Fun with elementary gates for quantum computation

In reference [1] it was shown that a set of gates that consists of all one-bit quantum gates and the two-bit exclusive-or gate is universal in the sense that all unitary operations on arbitrary many bits can be expressed as compositions of these gates.

In this notebook we are interested in constructing a gate V such that V^2=X (NOT) gate. Thanks to Lemma 4.3 from the reference [1] V gate can be easily expressed in terms of elementary gates. A cool thing about this is, if we have V gate, we can follow Lemma 6.1 from the same reference and construct a Toffoli gate.
We will now show how to bring these interesting efforts to fruition.

[1] https://arxiv.org/abs/quant-ph/9503016 (https://arxiv.org/abs/quant-ph/9503016)

Import needed libraries

In [1]:
```
%matplotlib inline
from qiskit import QuantumRegister, ClassicalRegister
# Importing standard Qiskit libraries and configuring account
from qiskit import QuantumCircuit, execute, Aer, IBMQ
from qiskit.compiler import transpile, assemble
from qiskit.tools.jupyter import *
from qiskit.visualization import *
# Loading your IBM Q account(s)
provider = IBMQ.load_account()
```

Here we define a function that returns counts for ploting histogram of the measurement results. The input for the function is the backend type, for example 'qasm_simulator'.

In [8]:
```
from qiskit.tools.visualization import plot_histogram

def get_counts(back_end):
    ## First, simulate the circuit
    simulator = Aer.get_backend(back_end)
    job = execute(circuit, backend=simulator, shots=1000000)
    result = job.result()

    ## Then, plot a histogram of the results
    counts = result.get_counts(circuit)
    return counts
```

V^2 = X (NOT gate)

Our constraint is that V^2 = X (NOT gate).

According to Lemma 4.1 from [1] every unitary 2 x 2 matrix can be expressed as

$$\begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix} \cdot \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \cdot \begin{pmatrix} cos(\theta/2) & sin(\theta/2) \\ -sin(\theta/2) & cos(\theta/2) \end{pmatrix} \cdot \begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix}$$

Since

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is an unitary matrix and we demand that V^2 = X, it is easy to see that $\alpha = -\beta = -\pi/2$, $\delta = \pi/4$, and $\theta = \pi/2$. This means that

$$V = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \cdot \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \cdot \begin{pmatrix} cos(\pi/4) & sin(\pi/4) \\ -sin(\pi/4) & cos(\pi/4) \end{pmatrix} \cdot \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = 1/\sqrt{2} \begin{pmatrix} e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{pmatrix}$$

If we use the following notation

$$R_y(\theta) = \begin{pmatrix} cos(\theta/2) & sin(\theta/2) \\ -sin(\theta/2) & cos(\theta/2) \end{pmatrix}$$

which is a rotation by θ around y axis,

$$R_z(\alpha) = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix},$$

a rotation by α around z axis, and

$$Ph(\delta) = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix},$$

a phase shift with respect to δ, we can express V as

$$V = Ph(\pi/4)R_z(-\pi/2)R_y(\pi/2)R_z(\pi/2).$$

According to Lemma 4.3 from [1] for any special unitary matrix W (i.e., with unity determinant and W is member of SU(2)), there exists matrices A, B, and C in SU(2) such that

$$ABC = I$$

and

$$AXBXC = W.$$

In our case

$$W = R_z(-\pi/2)R_y(\pi/2)R_z(\pi/2)$$

and

$$A = R_z(-\pi/2)R_y(\pi/4)$$

$$B = R_y(-\pi/4)R_z(0) = R_y(-\pi/4)$$

and

$$C = R_z(\pi/2).$$

Knowing these 3 matrices is very important since we can use them to make a **control V** gate according to Lemma 5.1 from the same [1] reference. We will now show how to do that.

Control V gate

Our V gate contains a phase gate so we need first to show how to implement the phase gate. It is easy to check that the following matrix product results in the phase gate.

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ e^{i\delta} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ e^{i\delta} & 0 \end{pmatrix} = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix}$$

This means that the phase gate can be implemented as

$$Ph(\delta) = U_1(\delta) X U_1(\delta) X.$$

where

$$U_1(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix}.$$

The phase gate can be coded as the following python function that takes 3 parameters: a quantum circuit, an angle and a qubit.

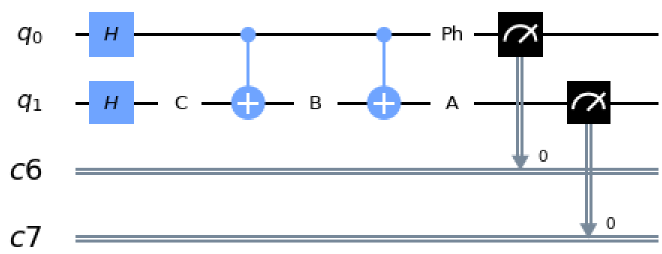
In [ ]:
```
#define phase gate
def Ph(circuit, theta, qubit):
    circuit.x(qubit)
    circuit.u1(theta, qubit)
    circuit.x(qubit)
    circuit.u1(theta, qubit)
```

We can also implement it as an instruction as in the code snippet below.

In [ ]:
```
#define phase gate as an instruction
#delta  is a global variable defined outside the gate
qc_ph = QuantumCircuit(1, name='Ph')
qc_ph.x(0)
qc_ph.u1(delta, 0)
qc_ph.x(0)
qc_ph.u1(delta, 0)
Ph = qc_ph.to_instruction()
```

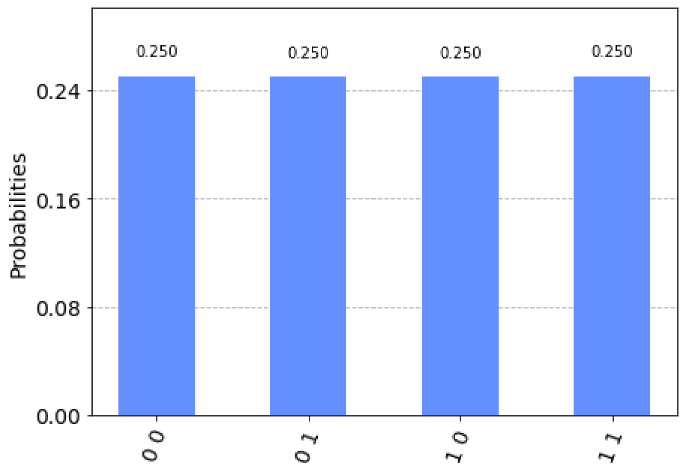
Essentially control V gate is implemented as the following circuit.

