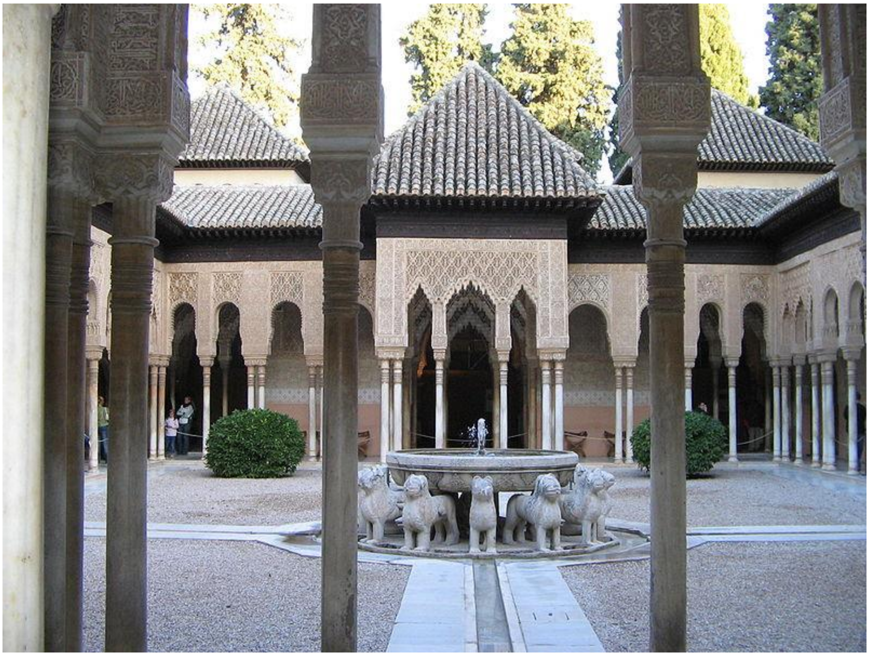
ON THE UNIFICATION PROBLEM IN PHYSICS – THE PAPER OF THEODOR KALUZA



Samir Lipovaca

Many thanks to my wife Besima for her translation of the poem on the basin of the Fountain of the Lions in the Alhambra, Granada.

“All that we are is the result of what we have thought.”

Buddha

... and when it rains you cannot
and I do understand the disappointment
of a cloud turned to common water.

In the 1174th Year of Our Lord, if I counted right.

Wisdom from a Bosnian medieval tombstone (*stećak*)

Contents

Preface

The aim of this book is to present the paper of Theodor Kaluza *On the Unification Problem in Physics*. The problem of unification in physics through a hyperspace (more than 3 spatial dimensions) is called the Kaluza-Klein miracle. The name came from the two original proponents of this unification method, the German mathematician Theodor Kaluza and the Swedish physicist Oscar Klein. There are two reasons for inspiration for this book. First, in the eternal struggle of good and evil, the world is today, it seems more than ever, shrouded in negativity and division. For example, when we observe royal penguins [1] showing co-operative behavior in order to face an incredibly sharp environment where cold winds can reach up to -60^0C, we cannot help but wonder if the human species is capable of exhibiting similar behavior. The penguins are attached to each other to avoid the wind and preserve heat. When the penguin warms up he will go to the periphery of the group so that other penguins can enjoy protection from brutal cold. It is almost impossible to imagine a picture in which the penguins would be replaced by people and people would show cooperative behavior in order to protect themselves from brutal cold. This non-cooperative irrationality has no justification because Cosmos is united on the fundamental level of forces and elementary particles. Therefore, to speak of unity is natural and this book should be understood as a protest against negativity and division. It is hoped that this book will allow the reader to feel at least some of this fundamental unity and that it will inspire him to nourish goodness in himself and toward others. In addition, to freely interpret the view of Leopold Infeld [2, page 309] about Einstein's popularity after the First World War, it is healthy to get away with abstract thought at least briefly from the sad and disappointing reality. After this, in a way, an idealistic reason, the other is practical. Generally, complete derivations of equations are not a priority in physics literature, but are left as a useful exercise for the reader to make them using hints and tips from the relevant text. The author of this book considers that the reader only if exposed to a complete derivation of equations can have a complete pleasure and enjoyment in the process of acquiring new knowledge. That is why in the book he provided complete derivations of all equations.

Introduction

The term Kaluza-Klein theory implies a unified theory of the field of gravity and electromagnetism built on the idea of the fifth dimension that is added to the usual dimensions: three dimensions of space and one of time. This theory is considered an important precursor to the string theory. The original hypothesis of the unification of gravity and electromagnetism based on the fifth-dimensional world was set by Theodor Kaluza.

In April 1919, Einstein received a letter [3, page 99] from Theodor Kaluza in which he asked Einstein to consider his idea of unifying gravity and electromagnetism in five dimensions. Two years later, in 1921, Kaluza's work entitled *On the Unification Problem in Physics* was published. Kaluza based his work on a fully developed general theory of relativity. In other words, it was a purely classical extension of the general theory of relativity to five dimensions. No quantum interpretation was considered. Kaluza left the basic principles of general relativity theory untouched and only added an extra dimension. Amazingly, with this incredible addition, Kaluza has discovered that it can reproduce both Einstein's equations of gravity and Maxwell's equations of electromagnetism with one set of relations.

Speaking in technical language [4], Kaluza's fifth-dimensional metric tensor has 15 components. 10 components are identified with a metric of four-dimensional space and time. The 4 components are identified with the electromagnetic vector potential and the remaining component with an unidentified scalar field, sometimes called a dilaton. In addition, Kaluza introduced an additional hypothesis known as the “condition of a cylinder", according to which no component of the fifth-dimensional metric tensor does depend on the fifth dimension. This hypothesis seems completely understandable since our entire physical experience in four dimensions does not point to the fifth dimension. Considering the scalar field, Kaluza required it to be a constant. With this condition, Kaluza reproduced Einstein's equations of gravity and Maxwell's equations of electromagnetism in a miraculous way.

Kaluza’s theory is a classical fifth-dimensional theory. In 1926, Oskar Klein gave a quantum-mechanical interpretation to this theory in accordance with the quantum mechanics of that time. In order to explain the "cylinder condition", Klein introduced the hypothesis that the fifth dimension is microscopic and essentially curled up. Based on the quantum of electric charge, Klein calculated the order of the size of the fifth dimension. The quantization of the electric charge could then be easily explained by the integer multiples of the fifth dimensional impulse. Namely, Klein suggested that the fifth dimension was closed and periodic. Then the electric charge can be identified by moving in the fifth dimension in terms of standing waves.

Kaluza’s theory was finally completed in the 1940s. Three independent research groups found complete field equations including the scalar field: Thiry in France, Jordan, Ludwig and Müller in Germany, and Scherrer in Switzerland. However, Kaluza's idea that the world could have more than 4 dimensions will remain forever proof of Kaluza’s incredible intuition that the fifth-dimensional space could be used to unify Einstein's theory of general relativity and Maxwell's theory of electromagnetism.

On the Unification Problem in Physics [5]

from Th. Kaluza

in Königsberg

(Submitted by Mr. Einstein on December 8, 1921; see above, page 859.)

If we leave *the electromagnetic four potential* q_μ aside, the fundamental metric tensor of the four-dimensional world $g_{\mu\nu}$ must be interpreted in the general theory of relativity as *the tensor potential of gravity* when we characterize world events.

The dualism of gravity and electricity, which also stays here, does not really take anything of the charming beauty of either theory, but it is calling for its overcoming by a completely *unified picture of the world*.

A surprisingly brave approach to the solution of this problem, which is one of the great favorite ideas of the human spirit, was undertaken by H. Weyl [1] ([1] Sitzungsber. d. Berl. Akad. 1918 page. 465.) a few years ago and he found in a further radical revision of the geometrical foundations, in addition to the tensor $g_{\mu\nu}$, another kind of metric fundamental vector, and interpreted it as the electromagnetic potential q_μ: The complete world metric is presented there as the common source of all natural phenomena.

The same goal will be sought here in a different way.

If we ignore the difficulties that accompanied the realization of this profound theory of H. Weyl, it would also be idealistic, an even more complete realization of the unification idea: The gravitational and electromagnetic fields arise from a single universal tensor. - I would now like to show that such a close union between the two forces in the world seems possible in principle.

* * *

The rotor shape of the electromagnetic field components $F_{\chi\lambda}$, but even more so the unmistakable formal correspondence in the structure of the gravitational and electromagnetic equations [2] ([2] For this, look also H. Thirring. Phys. Ztschr. 19 page. 204.) formally require the assumption that $\frac{1}{2}F_{\chi\lambda} = \frac{1}{2}(q_{\chi.\lambda} - q_{\lambda.\chi})$ [1] ([1] Indices separated by a dot should indicate differentiation according to the corresponding world parameters.) could somehow be equal to the truncated three-index quantities $\begin{bmatrix} i\lambda \\ \chi \end{bmatrix} = \frac{1}{2}\left(g_{i\chi.\lambda} + g_{\chi\lambda.i} - g_{i\lambda.\chi}\right)$. If one gives a room to this idea, one sees with great certainty the way, which at the beginning, is rather unattractive: Since in a four-dimensional world the additional three-index quantities cannot exist in addition to those already used for the components of the gravitational field, then the view of $F_{\chi\lambda}$ can hardly be different, so we are obliged to call for help from a new, *fifth dimension of the world*, which, perhaps, is a very unusual surprise.

Now, indeed, our previous physical experience stands because otherwise there is hardly any indication of such an extra world parameter, but it is also free to regard our space-time world as *a four dimensional part of* R_5; we just have to consider the fact that we never notice any changes other than space-time changes in the state variables, so we have to set their *derivatives with respect to the new parameter equal to zero*, or treat them to be small as they are of higher order ("cylinder condition"). The fear that this might undo the introduction of the fifth dimension is unfounded because of the integration of this world parameter into the three-index quantities.

