Real Analysis

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Naturals and Integers

Axiom 1.1 (Principle of mathematical induction)**.** Let $P(n)$ be an property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that we have the implication that whenever $P(n)$ is true, $P(n+1))$ is true. Then $P(n)$ is true for every natural number n.

Definition 1.1 (Peano Axioms)**.** The Peano axioms include the principle of induction and definition of equality, I just included the ones that define the natural numbers under this definition.

1. 0 is a natural number.
2. If n is a natural number, than $S(n)$ (e.g. $n++$) is a natural number.
3. For all n, if n is a natural number than $S(n) \neq 0$.
4. For all n, if $S(n) = S(m)$, than $n = m$.

Definition 1.2 (Addition)**.**

1. $0 + m = m$
2. $(n++) + m = (n+m)++$

Lemma 1.1 (Cancellation)**.** If $a + b = a + c$, then $b = c$.

Proof: Suppose $0 + b = 0 + c$. Then $b = c$. By inductive hypothesis, $a + b = a + c \Rightarrow b = c$. It follows that:

$$(a++) + b = (a+b)++$$
$$(a+b)++ = (a+c)++$$

$\square$

Lemma 1.2 (Commutivity of Addition)**.** For all natural numbers a, b, we have $a + b = b + a$.

Proof: First, we prove $n + m++ = (n+m)++$. $0 + m++ = (0+m)++$ and by inductive hypothesis $n + m++ = (n+m)++$. It follows:

$$\begin{aligned}(n++) + m++ &= (n + m++)++\\ &= ((n+m)++)++\\ &= ((n++)+m)++\end{aligned}$$

Next, we prove $0 + m = m + 0$. $0 + 0 = 0 + 0$, and by inductive hypothesis $n + 0 = 0 + n$. It follows:

$$\begin{aligned}(n++) + 0 &= (n+0)++ \\ &= (0+n)++ \\ &= (0+n++)\end{aligned}$$

Lastly, we prove commutivity of addition. $0 + m = m + 0$ and by inductive hypothesis $n + m = m + n$. It follows:

$$\begin{aligned}(n++) + m &= (n+m)++ \\ &= (m+n)++ \\ &= (m+n++)\end{aligned}$$

□

Definition 1.3 (Ordering of Natural Numbers)**.** Let n and m be natural numbers. We say n is greater than or equal to m, and write $n \geq m$ or $m \geq n$ iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m and write $n > m$ or $m < n$ iff $n \geq m$ and $n \neq m$.

Definition 1.4 (Strong Principle of Induction)**.** Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: If $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true.[1] Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

[1] $P(m_0)$ is true because for m_0 the hypothesis is vacuous.

Definition 1.5 (Multiplication)**.**

1. $0 \times m = 0$

2. $(n++) \times m = (n \times m) + m$

Lemma 1.3 (Multiplication is Commutative)**.** For any natural numbers a, b we have $a \times b = b \times a$.

Proof: $0 \times 0 = 0 \times 0$ and by inductive hypothesis $m \times 0 = 0 \times m$. It follows that:

$$\begin{aligned}(m++) \times 0 &= (m \times 0) + 0 \\ &= (0 \times m) + 0 \\ &= 0 \times m\end{aligned}$$

So $m \times 0 = 0 \times m$. Using this fact as a base case, by inductive hypothesis $n \times m = m \times n$. It follows that:

$$(n++) \times m = (n \times m) + m = (m \times n) + m$$

We see that $m \times n++ = (n++) + (n++) + \ldots + (n++)$, with $n++$ being added to itself m times as a consequence of the definition of multiplication. It follows from the commutivity of addition that:

$$\begin{aligned}(n++) + (n++) + \ldots + (n++) &= n + n + \ldots + n + (++) + (++) + \ldots + (++) \\ &= (m \times n) + (m \times 1) \\ &= (m \times n) + m \\ &= (n \times m) + m \\ &= (n++) \times m\end{aligned}$$

□

Lemma 1.4 (Multiplication is Distributive). $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$

Proof: $(b+c)a = ba + ca$ follows from the proof that $a(b+c) = ab + ac$ and the commutivity of multiplication. $a(b+0) = ab + 0 = ab + a0$, and by inductive hypothesis $a(b+c) = ab + ac$. It follows:

$$\begin{aligned}a(b+c++) &= a((b+c)++) \\ &= a(b+c) + a \\ &= ab + ac + a \\ ab + a(c++) &= ab + ac + a\end{aligned}$$

□

Definition 1.6 (Euclidean Algorithm). Let n be a natural number, and let q be a positive number. Then there exists natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.[2]

[2] We can divide a natural number n by q to obtain a quotient m and a remainder r.

