

# Topology

Samuel Lindskog

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## Open and closed sets

**Definition 1.1** (Metric). A *metric* on a set  $X$  is a real-valued function  $d$  on  $X \times X$  that has the following properties:

- (a) For all  $x, y \in X$ ,  $d(x, y) \geq 0$ .
- (b)  $d(x, y) = 0$  iff  $x = y$ .
- (c) For all  $x, y \in X$ ,  $d(x, y) = d(y, x)$ .
- (d) For all  $x, y, z \in X$ ,  $d(x, z) \leq d(x, y) + d(y, z)$ .

**Definition 1.2** (Metric space). A metric space  $(X, d)$  is a set  $X$  equipped with a metric  $d$  on  $X$ .

**Definition 1.3** (Subspace). If  $(X, d)$  is a metric space and  $Y$  is a subset of  $X$ , then the restriction  $d'$  of  $d$  to  $Y \times Y$  is a metric on  $Y$ , and  $(Y, d')$  is called a subspace of  $(X, d)$ .

*Remark.* Any set  $X$  can be made into a discrete metric space by associating with  $X$  the metric  $d$  defined by

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

**Definition 1.4** (Open ball). The open ball  $B(x, r)$  with center  $x \in X$  and radius  $r > 0$  is defined by

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

**Definition 1.5** (Interior point). Let  $Y$  be a subset of  $X$ . A point  $x \in X$  is an interior point of  $Y$  if there exists  $r > 0$  such that  $B(x, r) \subseteq Y$ . The set of interior points of  $Y$  is the interior of  $Y$ , and it is denoted by  $\text{int}(Y)$ .<sup>1</sup>

<sup>1</sup>  $\text{int}(Y) \subseteq Y$ .

**Definition 1.6** (Open subset). A subset  $Y$  of  $X$  is open if  $\text{int}(Y) = Y$ .

**Theorem 1.1.** Any open ball  $B(x, r)$  in a metric space  $X$  is an open subset of  $X$ .

*Proof:* Suppose  $y \in B(x, r)$ . Then  $d(x, y) < r$ , and  $0 < r - d(x, y)$ . Suppose  $z \in B(y, r - d(x, y))$ . It follows from the definition of a metric that  $d(x, z) \leq d(x, y) + d(y, z)$ , so  $d(x, z) \leq d(x, y) + (r - d(x, y)) = r$ , so  $z \in B(x, r)$ .  $\square$

**Theorem 1.2.** The union of a family of open subsets of a metric space  $X$  is an open subset of  $X$ .

*Proof:* Suppose  $\{U_\alpha\}_{\alpha \in A}$  a family of open subsets of  $X$ . If  $x \in \bigcup_{\alpha \in A} U_\alpha$ , then  $\exists \alpha (x \in U_\alpha)$ , so there exists an open ball  $B(x, r)$  such that  $B(x, r) \subseteq U_\alpha$ . Because  $x \in U_\alpha \Rightarrow x \in \bigcup_{\alpha \in A} U_\alpha$ , then  $B(x, r) \subseteq \bigcup_{\alpha \in A} U_\alpha$ .  $\square$

**Theorem 1.3.** A subset  $U$  of a metric space  $X$  is open iff  $U$  is a union of open balls in  $X$ .

*Proof:* Theorem 1.1 and 1.3 prove the left implication. If  $U$  is an open subset of  $X$ , then for all  $x \in U$ , there exists  $r(x) > 0$  such that  $B(x, r(x)) \subseteq U$ , so  $\bigcup_{x \in U} B(x, r(x)) = U$ .  $\square$

**Theorem 1.4.** The intersection of any finite number of open subsets of a metric space is open.

*Proof:* Suppose  $x \in \bigcap_{n=1}^m U_n$ , a finite union of open subsets of a metric space. Then for all  $n$ , there exists  $r(n) > 0$  such that  $B(x, r(n)) \subseteq U_n$ . Let  $r = \min(r(1) \dots r(m))$ . Then for all  $n$  we see  $B(x, r) \subseteq B(x, r(n))$  and thus  $B(x, r) \subseteq \bigcap_{n=1}^m U_n$ .  $\square$

**Theorem 1.5.** Let  $Y$  be a subspace of a metric space  $X$ . Then a subset  $U$  of  $Y$  is open in  $Y$  iff  $U = V \cap Y$  for some open subset  $V$  of  $X$ .

