Complex Variables

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Contents

All proofs are original work with hints taken occasionally :). Definitions, theorems, and other material contained within is partially or completely copied, or paraphrased from Complex Analysis by Theodore W. Gamelin.

1 Complex numbers

1.1 Fundamental definitions and identities

Definition 1.1 (Complex number)**.** A complex number is an expression with of the form $z = x + iy$, where x and y are real numbers.

Definition 1.2. Ever complex number $z \neq 0$ has a multiplicative inverse given by

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

Definition 1.3 (Modulus)**.** The modulus of a complex number $z = x + iy$ is the length of the vector (x, y), and is denoted $|z|$.

$$|z| = \sqrt{x^2 + y^2}.$$

Proposition 1.4. For $z, w \in \mathbb{C}$, it follows from the triangle inequality that

$$\begin{aligned} |z + w| &\leq |z| + |w| \\ |z - w| &\geq |z| - |w| \end{aligned}$$

Definition 1.5 (Multiplication)**.** $(x + iy)(u + iv) = xu - yv + i(xv + yu)$.

Definition 1.6 (Complex conjugate)**.** The complex conjugate of a complex number $z = x + iy$ is defined to be $\bar{z} = x - iy$.

Proposition 1.7. For $z, w \in \mathbb{C}$, the following identities hold:

$$\begin{aligned} &\bar{\bar{z}} = z \\ &\overline{z + w} = \bar{z} + \bar{w} \\ &\overline{zw} = \bar{z}\bar{w} \\ &\overline{z\bar{w}} = \bar{z}w \\ &|z| = |\bar{z}| \\ &|z|^2 = z\bar{z} \end{aligned}$$

Proposition 1.8. The real and imaginary parts of z can recovered from z by

$$\begin{aligned} \operatorname{Re} z &= (z + \bar{z})/2 \\ \operatorname{Im} z &= (z - \bar{z})/2i \end{aligned}$$

Definition 1.9 (Triangle inequality)**.** Suppose $a, b \in \mathbb{R}^n$, with $|a|$ the distance from a to 0 under the euclidean metric. Then

$$|a + b| \leq |a| + |b|.$$

Proof: If dot product of two vectors is zero, they are LI. Prove basis exists such that each vector dotted with all vectors in basis is zero (use nullity potentially). if a, b vectors such that $b \cdot a = 0$, then $a \cdot (a + b) = a \cdot a$. If $|a + b| < a$ then $a \cdot (a + b) < a \cdot a$, so $|a + b| \geq |a|$. $|a|, |b|$ are both geq than magnitude of their sides made of a scalar multiple of $a + b$. □

Proposition 1.10. Let $a, b \in \mathbb{C}$. Then

$$|a + b|^2 = |a|^2 + |b|^2 + a\bar{b} + b\bar{a} = |a|^2 + |b|^2 + 2\operatorname{Re} a\bar{b}.$$

Definition 1.11 (Cauchy's inequality)**.**

$$\left| \sum_{i=1}^{n} a_i b_i \right|^2 \leq \sum_{i=1}^{n} |a_i|^2 + \sum_{i=1}^{n} |b_i|^2$$

1.2 Polar representation

Definition 1.12 (Polar representation)**.** The polar representation of a complex number $z = x + iy$ is

$$re^{i\theta} = r(\cos\theta + i\sin\theta).$$

Here $r = |z|$. The *argument* of z is a multivalued function of θ, with

$$\arg z \in \{\theta + 2\pi k \mid k \in \mathbb{Z}\}.$$

The principle value of $\arg z$ denoted $\operatorname{Arg} z$ is the unique member of $\arg z$ such that $-\pi < \operatorname{Arg} z \leq \pi$.

Definition 1.13 (de Moiver's formulae)**.** The identies obtained by equating the imaginary and real parts of the expansions of $e^{in\theta}$ and $(e^{in})^\theta$ are known as de Moivre's formulae, e.g.

$$\begin{aligned}
&e^{2i\theta} = (e^{i\theta})^2\\
&\cos 2\theta + i\sin 2\theta = \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta\\
&\cos 2\theta = \cos^2\theta - \sin^2\theta\\
&\sin 2\theta = 2\cos\theta\sin\theta
\end{aligned}$$

Definition 1.14 (nth root)**.** A number $z \in \mathbb{C}$ is the nth root of $w \in \mathbb{C}$ if $z^n = w$. If $w = \rho e^{i\varphi} \neq 0$, then the nth roots of w are

$$\rho^{1/n} e^{i\varphi/n + 2\pi k/n}, \quad k = 0, 1, \ldots, n-1.$$

This is equivalent to multiplying $\rho^{1/n} e^{i\varphi/n}$ by the nth roots of unity, i.e. all nth roots of 1.

1.3 Exp, log, and power functions

Definition 1.15 (Extended complex plane)**.** The extended complex plane is the complex plane together with the point at infinity, denoted $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

Proposition 1.16. If $z \in \mathbb{C}$ with $z = x + iy$ then

$$e^z = e^x e^{iy} = e^x(\cos y + i\sin y).$$

2 Analytic Functions

2.1 Limits

Definition 2.1 (Limits)**.** If the limit of $f(z)$ as z approaches z_0 is w_0, this means that for all $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

This is written as $\lim_{z \to z_0} f(z) = w_0$.

Lemma 2.2. The limit of a function is unique.

Proof: Suppose $f : \mathbb{C} \to \mathbb{C}$ a function and that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z \to z_0} f(z) = w_1$ with $w_0 \neq w_1$. Because $w_0 \neq w_1$, $|w_0 - w_1| = L > 0$. If we take $\epsilon = L/2$, there exists $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - w_0| < L/2 \wedge |f(z) - w_1| < L/2$. Because $(z - w_1) + (w_0 - z) = w_0 - w_1$, by the triangle inequality $|w_0 - w_1| \leq |f(z) - w_0| + |f(z) - w_1| = L$, a contradiction. □

Theorem 2.3. *Suppose $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$, and that $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Then $\lim_{z\to z_0} f(z) = w_0$ iff*

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

Proof: do dis sheit □

Theorem 2.4. *Suppose that*

$$\lim_{z\to z_0} f(x) = w_0 \text{ and } \lim_{z\to z_0} F(z) = W_0.$$

Then the following is true:

$$\begin{aligned}
&\lim_{z\to z_0} [f(z) + F(z)] = w_0 + W_0, \\
&\lim_{z\to z_0} [f(z)F(z)] = w_0 W_0, \\
&\lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}, \quad W_0 \neq 0.
\end{aligned}$$

Proof: prove dis □