Differential Equations

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First-Order Differential Equations

Helpful terms and theorems

Definition 1.1 (Order)**.** The order of a differential equation is the order of the highest dericatice appearing in the equation.

Definition 1.2 (Normal form)**.** The normal form of a first-order equation is a function f which relates a function $x = x(t)$ with its first derivative.

$$x' = f(t, x).$$

A function $x = x(t)$ is a solution of this equation on the time interval $I : a < t < b$ if it is differentiabe on I and, when substituted into the equation, it satisfies the equation identically for every $t \in I$, i.e.

$$x'(t) = f(t, x(t)), \text{ for every } t \in I.$$

In other words to check if a function is a solution, substitute the function in question into the differential equation and check that it reduces to an identity.

Definition 1.3 (Initial value problem)**.** Subjecting a differential equation involving $x(t)$ and its derivatives to a condition $x(t_0) = x_0$ is called an initial value problem. The interval of existance of an IVP is the largest time interval where the solution is valid.[1]

[1] A solution to an IVP is called a particular solution.

Definition 1.4 (General solution)**.** The infinite set of solutions of a first-order equation is called the general solution of the equation.

Definition 1.5 (Nullclines and isoclines)**.** The sets of points (t, x) where the slope field is zero are called nullclines[2], i.e. where

[2] These constant solutions $f(t, x) = k$ for some constant k are called equilibrium solutions.

$$x' = f(t, x) = 0.$$

The set of points t, x where $f(t, x) = k$ for some constant k are called isoclines. When we say set of points, we mean the non-empty pre image $\{k\}$.

Theorem 1.1 (Fundamental theorem of calculus)**.**

$$\frac{d}{dt}\int_a^t g(s)ds = g(t)$$

Theorem 1.2 (Existence and uniqueness)**.** Assume the function $f(t,x)$ and its partial derivative $f_x(t,x)$ are continuous in a rectangle $a < t < b, c < x < d$. Then, for any value t_0 in $a < t < b$ and x_0 in $c < x < d$, the initial value problem

$$x' = f(t,x)$$
$$x(t_0) = x_0$$

has a unique solution valid on some open interval $a < \alpha < t < \beta < b$ containing t_0.

Definition 1.6 (Integral curve)**.** After simplifying a differential equation so that it is in terms of x and t, we obtain a one-parameter family of curves $\phi(t,x) = C$ in the t, x plane, consisting of the pre-images of $\phi(t,x)$ under {C}. These so-called integral curves define implicit solutions of the equation. Explicit solutions are the curves for particular values of C.

Seperable equations

Definition 1.7 (Seperable equation)**.** A differential equation of the form

$$\frac{dx}{dt} = f(x)g(t)$$

is called a seperable equation. We can obtain x through the following procedure:

$$\begin{aligned} \frac{dx}{dt} &= f(x)g(t) \\ \int \frac{1}{f(x)}\frac{dx}{dt}dt &= \int g(t)dt \\ \int \frac{1}{f(x)}dx &= \int g(t)dt + C \end{aligned}$$

The final form of the seperable equation is made possible by the chain rule, and a helpful step forward towards finding the solution is

$$e^{\int g(t)dt} = f(x) + c.$$

Equations where x' is related to a non-identity function of x can not utilize the quick natural log method in equation 1.

Definition 1.8 (Homogeneous equation)**.** A first-order linear differential equation is called homogeneous[3] if it is of the form

$$x' + p(t)x = 0.$$

The solution is

$$x = Ce^{-\int p(t)dt}. \tag{1}$$

[3] A homogenous equation is seperable.

Definition 1.9 (Autonomous equation)**.** An autonomous differential equation is a differential equation with no explicit time dependence, i.e.

$$\frac{dx}{dt} = f(x).$$

As described above, constant solutions to an autonomous equation are called steady-state or equilibrium solutions.

Definition 1.10 (Stable and unstable equilibrium)**.** For stable equilibrium solutions, solutions with values of x close to the phase-line converge to the phase line. For unstable equilibrium solutions are not stable.[4] The roots of $f(x) = 0$ are the equilibrium solutions.

[4] If solutions near the phase line converge or diverge depending on how they approach, the solution is semi-stable. If all perturbations converge to the phase line, the solution is globally stable.

Theorem 1.3. Let x^* be an isolated critical point, or equilibrium, for the autonomous equation

$$\frac{dx}{dt} = f(x).$$

If $f'(x^*) < 0$, then x^* is stable. If $f'(x^*) > 0$, then x^* is unstable. If $f'(x^*) = 0$ then higher derivatives must be analysed to find information about stability.

