

Real Analysis

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1 The natural numbers

1.1 Peano axioms

Definition 1.1 (Peano axioms). Using $++$ as the successor operation, the natural numbers are defined as follows:

- (a) 0 is a natural number.
- (b) If n is a natural number, then $n++$ is also a natural number.
- (c) For all natural numbers n , $n++ \neq 0$.

Definition 1.2 (Addition of natural numbers). Let m be a natural number. $0 + m := m$ and $(n++) + m := (n + m)++$.

Proposition 1.3. There is only one zero, i.e. for $a \in \mathbb{N}$ if $0 + a = 0' + a = a$, then $0 = 0'$.

Proof: Suppose $0 \neq 0'$. Then 0 is a successor of $0'$ or $0'$ is a successor of 0. Because no successor of a natural number is 0, this is impossible. \square

Proposition 1.4. $m + 0 = m$.

Proof: Let $n \in \mathbb{N}$. $0 + 0 := 0$, so by inductive hypothesis $n + 0 = n$. $(n++) + 0 := (n + 0)++$, and from the inductive hypothesis equals $n++$. \square

Lemma 1.5. For any natural numbers n and m , $n + (m++) = (n + m)++$.

Proof: Suppose $n, m \in \mathbb{N}$. $0 + (m++) := m++ = (0 + m)++$. By inductive hypothesis $n + (m++) = (n + m)++$. From the definition of addition $(n++) + (m++) = (n + (m++))++$ and from the inductive hypothesis $n + (m++) = (n + m)++$ so we have

$$\begin{aligned}(n++) + (m++) &= (n + (m++))++ \\ &= ((n + m)++)++ \\ &= ((n++) + m)++\end{aligned}$$

\square

Proposition 1.6 (Commutativity of addition). For $n, m \in \mathbb{N}$, $n + m = m + n$.

Proof: Let $n, m \in \mathbb{N}$. From proposition 1.4, $0 + m = m + 0$, so by inductive hypothesis $n + m = m + n$. $(n++) + m = (n + m)++$ and from inductive hypothesis this equals $(m + n)++$. From lemma 1.5, this equals $m + (n++)$. \square

Proposition 1.7. If $a, b \in \mathbb{N}$ and $a + b = a$, then $b = 0$.

Proof: Suppose $a, b \in \mathbb{N}$ with $a + b = a$. \square

Proposition 1.8 (Associativity of addition). Let $a, b, c \in \mathbb{N}$. Then $(a + b) + c = a + (b + c)$.

Proof: Suppose $a, b \in \mathbb{N}$. From here we utilize the definition of addition, and commutivity of addition for the rest of the proof. It follows that $(a + b) + 0 = a + b = a + (b + 0)$. By inductive hypothesis suppose $(a + b) + c = a + (b + c)$ for $c \in \mathbb{N}$. Then

$$\begin{aligned}(a + b) + c++ &= [(a + b) + c]++ \\ &= [a + (b + c)]++ \\ &= a + (c + b)++ \\ &= a + [(c++) + b] \\ &= a + (b + c++)\end{aligned}$$

\square

Proposition 1.9 (Cancellation law). Let $a, b, c \in \mathbb{N}$. If $a + b = a + c$, then $b = c$.

Proof: If $0 + b = 0 + c$ then from the definition of addition $b = c$. By inductive hypothesis for any $n \in \mathbb{N}$, $n + b = n + c$. $(n++) + b = (n + b)++$ and $(n++) + c = (n + c)++$, so from the inductive hypothesis and the axioms of natural numbers, $(n++) + b = (n++) + c$. \square

Definition 1.10 (Positive natural number). A natural number n is said to be positive iff it is not 0.

Definition 1.11 (Ordering of natural numbers). Let $n, m \in \mathbb{N}$. We write $n \geq m$ or $m \leq n$ iff $n = m + a$ for some $a \in \mathbb{N}$.

Proposition 1.12. If a or b are not zero, then $a + b \neq 0$.

Proof: Suppose $a, b \in \mathbb{N}$ with $b \neq 0$. If $a = 0$ then $a + b = 0 + b = b \neq 0$. If $a \neq 0$, because no natural number has zero as a successor it follows from the definition of addition that $a + b \neq 0$. \square

Proposition 1.13 (Trichotomy of order for natural numbers). Let $a, b \in \mathbb{N}$. Then exactly one of the following statements is true: $a < b$, $a = b$, $a > b$.