* * *

So we go into R_5 and apply Einstein's approaches to it; x^0 as the new parameter enters in addition to the usual x^1 do x^4. If g_{rs} [2] ([2] Latin indices should always run from 0, Greek only from 1 to 4.) denotes the fundamental metric tensor of this R_5, then by virtue of the cylinder condition, the three-index quantities $\begin{bmatrix} ik \\ l \end{bmatrix}$, denoted here by $-\Gamma_{ikl}$, become:

$$2\Gamma_{\chi\lambda\mu} = g_{\chi\lambda.\mu} - g_{\lambda\mu.\chi} - g_{\mu\chi.\lambda} \text{ (as before)},$$

$$2\Gamma_{0\chi\lambda} = g_{0\chi.\lambda} - g_{0\lambda.\chi}, 2\Gamma_{\chi\lambda 0} = -\left(g_{0\chi.\lambda} + g_{0\lambda.\chi}\right), \tag{1}$$

$$2\Gamma_{00\chi} = g_{00.\chi}, 2\Gamma_{0\chi 0} = -g_{00.\chi}, 2\Gamma_{000} = 0.$$

The result initially encourages little: Although the $\Gamma_{0\chi\lambda}$ really do appear in rotor shape, the ten $\Gamma_{\chi\lambda 0}$ which according to the intended interpretation must also be of an electric nature, threaten to block the way. Nevertheless, we pursue it and put it in order to obtain the $\Gamma_{0\chi\lambda}$ proportional to the $F_{\chi\lambda}$:

$$g_{0\chi} = 2\alpha q_\chi, g_{00} = 2g, \tag{2}$$

so that the *fundamental metric tensor* of R_5 becomes essentially the *tensor potential of gravity* limited by the *electromagnetic four-potential*; The role of the component g in the corner remains for the time being undetermined. If one writes $\Sigma_{\chi\lambda}$ for the sums $q_{\chi.\lambda} + q_{\lambda.\chi}$ corresponding to $F_{\chi\lambda}$ then one has:

$$\Gamma_{0\chi\lambda} = \alpha F_{\chi\lambda}\,, \Gamma_{\chi\lambda 0} = -\alpha\Sigma_{\chi\lambda}\,, \Gamma_{00\chi} = -\Gamma_{0\chi 0} = g_{.\chi}. \quad (3)$$

By the electromagnetic field $F_{\chi\lambda}$, its "associated" field $\Sigma_{\chi\lambda}$ as well as the gradient [1] ([1] Four-dimensionally reinterpreted.) of g are thus exhausted the thirty five new three-index quantities (five of which vanish). From the comprehensive equation

$$(\Gamma_{ikl} + \Gamma_{kli} + \Gamma_{lik})_{.m} = \Gamma_{mik.l} + \Gamma_{mkl.i} + \Gamma_{mli.k} \quad (4)$$

the known relations arise from the cylinder condition:

$$F_{\chi\lambda.\mu} + F_{\lambda\mu.\chi} + F_{\mu\chi.\lambda} = 0 \text{ i } g_{.\chi\lambda} = g_{.\lambda\chi}. \quad (4a)$$

* * *

We now restrict, as usual, the parameter selection by $g = |g_{rs}| = -I$ and let (Approximation I) the g_{rs} differ only slightly from the "Euclidean" values $-\delta_{rs}$. With $\Gamma^{l}_{ik} = -\left\{ {ik \atop l} \right\} = -\Gamma_{ikl}$ the components of the *four-index tensors* of interest here then become:

$$\{\chi\lambda, \mu 0\} = \alpha F^{\lambda}_{\chi.\mu}\,, \{\chi 0, 0\lambda\} = -g_{.\chi\lambda}\,, \quad (5)$$

$$\{\chi\lambda, 00\} = \{\chi 0, 00\} = \{00, 00\} = 0.$$

Here, fortunately, the associated field from Eq. (3) does not occur: Of electrical quantities, it is the *field derivatives alone* which determine the curvature of the R_5. If one forms the fully contracted tensor $R_{ik} = \{ir, rk\}$, then by virtue of our assumptions (in a well-known way of writing):

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu.\rho} \text{ (as before),} \tag{6}$$

$$R_{0\mu} = -\alpha\Delta\mathscr{w}_{\mu}F = -\alpha F_{\mu\rho.\rho},$$

$$R_{00} = -\square\mathscr{g}.$$

(**Note**: $-\alpha F_{\mu\rho.\rho}$ is not a part of Kaluza's paper, but we added it to clarify $-\alpha\Delta\mathscr{w}_{\mu}F$ term)

The fifteen components of the curvature tensor on the left-hand side thus decay into: 1. the old *field equations of gravitation*, 2. the *basic electromagnetic equations*, 3. a Poisson equation for the $\mathscr{g}$, which is still without interpretation. Therein lies a first justification for our approach and for the hope to recognize gravitation and electricity as manifestations of a universal field.

* * *

For the *energy tensor* of matter, dominating the right-hand sides of the field equations, the following applies in R_5 in approximation I:

$$T_{ik} = T^{ik} = \mu_0 u^i u^k \tag{7}$$

(μ_0 = rest mass density, $u^r = \frac{dx^r}{ds}$, $ds^2 = g_{lm} dx^l dx^m$).

Since now (for all three types of field equations) $R_{0\mu} = -\chi T_{0\mu}$, the Maxwell equations demand, according to (6), for the components of the four-current:

$$I^\mu = \rho_0 v^\mu = \frac{\chi}{\alpha} T_{0\mu} = \frac{\chi}{\alpha} \mu_0 u^0 u^\mu \tag{8}$$

(ρ_0 = rest charge density, $v^\rho = \frac{dx^\rho}{d\sigma}, d\sigma^2 = g_{\lambda\mu} dx^\lambda dx^\mu$);

the space-time energy tensor is thus essentially determined by the current density.

We start the further investigation first under the assumption: $u^0, u^1, u^2, u^3 \ll 1, u^4 \sim 1$ (Approximation II). In addition to a low velocity, this farther causes a very low specific charge $\frac{\rho_0}{\mu_0}$ of the moving matter; because then $d\sigma^2 \sim ds^2, v^\rho \sim u^\rho$ becomes, it follows from (8), if one sets [1] ([1] For the sake of the equations of motion; see the next section.) $\alpha = \sqrt{\frac{\chi}{2}} = 3.06x10^{-14}$:

$$\rho_0 = \frac{\chi}{\alpha} \mu_0 u^0 = 2\alpha \mu_0 u^0 \ll \mu_0. \tag{8a}$$

Above all, this equation teaches us that in this case we may understand the electric charge essentially as the fifth component of the momentum which is "inclined" to the spaces x^0 = const. of the "moving" mass: A further fusion of two otherwise heterogeneous basic concepts appears complete.

Since, finally, in approximation II, $T_{00}, T_{11}, T_{22,} T_{33} \sim 0$, according to (7):

$$T = g^{ik} T_{ik} = -T_{44} = -\mu_0, \tag{9}$$

thus for the usual form of the field equations of the first kind:

$$R_{00} = -R_{44} = \frac{\chi}{2}\mu_0. \tag{10}$$

Thus, according to (6), the corner potential g turns out essentially as a negative gravitational potential, while $\mathfrak{G} = \frac{g_{44}}{2}$ retains the old meaning.

* * *

Having thus in a satisfactory manner the governing quantities of the field equations, the question arises as to whether the "geodesic" equations of motion in R_5:

$$\dot{u}^l = \frac{du^l}{ds} = \Gamma^l_{rs} u^r u^s \tag{11}$$

now also represent the motion of charged matter in the gravitational and electromagnetic field in an experience-faithful manner. In Approximation II this is readily the case: Because of the interchangeability of ds and $d\sigma$ one obtains according to (3):

$$\bar{v}^\lambda = \frac{dv^\lambda}{d\sigma} = \Gamma^\lambda_{\rho\sigma} v^\rho v^\sigma + 2\alpha F^\lambda_\chi u^0 v^\chi - g_{.\lambda} {u^0}^2, \qquad (11a)$$

i.e. because of the smallness of the term with ${u^0}^2$ for the ponderomotive force density one obtains:

$$\pi^\lambda = \mu_0 \bar{v}^\lambda = \Gamma^\lambda_{\rho\sigma} T^{\rho\sigma} + F^\lambda_\chi I^\chi$$

$$(\alpha = \sqrt{\frac{\chi}{2}} \text{ taken; see (8a)).} \qquad (12)$$

The total force thus splits by itself into a gravitational and an electromagnetic part of the usual form.

Finally, there remains for the 0-component of (11) only:

$$\dot{u}^0 = \alpha \Sigma_{44} = 2\alpha q_{4.4}\,, \qquad (11b)$$

so that, in the Approximation II, the conditional quasi-static $\frac{d}{dx^4}\left(\frac{\rho_0}{\mu_0}\right) = 2\chi q_{4.4}$ [1] ([1] See (8a).) becomes small in higher order: The required constancy of ρ_0 thus appears guaranteed.

Also for the equations of motion, therefore, the "associated" field remains meaningless in our approximation.

* * *

If Approximation II corresponds to reality, then the above desired unification theory would be satisfactorily carried out in its main features: *A single potential tensor* generates a *universal field* which under ordinary conditions splits into a *gravitational* and an *electrical part*.

Now, however, the matter in its last building blocks is at least not at all weakly charged; its “Rest in the Big” stands, to talk to H. Weyl, opposite to its “Unrest in the Small”, and this applies, in the above view, especially for the new world parameter x^0: For the electron or the H-nucleus $\frac{\rho_0}{\mu_0}$ indeed is not small and thus the “velocity”-component u^0 is not at all small! Thus, in the form determined by Approximation II, the theory can, at most roughly, phenomenologically *summarize the macro physical events,* and thus it follows the central question of its applicability to precisely those elementary particles.

However, if one tries to describe the *movement of the electron* by a geodesic in R_5, one immediately encounters a serious *difficulty*[1] ([1] I owe a hint of this discrepancy in the development of the above approaches to the valuable interest of Mr. Einstein) that threatens to collapse the erected structure. It consists briefly, that for the electron because of $\frac{e}{m} = 1.77x10^7$ (reduced to light second) u^0 becomes, with the rigid application of the earlier assumptions, so enormously large, that the last term in (11a) instead of disappearing, assumes a value that exceeds everything and defies all experience, even if everything else formally remains the same. Now, although the transition to large u^0 anyway requires modifications (so eliminates the substitutability of ds by $d\sigma$), after all it hardly seems possible to carry out the theory solely in the old framework without a new hypothesis.

On the other hand, I believe - with all reservations – to see in the *following direction* a way open, which, if it leads to the goal, would develop a more satisfactory point of view.

Since at not too large velocities of the field-producing matter, also for any u^0, $R_{00} \sim -R_{44}$ remains, the two gravitational terms in (11a) assume, with appropriate determination of the still completely irrelevant reality character of x^0, opposite signs, and it then appears, by abandoning the already somewhat questionable gravitational constant χ that a reconciliation of the conflicting orders of magnitude come about, where gravitation remains as a kind of the difference effect. This path captivates with the prospect of being able to assign the role of a *statistical quantity* to that constant. But at the moment the consequences of this hypothesis still cannot be examined sufficiently; there are also still other possibilities to examine. After all, the sphinx of modern physics, quantum theory, threatens every approach that demands universal validity.

* * *

Despite full acknowledgment of the described physical as well as the epistemological difficulties that pile up before the view developed here, it will be difficult to believe that in all those relationships which cannot be outperformed in terms of *formal unity*, only a capricious coincidence drives it’s tempting game. But if it should be confirmed that there is more behind the presumed correlations than just an empty formalism, this would definitely mean a new triumph for Einstein's general theory of relativity, whose appropriate *application to a five-dimensional world* comes into question here.

Discussion

Kaluza begins his paper with a statement of a fundamental similarity between gravity and electromagnetism. Both forces emerge from the corresponding potentials: electromagnetic from the *four potential* q_μ and gravitational from the fundamental metric tensor $g_{\mu\upsilon}$ which must be interpreted in the general theory of relativity as the *tensor potential of gravity.* In essence, both of these potentials have the same mathematical identity: both are tensors and Kaluza call this mathematical identity the dualism of gravity and electricity. By introducing this concept of the dualism of gravity and electricity into discussion, there is no return back, as the concept naturally stimulates the feelings of a more fundamental picture from which both forces would emerge. Basically, we should focus on this more fundamental picture, not on the concept of the dualism of gravity and electricity, and this is the message of Kaluza when he says that the concept calls for its overcoming by a completely *unified picture of the world*.

