Topology

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Open and closed sets

Definition 1.1 (Metric)**.** A *metric* on a set X is a real-valued function d on $X \times X$ that has the following properties:

(a) For all $x, y \in X$, $d(x, y) \geq 0$.

(b) $d(x, y) = 0$ iff $x = y$.

(c) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(d) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1.2 (Metric space)**.** A metric space (X, d) is a set X equipped with a metric d on X.

Definition 1.3 (Subspace)**.** If (X, d) is a metric space and Y is a subset of X, then the restriction d' of d to $Y \times Y$ is a metric on Y, and (Y, d') is called a subspace of (X, d).

Remark. Any set X can be made into a discreet metric space by associating with X the metric d defined by

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Definition 1.4 (Open ball)**.** The open ball $B(x, r)$ with center $x \in X$ and radius $r > 0$ is defined by

$$B(x, r) = \{y \in X | d(x, y) < r\}.$$

Definition 1.5 (Interior point)**.** Let Y be a subset of X. A point $x \in X$ is an interior point of Y if there exists $r > 0$ such that $B(x, r) \subseteq Y$. The set of interior points of y is the interior of Y, and it is denoted by $\text{int}(Y)$.[1]

[1] $\text{int}(Y) \subseteq Y$.

Definition 1.6 (Open subset)**.** A subset Y of X is open if $\text{int}(Y) = Y$.

Theorem 1.1. Any open ball $B(x, r)$ in a metric space X is an open subset of X

Proof: Suppose $y \in B(x, r)$. Then $d(x, y) < r$, and $0 < r - d(x, y)$. Suppose $z \in B(y, r - d(x, y))$. If follows from the definition of a metric that $d(x, z) \leq d(x, y) + d(y, z)$, so $d(x, z) \leq d(x, y) + (r - d(x, y)) = r$, so $z \in B(x, r)$. $\square$

Theorem 1.2. The union of a family of open subsets of a metric space X is an open subset of X.

Proof: Suppose $\{U_\alpha\}\ \alpha \in A$ a family of open subsets of X. If $x \in \bigcup_{\alpha \in A} U_\alpha$, then $\exists \alpha\ (x \in U_\alpha)$, so there exits an open ball $B(x, r)$ such that $B(x, r) \subseteq U_\alpha$. Because $x \in U_\alpha \Rightarrow x \in \bigcup_{\alpha \in A} U_\alpha$, then $B(x, r) \subseteq \bigcup_{\alpha \in A} U_\alpha$. □

Theorem 1.3. A subset U of a metric space X is open iff U is a union of open balls in X.

Proof: Theorem 1.1 and 1.3 prove the left implication. If U is an open subset of X, then for all $x \in U$, there exists $r(x) > 0$ such that $B(x, r(x)) \in U$, so $\bigcup_{x \in U} B(x, r(x)) = U$. □

Theorem 1.4. The intersection of any finite number of open subsets of a metric space is open.

Proof: Suppose $x \in \bigcap_{n=1}^{m} U_n$, a finite union of open subsets of a metric space. Then for all n, there exists $r(n) > 0$ such that $B(x, r(n)) \in U_n$. Let $r = \min(r(1) \dots r(m))$. Then for all $r(n)$ we see $B(x, r) \subseteq B(x, r(n))$ and thus $B(x, r) \subseteq \bigcap_{n=1}^{m} U_n$. □

Theorem 1.5. Let Y be a subspace of a metric space X. Then a subset U of Y is open in Y iff $U = V \cap Y$ for some open subset V of X.

Proof: Suppose $x \in V \cap Y$. Then there exists an open ball in X with radius $r(x)$ such that $B(x, r(x)) \subseteq V$, and $x \in Y$. Because $Y \subseteq X$ we see that $Y \cap B(x, r(x)) = \{y \in X \cap Y | d(x, y) < r(x)\} = \{y \in Y | d(x, y) < r(x)\}$, by definition an open ball in Y. Trivially $V \cap Y \subseteq \bigcap_{x \in V \cap Y} Y \cap B(x, r(x))$ and by definition the reverse is true.

