Nonlinear Dynamics

Samuel Lindskog

April 4, 2025

Contents

1 Flows on the line

1.1 Introduction

Definition 1.1 (Fixed points)**.** A fixed point on a phase diagram is a point in which there is no flow, i.e. $x' = 0$. Fixed points represent equilibrium solutions, and are denoted with an asterisk x^*.

Definition 1.2 (Phase point)**.** A phase point is an imaginary particle placed at a point x_0 from which we can observe how it is carried along with the "flow". As time increases, the phase point moves along the x-axis according to some function $x(t)$. $x(t)$ is called the trajectory based at x_0.

Theorem 1.3. *Consider the IVP*

$$\begin{aligned} x' &= f(x), \\ x(0) &= x_0. \end{aligned}$$

If $f(x)$ and $f'(x)$ are continuous on an open interval R of the x-axis, and $x_0 \in R$, then the initial value problem has a unique solution on some time interval $-\tau, \tau$ about $t = 0$.

Remark. In a first-order system, trajectories can either approach a fixed point, or diverge to infinity. Trajectories are forced to increase or decrease monotonically because x' can not hold two values for the same x. This means that phase points never 'overshoot' a fixed point to which its path converges. Therefore there are no periodic solutions to $x' = f(x)$.

Definition 1.4 (Potentials)**.** In a first-order system $x' = f(x)$, the potential function $V(x)$ is defined by

$$f(x) = -\frac{dV}{dx}$$

Remark. Using the chain rule, we can see

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dx}\frac{dx}{dt} \\ &= -\left(\frac{dV}{dx}\right)^2 \\ &\leq 0 \end{aligned}$$

Therefore potential decreases or stays constant along trajectories.

Proposition 1.5 (Euler's method)**.** Suppose $x' = f(x)$ a one-dimensional dynamical system. Eulers method is a way of estimating $x(t)$ at discreet times spaced Δt apart. We define x_n to be the approximate value of $x(t)$ at $n\Delta t$ by choosing a starting point x_0, and using the following recursive definition to find any x_n:

$$x_{n+1} = x_n + f(x_n)\Delta t.$$

1.2 Bifurcations

Definition 1.6 (Bifurcation)**.** A bifurcation is a change in the qualitative structure of the flow caused by changing a parameter in an equation. The values at which bifurcations occur are called bifurcation points.

Remark. Bifurcation diagrams plot fixed points on the vertical axis against parameter values r on the horizontal axis. Dashed lines represent unstable fixed points, where solid lines represent stable ones.

Definition 1.7 (Saddle-node bifurcation)**.** This is a bifurcation presents as fixed points colliding and annihilating as a parameter is varied. An example of this is increasing parameter r in the equation $x' = r + x^2$. When $r < 0$ this equation has two zeros in the phase plane and thus two fixed points. When $r = 0$ $x(t)$ has one phase point and when $r > 0$ there are no phase points. Saddle-node bifurcation has normal form

$$x' = r + x^2 \text{ or } x' = r - x^2.$$

Definition 1.8 (Normal form)**.** The normal form of a bifurcation is the prototypical presentation of that bifurcation. For example, the partial taylor expansion of a function with a saddle-node bifurcation at $x = x^*$ and $r = r_c$ presents as the normal form of a saddle bifurcation

$$f(x^*, r_c) + (x - x^*)\frac{\partial f}{\partial x}(x^*, r_c) + (r - r_c)\frac{\partial f}{\partial r}(x^*, r_c) + \frac{1}{2}(x - x_0)^2\frac{\partial^2 f}{\partial x^2}(x^*, r_c).$$

Because $\frac{\partial f}{\partial x}(x^*, r_c) = 0$ and $f(x^*, r_c) = 0$ at the bifurcation point, this equation can then be written in normal form

$$(r - r_c)\frac{\partial f}{\partial r}(x^*, r_c) + \frac{1}{2}(x - x^*)^2\frac{\partial^2 f}{\partial x^2}(x^*, r_c).$$

Definition 1.9 (Transcritical bifurcation)**.** A fixed point that exists for all values of a parameter, but whoes stablility changes depending on that parameter, is said to undergo transcritical bifurcation. The normal form of a transcritical bifurcation is

$$x' = rx - x^2.$$

Example 1.10. Show that the first-order system $x' = x(1 - x^2) - a(1 - e^{-bx})$ undergoes a transcritical bifurcation at $x = 0$ when the parameters a, b satisfy a certain equation.

