Calculus: Early Transcendentals Notes

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July 24, 2025

Contents

Chapter 1

Algebra

1.1 Fundamentals

1.1.1 Real numbers and absolute value

Remark. TODO: binomial theorem, cosh sinh, complete the square, difference of squares, partial fractions, 2.3 epsilon delta proofs, sunthetic division/polynomial long division, conjugate multiplication

Chapter 2

Calculus

2.1 Chapter 1: Functions

2.1.1 Functions and their graphs

Definition 2.1.1 (Function)**.** A function $f : D \to Y$ is a rule that assigns a unique element $f(x) \in Y$ to each element $x \in D$. The set D is called the domain, and the set $f(D) \subseteq Y$ is called the range.

Proposition 2.1.2 (Vertical line test)**.** A funciton f can have only one value $f(x)$ for each x in it's domain.

Definition 2.1.3 (Increasing function)**.** Let f be a function defined on an interval I, and let x_1 and x_2 be any two points in I.

(a) If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is said to be increasing on I.

(b) If $f(x_2) < f(x_1)$ whenvever $x_2 < x_1$, then f is said to be decreasing on I.

Definition 2.1.4 (Polynomial)**.** A function p is a polynomial if it is of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

Where n is a nonnegative coefficient. The greatest n value for which a_n is nonzero is called the degree of the polynomial. A polynomial with one term is called a monomial

Definition 2.1.5 (Even and odd functions)**.** A function $y = f(x)$ is even if

$$f(-x) = f(x),$$

and is odd if

$$f(-x) = -f(x)$$

for all x in the domain of f. The graph of an even function is symmetric about the y-axis, and an odd function is symmetric about the origin.

Remark. The names even and odd come from powers of x. Even-degree monomials of x are even, and odd-degree monomials of x are odd.

Definition 2.1.6 (Linear function)**.** A linear function is a function of the form

$$f(x) = mx + b$$

for constants m, b. In other words, a linear function is a polynomial of degree 1.

Definition 2.1.7 (Quadratic)**.** A quadratic function is a polynomial of degree 2. A cubic function is a function of degree 3.

Definition 2.1.8 (Power function)**.** A power function a function of the form

$$f(x) = x^a$$

for some constant a.

Definition 2.1.9 (Rational function)**.** A rational function is a quotient or ratio $f(x) = p(x)/q(x)$, where p, q are polynomials and $q(x) \neq 0$.

Definition 2.1.10 (Algebraic function)**.** Any function constructed from polynomials p_k using algebraic operations $+, -, \times, /, (p_k)^a$ is an algebraic function.

Definition 2.1.11 (Exponential function)**.** Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant $a \neq 1$, are called exponential functions.

Definition 2.1.12 (Logarithic function)**.** There are the functions $f(x) = \log_a x$, where $a \neq 1$ is a positive constant.

Definition 2.1.13 (Transcendental functions)**.** These are functions that are not algebraic. They include trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many more.

2.1.2 Combining Functions; Shifting and Scaling Graphs

Definition 2.1.14 (Operations on functions)**.** Arithmetic operations on functions are defined as follows:

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x). \\
(f-g)(x) &= f(x) - g(x). \\
(fg)(x) &= f(x)g(x). \\
(f/g)(x) &= f(x)/g(x) \quad \text{where } g(x) \neq 0.
\end{aligned}$$

Definition 2.1.15 (Composite function)**.** If f and g are functions, the composite function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)).$$

Proposition 2.1.16. To shift a function f vertically we can add a constant k, e.g.

$$f(x) + k.$$

To shift a function horizontally, we can add a constant to the functions input, e.g.

$$f(x + k).$$

2.2 Chapter 2: Limits and continuity

2.2.1 Rates of change and tangent lines to curves

Definition 2.2.1 (Average rate of change)**.** The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$

Definition 2.2.2 (Secant line)**.** A line joining two points of a curve is called a secant line.

Remark. The average rate of change of f from x_1 to x_2 is the slope of the secant line between these points.

Definition 2.2.3 (Tangent line)**.** The qualitative definition of a tangent line is the line that grazes the curve at some point on the graph.

Definition 2.2.4 (Instantaneous rate of change)**.** The instantaneous rate of change of a graph at a point p is the slope of the tangent line at point p.

