HW2

Samuel Lindskog

February 6, 2025

Problem 1

Let a be a positive number. Then there exists exactly one natural number b such that $b++ = a$.

Proof: Suppose $a, b, c \in \mathbb{N}$, and suppose to the contrary that $b \neq c$ with $b++ = a$ and $c++ = a$. By the trichotomy of order for natural numbers (proposition 2.2.13), wlog $b > c$ and thus there exists $n \in \mathbb{N}^+$ such that $c + n = b$. From the definition of addition, $b++ = (c+n)++ = (c++) + n$ so $b++ > c++$, a contradiction. $\square$

Problem 2

Let $a, b, c, d \in \mathbb{N}$. If $a < b$ and $c < d$ then $a + c < b + d$.
Proof: Let $a, b, c, d \in \mathbb{N}$ with $a < b$ and $c < d$. Then there exists $x, y \in \mathbb{N}^+$ such that $a + x = b$ and $c+y = d$. Therefore $a+x+c+y = b+d$ and by commutivity and associativity of addition (propositions 2.2.4 and 2.2.5) $a + c + (x + y) = b + d$. By proposition 2.2.8, $x + y$ is positive so $a + c < b + d$. $\square$

Problem 3

Let n, m be natural numbers. Then $n \times m = 0$ iff at least one of n, m is equal to zero.
Proof: Let $n, m \in \mathbb{N}$ with $n = 0$. Then from the definition of multiplication, and commutivity of multiplication (lemma 2.3.2), $n \times m = m \times n = 0$. Redefine $n, m \in \mathbb{N}$ and let $n \times m = 0$. Suppose to the contrary that n and m are nonzero. Then from the definition of multiplication, $n \times m = m + m + \ldots + m$ with n m's added together. By proposition 2.2.8 this is a positive quantity, a contradiction. $\square$