Homework 3

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Problem 1

False. Suppose $A \subset \mathbb{Z}$ a proper subset of $\mathbb{Z}$, let $\mathbb{Z}_{even}$ be the set of all even integers, and $\mathbb{Z}_{odd}$ be the set of all odd integers. Let $f : \mathbb{Z} \to \mathbb{Z}$ with $f(\mathbb{Z}_{even}) = A$ and $f(\mathbb{Z}_{odd}) = A^c$, and let $g : \mathbb{Z} \to \mathbb{Z}$ with $g(\mathbb{Z}_{odd}) = A$ and $g(\mathbb{Z}_{even}) = A^c$. Iff $x \in \mathbb{Z}$, then $x \in A \cup A^c$, so $f^{-1}(\{x\}) \cap \mathbb{Z} \neq \emptyset$ and $g^{-1}(\{x\}) \cap \mathbb{Z} \neq \emptyset$, so f and g are both onto. Let $h : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$h(k) = \begin{cases} f(k) & \text{if } k \text{ is even} \\ g(k) & \text{if } k \text{ is odd} \end{cases}$$

Then if k is even $h(k) = f(k) \in A$, and if k is odd $h(k) = g(k) \in A$, so $h(\mathbb{Z}) \subseteq A$. But then $\mathbb{Z} \setminus h(\mathbb{Z}) \neq \emptyset$, so h is not onto.

Problem 2

Let $S = [0, \infty)$, $T = \mathbb{R}$, and $f : S \to T$ defined by

$$f(x) = \begin{cases} x - \lfloor \frac{x}{2} \rfloor & \text{if } \lceil x \rceil \text{ is odd} \\ -x + \lceil \frac{x}{2} \rceil & \text{if } \lceil x \rceil \text{ is even} \end{cases}$$

Problem 3

Assuming $0 \notin \mathbb{N}$, let $S = \mathbb{N} \times \{0, 1\}$, $T = \mathbb{Z}$, and $(x, y) \in S$. Let $f : S \to T$ be defined by

$$f\big((x, y)\big) = \begin{cases} x & \text{if } y = 0 \\ -x + 1 & \text{if } y = 1 \end{cases}$$

Problem 4

(a) By definition, $f^{-1}\big(\{f(a)\}\big) = \big\{x \in A | f(x) \in \{f(a)\}\big\}$, and because $f(a) \in \{f(a)\}$, $a \in f^{-1}\big(\{f(a)\}\big)$.

(b) If $c \neq a$ and $c \in f^{-1}\big(\{f(a)\}\big)$, then as described above, $f(c) \in \{f(a)\}$. Because $f(a)$ is the only element of $\{f(a)\}$, this implies $f(c) = f(a)$, and because $c \neq a$, f is not one-to-one.

Problem 5

Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions, where f is onto and $g \circ f$ is one-to-one. This implies f is one-to-one, because if $x, y \in \mathbb{R}$ with $x \neq y$ and $f(x) = f(y)$, then $g\big(f(x)\big) = g\big(f(y)\big)$, a contradiction. Suppose to the contrary that g is not one-to-one. Then there exists $a, b \in \mathbb{R}$, with $a \neq b$ such that $g(a) = g(b)$. Because f is onto and one-to-one, there exists $a', b' \in \mathbb{R}$ with $a' \neq b'$ such that $f(a') = a$ and $f(b') = b$. But then $g\big(f(a')\big) = g\big(f(b')\big)$, a contradiction.