Proof: Suppose $a, b \in \mathbb{N}$ and $a < b$. Then for some $c \in \mathbb{N}$, $a = b + c$ with $b \neq a$. If $c = 0$ then $a = b$, a contradiction. If $b < a$, then for some $d \in \mathbb{N}$, $b = a + d$ with $a \neq b$. If $d = 0$ then $a = b$, a contradiction. Because $b = b + d + c$ and $c, d \neq 0$, it follows from commutivity and propositions 1.12 and 1.3 that this is impossible. Therefore wlog if $a < b$ then a is not greater than or equal to b . Suppose $a = b$. If $a < b$ then $a = b + c$ for some $c \in \mathbb{N}$ with $b \neq c$, a contradiction. Therefore wlog if $a = b$ then a is not less than or greater than b . \square

Proposition 1.14 (Strong principle of induction). Let $m_0, m, m' \in \mathbb{N}$, and let $P(x)$ be a property of arbitrary $x \in \mathbb{N}$. Suppose that for each $m \geq m_0$ the following implication holds:

$$\left(\forall m' \in [m_0, m), P(m') \right) \Rightarrow P(m).$$

Then we can conclude $P(m)$ is true for all natural numbers $m \geq m_0$.

1.2 Multiplication

Definition 1.15 (Multiplication of natural numbers). Let m be a natural number. $0 \times m := 0$ and $(n++) \times m := (n \times m) + m$.

Proposition 1.16. $m \times 0 = 0$.

Proof: From the definition of multiplication, $0 \times 0 = 0$. By inductive hypothesis suppose $m \times 0 = 0$. Then $(m++) \times 0 = (m \times 0) + 0 = 0$. \square

Proposition 1.17. For $n, m \in \mathbb{N}$, $n \times (m++) = (n \times m) + n$.

Proof: Let $n, m \in \mathbb{N}$. $0 \times (m++) = 0 = (0 \times m) + 0$. By inductive hypothesis, $(n \times (m++)) = (n \times m) + n$. It follows that

$$\begin{aligned} (n++) \times (m++) &= (n \times (m++)) + (m++) \\ &= (n \times m) + n + (m++) \\ &= (n \times m) + m + (n++) \\ &= ((n++) \times m) + (n++) \end{aligned}$$

\square

Proposition 1.18. For $m \in \mathbb{N}$, $1m = m$.

Proof: If $m \in \mathbb{N}$ $0 \times m = 0$. Then $(0++) \times m = 1 \times m = 0 + m = m$. \square

Lemma 1.19 (Commutivity of multiplication). Let $n, m \in \mathbb{N}$. Then $n \times m = m \times n$.

Proof: Let $n, m \in \mathbb{N}$. $0 \times m = m \times 0 = 0$. By inductive hypothesis, $n \times m = m \times n$. It follows from proposition 1.17 that

$$\begin{aligned} (n++) \times m &= (n \times m) + m \\ &= (m \times n) + m \\ &= m \times (n++) \end{aligned}$$

\square

Proposition 1.20 (Distributive law). For any natural numbers a, b, c , we have $a(b + c) = ab + ac$.

Proof: TODO □

Proposition 1.21 (Associativity of multiplication). If $a, b, c \in \mathbb{N}$ then $(a \times b) \times c = a \times (b \times c)$.

Proof: TODO □

Proposition 1.22. If $a, b \in \mathbb{N}^+$, then $ab \neq 0$.

Proof: Let $a \in \mathbb{N}^+$. By proposition 1.18 $1a = a$ and a is positive. By inductive hypothesis if $n \in \mathbb{N}^+$ then na is positive. $n++$ is a successor to n , and no successor of a natural number is zero, so $n++$ is positive. $(n++)a = na + a$. Both na and a are positive and by proposition 1.12, $na + a$ is positive and thus not zero. □

Proposition 1.23. If a, b are natural numbers such that $a < b$, and c is positive, then $ac < bc$.

Corollary 1.24. Let $a, b, c \in \mathbb{N}$ such that $ac = bc$ and c is non-zero. Then $a = b$.

Proposition 1.25 (Euclid's division lemma). Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.

Definition 1.26 (Exponentiation for natural numbers). Let $m \in \mathbb{N}$. $m^0 := 1$, and $m^{n++} = m^n \times m$.

2 Set theory

2.1 Fundamentals

Definition 2.1 (Axioms of sets).