Remark. As a recap, both homogenous and autonomous equations are seperable, but seperable equations are not necessarily either of these. Their forms are

1. $x' + f(x)g(t) = 0 \qquad$ (seperable)
2. $x' + p(t)x = 0 \qquad$ (homogenous)
3. $x' + f(x) = 0 \qquad$ (autonomous)

It should also be noted that seperable equations are not necessarily linear.

Remark. A technique for solving seperable IVP's is to unilize definite integrals during the integration step. This involves taking the definite integral of $f(x)$ starting at x_0, and the definite integral of $g(t)$ starting at t_0, i.e.

$$\int_{x_0}^{x} \frac{1}{f(y)} dy = \int_{t_0}^{t} g(s) ds.$$

This works by adjusting the constant of integration of both sides so that they are equal under the initial condition $x(t_0) = x_0$.

Non-seperable equations

Definition 1.11 (Linear equation)**.** A differential equation of the form

$$x' + p(t)x = q(t) \tag{2}$$

is called a first-order linear equation[5]. If a first-order equation can not be put into this form, the equation is called nonlinear.

[5] This is also called the normal form of a first-order linear equation.

Definition 1.12 (Forcing term)**.** The term $q(t)$ in equation 2 is called the forcing term, or source term.

Definition 1.13 (Integrating factor)**.** A function $\mu(t)$ exists such that

$$\mu(t)(x' + p(t)x) = (\mu(t)x)'.$$

The function $\mu(t)$ is called an integrating factor and is given by

$$\mu(t) = e^{\int p(t)dt}$$

This can be used to solve linear equations by multiplying both sides by the integrating factor.

Theorem 1.4 (Structure)**.** Consider the normal form of a first order linear equation

$$x' + p(t)x = q(t).$$

The general solution $x(t)$ is the sum of the general solution to the homogeneous equation plus any solution to the nonhomogeneous equation. i.e.

$$x(t) = x_h(t) + x_p(t),$$

where

$$x_h(t) = Ce^{-P(t)}, \quad x_p(t) = e^{-P(t)} \int q(t)e^{P(t)}dt.$$

Therefore, the solution consists of two parts, the transient (homogenous) solution $x_h(t)$ and the steady-state (particular) solution $x_p(t)$.

Definition 1.14 (Bifurcation)**.** Bifurcation is said to occur when there is a significant change in the character of the equilibrium solutions, as the bifurcation parameter h changes. Such a parameter could be the harvesting rate of a fish population in an environment with a set carrying capacity. Bifurcation diagrams plot the equilibium solutions x^* on the y-axis vs the bifurcation parameter h on the x-axis.

Second-order linear equations

Definition 1.15 (Linear equation)**.** The normal form of a second-order linear differential equation is

$$ax'' + bx' + cx = f(t).$$

In some equations b is the damping coefficient, and c the spring constant.

Homogeneous equations

Definition 1.16 (Homogeneous linear equation with constant coefficients)**.**

$$ax'' + bx' + cx = 0. \tag{3}$$

Definition 1.17 (Hooke's law)**.** Let x be displacement from equilibrium and k be the spring constant. Then

$$F_s = -kx$$

Definition 1.18 (Spring-mass equation)**.** The spring-mass equation relates the acceleration of a mass on a spring with the force applied by the spring given by Hooke's law:

$$mx'' = -kx.$$

For initial conditions $x(0) = x_0$ and $x'(0) = 0$ we find $x(t)$ is

$$x(t) = x_0 \cos \sqrt{k/m} t.$$

Definition 1.19 (Damped Oscillator)**.** If there is friction as the mass moves, the frictional force is a function of the velocity x' and the damping coefficient γ

$$F_d = -\gamma x'.$$

Therefore the equation of motion is

$$mx'' = -\gamma x' - kx.$$

Remark. The damped spring-mass equation has the form[6]

$$ax'' + bx' + cx = 0.$$

[6] An equation of this form is called a homogenous linear equation with constant coefficients.

For such an equation, there are always exactly two independent solutions $x_1(t)$ and $x_2(t)$, and so the general solution $\phi(t)$is of the form

$$\phi(t) = c_1 x_1(t) + c_2 x_2(t).$$

Definition 1.20 (Characteristic equation)**.** To solve equation 3, first note that $x(t) = e^{\lambda t}$ for some constant λ. Substituting $e^{\lambda t}$, we can solve for λ with the characteristic equation[7]

[7] The roots of this equation are called eigenvalues.

$$a\lambda^2 + b\lambda + c = 0.$$

The values of λ can be real or complex. If $b^2 - 4ac > 0$, then there are two real unequal eigenvalues, and hence there are two indpendent solutions, so the general solution is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

In this case, if $|\lambda_1| = |\lambda_2|$, then the general solution is

$$x(t) = c_1 e^{\alpha t} + c_2 e^{-\alpha t}$$

Which is exponential. If $|\lambda_1| = a$, this equation can be written in terms of hyperbolic functions cosh and sinh as

$$x(t) = c_1 \cosh at + c_2 \sinh at$$

If $b^2 - 4ac = 0$ then the general solution is

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

If $b^2 - 4ac < 0$ then the eigenvalues are complex.