Kaluza then refers to the approach to the solution of this problem of unification by Hermann Weyl [6]. According to

Einstein, the phenomena of gravity can be attributed to the world's metric and the laws through which matter and metrics interact are nothing else but gravity laws. While the gravitational potentials are components of an invariant quadratic differential form, electromagnetic phenomena are controlled by four potentials whose components are components of an invariant linear differential form. This fundamental similarity was most likely led Weyl to look for a geometry that, when applied to the world, would explain not only gravitational phenomena but also electromagnetic phenomena. All physical quantities have a sense in the geometry of the world. The action appears from the beginning as a pure number and leads, in essence, to one unified universal law. It even allows us to understand in a certain sense why the world is four-dimensional. Through the action as a pure number, the Pythagoreans' belief that all is number, in some way seems to be justified. The metric connection of the space depends not only on the quadratic differential form, but also on the linear differential form. For the analytical representation of the geometry forms $g_{ik}dx_i dx_k$ and $\phi_i dx_i$ are on the same footing as

$\lambda g_{ik} dx_i dx_k$ and $\phi_i dx_i + d(ln\lambda)$ where λ is an arbitrary position function. An invariant quantity is an anti-symmetric tensor with components $F_{ik} = \frac{\partial \phi_i}{\partial x_k} - \frac{\partial \phi_k}{\partial x_i}$. Namely, ϕ_i transforms as

$$\phi_i \longrightarrow \phi_i + \frac{1}{\lambda}\frac{\partial \lambda}{\partial x_i}$$

which leads that $\frac{\partial \phi_i}{\partial x_k}$ transforms as

$$\frac{\partial \phi_i}{\partial x_k} \longrightarrow \frac{\partial \phi_i}{\partial x_k} - \frac{1}{\lambda^2}\,\frac{\partial \lambda}{\partial x_k}\frac{\partial \lambda}{\partial x_i} + \frac{1}{\lambda}\frac{\partial^2 \lambda}{\partial x_k \partial x_i}$$

from which it can be seen that F_{ik} is invariant to λ transformations. It is very suggestive here to interpret ϕ_i as an electromagnetic potential and a tensor F_{ik} as an electromagnetic field tensor. Thus, Weyl, as Kaluza says, "found in a further radical revision of the geometrical foundations, in addition to the tensor $g_{\mu\nu}$, another kind of metric fundamental vector, and interpreted it as the electromagnetic potential q_μ: The complete world metric is presented there as the common source of all natural phenomena." Unfortunately, this approach to solving the problem of unifying gravity and electromagnetism was not successful. 32 years after the publication of the paper *Gravity and Electricity* 1918 [6], Weyl stated that "There holds, as we know now, a principle of gauge invariance in nature; but it does not connect the electromagnetic potentials ϕ_i, as I had assumed, with Einstein's gravitational potentials g_{ik}, but ties them to the four components of the wave field ψ by which Schr$\ddot{o}$dinger and Dirac taught us to represent the electron.”

Kaluza also seeks a solution to this unification problem, but in another way. Namely, his idea is that the gravitational and electromagnetic fields arise from a single universal tensor. In his paper he wants to show that such a close union between the two forces in the world appears to be possible in principle. A formal correspondence in the construction of gravitational and electromagnetic equations temptingly indicates that both fields could have a common source. For example, in the approximation of a weak gravitational field, the gravitational potential satisfies the Poisson equation (see Part A of the Appendix)

$$\nabla^2 V = 4\pi\rho \ ,$$

exactly the same equation that the electrostatic potential satisfies

$$\nabla^2 \phi = -4\pi\rho \ .$$

On the other hand, the rotor shape of the electromagnetic field components $F_{\chi\lambda} = q_{\chi.\lambda} - q_{\lambda.\chi}$ cannot be neglected so that Kaluza introduces the assumption $\frac{1}{2}F_{\chi\lambda} = \frac{1}{2}(q_{\chi.\lambda} - q_{\lambda.\chi})$ where the factor $\frac{1}{2}$ is in a formal correspondence with the same factor in the three-index quantities $\begin{bmatrix} i\lambda \\ \chi \end{bmatrix} = \frac{1}{2}\left(g_{i\chi.\lambda} + g_{\chi\lambda.i} - g_{i\lambda.\chi}\right)$. The problem is that in the four dimensional world there can be no additional three-index quantities that would represent $\frac{1}{2}F_{\chi\lambda}$. So we stayed with one very radical idea how to solve this problem: in essence, we have no choice but to call for a new, *fifth dimension of the world* in order to generate additional three-index quantities. The idea is that we write the three-index quantities $\begin{bmatrix} i\lambda \\ \chi \end{bmatrix}$ in 5 dimensions, hoping that in this way the gravitational and electromagnetic fields will result from one universal tensor. Of course, our entire physical experience is such that we do not have any feelings or hints for the fifth dimension. But also nothing does limit us to look at our spatial-temporal world as a *four dimensional part* of the fifth-dimensional world R_5. To reconcile these two opposing views, we must take into account the fact that we never notice any changes other than space-time changes in the state variables, so we have to set their derivatives with respect to the fifth dimension parameter equal to zero, or treat them to be small as they are of higher orders. This is a "condition of the cylinder" that allows the common coexistence of these two opposing views. The fear that this could undo the introduction of the fifth dimension is unfounded because the fifth-dimensional parameter is treated in the three-index quantities in the same way as the usual parameters of space and time in four dimensions.

So we consider one fifth-dimensional space R_5 in which besides the usual parameters x^1 to x^4 (3 spatial and one time dimension) we have an additional dimension x^0 as a new parameter. g_{rs} is a fundamental metric tensor in this fifth-dimensional space which is defined in the same way as the fundamental metric tensor in the general theory of relativity. Analogy with the general theory of relativity is complete, only we require that the relevant tensors "live" in five dimensions instead of four. Thus, the term for Christoffel Symbols of the First Kind in 5 dimensions is the same as for four dimensions, except that the index values go from 0 to 4,

$$\Gamma_{ikl} = \frac{1}{2}(g_{ik.l} - g_{kl.i} - g_{li.k})$$

where $i, k, l = 0{,}1{,}2{,}3{,}4$. Kaluza denotes three-index quantities $\begin{bmatrix} ik \\ l \end{bmatrix}$ with $-\Gamma_{ikl}$. In part a of the Appendix we showed a detailed derivation of the equations of the set (1) based on the condition of the cylinder. One encouraging feature of the set of equations (1) is that $\Gamma_{0\chi\lambda}$ is actually appearing in a rotor form. However, ten $\Gamma_{\chi\lambda 0}$ which also have to be of an electric nature, are not in the rotor form. Nevertheless, since $\Gamma_{0\chi\lambda}$ are in the rotor form, they must be proportional to the components of the electromagnetic tensor $F_{\chi\lambda}$ so we assume equalities (2). Equalities (2) essentially mean that the *fundamental metric tensor* of R_5 is basically a *tensor potential of gravity* that is corrected by an *electromagnetic four potential.* The role of the pure fifth-dimensional component g_{00} is unclear so far. It is interesting to point out here that in order to continue this way, we must accept that in addition to the anti-symmetric electromagnetic tensor $F_{\chi\lambda}$ we have a new unexpected symmetric electromagnetic tensor $\Sigma_{\chi\lambda} = q_{\chi.\lambda} + q_{\lambda.\chi}$, which in a way is the opposite of the electromagnetic tensor $F_{\chi\lambda}$, and which together with it creates a balanced view on the whole problem. In part b of the Appendix we have shown a detailed derivation of the equations of the set (3) based on the tensors $F_{\chi\lambda}$ and $\Sigma_{\chi\lambda}$ and the gradient of the fifth-dimensional component g_{00}.

The antisymmetry of the electromagnetic tensor $F_{\chi\lambda}$ requires 16 different quantities $\Gamma_{0\chi\lambda}$, while the symmetry of the tensor $\Sigma_{\chi\lambda}$ requires 10 different quantities $\Gamma_{\chi\lambda 0}$. The four-dimensional gradient of $\mathcal{g}$ defines 4 different quantities $\Gamma_{00\chi}$ and 4 different quantities $\Gamma_{0\chi 0}$. Additionally, $\mathcal{g}$ by g_{00} also determines the quantity Γ_{000}. Thus, with the tensors $F_{\chi\lambda}$ and $\Sigma_{\chi\lambda,}$ as well as the four-dimensional gradient of $\mathcal{g}$, 35 new three-index quantities are determined out of which 5 disappear since $F_{\chi\chi} = 0$ and $\Gamma_{000} = 0$. Parts c and d of the Appendix contain detailed derivation of the set of equations (4) and (4a). And so completely unexpectedly we arrive at the compact record of the two Maxwell equations

$$\frac{1}{c}\frac{\partial \vec{H}}{\partial t} = -\vec{\nabla} \times \vec{E}$$

$$\vec{\nabla}\vec{H} = 0.$$

This is one very promising sign that we are probably on the right path of unifying gravity and electromagnetism. In this sense, to reduce the complexity of the problem with respect to explicit writing of the tensor equations, we assume that g_{rs} deviates only a little from the "euclidean" value $-\delta_{rs}$ so that the determinant of g_{rs} in the first approximation is equal to the negative unit matrix. Part e of the Appendix contains all the details regarding the derivation of the equations of the set (5), where we can see why the symmetric electromagnetic tensor $\Sigma_{\chi\lambda}$ does not appear in these equations, so that from the electric quantities only the electromagnetic tensor $F_{\chi\lambda}$ contributes to the curvature of R_5. Finally, in part f of the Appendix, we explicitly did write out in the first approximation the curvature tensor components, where something unusual and exciting happened. Not only did we get the usual equations of the gravitational field, but we also got electromagnetic fundamental equations, which is exactly what we were looking for. But, as we see, there is a price for all of this, expressed in the additional Poisson equation for g for which at this moment we have no interpretation. Nonetheless, fifteen components of the curvature tensor clearly show that gravity and electricity could be understood as manifestations of a single, universal field.

Kaluza is now introducing the energy tensor of matter in the game that does rule the righthand sides of the field equations. In section g of the Appendix we repeated the concepts related to the energy tensor of matter and showed that in the approximation I we have $T_{ik} = T^{ik} = \mu_0 u^i u^k$.

It seems that Kaluza implicitly assumed this form of Einstein's equations in 5 dimensions:

$$R_{ik} - \frac{1}{2} g_{ik} R = -\chi T_{ik}$$

with a negative sign on the right, since in the paper it is stated that $R_{0\mu} = -\chi T_{0\mu}$. Having in mind the approximation I, $R_{0\mu} \approx -\alpha F_{\mu\rho.\rho} \sim -\alpha I_\mu = -\chi T_{0\mu}$ and given that the components of the four-current are $\rho_0 v_\mu$, we find that

$$I_\mu = \rho_0 v_\mu = \frac{\chi}{\alpha} T_{0\mu} = \frac{\chi}{\alpha} \mu_0 u_0 u_\mu.$$

Since in the approximation I, $T_{ik} = T^{ik}$, this is exactly Kaluza's equation (8): $I^\mu = \rho_0 v^\mu = \frac{\chi}{\alpha} T_{0\mu} = \frac{\chi}{\alpha}\mu_0 u^0 u^\mu$. We see that the spatial-temporal components of the energy tensor do depend on the components of the four-current.

