Real Analysis

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Contents

1 The Natural Numbers

1.1 Peano Axioms

Definition 1.1 (Peano axioms)**.** Using ++ as the successor operation, the natural numbers are defined as follows:

(a) 0 is a natural number.

(b) If n is a natural number, then $n++$ is also a natural number.

(c) For all natural numbers n, $n++ \neq 0$.

Definition 1.2 (Addition of natural numbers)**.** Let m be a natural number. $0 + m := m$ and $(n++) + m := (n + m)++$.

Proposition 1.3. There is only one zero, i.e. for $a \in \mathbb{N}$ if $0 + a = 0' + a = a$, then $0 = 0'$.

Proof: Suppose $0 \neq 0'$. Then 0 is a successor of $0'$ or $0'$ is a successor of 0. Because no successor of a natural number is 0, this is impossible. □

Proposition 1.4. $m + 0 = m$.

Proof: Let $n \in \mathbb{N}$. $0+0 := 0$, so by inductive hypothesis $n+0 = n$. $(n++)+0 := (n+0)++$, and from the inductive hypothesis equals $n++$. □

Lemma 1.5. For any natural numbers n and m, $n + (m++) = (n + m)++$.

Proof: Suppose $n, m \in \mathbb{N}$. $0 + (m++) := m++ = (0 + m)++$. By inductive hypothesis $n+(m++) = (n+m)++$. From the definition of addition $(n++)+(m++) = (n+(m++))++$ and from the inductive hypothesis $n + (m++) = (n + m)++$ so we have

$$\begin{aligned}(n++) + (m++) &= (n + (m++))++ \\ &= ((n + m)++)++ \\ &= ((n++) + m)++\end{aligned}$$

□

Proposition 1.6 (Commutivity of addition)**.** For $n, m \in \mathbb{N}$, $n + m = m + n$.

Proof: Let $n, m \in \mathbb{N}$. From proposition 1.4, $0 + m = m + 0$, so by inductive hypothesis $n + m = m + n$. $(n++) + m = (n + m)++$ and from inductive hypothesis this equals $(m + n)++$. From lemma 1.5, this equals $m + (n++)$. □

Proposition 1.7. If $a, b \in \mathbb{N}$ and $a + b = a$, then $b = 0$.

Proof: Suppose $a, b \in \mathbb{N}$ with $a + b = a$. □

Proposition 1.8 (Associativity of addition)**.** Let $a, b, c \in \mathbb{N}$. Then $(a+b)+c = a+(b+c)$.

Proof: Suppose $a, b \in \mathbb{N}$. From here we utilize the definition of addition, and commutivity of addition for the rest of the proof. It follows that $(a + b) + 0 = a + b = a + (b + 0)$. By inductive hypothesis suppose $(a + b) + c = a + (b + c)$ for $c \in \mathbb{N}$. Then

$$\begin{aligned}(a + b) + c++ &= [(a + b) + c]++ \\ &= [a + (b + c)]++ \\ &= a + (c + b)++ \\ &= a + [(c++) + b] \\ &= a + (b + c++)\end{aligned}$$

□

Proposition 1.9 (Cancellation law)**.** Let $a, b, c \in \mathbb{N}$. Iff $a + b = a + c$, then $b = c$.

Proof: If $0 + b = 0 + c$ then from the definition of addition $b = c$. By inductive hypothesis for any $n \in \mathbb{N}$, $n+b = n+c$. $(n++)+b = (n+b)++$ and $(n++)+c = (n+c)++$, so from the inductive hypothesis and the axioms of natural numbers, $(n++) + b = (n++) + c$. □

Definition 1.10 (Positive natural number)**.** A natural number n is said to be positive iff it is not 0.

Definition 1.11 (Ordering of natural numbers)**.** Let $n, m \in \mathbb{N}$. We write $n \geq m$ or $m \geq n$ iff $n = m + a$ for some $a \in \mathbb{N}$.

Proposition 1.12. If a or b are not zero, then $a + b \neq 0$.

Proof: Suppose $a, b \in \mathbb{N}$ with $b \neq 0$. If $a = 0$ then $a + b = 0 + b = b \neq 0$. If $a \neq 0$, because no natural number has zero as a successor it follows from the definition of addition that $a + b \neq 0$. □

Proposition 1.13 (Trichotomy of order for natural numbers)**.** Let $a, b \in \mathbb{N}$. Then exactly one of the following statements is true: $a < b$, $a = b$, $a > b$.

Proof: Suppose $a, b \in \mathbb{N}$ and $a < b$. Then for some $c \in \mathbb{N}$, $a = b + c$ with $b \neq a$. If $c = 0$ then $a = b$, a contradiction. If $b < a$, then for some $d \in \mathbb{N}$, $b = a + d$ with $a \neq b$. If $d = 0$ then $a = b$, a contradiction. Because $b = b + d + c$ and $c, d \neq 0$, it follows from commutivity and propositions 1.12 and 1.3 that this is impossible. Therefore wlog if $a < b$ then a is not greater than or equal to b. Suppose $a = b$. If $a < b$ then $a = b + c$ for some $c \in \mathbb{N}$ with $b \neq c$, a contradiction. Therefore wlog if $a = b$ then a is not less than or greater than b. □

Proposition 1.14 (Strong principle of induction)**.** Let $m_0, m, m' \in \mathbb{N}$, and let $P(x)$ be a property of arbitrary $x \in \mathbb{N}$. Suppose that for each $m \geq m_0$ the following implication holds:

$$\left(\forall m' \in [m_0, m),\ P(m')\right) \Rightarrow P(m).$$

Then we can conclude $P(m)$ is true for all natural numbers $m \geq m_0$.

1.2 Multiplication

Definition 1.15 (Multiplication of natural numbers)**.** Let m be a natural number. $0 \times m :=$ 0 and $(n++) \times m := (n \times m) + m$.

Proposition 1.16. $m \times 0 = 0$.

Proof: From the definition of multiplication, $0 \times 0 = 0$. By inductive hypothesis suppose $m \times 0 = 0$. Then $(m++) \times 0 = (m \times 0) + 0 = 0$. □

Proposition 1.17. For $n, m \in \mathbb{N}$, $n \times (m++) = (n \times m) + n$.

Proof: Let $n, m \in \mathbb{N}$. $0 \times (m{+}{+}) = 0 = (0 \times m) + 0$. By inductive hypothesis, $(n \times (m{+}{+})) = (n \times m) + n$. It follows that

$$\begin{aligned}(n++) \times (m++) &= (n \times (m++)) + (m++) \\ &= (n \times m) + n + (m++) \\ &= (n \times m) + m + (n++) \\ &= ((n++) \times m) + (n++)\end{aligned}$$

□

Proposition 1.18. For $m \in \mathbb{N}$, $1m = m$.

Proof: If $m \in \mathbb{N}$ $0 \times m = 0$. Then $(0++) \times m = 1 \times m = 0 + m = m$. □

Lemma 1.19 (Commutivity of multiplication)**.** Let $n, m \in \mathbb{N}$. Then $n \times m = m \times n$.

Proof: Let $n, m \in \mathbb{N}$. $0 \times m = m \times 0 = 0$. By inductive hypothesis, $n \times m = m \times n$. It follows from proposition 1.17 that

$$\begin{aligned}(n++) \times m &= (n \times m) + m \\ &= (m \times n) + m \\ &= m \times (n++)\end{aligned}$$

□

Proposition 1.20 (Distributive law)**.** For any natural numbers a, b, c, we have $a(b + c) = ab + ac$.

Proof: TODO □

Proposition 1.21 (Associativity of multiplication)**.** If $a, b, c \in \mathbb{N}$ then $(a \times b) \times c = a \times (b \times c)$.

Proof: TODO □

Proposition 1.22. If $a, b \in \mathbb{N}^+$, then $ab \neq 0$.

Proof: Let $a \in \mathbb{N}^+$. By proposition 1.18 $1a = a$ and a is positive. By inductive hypothesis if $n \in \mathbb{N}^+$ then na is positive. $n++$ is a successor to n, and no successor of a natural number is zero, so $n++$ is positive. $(n++)a = na + a$. Both na and a are positive and by proposition 1.12, $na + a$ is positive and thus not zero. □

Proposition 1.23. If a, b are natural numbers such that $a < b$, and c is positive, then $ac < bc$.