In [9]:
```
#define some constants
pi = 3.14159
delta = pi/4

#define A gate as an instruction
qc_A = QuantumCircuit(1, name='A')
qc_A.ry(pi/4, 0)
qc_A.rz(-pi/2, 0)
A = qc_A.to_instruction()

#define B gate as an instruction
qc_B = QuantumCircuit(1, name='B')
qc_B.ry(-pi/4, 0)
B = qc_B.to_instruction()

#define C gate as an instruction
qc_C = QuantumCircuit(1, name='C')
qc_C.rz(pi/2, 0)
C = qc_C.to_instruction()

#define phase gate as an instruction
qc_ph = QuantumCircuit(1, name='Ph')
qc_ph.x(0)
qc_ph.u1(delta, 0)
qc_ph.x(0)
qc_ph.u1(delta, 0)
Ph = qc_ph.to_instruction()

#now assemble respective quantum circuit
## Define a 2-qubit quantum circuit
q = QuantumRegister(2, 'q')
circuit = QuantumCircuit(q)
## We need also 4 classical registers
c = [ ClassicalRegister(1) for _ in range(2) ]
circuit = QuantumCircuit(q)
for register in c:
    circuit.add_register(register)

#initial state of an equal superposition of |0> and |1> states
#so that we have all possible inputs of 2 qubits
circuit.h(q[0])
circuit.h(q[1])

circuit.append(C,[1])    # C gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(B, [1])   # B gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(A, [1])   # A gate acting on q[1]
circuit.append(Ph, [0])  # phase gate acting on q[0] (it really does not matter which qubit we pick)

#finish with measurements
circuit.measure(q[0], c[0])
circuit.measure(q[1], c[1])

## Draw the circuit
%matplotlib inline
circuit.draw(output="mpl")
```



There are 4 possible, equally probable, outputs: 00, 01, 10, or 11 with 1/4 probability.

In [10]:
```
plot_histogram(get_counts('qasm_simulator'))
```



Math part

To understand why we are getting these outputs we need to work out respective math.
Our input state to the control V gate is a state of an equal superposition of 2 qubits.
We start with 2 qubits in 0 state and apply Hadammard gate to each qubit resulting in

$$\Psi_i = H|0> H|0>= 1/\sqrt{2}(|0> + |1>)1/\sqrt{2}(|0> + |1>) = 1/2(|00> + |01> + |10> + |11>).$$

It is not difficult to see that the following are matrix representations of A, B, and C gates.

$$A = \begin{pmatrix} e^{-i\pi/4}cos(\pi/8) & e^{-i\pi/4}sin(\pi/8) \\ -e^{i\pi/4}sin(\pi/8) & e^{i\pi/4}cos(\pi/8) \end{pmatrix}$$
$$B = \begin{pmatrix} cos(\pi/8) & -sin(\pi/8) \\ sin(\pi/8) & cos(\pi/8) \end{pmatrix}$$
$$C = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

Having this information we can easily deduce how these gates act on |0> and |1> qubit states.

$$A|0>= e^{-i\pi/4}cos(\pi/8)|0> - e^{i\pi/4}sin(\pi/8)|1>$$

$$A|1>= e^{-i\pi/4}sin(\pi/8)|0> + e^{i\pi/4}cos(\pi/8)|1>$$

$$B|0>= cos(\pi/8)|0> + sin(\pi/8)|1>$$

$$B|1>= -sin(\pi/8)|0> + cos(\pi/8)|1>$$

$$C|0>= e^{i\pi/4}|0>$$

$$C|1>= e^{-i\pi/4}|1>.$$

Now, we can analyze the circuit. After the C gate, the state of 2 qubits is

$$\Psi_c = 1/2(e^{i\pi/4}|00> + e^{-i\pi/4}|01> + e^{i\pi/4}|10> + e^{-i\pi/4}|11>).$$

Then comes the first CNOT gate which flips the second qubit state only if the first qubit is in |1> state resulting in

$$\Psi_{x1} = 1/2(e^{i\pi/4}|00> + e^{-i\pi/4}|01> + e^{i\pi/4}|11> + e^{-i\pi/4}|10>).$$

After the B gate, the state is

$$\begin{gathered}\Psi_b = 1/2[e^{i\pi/4}|0> (cos(\pi/8)|0> + sin(\pi/8)|1>) + \\ e^{-i\pi/4}|0> (-sin(\pi/8)|0> + cos(\pi/8)|1>) + \\ e^{i\pi/4}|1> (-sin(\pi/8)|0> + cos(\pi/8)|1>) + \\ e^{-i\pi/4}|1> (cos(\pi/8)|0> + sin(\pi/8)|1>)]\end{gathered}$$

which simplifies to

$$\begin{gathered}\Psi_b = 1/2[|00> (e^{i\pi/4}cos(\pi/8) - e^{-i\pi/4}sin(\pi/8)) + \\ |01> (e^{i\pi/4}sin(\pi/8) + e^{-i\pi/4}cos(\pi/8)) + \\ |10> (-e^{i\pi/4}sin(\pi/8) + e^{-i\pi/4}cos(\pi/8)) + \\ |11> (e^{i\pi/4}cos(\pi/8) + e^{-i\pi/4}sin(\pi/8))].\end{gathered}$$

After the second CNOT gate, the 2 qubits state is

$$\Psi_{x2} = 1/2[|00>(e^{i\pi/4}cos(\pi/8) - e^{-i\pi/4}sin(\pi/8)) + \\ |01>(e^{i\pi/4}sin(\pi/8) + e^{-i\pi/4}cos(\pi/8)) + \\ |11>(-e^{i\pi/4}sin(\pi/8) + e^{-i\pi/4}cos(\pi/8)) + \\ |10>(e^{i\pi/4}cos(\pi/8) + e^{-i\pi/4}sin(\pi/8))].$$

Finally comes the A gate resulting in

$$\Psi_a = 1/2[|0>(e^{i\pi/4}cos(\pi/8) - e^{-i\pi/4}sin(\pi/8))(e^{-i\pi/4}cos(\pi/8)|0> - e^{i\pi/4}sin(\pi/8)|1>) + \\ |0>(e^{i\pi/4}sin(\pi/8) + e^{-i\pi/4}cos(\pi/8))(e^{-i\pi/4}sin(\pi/8)|0> + e^{i\pi/4}cos(\pi/8)|1>) + \\ |1>(-e^{i\pi/4}sin(\pi/8) + e^{-i\pi/4}cos(\pi/8))(e^{-i\pi/4}sin(\pi/8)|0> + e^{i\pi/4}cos(\pi/8)|1>) + \\ |1>(e^{i\pi/4}cos(\pi/8) + e^{-i\pi/4}sin(\pi/8))(e^{-i\pi/4}cos(\pi/8)|0> - e^{i\pi/4}sin(\pi/8)|1>)]$$

which simplifies to

$$\Psi_a = 1/2[|00> + |01> + |10>(cos(\pi/4) + e^{-i\pi/2}sin(\pi/4)) + |11>(cos(\pi/4) - e^{i\pi/2}sin(\pi/4))].$$