- (a) (Sets are objects) If A is a set, then A is also an object. In particular, given two sets A and B , it is meaningful to ask whether A is also an element of B .
- (b) (Equality of sets) Two sets A and B are equal iff every element of A is an element of B and vice versa.
- (c) (Empty set) There exists a set known as the empty set, denoted \emptyset , which contains no elements. In other words, for all objects x we have $x \notin \emptyset$.
- (d) (Singleton sets) If a is an object, then there exists a set $\{a\}$ whose only element is a , i.e. for every object y we have $y \in \{a\}$ iff $y = a$. $\{a\}$ is referred to as a singleton set.
- (e) (Pairwise union) Given any two sets A and B , there exists a set $A \cup B$, called the union of A and B , which consists of all the elements which belong to A or B . In other words,

$$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B).$$

- (f) (Axiom of specification) Let A be a set, and for each $x \in A$ let $P(x)$ be a property pertaining to x . Then there exists a set $\{x \in A \mid P(x)\}$ whose elements are precisely the elements x in A for which $P(x)$ is true.
- (g) (Replacement) Let A be a set. For any object $x \in A$ and any object y , suppose we have a property $P(x, y)$ that is true for at most one y for each $x \in A$. Then

$$z \in \{y \mid P(x, y), x \in A\} \Leftrightarrow P(x, z).$$

- (h) (Infinity) There exists a set \mathbb{N} , whose elements are called natural numbers, as well as an object $0 \in \mathbb{N}$, and an object $N++$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms hold.
- (i) (Universal specification) DANGER - Suppose for every object x we have a property $P(x)$. Then there exists a set $\{x \mid P(x)\}$.
- (j) (Regularity). If A is a non-empty set, then there is at least one element x of A which is either not a set, or is disjoint from A .

- (k) (Power set) Let X and Y be sets. Then there exists a set, denoted Y^X , which consists of all the functions from X to Y , thus

$$f \in Y^X \Leftrightarrow f \text{ is a function from } X \text{ to } Y.$$

- (l) (Union) Let A be a set whose elements are all sets. Then there exists a set $\bigcup A$ defined

$$x \in \bigcup A = \{x \mid \exists S \in A, x \in S\}.$$

Remark. The axioms of set theory introduced, excluding universal specification, are known as the Zermelo-Fraenkel axioms of set theory.

Lemma 2.2 (Single choice). Let A be a non-empty set. Then there exists an object x such that $x \in A$.

Proof: Suppose there does not exist any object x such that $x \in A$. Simultaneously $x \notin \emptyset$, so $x \in A \Leftrightarrow x \in \emptyset$ and $A = \emptyset$, a contradiction. \square

Definition 2.3 (Subset). Let A, B be sets. We say that A is a subset of B , denoted $A \subseteq B$, iff every element of A is also an element of B . We say that A is a proper subset of B , denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Theorem 2.4. Let A be a set. Then $\emptyset \subseteq A$.

Proof: If $\emptyset \subseteq A$ then for all objects x ,

$$x \in \emptyset \Rightarrow x \in A.$$

This is vacuously true because there does not exist x such that $x \in \emptyset$. \square

Definition 2.5 (Intersection). The intersection $S_1 \cap S_2$ of two sets is the set

$$S_1 \cap S_2 = \{x \mid x \in S_1 \wedge x \in S_2\}.$$

Definition 2.6 (Union). The union $S_1 \cup S_2$ of two sets is the set

$$S_1 \cup S_2 = \{x \mid x \in S_1 \vee x \in S_2\}.$$

Definition 2.7 (Disjoint). Two sets are disjoint if $A \cap B = \emptyset$.

Definition 2.8 (Difference set). If A and B are sets, the set $A \setminus B$ is the set A with any elements of B removed, i.e.

$$A \setminus B := \{x \mid x \in A \wedge x \notin B\}.$$

Proposition 2.9. Let A, B, C be subsets of set X .

- (a) (Minimal element) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.
- (b) (Maximal element) $A \cup X = X$ and $A \cap X = A$.
- (c) (Identity) $A \cap A = A$ and $A \cup A = A$.
- (d) (Commutativity) $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
- (e) (Associativity) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.
- (f) (Distributivity) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- (g) (Partition) $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$.
- (h) (De Morgan Laws) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Definition 2.10 (Ordered pair). If x and y are any objects, we define the ordered pair (x, y) to be a new object which consists of x as its "first component" and y as its "second component". Two ordered pairs x, y and x', y' are equal if

$$x = x', \quad y = y'.$$

Definition 2.11 (Cartesian product). Let A, B be sets. Then the cartesian product of A and B , written $A \times B$, is

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Definition 2.12 (Ordered n -tuple). Let n be a natural number. An ordered n -tuple $(x_i)_{1 \leq i \leq n}$ is a collection of objects x_i , one for every natural number i between 1 and n . Two ordered n -tuples $(x_i)_{1 \leq i \leq n}$ and $(y_i)_{1 \leq i \leq n}$ are said to be equal iff $x_i = y_i$ for all $1 \leq i \leq n$.

