"Calculus: Early Trancendentals" Notes

Samuel Lindskog

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Limits

Definition 1.1 (Average Rate of Change)**.** The average rate of change of $y = f(x)$ with respect to x over the interval (x_1, x_2) is:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

Theorem 1.1 (Limit Laws)**.** If L, M, c, k are real numbers and $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$, then:

Sum rule:	$\lim_{x\to c}(f(x) + g(x)) = L + M$
Difference Rule:	$\lim_{x\to c}(f(x) - g(x)) = L - M$
Constant Multiple Rule:	$\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
Product Rule:	$\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
Quotient Rule:	$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M},\ M \neq 0$
Power Rule:	$\lim_{x\to c}\left[f(x)\right]^n = L^n,\ n$ a positive integer
Root Rule:	$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L},$ n a positive integer

(For the root rule, if n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

Theorem 1.2 (Sandwich theorem)**.** Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some neighborhood of c. Suppose also that:

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = L$$

Then $\lim_{x\to c} f(x) = L$.

Definition 1.2 (Limit)**.** Let $f(x)$ be defined on an open neighborhood of c not necessarily including c. We say that the limit of $f(x)$ as x approaches c is the number L if for every number $\epsilon > 0$, there exits a corresponding number $\delta > 0$ such that:

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

Extreme Values

Definition 1.3 (Absolute maximum and minimum)**.** Let f be a function with domain D. Then f has an absolute maximum value on D at point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an absolute minimum value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Theorem 1.3 (Extreme value theorem)**.** If f is continuous on a finite closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$.

Definition 1.4 (Local maximum and minimum)**.** A function f has a local maximum value (or minimum) at a point c within its domain D if $f(x) \leq f(c)$ (or $f(x) \geq f(c)$) for all $x \in D$ lying in some open interval containing c.[1]

[1] Local extrema are also called relative extrema.

Remark. A set of all local maxima (minima) will automatically include the absolute maximum (minimum) if there is one.

Theorem 1.4 (First derivative theorem for local extreme values)**.** If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

Remark. The only places where a function f can possibly have an extreme value (local or global)are

1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f

Definition 1.5 (Critical point)**.** An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.