Definition 1.7 (Exponentiation). Let m, n be natural numbers.

$$\begin{aligned}m^0 &= 1 \\ m^{n++} &= m^n \times m\end{aligned}$$

Definition 1.8 (Integers). An integer is an expression of the form $a—b$, with $a, b \in \mathbb{N}$. Two integers are considered to be equal, $a—b = c—d$ iff $a + d = c + b$.[3]

[3] The symbol "—" is a meaningless placeholder. When subtraction is defined we will see that $a—b = a - b$.

Definition 1.9 (Sum and Product of Integers). The sum of two integers, $a—b + c—d$ is defined:

$$(a—b) + (c—d) = (a+c)—(b+d)$$

The product of two integers, $a—b \times c—d$ is defined:

$$(a—b) \times (b—c) = (ac+bd)—(ad+bc)$$

Definition 1.10 (Negation of integers). If $(a—b)$ is an integer, we define the negation $-(a—b)$ to be the integer $b—a$.

Remark. We may identify the natural numbers with integers by setting $n \equiv n—0$;

Theorem 1.1 (Trichotomy of integers)**.** Let x be an integer. Then exactly one of the following three statments is true: (a) x is zero; (b) x is equal to a positive natural number; or (c) x is the negation $-n$ of a positive natural number n.[4]

[4] If n is a positive natural number, we call $-n$ a negative integer.

Definition 1.11 (Subtraction)**.** The subtraction $x - y$ of two integers is defined:

$$x - y = x + (-y)$$

Remark. The integers form a commutative ring (adhere to familiar laws of algebra), and are well-defined. For more info see algebra notes.[5]

[5] By well defined we mean that equal inputs produce equal outputs.

Definition 1.12 (Ordering of the Integers)**.** Let n and m be integers. We say that n is greater than or equal to m iff $n = m + a$ for some natural number a. We say that n is strictly greater than m iff $n \geq m$ and $n \neq m$.

Definition 1.13 (Rational Numbers)**.** A rational number is an expression of the form $a//b$, where a and b are integers and $b \neq 0$; $a//0$ is not considered to be a natural number. Two rational numbers are considered to be equal, $a//b = c//d$, iff $ad = cb$.

Definition 1.14 (Sum, Product, and Negation of Rational Numbers)**.** If $a//b$ and $c//d$ are rational numbers, we define their sum:

$$(a//b) + (c//d) = (ad + bc)//(bd)$$

Their product:

$$(a//b) \times (c//d) = (ac)//(bd)$$

And the negation:

$$-(a//b) = (-a)//b$$

Remark. The set of rationals form a field, a stronger classification than a commutative ring. The rationals are also well defined.

Definition 1.15 (Reciprocal of Rationals)**.** If $x = a//b$ is a non-zero rational number, the we define the reciprocal x^{-1} of x to be the rational number:

$$x = a//b$$
$$x^{-1} = b//a$$

Definition 1.16 (Quotient of Rational Numbers)**.** The quotient x/y of two rational numbers x and y provided that y is non-zero, is given by the formula:

$$x/y = x \times y^{-1}$$

Definition 1.17. A rational number x is said to be positive iff we have $x = a/b$ for some positie integers a and b. It is said to be *negative* iff we have $x = -y$ for some positive rational y.

Definition 1.18 (Trichotomy of Rationals)**.** Let x be a rational number. Then exactly one of the following three statements is true: (a) x is equal to 0. (b) x is a positive rational number. (c) x is a negative rational number.

Definition 1.19 (Ordering of rationals)**.** Let x and y be rational numbers. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - y$ is a negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$.

Definition 1.20 (Absolute Value)**.** If x is a rational number, the absolute value $|x|$ of x is defined as follows. If x is positive, then $|x| = x$. If x is negative, then $|x| = -x$. If x is zero, then $|x| = 0$.

Definition 1.21 (Distance)**.** Let x and y be rational numbers. The quantity $|x - y|$ is called the distance between x and y and is sometimes denoted $d(x, y)$.