Kaluza assumes a small speed (Approximation II): $u^0, u^1, u^2, u^3 \ll 1, u^4 \sim 1$. It is easy to see that this means that $d\sigma^2 \sim ds^2$ and $v^\rho \sim u^\rho$. Namely, if we go from the expression for ds^2 and we do explicitly write it, we find

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0\mu}dx^0 dx^\mu + d\sigma^2.$$

If we divide both sides with ds^2 we find further

$$1 = g_{00}(\frac{dx^0}{ds})^2 + 2g_{0\mu}\frac{dx^0}{ds}\frac{dx^\mu}{ds} + \frac{d\sigma^2}{ds^2}$$

or

$$1 = g_{00}(u^0)^2 + 2g_{0\mu}u^0 u^\mu + \frac{d\sigma^2}{ds^2}.$$

Fully writing the first two terms in the approximation II gives

$g_{00}(u^0)^2 + 2g_{01}u^0u^1 + 2g_{02}u^0u^2 + 2g_{03}u^0u^3 + 2g_{04}u^0u^4 \approx$

$g_{00}(u^0)^2 + 2g_{01}u^0u^1 + 2g_{02}u^0u^2 + 2g_{03}u^0u^3 + 2g_{04}u^0$

since $u^4 \sim 1$. Since $u^0, u^1, u^2, u^3 \ll 1$ these 5 terms are negligible, so $1 \approx \frac{d\sigma^2}{ds^2}$ or $d\sigma^2 \sim ds^2$. This means that $d\sigma \sim ds$ so that

$v^\rho = \frac{dx^\rho}{d\sigma} \sim \frac{dx^\rho}{ds} = u^\rho$. We can easily understand why in the small-speed approximation $u^4 \sim 1$. In the special theory of relativity we do know that $d\sigma^2 = (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$. If we divide both sides with $d\sigma^2$, we find

$$1 = (\frac{dx^4}{d\sigma})^2 - (\frac{dx^1}{d\sigma})^2 - (\frac{dx^2}{d\sigma})^2 - (\frac{dx^3}{d\sigma})^2 =$$

$$= (u^4)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2$$

and since $u^1, u^2, u^3 \ll 1$, this means that $(u^4)^2 \sim 1$ or $u^4 \sim 1$. Returning to the equation (8), since $v^\rho \sim u^\rho$, we find that $\rho_0 u^\mu \sim \frac{\chi}{\alpha}\mu_0 u^0 u^\mu$, that is, $\rho_0 \sim \frac{\chi}{\alpha}\mu_0 u^0$, which is the first part of the equation (8a). To get to the second part of the equation (8a) we must use the equations of motion how Kaluza states in his paper, from which follows the relation between α and χ, namely $\alpha = \sqrt{\frac{\chi}{2}}$. We will show that now.

We start from the geodetic equations of motion (11) in R_5: $\dot{u^l} = \frac{du^l}{ds} = \Gamma^l_{rs} u^r u^s$. Since in the approximation II, $d\sigma \sim ds$ and $v^\rho \sim u^\rho$, we find that

$$\frac{dv^\lambda}{d\sigma} = \Gamma^\lambda_{rs} u^r u^s = \Gamma^\lambda_{0s} u^0 u^s + \Gamma^\lambda_{\rho s} v^\rho u^s =$$

$$= \Gamma^\lambda_{00} {u^0}^2 + \Gamma^\lambda_{0\chi} u^0 v^\chi + \Gamma^\lambda_{\rho 0} v^\rho u^0 + \Gamma^\lambda_{\rho\sigma} v^\rho v^\sigma.$$

By writing further we do obtain

$$\frac{dv^\lambda}{d\sigma} = \Gamma^\lambda_{\rho\sigma} v^\rho v^\sigma - \Gamma_{0\chi\lambda} u^0 v^\chi - \Gamma_{\chi 0\lambda} v^\chi u^0 - \Gamma_{00\lambda} {u^0}^2$$

so that due to the relations (3)

$$\frac{dv^\lambda}{d\sigma} = \Gamma^\lambda_{\rho\sigma} v^\rho v^\sigma - 2\alpha F_{\chi\lambda} u^0 v^\chi - g_{.\lambda} {u^0}^2 \, .$$

In the approximation I is $g^{\lambda\mu} F_{\chi\mu} = F^\lambda_\chi \approx -\delta^{\lambda\mu} F_{\chi\mu} = -F_{\chi\lambda}$ so that finally we get the relation (11a)

$$\bar{v}^\lambda = \frac{dv^\lambda}{d\sigma} = \Gamma^\lambda_{\rho\sigma} v^\rho v^\sigma + 2\alpha F^\lambda_\chi u^0 v^\chi - g_{.\lambda} {u^0}^2.$$

For the ponderomotive force density, then we find, neglecting the term with ${u^0}^2$ in the approximation II and, of course, having in mind small velocities,

$$\pi^\lambda = \mu_0 \bar{v}^\lambda = \Gamma^\lambda_{\rho\sigma} T^{\rho\sigma} + F^\lambda_\chi I^\chi = \Gamma^\lambda_{\rho\sigma} \mu_0 v^\rho v^\sigma + 2\alpha F^\lambda_\chi \mu_0 u^0 v^\chi$$

from which it follows that $I^\chi = 2\alpha\mu_0 u^0 v^\chi = \rho_0 v^\chi$. Since we have shown that $\rho_0 \sim \frac{\chi}{\alpha}\mu_0 u^0$ in the approximation II, it does finally follow that $2\alpha = \frac{\chi}{\alpha}$ or $\alpha = \sqrt{\frac{\chi}{2}}$. $F^\lambda_\chi I^\chi$ term is precisely the Lorentz force which causes the motion of the charged matter to deviate from the geodetic line.

We are now in a position to show the second part of the equation (8a). Kaluza explicitly states in the paper that

$$\alpha = \sqrt{\frac{\chi}{2}} = 3.06x10^{-14}$$

so that

$$\chi = 2\alpha^2 = 1.87x10^{-27}.$$

This value is not surprising since Kaluza simply assumed the same form of Einstein's equations, only in 5 dimensions. If we recall Einstein's equations of the four dimensional space and time in the presence of matter

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta}$$

where Einstein's gravitational constant is $\kappa = \frac{8\pi G}{c^4}$, Kaluza assumed that the energy tensor of matter in 5 dimensions had the same form as in the four dimensional space and time with the same binding constant, i.e. $\kappa = \chi$. Indeed, if we put in κ the values of Newton's gravitational constant G and the speed of light c,

$$G = 6.67408x10^{-11} m^3 kg^{-1} s^{-2}$$

$$c = 299792458 m s^{-1}$$

we find that $\kappa = 1.866x10^{-27} \frac{cm}{g} \approx 1.87x10^{-27} \frac{cm}{g}$.

Since $\chi = 2\alpha^2$ we have $\rho_0 = 2\alpha\mu_0 u^0$ so that, due to the tinyness of α and $u^0 \ll 1$, it follows that $\rho_0 \ll \mu_0$. So Kaluza did show that if we assume the small velocities and that the energy tensor of matter in 5 dimensions has the same form as in the four dimensional space and time with the same binding constant, we must understand the electric charge in essence as the fifth component of the impulse ($\rho_0 \sim \frac{\chi}{\alpha}\mu_0 u^0$) and that the rest charge density is much smaller than the rest mass density ($\rho_0 \ll \mu_0$). Then the electric charge can be identified with a motion in the fifth dimension, one completely unexpected conclusion.

Returning to the energy tensor of matter, we see that in the approximation II

$$T_{00} = T^{00} = \mu_0 u^0 u^0 \sim 0$$

$$T_{11} = T^{11} = \mu_0 u^1 u^1 \sim 0$$

$$T_{22} = T^{22} = \mu_0 u^2 u^2 \sim 0$$

$$T_{33} = T^{33} = \mu_0 u^3 u^3 \sim 0$$

since $u^0, u^1, u^2, u^3 \ll 1$. This means that the trace of the energy tensor is

$$T = g^{ik}T_{ik} \approx -\delta^{ik}T_{ik} = -T_{ii} \approx -T_{44} = -\mu_0 u^4 u^4 \approx -\mu_0$$

since $u^4 \sim 1$ in the approximation II. If we recall Einstein's equations of the four-dimensional space and time

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta}$$

and multiply both sides by $g^{\alpha\beta}$, we find

$$R_{\alpha}^{\alpha} - \frac{1}{2} g_{\alpha}^{\alpha} R = \kappa T_{\alpha}^{\alpha}.$$

Since $R = R_{\alpha}^{\alpha}$, $g_{\alpha}^{\alpha} = 4$ and $T = T_{\alpha}^{\alpha}$ we get $-R = \kappa T$ where

$$T = T^{11} + T^{22} + T^{33} + T^{44} \approx T^{44} = \mu_0 \, .$$

We find that $R = -\chi\mu_0$. In part h of the Appendix we have shown that the electromagnetic field does not contribute to the curvature of the four-dimensional space and time, so that R in the fifth-dimensional form of Einstein's equations which Kaluza assumed

$$R_{ik} - \frac{1}{2} g_{ik} R = -\chi T_{ik}$$

is exactly equal to $-\chi\mu_0$. For R_{00} in the approximation II we find

$$R_{00} - \frac{1}{2} g_{00} R = -\chi T_{00} = 0.$$

The left side is due to the approximation I equal to

$$R_{00} - \tfrac{1}{2}(-\delta_{00})R = R_{00} + \tfrac{1}{2}R$$

so that $R_{00} = -\frac{1}{2}R = -\frac{1}{2}(-\chi\mu_0) = \frac{1}{2}\chi\mu_0$. In the same way, we find R_{44}

$$R_{44} - \frac{1}{2} g_{44} R = -\chi T_{44} = -\chi\mu_0.$$

The left side is $R_{44} + \frac{1}{2}R$ so that

$$R_{44} = -\tfrac{1}{2}R - \chi\mu_0 = \tfrac{1}{2}\chi\mu_0 - \chi\mu_0 = -\tfrac{1}{2}\chi\mu_0.$$

This was precisely the derivation of the Kaluza's equation (10)

$$R_{00} = -R_{44} = \frac{\chi}{2}\mu_0.$$

If we now recall the equations (6), that $R_{00} = -\square \mathscr{g}$, this means that

$$\square \mathscr{g} = -\frac{\chi}{2}\mu_0,$$

i.e. that $\mathscr{g}$ potential is essentially a negative gravitational potential. Kaluza says that $\mathfrak{G} = \frac{g_{44}}{2}$ retains the old meaning. Namely, $\frac{g_{44}}{2}$ in our notation is the same as $\frac{g_{00}}{2} \approx \frac{1}{2} + V$ and in section j of the Appendix we have shown that V is Newton's potential.

On pages 34 and 35 we have shown that the equations of motion (11) in R_5 represent the motion of the charged matter in the gravitational and electromagnetic field. It is interesting now to consider the 0-component of the equations of motion (11)

$$\dot{u}^0 = \frac{du^0}{ds} = \Gamma^0_{rs} u^r u^s.$$

In the approximation II, $u^0, u^1, u^2, u^3 \ll 1, u^4 \sim 1$ so that

$$\dot{u}^0 = \Gamma^0_{0s} u^0 u^s + \Gamma^0_{\alpha s} u^\alpha u^s =$$

$$= \Gamma^0_{00} u^0 u^0 + \Gamma^0_{0\alpha} u^0 u^\alpha + \Gamma^0_{\alpha 0} u^\alpha u^0 + \Gamma^0_{\alpha\beta} u^\alpha u^\beta =$$

$$= -\Gamma_{0\alpha 0}\, u^0 u^\alpha - \Gamma_{\alpha 00} u^\alpha u^0 - \Gamma_{\alpha\beta 0} u^\alpha u^\beta =$$

$$= \frac{1}{2} g_{00.\alpha} u^0 u^\alpha + \frac{1}{2} g_{00.\alpha} u^\alpha u^0 + \alpha \Sigma_{\alpha\beta} u^\alpha u^\beta =$$

$$= g_{00.\alpha} u^0 u^\alpha + \alpha \Sigma_{\alpha\beta} u^\alpha u^\beta \approx \alpha \Sigma_{44}.$$

Let us now recall that $\rho_0 = 2\alpha\mu_0 u^0 \ll \mu_0$ (the equation (8a)). From this relation it follows that

$$\frac{\rho_0}{\mu_0} = 2\alpha u^0$$

so that $\frac{d}{dx^4}\left(\frac{\rho_0}{\mu_0}\right) = 2\alpha \frac{du^0}{dx^4}$. In the approximation II we have $u^4 = \frac{dx^4}{ds} \sim 1$ which means that $dx^4 \sim ds$ so that $\frac{du^0}{dx^4} \sim \frac{du^0}{ds} = \dot{u}^0$ and

$$\frac{d}{dx^4}\left(\frac{\rho_0}{\mu_0}\right) = 2\alpha(\alpha\Sigma_{44}) = 2\alpha^2\Sigma_{44}.$$

As $\Sigma_{44} = q_{4.4} + q_{4.4} = 2q_{4.4}$, we finally get that

$$\dot{u}^0 \approx 2\alpha q_{4.4}$$

and

$$\frac{d}{dx^4}\left(\frac{\rho_0}{\mu_0}\right) = 4\alpha^2 q_{4.4} = 4\frac{\chi}{2}q_{4.4} = 2\chi q_{4.4}.$$

We have seen that 10^{-27} is the order of magnitude for χ so that $\frac{d}{dx^4}\left(\frac{\rho_0}{\mu_0}\right) \approx 0$ and the required constantity of ρ_0 seems really guaranteed. It is also interesting that the "associated" field $\Sigma_{\chi\lambda}$ does not enter the equations of motion

$$\frac{dv^\lambda}{d\sigma} = \Gamma^\lambda_{\rho\sigma}v^\rho v^\sigma - 2\alpha F_{\chi\lambda}u^0 v^\chi - g_{.\lambda}{u^0}^2$$

so it remains without meaning in the approximation II. If the approximation II corresponds to reality then the desired unification theory is realized in its main characteristics in a satisfactory way. Namely, one tensor potential creates a universal field which, at the level of our everyday experience, is divided into a gravitational and electromagnetic part.

