HW3

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Problem 1

Show $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Proof: Suppose A, B are sets and $X = A \cup B$. If $a \in X \setminus A$, it follows from the definition of intersection, union, and difference sets that

$$\begin{aligned} a \in X \setminus A &\Leftrightarrow a \in (A \cup B) \setminus A \\ &\Leftrightarrow (a \in A \vee a \in B) \wedge a \notin A \\ &\Leftrightarrow (a \in A \wedge a \notin A) \vee (a \in B \wedge a \notin A) \\ &\Leftrightarrow a \in B \wedge a \notin A \\ &\Leftrightarrow a \in B \setminus A. \end{aligned}$$

Therefore wlog $X \setminus A = B \setminus A$ and $X \setminus B = A \setminus B$. Because $A \subseteq X$ and $B \subseteq X$, we can use De Morgan's laws to see $(A \setminus B) \cup (B \setminus A) = (X \setminus B) \cup (X \setminus A) = X \setminus (A \cap B) = (A \cup B) \setminus (A \cap B)$. □

Problem 2

Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are bijective, then so is $g \circ f$, and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof: Suppose $f : A \to B$ and $g : B \to C$ are bijective functions. If $a, a' \in A$ with $a' \neq a$, it follows from the bijectivity of f and g that $f(a) \neq f(a')$ and thus $g(f(a)) \neq g(f(a'))$. Therefore $g \circ f$ is injective. Because f and g are surjective, $g^{-1}(C) = B$ and $f^{-1}(g^{-1}(C)) = A$, so $g \circ f$ is surjective and thus bijective. Every value in the codomain of f and g is mapped to by exactly one element in the domain of their respective functions. As a result of this the inverse functions of f and g exist and are given by

$$\begin{aligned} f^{-1} &: B \to A, \quad f(a) \mapsto a, \\ g^{-1} &: C \to B, \quad g(b) \mapsto b. \end{aligned}$$

These functions are injective as stated above, and surjective due to the fact that f and g are defined on their entire domain. Following the reasoning above, $f^{-1} \circ g^{-1}$ is a bijective function, with $g(f(f^{-1}(g^{-1}(c)))) = c$ for $c \in C$. Therefore $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

Problem 3

Let X, Y, Z be sets. Then X has equal cardinality with X. If X has equal cardinalit with Y, then Y has equal cardinality with X. If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.

Proof: Suppose for a function $f : A \to B$ an inverse $f^{-1} : B \to A$ exists. If f not injective, then there exists $a, a' \in A$ with $a \neq a'$ such that $f(a) = f(a')$. But $f^{-1}(f(a)) \neq f^{-1}(f(a'))$, which contradicts the definition of a function. Because f^{-1} is defined on every $b \in B$, and is the unique element for which f maps to from some $a'' \in A$, then f surjective, and is thus bijective

The inverse function of the identity function on X is the identity function, therefore a bijection exists between X and X and $|X| = |X|$. If X has equal cardinality with Y, then a bijection f exists from X to Y, and thus an inverse function f^{-1} exists from Y to X. This inverse function maps each element in the codomain of the original function to a unique element in the domain from which f maps to that codomain element. This makes f^{-1} injective, and surjectivity follows from the fact that f is defined on its entire domain. Therefore $f^{-1} Y \to X$ is bijective and $|Y| = |X|$. If $|X| = |Y| = |Z|$, the composition of bijective functions is bijective, so bijective functions f from X to Y and g from Y to Z exist. Define a function $g \circ f : X \to Z$ by $x \mapsto g(f(x))$. It follows from problem 2 that this is a bijective function, so $|X| = |Z|$. $\square$