Differential Equations

Samuel Lindskog

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First-Order Differential Equations

Definition 1.1 (Order)**.** The order of a differential equation is the order of the highest dericatice appearing in the equation.

Definition 1.2 (Normal form)**.** The normal form of a first-order equation is a function f which relates a function $x = x(t)$ with its first derivative.

$$x' = f(t, x).$$

A function $x = x(t)$ is a solution of this equation on the time interval $I : a < t < b$ if it is differentiabe on I and, when substituted into the equation, it satisfies the equation identically for every $t \in I$, i.e.

$$x'(t) = f(t, x(t)), \text{ for every } t \in I.$$

In other words to check if a function is a solution, substitute the function in question into the differential equation and check that it reduces to an identity.

Definition 1.3 (Initial value problem)**.** Subjecting a differential equation involving $x(t)$ and its derivatives to a condition $x(t_0) = x_0$ is called an initial value problem. The interval of existance of an IVP is the largest time interval where the solution is valid.[1]

[1] A solution to an IVP is called a particular solution.

Definition 1.4 (General solution)**.** The infinite set of solutions of a first-order equation is called the general solution of the equation.

Definition 1.5 (Nullclines and isoclines)**.** The sets of points (t, x) where the slope field is zero are called nullclines[2], i.e. where

$$x' = f(t, x) = 0.$$

[2] These constant solutions $f(t, x) = k$ for some constant k are called equilibrium solutions.

The set of points t, x where $f(t, x) = k$ for some constant k are called isoclines.

Theorem 1.1 (Fundamental theorem of calculus)**.**

$$\frac{d}{dt}\int_a^t g(s)ds = g(t)$$

Definition 1.6 (Seperable equation)**.** A differential equation of the form

$$x' = f(x)g(t)$$

is called a seperable equation. We can obtain x through the following procedure:

$$\frac{dx}{dt} = f(x)g(t)$$
$$\int \frac{1}{f(x)}\frac{dx}{dt}dt = \int g(t)dt$$
$$\int \frac{1}{f(x)}dx = \int g(t)dt + C$$

The final form of the seperable equation is made possible by the chain rule, and a helpful step forward towards finding the solution is

$$e^{\int g(t)dt} = f(x) + c.$$

Definition 1.7 (Linear equation)**.** A differential equation of the form

$$x' + p(t)x = q(t)$$

is called a first-order linear equation[3]. If a first-order equation can not be put into this form, the equation is called nonlinear.

[3] This is also called the normal form of a first-order linear equation.

Definition 1.8 (Homogeneous equation)**.** A first-order linear differential equation is called homogeneous[4] if it is of the form

$$x' + p(t)x = 0.$$

[4] A homogenous equation is seperable.

The solution is

$$Ce^{-\int p(t)dt}.$$

Definition 1.9 (Integrating factor)**.** A function $\mu(t)$ exists such that

$$\mu(t)(x' + p(t)x) = (\mu(t)x)'.$$

The function $\mu(t)$ is called an integrating factor and is given by

$$\mu(t) = e^{\int p(t)dt}$$

This can be used to solve linear equations by multiplying both sides by the integrating factor.

Definition 1.10 (Autonomous equation)**.** An autonomous differential equation is a differential equation with no explicit time dependence, i.e.

$$\frac{dx}{dt} = f(x).$$

As described above, constant solutions to an autonomous equation are called steady-state or equilibrium solutions.

Definition 1.11 (Stable and unstable equilibrium)**.** For stable equilibrium solutions, solutions with values of x close to the phase-line converge to the phase line. For unstable equilibrium solutions are not stable.[5] The roots of $f(x) = 0$ are the equilibrium solutions.

[5] If solutions near the phase line converge or diverge depending on how they approach, the solution is semi-stable. If all perturbations converge to the phase line, the solution is globally stable.

Theorem 1.2. Let x^* be an isolated critical point, or equilibrium, for the autonomous equation

$$\frac{dx}{dt} = f(x).$$

If $f'(x^*) < 0$, then x^* is stable. If $f'(x^*) > 0$, then x^* is unstable. If $f'(x^*) = 0$ then higher derivatives must be analysed to find information about stability.

Theorem 1.3 (Existence and uniqueness)**.** Assume the function $f(t, x)$ and its partial derivative $f_x(t, x)$ are continuous in a rectangle $a < t < b, c < x < d$. Then, for any value t_0 in $a < t < b$ and x_0 in $c < x < d$, the initial value problem

$$x' = f(t, x)$$
$$x(t_0) = x_0$$

has a unique solution valid on some open interval $a < \alpha < t < \beta < b$ containing t_0.

Second-order linear equations

Remark. Chapter two in the book deals with second-order differential equations of the form

$$ax'' + bx' + cx = f(t). \tag{1}$$

Definition 1.12 (Hooke's law)**.** Let x be displacement from equilibrium and k be the spring constant. Then

$$F_s = -kx$$

Definition 1.13 (Spring-mass equation)**.** The spring-mass equation relates the acceleration of a mass on a spring with the force applied by the spring given by Hooke's law:

$$mx'' = -kx.$$

For initial conditions $x(0) = x_0$ and $x'(0) = 0$ we find $x(t)$ is

$$x(t) = x_0 \cos \sqrt{k/m} t.$$

Definition 1.14 (Damped Oscillator)**.** If there is friction as the mass moves, the frictional force is a function of the velocity x' and the damping coefficient γ

$$F_d = -\gamma x'.$$

Therefore the equation of motion is

$$mx'' = -\gamma x' - kx.$$

Remark. The damped spring-mass equation has the form[6]

$$ax'' + bx' + cx = 0.$$

For such an equation, there are always exactly two independent solutions $x_1(t)$ and $x_2(t)$, and so the general solution $\phi(t)$is of the form

$$\phi(t) = c_1 x_1(t) + c_2 x_2(t).$$

Definition 1.15 (Characteristic equation)**.** To solve equation 1, first note that $x(t) = e^{\lambda t}$ for some constant λ. Substituting $e^{\lambda t}$, we can solve for λ with the characteristic equation[7]

$$a\lambda^2 + b\lambda + c = 0.$$

The values of λ can be real or complex. If $b^2 - 4ac > 0$, then there are two real unequal eigenvalues, and hence there are two indpendent solutions, so the general solution is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

In this case, if $|\lambda_1| = |\lambda_2|$, then the general solution is

$$x(t) = c_1 e^{\alpha t} + c_2 e^{-\alpha t}$$

Which is exponential. If $|\lambda_1| = a$, this equation can be written in terms of hyperbolic functions cosh and sinh as

$$x(t) = c_1 \cosh at + c_2 \sinh at$$

If $b^2 - 4ac = 0$ then the general solution is

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

If $b^2 - 4ac < 0$ then the eigenvalues are complex.

Definition 1.16 (Euler's formula)**.**

$$e^{i\beta t} = \cos \beta t + i \sin \beta t.$$

Theorem 1.4. If $x(t) = g(t) + ih(t)$ is a complex-valued solution of differential equation 1, then its real and imaginary parts $x_1(t) = g(t)$ and $x_2(t) = h(t)$ are real-valued solutions.

Remark. As a consequence of theorem 1.4, if $\lambda_1 = \alpha + i\beta$, then the general solution to equation 1 is

$$x(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t.$$

If $\alpha < 0$, these solutions represent decaying oscillations, and if $\alpha > 0$ then these solutions represent growing oscillations. If $\alpha = 0$ then the solutions are purely oscillatory with frequency β and period $2\pi/\beta$.

[6] An equation of this form is called a homogenour linear equation with constant coefficients.

[7] The roots of this equation are called eigenvalues.