z_n - z| < \epsilon\big)$$

When a sequence has a limit, it is said to converge.

Theorem 5.2. *Suppose that $z_n = x_n + iy_n$. Then $\lim_{n\to\infty} z_n = z$ iff*

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y.$$

Definition 5.3 (Convergence of series)**.** An infinite series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \dots$$

of complex numebrs converges to the sum S if the sequence

$$S_N = \sum_{n=1}^{N} z_n$$

of partial sums converges to S. We then write

$$\sum_{n=1}^{\infty} z_n = S.$$

Theorem 5.4. *Suppose that $z_n = x_n + iy_n$ and $S = X + iY$. Then*

$$\sum_{n=1}^{\infty} z_n = S \Leftrightarrow \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

Corollary 5.5. If a series of complex numbers converges, the nth term converges to zero as n tends to infinity.

5.2 Taylor series

Theorem 5.6. *Suppoe that a function f is analytic throughout a disk $|z-z_0| < R_0$, centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad (|z-z_0| < R_0),$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad n \in \mathbb{Z}^+.$$

Definition 5.7 (Maclaurin series)**.** A Maclaurin series is a Taylor series centered at $z_0 = 0$.

Theorem 5.8. *Suppose that a function f is analytic throughout an annular domain $R_1 < |z-z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad (R_1 < |z-z_0| < R_2).$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \quad n = 0, 1, \ldots,$$

and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{-n+1}} \quad n = 1, 2, \ldots.$$

5.3 Absolute and uniform convergence

Definition 5.9 (Absolute convergence)**.** A series of complex numbers converges absolutely if the series of absolute values of those numbers converges.

Theorem 5.10. *If a power series*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

converges when $z = z_1$ with $z_1 \neq z_0$, then it is absolutely convergent at each point z in the open disk $|z-z_0| < R_1$ where $R_1 = |z_1 - z_0|$.

Remark. The greatest circle centered at z_0 such that the above series converges is called the circle of convergence.

Remark. when proving this, go over uniform convergence.

Theorem 5.11. *A power series*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

represents a continuous function $S(z)$ at each point inside its circle of convergence $|z-z_0| = R$.

5.4 Integration and differentiation of power series

Theorem 5.12. *Let C denote any contour interior to the circle of convergence of the power series*

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \tag{3}$$

and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C, i.e.

$$\int_C g(z)S(z)dz = \sum_{n=1}^{\infty} a_n \int_C g(z)(z-z_0)^n dz.$$

Corollary 5.13. The sum $S(z)$ of power series in equation 3 is analytic at each point z interior to the circle of convergence of that series.

Theorem 5.14. *The power series in equation 3 can be differentiated term by term, i.e. at each point z interior to the circle of convergence of that series,*

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Theorem 5.15. *If a series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges to $f(z)$ at all points interior to some circle $|z - z_0| = R$, then it is the Taylor series expansion for f in powers of $z - z_0$.

Theorem 5.16. *If a series*

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z - z_0$ for that domain.

5.5 Multiplication and division of power series

Remark. read 67

6 Residues and Poles

6.1 Residues

Definition 6.1. The residue of f at a point z_0 is the coefficient b_1 for the Laurent series valid in a deleted neighborhood of z_0, i.e.

$$2\pi i \mathrm{Res}_{z=z_0} f(z) = \int_C f(z) dz.$$

The residue at infinity is b_1 term for the Laurent series valid as $|z| \to \infty$.

Remark. The Cauchy-Goursat theorem implies that if f is analytic inside and on a simple closed curve C, the residue of f at any point interior to C must be zero. Alternatively, if f is analytic, all coefficients on the principle part of the laurent series must be zero due to the integrand for these coefficients becoming analytic. It should be noted that a functions analyticity on a simply connected domain is a sufficient but not necessary condition for the integrals of simple closed curves being zero on that domain.

6.2 Classifying singular points

Definition 6.2 (Singular point)**.** A point z_0 is called a singular point of f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.

Definition 6.3 (Isolated point)**.** A singular point is said to be isolated if there is a deleted neighborhood of z_0 throughout which f is analytic.

Theorem 6.4. *Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k ($k \in \mathbb{Z}^+$) inside C, then*

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z).$$

Definition 6.5 (Types of singular points)**.** If the principle part of the Laurent series at an isolated singular point has a minimum degree $-m$, then pole is called a pole of order m. A pole of order $m = 1$ is referred to as a simple pole.

If the principle part of the Laurent series at an isolated singular point has all zero coefficients, the singularity at z_0 is said to be removable.

If an infinite number of the coefficients in the principle part are nonzero, z_0 is said to be an essential singular point of f.

Theorem 6.6. *An isolated singular point z_0 of a function f is a pole of order m iff $f(z)$ can be written in the form*

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where $\phi(z)$ is analytic and nonzero at z_0. Moreover,

$$Res_{z=z_0} f(z) = \phi(z_0), \text{ if } m = 1$$

and

$$Res_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \text{ if } m \geq 2.$$

Definition 6.7 (Zeros of analytic functions)**.** Suppose a function f is analytic at a point z_0. If $f(z_0) = 0$, and if there is a positive integer m with $f^{(m)}(z_0) \neq 0$ such that $n > m$ implies $f^{(n)}(z_0) = 0$, then f is said to have a zero of order m at z_0.

Theorem 6.8. *Let a function f be analytic at a point z_0. It has a zero of order m at z_0 iff there is a function g, which is analytic and nonzero at z_0, such that*

$$f(z) = (z-z_0)^m g(z).$$

Theorem 6.9. *Suppose that*

(a) *Two functions p and q are analytic at a point z_0;*

(b) *$p(z_0) \neq 0$ and q has a zero of order m at z_0.*

Then the quotient $p(z)/q(z)$ has a pole of order m at z_0.

Theorem 6.10. *Let two functions p and q be analytic at a point z_0. If*

$$p(z_0) \neq 0, \; q(z_0) = 0, \; q'(z_0) \neq 0,$$

then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$Res_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

7 Mappings by elementary functions

Definition 7.1 (Bilinear transformation)**.** The transformation

$$w = \frac{az+b}{cz+d}, \; (ad-bc \neq 0),$$

where $a, b, c, d \in \mathbb{C}$, is called a linear fractional transformation. This is a bijective mapping from the extended z-plane to the extended w-plane. Solving this equation for z yields

$$z = \frac{-dw+b}{cw-a}, \; (ad-bc \neq 0).$$

The equation

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Defines a linear fractional transformation that maps z_1, z_2, z_3 in the finite z plane onto distinct points w_1, w_2, w_3 respectively in the finite w plane.