HW 6

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Proposition 0.1. If B is the set of upper bounds of a set R, and Y is the set of convergent points of all convergent sequences in B, $B = Y$

Proof: Suppose $(b_n)_{n=0}^{\infty}$ is a sequence in B, the set of all upper bounds of R, and let $(b_n)_{n=0}^{\infty}$ converge to $L \in \mathbb{R}$. Then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies $|a_n - L| < \epsilon$. If there exists $r \in R$ such that $r > L$, then $r = L + a$ for some $a \in \mathbb{R}^+$. But there exists $M \in \mathbb{N}$ such that $n > M$ implies $|b_n - L| < a/2$, so

$$L - a/2 < b_n < L + a/2 < L + a < c$$
$$b_n < c,$$

a contradiction. Therefore $Y \subseteq B$.

If $b \in B$, the sequence $(b_n = b)_{n=0}^{\infty}$ is ϵ-close to b for all $\epsilon > 0$, and thus converges to b. Thus $B \subseteq Y$. □

Problem 1

(a) If a nonempty set $R \subseteq \mathbb{R}$ has an upper bound then it has a least upper bound (supremum).

Proof: Let R be a nonempty set with an upper bound. Let B be the set of all upper bounds of R. Suppose to the contrary that for all decreasing sequences $(b_n)_{n=0}^{\infty}$ in B,

$$\exists b \in B,\ \forall N \in \mathbb{N},\ \exists n \in \mathbb{N}\big(n > N \wedge b_n > b\big). \tag{1}$$

The constant sequence $(b_n = b)_{n=0}^{\infty}$ in B contradicts claim (1). Thus there exists a decreasing sequence $(b'_n)_{n=1}^{\infty}$ in B such that

$$\forall b \in B,\ \exists N \in \mathbb{N},\ \forall n \in \mathbb{N},\ \big(n > N \Rightarrow b'_n \leq b\big). \tag{2}$$

It follows from proposition 6.3.8 (Tao) that any decreasing sequence in $\mathbb{R}$ bounded below converges. Because for all $b \in B$ and all $r \in R$, $b > r$, B is bounded below. Thus $(b'_n)_{n=0}^{\infty}$ converges (by proposition 0.1) to some $l \in B$. Because the sequence is decreasing, for all $n \in \mathbb{N}$, $b'_n \geq l$. It then follows from equation (2) that for all $b \in B$, $l \leq b$. Thus $\sup R$ exists. □

(b) If a nonempty subset of $\mathbb{R}$ has an infimum, then it is bounded.

Proof: Let $R \subseteq \mathbb{R}$, and $l = \inf R$. Then R is bounded below by l, i.e.

$$\forall r \in R,\ r > l.$$

If there exists $l' \in \mathbb{R}$ with $l' \geq 0$ such that $\forall r \in R, |r| \leq l'$, then

$$l' < r < l'$$

But r can be arbitrarily large, so this is not necessarily true. Therefore the statement is false. A correct statement could be ”If a nonempty set of $\mathbb{R}$ has an infimum and a supremum, then it is bounded.” □

(c) Every nonempty bounded subset of $\mathbb{R}$ has a maximum and a minimum.

Proof: If $R = [0, 1)$, then for all $\epsilon > 0$ there exists $r \in R$ such that $|1 - r| < \epsilon$, and $1 \notin R$. Because for all $r \in R$ we have $r < 1$, $|1 - r| = 1 - r > 0$. Therefore for any r, there exists $a = r + (1 - r)/2 = r/2 + 1/2 < 1$, so $a > r$ and $a \in \mathbb{R}$. $|r| < 2$ for all $r \in R$, so R is bounded. Therefore, the bounded set R has no maximum and the statement is false. □

(d) Let S be a nonempty subset of $\mathbb{R}$. If $m = \inf S$ and $m' < m$ then m' is a lower bound of S.

Proof: Because $m = \inf S$, m is a lower bound for S, and thus for all $s \in S$, $s \geq m$. Because $m' < m$, it follows from elementary properties of the ordering of the reals that for all $s \in S$, $s \geq m > m'$, so the statement is true. □

Problem 2

(a) The interval $I = (0, 4]$ has supremum 4, and infimum 0. For any upper (lower) bound smaller (greater) than 4 (0), there exists $i \in I$ such that i is greater (lesser) than that upper (lower) bound. The maximum is 4, and no minimum, because supremum S is an element of S, but infimum S is not. The set is bounded because it is both bounded above and below by its supremum and infimum

(b) The set $A = \{1/n \,|\, n > 0,\, n \in \mathbb{N}\}$ has supremum 1, and an infimum 0. The supremum of A is 1 because 1 is the maximum value of A. As n increases, $1/n$ is strictly larger than zero and becomes arbitrarily close to *zero*. Thus for any number a greater than 0, there exists n such that $1/n < a$, so zero is the infimum. $0 \notin A$, so the set has no minimum. The set is bounded because it is both bounded above and below by its supremum and infimum.