Corollary 1.24. Let $a, b, c \in \mathbb{N}$ such that $ac = bc$ and c is non-zero. Then $a = b$.

Proposition 1.25 (Euclid's division lemma)**.** Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.

Definition 1.26 (Exponentiation for natural numbers)**.** Let $m \in \mathbb{N}$. $m^0 := 1$, and $m^{n++} = m^n \times m$.

2 Set Theory

2.1 Fundamentals

Definition 2.1 (Axioms of sets)**.**

(a) (Sets are objects) If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.

(b) (Equality of sets) Two sets A and B are equal iff every element of A is an element of B and vice versa.

(c) (Empty set) There exists a set known as the empty set, denoted $\emptyset$, which contains no elements. In other words, for all objects x we have $x \notin \emptyset$.

(d) (Singleton sets) If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e. for every object y we have $y \in \{a\}$ iff $y = a$. $\{a\}$ is referred to as a singleton set.

(e) (Pairwise union) Given any two sets A and B, there exists a set $A \cup B$, called the union of A and B, which consists of all the elements which belong to A or B. In other words,
$$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B).$$

(f) (Axiom of specification) Let A be a set, and for each $x \in A$ let $P(x)$ be a property pertaining to x. Then there exists a set $\{x \in A \mid P(x)\}$ whose elements are precisely the elements x in A for which $P(x)$ is true.

(g) (Replacement) Let A be a set. For any object $x \in A$ and any object y, suppose we have a property $P(x, y)$ that is true for at most one y for each $x \in A$. Then
$$z \in \{y \mid P(x, y),\, x \in A\} \Leftrightarrow P(x, z).$$

(h) (Infinity) There exists a set $\mathbb{N}$, whose elements are called natural numbers, as well as an object $0 \in \mathbb{N}$, and an object $N++$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms hold.

(i) (Universal specification) DANGER - Suppose for every object x we have a property $P(x)$. Then there exists a set $\{x \mid P(x)\}$.

(j) (Regularity). If A is a non-empty set, then there is at least one element x of A which is either not a set, or is disjoint from A.

(k) (Power set) Let X and Y be sets. Then there exists a set, denoted Y^X, which consists of all the functions from X to Y, thus

$$f \in Y^X \Leftrightarrow f \text{ is a function from } X \text{ to } Y.$$

(l) (Union) Let A be a set whose elements are all sets. Then there exists a set $\bigcup A$ defined

$$x \in \bigcup A = \{x \,|\, \exists S \in A,\, x \in S\}.$$

Remark. The axioms of set theory introduced, excluding universal specification, are known as the Zermelo-Fraenkel axioms of set theory.

Lemma 2.2 (Single choice)**.** Let A be a non-empty set. Then there exists an object x such that $x \in A$.

Proof: Suppose there does not exist any object x such that $x \in A$. Simultaneously $x \notin \emptyset$, so $x \in A \Leftrightarrow x \in \emptyset$ and $A = \emptyset$, a contradiction. □

Definition 2.3 (Subset)**.** Let A, B be sets. We say that A is a subset of B, denoted $A \subseteq B$, iff every element of A is also an element of B. We say that A is a proper subset of B, denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Theorem 2.4. *Let A be a set. Then $\emptyset \subseteq A$.*

Proof: If $\emptyset \subseteq A$ then for all objects x,

$$x \in \emptyset \Rightarrow x \in A.$$

This is vacuously true because there does not exist x such that $x \in \emptyset$. □

Definition 2.5 (Intersection)**.** The intersection $S_1 \cap S_2$ of two sets is the set

$$S_1 \cap S_2 = \{x \,|\, x \in S_1 \wedge x \in S_2\}.$$

Definition 2.6 (Union)**.** The union $S_1 \cup S_2$ of two sets is the set

$$S_1 \cup S_2 = \{x \,|\, x \in S_1 \vee x \in S_2\}.$$

Definition 2.7 (Disjoint)**.** Two sets are disjoint if $A \cap B = \emptyset$.

Definition 2.8 (Difference set)**.** If A and B are sets, the set $A \setminus B$ is the set A with any elements of B removed, i.e.

$$A \setminus B := \{x \,|\, x \in A \wedge x \notin B\}.$$

Proposition 2.9. Let A, B, C be subsets of set X.

(a) (Minimal element) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

(b) (Maximal element) $A \cup X = X$ and $A \cap X = A$.

(c) (Identity) $A \cap A = A$ and $A \cup A = A$.

(d) (Commutativity) $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

(e) (Associativity) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

(f) (Distributivity) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(g) (Partition) $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$.

(h) (De Morgan Laws) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Definition 2.10 (Ordered pair)**.** If x and y are any objects, we define the ordered pair (x, y) to be a new object which consists of x as its "first component" and y as its "second component". Two ordered pairs x, y and x', y' are equal if

$$x = x', \quad y = y'.$$

Definition 2.11 (Cartesian product)**.** Let A, B be sets. Then the cartesian product of A and B, written $A \times B$, is

$$A \times B = \{(a, b) \,|, a \in A,\, b \in B\}.$$

Definition 2.12 (Ordered n-tuple)**.** Let n be a natural number. An ordered n-tuple $(x_i)_{1 \leq i \leq n}$ is a collection of objects x_i, one for every natural number i between 1 and n. Two ordered n-tuples $(x_i)_{1 \leq i \leq n}$ and $(y_i)_{1 \leq i \leq n}$ are said to be equal iff $x_i = y_i$ for all $1 \leq i \leq n$.

Definition 2.13 (n-fold Cartesian product)**.** If $(X_i)_{1 \leq i \leq n}$ is an ordered n-tuple of sets, their Cartisian product $\prod_{i=1}^{n} X_i$ is defined

$$\prod_{i=1}^{n} X_i = \{(x_i)_{1 \leq i \leq n} \,|\, x_i \in X_i\}.$$

Definition 2.14 (Indexed family)**.** If for each element $j \in J$ with $J \neq \emptyset$, there corresponds a set A_j, then

$$\mathscr{A} = \{A_j \,|\, j \in J\}.$$

Is called an indexed family of sets with J as the index set. If $J = \{1, 2, \ldots, n\}$ we may index the set similarly to sum notation.

Definition 2.15 (Union and intersection of indexed family)**.** The union of all sets in an indexed family $\mathscr{A}$ with index set J is

$$\bigcup_{j \in J} A_j = \{x \,|\, \exists A_j \in \mathscr{A},\, x \in A_j\}.$$

The intersection of all sets in $\mathscr{A}$ is

$$\bigcap_{j \in J} A_j = \{x \,|\, \forall A_j \in \mathscr{A},\, x \in A_j\}.$$

Lemma 2.16 (Finite choice)**.** Let $n \geq 1$ be a natural number, and for each natural number $1 \leq i \leq n$, let X_i be a non-empty set. Then there exists an n-tuple $(x_i)_{1 \leq i \leq n}$ such that $x_i \in X_i$ for all $1 \leq i \leq n$. In other words if each X_i is non-empty, then its n-fold cartesian product is nonempty.

Proof: Let $\mathscr{A} = \{A_i \,|\, 1 \leq i \leq n\}$ with $n \in \mathbb{N}$ be an indexed family of nonempty sets. It follows from lemma 2.2 that for each A_i, $1 \leq i \leq n$, there exists $a_i \in A_i$. Using this fact, define an ordered n-tuple $(a_i)_{1 \leq i \leq n}$. $\square$

Definition 2.17 (Upper and lower bound)**.** Let $S \subseteq \mathbb{R}$. If there exists a real number m such that $m \geq s$ for all $s \in S$, then m is called an upper bound of S, and we say that S is bounded above. If $m \leq s$ for all $s \in S$, then m is a lower bound of S and S is bounded below. The set S is said to be bounded if it is bounded above and bounded below.

Definition 2.18 (Maximum and minimum)**.** If an upper bound m of S is a member of S, then m is called the maximum of S, and we write $m = \max S$. If a lower bound of S is a member of S, then it is called the minimum of S, and we write $m = \min S$.

Definition 2.19 (Supremum and infimum)**.** Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then the least upper bound of S is called its supremum, denoted $\sup S$. Therefore $m = \sup S$ iff

(a) $m \geq s$ for all $s \in S$.

(b) If $m' < m$, then there exists $s' \in S$ such that $s' > m'$.