Definition 1.22 (Triangle Inequality for Rationals)**.** If a, b are rational numbers, then:

$$|a + b| \leq |a| + |b|$$

Proof: Suppose $a, b \in \mathbb{Q}$. If a, b are both positive or zero:

$$|a| + |b| = a + b = |a + b|$$

If a, b are both negative:

$$|a| + |b| = -a - b = |a + b|$$

Wlog we shall prove the case if a is negative and b is positive or zero. Suppose $e, g \geq 0$ and $f, h > 0$. Then we can represent $a = \frac{-e}{f}$ and $b = \frac{g}{h}$, so:

$$\begin{aligned} a + b &= \frac{-eh + gf}{fh} \\ |a| + |b| &= \frac{eh + gf}{fh} \end{aligned}$$

In the case $a + b \geq 0$:

$$\begin{aligned} |a + b| &= \frac{-eh + gf}{fh} \\ &\leq \frac{-eh + gf}{fh} + \frac{2eh}{fh} \\ &= \frac{eh + gf}{fh} \\ &= |a| + |b| \end{aligned}$$

In the case $a+b<0$:

$$\begin{aligned}|a+b| &= \frac{eh-gf}{fh} \\ &\leq \frac{eh-gf}{fh} + \frac{2gf}{fh} \\ &= \frac{eh+gf}{fh} \\ &= |a|+|b|\end{aligned}$$

so $|a+b| \leq |a|+|b|$. □

Definition 1.23 (ϵ-closeness)**.** Let $\epsilon > 0$ be a rational number, and let x, y be rational numbers. We say that y is ϵ-close to x iff we have $d(y,x) \leq \epsilon$.

Corollary 1.1. Let $\epsilon, \delta > 0$. If x and y are $\epsilon - close$ and z and w are $\delta - close$, then xz and yw are $(\epsilon|z| + \delta|x| + \epsilon\delta)$-close.

Proof: First, we can represent $y = x + a$ with $|a| < \epsilon$, and $w = z + b$ with $|b| < \delta$. It follows:

$$\begin{aligned}yw = (x+a)(z+b) &= xz + bx + az + ab \\ d(xz, yw) &= |bx + az + ab| \\ &= |b||x| + |a||z| + |ab| \\ &\leq \delta|x| + \epsilon|z| + \delta\epsilon\end{aligned}$$

□

Definition 1.24 (Exponentiation to a natural number)**.** Let x be a rational number.

$$\begin{aligned}x^0 &= 1 \\ x^{n+1} &= x^n \times x\end{aligned}$$

Definition 1.25 (Exponentiation to a negative number)**.** Let x be a non-zero rational number. Then for any negative integer $-n$, we define $x^{-n} = 1/x^n$.

Da Reals

Definition 1.26 (Sequences)**.** Let m be an integer. A sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is any function from the set $\{n \in \mathbb{Z} \mid n \geq m\}$ to $\mathbb{Q}$.

Definition 1.27 (Bounded Sequences)**.** Let $M \geq 0$ be a rational. A finite sequence $a_1, a_2, \ldots, a_n$ is bounded by M iff $|a_i| \leq M$ for all $1 \leq i \leq n$. An infinite sequence $(a_n)_{n=1}^{\infty}$ is bounded by M iff $|a_i| \leq M$ for all $i \geq 1$.

Lemma 1.5 (Finite Sequences are Bounded)**.**

Proof: Suppose $(a_n)_{n=1}^{k}$ an arbitrary finite sequence. If $k = 1$, then $|a_n| \leq |a_1|$ for all n. By induction hypothesis, the sequence $(a_n)_{n=1}^{k}$ is bounded, and therefore there exists M such that $|a_n| \leq M$ for all n. It follows from the properties of ordering that for all $n \leq k$, $|a_n| < M + |a_{k+1}|$ and $|a_{k+1}| < M + |a_{k+1}|$. Therefore $(a_n)_{n=1}^{k+1}$ is bounded. □

Definition 1.28 (Cauchy Sequence)**.** A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is said to be a cauchy sequence iff for every rational $\epsilon > 0$, there exists an $N \geq 0$ such that $d(a_j, a_k) \leq \epsilon$ for all $j, k \geq N$.