Definition 2.13 (n -fold Cartesian product). If $(X_i)_{1 \leq i \leq n}$ is an ordered n -tuple of sets, their Cartesian product $\prod_{i=1}^n X_i$ is defined

$$\prod_{i=1}^n X_i = \{(x_i)_{1 \leq i \leq n} \mid x_i \in X_i\}.$$

Definition 2.14 (Indexed family). If for each element $j \in J$ with $J \neq \emptyset$, there corresponds a set A_j , then

$$\mathcal{A} = \{A_j \mid j \in J\}.$$

Is called an indexed family of sets with J as the index set. If $J = \{1, 2, \dots, n\}$ we may index the set similarly to sum notation.

Definition 2.15 (Union and intersection of indexed family). The union of all sets in an indexed family \mathcal{A} with index set J is

$$\bigcup_{j \in J} A_j = \{x \mid \exists A_j \in \mathcal{A}, x \in A_j\}.$$

The intersection of all sets in \mathcal{A} is

$$\bigcap_{j \in J} A_j = \{x \mid \forall A_j \in \mathcal{A}, x \in A_j\}.$$

Lemma 2.16 (Finite choice). Let $n \geq 1$ be a natural number, and for each natural number $1 \leq i \leq n$, let X_i be a non-empty set. Then there exists an n -tuple $(x_i)_{1 \leq i \leq n}$ such that $x_i \in X_i$ for all $1 \leq i \leq n$. In other words if each X_i is non-empty, then its n -fold cartesian product is nonempty.

Proof: Let $\mathcal{A} = \{A_i \mid 1 \leq i \leq n\}$ with $n \in \mathbb{N}$ be an indexed family of nonempty sets. It follows from lemma 2.2 that for each A_i , $1 \leq i \leq n$, there exists $a_i \in A_i$. Using this fact, define an ordered n -tuple $(a_i)_{1 \leq i \leq n}$. \square

2.2 Functions

Definition 2.17 (Relation). Let A, B be sets. A relation between A and B is a subset of $A \times B$.

Definition 2.18 (Equivalence relation). An equivalence relation on a set S is a relation such that for all $x, y, z \in S$, the relation satisfies the following properties:

- (a) (Reflexive property) xRx .
- (b) (Symmetric property) $xRy \Rightarrow yRx$.
- (c) (Transitive property) $xRy \wedge yRx \Rightarrow xRz$.

Definition 2.19 (Partition). A partition of a set S is a collection \mathcal{P} of nonempty subsets of S that are pairwise disjoint, and whose union is S , i.e.

- (a) $A = \bigcup \mathcal{P}$.
- (b) $\forall A, B \in \mathcal{P}, A \neq B \Rightarrow A \cap B = \emptyset$.

Definition 2.20 (Function). A function from A to B , denoted $f : A \rightarrow B$ is a nonempty relation $f \subseteq A \times B$ that satisfies the following properties:

- (a) (Existence) $\forall a \in A, \exists b \in B, (a, b) \in f$.

(b) (Uniqueness) $(a, b) \in f \wedge (a, c) \in f \Rightarrow b = c$.

Set A is called the domain of f , and set B is called the codomain. The range of f is $f(A)$, i.e. $\{b \in B \mid (a, b) \in f\}$.

Definition 2.21 (Equality of functions). Two functions $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$ are equal if their domains and codomains are equal, and furthermore that $f(x) = g(x)$ for all $x \in X$.

Definition 2.22 (Composition). Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions such that the codomain of f is the same set as the domain of g . Then the composition $g \circ f : X \rightarrow Z$ of the two functions g and f is the function defined by the formula

$$(g \circ f)(x) = g(f(x)).$$

Lemma 2.23. Let $f : Z \rightarrow W$, $g : Y \rightarrow Z$, and $h : X \rightarrow Y$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.