If S is bounded below, then the greatest lower bound of S is called its infimum and is denoted by $\inf S$.

Theorem 2.20 (Archimedean property)**.** *For each $x > 0$, there exists $n \in \mathbb{N}$ such that $0 < 1/n < x$.*

2.2 Functions

Definition 2.21 (Relation)**.** Let A, B be sets. A relation between A and B is an subset of $A \times B$.

Definition 2.22 (Equivalence relation)**.** An equivalence relation on a set S is a relation such that for all $x, y, z \in S$, the relation satisfies the following properties:

(a) (Reflexive property) xRx.

(b) (Symmetric property) $xRy \Rightarrow yRx$.

(c) (Transitive property) $xRy \wedge yRx \Rightarrow xRz$.

Definition 2.23 (Partition)**.** A partition of a set S is a collection $\mathscr{P}$ of nonempty subsets of S that are pairwise disjoint, and whose union is S, i.e.

(a) $A = \bigcup \mathscr{P}$.

(b) $\forall A, B \in \mathscr{P}$, $A \neq B \Rightarrow A \cap B = \emptyset$.

Definition 2.24 (Function)**.** A function from A to B, denoted $f : A \to B$ is a nonempty relation $f \subseteq A \times B$ that satisfies the following properties:

(a) (Existence) $\forall a \in A$, $\exists b \in B$, $(a, b) \in f$.

(b) (Uniqueness) $(a, b) \in f \wedge (a, c) \in f \Rightarrow b = c$.

Set A is called the domain of f, and set B is called the codomain. The range of f is $f(A)$, i.e. $\{b \in B \,|\, (a, b) \in f\}$.

Definition 2.25 (Equality of functions)**.** Two functions $f : X \to Y$ and $g : X' \to Y'$ are equal if their domains and codomains are equal, and furthermore that $f(x) = g(x)$ for all $x \in X$.

Definition 2.26 (Composition)**.** Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that the codomain of f is the same set as the domain of g. Then the composition $g \circ f : X \to Z$ of the two functions g and f is the function defined by the formula

$$(g \circ f)(x) = g(f(x)).$$

Lemma 2.27. Let $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.
Proof: $g \circ h$ is a function from X to Z, and $f \circ g$ is a function from $Y \to W$, so $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are functions from X to W. It follows from the definition of function composition that

$$\begin{aligned}
(f \circ (g \circ h))(x) &= f((g \circ h)(x)) \\
&= f(g(h(x))) \\
&= (f \circ g)(h(x)) \\
&= ((f \circ g) \circ h)(x)
\end{aligned}$$

□

Definition 2.28 (Injective)**.** A function $f : X \to Y$ is injective (one-to-one) if for $x, x' \in X$,

$$x \neq x' \to f(x) \neq f(x')$$

Definition 2.29 (Surjective)**.** A function $f : X \to Y$ is surjective (onto) if

$$\forall y \in Y, \, \exists x \in X, \, f(x) = y.$$

Definition 2.30 (Bijective)**.** A function is bijective (invertible) if it is injective and surjective.

Proposition 2.31. Let $f : A \to B$ and $g : B \to C$. Then

(a) If f and g are surjective, then $g \circ f$ is surjective.

(b) If f and g are injective, then $g \circ f$ is injective.

(c) If f and g are bijective, then $g \circ f$ is bijective.

Lemma 2.32. If $f : X \to Y$ is bijective then f is invertible. In other words for all $y \in Y$ there exists a unique $x \in X$ denoted $f^{-1}(y)$ such that $f(x) = y$. Therefore the inverse of f, $f^{-1} : Y \to X$ exists and is defined

$$f^{-1}(y) = x.$$

Definition 2.33 (Identity function)**.** A function defined on a set A that maps each element in A onto itself is called the identity function on A, and is denoted i_A.

Proposition 2.34. Let $f : A \to B$ be bijective. Then

(a) $f^{-1} : B \to A$ is bijective.

(b) $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Theorem 2.35. *Let $f : A \to B$ and $g : A \to B$ be bijective. Then the composition $g \circ f : A \to C$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Definition 2.36 (Image)**.** If $f : X \to Y$ is a function from X to Y, and $S \subseteq X$, we define the image of S under f, $f(S)$ to be the set

$$f(S) = \{f(x) \,|\, x \in S\}.$$

Definition 2.37 (Inverse image)**.** If U is a subset of Y, we define the set $f^{-1}(U)$ to be the set

$$f^{-1}(U) = \{x \in X \,|\, f(x) \in U\}.$$

We call $f^{-1}(U)$ the inverse image of U.

Theorem 2.38. *Suppose that $f : A \to B$, let C, C_1, C_2 be subsets of A, and let D, D_1, D_2 be subsets of B. Then the following hold:*

(a) $C \subseteq f^{-1}(f(C))$.

(b) $f(f^{-1}(D)) \subseteq D$.

(c) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.

(d) $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.

(e) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.

(f) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$.

(g) $f^{-1}(B \setminus D) = A \setminus f^{-1}(D)$.

Proof: test □

Proposition 2.39. If X, Y are sets and $f : X \to Y$ then $f(X) \subseteq Y$.

Proof: $y \in f(X)$ implies $y \in \{y \,|\, (x, y) \in f\}$ and f is a subset of $X \times Y$, so it follows from the definition of the cartesian product that $y \in Y$. □

Lemma 2.40. Let X be a set. Then the set

$$\{Y \,|\, Y \subseteq X\}$$

Is a set.

Proof: Let X be a set and $A \subseteq X$ with $A \neq \emptyset$. Then there exists $p \in A$, and we can define a function $f : X \to A$ with $x \in X$ by

$$f(x) = \begin{cases} x \in A & f(x) = x \\ x \notin A & f(x) = p \end{cases}$$

Thus for all $a \in f(X)$, $a \in A$ or $a = p \in A$, so $f(X) \subseteq A$. Next, for all $x \in X$, $(x, f(x)) \in f(X)$. Because for all $a \in A$ we have $a \in X$ then for all $a \in A$, $(a, f(a)) = (a, a) \in f(X)$ so from the definition of an image $A \subseteq f(X)$. Thus $A = F(X)$. From the power set axiom in definition 2.1,

$$\{f : X \to A \,|\, A \subseteq X \land A \neq \emptyset\} \subseteq X^X$$

From replacement, pairwise union, and singleton set axioms in definition 2.1, we can define a set $\mathrm{P}(X)$ that is the union of all images of functions in X^X, and $\{\emptyset\}$. As established above, all nonempty subsets of X are included in this set, and from proposition 2.39 all images of functions in X^X are subsets of X. □

Definition 2.41 (Power set)**.** For a set X, the set $\{Y \,|\, Y \subseteq X\}$ is called the power set of X, and is denoted $\mathrm{P}(X)$ or 2^X.

Definition 2.42 (Cardinality)**.** We say that two sets X and Y have equal cardinality iff there exists a bijection $f : X \to Y$ from X to Y.

Proposition 2.43. Let X, Y, Z be sets.

(a) X has equal cardinality with X.

(b) If X has equal cardinality with Y, then Y has equal cardinality with X.

(c) If X has equal cardinality Y and Y has equal cardinality with Z, then X has equal cardinality with Z

Proof: □

Definition 2.44 (Cardinality n)**.** Let n be a natural number. A set X is said to have cardinality n, if it has equal cardinality with $\{\in \mathbb{N} \,|\, 1 \leq i \leq n\}$. In this case we say that X has n elements.

Lemma 2.45. Suppose that $n \geq 1$, and set X has cardinality n. Then X is non-empty, and if x is any element of X, then the set $X - \{x\}$ has cardinality $n - 1$.

Proposition 2.46. Let X be a set with some cardinality n. Then X cannot have any other cardinality, i.e. X cannot have cardinality m for any $m \neq n$.

Definition 2.47 (Finite set)**.** A set is finite iff it has cardinality n for some natural number n; otherwise, the set is called infinte.

Theorem 2.48. *The set of natural numbers is infinite.*

3 Integers and Rationals

3.1 The Integers

Definition 3.1 (Integers)**.** An integer is an expression of the form $a - b$, where a and b are natural numbers. Two integers are considered to be equal, $a - b = c - d$, iff $a + d = c + b$. The set of all integers is denoted $\mathbb{Z}$.