2.2.2 Limit of a function and limit laws

Definition 2.2.5 (Limit)**.** Let $f(x)$ be defined on an open interval about c. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x \to c} f(x) = L,$$

if for every number $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Definition 2.2.6 (Limit laws)**.** If L, M, c, k are real numbers and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

Sum Rule:	$\lim_{x \to c}(f(x) + g(x)) = L + M$
Difference Rule:	$\lim_{x \to c}(f(x) - g(x)) = L - M$
Constant Multiple Rule:	$\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
Product Rule:	$\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
Quotient Rule:	$\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
Power Rule:	$\lim_{x \to c}[f(x)]^n = L^n, \quad n \in \mathbb{Q}^+$

If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.

Theorem 2.2.7. *If $P(x)$ is some polynomial $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, then*

$$\lim_{x \to c} P(x) = P(c).$$

Theorem 2.2.8. *If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then*

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Theorem 2.2.9 (Sandwich theorem)**.** *Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that*

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Theorem 2.2.10. *If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and if the limits of f and g both exist as x approaches c, then*

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

2.2.3 One-sided limits

Definition 2.2.11 (Right limit)**.** Assume the domain of f contains an interval (c, d) to the right of c. We say that $f(x)$ has a right-handed limit L at c and write

$$\lim_{x \to c^+} f(x) = L$$

if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$c < x < c + \delta |f(x - L)| < \epsilon.$$

2.2.4 Continuity

Definition 2.2.12 (Continuity)**.** Let c be a real number that is in the interval of the domain of a function f. f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

f is right-continuous at c if

$$\lim_{x \to c^+} f(x) = f(c.)$$

f is left-continuous at c if

$$\lim_{x \to c^-} f(x) = f(c).$$

Remark. If a function is not continuous at a point c of its domain, we say that f is discontinuous at c, and that f has a discontinuity at c.

Proposition 2.2.13 (Continuity test)**.** A function $f(x)$ is continuous at a point $x = c$ iff it meets the following three conditions:

(a) $f(c)$ exists.

(b) $\lim_{x\to c} f(x)$ exists.

(c) $\lim_{x\to c} f(x) = f(c)$.

Definition 2.2.14 (Continuous function)**.** A function is continous if it is continous at every point in its domain.

Theorem 2.2.15. *If the function f and g are continous at $x = c$, then the following algebraic combination are continous at $x = c$.*

$$
\begin{aligned}
&f + g \\
&f - g \\
&k \cdot f,\ k \in \mathbb{R} \\
&f \cdot g \\
&f/g,\ g(c) \neq 0 \\
&f^n,\ n \in \mathbb{N}^+ \\
&f^{1/n},\ \ \textit{if defined on an interval containing } c.
\end{aligned}
$$

Theorem 2.2.16. *If $\lim_{x\to c} f(x) = b$ and g is continous at the point b, then*

$$\lim_{x\to c} g(f(x)) = g(b).$$

Theorem 2.2.17 (Intermediate value theorem for continuous functions)**.** *If f is a continous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c \in [a, b]$.*

2.2.5 Limits involving infinity

Definition 2.2.18 (Limits approaching infinity)**.** We say that $f(x)$ has the limit L as x approaches infinity and write

$$\lim_{x\to\infty} f(x) = L$$

if, for all $\epsilon > 0$ there exists M such that for all x in the domain of f

$$x > M \Rightarrow |f(x) - L| < \epsilon$$

Theorem 2.2.19. *Theorem 2.2.6 applies to limits that appraoch infinity.*

Definition 2.2.20 (Horizontal asymptote)**.** A line $y = b$ is a horizontal asymptote of a graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b.$$

Definition 2.2.21 (Infinite limits)**.** We say that $f(x)$ approaches infinity as x approaches c and write

$$\lim_{x\to c} = \infty,$$

if for every number $B > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow f(x) > B.$$

Definition 2.2.22. A line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty.$$

2.3 Chapter 3: Derivatives

2.3.1 Tangent lines and the derivative

Definition 2.3.1 (Tangent line)**.** The slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h},$$

provided the limit exists. The tangent line to the curve at P is the line through P with this slope.

Definition 2.3.2 (Derivative)**.** The derivative of a function f at a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h},$$

provided this limit exists.

