Homework 2

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Problem 1

1. Show that f_n converges to 0 pointwisely

   Proof: For all x in $(0, 1)$, for all real $\epsilon > 0$ there exists an integer $N = \lceil \log_x \epsilon \rceil$ such that for all integers $n > N$ we have $d(x^n, 0) < \epsilon$. $\square$

2. Does f_n converge to 0 uniformly?

   Proof: No. Suppose to the contrary that $\{f_n\}$ converges uniformly. Then $\forall \epsilon > 0, \exists N \in \mathbb{Z}$ such that $n > N$ implies $\forall x \in X, d(f_n(x), 0) < \epsilon$. But there exists ϵ in $(0, 1)$ such that $\forall N \in \mathbb{Z}$, there exists $x_n = \epsilon^{1/(n+1)}$ with $n > 0, N$ such that $(x_n)^n > \epsilon$. $\square$

Problem 2

Let Y be a subspace of X and let S be a subset of Y. show that the closure of S in Y coincides with $\overline{S} \cap Y$ where $\overline{S}$ is the closure of S in X.

Proof: Suppose y is in the closure of S in Y. Then for all $r > 0$ there exists $B(y, r) \cap S \neq 0$. Then a sequence $\{y_n\}_{n=1}^{\infty}$ exists with $y_n \in Y$ such that $\forall r > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies $d(y_n, y) < r$. Therefore $\{y_n\}$ is a sequence in Y which converges to y and following theorem 1.11, because $y \in X$ this implies $y \in \overline{S} \wedge y \in Y$, which is logically equivalent to $y \in \overline{S} \cap Y$. $\square$

Problem 3

A sequence $\{x_k\}_{k=1}^{\infty}$ in a metric space (X, d) is a fast cauchy sequence if

$$\sum_{k=1}^{\infty} d(x_k, x_{k+1}) < \infty.$$

Show that a fast Cauchy sequence is a Cauchy sequence.

Proof: If $\{x_k\}_{k=1}^{\infty}$ a fast sequence then the sum of all x_k is a finite real number. Then there exists $a \in \mathbb{R}$ with $a > 0$, and $N \in \mathbb{N}$ such that

$$a - \lim_{n\to\infty} \sum_{k=1}^{n} d(x_k, x_{k+1}) = 0 \tag{1}$$

$$a - \sum_{k=1}^{N} d(x_k, x_{k+1}) = \lim_{n\to\infty} \sum_{N+1}^{n} d(x_k, x_{k+1}). \tag{2}$$

It follows from the definition of a metric that for $l, m \in \mathbb{N}$ with $l, m > N$

$$a - \sum_{k=1}^{N} d(x_k, x_{k+1}) \geq d(x_l, x_m). \tag{3}$$

Following equation one, we can establish that for all $\epsilon > 0$, with $\epsilon > a - \sum_{k=1}^{N} d(x_k, x_{k+1})$, there exists N such that $l, m > N$ implies $d(x_l, x_m) < \epsilon$, and thus $\{x_k\}$ is Cauchy. $\square$