Remark. The use of $-$ is purely notational (until subtraction is defined). $a - b$ can be interpreted as an ordered pair in $\mathbb{N} \times \mathbb{N}$.

Definition 3.2 (Integer addition)**.** The sum of two integers $(a - b) + (c - d)$ is defined by the formula

$$(a - b) + (c - d) = (a + c) - (c + d)$$

Definition 3.3 (Integer multiplication)**.** The product of two integers $(a - b) \times (c - d)$ is defined by the formula

$$(a - b) \times (c - d) = (ac + bd) - (ad + bc).$$

Remark. We may identify the integers with natural numbers by setting $n \equiv n - 0$. Definitions of equality and previously defined operations remain consistent with each other.

Proposition 3.4. If $a, b \in \mathbb{Z}$ and $a + b = b$ then $a = 0$.

Lemma 3.5. Addition and multiplication are well defined.

Definition 3.6 (Negation of integers)**.** If $(a-b)$ is an integer, we define the negation $-(a-b)$ to be the integer $b-a$.

Lemma 3.7 (Trichotomy of integers)**.** Let x be an integer. Then either x is zero, equal to a positive natural number, or x negated is a positive natural number.

Definition 3.8 (Positive integer)**.** If n is a positive natural number, we call n a positive integer, and $-n$ a negative integer.

Proposition 3.9 (Integer laws for algebra)**.** Let x, y, z be integers. Then the following identities hold:

$$\begin{aligned}
&x + y = y + x \\
&(x + y) + z = x + (y + z) \\
&x + 0 = 0 + x = x \\
&x + (-x) = 0 \\
&xy = yx \\
&(xy)z = x(yz) \\
&1x = x \\
&x(y + z) = xy + xz
\end{aligned}$$

Proposition 3.10. If $a, b \in \mathbb{Z}$ with $a, b > 0$, then $ab > 0$.

Proof: If $a, b \in \mathbb{Z}$ with $a, b > 0$, then for some $x, y \in \mathbb{N}^+$, $a = x - 0$ and $b = y - 0$. Thus $ab = (xy + 0) = (0 + 0) = xy - 0$. Because $x, y \neq 0$, by proposition 1.22 $xy > 0$ so from the definition of a positive integer $ab > 0$. □

Proposition 3.11. If $a, b \in \mathbb{Z}$ with $a, b > 0$, then $a + b > 0$.

Proof: If $a, b \in \mathbb{Z}^+$, then for some $x, y \in \mathbb{N}^+$ we have $a = x - 0$ and $b = y - 0$, so $a + b = ((x + y) - 0)$. It follows from proposition 1.12 that $x + y > 0$ so from the definition of a positive integer, $a + b > 0$. □

Proposition 3.12. If $x \in \mathbb{Z}$ with $x = (a - b)$ then $-1 \cdot (a - b) = -(a - b)$.

Proof: $-1 \cdot (a - b) = (0 - 1) \cdot (a - b) = (0a + b) - (a + 0b) = -(a - b)$. □

Proposition 3.13 (Integers have no zero divisors)**.** If a, b are integers such that $ab = 0$, then $a = 0$ or $b = 0$.

Corollary 3.14 (Cancellation law)**.** If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.

Proof: Let $a, b, c \in \mathbb{Z}$ with $c =\neq 0$. If $a = 0$ it follows from proposition 3.13 that $ac = 0$ so $bc = 0$ and thus $b = 0$, so $a = b$. If $a \neq 0$, suppose to the contrary that $b \neq a$. It follows from proposition 3.4 that there exists $d \in \mathbb{Z}$ with $d \neq 0$ such that $a + d = b$. Using laws for algebra we see that $ac = ac + dc$. By proposition 3.13 $dc \neq 0$, a contradiction by proposition 3.4. Therefore $a = b$. □

Definition 3.15 (Ordering of integers)**.** Let $n, m \in \mathbb{Z}$. We say that n is greater than or equal to m and write $n \geq m$ or $m \leq n$ iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m and write $n > m$ or $m < n$ iff $n \geq m$ and $n \neq m$.

3.2 The Rationals

Definition 3.16 (Rational number)**.** A rational number is an expression of the form $a//b$, where a and b are integers and $b \neq 0$. Two rational numbers are equal, $a//b = c//d$, iff $ad = bc$. The set of all rational numbers is denoted $\mathbb{Q}$.

Remark. We may indentify the rationals with natural numbers by setting $n//1 \equiv n$.

Definition 3.17 (Addition of rationals)**.** If $a//b$ and $c//d$ are rationals, their sum is

$$(a//b) + (c//d) = (ad + bc)//(bd).$$

Definition 3.18 (Product of rationals)**.** If $a//b$ and $c//d$ are rationals, their product is

$$(a//b) \cdot (c//d) = (ac)//(bd).$$

Definition 3.19 (Negation of rationals)**.** The negation of a rational $(a//b)$, denoted $=$ $(a//b)$ is

$$-(a//b) = (-a//b).$$

Definition 3.20 (Reciprocal of rationals)**.** If $x = a//b$ is a non-zero rational number, then the reciprocal of x^{-1} of x is defined

$$x^{-1} = b//a.$$

Lemma 3.21. The sum, product, negation, and reciprocal operations on rational numbers are well-defined.

Proposition 3.22. The negation of the negation of $x \in \mathbb{Q}$ is x.

Proof: The negation of the negation of an integer $x = (a - b)$ is $- - (a - b) = -(b - a) = (a - b)$ so $- - x = x$. The negation of the negation of a rational number $y = (c//d)$ is $- - (c//d) = -(-c//d) = (- - c//d) = c//d$. □

Definition 3.23 (Quotient)**.** The quotient of two rationals x and y with $y \neq 0$, denoted x/y, is

$$x/y = x \times y^{-1}.$$

Definition 3.24 (Subtraction)**.** The difference of two rationals x and y, denoted $x - y$, is defined

$$x - y = x + (-y).$$

Definition 3.25 (Positive rational number)**.** A rational number x is said to be positive iff we have $x = a/b$ for some positive integers a and b. It is said to be negative iff $x = -y$ for some positive rational y.

Definition 3.26 (Ordering of rationals)**.** Let $x, y \in \mathbb{Q}$. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - y$ is a positive negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$, and $x \leq y$ iff either $x < y$ or $x = y$.

Proposition 3.27. $x \in \mathbb{Q}$ is positive iff $x > 0$, and negative iff $x < 0$.

Proof: If $x = a//b$ is a positive rational number then $a, b > 0$. Because $0 = 0//d$ for some $d \in \mathbb{N} \setminus \{0\}$, $x - 0 = x + 0 = ad//bd = a//b$ and thus $x > 0$. If $x > 0$ then $x - 0$ is positive. Because $0 = 0//d$ for some $d \in \mathbb{N} \setminus \{0\}$ we have $x - 0 = x + 0 = ad//bd = a//b$, which is positive. □

Proposition 3.28 (Laws of algebra for rationals)**.** Let x, y, z be rationals. Then the following laws of algebra hold:

$$\begin{aligned}
&x + y = y + x \\
&(x + y) + z = x + (y + z) \\
&x + 0 = x \\
&x + (-x) = 0 \\
&xy = yx \\
&(xy)z = d(yz) \\
&1x = x \\
&x(y + z) = xy + xz
\end{aligned}$$

Proposition 3.29. $-1x = -x$.

Proof: If $x = a//b$ then $-1 \cdot x$ is $(-1//1) \cdot (a//b) = -1a//b$. From proposition 3.12, $-1a = -a$ so $-1a//b = -(a//b) = -x$. □

Proposition 3.30. If $a, b \in \mathbb{Q}$ with $a > 0$ and $b > 0$ then $a + b > 0$.

Proof: Suppose $a, b \in \mathbb{Q}$ with $a, b > 0$. It follows from the definition of positive rational number that for some positive $x, y, z, w \in \mathbb{Z}$, $a = x//y$ and $b = z//w$, so $ab = xw + zy/yw$. By proposition 3.10 $xw, zy, yw > 0$, so by 3.11, $a + b > 0$. □

Lemma 3.31 (Trichotomy of rationals)**.** Let x be a rational number. Then exactly one of the following three statements is true:

(a) $x = 0$.

(b) x is positive.

(c) x is negative.