Proof: $g \circ h$ is a function from X to Z , and $f \circ g$ is a function from $Y \rightarrow W$, so $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are functions from X to W . It follows from the definition of function composition that

$$\begin{aligned} (f \circ (g \circ h))(x) &= f((g \circ h)(x)) \\ &= f(g(h(x))) \\ &= (f \circ g)(h(x)) \\ &= ((f \circ g) \circ h)(x) \end{aligned}$$

□

Definition 2.24 (Injective). A function $f : X \rightarrow Y$ is injective (one-to-one) if for $x, x' \in X$,

$$x \neq x' \rightarrow f(x) \neq f(x')$$

Definition 2.25 (Surjective). A function $f : X \rightarrow Y$ is surjective (onto) if

$$\forall y \in Y, \exists x \in X, f(x) = y.$$

Definition 2.26 (Bijective). A function is bijective (invertible) if it is injective and surjective.

Proposition 2.27. Let $f : A \rightarrow B$ and $g : B \rightarrow C$. Then

- (a) If f and g are surjective, then $g \circ f$ is surjective.
- (b) If f and g are injective, then $g \circ f$ is injective.
- (c) If f and g are bijective, then $g \circ f$ is bijective.

Lemma 2.28. If $f : X \rightarrow Y$ is bijective then f is invertible. In other words for all $y \in Y$ there exists a unique $x \in X$ denoted $f^{-1}(y)$ such that $f(x) = y$. Therefore the inverse of f , $f^{-1} : Y \rightarrow X$ exists and is defined

$$f^{-1}(y) = x.$$

Definition 2.29 (Identity function). A function defined on a set A that maps each element in A onto itself is called the identity function on A , and is denoted i_A .

Proposition 2.30. Let $f : A \rightarrow B$ be bijective. Then

- (a) $f^{-1} : B \rightarrow A$ is bijective.
- (b) $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Theorem 2.31. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be bijective. Then the composition $g \circ f : A \rightarrow C$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Definition 2.32 (Image). If $f : X \rightarrow Y$ is a function from X to Y , and $S \subseteq X$, we define the image of S under f , $f(S)$ to be the set

$$f(S) = \{f(x) \mid x \in S\}.$$

Definition 2.33 (Inverse image). If U is a subset of Y , we define the set $f^{-1}(U)$ to be the set

$$f^{-1}(U) = \{x \in X \mid f(x) \in U\}.$$

We call $f^{-1}(U)$ the inverse image of U .

Proposition 2.34. If X, Y are sets and $f : X \rightarrow Y$ then $f(X) \subseteq Y$.

Proof: $y \in f(X)$ implies $y \in \{y \mid (x, y) \in f\}$ and f is a subset of $X \times Y$, so it follows from the definition of the cartesian product that $y \in Y$. \square

Lemma 2.35. Let X be a set. Then the set

$$\{Y \mid Y \subseteq X\}$$

Is a set.

Proof: Let X be a set and $A \subseteq X$ with $A \neq \emptyset$. Then there exists $p \in A$, and we can define a function $f : X \rightarrow A$ with $x \in X$ by

$$f(x) = \begin{cases} x \in A & f(x) = x \\ x \notin A & f(x) = p \end{cases}$$

Thus for all $a \in f(X)$, $a \in A$ or $a = p \in A$, so $f(X) \subseteq A$. Next, for all $x \in X$, $(x, f(x)) \in f(X)$. Because for all $a \in A$ we have $a \in X$ then for all $a \in A$, $(a, f(a)) = (a, a) \in f(X)$ so from the definition of an image $A \subseteq f(X)$. Thus $A = f(X)$. From the power set axiom in definition 2.1,

$$\{f : X \rightarrow A \mid A \subseteq X \wedge A \neq \emptyset\} \subseteq X^X$$

From replacement, pairwise union, and singleton set axioms in definition 2.1, we can define a set $P(X)$ that is the union of all images of functions in X^X , and $\{\emptyset\}$. As established above, all nonempty subsets of X are included in this set, and from proposition 2.34 all images of functions in X^X are subsets of X . \square

Definition 2.36 (Power set). For a set X , the set $\{Y \mid Y \subseteq X\}$ is called the power set of X , and is denoted $P(X)$ or 2^X .

Definition 2.37 (Cardinality). We say that two sets X and Y have equal cardinality iff there exists a bijection $f : X \rightarrow Y$ from X to Y .

Proposition 2.38. Let X, Y, Z be sets.

- (a) X has equal cardinality with X .
- (b) If X has equal cardinality with Y , then Y has equal cardinality with X .
- (c) If X has equal cardinality Y and Y has equal cardinality with Z , then X has equal cardinality with Z

Proof:

\square

Definition 2.39 (Cardinality n). Let n be a natural number. A set X is said to have cardinality n , if it has equal cardinality with $\{i \in \mathbb{N} \mid 1 \leq i \leq n\}$. In this case we say that X has n elements.