We finish with the Phase gate which just adds a phase factor to Ψ_a resulting in

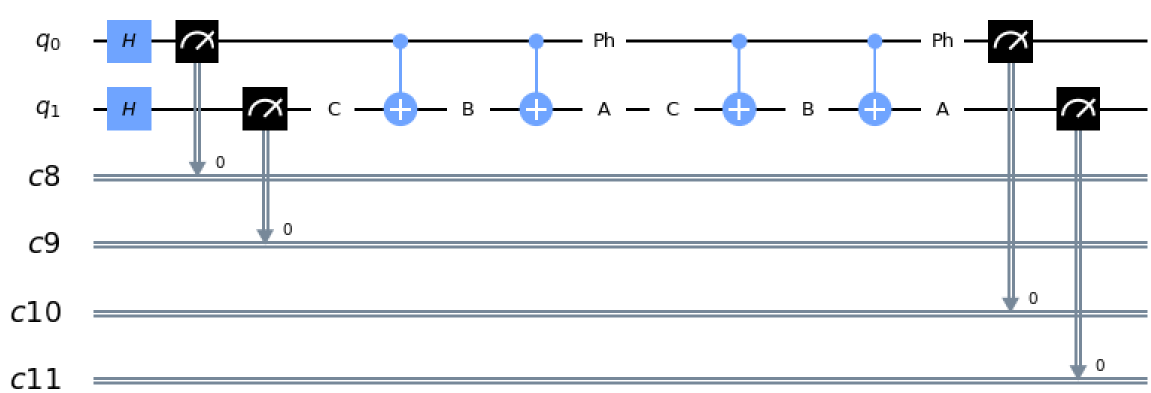
$$\Psi_{out} = e^{i\pi/4}/2[|00> + |01> + |10>(1-i)/\sqrt{2} + |11>(1-i)/\sqrt{2}].$$

As we can see our math confirms 4 possible, equally probable, outputs: 00, 01, 10, or 11 with 1/4 probability.

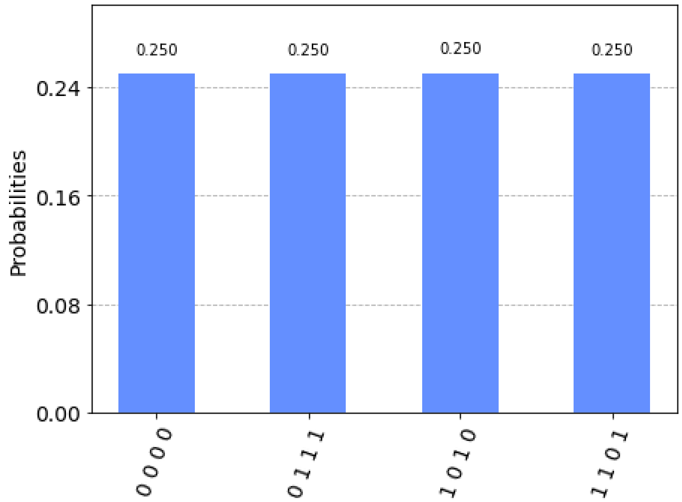
2 times in a row **control** V gate = CNOT

Now if we apply 2 times in a row **control** V gate we should effectively have a **control** X gate which is CNOT gate. The circuit below shows exactly that.

In [11]:
```
## Define a 2-qubit quantum circuit
q = QuantumRegister(2, 'q')
circuit = QuantumCircuit(q)
## We need also 4 classical registers
c = [ ClassicalRegister(1) for _ in range(4) ]
circuit = QuantumCircuit(q)
for register in c:
    circuit.add_register(register)

#initial state of an equal superposition of |0> and |1> states
#so that we have all possible inputs of 2 qubits
circuit.h(q[0])
circuit.h(q[1])

circuit.measure(q[0], c[0])
circuit.measure(q[1], c[1])

circuit.append(C,[1])    # C gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(B, [1])   # B gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(A, [1])   # A gate acting on q[1]
circuit.append(Ph, [0])  # phase gate acting on q[0] (it really does not matter which qubit we pick)

circuit.append(C,[1])    # C gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(B, [1])   # B gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(A, [1])   # A gate acting on q[1]
circuit.append(Ph, [0])  # phase gate acting on q[0] (it really does not matter which qubit we pick)

#finish with measurements
circuit.measure(q[0], c[2])
circuit.measure(q[1], c[3])

## Draw the circuit
%matplotlib inline
circuit.draw(output="mpl")
```



In [12]:
```
plot_histogram(get_counts('qasm_simulator'))
```



2 times in a row V gate = X

Using the expression for V gate

$$V = Ph(\pi/4)R_z(-\pi/2)R_y(\pi/2)R_z(\pi/2)$$

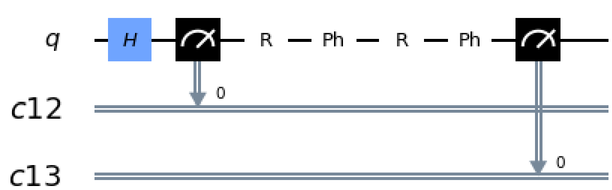
we can now show when we apply twice in a row V gate we effectively get X gate. For that purpose we define R gate as

$$R = R_z(-\pi/2)R_y(\pi/2)R_z(\pi/2)$$

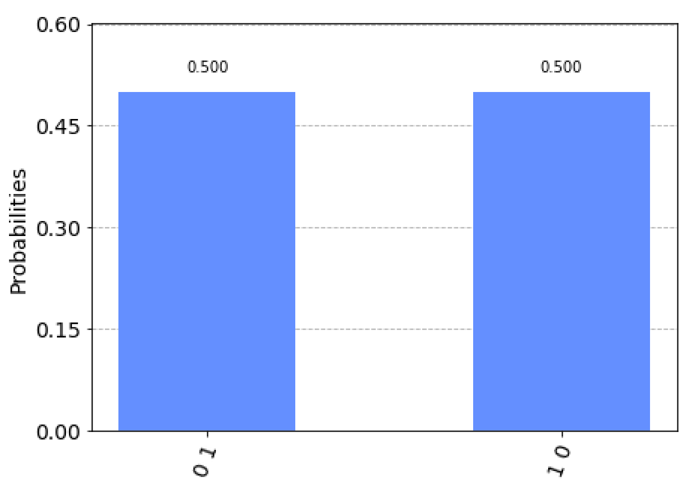
so essentially V gate in the circuit will be recognized as

