HW 5

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Problem 1

Show that if a sequence of rational numbers converges to a rational number then the sequence is Cauchy.

Proof: Let $(a_n)_{n=1}^{\infty}$ be a sequence of rational numbers. If $(a_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{Q}$, then for all $\epsilon > 0$ there exists $N > 0$ such that $n > N \Rightarrow |a_n - L| < \epsilon$. Choose $M \in \mathbb{N}$ such that $n > M$ implies $|a_n - L| < \epsilon/2$. It follows that for $i, j > M$,

$$\begin{aligned}
\epsilon = \frac{\epsilon}{2} + \frac{\epsilon}{2} &> |a_i - L| + |a_j - L| \\
&= |a_i - L| + |-(a_j - L)| \\
&\geq |a_i - L + (-(a_j - L))| \\
&= |a_i - a_j|
\end{aligned}$$

Thus for all $\epsilon > 0$ there exists M such that $i, j > M$ implies $|a_i - a_j| < \epsilon$ and $(a_n)_{n=1}^{\infty}$ is Cauchy. $\square$

Problem 2

Section 5.1: Exercise 5.1.1

Proof: Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence, so

$$\forall \epsilon \in \mathbb{Q}^+,\ \exists N \in \mathbb{N},\ \big(i, j > N \Rightarrow |a_i - a_j| < \epsilon\big).$$

Therefore choose $M \in \mathbb{N}$ such that $i, j > M$ implies $|a_i - a_j| < 1$. Then the sequence $(a_n)_{n=1}^{M+1}$ is finite and thus by lemma 5.1.14 there exists $L \in \mathbb{Q}$ such that for all $1 \leq n \leq M+1$, $|a_n| < L$. Therefore there exists $b \in \mathbb{Q}^+$ such that $a_{M+1} + b = L$, i.e. $a_{M+1} = L - b$. As a consequence of the fact that $|a_{M+1}|$ is nonnegative, $0 < b \leq L$ and $L > 0$.

Suppose to the contrary that for some $k \in \mathbb{N}^+$ with $k > M$, $|a_k| \geq L + 1$, i.e. $|a_k| = L + 1 + l$ for some $l \in \mathbb{Q}^+ \cup \{0\}$. Utilizing the properties of absolute value, the fact that two nonnegative numbers added are nonnegative, and the facts

$$\begin{gathered}
L, b, 1 > 0, \\
L > L - b \geq 0, \\
l \geq 0,
\end{gathered}$$

we prove by cases that $|a_k - a_{M+1}| > 1$, a contradiction:

(a)

$$\begin{aligned}
|a_k - a_{M+1}| &= |(L + 1 + l) + (L - b)| = L + L - b + 1 + l \geq 1 + L > 1 \\
&= |-(L + 1 + l) - (L - b)| = |(L + 1 + l) + (L - b)| > 1
\end{aligned}$$

(b)

$$\begin{aligned}
|a_k - a_{M+1}| &= |(L + 1 + l) - (L - b)| = |1 + l + b| \geq 1 + b > 1 \\
&= |-(L + 1 + l) + (L - b)| = |(L + 1 + l) - (L - b)| > 1
\end{aligned}$$

Therefore $|a_k| < L + 1$. $\square$

Problem 3

Section 5.3: Exercise 5.2.1

Proof: Suppose $(a_n)_{n=0}^{\infty}$ is Cauchy, and $(b_n)_{n=1}^{\infty}$ is an equivalent sequence. It follows that

$$\forall \epsilon \in \mathbb{Q}^+, \exists N \in \mathbb{N}, \big(n \geq N \Rightarrow |a_n - b_n| < \epsilon\big)$$

and

$$\forall \epsilon \in \mathbb{Q}^+, \exists M \in \mathbb{N}, \big(i, j \geq M \Rightarrow |a_i - a_j| < \epsilon\big).$$

Therefore, given $\epsilon > 0$, if $i, j \geq \max\{N, M\}$ then

$$\begin{aligned} |b_i - b_j| &= |b_i - a_i + a_i - a_j + a_j - b_j| \\ &\leq |b_i - a_i| + |a_i - a_j| + |a_j - b_j| \\ &= 3\epsilon. \end{aligned}$$