Definition 2.3.3 (Differentiability)**.** A function $y = f(x)$ is differentiable on an open interval if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the right/left hand limits of the derivative exist on the left/right side of the interval respectively.

Theorem 2.3.4. *If f has a derivative at $x = c$, then f is continuous at $x = c$.*

2.4 Chapter 10: Parametric equations

2.4.1 Parametrizations of plane curves

Definition 2.4.1 (Parametric curve)**.** If x and y are given as functions of t

$$x = f(t), \quad y = g(t),$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ is a parametric curve. These equations are called parametric equations for the curve.

Remark. The variable t is a parameter for the curve, and its domain I is the parameter interval. When we give parametric equations for a curve, we say that we have parameterized the curve. The equations and interval together constitute a parametrization of the curve.

2.4.2 Calculus with parametric curves

Remark. A parametrized curve $x = f(t)$ and $y = g(t)$ is differentiable at t if f and g are differentiable at t.

2.5 Chapter 12: Vector-valued functions

2.5.1 Derivatives of vector functions

Definition 2.5.1 (Limit)**.** Let $r : \mathbb{R} \to \mathbb{R}^n$ be a function. We say that r has limit L as t approaches t_0, denoted $\lim_{t\to t_0} r(t) = L$, If

$$\forall \epsilon > 0, \exists \delta > 0, \big(|t - t_0| < \delta \Rightarrow |r(t) - L| < \epsilon\big).$$

Definition 2.5.2 (Continuity)**.** A vector-valued function $r(t)$ is continous at a point $t = t_0$ in its domain if $\lim_{t\to t_0} r(t) = r(t_0)$. The function is continuous if it is continuous at every point in it's domain.

Definition 2.5.3. The vector function $r(t) = f(t)i + g(t)j + h(t)k$ has a derivative at t if f, g, h have derivatives at t. The derivative is the vector function

$$r'(t) = \frac{\mathrm{d}f}{\mathrm{d}t}i + \frac{\mathrm{d}g}{\mathrm{d}t}j + \frac{\mathrm{d}h}{\mathrm{d}t}k.$$

Definition 2.5.4 (Velocity, acceleration)**.** If r is the position vector, the $\frac{\mathrm{d}r}{\mathrm{d}t}$ is the velocity vector and $\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}$ is the acceleration vector.

Definition 2.5.5 (Indefinite integral)**.** The indefinite integral of r with respect to t is the set of all antiderivatives of r. If R is any antiderivate of r, then

$$\int r(t)dt = R(t) + C.$$

Definition 2.5.6 (Definite integral)**.** If the components of $r(t) = f(t)i + g(t)j + h(t)k$ are integrable over $[a, b]$, then so is r, and the definite integral of r from a to b is

$$\int_a^b r(t)dt = \left(\int_a^b f(t)dt\right)i + \left(\int_a^b g(t)dt\right)j + \left(\int_a^b h(t)dt\right)k.$$

2.5.2 Arc length in space

Definition 2.5.7 (Length)**.** The length of a smooth curve $r(t) = x(t)i + y(t)j + z(t)k$ for $a \le t \le b$ as t increases on this interval is

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, dt.$$

In other words, if v is the first derivative of r,

$$L = \int_a^b |v| dt.$$

2.5.3 Velocity and acceleration in polar coordinates

2.6 Chapter 13: Partial derivatives

2.6.1 Gradient vectors

Definition 2.6.1 (Directional derivative)**.** The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $u = u_1 i + u_2 j$ is the number

$$\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)_{u,P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s},$$

provided the limit exists.

Definition 2.6.2 (Gradient vector)**.** The gradient vector of $f(x, y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j.$$

Theorem 2.6.3. *If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then*

$$\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)_{u,P_0} = \nabla f|_{P_0} \cdot u.$$

Proposition 2.6.4. The derivative along a path in the domain of f, parameterized by r is

$$\frac{\mathrm{d}}{\mathrm{d}t} f(r(t)) = \nabla f(r(t)) \cdot r'(t).$$

2.7 Chapter 14: Multiple integrals

2.7.1 Double integrals over rectangles

Theorem 2.7.1 (Fubini's theorem)**.** *Let $f(x, y)$ be continuous on a region R.*