However, there is a problem. The matter is not at all weakly charged in its last building blocks. Following Weyl's expression, its "Rest in the Big" is opposed to its "Unrest in the Small". At the level of everyday experience, the matter seems continuous to us, without sudden fluctuations, but the situation at the level of elementary particles is completely opposite. This is especially true for the new world parameter x^0. Namely, for electron or proton, $\frac{\rho_0}{\mu_0}$ is not small, and therefore u^0 component of the "velocity" is not at all small. Let's remember that $u^0 = \frac{1}{2\alpha}\frac{\rho_0}{\mu_0}$ and $\alpha = 3.06x10^{-14}$. This means that in the form determined by the approximation II, the theory can only describe roughly macroscopic phenomena and there is a fundamental problem of its applicability to elementary particles.

Namely, if we try to describe the motion of an electron using a geodetic line in $R5$, then u^0 does become enormously large that the last member in (11a) can no longer be ignored. This is precisely because for an electron, according to Kaluza,

$$\frac{\rho_0}{\mu_0} = \frac{e}{m} = 1.77x10^7$$

so $u^0 = 2.89x10^{20}$. In the CGS system, the electromagnetic unit for electric charge is emu and the relation between that unit and the electrostatic unit for the electric charge (statC) is $1statC = \frac{1}{c}emu$ where c denotes the speed of light. Since $1C = 3x10^9 statC = (3x10^9)\left(\frac{1}{3x10^{10}}\right)emu = \frac{1}{10}emu$ where we used $3x10^{10}$ value for the speed of light in the CGS system, for the electric charge of the electron e we find that

$$e = 1.6021917x10^{-19}C = (1.6021917x10^{-19})\left(\frac{1}{10}\right)emu$$

$$= 1.6021917x10^{-20}emu.$$

Since the mass of the electron m is equal to $9.109558x10^{-28}g$, we find that

$$\frac{e}{m} = 1.7588028x10^7 \frac{emu}{g} \approx 1.76x10^7 \frac{emu}{g},$$

which is very close to the Kaluza’s value. We do see that Kaluza under the term "reduced to light second" understood the use of an electromagnetic unit for the electric charge in calculating the $\frac{e}{m}$ quantity. Definitely, switching to a large u^0 requires modifications, since $d\sigma \sim ds$ is no longer sustainable, so it is not possible to implement a theory without new hypotheses only in the old framework.

Kaluza believes that he sees a solution to the problem in some other direction. He bases his argument on the equation (10)

$$R_{00} = -R_{44} = \frac{\chi}{2}\mu_0.$$

We have derived this equation under the assumption of approximations I and II. Namely, in the approximation II (a small velocity) we have shown that $R = -\chi\mu_0$. Including this expression in the Einstein's fifth-dimensional equation, under the assumption of approximations I (that g_{rs} deviates only a little from the "euclidean" value $-\delta_{rs}$) and II, we have concluded that $R_{00} = \frac{1}{2}\chi\mu_0$. In the same way we found for R_{44} that $R_{44} = -\frac{1}{2}\chi\mu_0$. Then it is to be expected that at not too large velocities of the matter producing a field and for any u^0, R_{00} remains approximately equal to $-R_{44}$, i.e. $R_{00} \sim -R_{44}$. This has the consequence that the two gravitational terms in (11a) assume the opposite signs. Namely, according to (6), $R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu.\rho}$ so that $R_{44} = \Gamma^{\rho}_{44.\rho}$. On the other hand it is to be expected that at not too large velocities of the matter the first gravitational term in (11a) is

$$\Gamma^{\lambda}_{\rho\sigma} v^{\rho} v^{\sigma} \sim \Gamma^{\lambda}_{44} (v^4)^2 \sim \Gamma^{\lambda}_{44} {u^4}^2,$$

while the other term is $-g_{.\lambda} {u^0}^2 = -\Gamma_{00\lambda} {u^0}^2 = \Gamma^{\lambda}_{00} {u^0}^2$. In section f of the Appendix we have shown that $R_{00} \approx -g_{.\rho\rho} = (-\Gamma_{00\rho})_{.\rho} = \Gamma^{\rho}_{00.\rho}$. So $R_{00} \sim -R_{44}$ means that

$$\Gamma^{\rho}_{00.\rho} \sim -\Gamma^{\rho}_{44.\rho}$$

with a trivial solution $\Gamma^{\rho}_{00} \sim -\Gamma^{\rho}_{44}$. This solution clearly shows that the two gravitational terms in (11a) presume the opposite signs

$$\Gamma^{\lambda}_{\rho\sigma} v^{\rho} v^{\sigma} - g_{.\lambda} {u^{0}}^{2} \sim \Gamma^{\lambda}_{44} {u^{4}}^{2} + \Gamma^{\lambda}_{00} {u^{0}}^{2}$$

in which gravity remains a kind of the difference effect. This explanation assumes that the realistic character of x^0 is still completely irrelevant, i.e. that the "cylinder condition" is still valid and that the value of the constant χ is based on the fact that Kaluza did simply assume the same form of Einstein's equations, only in 5 dimensions. Kaluza assigns to this constant a role of a statistical quantity. At the time of writing his paper, Kaluza did not have time to investigate the consequences of the above solution sufficiently. He implies that other possibilities should also be taken into account and is aware that the last word could come from the quantum theory.

Kaluza ends the paper with the conclusion that it is hard to believe that only a coincidence can explain a set of relations that faithfully does reproduce both Einstein's equations of gravity and

Maxwell's equations of electromagnetism. If, as he says, "it should be confirmed that there is more behind the presumed correlations than just an empty formalism", then that would be another success for Einstein's theory of relativity, its purely classical extension to five dimensions.

In the end

Kaluza was not considered to be sufficiently worthy to be promoted as can be seen from the fact that he was a Privatdozent at the University of Königsberg for 20 years [8]. Most academics were promoted after a few years as Privatdozent. All the weight of this incredible situation is illustrated by the fact that the position of the Privatdozent at that time was practically a position without money.

Kaluza did not lose his motivation and when he was writing to Einstein in April 1919 and told him about his ideas to unite Einstein's theory of gravity and Maxwell's electromagnetic theory he still did lecture at Königsberg. Einstein encouraged him to publish his original ideas what he did in 1921, publishing the paper entitled *On the Unification Problem in Physics*. Einstein presented the paper on December 8 that same year. In November 1926, Einstein wrote that Kaluza is a mathematician who came up with an extraordinary idea trying to help Kaluza to be promoted. Only in 1929, exactly 20 years after being appointed as a Privatdozent, Kaluza became a professor in Kiel. In 1935 Kaluza became a full professor at the University of Göttingen where he remained until his death in 1954, two months before retirement.

He did not accept the national socialist ideology [9] and, therefore, his appointment as a professor in Göttingen was problematic and possible only thanks to the protection of his colleague Helmut Hasse. Kaluza was extremely versatile. He spoke or wrote 17 languages [10] including Hebrew, Hungarian, Arabic and Lithuanian. He even wrote letters in Arabic, his favorite language. He did not emphasize the depth of his knowledge of mathematics, physics, astronomy, chemistry, biology, law, philosophy and literature. He was an unusually modest person and it is said that in the thirties he learned by himself to swim by reading a book about it and that he did swim at the first attempt.

As we have already said, in April 1919, Kaluza discovered that Maxwell's equations appear completely spontaneously when Einstein's equations of general relativity are solved in five dimensions. This insight is remembered as Kaluza-Klein Theory. At the beginning of the book, we said that this is a classic unified field theory of gravity and electromagnetism, built on the idea of the fifth dimension, which is added to the usual dimensions: three dimensions of space and one of time, and that this theory is considered an important forerunner of the string theory. Klein gave this theory a quantum-mechanical interpretation in accordance with the quantum mechanics knowledge of that time. This theory has been neglected for years because attention has been focused on quantum mechanics, so that Kaluza's idea that the fundamental forces can be explained by additional dimensions only emerged again in the development of the string theory. It is important to note that in 1914, Gunnar Nordström [9] introduced an additional spatial dimension in his theory of gravity, which allowed the coupling of electromagnetism with gravity. This was the first of the extra dimensional theories. However, it was experimentally confirmed that Nordström's theory of gravity was inferior to Einstein's theory of gravity because it did not predict the curvature of light that was observed during the solar eclipse of 1919. So Kaluza was rightly "the inventor of the fifth dimension".

Only in the mid-seventies of the last century interest [10] for additional dimensions was re-awakened. Two new forces of nature were discovered, weak and strong interaction, so the unification of all four forces of nature became a major challenge.

The Kaluza theory and the Klein idea of 1926 that the fifth dimension is curved into a small circle have become the basis of the so-called superstring theory. This theory has promised for the first time that it will reconcile quantum mechanics with the general theory of relativity. Small strings that by their vibrations produce all the elementary particles, make up the basic components of the physical world. However, in order for this theory to provide a consistent description of the nature, the space can not have only one, but six additional dimensions that “close-up” on themselves to form circles.

It is not questionable how much of the inspiration Kaluza felt at the moment when he discovered that Maxwell's equations appear completely spontaneously when Einstein's equations of general relativity are solved in five dimensions. Kaluza’s son [3, page 108] later described this moment of inspiration when Kaluza realized that the fifth dimension would allow space for an unified theory of nature: "He sat completely still for several seconds, and then he whispered very sharply and banged the table, and he stood up but remained completely motionless for several seconds. Then he began to hum the last part of an Aria of Figaro.”

Probably the same state of mind took the builders of the Fountain of the Lions in the Alhambra, when in their own way with pearly water flow they combined light with gravity. This excerpt from the reference 11 illustrates the basic idea: “The common guiding thought of all builders of the Alhambra was to create a place that will be an image of the paradise. For them, this meant achieving the unity of human endeavor with the elements of nature. That is why this whole space is a play of light and shadow, and water that symbolizes life is constantly on the move; at one moment it flows calmly, in another moment it bubbles or falls. Everything exudes with harmony and beauty that comes out from the knowledge of numbers and their proportions, so the Alhambra equally celebrates art and mathematics.” We have seen in equations (6) how mathematics, in one elegant, almost intimidating, and at the same time artistic way, did unify Maxwell's equations and the Einstein equations of the general theory of relativity.