Lemma 2.40. Suppose that $n \geq 1$, and set X has cardinality n . Then X is non-empty, and if x is any element of X , then the set $X - \{x\}$ has cardinality $n - 1$.

Proposition 2.41. Let X be a set with some cardinality n . Then X cannot have any other cardinality, i.e. X cannot have cardinality m for any $m \neq n$.

Definition 2.42 (Finite set). A set is finite iff it has cardinality n for some natural number n ; otherwise, the set is called infinite.

Theorem 2.43. The set of natural numbers is infinite.

3 Integers and rationals

3.1 The integers

Definition 3.1 (Integers). An integer is an expression of the form $a - b$, where a and b are natural numbers. Two integers are considered to be equal, $a - b = c - d$, iff $a + d = c + b$. The set of all integers is denoted \mathbb{Z} .

Remark. The use of $-$ is purely notational (until subtraction is defined). $a - b$ can be interpreted as an ordered pair in $\mathbb{N} \times \mathbb{N}$.

Definition 3.2 (Integer addition). The sum of two integers $(a - b) + (c - d)$ is defined by the formula

$$(a - b) + (c - d) = (a + c) - (b + d)$$

Definition 3.3 (Integer multiplication). The product of two integers $(a - b) \times (c - d)$ is defined by the formula

$$(a - b) \times (c - d) = (ac + bd) - (ad + bc).$$

Remark. We may identify the integers with natural numbers by setting $n \equiv n - 0$. Definitions of equality and previously defined operations remain consistent with each other.

Proposition 3.4. If $a, b \in \mathbb{Z}$ and $a + b = 0$ then $a = 0$.

Proof: prove □

Lemma 3.5. Addition and multiplication are well defined.

Definition 3.6 (Negation of integers). If $(a - b)$ is an integer, we define the negation $-(a - b)$ to be the integer $b - a$.

Lemma 3.7 (Trichotomy of integers). Let x be an integer. Then either x is zero, equal to a positive natural number, or x negated is a positive natural number.

Definition 3.8 (Positive integer). If n is a positive natural number, we call n a positive integer, and $-n$ a negative integer.

Proposition 3.9 (Integer laws for algebra). Let x, y, z be integers. Then the following identities hold:

$$\begin{aligned}x + y &= y + x \\(x + y) + z &= x + (y + z) \\x + 0 &= 0 + x = x \\x + (-x) &= 0 \\xy &= yx \\(xy)z &= x(yz) \\1x &= x \\x(y + z) &= xy + xz\end{aligned}$$

Proposition 3.10. If $a, b \in \mathbb{Z}$ with $a, b > 0$, then $ab > 0$.

Proof: If $a, b \in \mathbb{Z}$ with $a, b > 0$, then for some $x, y \in \mathbb{N}^+$, $a = x - 0$ and $b = y - 0$. Thus $ab = (xy + 0) = (0 + 0) = xy - 0$. Because $x, y \neq 0$, by proposition 1.22 $xy > 0$ so from the definition of a positive integer $ab > 0$. □

Proposition 3.11. If $a, b \in \mathbb{Z}$ with $a, b > 0$, then $a + b > 0$.

Proof: If $a, b \in \mathbb{Z}^+$, then for some $x, y \in \mathbb{N}^+$ we have $a = x - 0$ and $b = y - 0$, so $a + b = ((x + y) - 0)$. It follows from proposition 1.12 that $x + y > 0$ so from the definition of a positive integer, $a + b > 0$. □

Proposition 3.12. If $x \in \mathbb{Z}$ with $x = (a - b)$ then $-1 \cdot (a - b) = -(a - b)$.

Proof: $-1 \cdot (a - b) = (0 - 1) \cdot (a - b) = (0a + b) - (a + 0b) = -(a - b)$. □

Proposition 3.13 (Integers have no zero divisors). If a, b are integers such that $ab = 0$, then $a = 0$ or $b = 0$.

Corollary 3.14 (Cancellation law). If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.

Proof: Let $a, b, c \in \mathbb{Z}$ with $c \neq 0$. If $a = 0$ it follows from proposition 3.13 that $ac = 0$ so $bc = 0$ and thus $b = 0$, so $a = b$. If $a \neq 0$, suppose to the contrary that $b \neq a$. It follows from proposition 3.4 that there exists $d \in \mathbb{Z}$ with $d \neq 0$ such that $a + d = b$. Using laws for algebra we see that $ac = ac + dc$. By proposition 3.13 $dc \neq 0$, a contradiction by proposition 3.4. Therefore $a = b$. \square

Definition 3.15 (Ordering of integers). Let $n, m \in \mathbb{Z}$. We say that n is greater than or equal to m and write $n \geq m$ or $m \leq n$ iff we have $n = m + a$ for some natural number a . We say that n is strictly greater than m and write $n > m$ or $m < n$ iff $n \geq m$ and $n \neq m$.