Proof: $x = 0$ is a fixed point for all (a,b) so it is plausible that the point bifurcates transcritically. Using the second degree taylor expansion for x', we can estimate the behavior of this function near $x = 0$:

$$x' = (1 - ba)x + \frac{1}{2}(b^2 a)x^2 + O(x^3).$$

From this we see that transcritical bifurcation must occur when $ba = 1$, i.e. when the first derivative of x' with respect to x is zero. Using this equation we can estimate the location of fixed points near $ab = 1$ by simplifying the above equation to solve for x^*:

$$x^* \approx \frac{2(ba - 1)}{b^2 a}.$$

□

Example 1.11. Describe the behavior of the laser equation $n' = Gn(N_0 - \alpha n) - kn$.

Proof: The change in the number of photons n' is given by the rate of photon generation GnN minus the outflow rate kn. This equation has a normal form $X' = RX - X^2$. The rate of photon generation is proportional to the number of excited atoms $N = (N_0 - \alpha n)$. We can simplify the above equation into two forms which tell us about the nature of the fixed points as N_0 changes.

$$\begin{aligned} n' &= -G\alpha n^2 + (GN_0 - k)n \\ n' &= -G\alpha n\left(n - \frac{GN_0 - k}{G\alpha}\right) \end{aligned}$$

Clearly transcritical bifurcation of fixed point $n = 0$ occurs at $GN_0 - k = 0$. This occurs with our intuitive understanding that if the rate of photon egress is larger than photon generation for small n, then no laser action will take place. However if $GN_0 > k$ then the fixed point at $n = 0$ becomes unstable and a stable fixed point emerges, as given by the second equation. □

Definition 1.12 (Supercritical pitchfork bifurcation)**.** Pitchfork bifurcations occur in problems which have symmetry, i.e. fixed points appear and disappear in symmetrical pairs. The normal form of supercritical pitchfork bifurcation is

$$x' = rx - x^3.$$

Definition 1.13 (Subcritical pitchfork bifurcation)**.** Subcritical pitchfork bifurcation presents as a mirror image about the y-axis of supercritical bifurcation, but with unstable symmetrical fixed points with a stable fixed point at zero, that converge to an unstable fixed point at $x = 0$. The normal form of a subcritical pitchfork bifurcation is

$$x' = rx + x^3.$$

2 Linear systems

2.1 Classification of linear systems

Definition 2.1 (Closed orbit)**.** A trajectory which returns to its starting point forms a closed orbit.

Definition 2.2 (Attracting point)**.** We say that x^* is an attracting fixed point if all trajectories that start near x^* apporach it as $t \to \infty$. If x^* attracts all trajectories in the phase plane, it is called globally attracting.

Definition 2.3 (Liapunov stable)**.** If all trajectories that start sufficiently close to a fixed point x^* remain close to x^* for all time, the fixed point is Liapunov stable.

Definition 2.4 (Neutrally stable)**.** When a fixed point is Liapunov stable but not attracting, it is called neutrally stable.

Definition 2.5 (Stable/unstable)**.** If a fixed point is both Liapunov stable and attracting, it is stable. If a fixed point is neither of these it is unstable.

Definition 2.6 (Stable and unstable manifold)**.** For a saddle point x^*, the stable manifold is the set of initial conditions x_0 such that $\lim_{t\to\infty} x(t) = x^*$. The unstable manifold of x^* is the set of initial conditions such that $\lim_{t\to-\infty} x(t) = x^*$.

Definition 2.7 (Saddle point)**.** If a system has real eigenvalues of opposite sign, it is a saddle point.

Definition 2.8 (Stable and unstable node)**.** If a system has all positive eigenvalues, it is an unstable node. If it has all negative eigenvalues, it is a stable node.

Definition 2.9 (Spiral and center)**.** If the eigenvalues are complex with no real part, the fixed point is a center. If the eigenvalues are complex with nonzero real part, the fixed point is a spiral.

Definition 2.10 (Star and degenerate node)**.** If the eigenvalues are equal with unequal eigenvectors, the node is a star node. If there is only one eigenvector, the fixed point is a degenerate node.

3 Phase plane

Definition 3.1 (Conservative system)**.** Given a system $x' = f(x)$, a conserved quantity is a real-valued continuous function $E(x)$ that is constant on trajectories, i.e. $dE/dt = 0$. A conservative system is one for which a conserved quantity exists and is nonconstant on every open set.

Definition 3.2 (Homoclinic orbit)**.** A homoclinic orbit is a trajectory which starts and ends at the origin.

Definition 3.3 (Reversable system)**.** A reversible system is any second-order system that is invariant under $t \to -t$ and $y \to -y$. For example, any system of the form

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y), \end{aligned}$$

Where f is odd in y and g is even in y is reversible. Reversable systems are not necessarily conservative.