3.3 Absolute Value and Exponentiation

Definition 3.32 (Absolute value)**.** If x is a rational number, the absolute value $|x|$ of x is defined as follows:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Definition 3.33 (Distance)**.** The distance between $x, y \in \mathbb{Q}$, sometimes denoted $d(x, y)$, is

$$d(x, y) = |x - y|.$$

Proposition 3.34. For all $x \in \mathbb{Q}$, $|x| \geq 0$.

Proof: If $x \geq 0$ then $|x| = x$ so $|x| \geq 0$. If $x < 0$ then $|x| = -x$. By proposition 3.27 x is negative. Therefore there exists $y \in \mathbb{Q}^+$ such that $x = -y$, so by proposition 3.22 $-x = - - y = y$ and $-x$ is positive. By proposition 3.27, $-x > 0$. □

Proposition 3.35 (Triangle inequality)**.** For $x, y \in \mathbb{Q}$, $|x + y| \leq |x| + |y|$.

Definition 3.36 (ϵ-closeness)**.** Let $\epsilon > 0$ be a rational number, and let x, y be rational numbers. We say that y is ϵ-close to x iff $d(y, x) < \epsilon$.

Definition 3.37 (Exponentiation to a natural number)**.** Let x be a rational number. To raise x to the power 0, we define $x^0 = 1$ and for all $n \in \mathbb{N}$, $x^{n+1} = x^n \times x$.

Definition 3.38 (Exponentiation to a negative number)**.** Let x be a non-zero rational number. Then for any negative integer $-n$,

$$x^{-n} = 1/x^n.$$

Proposition 3.39. If x and y are two rationals such that $x < y$, then there exists a third rational z such that $x < z < y$.

Proposition 3.40. There does not exists any rational number x for which $x^2 = 2$.

4 Real Numbers

4.1 Cauchy Sequences

Remark. Many definitions here are repeated later. Ones given here are necessary for the construction of the real numbers.

Definition 4.1 (Sequences)**.** Let m be an integer. A sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is any function from the set $\{n \in \mathbb{Z} \mid n \geq m\}$ to $\mathbb{Q}$.

Definition 4.2 (Cauchy sequence)**.** A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is said to be a Cauchy sequence iff

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall j, k \in \mathbb{N}, \big(j, k \geq N \Rightarrow |a_j - a_k| < \epsilon\big).$$

Definition 4.3 (Bounded sequence)**.** Let $M \geq 0$ be rational. A finite seqeunce $a_1, a_2, \ldots$ is bounded by M iff for all $i \in \mathbb{N}$, $|a_i| \leq M$.

Lemma 4.4. Every finite sequence $a_1, a_2, \ldots, a_n$ is bounded by some $M \in \mathbb{Q}$.

Definition 4.5 (Equivalent sequences)**.** Two sequences (a_n) and (b_n) are equivalent iff

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall i \in \mathbb{N}, \big(i, j \geq N \Rightarrow |a_i - b_i| < \epsilon\big).$$

Definition 4.6 (Real numbers)**.** A real number is defined to be an object of the form $\text{LIM}_{n\to\infty} a_n$, where $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence of rational numbers. Two real numbers are said to be equivalent if the Cauchy sequences they contain are equivalent. In this context LIM is a formal limit.

Definition 4.7 (Real operations)**.** Let $x = \text{LIM}_{n\to\infty} a_n$ and $y = \text{LIM}_{n\to\infty} b_n$. Then

$$\begin{aligned}
&x + y = \text{LIM}_{n\to\infty}(a_n + b_n),\\
&xy = \text{LIM}_{n\to\infty}(a_n b_n),\\
&x^{-1} = \text{LIM}_{n\to\infty} a_n^{-1}\\
&x/y = x \cdot y^{-1},\ y \neq 0.
\end{aligned}$$

Definition 4.8 (Bounded away from zero)**.** A sequence (a_n) is said to be bounded away from zero iff there exists a rational number $c > 0$ such that $|a_n| \geq c$ for all $n \geq 1$.

Definition 4.9 (Positive real number)**.** A real number x is said to be positive iff it can be written as a real number for some Cauchy sequence positively bounded away from zero.

Definition 4.10 (Absolute value)**.** Let x be a real number. We define the absolute value $|x|$ of x to equal x if x is positive, $-x$ when x is negative, and 0 when x is zero.

Definition 4.11 (Ordering of reals)**.** Let x and y be real numbers. We say that x is greater than y iff $x - y$ is a positive real number, and $x < y$ if $x - y$ is a negative real number. We define $x \geq y$ iff $x > y$ or $x = y$.

Definition 4.12 (Archemedian property)**.** Let x be a real number, and let ϵ be a positive real number. Then there exists a positive integer M such that $M\epsilon > x$.

Definition 4.13 (Real exponentiation by an integer)**.** Let x be a real number. Then

$$\begin{aligned}
&x^0 = 1;\\
&x^{n+1} = x^n \cdot x.
\end{aligned}$$

If $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ is nonzero, we define

$$x^{-n} = 1/x^n.$$

Definition 4.14 (nth root)**.** Let $x \geq 0$ be a non-negative real, and let $n \geq 1$ be a positive integer. We define $x^{1/n}$ as

$$x^{1/n} = \sup\{y \in \mathbb{R} \,|\, y \geq 0 \wedge y^n \leq x\}.$$

Lemma 4.15. $x^{1/n}$ is a real number.

Definition 4.16 (Rational exponents)**.** Let $x > 0$ be a positive real number, and let $q = a/b$ be a rational number. Then

$$x^q = (x^{1/b})^a.$$

5 Sequences

5.1 Sequences

Definition 5.1 (Sequence)**.** A sequence is a function whose domain is the set $\mathbb{N}$ of natural numbers, and can denoted $(s_n)_{n=a}^{b}$ for $a \in \mathbb{N}, b \in \mathbb{N} \cup \{\infty\}$. (s_n) will be used here as shorthand for $(s_n)_{n=0}^{\infty}$.

Definition 5.2 (Limit of a sequence)**.** A sequence (s_n) is said to converge to $s \in \mathbb{R}$ iff

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \big(n \geq N \Rightarrow |s_n - s| < \epsilon\big).$$

If (s_n) converges to s, then s is called the limit of the sequence (s_n), and we write $\lim_{n\to\infty} s_n = s$. If a sequence does not converge, it is said to diverge.

Definition 5.3 (Real exponentiation)**.** Let $x > 0$ be real, and let α be a real number. We define the quantity x^α by

$$x^\alpha = \lim_{n\to\infty} x^{q_n},$$

where $(q_n)_{n=0}^\infty$ is any sequence of rational numbers converging to α.

Lemma 5.4 (Continuity of exponentiation)**.** Let $x > 0$, and let α be a real number. Let $(q_n)_{n=1}^\infty$ be any sequence of rational numbers converging to α. Then $(x^{q_n})_{n=1}^\infty$ is also a convergent sequence. Furthermore, if $(q'_n)_{n=1}^\infty$ is a sequence converging to α, then $(x^{q'_n})_{n=1}^\infty$ has the same limit as $(x^{q_n})_{n=1}^\infty$.

Definition 5.5 (Divergence to Infinity)**.** A sequence (s_n) is said to diverge to infinity, and we write $\lim_{n\to\infty} s_n = \infty$ if

$$\forall M \in \mathbb{R},\, \exists N \in \mathbb{N},\, \forall n \in \mathbb{N},\, \big(n \geq N \Rightarrow s_n > M\big).$$

Definition 5.6 (Bounded sequence)**.** A sequence (s_n) is said to be bounded if

$$\exists M \geq 0, \forall n \in \mathbb{N},\, \big(|s_n| \leq M\big).$$

Theorem 5.7. *If a sequence converges, it is bounded.*

Theorem 5.8. *If a sequence converges, its limit is unique.*

5.2 Monotone and Cauchy sequences

Theorem 5.9. *Suppose that (s_n) and (t_n) are convergent sequences with $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Then*

(a) $\lim s_n + t_n = s + t$.

(b) $\lim ks_n = ks$.

(c) $\lim s_n t_n = st$.

(d) $\lim s_n/t_n = s/t$ *iff* $t \neq 0$ *and* $\forall n \in \mathbb{N},\, t_n \neq 0$.