*Proof:* Suppose  $x \in V \cap Y$ . Then there exists an open ball in  $X$  with radius  $r(x)$  such that  $B(x, r(x)) \subseteq V$ , and  $x \in Y$ . Because  $Y \subseteq X$  we see that  $Y \cap B(x, r(x)) = \{y \in X \cap Y \mid d(x, y) < r(x)\} = \{y \in Y \mid d(x, y) < r(x)\}$ , by definition an open ball in  $Y$ . Trivially  $V \cap Y \subseteq \bigcap_{x \in V \cap Y} Y \cap B(x, r(x))$  and by definition the reverse is true.

To prove the converse, suppose  $x \in U$ . Then there exists an open ball in  $Y$  with radius  $r(x)$  such that  $B(x, r(x)) \subseteq U$ . It follows from conclusions reached above that if  $B'(x, r(x))$  is open in  $X$ , then  $B'(x, r(x)) \cap Y = B(x, r(x))$ . Let  $V = \bigcup_{x \in U} B'(x, r(x))$ . Then  $V \cap Y \subseteq U$ , and  $x \in U \Rightarrow x \in V$ .  $\square$

**Definition 1.7** (Adherent point). Let  $Y$  be a subset of a metric space  $X$ . A point  $x \in X$  is adherent to  $Y$  if for all  $r > 0$

$$B(x, r) \cap Y \neq \emptyset$$

**Definition 1.8** (Closure). The closure of  $Y$  denoted by  $\bar{Y}$ , consists of all points in  $X$  that are adherent to  $Y$ .<sup>2</sup>

$$^2 Y \subseteq \bar{Y}.$$

**Definition 1.9** (Closed subset). The subset  $Y$  is closed if  $Y = \bar{Y}$ .<sup>3</sup>

<sup>3</sup> The empty set  $\emptyset$  and  $X$  are closed subsets of  $X$ . Interestingly,  $X$  is also open in  $X$ .

**Theorem 1.6.** If  $Y$  is a subset of a metric space  $X$ , then the closure of  $Y$  is closed, i.e.

$$\overline{\bar{Y}} = \bar{Y}$$

*Proof:*  $\bar{Y}$  contains all  $x \in X$  such that for all  $r > 0$  in  $B(x, r) \cap Y \neq \emptyset$ . Let  $y \in X$  with  $B(y, r') \cap \bar{Y} \neq \emptyset$  for  $r' > 0$ . Suppose to the

contrary that there does not exist  $x \in X$  such that  $x = y$ . Then there exists  $a = \min(d(x, y)) > 0$  such that  $\forall x (x \notin B(y, a))$ , therefore  $B(y, a) \cap \bar{Y} = \emptyset$ , a contradiction.  $\square$

**Theorem 1.7.** A subset  $Y$  of a metric space  $X$  is closed iff the complement of  $Y$  is open.

*Proof:* If  $Y$  is closed, then  $Y$  contains all  $x \in X$  such that for all  $r > 0$ ,  $B(x, r) \cap Y \neq \emptyset$ . Therefore iff  $y \in Y^c$  the negation is true, i.e. there exists  $r' > 0$  such that  $B(y, r') \cap Y = \emptyset$ , and because  $Y^c \cup Y = X$  we have  $B(y, r') \subset Y^c$  and  $Y^c$  is open.  $\square$

**Theorem 1.8.** The intersection of any family of closed sets is closed. The union of any finite family of closed sets is closed.