□

Problem 4

Section 5.3: Exercise 5.3.2

Proof: First we prove the product of two reals is real. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be Cauchy sequences. Then (also utilizing the reverse triangle inequality) we can choose $N, M \in \mathbb{Q}$ such that

$$i, j > N \Rightarrow |a_i - a_j| < 1, \Rightarrow a_j < 1 + a_i \tag{1}$$
$$i, j > M \Rightarrow |b_i - b_j| < 1, \Rightarrow b_j < 1 + b_i \tag{2}$$

Then for $\delta \in \mathbb{Q}$ choose $L, O \in \mathbb{N}$ such that

$$\begin{aligned} i, j > L \Rightarrow |a_i - a_j| < \delta \\ i, j > O \Rightarrow |b_i - b_j| < \delta \end{aligned}$$

Let $C = \max\{N, M, L, O\}$, and fix $k \in \mathbb{N}$ such that $k > C$. If $i, j > C$, it follows from proposition 4.3.7 and equations 1 and 2 that

$$\begin{aligned} |a_i b_i - a_j b_j| &< \delta|b_i| + \delta|a_i| + \delta^2 \\ &= \delta(|b_i| + |a_i|) + \delta^2 \\ &< \delta(|a_k| + 1 + |b_k| + 1) + \delta^2 \\ &= \delta(|a_k| + |b_k| + 2) + \delta^2 \end{aligned}$$

Given $\epsilon \in \mathbb{Q}^+$, we may find δ (thus determining C) such that $|a_i b_i - a_j b_j| < \epsilon$ as follows:

$$\begin{aligned} \epsilon &= \delta(|a_k| + |b_k| + 2) + \delta^2 \\ 0 &= \delta^2 + \delta(|a_k| + |b_k| + 2) - \epsilon \\ \delta &= \frac{-(|a_k| + |b_k| + 2) + \sqrt{(|a_k| + |b_k| + 2)^2 + 4\epsilon}}{2} \end{aligned}$$

Therefore given $\epsilon > 0$, $i, j > C$ implies $|a_i b_i - a_j b_j| < \epsilon$, so $\mathrm{LIM}_{n\to\infty} a_n b_n$ is Cauchy, and thus multiplication of two reals is real.

Next, we prove that multiplication is well-defined. Suppose $x, x', y \in \mathbb{R}$ with

$$\begin{aligned} x &= \mathrm{LIM}_{n\to\infty} a_n \\ x' &= \mathrm{LIM}_{n\to\infty} a'_n \\ y &= \mathrm{LIM}_{n\to\infty} b_n, \end{aligned}$$

and $x = x'$. Because $(b_n)_{n=1}^{\infty}$ is Cauchy, choose $N \in \mathbb{N}$ (and use the reverse triangle inequality) such that

$$i, j > N \Rightarrow |b_i - b_j| < 1 \Rightarrow |b_i| < 1 + |b_j|.$$

Fix $k \in \mathbb{N}$ such that $k > N$. Because $x = x'$, choose $M \in \mathbb{N}$ such that for some $\epsilon > 0$,

$$i > M \Rightarrow |a_i - a'_i| < \frac{\epsilon}{1 + |b_k|}.$$

It follows from the properties of absolute value that the above equation implies

$$\begin{aligned} i > M &\Rightarrow |b_i||a_i - a'_i| < |b_i|\frac{\epsilon}{1 + |b_k|} \\ &\Rightarrow |a_i b_i - a' i b_i| < \frac{|b_i|\epsilon}{1 + |b_k|} \end{aligned}$$

Therefore if $C = \max\{N, M\}$ then $i > C$ implies $|b_i| < 1 + |b_k|$. It then follows from the above implication that

$$\begin{aligned} i > C \Rightarrow |a_i b_i - a'_i b_k| &< \frac{(1 + |b_k|)\epsilon}{1 + |b_k|} \\ &= \epsilon \end{aligned}$$

Thus xy and $x'y$ are equivalent. □

Problem 5

Negate mathematical statments involving quantifiers.