(a) *If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then*

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

Proposition 2.7.2. The area of a closed, bounded plane region R is

$$A = \iint_R dA.$$

2.8 Chapter 15: Integrals Vector fields

2.8.1 Line integrals of scalar functions

Definition 2.8.1 (Line integral)**.** If f is defined on a curve C given parametrically by $r(t) = g(t)i + h(t)j + k(t)k$ on $a \le t \le b$, then the line integral of f over C is

$$\int_C f(x,y,z)ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k,$$

and can be evaluated as

$$\int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),k(t))|v(t)|dt.$$

Proposition 2.8.2. If a piecewise smooth curve C is made by joining a finite number of smooth curves $C_1, \dots, C_n$ end to end, then the integral of a function over C is the sum of the integrals over the curves $C_1, \dots, C_n$.

2.8.2 Vector fields and line integrals

Definition 2.8.3 (Vector field)**.** A vector field is a function that assigns a vector to each point in it's domain.

Proposition 2.8.4. A vector field is continuous if it's component functions are continuous. It is differentiable if each of it's component functions are differentiable.

Definition 2.8.5 (Line integral)**.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field with continuous components defined along a smooth curve C parametrized by $r : \mathbb{R} \to \mathbb{R}^n$ on $a \le t \le b$. If T is the unit vector tangent to r at s, then the line integral of F along C is

$$\int_C F \cdot T ds = \int_C \left(F \cdot \frac{\mathrm{dr}}{\mathrm{ds}}\right) ds = \int_C F \cdot dr.$$

Remark. Note that the line integral of a vector field is different from the line integral in definition 2.8.1. In said definition, we are finding the area between the curve and the function. In the case of definition 2.8.5, we are evaluating the definite integral of a function whose gradient is the output of f, dotted with the derivative of the parametrized curve, dr/dt. In other words, we are taking the integral of the directional derivative of f at $r(t)$. It should be noted that F must be a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ because the dimension of the domain and range of the gradient must be equal.

Theorem 2.8.6. *Let $F = Mi + Nj + Pk$ be a vector field whose components are continuous througout an open connected region D in space. Then F is conservative if and only if F is a gradient field ∇f for a differentiable function f.*

Definition 2.8.7 (Curl)**.** Let $F = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k$ be a vector field. Then $\operatorname{curl} F$, called the curl of F, is

$$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)j + (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)k$$

For a two dimensional vector field $F = M(x,y) + N(x,y)$, $\operatorname{curl} F$ is

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)k.$$

Remark. Notice the first term describes counterclockwise rotation in the y-z plane relative to the normal i, the second describes counterclockwise rotation in the z-x plane relative to the normal j, and the third describes counterclockwise rotation in the x-y plane relative to the normal k.

Theorem 2.8.8. *Let F be a filed on an open simply connected domain whose component functions have continuous first partial derivatives. Then F is conservative iff $curl F = 0$.*

Definition 2.8.9 (Differential form)**.** come back?

2.8.3 Green's theorem in the plane

Definition 2.8.10 (Circulation density)**.** The circulation density of a vector field $F = Mi + Nj$ at the point (x, y) is the magnitude of the k-component of the curl of F:

$$\operatorname{curl} F \cdot k = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y},$$

Definition 2.8.11 (Divergence)**.** The divergence, or flux density, of a vector field $F = Mi + Nj$ at the point (x, y) is

$$\operatorname{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

Theorem 2.8.12 (Green's theorem)**.** *Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $F = Mi + Nj$ be a vector field with M and N having continuous first derivatives in an open region containing R. Then the counterclockwise circulation of F around C is*

$$\oint_C F \cdot T\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

Additionally, the outward flux of F across C is

$$\oint_C F \cdot n\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy.$$

2.8.4 Path independence

Definition 2.8.13 (Path independence)**.** Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent in D and the field $\mathbf{F}$ is conservative in D.

Theorem 2.8.14 (Fundamental theorem of line integrals)**.** *Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\boldsymbol{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\boldsymbol{F} = \nabla f$ on a domain D containing C. Then*

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(B) - f(A).$$

Corollary 2.8.15. If F is the gradient function of some differentiable function f, and C a closed curve, then

$$\oint_C F \cdot dr = 0.$$

Theorem 2.8.16. *A vector field F is conservative iff F is a gradient field ∇f for a differentiable function f.*