*Proof:* Let  $\{Y_\alpha\}$  be a family of closed sets in  $X$ , and  $\alpha \in A$ , the number of elements in  $\{Y_\alpha\}$ . Following the fact that a union of open subsets is open, and the intersection of finite open subsets is open, as well as the previous theorem, we see

$$\begin{aligned} X \setminus \bigcup_{\alpha \in A} Y_\alpha &= \bigcap_{\alpha \in A} X \setminus Y_\alpha \\ X \setminus \bigcap_{\alpha \in A} Y_\alpha &= \bigcup_{\alpha \in A} X \setminus Y_\alpha \end{aligned}$$

$\square$

**Definition 1.10** (Convergent sequence). A sequence  $\{x_n\}_{n=1}^\infty$  in a metric space  $X$  converges to  $x \in X$  if

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0$$

In this case,  $x$  is the limit of  $\{x_n\}$  and we write  $x_n \rightarrow x$ , or

$$\lim_{n \rightarrow \infty} x_n = x.$$

**Lemma 1.1.** The limit of a convergent sequence in a metric space is unique

*Proof:* Let  $\lim_{n \rightarrow \infty} x_n = x, y$  and suppose to the contrary that  $x \neq y$ . Then  $d(x, y) > 0$  and for all  $\epsilon > 0$  there exists  $\delta$  such that  $d(x_n, x)$  and  $d(x_n, y)$  are both less than  $\frac{\epsilon}{2}$ . But then if  $\epsilon < d(x, y)$  then  $d(x_n, x) + d(x_n, y) < d(x, y)$ , a contradiction.  $\square$

**Theorem 1.9.** Let  $Y$  be a subset of the metric space  $X$ , then  $x \in X$  is adherent to  $Y$  iff there is a sequence in  $Y$  that converges to  $x$ .

*Proof:* If  $x$  is adherent to  $Y$ , then  $\forall r > 0$ ,  $B(x, r) \cap Y \neq \emptyset$ , i.e. for all  $r$  there exists  $y \in Y$  such that  $d(x, y_n) < r$ . Using this fact we can construct a sequence that converges to  $x$ . Let  $y_n \in Y$ , and  $\{y_n\}$  be a sequence such that for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $n > N$  implies  $d(x, y) < \epsilon$ .

Let  $\{y_n\}$  be a sequence with  $y_n \in Y$ , and let  $x \in X$ . Let  $\{y_n\}$  be such that for all  $\epsilon > 0$ ,  $n \in \mathbb{N}$  with  $n > N$  implies  $d(x, y_n) < \epsilon$ . Then for all  $r > 0$  there exists  $r = \epsilon$  such that  $y_n \in B(x, r)$ , and thus  $B(x, r) \cap Y \neq \emptyset$  for all  $r > 0$ .  $\square$

### Completeness

**Definition 2.1** (Cauchy sequence). A sequence  $\{x_n\}_{n=1}^{\infty}$  in a metric space  $X$  is a Cauchy sequence if

$$\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0.$$

**Lemma 2.1.** A convergent sequence is a Cauchy sequence.<sup>4</sup>

*Proof:* Suppose  $\{x_n\}$  in  $X$  a sequence that converges to  $x$  in  $X$ .

Then

$$\forall \epsilon > 0, \exists n, m > N (d(x_n, x), d(x_m, x) < \epsilon).$$

If we choose  $N$  such that  $d(x_n, x), d(x_m, x) < \frac{\epsilon}{2}$  then

$$d(x_n, x) + d(x_m, x) < \epsilon \Rightarrow d(x_n, x_m) < \epsilon.$$

$\square$

**Lemma 2.2.** If  $\{x_n\}$  is a Cauchy sequence and if there is a subsequence  $\{x_{n_k}\}_{k=1}^{\infty}$  of  $\{x_n\}$  that converges to  $x$ , then  $\{x_n\}$  converges to  $x$ .

*Proof:* Suppose  $\{x_n\}$  a convergent sequence and  $\{x_{n_k}\}_{k=1}^{\infty}$  a subsequence which converges to  $x$  then

$$\forall \delta > 0, \exists N (n_k > N \Rightarrow d(x_{n_k}, x) < \delta)$$

$$\forall \epsilon > 0, \exists M (n > M \wedge n_k > M, N \Rightarrow d(x_n, x_{n_k}) < \epsilon).$$

Because  $d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon + \delta$  then  $d(x_n, x) < \epsilon + \delta$ .  $\square$

**Definition 2.2** (Complete metric space). A metric space  $X$  is complete if every Cauchy sequence in  $X$  converges.