And as the equations (6) are a reflection of a masterful usage of mathematics and the Fountain is only a reflection of a masterful usage of numbers and their ratio, art, and mathematics. It relies on 12 marble lions and every hour another lion throws water out of its mouth. To make the mood perfect, on the outer wall of this gem are the verses of Ibn Zamrak, the most famous poet of the Alhambra, with which we end this book. And who knows, maybe Kaluza still explores new dimensions in that other finite dimensionality, enjoying the beauty of this brushed pearl sprinkled with morning dew.

Poem of the Fountain of the Lions

Ibn Zamrak (1333 – 1393)

May the God, who gifted the imam Muhammad with the talent for decorating his villas, be blessed!

Are not miracles in this garden that God made, incomparable in its beauty, like this fountain where the transparent light is reflected from the edges woven with pearl seeds?

A brushed pearl sprinkled with morning dew.

Liquid silver flows over precious stones and flows even more luminous.
Solid and fluid are so alike that it's hard to know what marble and what the water is.

And which of them flows?

Don’t you see how water flows over the edges of the fountain?
As a lover who tries to hide tears from his beloved in vain.
And, in the truth, the fountain is only a cloud that quenches the thirst of the lions.

Just as the hand of the caliph in the morning gives away his blessings to the lions of war.
And those fearsome lions are calm out of respect for him.

Oh, the true successor of the Ansares, your sublime inheritance diminishes the firmness of the mountains!

May the peace of God be with you forever, and your joys numerous and your enemies disappointed.

Appendix

For a mathematically inclined reader, in this appendix we have collected detailed derivations of the various equations appearing in Kaluza's paper.

a. Derivation of the set of equations (1) on page 15

These equations are in fact explicit expressions for Christoffel Symbols of the First Kind in 5 dimensions. We begin with the definition of Γ_{ikl},

$$\Gamma_{ikl} = \frac{1}{2}(g_{ik.l} - g_{kl.i} - g_{li.k})$$

where $i, k, l = 0,1,2,3,4$. The same expression with Greek indices only is

$$\Gamma_{\chi\lambda\mu} = \frac{1}{2}(g_{\chi\lambda.\mu} - g_{\lambda\mu.\chi} - g_{\mu\chi.\lambda})$$

or $2\Gamma_{\chi\lambda\mu} = g_{\chi\lambda.\mu} - g_{\lambda\mu.\chi} - g_{\mu\chi.\lambda}$ where $\chi, \lambda, \mu = 1,2,3,4$. This is exactly the first equation of the set (1) on page 15. Similar to this equation, when the first index is 0,

$$2\Gamma_{0\lambda\mu} = g_{0\lambda.\mu} - g_{\lambda\mu.0} - g_{\mu0.\lambda}\,.$$

Based on the cylinder condition $g_{\lambda\mu.0} = 0$ and the symmetry of the fundamental metric tensor $g_{rs} = g_{sr}$ it follows that $2\Gamma_{0\lambda\mu} = g_{0\lambda.\mu} - g_{0\mu.\lambda}$ which is exactly the second equation of the set (1). We can also see that

$$2\Gamma_{\chi 0\mu} = g_{\chi 0.\mu} - g_{0\mu.\chi} - g_{\mu\chi.0} = g_{0\chi.\mu} - g_{0\mu.\chi} = 2\Gamma_{0\chi\mu}\,.$$

This is to be expected since Γ_{ikl} is symmetrical in the first two indices as can be seen from its definition above. Of course, this symmetry arises from the symmetry of the fundamental metric tensor g_{rs}. Continuing in the same way, the remaining components of Γ_{ikl} are simplified due to the cylinder condition:

$$2\Gamma_{\chi\lambda 0} = g_{\chi\lambda.0} - g_{\lambda 0.\chi} - g_{0\chi.\lambda} = -(g_{0\lambda.\chi} + g_{0\chi.\lambda})$$

$$2\Gamma_{00\mu} = g_{00.\mu} - g_{0\mu.0} - g_{\mu 0.0} = g_{00.\mu}$$

$$2\Gamma_{0\lambda 0} = g_{0\lambda.0} - g_{\lambda 0.0} - g_{00.\lambda} = -g_{00.\lambda} = 2\Gamma_{\lambda 00}$$

$$2\Gamma_{000} = g_{00.0} - g_{00.0} - g_{00.0} = 0\,.$$

This completes the derivation of the equations of the set (1) on page 15.

b. Derivation of the set of equations (3) on page 16

We start with the term for $\Gamma_{0\chi\lambda}$

$$2\Gamma_{0\chi\lambda} = g_{0\chi.\lambda} - g_{0\lambda.\chi} \,.$$

Based on the assumption that $g_{0\chi} = 2\alpha q_{\chi}$ we have $2\Gamma_{0\chi\lambda} = 2\alpha q_{\chi.\lambda} - 2\alpha q_{\lambda.\chi}$. Since $F_{\chi\lambda} = q_{\chi.\lambda} - q_{\lambda.\chi}$ we finally get $\Gamma_{0\chi\lambda} = \alpha F_{\chi\lambda}$ which is exactly the first equation of the set (3).

Since $2\Gamma_{\chi\lambda 0} = -(g_{0\chi.\lambda} + g_{0\lambda.\chi})$, similarly we obtain $\Gamma_{\chi\lambda 0} = -\alpha\Sigma_{\chi\lambda}$ where $\Sigma_{\chi\lambda} = g_{0\chi.\lambda} + g_{0\lambda.\chi}$. Finally, since we assume that $g_{00} = 2\mathcal{g}$ and since

$$2\Gamma_{00\chi} = -2\Gamma_{0\chi 0} = g_{00.\chi}$$

we find that $\Gamma_{00\chi} = -\Gamma_{0\chi 0} = \mathcal{g}_{.\chi}$.

This completes the derivation of the equations of the set (3).

c. Derivation of the equation (4) on page 16

If we explicitly write out the left side we get

$$(\Gamma_{ikl} + \Gamma_{kli} + \Gamma_{lik})_{.m} = \Gamma_{ikl.m} + \Gamma_{kli.m} + \Gamma_{lik.m} =$$

$$= \frac{1}{2}(g_{ik.l} - g_{kl.i} - g_{li.k})_{.m} + \frac{1}{2}(g_{kl.i} - g_{li.k} - g_{ik.l})_{.m}$$

$$+ \frac{1}{2}(g_{li.k} - g_{ik.l} - g_{kl.i})_{.m} =$$

$$= -\frac{1}{2}(g_{kl.im} + g_{li.km} + g_{ik.lm})$$

where the colored terms $g_{ik.l}, g_{li.k}$ and $g_{kl.i}$ cancel each other. If we now add and subtract the same terms in the last expression, which essentially does not change anything, we get

$$\frac{1}{2}(g_{mi.kl} - g_{ik.ml} - g_{km.il} + g_{mk.li} - g_{kl.mi} - g_{lm.ki}$$

$$+ g_{ml.ik} - g_{li.mk} - g_{im.lk})$$

where we colored in red the terms we did add and subtract, assuming that the order of the metric tensor derivations is irrelevant, for example $g_{ik.lm} = g_{ik.ml}$. These 9 terms can be further grouped as

$$\frac{1}{2}(g_{mi.k} - g_{ik.m} - g_{km.i})_{.l} + \frac{1}{2}(g_{mk.l} - g_{kl.m} - g_{lm.k})_{.i}$$

$$+ \frac{1}{2}(g_{ml.i} - g_{li.m} - g_{im.l})_{.k}$$

which is equal to $\Gamma_{mik.l} + \Gamma_{mkl.i} + \Gamma_{mli.k}$. We have thus shown that

$$(\Gamma_{ikl} + \Gamma_{kli} + \Gamma_{lik})_{.m} = \Gamma_{mik.l} + \Gamma_{mkl.i} + \Gamma_{mli.k}$$

which is exactly the equation (4).

d. Derivation of relations (4a) on page 16

Due to the cylinder condition, the equation (4) for m = 0 is reduced to

$$(\Gamma_{ikl} + \Gamma_{kli} + \Gamma_{lik})_{.0} = 0 = \Gamma_{0ik.l} + \Gamma_{0kl.i} + \Gamma_{0li.k} \,.$$

If we now limit the values of i, k, l to 1, 2, 3, 4 which is equal to the use of Greek indices, we find $\Gamma_{0\chi\lambda.\mu} + \Gamma_{0\lambda\mu.\chi} + \Gamma_{0\mu\chi.\lambda} = 0$. In part b of this Appendix we have shown that $\Gamma_{0\chi\lambda} = \alpha F_{\chi\lambda}$ so that we further have

$$\alpha F_{\chi\lambda.\mu} + \alpha F_{\lambda\mu.\chi} + \alpha F_{\mu\chi.\lambda} = \alpha\left(F_{\chi\lambda.\mu} + F_{\lambda\mu.\chi} + F_{\mu\chi.\lambda}\right) = 0$$

and since $\alpha \neq 0$ we finally get

$$F_{\chi\lambda.\mu} + F_{\lambda\mu.\chi} + F_{\mu\chi.\lambda} = 0$$

which is the first (4a) relation. This is in fact a compact record of the two Maxwell equations

$$\frac{1}{c}\frac{\partial\vec{H}}{\partial t} = -\vec{\nabla} \times \vec{E}$$

$$\vec{\nabla}\vec{H} = 0.$$

For $i = k = 0$ the equation (4) is reduced to

$$(\Gamma_{00l} + \Gamma_{0l0} + \Gamma_{l00})_{.m} = \Gamma_{m00.l} + \Gamma_{m0l.0} + \Gamma_{ml0.0} = \Gamma_{m00.l} \,,$$

where the last two terms vanish due to the cylinder condition. We have already shown that

$$\Gamma_{m00} = -\frac{1}{2} g_{00.m} = -\frac{1}{2} 2\mathcal{g}_{.m} = -\mathcal{g}_{.m}.$$

In the same way $\Gamma_{00l} = \mathcal{g}_{.l}$ and $\Gamma_{0l0} = -\mathcal{g}_{.l}$ so that

$$(\Gamma_{00l} + \Gamma_{0l0} + \Gamma_{l00})_{.m} = (\mathcal{g}_{.l} - \mathcal{g}_{.l} - \mathcal{g}_{.l})_{.m} = -\mathcal{g}_{.lm} =$$

$$= -\mathcal{g}_{.ml}$$

or $\mathcal{g}_{.lm} = \mathcal{g}_{.ml}$. This is, of course, true for Greek indices: $\mathcal{g}_{.\chi\lambda} = \mathcal{g}_{.\lambda\chi}$ which is the second (4a) relation. It is interesting to show now that the consistency of the equation (4) for $m = 0$ is exactly based on the second (4a) relation and the cylinder condition. For $m = i = 0$ case, we have

$$0 = \Gamma_{00k.l} + \Gamma_{0kl.0} + \Gamma_{0l0.k} = \mathcal{g}_{.kl} - \mathcal{g}_{.lk} = 0\,.$$

In the same way, for $m = k = 0$ and $m = l = 0$ cases, we find

$$0 = \Gamma_{0i0.l} + \Gamma_{00l.i} + \Gamma_{0li.0} = -\mathcal{g}_{.il} + \mathcal{g}_{.li} = 0$$

and

$$0 = \Gamma_{0ik.0} + \Gamma_{0k0.i} + \Gamma_{00i.k} = -\mathcal{g}_{.ki} + \mathcal{g}_{.ik} = 0\,.$$

