HW

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Definitions are partially or completely copied from "Analysis with an introduction to proof" by Steven Lay, or Tao. Propositions are original.

Definition 1.1 (bounded sequence)**.** Let S be a subset of $\mathbb{R}$. If there exists a $m \in \mathbb{R}$ such that $m \geq s$ for all $s \in S$, then m is an upper bound. If a set is bounded above and below, then the set is bounded.

Definition 1.2 (convergent sequence)**.** A sequence (s_n) is said to converge to the real number s provided that

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in N, \big(n \geq N \Rightarrow |s_n - s| < \epsilon\big).$$

Definition 1.3 (limit of a sequence)**.** If (s_n) is said to converges to $s \in \mathbb{R}$, then s is called the limit of the sequence.

Definition 1.4 (supremum)**.** Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then the least upper bound of S is called its supremum, and is denoted by $\sup S$. Thus $m = \sup S$ iff

(a) $\forall s \in S, m \geq s$;

(b) $m' < m \Rightarrow \exists s' \in S \wedge s' > m'$

Definition 1.5 (limsup)**.** Let S_n be a bounded sequence. A subsequential limit of (s_n) is any real number that is the limit of some subsequence of (s_n). If S is the set of all subsequential limits of s_n, then the limit superior of (s_n) is

$$\limsup s_n = \sup S.$$

Definition 1.6 (subsequence)**.** Let $(s_n)_{n=1}^{\infty}$ be a sequence and let $(n_k)_{k=1}^{\infty}$ be any sequence of natural numbers such that $n_1 < n_2 < n_3 < \ldots$. The sequence $(s_{n_k})_{k=1}^{\infty}$ is called a subsequence of $(s_n)_{n=1}^{\infty}$.

Definition 1.7. Let $x \geq 0$ be a non-negative real, and let $n \geq 1$ be a positive integer. We define $x^{1/n}$, also known as the nth rooth of x, by the formula

$$x^{1/n} := \sup\{y \in \mathbb{R} \,|\, y \geq 0 \wedge y^n \leq x\}.$$

Definition 1.8. Let $x > 0$ be a positive real number, and let q be a rational number. To define x^q, we write $q = a/b$ for some integer a and positive integer b, and define

$$x^q := (x^{1/b})^a.$$

Proposition 1.9. Let set $S \subseteq \mathbb{R}$ such that $\sup S$ ($\inf S$) exists and is equal to $L \in \mathbb{R}$. Then

$$\forall \epsilon > 0, \exists s \in S, \big(|L - s| < \epsilon\big).$$

Proof: Because $L = \sup S$, if $B = L - \epsilon$ for some $\epsilon > 0$, it follows from the definition of supremum that there exists $s \in S$ such that $s > B$. Because

$$L - B = L - (L - \epsilon) = \epsilon$$

and $B < s < L$, we have

$$0 < L - s < \epsilon$$

So $|L - s| < \epsilon$, as required. If $L = \inf S$ and $B = L + \epsilon$ for some $\epsilon > 0$, it follows from the definition of infimum that there exists $s \in S$ such that $s < B$ Because

$$L - B = L - (L + \epsilon) = \epsilon$$

and $L < s < B$, we have

$$-\epsilon < L - s < 0$$

So $|L - s| < \epsilon$, as required. □

Proposition 1.10. Let set $S \subseteq \mathbb{R}$ such that $\limsup S$ ($\liminf S$) exists and is equal to $L \in \mathbb{R}$. Then

$$\forall \epsilon > 0, \exists s \in S, \big(|L - s| < \epsilon\big).$$

Proof: If L is the limit of some subsequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$, then (a_{n_k}) converges to L, i.e.

