# Homework 3

Samuel Lindskog

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## Problem 1

Let $(X_1, d_1)$, $(X_2, d_2)$ be two metric spaces. Let $X = X_1 \times X_2$, with $(X, d)$ a metric space, and $d$ satisfying property (4.4), let $U_1, U_2$ be open subsets of $X_1, X_2$, and $x = (x_1, x_2) \in U_1 \times U_2$. There exists $r_1, r_2 > 0 \in \mathbb{R}$ such that $B(x_1, r_1) \subseteq U_1$ and $B(x_2, r_2) \subseteq U_2$. Suppose $n \in \mathbb{N}$ with $r_n > 0 \in \mathbb{R}$ and $\{x_n = (x_{1n}, x_{2n})\}_{n=1}^{\infty}$ a sequence in X such that $x_n \in B(x, r_n)$ and

$$\forall \epsilon > 0 \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow r_n \leq \epsilon.$$

Therefore $\lim_{n\to\infty} x_n = x$. Following property (4.4), $x_{1n}$ and $x_{2n}$ converge to $x_1$ and $x_2$ respectively, so

$$\exists M \in \mathbb{N} \text{ such that } n \geq M \Rightarrow d(x_{1n}, x_1) < r_1 \text{ and}$$
$$\exists L \in \mathbb{N} \text{ such that } n \geq L \Rightarrow d(x_{2n}, x_2) < r_2.$$

For all $\epsilon$, there exists $N$ arbitrarily large so that $N \geq \max(M, L)$, and thus if $n = N$, $x_n \in B(x, r_n)$ implies $x_{1n} \in B(x_1, r_1)$ and $x_{2n} \in B(x_2, r_2)$, so $B(x, r_n) \subseteq X$.