To prove the converse, suppose $x \in U$. Then there exits an open ball in Y with radius $r(x)$ such that $B(x, r(x)) \in U$. It follows from conclusions reached above that if $B'(x, r(x))$ is open in X, then $B'(x, r(x)) \cap Y = B(x, r(x))$. Let $V = \bigcup_{x \in U} B'(x, r(x))$. Then $V \cap Y \subseteq U$, and $x \in U \Rightarrow x \in V$. □

Definition 1.7 (Adherent point)**.** Let Y be a subset of a metric space X. A point $x \in X$ is adherent to Y if for all $r > 0$

$$B(x, r) \cap Y \neq \emptyset$$

Definition 1.8 (Closure)**.** The closure of Y denoted by $\overline{Y}$, consists of all points in X that are adherent to Y.[2]

[2] $Y \subseteq \overline{Y}$.

Definition 1.9 (Closed subset)**.** The subset Y is closed if $Y = \overline{Y}$.[3]

[3] The empty set $\emptyset$ and X are closed subsets of X. Interestingly, X is also open in X.

Theorem 1.6. If Y is a subset of a metric space X, then the closure of Y is closed, i.e.

$$\overline{\overline{Y}} = \overline{Y}$$

Proof: $\overline{Y}$ contains all $x \in X$ such that for all $r > 0$ in $B(x, r) \cap Y \neq \emptyset$. Let $y \in X$ with $B(y, r') \cap \overline{Y} \neq \emptyset$ for $r' > 0$. Suppose to the contrary that there does not exist $x \in X$ such that $x = y$. Then there exists $a = \min\big(d(x,y)\big) > 0$ such that $\forall x\,(x \notin B(y,a))$, therefore $B(y,a) \cap \overline{Y} = \emptyset$, a contradiction. □

Theorem 1.7. A subset Y of a metric space X is closed iff the complement of Y is open.

Proof: If Y is closed, then Y contains all $x \in X$ such that for all $r > 0$, $B(x,r) \cap Y \neq \emptyset$. Therefore iff $y \in Y^c$ the negation is true, i.e. there exists $r' > 0$ such that $B(y,r') \cap Y = \emptyset$, and because $Y^c \cup Y = X$ we have $B(y,r') \subset Y^c$ and Y^c is open. □

Theorem 1.8. The intersection of any family of closed sets is closed. The union of any finite family of closed sets is closed.

Proof: Let $\{Y_\alpha\}$ be a family of closed sets in X, and $\alpha \in A$, the number of elements in $\{Y_\alpha\}$. Following the fact that a union of open subsets is open, and the intersection of finite open subsets is open, as well as the previous theorem, we see

$$X \setminus \bigcup_{\alpha \in A} Y_\alpha = \bigcap_{\alpha \in A} X \setminus Y_\alpha$$
$$X \setminus \bigcap_{\alpha \in A} Y_\alpha = \bigcup_{\alpha \in A} X \setminus Y_\alpha$$

□

Definition 1.10 (Convergent sequence)**.** A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space X converges to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0$$

In this case, x is the limit of $\{x_n\}$ and we write $x_n \to x$, or

$$\lim_{n\to\infty} x_n = x.$$

Lemma 1.1. The limit of a convergent sequence in a metric space is unique

Proof: Let $lim_{n\to\infty} x_n = x, y$ and suppose to the contrary that $x \neq y$. Then $d(x,y) > 0$ and for all $\epsilon > 0$ there exits δ such that $d(x_n,x)$ and $d(x_n,y)$ are both less than $\frac{\epsilon}{2}$. But then if $\epsilon < d(x,y)$ then $d(x_n,x) + d(x_n,y) < d(x,y)$, a contradiction. □

Theorem 1.9. Let Y be a subset of the metric space X, then $x \in X$ is adherent to Y iff there is a sequence in Y that converges to x.

Proof: If x is adherent to Y, then $\forall r > 0$, $B(x,r) \cap Y \neq \emptyset$, i.e. for all r there exits $y \in Y$ such that $d(x,y_n) < r$. Using this fact we can construct a sequence that converges to x. Let $y_n \in Y$, and $\{y_n\}$ be a sequence such that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies $d(x,y) < \epsilon$.

