Advanced Calculus

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Test 1

Definition 1.1 (Relation)**.** A relation between A and B is any subset R of $A \times B$. If $(a, b) \in R$, then we say aRb.

Definition 1.2 (Equivalence Relation)**.** A relation R on a set S is an equivalence relation if it has the following properties for all x, y, z in S:

1. $xRx \qquad$ (reflexive property)
2. $xRy \Rightarrow yRx \qquad$ (Symmetric property)
3. $xRy \wedge yRz \Rightarrow xRz \qquad$ (Transitive property)

A partition of a set S is a collection $\mathscr{P}$ of nonempty subsets such that

1. $x \in S \Rightarrow x \in \bigcup_{A \in \mathscr{P}} A$
2. $\forall A, B \in \mathscr{P}, A \neq B \Rightarrow A \cap B = \varnothing$

Definition 1.3 (Function)**.** Let A and B be sets. A function from A to B is a nonempty relation $f \subseteq A \times B$ that satisfies the following two conditions:

1. $\forall a \in A, \exists b \in B, (a, b) \in f$
2. $(a, b) \in f \wedge (a, c) \in f \Rightarrow b = c$

Definition 1.4 (Upper bound)**.** Let $S \subseteq \mathbb{R}$. If there exists a real number m such that $m \geq s$ for all $s \in S$, then m is an upper bound of S

Definition 1.5 (Maximum)**.** If an upper bound m of S is a member of S, then m is called the maximum of S.

Definition 1.6 (Supremum)**.** Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then the least upper bound of S is called the supremum. Thus $m = \sup S$ iff

1. $\forall s \in S, m \geq s$
2. $m' < m \Rightarrow \exists s' \in S, s' > m'$

Axiom 1.1 (Completeness of $\mathbb{R}$)**.** Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound, i.e. $\sup S$ exists.

Definition 1.7 (Open and closed set)**.** Let $S \subseteq \mathbb{R}$. If $\operatorname{bd} S \subseteq S$, then S is said to be closed. If $\operatorname{bd} S \subset \mathbb{R} \setminus S$, then S is said to be open.

Definition 1.8 (Accumulation point)**.** Let S be a subset of $\mathbb{R}$. A point x in $\mathbb{R}$ is an accumulation point of S if every deleted neighborhood of x contains a point of S.

Test 2

Definition 1.9 (Convergence)**.** A sequence (s_n) is said to converge to the real number s provided that

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N \Rightarrow |s_n - s| < \epsilon.$$

If (s_n) converges to s, then s is called the limit of the sequence (s_n), and we write $\lim_{n\to\infty} s_n = s$. If a sequence does not converge it diverges.

Theorem 1.1. Let (s_n) and (a_n) be sequences of real numbers and let $s \in \mathbb{R}$. If for some $k > 0$ and some $m \in \mathbb{N}$ we have

$$|s_n - s| \leq k|a_n|, \quad \text{for all } n \geq m,$$

and if $\lim a_n = 0$, then it follows that $\lim s_n = s$.

Theorem 1.2. Every convergent sequence is bounded.

Theorem 1.3. If a sequence converges, its limit is unique.

Definition 1.10 (Monotone sequence)**.** A sequence (s_n) of real numbers is increasing if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$ and is decreasing if $s_n \geq s_{n+1}$ for all $n \in N$. A sequence is monotone if it is increasing or decreasing.

Definition 1.11 (Liminf and limsup)**.** Let (s_n) be a bounded sequence. A subsequential limit of (s_n) is any real number that is the limit of some subsequence of (s_n). If S is the set of all subsequential limits of (s_n), then we define the limit superior of (s_n) to be

$$\limsup s_n = \sup S.$$

The limit inferior of (s_n) is

$$\liminf s_n = \inf S.$$

Test 3

Definition 1.12 (Limit)**.** Let $f : D \to \mathbb{R}$, c be an accumulation point of D, and $x \in D$. We say that a real number L is a limit of f at c if

$$\forall \epsilon > 0\, \exists \delta > 0 \left(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon\right)$$

Definition 1.13 (Right-hand limit)**.** Let $f : \mathbb{R} \to \mathbb{R}$, a be an accumulation point of (a, b), and $x \in (a, b)$. L is a right hand limit of f, denoted $\lim_{x \to a^+} f(x) = L$ if $g : (a, b) \to \mathbb{R}$ with $g\big((a, b)\big) = f\big((a, b)\big)$ and $\lim_{x \to a} g(x) = L$.

Definition 1.14 (Continuity)**.** Let $f : D \to \mathbb{R}$ and let $c \in D$. We say that f is continuous at c if

$$\forall \epsilon > 0, \exists \delta > 0, \big(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon\big).$$

If f is continuous at each point of a subset S of D, then f is said to be continuous on S. If f is continuous on its domain D, then f is said to be a continuous function.