$$Ph(\pi/4)R.$$

In [13]:
```
#define R gate as an instruction that contains only rotations
qc_R = QuantumCircuit(1, name='R')
qc_R.rz(pi/2, 0)
qc_R.ry(pi/2, 0)
qc_R.rz(-pi/2, 0)
R = qc_R.to_instruction()

#now assemble respective quantum circuit
## Define a 1-qubit quantum circuit
q = QuantumRegister(1, 'q')
circuit = QuantumCircuit(q)
## We need also 2 classical registers
c = [ ClassicalRegister(1) for _ in range(2) ]
circuit = QuantumCircuit(q)
for register in c:
    circuit.add_register(register)

#initial state of an equal superposition of |0> and |1> states
#so that we have all possible inputs of 1 qubit
circuit.h(q[0])
circuit.measure(q[0], c[0])

circuit.append(R,[0])    # R gate acting on q[0]
circuit.append(Ph, [0])  # phase gate acting on q[0]
circuit.append(R,[0])    # R gate acting on q[0]
circuit.append(Ph, [0])  # phase gate acting on q[0]

#finish with measurements
circuit.measure(q[0], c[1])

## Draw the circuit
%matplotlib inline
circuit.draw(output="mpl")
```



In [14]:
```
plot_histogram(get_counts('qasm_simulator'))
```



As expected the input |0> state is changed to |1> and vice versa.

V^+ gate

Now, let's define V^+ gate. We start from the expression for V gate

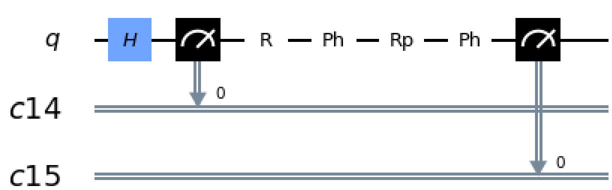
$$V = Ph(\pi/4)R_z(-\pi/2)R_y(\pi/2)R_z(\pi/2)$$

and apply Hermitian conjugation to it. We obtain

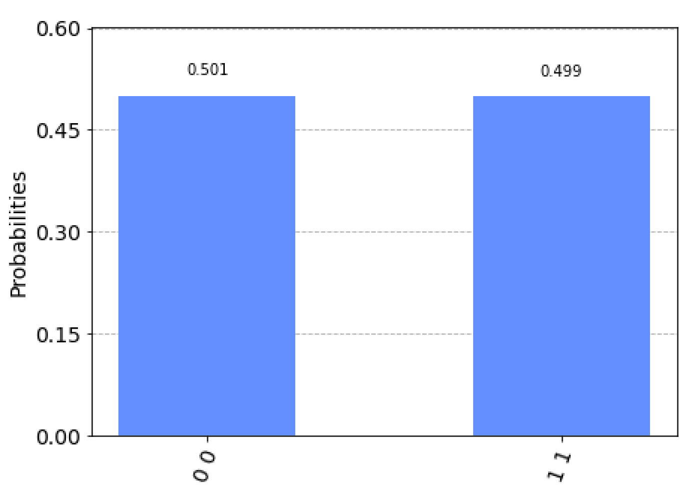
$$V^+ = R_z(-\pi/2)R_y(-\pi/2)R_z(\pi/2)Ph(-\pi/4).$$

Obviously, $VV^+ = V^+V = I$. The circuit below shows exactly that identity.

In [15]:
```
#define Rp gate as an instruction that contains only rotations
qc_Rp = QuantumCircuit(1, name='Rp')
qc_Rp.rz(pi/2, 0)
qc_Rp.ry(-pi/2, 0)
qc_Rp.rz(-pi/2, 0)
Rp = qc_Rp.to_instruction()

#now assemble respective quantum circuit
## Define a 1-qubit quantum circuit
q = QuantumRegister(1, 'q')
circuit = QuantumCircuit(q)
## We need also 2 classical registers
c = [ ClassicalRegister(1) for _ in range(2) ]
circuit = QuantumCircuit(q)
for register in c:
    circuit.add_register(register)

#initial state of an equal superposition of |0> and |1> states
#so that we have all possible inputs of 1 qubit
circuit.h(q[0])
circuit.measure(q[0], c[0])

delta = pi/4
circuit.append(R,[0])      # R gate acting on q[0]
circuit.append(Ph, [0])    # phase gate acting on q[0]
delta = -pi/4
circuit.append(Rp,[0])     # R gate acting on q[0]
circuit.append(Ph, [0])    # phase gate acting on q[0]

#finish with measurements
circuit.measure(q[0], c[1])

## Draw the circuit
%matplotlib inline
circuit.draw(output="mpl")
```



In [16]:
```
plot_histogram(get_counts('qasm_simulator'))
```



control V^+ gate

Let's constuct a circuit with **control** V^+ gate. We need to Hermitian conjugate **control** V gate. So