Definition 1.21 (Euler's formula)**.**

$$e^{i\beta t} = \cos \beta t + i \sin \beta t.$$

Theorem 1.5. If $x(t) = g(t) + ih(t)$ is a complex-valued solution of differential equation **??**, then its real and imaginary parts $x_1(t) = g(t)$ and $x_2(t) = h(t)$ are real-valued solutions.

Remark. As a consequence of theorem 1.5, if $\lambda_1 = \alpha + i\beta$, then the general solution to equation **??** is

$$x(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t.$$

If $\alpha < 0$, these solutions represent decaying oscillations, and if $\alpha > 0$ then these solutions represent growing oscillations. If $\alpha = 0$ then the solutions are purely oscillatory with frequency β and period $2\pi/\beta$.

Definition 1.22 (Phase-amplitude form)**.** The general solution

$$x(t) = c_1 \cos \beta t + c_2 \sin \beta t$$

can be written as

$$A \cos(\beta t - \rho).$$

The constants A and ρ are related to c_1 and c_2 by

$$A = \sqrt{c_1^2 + c_2^2}, \qquad \rho = \arctan \frac{c_2}{c_1}.$$

A is the amplitude and ρ is the phase. If $c_1 < 0$, then we add π to ρ.

Definition 1.23 (Damping)**.** Suppose the motion of a mass-spring system is governed by the equation

$$mx'' + \gamma x' + kx = 0, \qquad m, \gamma, k > 0.$$

If $\gamma^2 - 4mk > 0$, the eigenvalues are real, distinct, negative, and the sytem is overdamped. If $\gamma^2 = 4mk$, the eigenvalues are real, equal, and negative, and the system is critically damped. If $\gamma^2 - 4mk < 0$, the eigenvalues are complex, have negative real part, and the system is underdamped.[8]

[8] If the system is not underdamped, we say it decays without oscillations. If it is underdamped, we say it oscillates with decay.

Definition 1.24 (Envelope)**.** An envelope of a planar family of curves is a curve that is tangent to each member of the family at some point

Nonhomogeneous equations

Definition 1.25 (Nonhomogeneous equation)**.** A nonhomogeneous equation is of the form

$$ax'' + bx' + cx = f(t).$$

The term $f(t)$ is called the forcing term.

Theorem 1.6 (Structure theorem)**.** The general solution of the nonhomogeneous equation 1.25 is given by the sum of the general solution to the homogeneous equation 3 and any specific solution to the nonhomogeneous equation. In other words

$$x(t) = c_1 x_1(t) + c_2 x_1(t) + x_p(t).$$

Definition 1.26 (Undetermined coefficients)**.** Guess the form of $x_p(t)$ from the form of the source term $f(t)$. Some guesses include

Form of source function $f(t)$	Trial form of particular solution $x_p(t)$
α	A
$\alpha^{\beta t}$	$Ae^{\beta t}$
polynomial of degree n	$A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0$
$\alpha \sin \omega t$; $\alpha \cos \omega t$	$A \sin \omega t + B \cos \omega t$
$\alpha e^{rt} \sin \omega t$; $\alpha e^{rt} \cos \omega t$	$e^{rt}(A \sin \omega t + B \cos \omega t)$

If a term in the initial guess for a particular solution x_p is not linearly independent from the homogeneous solution, then modify the guess by multiplying by the smallest power of t that eliminates linear dependence.

Definition 1.27 (Beats)**.** A system exhibits the phenomenon of beats when a high frequency is modulated by a low frequency. This occurs when the frequency of the homogeneous equation is different then that of the forcing function. In undamped systems

$$x'' + \omega_0 t = Acos(\omega t)$$

Beats occur when $\omega_0^2 \neq \omega$. Otherwise resonance occurs.

Laplace transforms

Definition 1.28 (Laplace transform)**.** Let $x = x(t)$ be a function defined on the interval $0 \leq t \leq \infty$. The Laplace transform of $x(t)$ is the function $X(s)$ defined by

$$X(s) = \int_0^\infty x(t) e^{-st} dt,$$

Provided the improper integral exists, meaning

$$\lim_{b\to\infty}\int_0^b x(t)e^{-st}dt \text{ exists.}$$

Often, the Laplace transform is repersented in function notation,

$$\mathcal{L}[x(t)](s) = X(s) \text{ or } \mathcal{L}[x] = X(s).$$

In this context, t, x are called time domain variables, and s, X are called transform domain variables.

