Partial Differential Equations

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1 Heat Equation

1.1 One-dimensional heat equation

Definition 1.1 (Thermal energy density)**.** $e(x,t)$ represents thermal energy density of a one-dimensional rod with heat energy flowing only longitudinally. Heat energy in a small slice of width Δx is therefore $e(x,t)A\Delta x$.

Definition 1.2 (Heat flux)**.** $\phi(x,t)$ represents heat flux, and is the amount of thermal energy per unit time flowing to the right per unit surface area.

Definition 1.3 (Conservation of heat energy)**.** The conservation of heat energy equation is

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.$$

Q is heat energy per unit volume generated per unit time.

Proof: The change in thermal energy with respect to time of a slice in a system with one-dimensional heat flux is given by the equation

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b e\,dx = \phi(a) - \phi(b) + \int_a^b Q(x)\,dx \\
&\int_a^b \frac{\partial e}{\partial t} dx = \int_a^b \Big(-\frac{\partial \phi}{\partial x} + Q(x)\Big) dx \\
&\int_a^b \Big(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q(x)\Big) dx = 0 \\
&\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x)
\end{aligned}$$

□

Proposition 1.4. The relation between thermal energy density and temperature u is given by the following equation:

$$e(x,t) = c(x)\rho(x)u(x,t).$$

ρ is mass density, and c is specific heat.

Definition 1.5 (Fourier's law of heat conduction)**.** Heat flux is related to temperature by the following equation, known as Fourier's law of heat conduction:

$$\phi = -K_0 \frac{\partial u}{\partial x}.$$

K_0 is the thermal conductivity of the material. Thermal diffusivity k is given by the equation

$$k = \frac{K_0}{c\rho}.$$

Proposition 1.6. Subbing in Fourier's law of heat conduction, the conservation of heat energy equation becomes

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q(x).$$

Definition 1.7 (Newton's law of cooling)**.** Newton's law of cooling states that heat flux at the left boundary is negatively proportional to the difference in temperature between an outside medium and the left side of a one-dimensional system, i.e.

$$-K_0(0)\frac{\partial u}{\partial x} = -H\big(u(0,t) - u_B(t)\big).$$

H is the heat transfer coefficient, or the convection coefficient. For the right boundary, H does not have a negative sign.

Proposition 1.8. If a boundary is perfecty insulated, heat flux at the boundary is zero.

Definition 1.9 (Perfect thermal contact)**.** Two one-dimensional rods are said to be in perfect thermal contact if temperature is continuous at the boundary and thermal flux is the the same as x approaches the boundary in both rods.

Definition 1.10 (Equilibrium temperature distribution)**.** In solving differential equations involving heat transfer, we are often interested in the equilibrium temprature distribution. This is the temperature distribution a system will reach regardless of the initial temperature distribution. The answer will be a function of x only.

Proposition 1.11. Because of conservation of energy, a perfectly insulated rod with no internal energy source will maintain the total thermal energy given by its initial conditions $f(x)$ as it approaches equilibrium. Thus the equilibrium temperature distribution must be a constant $u(x)$ given by the equation

$$u(x) = \int_0^L f(x)dx.$$

2 Seperation of variables

Definition 2.1 (Linear operator)**.** A linear operator L satisfies the linearity property if for $c_1, c_2 \in \mathbb{R}$ and $u_1, u_2 \in \{f \mid f : \mathbb{R} \to \mathbb{R}\}$

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2).$$

Definition 2.2 (Linear equation)**.** A linear equation for the unknown function u with linear operator L is

$$L(u) = f.$$

If $f = 0$, then this is a linear homogeneous equation.

Proposition 2.3. It follows from the definition of a linear operator L that $L(0) = 0$. Therefore 0 is a solution to any linear homogeneous equation, also known as the trivial solution.

Definition 2.4 (Principle of superposition)**.** If u_1 and u_2 satisfy a linear homogeneous equation, then any linear combination of these solutions is itself a solution.

