Homework 5

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Problem 1

Suppose suppose $S \subseteq \mathbb{R}$ a nonempty set bounded above, and $x = \sup(S)$. Let $N(x, \epsilon)$, with $\epsilon > 0$ be an arbitrary ϵ-neighborhood of x. If $a \in N(x, \epsilon)$, then

$$x - \epsilon < a < x + \epsilon, \text{ i.e.}$$
$$a \in (x - \epsilon, x + \epsilon).$$

If $a < x$, it follows from the definition of supremum that there exists $s \in S$ such that $a < s < x$, and thus $s \in N(x, \epsilon)$. If $a > x$, then because all elements of S are less than x, a is not is S and thus $a \in \mathbb{R} \setminus S$. Thus for all ϵ, the intersection of $N(x, \epsilon)$ with S as well as its intersection with $\mathbb{R} \setminus S$ are nonempty, so x is a boundary point of S.

Problem 2

Suppose $S \subseteq T \subseteq \mathbb{R}$. If x is an interior point of S, then there exists $\epsilon > 0$ such that $N(x, \epsilon) \subseteq S$. Because $S \subseteq T$, it follows that $N(x, \epsilon) \subseteq T$, so x is an interior point of T, i.e. $\text{int}(S) \subseteq T$.

Problem 3

(a) $S = (1, 4)$. The interior of S is equal to S because the interval is open.

(b) $S = [0, 1)$. The interior of S is $(0, 1)$.

Problem 4

(a) $S = (-\sqrt{2}, -1) \cup (1, \sqrt{2})$. The interior of S is equal to S because the interval is open.

(b) $S = (-1, 1)$. The interior of S is equal to S similarly to the above.

Problem 5

Assuming $n \in \mathbb{N}$, an open cover for S such that no finite subcover of S is contained in S is

$$\left\{ N_n\left(x,\ \frac{1}{2n} - \frac{1}{2(n+1)}\right) \;\middle|\; n \in \mathbb{N} \right\}$$

The center point of each neighborhood N_n is positioned such that the N_{n-1} neighborhood includes all points closer to it's center than the center of N_n. Therefore, if any element of the open cover N_k with $k \in \mathbb{N}$ is removed, then the k^{th} element of S will not be in the union of the subcover.