Let $\{y_n\}$ be a sequence with $y_n \in Y$, and let $x \in X$. Let $\{y_n\}$ be such that for all $\epsilon > 0$, $n \in \mathbb{N}$ with $n > N$ implies $d(x, y_n) < \epsilon$. Then for all $r > 0$ there exists $r = \epsilon$ such that $y_n \in B(x, r)$, and thus $B(x, r) \cap Y \neq \emptyset$ for all $r > 0$. □

Completeness

Definition 2.1 (Cauchy sequence)**.** A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space X is a Cauchy sequence if

$$\lim_{m,n \to \infty} d(x_n, x_m) = 0.$$

Lemma 2.1. A convergent sequence is a cauchy sequence.[4]

Proof: Suppose $\{x_n\}$ in X a sequence that converges to x in X. Then

$$\forall \epsilon > 0, \exists n, m > N \left(d(x_n, x), d(x_m, x) < \epsilon \right).$$

If we choose N such that $d(x_n, x), d(x_m, x) < \frac{\epsilon}{2}$ then

$$d(x_n, x) + d(x_m, x) < \epsilon \Rightarrow d(x_n, d_m) < \epsilon.$$

□

[4] In a complete metric space the reverse is true.

Lemma 2.2. If $\{x_n\}$ is a Cauchy sequence and if there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ that converges to x, then $\{x_n\}$ converges to x.

Proof: Suppose $\{x_n\}$ a convergent sequence and $\{x_{n_k}\}_{k=1}^{\infty}$ a subsequence which converges to x then

$$\forall \delta > 0, \exists N \left(n_k > N \Rightarrow d(x_{n_k}, x) < \delta \right)$$
$$\forall \epsilon > 0, \exists M \left(n > M \wedge n_k > M, N \Rightarrow d(x_n, x_{n_k}) < \epsilon \right).$$

Because $d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon + \delta$ then $d(x_n, x) < \epsilon + \delta$. □

Definition 2.2 (Complete metric space)**.** A metric space X is complete if every cauchy sequence in X converges.

Theorem 2.1. A complete subspace Y of a metric space X is closed in X

Proof: If $x \in \overline{Y}$, then $\forall r > 0, \exists B(x, r)$ such that $B(x, r) \cap Y \neq \emptyset$, so $\exists y \in Y$ such that $d(x, y) < r$. It follows there exists a Cauchy sequence $\{y_n\}$ in Y with limit x such that $\forall r, \exists N \ (n > N \Rightarrow d(x, y_n) < r)$. And because every Cauchy sequence in Y converges, $x \in Y$ and $\overline{Y} = Y$. □

Definition 2.3 (Uniform convergence)**.** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from a set S to a metric space X and let f be a function from S to X. The sequence $\{f_n\}$ converges uniformly to f on S if for each $\epsilon > 0$ there exists an integer N such that $d(f_n(s), f(s)) < \epsilon$ for all integers $n \geq N$ and for all $s \in S$.

Definition 2.4. A sequence $\{f_n\}$ of functions from S to X is a Cauchy sequence of functions if for each $\epsilon > 0$ there exists an integer N such that

$$d(f_n(s), f_m(s)) < \epsilon, \qquad \text{all } s \in S,\ n, m \geq N.$$

Theorem 2.2. Let S be a set, and let X be a complete metric space. If $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence of functions from S to X, then there exists a function f from S to X such that $\{f_n\}$ converges uniformly to f

Proof: If $\{x_n\}$ a Cauchy sequence in a complete metric space X, then $\{x_n\}$ converges. Therefore, for each $s \in S$, there exists $a_s \in X$ such that $lim_{n\to\infty} f_n(s) = a_s$. Let a f be a function from S to X defined by $f(s) = a_s$. It follows from the definition of a Cauchy sequence of functions that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $s \in S$, $n > N$ implies $d(f_n(s), f(s)) < \epsilon$, so $\{f_n\}$ converges uniformly. □

Definition 2.5 (Dense subsets)**.** A subset T of a metric space X is dense in X if $\overline{T} = X$.