The remaining cases are even more obvious. For $m = i = k = 0$ we obtain

$$0 = \Gamma_{000.l} + \Gamma_{00l.0} + \Gamma_{0l0.0} = 0\,.$$

In the same way, for $m = i = l = 0$ and $m = k = l = 0$ we find

$$0 = \Gamma_{00k.0} + \Gamma_{0k0.0} + \Gamma_{000.k} = 0$$

and

$$0 = \Gamma_{0i0.0} + \Gamma_{000.i} + \Gamma_{00i.0} = 0\,.$$

Finally, when all indices are equal to 0, $m = k = l = i = 0$,

$$0 = \Gamma_{000.0} + \Gamma_{000.0} + \Gamma_{000.0} = 0\,.$$

e. Derivation of the *other four-index tensors* of interest (5) on page 16

We begin with the Riemann-Christoffel tensor in 5 dimensions

$$R^i_{jkl} = \Gamma^i_{jk.l} - \Gamma^i_{jl.k} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}.$$

Its relation with Ricci tensor can be expressed with $R^i_{jki} = R_{jk} = \{ji, ik\}$. In general, $R^i_{jkl} = \{ji, lk\}$. Now we will show that in the approximation I, $\{\chi\lambda, \mu 0\} = \alpha F^{\lambda}_{\chi.\mu}$. According to the definition,

$$\{\chi\lambda, \mu 0\} = R^{\lambda}_{\chi 0\mu} = \Gamma^{\lambda}_{\chi 0.\mu} - \Gamma^{\lambda}_{\chi\mu.0} + \Gamma^m_{\chi\mu}\Gamma^{\lambda}_{m0} - \Gamma^m_{\chi 0}\Gamma^{\lambda}_{m\mu} =$$

$$= \Gamma^{\lambda}_{\chi 0.\mu} + \Gamma^m_{\chi\mu}\Gamma^{\lambda}_{m0} - \Gamma^m_{\chi 0}\Gamma^{\lambda}_{m\mu}$$

where the second term on the right side disappears due to the cylinder condition. Explicitly writing out the sum by the index m as a sum by the index 0 and the Greek index β, we find further

$$\{\chi\lambda, \mu 0\} = \Gamma^{\lambda}_{\chi 0.\mu} + \Gamma^0_{\chi\mu}\Gamma^{\lambda}_{00} + \Gamma^{\beta}_{\chi\mu}\Gamma^{\lambda}_{\beta 0} - \Gamma^0_{\chi 0}\Gamma^{\lambda}_{0\mu} - \Gamma^{\beta}_{\chi 0}\Gamma^{\lambda}_{\beta\mu}\,.$$

Now we need to calculate each term on the right side. Since $\Gamma^{\lambda}_{\chi 0} = g^{\lambda\beta}\Gamma_{\chi 0\beta} = g^{\lambda\beta}\alpha F_{\chi\beta} = \alpha F^{\lambda}_{\chi}$, where we used the equality $2\Gamma_{\chi 0\lambda} = 2\Gamma_{0\chi\lambda} = 2\alpha F_{\chi\lambda}$ which we previously explicitly derived, we finally find $\Gamma^{\lambda}_{\chi 0.\mu} = \alpha F^{\lambda}_{\chi.\mu}$.

For the second term $\Gamma^0_{\chi\mu}\Gamma^\lambda_{00}$ we get in the approximation I, where g_{rs} deviates only a little from the "Euclidean" value $-\delta_{rs}$,

$$\Gamma^0_{\chi\mu}\Gamma^\lambda_{00} = g^{0k}\Gamma_{\chi\mu k}g^{\lambda p}\Gamma_{00p} \approx \left(-\Gamma_{\chi\mu 0}\right)\left(-\Gamma_{00\lambda}\right) =$$

$$= -\alpha\Sigma_{\chi\mu}g_{.\lambda} = -\alpha g_{.\lambda}\Sigma_{\chi\mu} .$$

Namely, in the approximation I, $g_{rs} = -\delta_{rs} + h_{rs,}$ where $h_{rs} \ll 1$. The contravariant tensor g^{rs} must satisfy the condition $g_{mn}g^{ns} = \delta^s_m = \{1\ if\ s = m, 0\ if\ s \neq m\}$. It is easy to see that the condition is satisfied in the approximation I if we assume that $g^{ns} = -\delta^{ns} + h^{ns}$ where $h^{ns} = -h_{ns}$. We have

$$g_{mn}g^{ns} = (-\delta_{mn} + h_{mn})(-\delta^{ns} + h^{ns}) = \delta_{mn}\delta^{ns} -$$

$$-\delta_{mn}h^{ns} - h_{mn}\delta^{ns} + h_{mn}h^{ns} .$$

The last term $h_{mn}h^{ns}$ can be ignored in the approximation I so that

$$g_{mn}g^{ns} \approx \delta^s_m - h^{ms} - h_{ms} = \delta^s_m - (-h_{ms}) - h_{ms} =$$

$$= \delta^s_m + h_{ms} - h_{ms} = \delta^s_m .$$

We have thus shown that in the approximation I, $g^{rs} \approx -\delta^{rs}$ which is precisely the reason why $g^{0k}\Gamma_{\chi\mu k}g^{\lambda p}\Gamma_{00p} \approx \left(-\Gamma_{\chi\mu 0}\right)\left(-\Gamma_{00\lambda}\right)$.

In the same way we find for the remaining terms in the approximation I

$$\Gamma^{\beta}_{\chi\mu}\Gamma^{\lambda}_{\beta 0} = g^{\beta k}\Gamma_{\chi\mu k}g^{\lambda p}\Gamma_{\beta 0 p} \approx (-\Gamma_{\chi\mu\beta})(-\Gamma_{\beta 0\lambda}) =$$

$$= \Gamma_{\chi\mu\beta}\alpha F_{\beta\lambda} = \alpha F_{\beta\lambda}\Gamma_{\chi\mu\beta}$$

$$\Gamma^{0}_{\chi 0}\Gamma^{\lambda}_{0\mu} = g^{0k}\Gamma_{\chi 0k}g^{\lambda p}\Gamma_{0\mu p} \approx (-\Gamma_{\chi 00})(-\Gamma_{0\mu\lambda}) =$$

$$= -g_{.\chi}\alpha F_{\mu\lambda} = -\alpha g_{.\chi}F_{\mu\lambda}$$

$$\Gamma^{\beta}_{\chi 0}\Gamma^{\lambda}_{\beta\mu} = g^{\beta k}\Gamma_{\chi 0k}g^{\lambda p}\Gamma_{\beta\mu p} \approx (-\Gamma_{\chi 0\beta})(-\Gamma_{\beta\mu\lambda}) =$$

$$= \alpha F_{\chi\beta}\Gamma_{\beta\mu\lambda}$$

so that $\{\chi\lambda, \mu 0\} \approx \alpha F^{\lambda}_{\chi\cdot\mu} - \alpha g_{.\lambda}\Sigma_{\chi\mu} + \alpha F_{\beta\lambda}\Gamma_{\chi\mu\beta} + + \alpha g_{.\chi}F_{\mu\lambda} - \alpha F_{\chi\beta}\Gamma_{\beta\mu\lambda}$.

Compared to $\alpha F^{\lambda}_{\chi\cdot\mu}$ in the approximation I all other terms are negligible, so we finally find $\{\chi\lambda, \mu 0\} \approx \alpha F^{\lambda}_{\chi\cdot\mu}$.

We can also show that $\{\chi 0,0\lambda\} = -g_{.\chi\lambda}$.

$$\{\chi 0,0\lambda\} = R^{0}_{\chi\lambda 0} = \Gamma^{0}_{\chi\lambda.0} - \Gamma^{0}_{\chi 0.\lambda} + \Gamma^{m}_{\chi 0}\Gamma^{0}_{m\lambda} - \Gamma^{m}_{\chi\lambda}\Gamma^{0}_{m0} =$$

$$= -\Gamma^{0}_{\chi 0.\lambda} + \Gamma^{0}_{\chi 0}\Gamma^{0}_{0\lambda} + \Gamma^{\beta}_{\chi 0}\Gamma^{0}_{\beta\lambda} - \Gamma^{0}_{\chi\lambda}\Gamma^{0}_{00} - \Gamma^{\beta}_{\chi\lambda}\Gamma^{0}_{\beta 0}$$

Continuing we find

$$\Gamma^{0}_{\chi 0} = g^{0\alpha}\Gamma_{\chi 0\alpha} = -\Gamma_{\chi 00} = -(-g_{.\chi}) = g_{.\chi}$$

so that $\Gamma^{0}_{\chi 0.\lambda}$=$g_{.\chi\lambda}$. For the remaining terms we get

$$\Gamma^0_{\chi 0}\Gamma^0_{0\lambda} = (-\Gamma_{\chi 00})(-\Gamma_{0\lambda 0}) = (-g_{.\chi})(-g_{.\lambda}) = g_{.\chi}g_{.\lambda}$$

$$\Gamma^\beta_{\chi 0}\Gamma^0_{\beta\lambda} = (-\Gamma_{\chi 0\beta})(-\Gamma_{\beta\lambda 0}) = (\alpha F_{\chi\beta})(-\alpha\Sigma_{\beta\lambda}) =$$

$$= -\alpha^2 F_{\chi\beta}\Sigma_{\beta\lambda}$$

$$\Gamma^\beta_{\chi\lambda}\Gamma^0_{\beta 0} = (-\Gamma_{\chi\lambda\beta})(-\Gamma_{\beta 00}) = \Gamma_{\chi\lambda\beta}\Gamma_{\beta 00} = \Gamma_{\chi\lambda\beta}(-g_{.\beta}) =$$

$$= -g_{.\beta}\Gamma_{\chi\lambda\beta}$$

and finally

$$\{\chi 0,0\lambda\} = -g_{.\chi\lambda} + g_{.\chi}g_{.\lambda} - \alpha^2 F_{\chi\beta}\Sigma_{\beta\lambda} + g_{.\beta}\Gamma_{\chi\lambda\beta}$$

so that in the approximation I, $\{\chi 0,0\lambda\} \approx -g_{.\chi\lambda}$ ignoring all other terms in comparison with $g_{.\chi\lambda}$.

The remaining components of the *other four-index tensors* of interest are exact, which can easily be shown.

$$\{\chi\lambda,00\} = R^\lambda_{\chi 00} = \Gamma^\lambda_{\chi 0.0} - \Gamma^\lambda_{\chi 0.0} + \Gamma^m_{\chi 0}\Gamma^\lambda_{m0} - \Gamma^m_{\chi 0}\Gamma^\lambda_{m0} = 0$$

$$\{\chi 0,00\} = R^0_{\chi 00} = \Gamma^0_{\chi 0.0} - \Gamma^0_{\chi 0.0} + \Gamma^m_{\chi 0}\Gamma^0_{m0} - \Gamma^m_{\chi 0}\Gamma^0_{m0} = 0$$

$$\{00,00\} = R^0_{000} = \Gamma^0_{00.0} - \Gamma^0_{00.0} + \Gamma^m_{00}\Gamma^0_{m0} - \Gamma^m_{00}\Gamma^0_{m0} = 0 .$$

f. Derivation of relations (6) page 17

We begin with the expression for Ricci tensor $R_{ik} = \{ir, rk\} =$

$= R^r_{ikr} = \Gamma^r_{ik.r} - \Gamma^r_{ir.k} + \Gamma^m_{ir}\Gamma^r_{mk} - \Gamma^m_{ik}\Gamma^r_{mr}$.