$$W^+ = R_z(-\pi/2)R_y(-\pi/2)R_z(\pi/2)$$

$$A^+ = R_y(-\pi/4)R_z(\pi/2)$$

$$B^+ = R_y(\pi/4)$$

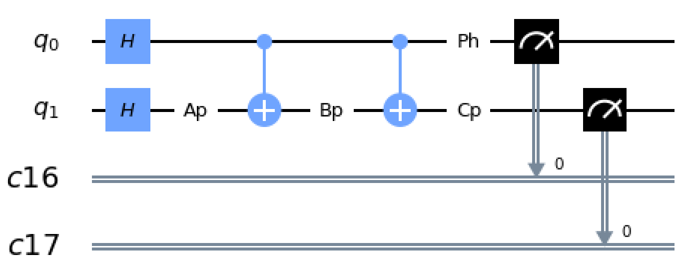
$$C^+ = R_z(-\pi/2).$$

$$C^+B^+A^+ = I$$

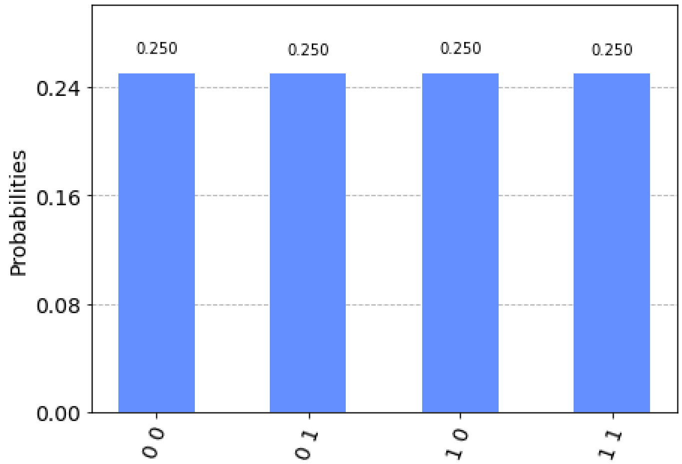
and

$$C^+XB^+XA^+ = W^+.$$

In [17]:
```
#define Ap gate as an instruction
qc_Ap = QuantumCircuit(1, name='Ap')
qc_Ap.rz(pi/2, 0)
qc_Ap.ry(-pi/4, 0)
Ap = qc_Ap.to_instruction()

#define Bp gate as an instruction
qc_Bp = QuantumCircuit(1, name='Bp')
qc_Bp.ry(pi/4, 0)
Bp = qc_Bp.to_instruction()

#define Cp gate as an instruction
qc_Cp = QuantumCircuit(1, name='Cp')
qc_Cp.rz(-pi/2, 0)
Cp = qc_Cp.to_instruction()

#now assemble respective quantum circuit
## Define a 2-qubit quantum circuit
q = QuantumRegister(2, 'q')
circuit = QuantumCircuit(q)
## We need also 4 classical registers
c = [ ClassicalRegister(1) for _ in range(2) ]
circuit = QuantumCircuit(q)
for register in c:
    circuit.add_register(register)

#initial state of an equal superposition of |0> and |1> states
#so that we have all possible inputs of 2 qubits
circuit.h(q[0])
circuit.h(q[1])

circuit.append(Ap,[1])    # C gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(Bp, [1])   # B gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(Cp, [1])   # A gate acting on q[1]
delta=-pi/4
circuit.append(Ph, [0])  # phase gate acting on q[0] (it really does not matter which qubit we pick)

#finish with measurements
circuit.measure(q[0], c[0])
circuit.measure(q[1], c[1])

## Draw the circuit
%matplotlib inline
circuit.draw(output="mpl")
```



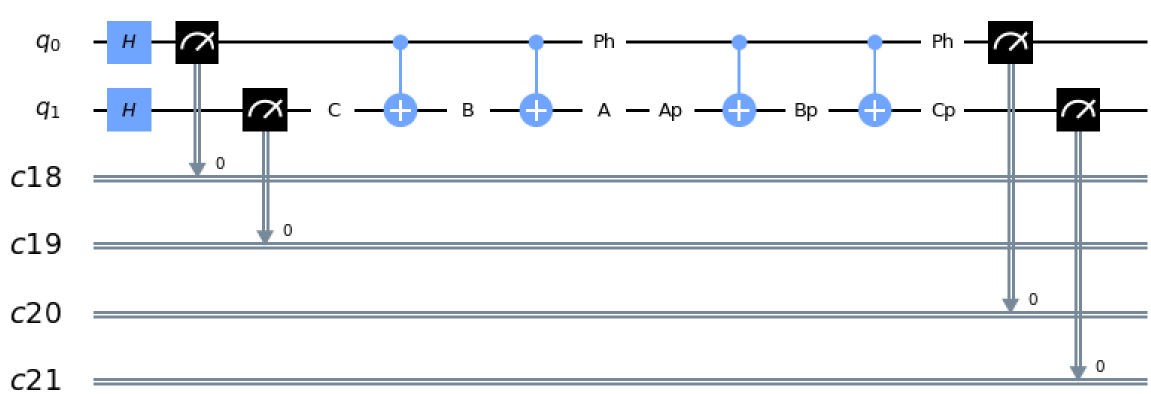
In [18]:
```
plot_histogram(get_counts('qasm_simulator'))
```



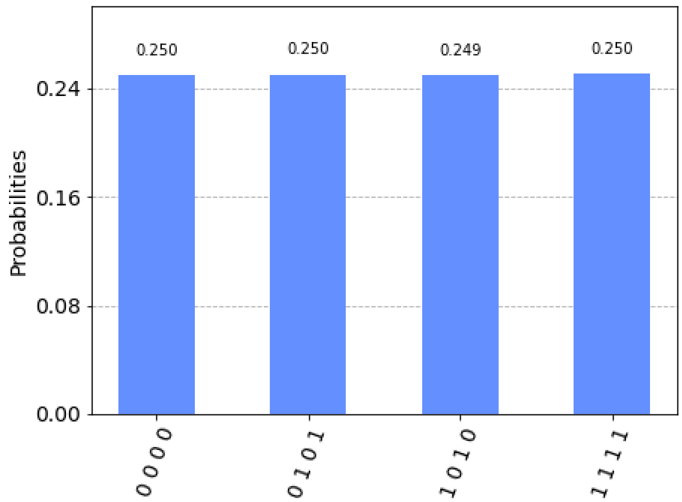
(control V)(control V^+) = I

If we apply together **control** V and **control** V^+ gates we should get identity gate as the circuit below shows.

In [19]:
```
#now assemble respective quantum circuit
## Define a 2-qubit quantum circuit
q = QuantumRegister(2, 'q')
circuit = QuantumCircuit(q)
## We need also 4 classical registers
c = [ ClassicalRegister(1) for _ in range(4) ]
circuit = QuantumCircuit(q)
for register in c:
    circuit.add_register(register)

#initial state of an equal superposition of |0> and |1> states
#so that we have all possible inputs of 2 qubits
circuit.h(q[0])
circuit.h(q[1])
circuit.measure(q[0], c[0])
circuit.measure(q[1], c[1])

circuit.append(C,[1])    # C gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(B, [1])   # B gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(A, [1])   # A gate acting on q[1]
delta=pi/4
circuit.append(Ph, [0])  # phase gate acting on q[0] (it really does not matter which qubit we pick)

circuit.append(Ap,[1])    # C gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(Bp, [1])   # B gate acting on q[1]
circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]
circuit.append(Cp, [1])   # A gate acting on q[1]
delta=-pi/4
circuit.append(Ph, [0])  # phase gate acting on q[0] (it really does not matter which qubit we pick)

#finish with measurements
circuit.measure(q[0], c[2])
circuit.measure(q[1], c[3])

## Draw the circuit
%matplotlib inline
circuit.draw(output="mpl")
```