Theorem 2.3 (Baire Category Theorem)**.** Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of dense open subsets of a complete metric space X. Then $\bigcap_{n=1}^{\infty} U_n$ is also dense in X.

Proof: We shall prove that $\bigcap_{n=1}^{\infty} U_n$ is dense in X by showing that for any ball with $r > 0$ and $x \in X$ there exists $y \in \bigcap_{n=1}^{\infty} U_n$ such that $y \in B(x, r)$.

If $\epsilon > 0$ there exists $y_1 \in U_1$ such that $y_1 \in B(x, \epsilon)$. Because $B(x, \epsilon)$ and U_1 are both open, there exists $1 > r_1 > 0$ such that $B(y_1, r_1) \subseteq U_1 \cap B(x, \epsilon)$ and by shrinking r_1 we have $\overline{B(y_1, r_1)} \subseteq U_1 \cap B(x, \epsilon)$. This procedure can be repeated by replacing $B(x, \epsilon)$ by $B(y_1, r_1)$, and finding $y_2 \in U_2 \cap B(y_1, r_1)$ with $1/2 > r_2 > 0$ such that $\overline{B(y_2, r_2)} \subseteq U_2 \cap B(y_1, r_1)$.[5] We can then define a cauchy sequence $\{y_n\}_{n=1}^{\infty}$ with each y_n satisfying $\overline{B(y_n, r_n)} \subseteq U_n \cap B(y_{n-1}, r_{n-1})$ with $1/n > r_n > 0$. Because X is complete, we know that $\lim_{n\to\infty} y_n = y$ with $y \in X$. If $y \notin \bigcap_{n=1}^{\infty} U_n$ then there exists n such that $y \notin B(y_n, r_n)$. If $m > n$ Then $y_m \in \overline{B(y_m, r_m)} \cap \overline{B(y_n, r_n)}$. By theorems 1.9 and 1.6, the limit of any convergent sequence in $\overline{B(y_m, r_m)}$ is in $\overline{B(y_m, r_m)}$, it follows that $y \in B(y_n, r_n)$, a contradiction. Therefore $y \in \bigcap_{n=1}^{\infty} U_n$. □

[5] Such y_2, r_2 exist because $y_1 \in X$ and U_2 is dense in X, so for every r_1-ball of y_1 there exists $y_2 \in U_2$ such that y_2 is in this ball.

Definition 2.6 (Nowhere dense)**.** A subset Y of X is nowhere dense if $\overline{Y}$ has no interior points, that is, if

$$\text{int}(\overline{Y}) = \emptyset.$$

Products of metric spaces

The properties and metric definitions that follow are numbered after the properties in the Gamelin "Introduction to Topology book". Let $(X_1, d_1), \ldots, (X_n, d_n)$ be metric spaces. The product set $X = X_1 \times \ldots \times X_n$ consists of all n-tuples $(x_1, \ldots, x_n)$, where $x_k \in X_k$, $1 \leq k \leq n$.

(4.1) $d(x,y) = \left[d_1(x_1,y_1)^2 + \ldots + d_n(x_n,y_n)^2\right]^{1/2}$.

(4.2) $\max\left(d_1(x_1,y_1), \ldots, d_n(x_n,y_n)\right)$.

(4.3) $d(x,y) = d(x_1,y_1) + \ldots + d_n(x_n,y_n)$.

(4.4) A sequence $\{x^j = (x_k^j)\}_{j=1}^{\infty}$ converges to $x = (x_1, \ldots, x_n)$ in X iff for each k the sequence of component entries $\{x_k^j\}_{j=1}^{\infty}$ converges to x_k in X_k.

(4.5) $d_k(x_k, y_k) \leq d(x,y), \qquad x, y \in X, 1 \leq K \leq n.$

Theorem 3.1. Suppose that d is a metric on $X = X_1 \times \ldots \times X_n$ that satisfies property 4.4. Then the open sets in (X, d) are the unions of product sets of the form $U_1 \times \ldots \times U_n$, where U_j is an open subset of X_j, $1 \leq j \leq n$.