Proof:

(a) A sequence a_n is not Cauchy.

$\exists \epsilon > 0$, $\forall N \in \mathbb{N}$, $\exists i, j \in \mathbb{N}$, $(i, j > N \wedge |a_i - a_j| \geq \epsilon)$.

(b) Two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are not equivalent.

$\exists \epsilon > 0$, $\forall N \in \mathbb{N}$, $\exists i \in \mathbb{N}$, $(i > N \wedge |a_i - b_i| \geq \epsilon)$.

(c) A sequence $(a_n)_{n=1}^{\infty}$ is not convergent to L.

$\exists \epsilon > 0$, $\forall N \in \mathbb{N}$, $\exists i \in \mathbb{N}$, $(i > N \wedge |a_i - L| \geq \epsilon)$.

(d) A sequence $(a_n)_{n=1}^{\infty}$ is not bounded.

$\forall L > 0$, $\exists n \in \mathbb{N}$, $|a_n| > L$.

□

Problem 6

Show that for all $x, y, z \in \mathbb{R}$:

(a) $1 \cdot x = x$.

Proof: Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence. It follows from algebraic rules for rational numbers that for any rational number r we have $1 \cdot r = r$. Thus $\mathrm{LIM}_{n\to\infty} a_n \cdot \mathrm{LIM}_{n\to\infty} 1 = \mathrm{LIM}_{n\to\infty}(a_n \cdot 1) = \mathrm{LIM}_{n\to\infty} a_n$. □

(b) $y - y = 0$.

Proof: Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence. It follows from algebraic rules for rational numbers that for any rational number r we have $r - r = 0$. Thus $\mathrm{LIM}_{n\to\infty} a_n - \mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty}(a_n - a_n) = \mathrm{LIM}_{n\to\infty} 0$. □

(c) If $z \neq 0$ then $z \cdot \frac{1}{z} = 1$.

Proof: Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence. It follows from algebraic rules for rational numbers that for any rational number $r \neq 0$ we have $r \cdot r^{-1} = 1$. Thus $\mathrm{LIM}_{n\to\infty} a_n \cdot \mathrm{LIM}_{n\to\infty} a_n^{-1} = \mathrm{LIM}_{n\to\infty}(a_n \cdot a_n^{-1}) = \mathrm{LIM}_{n\to\infty} 1$. □

(d) $(x + y)z = xz + yz$.

Proof: Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$ be a Cauchy sequences. It follows from the distributive property of rational multiplication that for any rational numbers x, y, z we have $(x + y)z = xz + yz$. Thus $(\mathrm{LIM}_{n\to\infty} a_n + \mathrm{LIM}_{n\to\infty} b_n) \cdot \mathrm{LIM}_{n\to\infty} c_n = \mathrm{LIM}_{n\to\infty}(a_n + b_n) \cdot \mathrm{LIM}_{n\to\infty} c_n = \mathrm{LIM}_{n\to\infty}(a_n + b_n)c_n = \mathrm{LIM}_{n\to\infty}(a_n c_n + b_n c_n) = \mathrm{LIM}_{n\to\infty} a_n c_n + \mathrm{LIM}_{n\to\infty} b_n c_n$. □

(e) If $x < y$ then $x + z < y + z$.

Proof: Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$ be a Cauchy sequences. It follows from the properties of order of the rationals that for any rational numbers x, y, z we have $x < y \to x + z < y + z$. Thus $(\mathrm{LIM}_{n\to\infty} a_n + \mathrm{LIM}_{n\to\infty} b_n) \cdot \mathrm{LIM}_{n\to\infty} c_n = \mathrm{LIM}_{n\to\infty}(a_n + b_n) \cdot \mathrm{LIM}_{n\to\infty} c_n = \mathrm{LIM}_{n\to\infty}(a_n + b_n)c_n = \mathrm{LIM}_{n\to\infty}(a_n c_n + b_n c_n) = \mathrm{LIM}_{n\to\infty} a_n c_n + \mathrm{LIM}_{n\to\infty} b_n c_n$. □

Bonus 5.4.4

Bonus 5.4.3