In [20]:
```
plot_histogram(get_counts('qasm_simulator'))
```

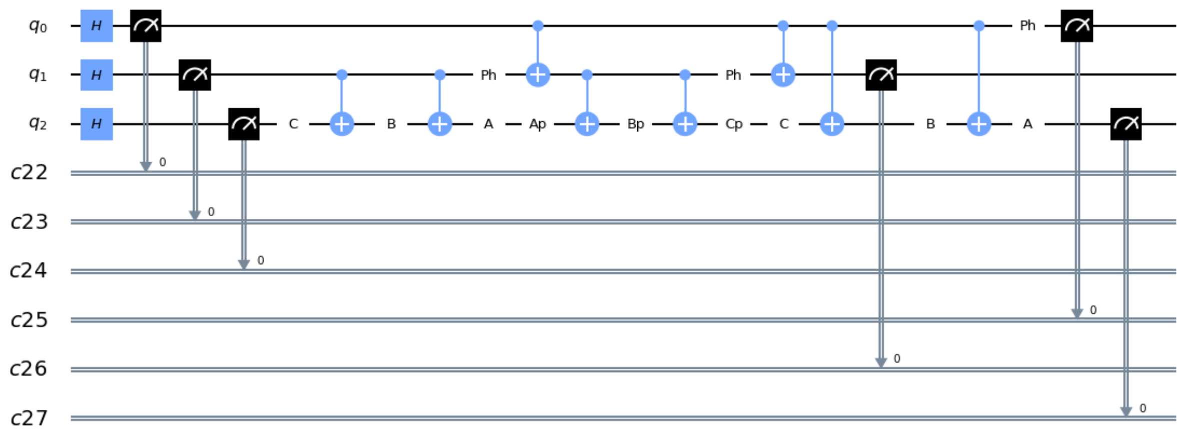


As expected qubits states do not change: 00->00, 10->10, 01->01, and 11->11.

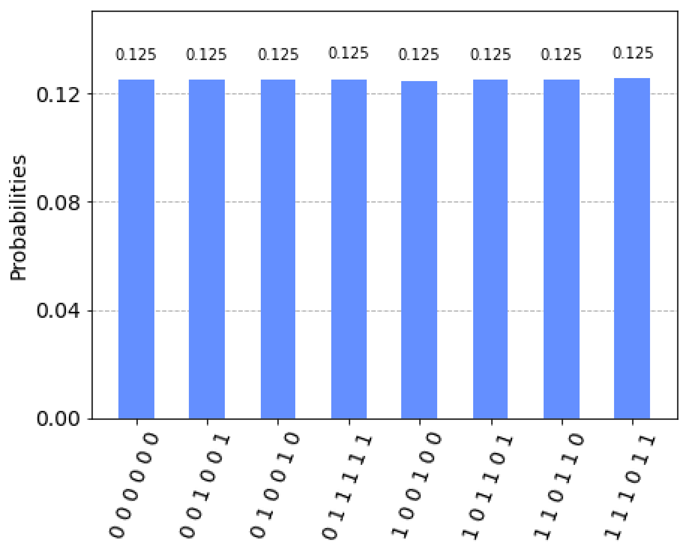
Toffoli gate

Now we can follow Lemma 6.1 from the reference [1] and construct a Toffoli gate.

In [21]:
```
#now assemble respective quantum circuit
## Define a 3-qubit quantum circuit
q = QuantumRegister(3, 'q')
circuit = QuantumCircuit(q)
## We need also 3 classical registers
c = [ ClassicalRegister(1) for _ in range(6) ]
circuit = QuantumCircuit(q)
for register in c:
    circuit.add_register(register)

#initial state of an equal superposition of |0> and |1> states
#so that we have all possible inputs of 3 qubits
circuit.h(q[0])
circuit.h(q[1])
circuit.h(q[2])
circuit.measure(q[0], c[0])
circuit.measure(q[1], c[1])
circuit.measure(q[2], c[2])

# control V gate between q[1] and q[2]
circuit.append(C,[2])    # C gate acting on q[2]
circuit.cnot(q[1], q[2]) # CNOT between q[1] and q[2]
circuit.append(B, [2])   # B gate acting on q[2]
circuit.cnot(q[1], q[2]) # CNOT between q[1] and q[2]
circuit.append(A, [2])    # A gate acting on q[2]
delta=pi/4
circuit.append(Ph, [1])  # phase gate acting on q[1]

circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]

# control V+ gate between q[1] and q[2]
circuit.append(Ap,[2])    # Ap gate acting on q[2]
circuit.cnot(q[1], q[2])  # CNOT between q[1] and q[2]
circuit.append(Bp, [2])   # Bp gate acting on q[2]
circuit.cnot(q[1], q[2])  # CNOT between q[1] and q[2]
circuit.append(Cp, [2])   # Cp gate acting on q[2]
delta=-pi/4
circuit.append(Ph, [1])  # phase gate acting on q[1]

circuit.cnot(q[0], q[1]) # CNOT between q[0] and q[1]

# control V gate between q[0] and q[2]
circuit.append(C,[2])    # C gate acting on q[2]
circuit.cnot(q[0], q[2]) # CNOT between q[0] and q[2]
circuit.append(B, [2])   # B gate acting on q[2]
circuit.cnot(q[0], q[2]) # CNOT between q[0] and q[2]
circuit.append(A, [2])    # A gate acting on q[2]
delta=pi/4
circuit.append(Ph, [0])  # phase gate acting on q[0]

#finish with measurements
circuit.measure(q[0], c[3])
circuit.measure(q[1], c[4])
circuit.measure(q[2], c[5])

## Draw the circuit
%matplotlib inline
circuit.draw(output="mpl")
```



In [22]:
```
plot_histogram(get_counts('qasm_simulator'))
```

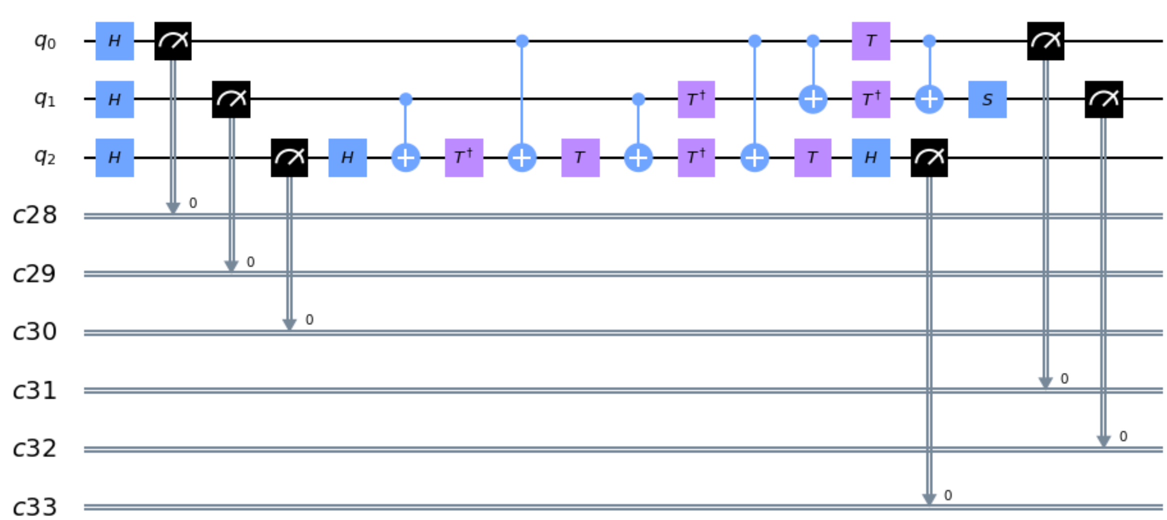


The Toffoli gate fips the q2 qubit state only when both q0 and q1 qubits are in the state |1>. The fourth bar corresponds to the case when the q2 qubit state is flipped from |1> to |0> and the eight bar corresponds to the case when the q2 qubit state is flipped from |0> to |1>.
This is just one of many possible implementations of the Toffoli gate.
For example, below is the implementation from the reference [2].