3.2 The rationals

Definition 3.16 (Rational number). A rational number is an expression of the form a/b , where a and b are integers and $b \neq 0$. Two rational numbers are equal, $a/b = c/d$, iff $ad = bc$. The set of all rational numbers is denoted \mathbb{Q} .

Remark. We may identify the rationals with natural numbers by setting $n/1 \equiv n$.

Definition 3.17 (Addition of rationals). If a/b and c/d are rationals, their sum is

$$(a/b) + (c/d) = (ad + bc)/(bd).$$

Definition 3.18 (Product of rationals). If a/b and c/d are rationals, their product is

$$(a/b) \cdot (c/d) = (ac)/(bd).$$

Definition 3.19 (Negation of rationals). The negation of a rational (a/b) , denoted $-(a/b)$ is

$$-(a/b) = (-a/b).$$

Definition 3.20 (Reciprocal of rationals). If $x = a/b$ is a non-zero rational number, then the reciprocal of x^{-1} of x is defined

$$x^{-1} = b/a.$$

Lemma 3.21. The sum, product, negation, and reciprocal operations on rational numbers are well-defined.

Proposition 3.22. The negation of the negation of $x \in \mathbb{Q}$ is x .

Proof: The negation of the negation of an integer $x = (a - b)$ is $- - (a - b) = -(b - a) = (a - b)$ so $- - x = x$. The negation of the negation of a rational number $y = (c/d)$ is $- - (c/d) = -(-c/d) = (- - c/d) = c/d$. \square

Definition 3.23 (Quotient). The quotient of two rationals x and y with $y \neq 0$, denoted x/y , is

$$x/y = x \times y^{-1}.$$

Definition 3.24 (Subtraction). The difference of two rationals x and y , denoted $x - y$, is defined

$$x - y = x + (-y).$$

Definition 3.25 (Positive rational number). A rational number x is said to be positive iff we have $x = a/b$ for some positive integers a and b . It is said to be negative iff $x = -y$ for some positive rational y .

Definition 3.26 (Ordering of rationals). Let $x, y \in \mathbb{Q}$. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - y$ is a positive negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$, and $x \leq y$ iff either $x < y$ or $x = y$.

Proposition 3.27. $x \in \mathbb{Q}$ is positive iff $x > 0$, and negative iff $x < 0$.

Proof: If $x = a/b$ is a positive rational number then $a, b > 0$. Because $0 = 0/d$ for some $d \in \mathbb{N} \setminus \{0\}$, $x - 0 = x + 0 = ad/bd = a/b$ and thus $x > 0$. If $x > 0$ then $x - 0$ is positive. Because $0 = 0/d$ for some $d \in \mathbb{N} \setminus \{0\}$ we have $x - 0 = x + 0 = ad/bd = a/b$, which is positive. \square

Proposition 3.28 (Laws of algebra for rationals). Let x, y, z be rationals. Then the following laws of algebra hold:

$$\begin{aligned}x + y &= y + x \\(x + y) + z &= x + (y + z) \\x + 0 &= x \\x + (-x) &= 0 \\xy &= yx \\(xy)z &= d(yz) \\1x &= x \\x(y + z) &= xy + xz\end{aligned}$$

Proposition 3.29. $-1x = -x$.

Proof: If $x = a/b$ then $-1 \cdot x$ is $(-1/1) \cdot (a/b) = -1a/b$. From proposition 3.12, $-1a = -a$ so $-1a/b = -(a/b) = -x$. \square

Proposition 3.30. If $a, b \in \mathbb{Q}$ with $a > 0$ and $b > 0$ then $a + b > 0$.

Proof: Suppose $a, b \in \mathbb{Q}$ with $a, b > 0$. It follows from the definition of positive rational number that for some positive $x, y, z, w \in \mathbb{Z}$, $a = x/y$ and $b = z/w$, so $ab = xw + zy/yw$. By proposition 3.10 $xw, zy, yw > 0$, so by 3.11, $a + b > 0$. \square

Lemma 3.31 (Trichotomy of rationals). Let x be a rational number. Then exactly one of the following three statements is true:

- (a) $x = 0$.
- (b) x is positive.
- (c) x is negative.