Theorem 5.10. *Let (s_n) be a sequence of positive numbers. Then $\lim_{n\to\infty} s_n = \infty$ iff $\lim_{n\to\infty} 1/s_n = 0$.*

Definition 5.11 (Increasing sequence)**.** A sequence (s_n) of real numbers is increasing if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$. A sequence is monotone if it is increasing or decreasing.

Theorem 5.12. *A monotone sequence is convergent iff it is bounded.*

Definition 5.13 (Cauchy sequence)**.** A sequence (s_n) of real numbers is a Cauchy sequence if

$$\forall \epsilon > 0,\, \exists N \in \mathbb{N},\, \forall n, m \in \mathbb{N},\, \big(m, n \geq N \Rightarrow |s_n - s_m| < \epsilon\big).$$

Lemma 5.14. Every convergent sequence is a Cauchy sequence

Lemma 5.15. Every Cauchy sequence is bounded.

Theorem 5.16. *A sequence of real numbers is convergent iff it is a Cauchy sequence.*

Definition 5.17 (Subsequence)**.** Let $(s_n)_{n=1}^\infty$ be a sequence and let $(n_k)_{k=1}^\infty$ be any sequence of natural numbers such that $n_1 < n_2 < n_2 < \ldots$. The sequence $(s_{n_k})_{k=1}^\infty$ is called a subsequence of $(s_n)_{n=1}^\infty$.

Theorem 5.18. *If a sequence converges to a real number s, then every subsequence of (s_n) also converges to s.*

Theorem 5.19. *Every bounded sequence has a convergent subsequence.*

5.3 Limit superior and inferior

Definition 5.20 (Limsup and liminf)**.** A subsequential limit of (s_n) is any real number that is the limit of some subsequence of (s_n). If S is the set of all subsequential limits of (s_n), then we define the limit superior of (s_n) to be

$$\limsup s_n = \sup S.$$

The limit inferior is defined

$$\liminf s_n = \inf S.$$

Theorem 5.21. *Let (s_n) be a bounded sequence and let $m = \limsup s_n$. Then the following properties hold:*

(a) For every $\epsilon > 0$ there exists a natural number N such that $n \geq N$ implies that $s_n < m + \epsilon$.

(b) For every $\epsilon > 0$ there exists an integer $k > i$ such that $s_k > m - \epsilon$.

Theorem 5.22. *Suppose that (r_n) converges to a positive number r and (s_n) is a bounded sequence. Then*

$$\limsup r_n s_n = r \limsup s_n$$

6 Series

6.1 Convergence tests

Definition 6.1 (Convergence of series)**.** Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, we define the Nth partial sum S_N of this series to be

$$S_N = \sum_{n=m}^{N} a_n.$$

If the sequence $(S_N)_{N=m}^{\infty}$ converges to L, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is convergent, and converges to L. We also write $L = \sum_{n=m}^{\infty} a_n = L$. If the parial sums S_N diverge, we say the infinite series $\sum_{n=m}^{\infty} a_n$ is divergent, and do not assign any real number to it.

Proposition 6.2. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. $\sum_{n=m}^{\infty} a_n$ converges iff for every real number $\epsilon > 0$, there exists an integer $N \geq m$ such that for all $p, q \geq N$,

$$\left| \sum_{n=p}^{q} a_n \right| \leq \epsilon.$$

Corollary 6.3. Let $\sum_{n=m}^{\infty} a_n$ be a convergent series of real numbers. Then we must have $\lim_{n\to\infty} a_n = 0$.

Definition 6.4 (Absolute convergence)**.** Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. We say that this series is absolutely convergent iff the series $\sum_{n=m}^{\infty} |a_n|$ is convergent.

Proposition 6.5. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If the series is absolutely convergent, then it is also convergent. Furthermore,

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n|.$$

Proposition 6.6 (Alternating series test)**.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which are non-negative and decreasing, thus $a_n \geq 0$ and $a_n \geq a_{n+1}$ for every $n \geq m$. Then the series

$$\sum_{n=m}^{\infty} (-1)^n a_n$$

is convergent iff the sequence a_n converges to 0 as $n \to \infty$.

Proof: The sequence of partial sums is Cauchy, thus converges. □

Proposition 6.7. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of non-negative real numbers. Then this series is convergent iff there is a real number M such that for all $N \geq m$,

$$\sum_{n=m}^{N} a_n \leq M.$$

Corollary 6.8 (Comparison test)**.** Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two formal series of real numbers, and suppose that $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, and

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Definition 6.9 (Geometric series)**.** The geometric series is defined

$$\sum_{n=0}^{\infty} x^n,$$

Where $x \in \mathbb{R}$.

Lemma 6.10. Let x be a real number. If $|x| \geq 1$, then the series $\sum_{n=0}^{\infty} x^n$ is divergent. If $|x| < 1$, then the series is absolutely convergent, and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Proposition 6.11. Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of non-negative real numbers. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent iff

$$\sum_{k=0}^{\infty} 2^k a_{2^k}$$

is convergent.

Corollary 6.12. Let $q > 0$ be a real number. Then the series $\sum_{n=1}^{\infty} \frac{1}{n^q}$ is convergent when $q > 1$ and divergent when $q \leq 1$.

Theorem 6.13 (Root test)**.** *Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $\alpha = \limsup |a_n|^{1/n}$.*

(a) If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent.

(b) If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is not convergent.

(c) If $\alpha = 1$, we cannot assert any conclusion.

7 Topology Shit

7.1 Heine-Borel Theorem

Definition 7.1. Let $\epsilon > 0$. A neighborhood of x is a set of the form

$$N(x;\epsilon) = \{y \in \mathbb{R} \mid |x-y| < \epsilon\},$$

where ϵ is referred to as the radius.

Definition 7.2 (Deleted neighborhood)**.** Let $x \in \mathbb{R}$ and $\epsilon > 0$. A deleted neighborhood of x is the set

$$N^*(x;\epsilon) = N(x;\epsilon) \setminus \{x\},$$

i.e.

$$N^*(x;\epsilon) = \{y \in \mathbb{R} \mid 0 < |x-y| < \epsilon\}.$$

Definition 7.3 (Interior point)**.** Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is an interior point of S if there exists a neighborhood N of x such that $N \subseteq S$.

Definition 7.4 (Boundary point)**.** If for every neighorhood N of x we have $N \cap S \neq \emptyset$ and $N \cap (\mathbb{R} \setminus S) \neq \emptyset$, then x is a boundary point of S.

Definition 7.5 (Adherent point)**.** Let $X \subseteq \mathbb{R}$, and let $y \in \mathbb{R}$. We say that y is an adherent point of X iff

$$\forall \epsilon > 0, \exists x \in X, \big(|x - y| < \epsilon\big).$$

Definition 7.6 (Accumulation point)**.** Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is an accumulation point of S if every deleted neighborhood of x contains a point of S.

Definition 7.7 (Isolated point)**.** We say that x is an isolated point of X if $x \in X$ and there exists some $\epsilon > 0$ such that $|x - y| > \epsilon$ for all $y \in X \setminus \{x\}$.

Definition 7.8 (Closure)**.** Let $X \subseteq \mathbb{R}$. The closure of X, denoted $\overline{X}$ is defined to be the set of all adherent points of X.

Lemma 7.9. Let $X \subseteq \mathbb{R}$. The set of all convergent points of sequences in X is the closure of X.

Theorem 7.10 (Heine-Borel)**.** *Let X be a subset of $\mathbb{R}$. Then the following statements are equivalent:*

(a) X is closed and bounded.

(b) Given any sequence $(a_n)_{n=0}^{\infty}$ of real numbers which takes values in X, there exists a subsequence $(a_{n_j})_{j=0}^{\infty}$ of the original sequence which converges to some number L in X.

8 Continuous Functions on $\mathbb{R}$

8.1 Limits of Functions

Definition 8.1 (Extended real numbers)**.** The extended real number system $\mathbb{R} \cup \{\infty, -\infty\}$ is denoted $\mathbb{R}^*$. An extended real number is said to be finte iff it is a real number, and infinite iff it is equal to $\pm\infty$.

Definition 8.2 (Intervals)**.** Let $a, b \in \mathbb{R}^*$. Then the closed interval $[a, b]$ is the set

$$\{x \in \mathbb{R}^* \mid a \leq x \leq b\}.$$

The open interval (a, b) is the set

$$\{x \in \mathbb{R}^* \mid a < x < b\}.$$

Definition 8.3 (Limit point)**.** Let $X \subseteq \mathbb{R}$. We say that $x \in \mathbb{R}$ is a limit point of X iff it is an adherent point of $X \setminus \{x\}$.