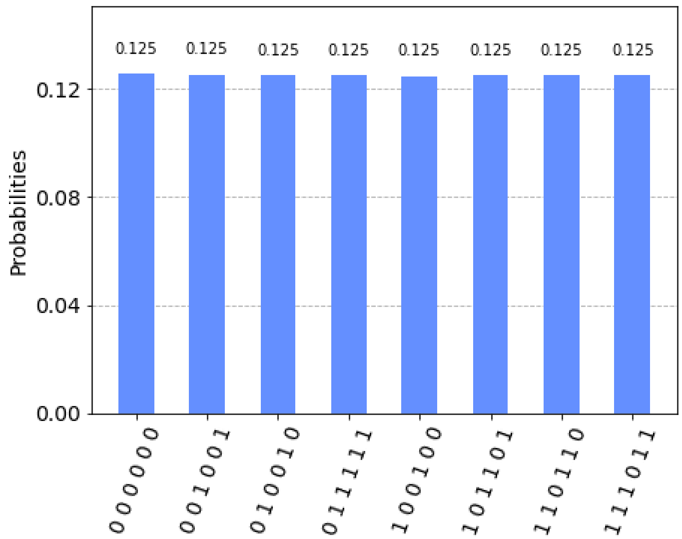
[2] https://arxiv.org/abs/1703.10793 (https://arxiv.org/abs/1703.10793)

Implementing a Toffoli gate with 10 single-qubit gates and 6 CNOT gates

In [23]:
```
## Define a 3-qubit quantum circuit
q = QuantumRegister(3, 'q')
circuit = QuantumCircuit(q)
## We need also 6 classical registers
c = [ ClassicalRegister(1) for _ in range(6) ]
circuit = QuantumCircuit(q)
for register in c:
    circuit.add_register(register)

#initial state of an equal superposition of |0> and |1> states
circuit.h(q[0]) #Hadammard on q[0]
circuit.h(q[1]) #Hadammard on q[1]
circuit.h(q[2]) #Hadammard on q[2]

#measure q0, q1, and q2 before the Toffoli gate (to display the input values to the Toffoli gate)
circuit.measure(q[0], c[0])
circuit.measure(q[1], c[1])
circuit.measure(q[2], c[2])

circuit.h(q[2]) #Hadammard on q[2]
circuit.cx(q[1],q[2]) #CNOT between q1 and q2
circuit.tdg(q[2]) #Hermitian conjugate T gate on q[2]
circuit.cx(q[0],q[2]) #CNOT between q0 and q2
circuit.t(q[2]) #T gate on q[2]
circuit.cx(q[1],q[2]) #CNOT between q1 and q2
circuit.tdg(q[1]) #Hermitian conjugate T gate on q[1]
circuit.tdg(q[2]) #Hermitian conjugate T gate on q[2]
circuit.cx(q[0],q[2]) #CNOT between q0 and q2
circuit.cx(q[0],q[1]) #CNOT between q0 and q1
circuit.t(q[0]) #T gate on q[0]
circuit.tdg(q[1]) #Hermitian conjugate T gate on q[1]
circuit.t(q[2]) #T gate on q[2]
circuit.cx(q[0],q[1]) #CNOT between q0 and q1
circuit.s(q[1]) #T gate on q[1]
circuit.h(q[2]) #Hadammard on q[2]

## Finish off with the measurements
circuit.measure(q[0], c[3])
circuit.measure(q[1], c[4])
circuit.measure(q[2], c[5])

## Draw the circuit
%matplotlib inline
circuit.draw(output="mpl")
```

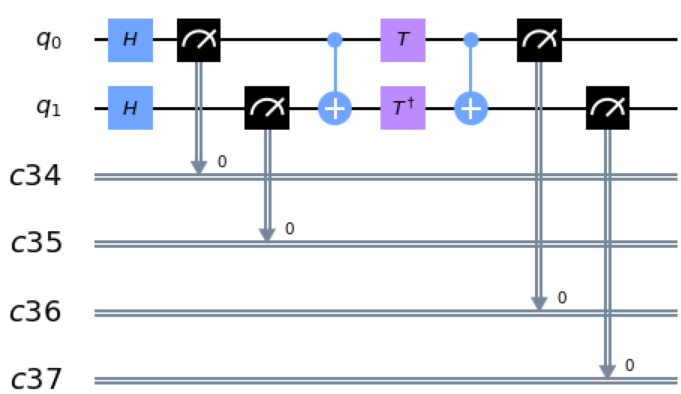


In [24]:
```
plot_histogram(get_counts('qasm_simulator'))
```

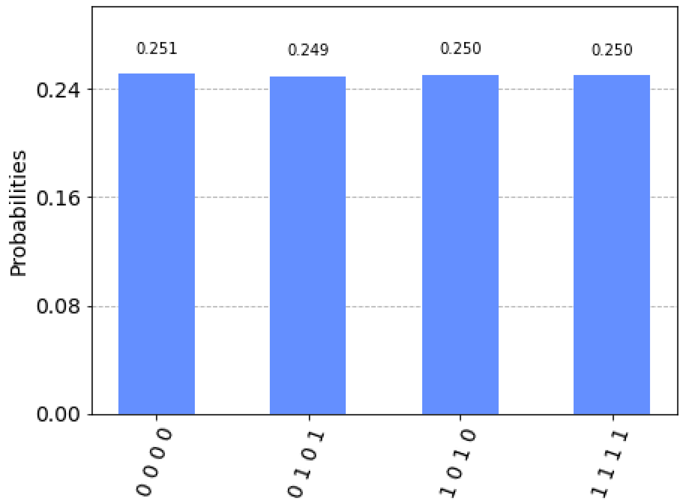


The circuit below does not change the states of q0 and q1 qubits!