Proof: Suppose that U an open subset of X and $y = (y_1, \ldots, y_n) \in U$. If $1 \leq m \leq \infty$ and $1 \leq k \leq n$, because each $y_k \in B(y_k, 1/m)$ it follows that y is an element of the product of open balls $B(y_1, 1/m) \times \ldots \times B(y_n, 1/m)$.[6] Suppose to the contrary that there does not exist $\epsilon > 0$ such that $B(y_1, \epsilon) \times \ldots \times B(y_n, \epsilon) \subseteq U$. Then for all m there exist $x^m = (x_1^m, \ldots, x_n^m) \in U^c$ such that $x^m \in B(y_1, 1/m) \times \ldots \times B(y_n, 1/m)$ i.e. for all k, $x_k^m \in B(y_k, 1/m)$. It follows

$$\lim_{m \to \infty} d_k(x_k^m, y_k) = 0.$$

But following property 4.4 this means

$$\lim_{m \to \infty} d(x^m, y) = 0.$$

It follows that because $x^m \in U^c$, $y \in \overline{U^c}$. U^c is closed so $y \in U^c$, a contradiction. Therefore each $y \in U$ is contained in a subset of U which is the product of open balls in X_k. □

[6] Note that these could be closed balls as well. Open balls are used to satisfy the proof.

Theorem 3.2. Let $(X_1, d_1), \ldots, (X_n, d_n)$ be complete metric spaces. Let d be a metric on $X = X_1 \times \ldots \times X_n$ that satisfies (4.4) and (4.5). Then (X, d) is complete.

Corollary 3.1. The n-dimensional Euclidean space $\mathbb{R}^n$, with the usual metric

$$|x - y| = \left[(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2\right]^{1/2}, \qquad x, y \in \mathbb{R}^n,$$

Is complete.

Compactness

Definition 3.1 (Cover). A family $\{U_\alpha\}_{\alpha \in A}$ of sets is said to cover a set S if S is contained in the union of the U_α's.

Definition 3.2 (Open cover). An open cover of a metric space X is a family of open subsets of X that covers X.

Definition 3.3 (Compactness). A metric space X is compact if every open cover has a finite subcover.

Definition 3.4 (Totally bounded). A metric space X is totally bounded if for each $\epsilon > 0$, there exists a finite number of open balls of radius ϵ that cover X.

Theorem 3.3. The following are equivalent for a metric space X:

1. X is compact.
2. Every sequence in X has a convergent subsequence.
3. X is totally bounded and complete.

Definition 3.5 (Bounded). A metric space X is bounded if there exists $b > 0$ such that $d(x, y) < b$ for all $x, y \in X$.

Lemma 3.1. A totally bounded metric space is bounded.

Lemma 3.2. Any subspace of a totally bounded metric space is totally bounded.

Lemma 3.3. A subset E of $\mathbb{R}^n$ is totally bounded iff E is bounded.

Theorem 3.4 (Heine-Borel theorem). The following are equivalent for a subspace E of $\mathbb{R}^n$.

1. E is compact.
2. Every sequence in E has a convergent subsequence.
3. E is closed and bounded.

Theorem 3.5. Let X be a totally bounded metric space. Then every sequence in X has a Cauchy subsequence.

Definition 3.6 (Seperability). A metric space X is seperable if there is a dense subset of X that is countable. In other words, X is seperable iff there is a sequence $\{x_j\}_{j=1}^{\infty}$ in X that is dense in X.

Theorem 3.6. A subspace of a separable metric space is separable.

Definition 3.7 (Base). A base of open sets for a metric space X is a family $\mathscr{B}$ of open subsets of X such that every open subset of X is the union of sets in $\mathscr{B}$.

Lemma 3.4. A family $\mathscr{B}$ of open subsets of a metric space X is a base of open sets iff for each $x \in X$ and each open neighborhood U of x, there exists $V \in \mathscr{B}$ such that $x \in V$ and $V \subseteq U$.

Definition 3.8 (Second-countable)**.** A metric is second-countable if there is a base of open sets that is at most countable.

Theorem 3.7. A metric space is second-countable iff it is seperable.

Theorem 3.8 (Lindelof's theorem)**.** Suppose the metric space X is second-countable. Then every open cover of X has a countable subcover.

Theorem 3.9. A compact metric space is seperable and second-countable.