HW 9

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April 25, 2025

Problem 1

First, we prove that a function $f : \mathbb{R} \to \mathbb{R}$ defined by a real variable x taken to a nonnegative integer power is continuous. Let $c \in \mathbb{R}$. If $f = x^0$, then by definition $f = 1$. Thus $|f(x) - f(c)| = |1 - 1| = 0$, so

$$\forall \epsilon > 0,\, \exists \delta > 0,\, \forall x \in \mathbb{R},\, \big(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon\big)$$

Is always true. By inductive hypothesis, suppose x^n is continuous for some $n \in \mathbb{N}$. Then $\lim_{x\to c} x^n = c^n$ for all $c \in \mathbb{R}$. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x$ is continuous because $|x - c| = |f(x) - f(c)|$, thus for all $\epsilon > 0$, $|x - c| < \epsilon \Rightarrow |f(x) - f(c) < \epsilon|$. It follows from limit laws in the text and the continuity of both x^n and x that

$$\begin{aligned}
\lim_{x\to c} x^{n+1} &= \lim_{x\to c} x^n \cdot x \\
&= \lim_{x\to c} x^n \cdot \lim_{x\to c} x \\
&= c^n \cdot c \\
&= c^{n+1}.
\end{aligned}$$

This closes the induction. If follows from proposition that a continuous function multiplied by a constant is continuous. Thus

$$\sum_{i=0}^{n} c_i x^i$$

Is a finite linear combination of continuous functions, and is thus continuous by proposition.