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall k \in \mathbb{N}, \big(k > N \Rightarrow |L - a_{n_k}| < \epsilon\big).$$

In other words, for every subsequential limit in the set of subsequential limits S, there exists an element of the subsequence, and thus an element of the sequence, which is ϵ-close to this limit. It follows from proposition 1.9 that there exists $s \in S$ which is $\epsilon/2$-close to $\sup S$, and an element of (a_n) which is $\epsilon/2$ close to s, so s is ϵ close to $\sup S$. □

Exercise 6.4.4

Suppose that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are two sequences of real numbers such that $a_n \leq b_n$ for all $n \geq m$. Then we have the inequalities

$$\sup(a_n)_{n=m}^{\infty} \leq \sup(b_n)_{n=m}^{\infty} \tag{1}$$
$$\inf(a_n)_{n=m}^{\infty} \leq \inf(b_n)_{n=m}^{\infty} \tag{2}$$
$$\limsup(a_n)_{n=m}^{\infty} \leq \limsup(b_n)_{n=m}^{\infty} \tag{3}$$
$$\liminf(a_n)_{n=m}^{\infty} \leq \liminf(b_n)_{n=m}^{\infty} \tag{4}$$

Proof: We prove these statements by contradiction. Suppose to the contrary exclusively either

$$\begin{aligned}
&\sup(a_n)_{n=1}^{\infty} = L > \sup(b_n)_{n=1}^{\infty} = M;\\
&\inf(a_n)_{n=1}^{\infty} = L > \inf(b_n)_{n=1}^{\infty} = M;\\
&\limsup(a_n)_{n=1}^{\infty} = L > \limsup(b_n)_{n=1}^{\infty} = M;\\
&\liminf(a_n)_{n=1}^{\infty} = L > \liminf(b_n)_{n=1}^{\infty} = M;\\
&\limsup(a_n)_{n=1}^{\infty} = L > \liminf(b_n)_{n=1}^{\infty} = M. \quad \text{\# for exercise 6.4.5}
\end{aligned}$$

Because $L > M$, there exists $c \in \mathbb{R}^+$ such that $L = M + c$. It follows from proposition 1.9 or 1.10 that there exists $a \in (a_n)$ such that a is $c/2$-close to L, and there exists $b \in (b_n)$ such that b is $c/2$-close to M. Because $L - c/2 < a < L + c/2$, we have $M + c - c/2 < a < M + c + c/2$, so $a > M + c/2$. But $M - c/2 < b < M + c/2$ so $b < M + c/2$ and $b < a$, contradicting the fact that $a \leq b$. □

Exercise 6.4.5

Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ be sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all $n \geq m$. Suppose also that $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ both converge to the same limit L. Then $(b_n)_{n=m}^{\infty}$ is also convergent to L.

Proof: It follows from Exercise 6.4.4 that $\limsup(c_n) \leq \liminf(b_n)$ and $\limsup(b_n) \leq \liminf(a_n)$. It follows from Tao proposition 6.4.12 that $\liminf(a_n) = c = \limsup(c_n)$. Thus $\limsup(b_n) = \liminf(b_n) = c$, and by the same proposition (b_n) converges to c. $\square$

Exercise 6.5.3

For any $x > 0$, we have $\lim_{n\to\infty} x^{1/n} = 1$.
Proof: It follows from the definition of an nth root that $\lim_{n\to\infty} x^{1/n}$ is equivalent to

$$\lim_{n\to\infty} \sup\{y \in \mathbb{R} \,|\, y^n \leq x\}.$$

If $L(n) = \sup\{y \in \mathbb{R} \,|\, y^n \leq x\}$, and $L(N) < L(N+1)$, is nonzero, it follows from proposition 1.9 that for any $\epsilon > 0$ there exists $l \in L_n$ such that $|L - l| < \epsilon$. But then $\square$

Exercise 6.6.5

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers, and let L be a real number. Then the following two statements are logically equivalent:

(a) The sequence $(a_n)_{n=1}^{\infty}$ converves to L.

(b) Every subsequence of $(a_n)_{n=1}^{\infty}$ converges to L.