2.1 Method of seperation of variables

The method of seperation of variables attempts to determine solutions of linear homogeneous equations using the ansatz

$$u(x,t) = \phi(x)G(t).$$

We begin by solving the following BVP:

$$\begin{aligned}
&\text{PDE: } \frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0 \\
&\text{BC: } u(0,t) = 0,\ u(L,t) = 0 \\
&\text{IC: } u(x,0) = f(x)
\end{aligned}$$

This is a linear homogenous differential equation with linear homogenous boundary conditions. Homogeniety of boundary conditions and the properties of linear operators is usually necessary because it allows us to take advantage of the principle of superposition. To begin, sub in product solution $u(x,t) = \phi(x)G(t)$:

$$\begin{aligned}
\phi(x)\frac{\mathrm{d}G}{\mathrm{d}t} &= k\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}G(t) \\
\frac{1}{kG}\frac{\mathrm{d}G}{\mathrm{d}t} &= \frac{1}{\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = -\lambda
\end{aligned}$$

The latter equation is reached as a result of the fact that the derivative operation is a linear operator. A consequence of this is that we can equate a linear combination of nth derivatives of u with respect to x with nth derivatives of u with respect to t, while maintaining seperation between independent variables established by the product equation $u(x,t) = \phi(x)G(t)$. Because x and t are independent variables, this equation seperated by its independent parts must be a constant. This yields two ODEs in x and t:

$$\begin{aligned}
\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} &= -\lambda\phi \\
\frac{\mathrm{d}G}{\mathrm{d}t} &= -\lambda kG
\end{aligned}$$

From here, boundary conditions determine solutions to our two ODEs. The homogeniety of the boundary conditions allows us to superimpose all possible solutions of u, which through the nature of our solutions and fourier series means we can represent a wide range of initial conditions. To find all possible solutions for u, we analyse the effect different values of λ have on our ODE solutions. If $\lambda > 0$, solutions for $\phi(x)$ are of the form

$$\begin{aligned}
&\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = -\lambda\phi, \quad \phi(0) = 0, \phi(L) = 0 \\
&\phi(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x),
\end{aligned}$$

and solutions for $G(t)$ are of the form

$$\begin{aligned}
&\frac{\mathrm{d}G}{\mathrm{d}t} = -k\lambda G \\
&G(t) = C_3 e^{-k\lambda t}.
\end{aligned}$$

From here, we see that values of $\lambda > 0$ that meet our boundary conditions are of the form $\lambda = (n\pi/L)^2$, $n \in \mathbb{N} \setminus \{0\}$, and that $C_1 = 0$ for ϕ. Therefore if $\lambda > 0$,

$$u(x,t) = B \sin\left(\frac{n\pi}{L}x\right) e^{-k(n\pi/L)^2 t}.$$

If $\lambda = 0$ then $\phi(x) = C_1 x + C_2$ and $G(t) = C_3$, so to meet boundary conditions $u(x,t) = 0$. If $\lambda < 0$ then $\phi(x)$ and $G(t)$ are both exponentials multiplied by arbitrary constants, so to meet boundary conditions $u(x,t) = 0$. Therefore for this BVP, nontrivial solutions exist for values of $\lambda > 0$. Such λ are called eigenvalues, and their corresponding solutions are called eigenfunctions.

2.2 Superposition

Using the principle of superposition, we see that for solutions $u_1, \ldots, u_M$, any linear combination of these solutions is itself a solution. Utilizing the theory of Fourier series, we know that

(a) Most functions can be approximated by a finite linear combination of $\sin n\pi x/L$.

(b) This approximation becomes arbitrarily accurate as M increases.

(c) As M approaches infinity, the resulting infinite series converges to $f(x)$, with some restrictions on $f(x)$.

Thus for most initial conditions $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}. \tag{1}$$

Proposition 2.5 (Orthogonality relation)**.** Functions $\sin\frac{n\pi x}{L}$ and cos satisfy the following relation:

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx = \begin{cases} 0, & n \neq m \\ L/2, & n = m \neq 0 \\ L, & n = m = 0 \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = 0$$

We can use proposition 2.5 to calculate the coefficient B_m in equation 1 by way of the following equation:

$$B_m = \frac{2}{L}\int_0^L f(x) \sin\frac{m\pi x}{L} dx.$$