Definition 8.4 (Algebra of functions)**.** Given two functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$:

$$\begin{aligned}
(f + g)(x) &= f(x) + g(x), \\
(f - g)(x) &= f(x) - f(x), \\
(fg)(x) &= f(x)g(x), \\
(f/g)(x) &= f(x)/g(x), \\
(cf)(x) &= cf(x).
\end{aligned}$$

Definition 8.5 (Limit of a function)**.** Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. We say that a real number L is a limit of f at c if

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in D, \big(0 < |x - c| < \delta \Rightarrow |f(x - L)| < \epsilon\big).$$

Remark. The effect of c being an accumulation point is that limits must be unique. If c wasn't an accumulation point of D, it would be vacuously true that every number is a limit for f at c. $|x - c| > 0$ specifies that we are focusing on f's approach to L as x approaches c, and not on the value of f at c.

Theorem 8.6. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x\to c} f(x) = L$ iff for each neighborhood V of L there exists a deleted neighborhood U of c such that $f(U \cap D) \subseteq V$.*

Theorem 8.7. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x\to c} f(x) = L$ iff for every sequence (s_n) in D that converges to c with $s_n \neq c$ for all n, the sequence $(f(s_n))$ converges to L.*

Corollary 8.8. Limits are unique.

Theorem 8.9. *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, and let c be an accumulation point of D. If $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$ and $k \in \mathbb{R}$, then*

$$\lim_{x\to c}(f+g)(x) = L + M,$$
$$\lim_{x\to c}(fg)(x) = LM,$$
$$\lim_{x\to c}(f/g)(x) = L/M, \text{ if } \forall x \in D,\ g(x) \neq 0 \text{ and } M \neq 0.$$

Definition 8.10 (Right and Left limits)**.** Suppose $f : D \to \mathbb{R}$ with c an accumulation point of D and $\lim_{x\to c} f(x) = L$ for some $L \in \mathbb{R}$. The right hand limit of f at c is the limit of f restricted to some domain (c, d) with $d > c$ as $x \to c$. The left hand limit of f at c is the limit of f restricted to some domain $(-d, c)$ as $x \to c$.

8.2 Continuous functions

Definition 8.11 (Continuity)**.** Let $f : D \to \mathbb{R}$ and let $c \in D$. We say that f is continuous at c iff

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in D,\ \big(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon\big).$$

If f is continuous at each point of a subset S of D, then f is said to be continuous on S. If f is continuous on its domain D, then f is said to be a continuous function.

Theorem 8.12. *Let $f : D \to \mathbb{R}$ and let $c \in D$. Then the following three conditions are equivalent:*

(a) f is continuous at c.

(b) If (x_n) is any sequence in D such that (x_n) converges to c, then $\lim_{n\to\infty} f(x_n) = f(c)$.

(c) For every neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(U \cap D) \subseteq V$.

Theorem 8.13. *Let f and g be functions from D to $\mathbb{R}$, and let $c \in D$. Suppose that f and g are continuous at c. Then*

(a) $f + g$ and fg are continuous at c.

(b) f/g is continuous at c if $g(c) \neq 0$.

Theorem 8.14. *Let $F : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. If f is continuous at a point $c \in D$ and g is continuous at $f(c)$, then the composition $g \circ f : D \to \mathbb{R}$ is continuous at c.*

Theorem 8.15. *A function $f : D \to \mathbb{R}$ is continuous on D iff for every open set G in $\mathbb{R}$ there exists an open set H such that $H \cap D = f^{-1}(G)$.*

Corollary 8.16. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous iff $f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$.

Definition 8.17 (Lipschitz continuity)**.** If $|f(x) - f(y)| \leq M|x - y|$ for some $M > 0$, the function is called Lipschitz continuous.

8.3 Properties of continous functions

Theorem 8.18. *Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.*

Corollary 8.19. Let D be a compact subset of $\mathbb{R}$, and suppose that $f : D \to \mathbb{R}$ is continuous. Then f assumes minimum and maximum values on D.

Theorem 8.20 (Intermediate value theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then if $f(a) < k < f(b)$ or $f(b) < k < f(a)$, then there exists $c \in (a, b)$ such that $f(c) = k$.*

Theorem 8.21. *Let I be a compact interval, and suppose that $f : I \to \mathbb{R}$ is a continuous function. Then the set $f(I)$ is a compact interval.*

8.4 Uniform continuity

Definition 8.22. Let $f : D \to \mathbb{R}$. We say that f is uniformly continuous on D if

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x, y \in D,\ \big(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon\big).$$

Theorem 8.23. *Suppose $f : D \to \mathbb{R}$ is continuous on a compact set D. Then f is uniformly continuous on D.*

Theorem 8.24. *Let $f : D \to \mathbb{R}$ be uniformly continuous on D and suppose that (x_n) is a Cauchy sequence in D. Then $(f(x_n))$ is a Cauchy sequence.*

Theorem 8.25. *A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b) iff it can be extended to a function $\overline{f}$ that is continuous on $[a, b]$.*

9 Differentiation

9.1 Differentiation

Definition 9.1 (Differentiation)**.** Let I be an interval containing a point c, and let $f : I \to \mathbb{R}$. We say that f is differentiable at c if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite. We denote the derivative of f at c by $f'(c)$. If f is differentiable at each point of the set $S \subseteq I$, then f is said to be differentiable on S, and the function $f' : S \to \mathbb{R}$ is called the derivative of f on S.

Theorem 9.2. *If $f : I \to \mathbb{R}$ is differentiable at a point $c \in I$, then f is continuous at c.*

Proof: Let $F : I \to \mathbb{R}$ be differentiable at $c \in I$. Then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \tag{1}$$

for some $f'(c) \in \mathbb{R}$. If f is not continuous at c, then

$$\exists \epsilon > 0,\ \forall \delta > 0,\ \exists x \in I,\ \big(|x - c| < \delta \wedge |f(x) - f(c)| \geq \epsilon\big).$$

Thus there exists x with $|x - c| < \delta$, so

$$\left|\frac{f(x) - f(c)}{x - c}\right| = \frac{|f(x) - f(c)|}{|x - c|} \geq \frac{\epsilon}{\delta}.$$

Because ϵ/δ is arbitrarily large for small δ,

$$\left\{ \frac{f(x) - f(c)}{x - c} \,\middle|\, |x - c| < \delta \right\}$$

is unbounded for all $\delta > 0$. □

Theorem 9.3. *Suppose that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then the following identities hold:*

(a) *For $k \in \mathbb{R}$, kf is differentiable at c and*

$$(kf')(c) = k \cdot f'(c).$$

(b) *The function $f + g$ is differentiable at c and*

$$(f+g)'(c) = f'(c) + g'(c).$$

(c) *The function fg is differentiable at c and*

$$(fg)'(c) = f(c)g'(c) + g(c)f'(c).$$

(d) *If $g(c) \neq 0$, the function f/g is differentiable at c and*

$$(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}.$$

9.2 Differentiation theorems

Theorem 9.4 (Chain rule). *Let I and J be intervals in $\mathbb{R}$, $f : I \to \mathbb{R}$, and $g : J \to \mathbb{R}$, with $f(I) \subseteq J$ and $c \in I$. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c and*

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Theorem 9.5. *If f is differentiable on an open interval (a, b) and if f assumes it's maximum or minimum at a point $c \in (a, b)$, then $f'(c) = 0$.*

Theorem 9.6 (Rolle's theorem). *Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and such that $f(a) = f(b)$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.*

Theorem 9.7 (Mean value theorem). *Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 9.8 (IVT for derivatives). *Let f be differentiable on $[a, b]$ and suppose that k is a number between $f'(a)$ and $f'(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = k$.*

Theorem 9.9. *Suppose that f is differentiable on an interval I and $f'(x) \neq 0$ for all $x \in I$. Then f is injective, f^{-1} is differentiable of $f(I)$, and*

$$(f^{-1})'(y) = \frac{1}{f'(x)},$$

where $y = f(x)$.

