Homework 4

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Problem 1

Let $E \subseteq \mathbb{R}^n$ be bounded. Then there exists $b \in \mathbb{R}$ with $b > 0$ such that for all $f, g \in E$ we have $d(f, g) < b$. Let $x = (x_1, \dots, x_n) \in E$ with $x_{mabs} = \max\bigl(\bigl\{|x_i|\bigr\}_{i=1}^n\bigr)$ and $l = b + x_{mabs}$. Suppose to the contrary that $E \not\subseteq [-l, l]^n$. Then there exists $y = (y_1, \dots, y_n) \in E$ such that $y \notin [-l, l]^n$, so there exists $k \in \mathbb{N}$ with $k \leq n$ such that $|y_k| > l$. It follows that

$$\begin{aligned} d(x, y) &= \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} \\ &\geq \sqrt{(x_k - y_k)^2}. \end{aligned}$$

Because $|y_k| = l + a$ for some $a \in \mathbb{R}$ with $a > 0$, it follows that

$$\begin{aligned} \sqrt{(x_k - y_k)^2} &= \bigl|x_k \pm (b + x_{mabs} + a)\bigr| \\ &\geq \bigl| \pm (b + a)\bigr| \\ &> b. \end{aligned}$$

This contradicts the fact that $d(x, y) < b$.

Problem 2

Suppose $[-l, l]^n \subseteq \mathbb{R}^n$ with $l > 0$, let $c = \lceil \frac{ln}{\epsilon} \rceil$, let $k \in \mathbb{N}$ with $k \leq n$, let $\epsilon \in \mathbb{R}$ with $\epsilon > 0$, and A a set of n-tuples with

$$A = \{(a_1, \dots, a_n) \mid a_k = \frac{i\epsilon}{n},\ -c \leq i \leq c,\ i \in \mathbb{Z}\}$$

Suppose $x = (x_1, \dots, x_n) \in [-l, l]^n$. For all x there exists $y = (y_1, \dots, y_n) \in A$ such that for all $j \in \mathbb{N}$ with $j \leq n$ we have component x_j of x and component y_j of y with $|x_j - y_j| < \frac{\epsilon}{n}$. It follows from the triangle inequality that $d(x, y) < \epsilon$, so $x \in B(y, \epsilon)$, and $\bigcap_{\alpha \in A} B(\alpha, \epsilon)$ a cover for $[-l, l]$, and therefore is totally bounded.

Problem 3

Because $\mathbb{Q}$ is equinumerous with $\mathbb{N}$, and thus $\mathbb{Q}^n$ is equinumerous with $\mathbb{N}$, it suffices to show that each open set in $\mathbb{R}^n$ is the union of rational-radius open balls centered at elements in $\mathbb{Q}^n$. Because each real number can be expressed as the limit of a series of rational numbers, $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. If U is an open set in $\mathbb{R}^n$, then for each $y_1 \in U$ there exists $r_1 \in \mathbb{R}, r_1 > 0$ such that $B(y_1, r_1) \subseteq U$. But because $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ there exists $q \in \mathbb{Q}^n$ such that $d(y_1, q) < r_1/2$, and thus if $r_2 \in \mathbb{Q}$ and $d(y_1, q) < r_2 < r_1/2$ then $y_1 \in B(q, r_2) \subseteq B(y_1, r_1)$. Therefore U, and by extension all open sets in $\mathbb{R}^n$, are the union of rational-radius open balls centered at elements in $\mathbb{Q}^n$.