# *Topology*

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## *Open and closed sets*

**Definition 1.1** (Metric)**.** A *metric* on a set $X$ is a real-valued function $d$ on $X \times X$ that has the following properties:

(a) For all $x, y \in X$, $d(x,y) \geq 0$.

(b) $d(x,y) = 0$ iff $x = y$.

(c) For all $x, y \in X$, $d(x,y) = d(y,x)$.

(d) For all $x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$.

**Definition 1.2** (Metric space)**.** A metric space $(X, d)$ is a set $X$ equipped with a metric $d$ on $X$.

**Definition 1.3** (Subspace)**.** If $(X, d)$ is a metric space and $Y$ is a subset of $X$, then the restriction $d'$ of $d$ to $Y \times Y$ is a metric on $Y$, and $(Y, d')$ is called a subspace of $(X, d)$.

*Remark.* Any set $X$ can be made into a discreet metric space by associating with $X$ the metric $d$ defined by

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

**Definition 1.4** (Open ball)**.** The open ball $B(x,r)$ with center $x \in X$ and radius $r > 0$ is defined by

$$B(x,r) = \{y \in X | d(x,y) < r\}.$$

**Definition 1.5** (Interior point)**.** Let $Y$ be a subset of $X$. A point $x \in X$ is an interior point of $Y$ if there exists $r > 0$ such that $B(x,r) \subseteq Y$. The set of interior points of $y$ is the interior of $Y$, and it is denoted by $\text{int}(Y)$.[1]

[1] $\text{int}(Y) \subseteq Y$.

**Definition 1.6** (Open subset)**.** A subset $Y$ of $X$ is open if $\text{int}(Y) = Y$.

**Theorem 1.1.** Any open ball $B(x,r)$ in a metric space $X$ is an open subset of $X$

*Proof:* Suppose $y \in B(x,r)$. Then $d(x,y) < r$, and $0 < r - d(x,y)$. Suppose $z \in B(y, r - d(x,y))$. If follows from the definition of a metric that $d(x,z) \leq d(x,y) + d(y,z)$, so $d(x,z) \leq d(x,y) + (r - d(x,y)) = r$, so $z \in B(x,r)$. □

**Theorem 1.2.** The union of a family of open subsets of a metric space $X$ is an open subset of $X$.

*Proof:* Suppose $\{U_\alpha\}\ \alpha \in A$ a family of open subsets of $X$. If $x \in \bigcup_{\alpha \in A} U_\alpha$, then $\exists \alpha\ (x \in U_\alpha)$, so there exits an open ball $B(x,r)$ such that $B(x,r) \subseteq U_\alpha$. Because $x \in U_\alpha \Rightarrow x \in \bigcup_{\alpha \in A} U_\alpha$, then $B(x,r) \subseteq \bigcup_{\alpha \in A} U_\alpha$. □

**Theorem 1.3.** A subset $U$ of a metric space $X$ is open iff $U$ is a union of open balls in $X$.

*Proof:* Theorem 1.1 and 1.3 prove the left implication. If $U$ is an open subset of $X$, then for all $x \in U$, there exists $r(x) > 0$ such that $B(x, r(x)) \in U$, so $\bigcup_{x \in U} B(x, r(x)) = U$. □

**Theorem 1.4.** The intersection of any finite number of open subsets of a metric space is open.

*Proof:* Suppose $x \in \bigcap_{n=1}^{m} U_n$, a finite union of open subsets of a metric space. Then for all $n$, there exists $r(n) > 0$ such that $B(x, r(n)) \in U_n$. Let $r = \min(r(1) \ldots r(m))$. Then for all $r(n)$ we see $B(x,r) \subseteq B(x, r(n))$ and thus $B(x,r) \subseteq \bigcap_{n=1}^{m} U_n$. □

**Theorem 1.5.** Let $Y$ be a subspace of a metric space $X$. Then a subset $U$ of $Y$ is open in $Y$ iff $U = V \cap Y$ for some open subset $V$ of $X$.

*Proof:* Suppose $x \in V \cap Y$. Then there exists an open ball in $X$ with radius $r(x)$ such that $B(x, r(x)) \subseteq V$, and $x \in Y$. Because $Y \subseteq X$ we see that $Y \cap B(x, r(x)) = \{y \in X \cap Y | d(x,y) < r(x)\} = \{y \in Y | d(x,y) < r(x)\}$, by definition an open ball in $Y$. Trivially $V \cap Y \subseteq \bigcap_{x \in V \cap Y} Y \cap B(x, r(x))$ and by definition the reverse is true.

To prove the converse, suppose $x \in U$. Then there exits an open ball in $Y$ with radius $r(x)$ such that $B(x, r(x)) \in U$. It follows from conclusions reached above that if $B'(x, r(x))$ is open in $X$, then $B'(x, r(x)) \cap Y = B(x, r(x))$. Let $V = \bigcup_{x \in U} B'(x, r(x))$. Then $V \cap Y \subseteq U$, and $x \in U \Rightarrow x \in V$. □

**Definition 1.7** (Adherent point)**.** Let $Y$ be a subset of a metric space $X$. A point $x \in X$ is adherent to $Y$ if for all $r > 0$

$$B(x,r) \cap Y \neq \emptyset$$

**Definition 1.8** (Closure)**.** The closure of $Y$ denoted by $\overline{Y}$, consists of all points in $X$ that are adherent to $Y$.[2]

**Definition 1.9** (Closed subset)**.** The subset $Y$ is closed if $Y = \overline{Y}$.[3]

[2] $Y \subseteq \overline{Y}$.

[3] The empty set $\emptyset$ and $X$ are closed subsets of $X$. Interestingly, $X$ is also open in $X$.