Homework

Samuel Lindskog

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Problem 1

Let $n \in \mathbb{N}$ be a positive integer. Suppose $x_1, \dots, x_n$, $y_1, \dots, y_n$, $z_1, \dots, z_n \in \mathbb{R}$. Show that

$$\sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}.$$

Proof: We shall prove by induction on n. Suppose $n = 1$. Then

$$\begin{aligned} \sqrt{(x_i - y_i)^2} + \sqrt{(y_i - z_i)^2} &= (x_i - y_i) + (y_i - z_i) \\ &= (x_i - z_i) \\ &= \sqrt{(x_i - z_i)^2} \end{aligned}$$

Suppose by inductive hypothesis that

$$\sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}.$$

It follows that $\sum_{i=1}^{n}(x_i - z_i) \leq \sum_{i=1}^{n}(x_i - y_i) + \sum_{i=1}^{n}(x_i - z_i)$. Suppose $x, y, z, w \in \mathbb{R}$. Then

$$\begin{aligned} \sqrt{x^2 + w^2} + \sqrt{y^2 + z^2} &\geq \sqrt{(x + y)^2 + (w + z)^2} \\ x^2 + y^2 + w^2 + z^2 + 2\sqrt{x^2 + w^2}\sqrt{y^2 + z^2} &\geq x^2 + y^2 + w^2 + z^2 + 2xy + 2wz \end{aligned}$$

□

Problem 2

Proof: For all $(x, y) \in (0, 1) \times (0, 1)$, $0 < x, y < 1$. So, for any (x, y) if $r = \min\big(|x - 1|, x, |y - 1|, y\big)$ and $(z, w) \in B((x, y), r)$, then $\sqrt{(x - z)^2 + (y - w)^2} < \frac{r}{2}$. Therefore $z = x \pm \frac{r}{2}$ and $w = y \pm \frac{r}{2}$, so $(z, w) \in U$. □

Proof: Suppose $(x, y) \in \{(0, 1) \times (0, 1)\}$. Then $d\big((x, y), (0, 0)\big) = \sqrt{x^2 + y^2}$. Because there exists $r > 0$ such that $B(x, r) \subset U$, if $r > \min\big(|x - 1|, x, |y - 1|, y\big)$, then $B(x, r) \not\subset U$. But because $x, y > 0$, we see that $d\big((x, y), (0, 0)\big) > r$, so $\exists a \in B\Big((0, 0), d\big((x, y), (0, 0)\big) - r\Big)$[1] such that $a \notin B((x, y), r)$ for any open ball. □

Problem 3

Proof: Let $X = \mathbb{R}$. If $U_i \in \{(\frac{-1}{i}, \frac{1}{i})\}_{i=1}^{\infty}$ is a sequence of open subsets, then $\cap_{i=1}^{\infty} U_i = 0$. Let $x \in U_i$, then $\forall \epsilon > 0, \exists N = \frac{2}{\epsilon}\ (i > N \Rightarrow |x| > \epsilon \Rightarrow x \notin U_N)$. So $\cap_{i=1}^{\infty} U_i = \{0\}$, and $\{0\}$ is a closed subset of X. □

[1] Not the empty set