3.3 Absolute value and exponentiation

Definition 3.32 (Absolute value). If x is a rational number, the absolute value $|x|$ of x is defined as follows:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Definition 3.33 (Distance). The distance between $x, y \in \mathbb{Q}$, sometimes denoted $d(x, y)$, is

$$d(x, y) = |x - y|.$$

Proposition 3.34. For all $x \in \mathbb{Q}$, $|x| \geq 0$.

Proof: If $x \geq 0$ then $|x| = x$ so $|x| \geq 0$. If $x < 0$ then $|x| = -x$. By proposition 3.27 x is negative. Therefore there exists $y \in \mathbb{Q}^+$ such that $x = -y$, so by proposition 3.22 $-x = -(-y) = y$ and $-x$ is positive. By proposition 3.27, $-x > 0$. \square

Proposition 3.35 (Triangle inequality). For $x, y \in \mathbb{Q}$, $|x + y| \leq |x| + |y|$.

Proof: \square

Definition 3.36 (ϵ -closeness). Let $\epsilon > 0$ be a rational number, and let x, y be rational numbers. We say that y is ϵ -close to x iff $d(y, x) < \epsilon$.

Definition 3.37 (Exponentiation to a natural number). Let x be a rational number. To raise x to the power 0, we define $x^0 = 1$ and for all $n \in \mathbb{N}$, $x^{n+1} = x^n \times x$.

Definition 3.38 (Exponentiation to a negative number). Let x be a non-zero rational number. Then for any negative integer $-n$,

$$x^{-n} = 1/x^n.$$

4 The real numbers

4.1 Sequences

5 Limits of sequences

Definition 5.1 (Cauchy Sequence of reals).

6 Series

Definition 6.1 (Convergence of series). Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, we define the N th partial sum S_N of this series to be

$$S_N = \sum_{n=m}^N a_n.$$

If the sequence $(S_N)_{N=m}^{\infty}$ converges to L , then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is convergent, and converges to L . We also write $L = \sum_{n=m}^{\infty} a_n = L$. If the partial sums S_N diverge, we say the infinite series $\sum_{n=m}^{\infty} a_n$ is divergent, and do not assign any real number to it.

Proposition 6.2. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. $\sum_{n=m}^{\infty} a_n$ converges iff for every real number $\epsilon > 0$, there exists an integer $N \geq m$ such that for all $p, q \geq N$,

$$\left| \sum_{n=p}^q a_n \right| \leq \epsilon.$$

Corollary 6.3. Let $\sum_{n=m}^{\infty} a_n$ be a convergent series of real numbers. Then we must have $\lim_{n \rightarrow \infty} a_n = 0$.

Definition 6.4 (Absolute convergence). Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. We say that this series is absolutely convergent iff the series $\sum_{n=m}^{\infty} |a_n|$ is convergent.

Proposition 6.5. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If the series is absolutely convergent, then it is also convergent. Furthermore,

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n|.$$

Proposition 6.6 (Alternating series test). Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which are non-negative and decreasing, thus $a_n \geq 0$ and $a_n \geq a_{n+1}$ for every $n \geq m$. Then the series

$$\sum_{n=m}^{\infty} (-1)^n a_n$$

is convergent iff the sequence a_n converges to 0 as $n \rightarrow \infty$.

Proof: The sequence of partial sums is Cauchy, thus converges. □

Proposition 6.7. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of non-negative real numbers. Then this series is convergent iff there is a real number M such that for all $N \geq m$,

$$\sum_{n=m}^N a_n \leq M.$$

Corollary 6.8 (Comparison test). Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two formal series of real numbers, and suppose that $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, and

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Definition 6.9 (Geometric series). The geometric series is defined

$$\sum_{n=0}^{\infty} x^n,$$

Where $x \in \mathbb{R}$.

Lemma 6.10. Let x be a real number. If $|x| \geq 1$, then the series $\sum_{n=0}^{\infty} x^n$ is divergent. If $|x| < 1$, then the series is absolutely convergent, and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Proposition 6.11. Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of non-negative real numbers. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent iff

$$\sum_{k=0}^{\infty} 2^k a_{2^k}$$

is convergent.

Corollary 6.12. Let $q > 0$ be a real number. Then the series $\sum_{n=1}^{\infty} \frac{1}{n^q}$ is convergent when $q > 1$ and divergent when $q \leq 1$.

Theorem 6.13 (Root test). Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $\alpha = \limsup |a_n|^{1/n}$.

- (a) If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent.
- (b) If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is not convergent.
- (c) If $\alpha = 1$, we cannot assert any conclusion.