In [25]:
```
## Define a 2-qubit quantum circuit
q = QuantumRegister(2, 'q')
circuit = QuantumCircuit(q)
## We need also 4 classical registers
c = [ ClassicalRegister(1) for _ in range(4) ]
circuit = QuantumCircuit(q)
for register in c:
    circuit.add_register(register)

#initial state of an equal superposition of |0> and |1> states
circuit.h(q[0]) #Hadammard on q[0]
circuit.h(q[1]) #Hadammard on q[1]

#measure q0 and q1 before the swap gate (to display the input values to the swap gate)
circuit.measure(q[0], c[0])
circuit.measure(q[1], c[1])


circuit.cx(q[0],q[1]) #CNOT between q0 and q1 (q0 is a control qubit)
circuit.t(q[0])       #T gate on q[0]
circuit.tdg(q[1])     #Hermitian conjugate T gate on q[1]
circuit.cx(q[0],q[1]) #CNOT between q0 and q1 (q0 is a control qubit)

## Finish off with the measurements
circuit.measure(q[0], c[2])
circuit.measure(q[1], c[3])

## Draw the circuit
%matplotlib inline
circuit.draw(output="mpl")
```

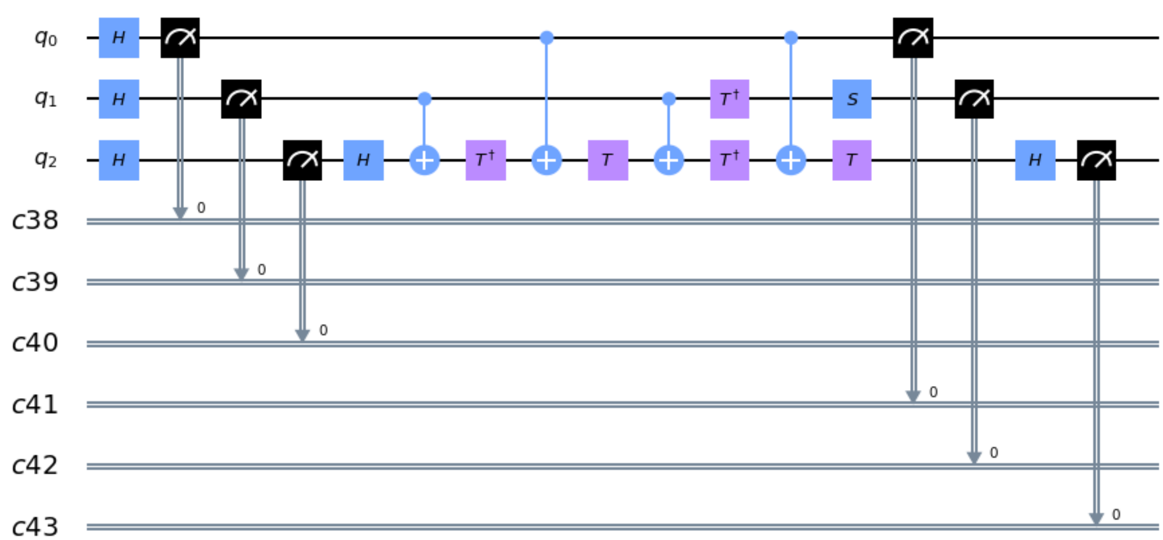


In [26]:
```
plot_histogram(get_counts('qasm_simulator'))
```

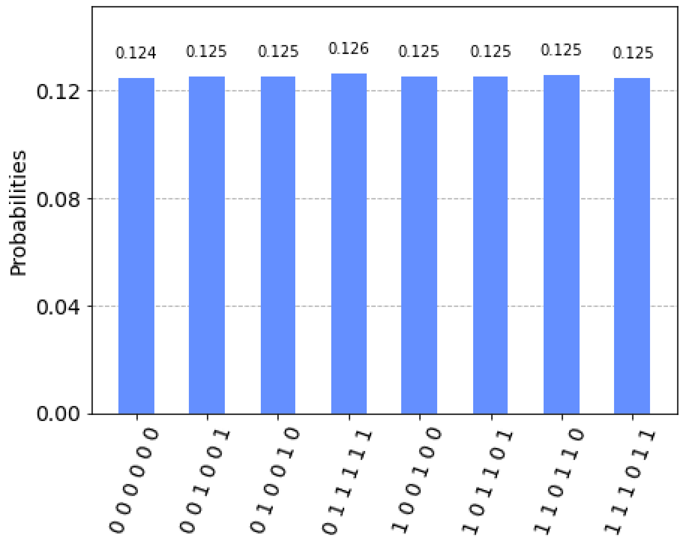


This information can be used to slightly simplify the above implementation of the Toffoli gate. Respective circuit is displayed below.

In [27]:
```python
## Define a 3-qubit quantum circuit
q = QuantumRegister(3, 'q')
circuit = QuantumCircuit(q)
## We need also 6 classical registers
c = [ ClassicalRegister(1) for _ in range(6) ]
circuit = QuantumCircuit(q)
for register in c:
    circuit.add_register(register)

#initial state of an equal superposition of |0> and |1> states
circuit.h(q[0]) #Hadammard on q[0]
circuit.h(q[1]) #Hadammard on q[1]
circuit.h(q[2]) #Hadammard on q[2]

#measure q0, q1, and q2 before the Toffoli gate (to display the input values to the Toffoli gate)
circuit.measure(q[0], c[0])
circuit.measure(q[1], c[1])
circuit.measure(q[2], c[2])

circuit.h(q[2]) #Hadammard on q[2]
circuit.cx(q[1],q[2]) #CNOT between q1 and q2
circuit.tdg(q[2]) #Hermitian conjugate T gate on q[2]
circuit.cx(q[0],q[2]) #CNOT between q0 and q2
circuit.t(q[2]) #T gate on q[2]
circuit.cx(q[1],q[2]) #CNOT between q1 and q2
circuit.tdg(q[1]) #Hermitian conjugate T gate on q[1]
circuit.tdg(q[2]) #Hermitian conjugate T gate on q[2]
circuit.cx(q[0],q[2]) #CNOT between q0 and q2
#circuit.cx(q[0],q[1]) #CNOT between q0 and q1
#circuit.t(q[0]) #T gate on q[0]
#circuit.tdg(q[1]) #Hermitian conjugate T gate on q[1]
circuit.t(q[2]) #T gate on q[2]
#circuit.cx(q[0],q[1]) #CNOT between q0 and q1
circuit.s(q[1]) #T gate on q[1]
circuit.h(q[2]) #Hadammard on q[2]

## Finish off with the measurements
circuit.measure(q[0], c[3])
circuit.measure(q[1], c[4])
circuit.measure(q[2], c[5])

## Draw the circuit
%matplotlib inline
circuit.draw(output="mpl")
```



In [28]:
```
plot_histogram(get_counts('qasm_simulator'))
```



As we can see the Toffoli gate behavior is demonstrated. The q2 qubit state is flipped only when both q0 and q1 qubits are in the state |1>.