Theorem 9.10 (Cauchy mean value theorem). *Let f and g be functions that are continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

Theorem 9.11 (L'Hospital's rule). *Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $c \in [a, b]$ and that $f(c) = g(c) = 0$. Suppose also that $g'(x) \neq 0$ for $x \in U$, where U is the intersection of (a, b) and some deleted neighborhood of c. If*

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L, \ L \in \mathbb{R},$$

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

Definition 9.12 (Limit at infinity)**.** Let $f : (a, \infty) \to \mathbb{R}$. We say that $L \in \mathbb{R}$ is the limit of f as $x \to \infty$, and write

$$\lim_{x\to\infty} f(x) = L,$$

if

$$\forall \epsilon > 0,\ \exists N > a,\ \forall x \in (a, \infty),\ \big(x > N \Rightarrow |f(x) - L| < \epsilon\big).$$

Definition 9.13. Let $f : (a, \infty) \to \mathbb{R}$. We say that f tends to ∞ as $x \to \infty$ and write

$$\lim_{x\to\infty} f(x) = \infty,$$

if

$$\forall \alpha \in \mathbb{R},\ \exists N > a,\ \forall x \in (a, \infty),\ \big(x > N \Rightarrow f(x) > \alpha\big).$$

Theorem 9.14 (L'Hospital's rule)**.** *Let f and g be differentiable on a, ∞. Suppose that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, and that $g'(x) \neq 0$ for $x \in (a, \infty)$. If*

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L,\ L \in \mathbb{R},$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

Theorem 9.15 (Taylor's theorem)**.** *Let f and its first n derivatives be continuous on $[a, b]$ and differentiable on (a, b), and let $x_0 \in [a, b]$. Then for each $x \in [a, b]$ with $x \neq x_0$ there exists a point c between x and x_0 such that*

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}.$$

10 Integration

10.1 Piecewise Constant Integrals

Definition 10.1 (Connected sets)**.** Let X be a subset of $\mathbb{R}$. We say that X is connected iff X is non-empty and whenever $x, y \in X$ with $x < y$, the bounded interval $[x, y]$ is a subset of X.

Definition 10.2 (Length of interval)**.** If I is a bounded interval, the length of I, denoted $|I|$, is defined as follows: If I is one of the intervals $[a, b], (a, b), [a, b), (a, b]$ for some real numbers $a < b$, then

$$|I| = b - a.$$

If I is a point or the empty set, $absI = 0$.

Definition 10.3 (Partition)**.** Let I be a bounded interval. A partition of I is a finite set P of bounded intervals contained in I such that $\bigcup P = I$ and $\bigcap P = \emptyset$.

Theorem 10.4. *Let I be a bounded interval, n be a natural number, and let P be a partition of I of cardinality n. Then*

$$|I| = \sum_{J\in P} |J|.$$

Definition 10.5 (Finer and coarser partitions)**.** Let I be a bounded interval, and let P and P' be two partitions of I. We say that P' is finer than P, or P is coarser than P', if for every J in P', there exists K in P such that $J \subseteq K$.

Definition 10.6 (Common refinement)**.** Let I be a bounded interval, and let P and P' be two partitions of I. We define the common refinement $P\#P'$ of P and P' to be the set

$$P\#P' = \{K \cap J \,|\, K \in P \wedge J \in P'\}.$$

Definition 10.7 (Constant function)**.** Let X be a subset of $\mathbb{R}$, and let $f : X \to R$ be a function. We say that f is constant iff there exists a real number c such that $f(x) = c$ for all $x \in X$. If $E \subseteq X$, we say that f is constant on E if the restriction $f|_E$ of f to E is constant.

Definition 10.8 (Piecewise constant)**.** Let I be a bounded interval, let $f : I \to \mathbb{R}$ be a function, and let P be a partition of I. We say that f is piecewise constant with respect to P iff for every $j \in P$, f is constant on J. We say that f is piecewise constant if there exists a partition of its domain with which it is constant relative to.

Definition 10.9 (Piecewise constant integral)**.** Let I be a bounded interval, and let P be a partition of I. Let $f : I \to \mathbb{R}$ be a function which is piecewise constant with respect to P. Then we define the piecewise constant integral $\int_{[P]} f$ of f with respect to the partition P by the formula

$$\int_{[P]} f = \sum_{J \in P} c_J |J|,$$

where for each $J \in P$ we let c_J be the constant value of f on J.

Definition 10.10 (Piecewise constant integral)**.** Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be a function which is piecewise constant function on I. Then we define the piecewise constant integral $\int_I f$ by the formula

$$\int_I f = \int_{[P]} f,$$

where P is any partition of I with respect to which f is piecewise constant. To explicitly denote we are taking the piecewise constant integral of a piecewise constant function, append p.c. to the integral. However this is usually clear through context.

10.2 Upper and Lower Riemann Integrals

Definition 10.11 (Majorization of functions)**.** Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$. We say that g majorizes f on I if we have $g(x) \geq f(x)$ for all $x \in I$, and that g minorizes f on I if $g(x) \leq f(x)$ for all $x \in I$.

Definition 10.12 (Upper and lower Riemann integrals)**.** Let $f : I \to \mathbb{R}$ be a bounded function defined on a bounded interval I. We define the upper Riemann integral $\overline{\int}_I f$ by the formula

$$\overline{\int}_I f = \inf \left\{ \int_I g \mid g \text{ majorizes } f \text{ and is piecewise constant} \right\},$$

and the lower Riemann integral $\underline{\int}_I f$ by the formula

$$\underline{\int}_I f = \sup \left\{ \int_I g \mid g \text{ minorizes } f \text{ and is piecewise constant} \right\}.$$

Lemma 10.13. Let $f : I \to \mathbb{R}$ be a function on a bounded interval I which is bounded by some real number M. Then we have

$$-M|I| \leq \underline{\int}_I f \leq \overline{\int}_I f \leq M|I|.$$

In particular, both the lower and upper Riemann integrals are real numbers.

Definition 10.14 (Riemann integral)**.** Let $f : I \to \mathbb{R}$ be a bounded function on a bounded interval I. If $\underline{\int}_I f = \overline{\int}_I f$, then we say that f is Riemann integrable on I and define

$$\int_I f = \underline{\int}_I f = \overline{\int}_I f.$$

Lemma 10.15. Let $f : I \to \mathbb{R}$ be a piecewise constant function of a bounded interval I. Then f is Riemann integrable, and $\int_I f = \text{p.c.} \int_I f$.

Definition 10.16 (Riemann sums)**.** Let $f : I \to \mathbb{R}$ be a bounded function on a bounded interval I, and let P be a partition of I. We define the upper Riemann sum $U(f, P)$ and lower Riemann sum $L(f, P)$ by

$$U(f, P) = \sum_{J \in P | j \neq \emptyset} \big(\sup_{x \in J} f(x) \big) |J|,$$

$$L(f, P) = \sum_{J \in P | j \neq \emptyset} \big(\inf_{x \in J} f(x) \big) |J|.$$

Proposition 10.17. Let $f : I \to \mathbb{R}$ be a bounded function on a bounded interval I. Then

$$\overline{\int}_I f = \int \{U(f,P) \,|\, P \text{ is a partition of I}\},$$

and

$$\underline{\int}_I f = \sup\{L(f,P) \,|\, P \text{ is a partition of I}\}.$$

Theorem 10.18. *Let I be a bounded interval, and let f be a function which is uniformly continuous on I. Then f is Riemann integrable.*

Corollary 10.19. Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable.

Definition 10.20 (Piecewise continuous)**.** Let I be a bounded interval, and let $f : I \to \mathbb{R}$. We say that f is piecewise continuous on I iff there exists a partition P of I such that $f|_J$ is continuous on J for all $J \in P$.

Proposition 10.21. Let $[a,b]$ be a closed and bounded interval and let $f : [a,b] \to \mathbb{R}$ be a monotone function. Then f is Riemann integrable on $[a,b]$.

Definition 10.22 (α-length)**.** Let I be a bounded interval, let X be an interval that is closed containing I, and let $\alpha : X \to \mathbb{R}$ be a monotone increasing function whenever $x, y \in X$ are such that $y \geq x$. Then we define the α-length $\alpha[I]$ of I be the following rules:

(a) If I is empty, then $\alpha[I] = 0$.

(b) If $I = \{a\}$ is a point, then $\alpha[I] = \lim_{x \to a^+ | x \in X} \alpha(x) - \lim_{x \to a^- | x \in X} \alpha(x)$, with the