Lemma 1.6 (Cauchy Sequences are Bounded)**.**

Proof: Suppose $(a_n)_{n=1}^{\infty}$ is Cauchy. Then, if $\epsilon \in \mathbb{Q}$ there exists $N \in \mathbb{N}$ such that $i, j \geq N$ implies that $|a_i - a_j| \leq \epsilon$. Because the sequence $(a_n)_{n=1}^{N}$ is finite, it is bounded by a number $\delta \in \mathbb{Q}$. Then $|a_N| \leq \delta$ and it follows that for all $i \geq N$,

$$\begin{aligned} |a_i - a_N| &\leq \epsilon \\ |a_i - a_N| + |a_N| &\leq \epsilon + \delta \\ |a_i| &< \epsilon + \delta \end{aligned}$$

But also $|a_n| \leq \epsilon + \delta$ for all $n \leq N$ so $|a_n| \leq \epsilon + \delta$ for all n. □

Definition 1.29 (Equivalent Sequences)**.** Two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are equivalent iff for each rational $\epsilon > 0$, there exists N such that $n \geq N$ implies $|a_n - b_n| \leq \epsilon$.[6]

[6] Symbolically, this means $\forall \epsilon \exists N \forall n (n \geq N \Rightarrow |a_n - b_n| \leq \epsilon)$.

Definition 1.30 (Real Numbers)**.** A real number is defined to be an object of the form $\text{LIM}_{n\to\infty} a_n$, where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Two real numbers $\text{LIM}_{n\to\infty} a_n$ and $\text{LIM}_{n\to\infty} b_n$ are said to be equal iff $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences.

Remark. Like with the definition of subtraction and rational numbers, $\text{LIM}_{n\to\infty} a_n$ is referred to as the formal limit of the sequence $(a_n)_{n=1}^{\infty}$.

Definition 1.31 (Addition of Reals)**.** Let $x = \text{LIM}_{n\to\infty} a_n$ and $y = \text{LIM}_{n\to\infty} b_n$ be real numbers. Then we define the sum $x + y$ to be $x + y = \text{LIM}_{n\to\infty}(a_n + b_n)$.

Lemma 1.7 (Sum of Cauchy Sequences is Cauchy)**.** Let $x = \text{LIM}_{n\to\infty} a_n$ and $y = \text{LIM}_{n\to\infty} b_n$ be real numbers. Then $x + y$ is also a real number.

Proof: Suppose a_n and b_n are Cauchy sequences. Then for all $\epsilon \in \mathbb{Q}$ there exists $N \in \mathbb{N}$ such that $i, j \geq N$ implies $|a_i - a_j| \leq \epsilon$, and for all $\delta \in \mathbb{Q}$ there exists $M \in \mathbb{N}$ such that $l, m \geq M$ implies

$|b_l - b_m| \leq \epsilon$. If follows that if $q, r \geq \max\{N, M\}$:

$$\begin{aligned}
|(a_q + b_q) - (b_r + a_r)| &= \left| a_q + b_q - \left(a_q + (-a_q + a_r) + b_q + (-b_q + b_r)\right) \right| \\
&= |(a_q - a_r) + (b_q - b_r)| \\
&\leq |a_q - a_r| + |b_q - b_r| \\
&\leq \epsilon + \delta
\end{aligned}$$

□

Definition 1.32 (Multiplication of Reals)**.** Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. Then we define the product xy to be $xy = \mathrm{LIM}_{n\to\infty} a_n b_n$.

Remark. The rationals can be embedded into the reals by identifying each rational number q with the real number $\mathrm{LIM}_{n\to\infty} q$.

Definition 1.33 (Sequences bounded away from zero)**.** A sequence $(a_n)_{n=1}^{\infty}$ of rational numbers is said to be bounded away from zero iff there exists a rational number $c > 0$ such that $|a_n| > c$ for all $n \geq 1$.

Lemma 1.8. Let x be a non-zero real number. Then $x = \mathrm{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is bounded away from zero.

Proof: If x is a non-zero real number, then $x = \mathrm{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$. Because $x \neq 0$, we know from the definition of equivalent sequences that $\exists\alpha\forall\beta\exists n(n \geq \beta \wedge |a_n| > \alpha)$.[7] Because a_n is Cauchy, if we choose $\epsilon < \frac{\alpha}{2}$, then $\exists\delta(i, j \geq \delta \Rightarrow |a_i - a_j| \leq \epsilon)$. It follows $\exists k(k \geq \delta \wedge |a_k| > \alpha)$.[8] If $l \geq \delta$, it follows:

$$\begin{aligned}
|a_k| &> \alpha \\
|a_l - a_k| &\leq \epsilon < \frac{\alpha}{2} \\
|a_l - a_k| &\neq |a_k| \\
a_l &= a_k + (a_l - a_k) \\
|a_l| &> \frac{\alpha}{2}
\end{aligned}$$