**Theorem 2.1.** A complete subspace  $Y$  of a metric space  $X$  is closed in  $X$

*Proof:* If  $x \in \bar{Y}$ , then  $\forall r > 0, \exists B(x, r)$  such that  $B(x, r) \cap Y \neq \emptyset$ , so  $\exists y \in Y$  such that  $d(x, y) < r$ . It follows there exists a Cauchy sequence  $\{y_n\}$  in  $Y$  with limit  $x$  such that  $\forall r, \exists N (n > N \Rightarrow d(x, y_n) < r)$ . And because every Cauchy sequence in  $Y$  converges,  $x \in Y$  and  $\bar{Y} = Y$ .  $\square$

**Definition 2.3** (Uniform convergence). Let  $\{f_n\}_{n=1}^{\infty}$  be a sequence of functions from a set  $S$  to a metric space  $X$  and let  $f$  be a function from  $S$  to  $X$ . The sequence  $\{f_n\}$  converges uniformly to  $f$  on  $S$  if for each  $\epsilon > 0$  there exists an integer  $N$  such that  $d(f_n(s), f(s)) < \epsilon$  for all integers  $n \geq N$  and for all  $s \in S$ .

<sup>4</sup> In a complete metric space the reverse is true.

**Definition 2.4.** A sequence  $\{f_n\}$  of functions from  $S$  to  $X$  is a Cauchy sequence of functions if for each  $\epsilon > 0$  there exists an integer  $N$  such that

$$d(f_n(s), f_m(s)) < \epsilon, \quad \text{all } s \in S, n, m \geq N.$$

**Theorem 2.2.** Let  $S$  be a set, and let  $X$  be a complete metric space. If  $\{f_n\}_{n=1}^{\infty}$  is a Cauchy sequence of functions from  $S$  to  $X$ , then there exists a function  $f$  from  $S$  to  $X$  such that  $\{f_n\}$  converges uniformly to  $f$ .

*Proof:* If  $\{x_n\}$  a Cauchy sequence in a complete metric space  $X$ , then  $\{x_n\}$  converges. Therefore, for each  $s \in S$ , there exists  $a_s \in X$  such that  $\lim_{n \rightarrow \infty} f_n(s) = a_s$ . Let a  $f$  be a function from  $S$  to  $X$  defined by  $f(s) = a_s$ . It follows from the definition of a Cauchy sequence of functions that for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $s \in S, n > N$  implies  $d(f_n(s), f(s)) < \epsilon$ , so  $\{f_n\}$  converges uniformly.  $\square$

**Definition 2.5** (Dense subsets). A subset  $T$  of a metric space  $X$  is dense in  $X$  if  $\bar{T} = X$ .

**Theorem 2.3** (Baire Category Theorem). Let  $\{U_n\}_{n=1}^{\infty}$  be a sequence of dense open subsets of a complete metric space  $X$ . Then  $\bigcap_{n=1}^{\infty} U_n$  is also dense in  $X$ .

*Proof:* We shall prove that  $\bigcap_{n=1}^{\infty} U_n$  is dense in  $X$  by showing that for any ball with  $r > 0$  and  $x \in X$  there exists  $y \in \bigcap_{n=1}^{\infty} U_n$  such that  $y \in B(x, r)$ .

