HW 8

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Proposition 1.1. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of non-negative real numbers. Then this series is convergent iff there is a real number M such that

$$\forall N \in \mathbb{Z}, \big(N \geq M \Rightarrow \sum_{n=m}^{N} a_n \leq M\big).$$

Corollary 1.2. Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two formal series of real numbers, and suppose that $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, and

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Problem 1

Prove use proposition 1.1 to prove 1.2.

Proof: Suppose for all $n \geq m$, $|a_n| \leq b_n$, and that the infinite series of b_n starting at $n = m$ converges to $L \in \mathbb{R}$. It follows that

$$\sum_{n=m}^{m} |a_n| = |a_n| \leq b_n = \sum_{n=m}^{m} b_n. \tag{1}$$

By inductive hypothesis, for all $N \geq m$,

$$\sum_{n=m}^{N} |a_n| \leq \sum_{n=m}^{N} b_n.$$

because $|a_{N+1}| \leq b_{N+1}$, it follows from the inductive hypothesis that

$$\sum_{n=m}^{N} |a_n| + |a_N + 1| = \sum_{n=m}^{N+1} |a_n| \leq \sum_{n=m}^{N} b_n + b_{N+1} = \sum_{n=m}^{N+1} b_n.$$

So

$$\sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n = L.$$

It follows from equation 1 that

$$\left|\sum_{i=m}^{m} a_n\right| = |a_n| \leq \sum_{i=m}^{m} |a_n| = |a_n|.$$

By inductive hypothesis, for all $N \geq m$,

$$\left|\sum_{i=m}^{N} a_n\right| \leq \sum_{i=m}^{N} |a_n|.$$

It follows from the triangle inequality and the inductive hypothesis that

$$\begin{aligned}
\left|\sum_{i=m}^{N+1} a_n\right| &= \left|\sum_{i=m}^{N} a_n + a_{N+1}\right| \\
&\le \left|\sum_{i=m}^{N} a_n\right| + |a_{N+1}| \\
&\le \sum_{i=m}^{N} |a_n| + |a_{N+1}| \\
&= \sum_{n=m}^{N+1} |a_n|.
\end{aligned}$$

Therefore,

$$\left|\sum_{n=m}^{\infty} a_n\right| \le \sum_{n=m}^{\infty} |a_n| \le \sum_{n=m}^{\infty} b_n.$$

If $\sum_{n=m}^{\infty} |a_n|$ did not converge, then by proposition 7.2.5 it would not be bounded. Therefore $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. $\square$

Problem 4

Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, and let L be a real number. Then the following statements are logically equivalent:

(a) f converges to L at x_0 in E

(b) For every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements E and converges to x_0, the sequence $(f(a_n))_{n=1}^{\infty}$ converges to L.

Proof: First we prove (a) implies (b). Suppose $\lim_{x\to x_0; x\in E} f(x) = L$, and let $(a_n)_{n=0}^{\infty}$ be a sequence of elements in E such that $\lim_{n\to\infty} a_n = x_0$. Then

$$\begin{aligned}
&\forall \epsilon > 0,\, \exists \delta > 0,\, \forall x \in E,\, \big(|x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon\big),\\
&\forall \epsilon' > 0,\, \exists N \in \mathbb{N},\, \forall n \in \mathbb{N},\, \big(n > N \Rightarrow |a_n - x_0| < \epsilon'\big).
\end{aligned}$$

Therefore,

$$\forall \epsilon > 0, \exists \delta > 0, \exists M \in \mathbb{N},\, \forall n \in \mathbb{N},\, \big(n > M \Rightarrow |a_n - x_0| < \delta \Rightarrow |f(a_n) - L| < \epsilon\big).$$

so

$$\forall \epsilon > 0,\, \exists M \in \mathbb{N},\, \forall n \in \mathbb{N},\, \big(n > M \Rightarrow |f(a_n) - L| < \epsilon\big),$$

and $(f(a_n))_{n=1}^{\infty}$ converges to L.

Next, we prove (b) implies (a) by contrapositive. Suppose

$$\exists \epsilon > 0,\, \forall \delta > 0,\, \exists x \in E,\, \big(|x - x_0| < \delta \wedge |f(x) - L| \geq \epsilon\big).$$

It follows that for some $\epsilon > 0$ there exists $a_n \in E$ such that $|a_n - x_0| < 1/n$ with $|f(a_n) - L| \geq \epsilon$. $(a_n)_{n=1}^{\infty}$ converges to x_0 because for all ϵ', $n = \lceil 1/\epsilon' \rceil$ implies $1 \leq n\epsilon'$ so $1/n \leq \epsilon'$. Because for all n, $|f(a_n) - L| \geq \epsilon$, the sequence $(f(a_n))_{n=1}^{\infty}$ does not converge to L, contradicting (b). □

Problem 5

Let f, g be functions defined from $\mathbb{R}$ to $\mathbb{R}$, and let a, b be real numbers. Show that if f and g are continuous at $x_0 \in \mathbb{R}$, then $af + bg$ is continuous at x_0.

Proof: If f, g are functions defined from $\mathbb{R}$ to $\mathbb{R}$, and are continuous on $\mathbb{R}$ at x_0, it follows from proposition 9.3.14 that

$$\begin{aligned}
\lim_{x\to x_0} af + bg &= \lim_{x\to x_0} af + \lim_{x\to x_0} bg\\
&= a \lim_{x\to x_0} f + b \lim_{x\to x_0} g\\
&= af(x_0) + bg(x_0),
\end{aligned}$$

so $af + bg$ is continuous at x_0. □

Problem 6

Let X and Y be subsets of R, and let $f : X \to Y$ and $g : Y \to \mathbb{R}$ be functions. Let x_0 be a point in X. If f is continuous at x_0, and g is continuous at $f(x_0)$, then the composition $g \circ f : X \to \mathbb{R}$ is continuous at x_0.

Proof: Because g is continuous at $f(x_0)$ and f is continuous at x_0,

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall y \in Y,\ \big(|y - f(x_0)| < \delta \Rightarrow |g(y) - g(f(x_0))| < \epsilon\big),$$
$$\forall \epsilon' > 0,\ \exists \delta' > 0,\ \forall x \in X,\ \big(|x - x_0| < \delta' \Rightarrow |f(x) - f(x_0)| < \epsilon'\big).$$

Therefore, there exists $\delta'' > 0$ such that $|x - x_0| < \delta''$ implies $|f(x) - f(x_0)| < \delta$ which implies $|g(f(x)) - g(f(x_0))| < \epsilon$, i.e.

$$\forall \epsilon > 0,\ \exists \delta'' > 0,\ \forall x \in X \big(|x - x_0| < \delta'' \Rightarrow |g \circ f(x) - g \circ f(x_0)| < \epsilon\big),$$

so $g \circ f$ is continuous at x_0. □

Problem 7

Let $n \geq 0$ be an integer, and for each $0 \leq i \leq n$ let c_i be a real number. Let $P : \mathbb{R} \to \mathbb{R}$ be the function

$$P(x) = \sum_{i=0}^{n} c_i x^i.$$

Show P is continuous.

Proof: test □