The last line can be proven by cases whether $a_l - a_k$ or a_k are negative or positive. So for all $l \geq \delta$ we have $|a_l| > \frac{\alpha}{2}$. An equivalent sequence bounded away from zero can now be easily constructed. □

Definition 1.34 (Reciprocals of Real Numbers)**.** Let x be a non-zero real number. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence bounded away from zero such that $x = \mathrm{LIM}_{n\to\infty} a_n$. Then we define the reciprocal x^{-1} by the formula $x^{-1} = \mathrm{LIM}_{n\to\infty} a_n^{-1}$.

Definition 1.35 (Division of Reals)**.** Division x/y of two real numbers x, y provided y is non-zero is given by the formula $x/y = x \times y^{-1}$.

[7] This is the negation of the definition of two equivalent sequences, with $b_n = 0$.

[8] This works because β is arbitrary.

Definition 1.36 (Positively Bounded Away From Zero)**.** Let $(a_n)_{n=1}^{\infty}$ be a sequence of rationals. We say that this sequence is positively bounded away from zero iff we have a positive rational $c > 0$ such that $a_n > c$ for all $n \geq 1$. You can fill in the blank for what negatively bounded away from zero means I hope.

Definition 1.37 (Positive and Negative Real Numbers)**.** A real number x is said to be positive or negative iff it can be written as $x = \text{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively or negatively bounded away from zero respectively.

Definition 1.38 (Absolute Value)**.** Let x be a real number. We define the absolute value $|x|$ of x to equal x if x is positive, $-x$ when x is negative, and 0 when x is zero.

Definition 1.39 (Ordering of Real Numbers)**.** Let x and y be real numbers. We say that x is greater than y iff $x - y$ is a positive real number, and x is less that y iff $x - y$ is a negative real number. We define $x \geq y$ iff $x > y$ or $x = y$, and similarly define $x \leq y$.

Lemma 1.9 (The Non-negative Reals are Closed)**.** Let $a_1, a_2, \ldots$ be a Cauchy sequence of non-negative rational numbers. Then $\text{LIM}_{n\to\infty} a_n$ is a non-negative real number.[9]

Proof: Suppose to the contrary that $\text{LIM}_{n\to\infty} a_n$ is a negative real number. Then there exists a Cauchy sequence b_n negatively bounded away from zero so that $\text{LIM}_{n\to\infty} a_n = \text{LIM}_{n\to\infty} b_n$. Therefore there exists a negative rational $-c$ such that $b_n < -c$ for all n. It follows from the hypothesis that for all n

$$\begin{aligned} a_n &> 0 \\ -b_n &> c \\ a_n - b_n &> c \end{aligned}$$

Because c is positive, $|a_n - b_n| > c$ for all n. Thus $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are not equivalent. □

[9] The title of the lemma builds on the fact that non-negative rationals are closed under addition.

Corollary 1.2. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be Cauchy sequences of rationals such that $a_n \geq b_n$ for all $n \geq 1$. Then $\text{LIM}_{n\to\infty} a_n \geq \text{LIM}_{n\to\infty} b_n$.

Proof: $\text{LIM}_{n\to\infty} a_n - \text{LIM}_{n\to\infty} b_n = \text{LIM}_{n\to\infty} a_n - b_n$. It follows from lemma 1.9 that because $a_n - b_n \geq 0$, $\text{LIM}_{n\to\infty} a_n - b_n$ is a non-negative real number. □

Lemma 1.10 (Archimedean Property)**.** Let x and ϵ be any positive real numbers. Then there exists a positive integer M such that $M\epsilon > x$.

Definition 1.40 (Upper Bound)**.** Let E be a subset of $\mathbb{R}$, and let M be a real number. We say that M is an upper bound for E, iff we have $x \leq M$ for every element x in E.

Definition 1.41 (Least Upper Bound)**.** Let E be a subset of $\mathbb{R}$, and let M be a real number. We say that M is a least upper bound for E iff:

1. M is an upper bound for E.

2. Any other upper bound M' for E must be larger than or equal to M.[10]

[10] Least upper bounds are unique.

Theorem 1.2 (Existence of Least Upper Bound)**.** Let E be a nonempty subset of $\mathbb{R}$. If E has an upper bound, then it must have exactly one least upper bound.