Real Analysis

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Contents

1 The natural numbers

1.1 Peano axioms

Definition 1.1 (Peano axioms)**.** Using ++ as the successor operation, the natural numbers are defined as follows:

(a) 0 is a natural number.

(b) If n is a natural number, then $n++$ is also a natural number.

(c) For all natural numbers n, $n++ \neq 0$.

Definition 1.2 (Addition of natural numbers)**.** Let m be a natural number. $0 + m \coloneqq m$ and $(n++) + m \coloneqq (n + m)++$.

Proposition 1.3. $m + 0 = m$.
Proof: Let $n \in \mathbb{N}$. $0 + 0 \coloneqq 0$, so by inductive hypothesis $n + 0 = n$. $(n++) + 0 \coloneqq (n + 0)++$, and from the inductive hypothesis equals $n++$. □

Lemma 1.4. For any natural numbers n and m, $n + (m++) = (n + m)++$.
Proof: Suppose $n, m \in \mathbb{N}$. $0+(m++) \coloneqq m++ = (0+m)++$. By inductive hypothesis $n+(m++) = (n+m)++$. From the definition of addition $(n++)+(m++) = (n+(m++))++$ and from the inductive hypothesis $n + (m++) = (n + m)++$ so we have

$$\begin{aligned}(n++) + (m++) &= (n + (m++))++ \\ &= ((n + m)++)++ \\ &= ((n++) + m)++\end{aligned}$$

□

Proposition 1.5 (Commutivity of addition)**.** For $n, m \in \mathbb{N}$, $n + m = m + n$.
Proof: Let $n, m \in \mathbb{N}$. From proposition 1.3, $0+m = m+0$, so by inductive hypothesis $n + m = m + n$. $(n++) + m = (n + m)++$ and from inductive hypothesis this equals $(m + n)++$. From lemma 1.4, this equals $m + (n++)$. □

Proposition 1.6 (Associativity of addition)**.** Let $a, b, c \in \mathbb{N}$. Then $(a+b)+c = a+(b+c)$.
Proof: exercise □

Proposition 1.7 (Cancellation law)**.** Let $a, b, c \in \mathbb{N}$. Iff $a + b = a + c$, then $b = c$.
Proof: If $0 + b = 0 + c$ then from the definition of addition $b = c$. By inductive hypothesis for any $n \in \mathbb{N}$, $n+b = n+c$. $(n++)+b = (n+b)++$ and $(n++)+c = (n+c)++$, so from the inductive hypothesis and the axioms of natural numbers, $(n++)+b = (n++)+c$. □

Definition 1.8 (Positive natural number)**.** A natural number n is said to be positive iff it is not 0.

Definition 1.9 (Ordering of natural numbers)**.** Let $n, m \in \mathbb{N}$. We write $n \geq m$ or $m \geq n$ iff $n = m + a$ for some $a \in \mathbb{N}$.

Proposition 1.10. Let $m_0, m, m' \in \mathbb{N}$, and let $P(x)$ be a property of arbitrary $x \in \mathbb{N}$. Suppose that for each $m \geq m_0$ the following implication holds:

$$\Big(\forall m' \in [m_0, m),\ P(m')\Big) \Rightarrow P(m).$$

Then we can conclude $P(m)$ is true for all natural numbers $m \geq m_0$.

1.2 Multiplication

Definition 1.11 (Multiplication of natural numbers)**.** Let m be a natural number. $0 \times m \coloneqq 0$ and $(n++) \times m \coloneqq (n \times m) + m$.

Lemma 1.12 (Commutivity of multiplication)**.** Let $n, m \in \mathbb{N}$. Then $n \times m = m \times n$.
Proof: exercise □

Lemma 1.13. Let $n, m \in \mathbb{N}$. Then $n \times m = 0$ iff n or m is zero.

Proof: exercise $\square$

Proposition 1.14 (Distributive law)**.** For any natural numbers a, b, c, we have $a(b + c) = ab + ac$.

Proposition 1.15 (Associativity of multiplication)**.** If $a, b, c \in \mathbb{N}$ then $(a \times b) \times c = a \times (b \times c)$.

Proposition 1.16. If a, b are natural numbers such that $a < b$, and c is positive, then $ac < bc$.

Corollary 1.17. Let $a, b, c \in \mathbb{N}$ such that $ac = bc$ and c is non-zero. Then $a = b$.

Proposition 1.18 (Euclid's division lemma)**.** Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.

Definition 1.19 (Exponentiation for natural numbers)**.** Let $m \in \mathbb{N}$. $m^0 := 1$, and $m^{n++} = m^n \times m$.