Theorem 1.7. The Laplace transform is a linear operation.

Remark. There are two conditions that guarantee existence of a Laplace transform for a function. first, we require that $f(t)$ not grow too fast, i.e. if $M > 0$ and r are constants then

$$|f(t)| \leq Me^{rt}$$

for all $t > t_0$. Second, we require that $f(t)$ be piecewise continuous on $0 \leq t < \infty$. This means that on any bounded subinterval of $0 \leq t < \infty$k we assume that $f(t)$ has at most a finite number of simple discontinuities, and any point of discontinuity $f(t)$ has finite left and right limits.

Definition 1.29 (Heaviside function)**.** We define the Heaviside function $H(t)$ by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Its translation by a units to the right is $H(t-a)$, or

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

A useful identity is

$$H(t-a) = \mathcal{L}^{-1}(\frac{1}{s}e^{-as}).$$

Definition 1.30 (Shift property)**.** The Laplace transform of a function times an exponential, $f(t)e^{at}$ is given by

$$\mathcal{L}[f(t)e^{at}] = F(s-a).$$

Definition 1.31 (Switching property)**.** The Laplace transform of a function $f(t)$ that switches on at $t = a$ is given by

$$\mathcal{L}^{-1}[e^{-as}F(s)] = H(t-a)f(t-a).$$

Linear systems

Remark. Consider the second order autonomous[9], homogenous equation

$$mx'' + \gamma x' + kx = 0. \tag{4}$$

This can be re-written in terms of x and $y = x'$:

$$my' = -kx - \gamma y \;\Rightarrow \begin{cases} x' = y \\ y' = -\frac{k}{m}x - \frac{\gamma}{m}y \end{cases}.$$

In fact, we can always reduce a second order linear equation of form 4 be defining $y = x'$ to the form

$$x' = ax + by$$
$$y' = cx + dy.$$

Systems of linear equations of this form[10] can be can be expressed with matrix multiplication as

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \mathbf{x'}(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$
$$\mathbf{x'} = A\mathbf{x}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Due to the nature of this system of differential equation x' and y' must be comprised of a combination of their antiderivatives x and y. The only function which satisfies this requirement is $\alpha \exp(\lambda t)$. Therefore $\mathbf{x}(t) = (\alpha \exp(\lambda t), \beta \exp(\lambda t))^T$. It follows that $\mathbf{x}'$ is linearly dependent with $\mathbf{x}$. Thus solutions for $\mathbf{x}$ are of the form $v \exp(\lambda t)$, where v is an eigenvector of A and λ its corresponding eigenvalue.

Remark. To find the eigenvalues for a 2×2 matrix, find the roots of the following equation:

$$\lambda^2 - (a+d)\lambda + (ad - cb) = 0.$$

In other words,

$$\lambda^2 - \text{tr}(A) + \det(A) = 0.$$

Eigenvectors and values determine the nature of possible solution.

Nonlinear systems

Remark. The phase diagram of a nonlinear system of equations $x' = f(x,y)$ and $y' = g(x,y)$ can be approximated by finding its equilibrium points, and then approximating the behavior near these equilibrium points using taylor series of each partial derivative.

$$\begin{pmatrix} (x - x_0)' \\ (y - y_0)' \end{pmatrix} = \begin{pmatrix} f'(x_0) & f'(y_0) \\ g'(x_0) & g'(y_0) \end{pmatrix} \begin{pmatrix} (x - x_0) \\ (y - y_0) \end{pmatrix}$$

[9] not time dependent

[10] y does not need to be the derivative of x

Definition 1.32 (Conservative force)**.** Obviously, energy is conserved in any closed physical system. For our purposes, we shall consider the system in question to be a mass m with position x, velocity x'. If a force $F = F(x)$ is applied to the mass, the potential energy V of the system can be expressed as a function of the amount of force applied over a distance, V(x). Because potential energy is usually a function of the distance over which a force is applied, if $F = F(x, x')$, this is typically interpreted[11] as a second force on the object which increases/decreases the kinetic energy of the mass. Therefore the force is not conservative. Mathematically, such a second force could be the friction on the mass as it moves, or negative friction. The relationship between newtons second law, and the total energy of a system with a conservative force is derived below:

$$
\begin{aligned}
&F(x) = mx'' \\
&x' = y, \quad y' = \frac{1}{m}F(x) \\
&F(x) = my' \\
&F(x) \cdot y = my' \cdot y \\
&F(x)dx = mydy \\
&\int F(x)dx + C = \int mydy = \frac{m}{2}y^2 \\
&\frac{m}{2}y^2 + V(x) = E
\end{aligned}
$$

[11] Alternatively, $F(x, x')$ could just be a more complicated forcing function.