If  $\epsilon > 0$  there exists  $y_1 \in U_1$  such that  $y_1 \in B(x, \epsilon)$ . Because  $B(x, \epsilon)$  and  $U_1$  are both open, there exists  $1 > r_1 > 0$  such that  $B(y_1, r_1) \subseteq U_1 \cap B(x, \epsilon)$  and by shrinking  $r_1$  we have  $\overline{B(y_1, r_1)} \subseteq U_1 \cap B(x, \epsilon)$ . This procedure can be repeated by replacing  $B(x, \epsilon)$  by  $B(y_1, r_1)$ , and finding  $y_2 \in U_2 \cap B(y_1, r_1)$  with  $1/2 > r_2 > 0$  such that  $\overline{B(y_2, r_2)} \subseteq U_2 \cap B(y_1, r_1)$ .<sup>5</sup> We can then define a cauchy sequence  $\{y_n\}_{n=1}^{\infty}$  with each  $y_n$  satisfying  $\overline{B(y_n, r_n)} \subseteq U_n \cap B(y_{n-1}, r_{n-1})$  with  $1/n > r_n > 0$ . Because  $X$  is complete, we know that  $\lim_{n \rightarrow \infty} y_n = y$  with  $y \in X$ . If  $y \notin \bigcap_{n=1}^{\infty} U_n$  then there exists  $n$  such that  $y \notin B(y_n, r_n)$ . If  $m > n$  Then  $y_m \in \overline{B(y_m, r_m)} \cap B(y_n, r_n)$ . By theorems 1.9 and 1.6, the limit of any convergent sequence in  $\overline{B(y_m, r_m)}$  is in  $\overline{B(y_m, r_m)}$ , it follows that  $y \in B(y_n, r_n)$ , a contradiction. Therefore  $y \in \bigcap_{n=1}^{\infty} U_n$ .  $\square$

<sup>5</sup> Such  $y_2, r_2$  exist because  $y_1 \in X$  and  $U_2$  is dense in  $X$ , so for every  $r_1$ -ball of  $y_1$  there exists  $y_2 \in U_2$  such that  $y_2$  is in this ball.

**Definition 2.6** (Nowhere dense). A subset  $Y$  of  $X$  is nowhere dense if  $\bar{Y}$  has no interior points, that is, if

$$\text{int}(\bar{Y}) = \emptyset.$$

### Products of metric spaces

The properties and metric definitions that follow are numbered after the properties in the Gamelin "Introduction to Topology book". Let  $(X_1, d_1), \dots, (X_n, d_n)$  be metric spaces. The product set  $X = X_1 \times \dots \times X_n$  consists of all  $n$ -tuples  $(x_1, \dots, x_n)$ , where  $x_k \in X_k$ ,  $1 \leq k \leq n$ .

$$(4.1) \quad d(x, y) = [d_1(x_1, y_1)^2 + \dots + d_n(x_n, y_n)^2]^{1/2}.$$

$$(4.2) \quad \max(d_1(x_1, y_1), \dots, d_n(x_n, y_n)).$$

$$(4.3) \quad d(x, y) = d_1(x_1, y_1) + \dots + d_n(x_n, y_n).$$

$$(4.4) \quad \text{A sequence } \{x^j = (x_k^j)\}_{j=1}^\infty \text{ converges to } x = (x_1, \dots, x_n) \text{ in } X \text{ iff for each } k \text{ the sequence of component entries } \{x_k^j\}_{j=1}^\infty \text{ converges to } x_k \text{ in } X_k.$$

$$(4.5) \quad d_k(x_k, y_k) \leq d(x, y), \quad x, y \in X, 1 \leq k \leq n.$$

**Theorem 3.1.** Suppose that  $d$  is a metric on  $X = X_1 \times \dots \times X_n$  that satisfies property 4.4. Then the open sets in  $(X, d)$  are the unions of product sets of the form  $U_1 \times \dots \times U_n$ , where  $U_j$  is an open subset of  $X_j$ ,  $1 \leq j \leq n$ .

*Proof:* Suppose that  $U$  an open subset of  $X$  and  $y = (y_1, \dots, y_n) \in U$ . If  $1 \leq m \leq \infty$ , Because  $y_n \in B(y_n, 1/m)$  it follows that  $y$  is an element of the product of open balls  $B(y_1, 1/m) \times \dots \times B(y_n, 1/m)$ . Suppose to the contrary that there does not exist  $\epsilon > 0$  such that  $B(y_1, \epsilon) \times \dots \times B(y_n, \epsilon) \subseteq U$ . Then for all  $m$  there exist  $x^m = (x_1^m, \dots, x_n^m) \in U^c$  such that for all  $k$ ,  $1 \leq k \leq n$ , we have  $x_k^m \in B(y_k, 1/m)$ , i.e.

