Homework 1

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Problem 1

(a) $\exists x \in A, \forall y \in B\,(y \neq x^2)$

(b) $x \in A \wedge x \notin B$

(c) $\forall x \in A\,(x \notin B)$

Problem 2

(a) $\forall x, y \in \mathbb{R}(x^2 = -y^2)$.

Proof: False. Counterexample is if $x, y = 2$ then $2^2 = 4$ and $-2^2 = -4$. □

(b) $\forall x \in \mathbb{N}, \exists y \in \mathbb{Z}(y^2 = x)$.

Proof: False. $2 \in \mathbb{N}$, and because $y = \sqrt{x}$, $y = \sqrt{2}$, and $y \notin \mathbb{Z}$. □

(c) $\forall y \in \mathbb{R}\,(3y = 0 \vee y \neq 0)$.

Proof: True. If $y = 0$, then $3y = 3 \cdot 0 = 0$. Or $y \neq 0$. □

(d) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}\,(xy = \pi)$.

Proof: True. Let x be arbitrary. If $x, y \in \mathbb{R}$ and $xy = \pi$, then $y = \frac{\pi}{x}$ and $y \in \mathbb{R}$. □

(e) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}\,(xy = 0)$.

Proof: True. Let x be arbitrary. If $x, y \in \mathbb{R}$, then if $y = 0$, $xy = x \cdot 0 = 0$. □

Problem 3

Proof: Suppose $x \in \mathbb{Q}$ and $|x - 3| < 1$. Then $-1 < x - 3 < 1$, so $2 < x < 4$. Because x is always positive $8 < x^3 < 64$ and $-4 < -x < -2$, so $4 < x^3 - x < 62$ and $4 \leq x^3 - x \leq 62$. □

Problem 4

Proof: Suppose $n, a \in \mathbb{Z}$. Case n is even, then $n = 2a$ for some $a \in \mathbb{Z}$. Then

$$\begin{aligned}\frac{1}{2}(n^2 - n^3) &= \frac{1}{2}\big((2a)^2 - (2a)^3\big)\\ &= \frac{1}{2}(4a^2 - 8a^3)\\ &= 2a^2 - 4a^3\end{aligned}$$

and because a is an integer, $\frac{1}{2}(n^2 - n^3)$ is an integer. Case n is odd, then $n = 2a + 1$ for some $a \in \mathbb{Z}$. Then

$$\begin{aligned}\frac{1}{2}(n^2 - n^3) &= \frac{1}{2}\big((2a+1)^2 - (2a+1)^3\big)\\ &= \frac{1}{2}(4a^2 + 4a + 1 + (2a+1)(4a^2 + 4a + 1))\\ &= \frac{1}{2}(2a+2)(4a^2 + 4a + 1)\\ &= (a+1)(4a^2 + 4a + 1)\end{aligned}$$

and because a is an integer, $\frac{1}{2}(n^2 - n^3)$ is an integer. □