Theorem 1.4. Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.

Proof: Let $\mathscr{B}$ be an open cover of $f(D)$, and let $U \in \mathscr{B}$. Suppose to the contrary that $f^{-1}(U)$ is not open in D. Then there exists a sequence (x_n) in $(f^{-1}(U))^c$ which converges to a point a in $f^{-1}(U)$. Because f is continuous we know that $\lim_{n \to \infty} f(x_n) = f(a)$, a contradiction because $(f(x_n))$ is a sequence in U^c. From this result define an open cover of D, $\mathscr{T} = \{f^{-1}(U) \mid U \in \mathscr{B}\}$. If a finite subcover of $\mathscr{T}$ exists, then clearly a finite subcover of $f(D)$ exists. $\square$

Theorem 1.5 (Intermediate value theorem)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. If k is any value between $f(a)$ and $f(b)$ then there exists $c \in (a, b)$ such that $f(c) = k$.

Definition 1.15 (Uniform continuity)**.** Let $f : D \to \mathbb{R}$. We say that f is uniformly continuous on D if

$$\forall \epsilon > 0, \exists \delta > 0 \, \big(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon\big)$$

Theorem 1.6. Suppose $f : D \to \mathbb{R}$ is continuous on compact set D. Then f is uniformly continuous on D.

Theorem 1.7. Let $f : D \to \mathbb{R}$ be uniformly continuous on D and suppose that (x_n) is a Cauchy sequence in D. Then $(f(x_n))$ is a Cauchy sequence.

Theorem 1.8. A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b) iff it can be extended to a function $\overline{f}$ that is continuous on $[a, b]$.

Definition 1.16 (differentiability)**.** Let f be a real-valued function defined on an open interval I containing the point c. We say that f is differentiable at c if the limit

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists and is finite. We denote the derivative of f at c by $f'(c)$ so that

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}.$$

If the function f is differentiable at each point of the set $S\subseteq I$, then f is said to be differentiable at each point of the set $S\subseteq I$, then f is said to be differentiable on S, and the function $f' : S\to\mathbb{R}$ is called the derivative of f on S.

Theorem 1.9. If f is differentiable on an open interval (a,b) and if f assumes its maximum or minimum at a point $c\in(a,b)$, then $f'(c)=0$.

Theorem 1.10 (Rolle's theorem)**.** Let f be a continuous function on $[a,b]$ that is differentiable on (a,b) and such that $f(a)=f(b)$. Then there exists at least one point c in (a,b) such that $f'(c)=0$.

Theorem 1.11 (MVT)**.** Let f be a continuous function on $[a,b]$ that is differentiable on (a,b). Then there exists at least one point $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Theorem 1.12 (IVT for derivatives)**.** Let f be differentiable on $[a,b]$ and suppose that k is a number between $f'(a)$ and $f'(b)$. Then there exists a point $c\in(a,b)$ such that $f'(c)=k$.

Theorem 1.13 (Cauchy MVT)**.** Let f and g be functions that are continuous on $[a,b]$ and differentiable on (a,b). Then there exists at least one point $c\in(a,b)$ such that

$$[f(b)-f(a)]g'(c)=[g(b)-g(a)]f'(c).$$

Theorem 1.14 (Chain rule)**.** Let I and J be intervals in $\mathbb{R}$, let $f : I\to\mathbb{R}$ and $g : j\to\mathbb{R}$, where $f(I)\subseteq J$, and let $c\in I$. If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g\circ f$ is differentiable at c and

$$(g\circ f)'(c)=g'(f(c))\cdot f'(c)$$

Theorem 1.15 (L'Hospital's rule)**.** Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $c\in[a,b]$ and that $f(c)=$

$g(c) = 0$. Suppose also that $g'(x) \neq 0$ for $x \in U$, where U is the intersection of (a, b) and some deleted neighborhood of c. If

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L \quad \text{with } L \in \mathbb{R}$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

Definition 1.17 (limit)**.** Let $f : (a, \infty) \to \mathbb{R}$. We say that $L \in \mathbb{R}$ is the limit of f as $x \to \infty$, and we write

$$\lim_{x \to \infty} f(x) = L,$$

provided that for all $\epsilon > 0$ there exists a real number $N > a$ such that $x > N$ implies that $|f(x) - L| < \epsilon$.

Definition 1.18. Let $f : (a, \infty) \to \mathbb{R}$. We say that f tends to ∞ as $x \to \infty$ and we write

$$\lim_{x \to \infty} f(x) = \infty,$$

provided that for all $\alpha \in \mathbb{R}$, there exists a real number $N > a$ such that $x > N$ implies that $f(x) > \alpha$.

Theorem 1.16 (lhop rule2)**.**

Theorem 1.17 (Taylor's theorem)**.**