$$\lim_{m \rightarrow \infty} d_k(x_k^m, y_k) = 0.$$

But following property 4.4 this means

$$\lim_{m \rightarrow \infty} d(x^m, y) = 0.$$

It follows that because  $x^m \in U^c$ ,  $y \in \overline{U^c}$ .  $U^c$  is closed, so it follows that  $y \in U^c$ , a contradiction.  $\square$

**Theorem 3.2.** Let  $(X_1, d_1), \dots, (X_n, d_n)$  be complete metric spaces. Let  $d$  be a metric on  $X = X_1 \times \dots \times X_n$  that satisfies (4.4) and (4.5). Then  $(X, d)$  is complete.

**Corollary 3.1.** The  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , with the usual metric

$$|x - y| = [(x_1 - y_1)^2 + \dots + (x_n - y_n)^2]^{1/2}, \quad x, y \in \mathbb{R}^n,$$

Is complete.

## Compactness

**Definition 3.1** (Cover). A family  $\{U_\alpha\}_{\alpha \in A}$  of sets is said to cover a set  $S$  if  $S$  is contained in the union of the  $U_\alpha$ 's.

**Definition 3.2** (Open cover). An open cover of a metric space  $X$  is a family of open subsets of  $X$  that covers  $X$ .

**Definition 3.3** (Compactness). A metric space  $X$  is compact if every open cover has a finite subcover.

**Definition 3.4** (Totally bounded). A metric space  $X$  is totally bounded if for each  $\epsilon > 0$ , there exists a finite number of open balls of radius  $\epsilon$  that cover  $X$ .

**Theorem 3.3.** The following are equivalent for a metric space  $X$ :

1.  $X$  is compact.
2. Every sequence in  $X$  has a convergent subsequence.
3.  $X$  is totally bounded and complete.

**Definition 3.5** (Bounded). A metric space  $X$  is bounded if there exists  $b > 0$  such that  $d(x, y) < b$  for all  $x, y \in X$ .

**Lemma 3.1.** A totally bounded metric space is bounded.

**Lemma 3.2.** Any subspace of a totally bounded metric space is totally bounded.

**Lemma 3.3.** A subset  $E$  of  $\mathbb{R}^n$  is totally bounded iff  $E$  is bounded.

**Theorem 3.4** (Heine-Borel theorem). The following are equivalent for a subspace  $E$  of  $\mathbb{R}^n$ .

1.  $E$  is compact.
2. Every sequence in  $E$  has a convergent subsequence.
3.  $E$  is closed and bounded.

**Theorem 3.5.** Let  $X$  be a totally bounded metric space. Then every sequence in  $X$  has a Cauchy subsequence.

**Definition 3.6** (Seperability). A metric space  $X$  is seperable if there is a dense subset of  $X$  that is countable. In other words,  $X$  is seperable iff there is a sequence  $\{x_j\}_{j=1}^\infty$  in  $X$  that is dense in  $X$ .

**Theorem 3.6.** A subspace of a separable metric space is separable.

**Definition 3.7** (Base). A base of open sets for a metric space  $X$  is a family  $\mathcal{B}$  of open subsets of  $X$  such that every open subset of  $X$  is the union of sets in  $\mathcal{B}$ .

**Lemma 3.4.** A family  $\mathcal{B}$  of open subsets of a metric space  $X$  is a base of open sets iff for each  $x \in X$  and each open neighborhood  $U$  of  $x$ , there exists  $V \in \mathcal{B}$  such that  $x \in V$  and  $V \subseteq U$ .

**Definition 3.8** (Second-countable). A metric is second-countable if there is a base of open sets that is at most countable.

**Theorem 3.7.** A metric space is second-countable iff it is separable.

**Theorem 3.8** (Lindelof's theorem). Suppose the metric space  $X$  is second-countable. Then every open cover of  $X$  has a countable subcover.

**Theorem 3.9.** A compact metric space is separable and second-countable.