For Greek indices we have

$$R_{\mu\nu} = \Gamma^r_{\mu\nu.r} - \Gamma^r_{\mu r.\nu} + \Gamma^m_{\mu r}\Gamma^r_{m\nu} - \Gamma^m_{\mu\nu}\Gamma^r_{mr}\,.$$

Explicitly writing out further the sums by the indexes m and r as sums by the index 0 and the Greek index we find

$$R_{\mu\nu} = \Gamma^0_{\mu\nu.0} + \Gamma^\rho_{\mu\nu.\rho} - \Gamma^0_{\mu 0.\nu} - \Gamma^\rho_{\mu\rho.\nu} + \Gamma^0_{\mu r}\Gamma^r_{0\nu} + \Gamma^\rho_{\mu r}\Gamma^r_{\rho\nu}$$

$$- \Gamma^0_{\mu\nu}\Gamma^r_{0r} - \Gamma^\rho_{\mu\nu}\Gamma^r_{\rho r} =$$

$$= \Gamma^\rho_{\mu\nu.\rho} - \Gamma^0_{\mu 0.\nu} - \Gamma^\rho_{\mu\rho.\nu} + \Gamma^0_{\mu 0}\Gamma^0_{0\nu} + \Gamma^0_{\mu\rho}\Gamma^\rho_{0\nu}$$

$$+ \Gamma^\rho_{\mu 0}\Gamma^0_{\rho\nu} + \Gamma^\rho_{\mu\beta}\Gamma^\beta_{\rho\nu} - \Gamma^0_{\mu\nu}\Gamma^0_{00} - \Gamma^0_{\mu\nu}\Gamma^\rho_{0\rho}$$

$$- \Gamma^\rho_{\mu\nu}\Gamma^0_{\rho 0} - \Gamma^\rho_{\mu\nu}\Gamma^\beta_{\rho\beta}$$

where we used the cylinder condition. Except for the first term, now we have to calculate the remaining terms individually. So we find for the second term

$$\Gamma^0_{\mu 0.\nu} = -\Gamma_{\mu 00.\nu} = -(-g_{.\mu}).\nu = g_{.\mu\nu}\,.$$

In order to calculate $\Gamma^\rho_{\mu\rho.\nu}$, it is useful to calculate Γ^r_{ir} in 5 dimensions first. We start with

$$\Gamma^r_{ir} = g^{rs}\Gamma_{irs} = g^{rs}\frac{1}{2}(g_{ir.s} - g_{rs.i} - g_{si.r}) =$$

$$= \frac{1}{2}(g^{rs}g_{ir.s} - g^{rs}g_{rs.i} - g^{rs}g_{si.r}) \,.$$

If in the third term we exchange indexes of summation $r \leftrightarrow s$, due to the symmetry of the metric tensor we obtain the first term

$$g^{rs}g_{si.r} = g^{sr}g_{ri.s} = g^{rs}g_{ir.s}$$

so that $\Gamma^r_{ir} = -\frac{1}{2}g^{rs}g_{rs.i}$. Using the known relation $g_{.i} = gg^{rs}g_{rs.i}$, which can be proven by differentiating the determinant g, which means that we have to differentiate each element g_{rs} and multiply with the gg^{rs} cofactor, we find that $\Gamma^r_{ir} = -\frac{1}{2}g^{-1}g_{.i} = -\frac{1}{2}(\ln(g))_{.i}$.

We are now in a position to calculate $\Gamma^{\rho}_{\mu\rho.\nu}$.

$$\Gamma^r_{\mu r} = -\frac{1}{2}(\ln(g))_{.\mu} = \Gamma^0_{\mu 0} + \Gamma^{\rho}_{\mu\rho} = -\Gamma_{\mu 00} + \Gamma^{\rho}_{\mu\rho} = \mathscr{g}_{.\mu} + \Gamma^{\rho}_{\mu\rho}$$

where we did explicitly write out the sum by the index r as a sum by the index 0 and the Greek index ρ. We find $\Gamma^{\rho}_{\mu\rho} = -\mathscr{g}_{.\mu} - \frac{1}{2}(\ln(g))_{.\mu}$ so that finally

$$\Gamma^{\rho}_{\mu\rho.\nu} = -\mathscr{g}_{.\mu\upsilon} - \frac{1}{2}(\ln(g))_{.\mu\upsilon} \,.$$

For the next three terms, by the usual procedure, we find

$$\Gamma^0_{\mu 0}\Gamma^0_{0\nu} = \left(-\Gamma_{\mu 00}\right)\left(-\Gamma_{0\nu 0}\right) = \mathcal{g}_{.\mu}\mathcal{g}_{.\upsilon}$$

$$\Gamma^0_{\mu\rho}\Gamma^{\rho}_{0\nu} = \left(-\Gamma_{\mu\rho 0}\right)\left(-\Gamma_{0\nu\rho}\right) = \left(-\alpha\Sigma_{\mu\rho}\right)\left(\alpha F_{\nu\rho}\right) =$$

$$= -\alpha^2 F_{\nu\rho}\Sigma_{\mu\rho}$$

$$\Gamma^{\rho}_{\mu 0}\Gamma^0_{\rho\upsilon} = \left(-\Gamma_{\mu 0\rho}\right)\left(-\Gamma_{\rho\nu 0}\right) = \Gamma_{\rho\nu 0}\Gamma_{0\mu\rho} = -\alpha^2 F_{\mu\rho}\Sigma_{\rho\upsilon}$$

where for the last term we used the fact that Γ_{ikl} is symmetric in the first two indices so that we could use the value of the previous term. $\Gamma^{\rho}_{\mu\beta}\Gamma^{\beta}_{\rho\nu}$ term we will not explicitly write out since nothing would be simplified by that. For the remaining terms, we find

$$\Gamma^0_{\mu\nu}\Gamma^{\rho}_{0\rho} = -\Gamma_{\mu\nu 0} \times 0 = 0$$

$$\Gamma^{\rho}_{\mu\nu}\Gamma^0_{\rho 0} = \Gamma^{\rho}_{\mu\nu}\left(-\Gamma_{\rho 00}\right) = \mathcal{g}_{.\rho}\Gamma^{\rho}_{\mu\nu}$$

$$\Gamma^{\rho}_{\mu\nu}\Gamma^{\beta}_{\rho\beta} = \Gamma^{\rho}_{\mu\nu}\left(-\mathcal{g}_{.\rho} - \frac{1}{2}(\ln(g))_{.\rho}\right) =$$

$$= -\Gamma^{\rho}_{\mu\nu}(\mathcal{g}_{.\rho} + \frac{1}{2}(\ln(g))_{.\rho})$$

where in the first of the remaining terms we used the fact that $\Gamma^{\rho}_{0\rho} = 0$ which easily follows from the expression for Γ^{r}_{ir} with $i = 0$.

Namely, $\Gamma^{r}_{0r} = -\frac{1}{2}(\ln(g))_{.0} = 0$ due to the cylinder condition, so that

$$\Gamma^r_{0r} = 0 = \Gamma^0_{00} + \Gamma^\rho_{0\rho} = \Gamma^\rho_{0\rho}\,.$$

Finally, for $R_{\mu\nu}$ we find

$$R_{\mu\nu} = \Gamma^\rho_{\mu\nu.\rho} - \mathcal{g}_{.\mu\nu} - \left(-\mathcal{g}_{.\mu\nu} - \frac{1}{2}(\ln(g))_{.\mu\nu}\right) + \mathcal{g}_{.\mu}\mathcal{g}_{.v}$$

$$- \alpha^2 F_{\nu\rho}\Sigma_{\mu\rho} - \alpha^2 F_{\mu\rho}\Sigma_{\rho\upsilon} + \Gamma^\rho_{\mu\beta}\Gamma^\beta_{\rho\nu} - \mathcal{g}_{.\rho}\Gamma^\rho_{\mu\nu}$$

$$+ \Gamma^\rho_{\mu\nu}\left(\mathcal{g}_{.\rho} + \frac{1}{2}(\ln(g))_{.\rho}\right).$$

The colored terms cancel each other so that we get

$$R_{\mu\nu} = \Gamma^\rho_{\mu\nu.\rho} + \frac{1}{2}(\ln(g))_{.\mu\nu} + \mathcal{g}_{.\mu}\mathcal{g}_{.v} - \alpha^2 F_{\nu\rho}\Sigma_{\mu\rho}$$

$$- \alpha^2 F_{\mu\rho}\Sigma_{\rho\upsilon} + \Gamma^\rho_{\mu\beta}\Gamma^\beta_{\rho\nu} + \frac{1}{2}\Gamma^\rho_{\mu\nu}(\ln(g))_{.\rho},$$

which in the approximation I is reduced to $R_{\mu\nu} \approx \Gamma^\rho_{\mu\nu.\rho}$, since all other terms are practically negligible compared to $\Gamma^\rho_{\mu\nu.\rho}$. This completes the derivation of the first relation of the set (6).

In the same way we find for $R_{0\mu}$,

$$R_{0\mu} = \Gamma^r_{0\mu.r} - \Gamma^r_{0r.\mu} + \Gamma^m_{0r}\Gamma^r_{m\mu} - \Gamma^m_{0\mu}\Gamma^r_{mr} =$$

$$= \Gamma^0_{0\mu.0} + \Gamma^\rho_{0\mu.\rho} - \Gamma^0_{00.\mu} - \Gamma^\rho_{0\rho.\mu} + \Gamma^0_{0r}\Gamma^r_{0\mu} + \Gamma^\rho_{0r}\Gamma^r_{\rho\mu}$$

$$- \Gamma^m_{0\mu}\left(-\frac{1}{2}(\ln(g))_{.m}\right)$$

where we explicitly did write out the sum by the index r as a sum by the index 0 and the Greek index ρ and used the term for Γ^{r}_{ir} for the case $i = m$. Taking into an account that the colored terms disappear due to the cylinder condition, using the expression for $\Gamma^{\rho}_{\mu\rho.\nu}$ for the case $\mu = 0, \nu = \mu$ and writing out the remaining sums by the indexes r and m as sums by the index 0 and the Greek index, we find further

$$R_{0\mu} = \Gamma^{\rho}_{0\mu.\rho} - \left(-g_{.0} - \frac{1}{2}(\ln(g))_{.0}\right)_{.\mu} + \Gamma^{0}_{00}\Gamma^{0}_{0\mu} + \Gamma^{0}_{0\rho}\Gamma^{\rho}_{0\mu} +$$

$$+\Gamma^{\rho}_{00}\Gamma^{0}_{\rho\mu} + \Gamma^{\rho}_{0\beta}\Gamma^{\beta}_{\rho\mu} \quad + \frac{1}{2}\Gamma^{m}_{0\mu}(\ln(g))_{.m} = \Gamma^{\rho}_{0\mu.\rho} + \Gamma^{0}_{0\rho}\Gamma^{\rho}_{0\mu} +$$

$$+\Gamma^{\rho}_{00}\Gamma^{0}_{\rho\mu} + \Gamma^{\rho}_{0\beta}\Gamma^{\beta}_{\rho\mu} + \frac{1}{2}\Gamma^{0}_{0\mu}(\ln(g))_{.0} + \frac{1}{2}\Gamma^{\rho}_{0\mu}(\ln(g))_{.\rho}$$

where the colored terms again disappear due to cylinder condition. Now, we have to calculate each term using the usual methods. For the first we find

$$\Gamma^{\rho}_{0\mu.\rho} = (-\Gamma_{0\mu\rho})_{.\rho} = -\alpha F_{\mu\rho.\rho}\,.$$

For the remaining we get

$$\Gamma^{0}_{0\rho}\Gamma^{\rho}_{0\mu} = (-\Gamma_{0\rho0})(-\Gamma_{0\mu\rho}) = g_{.\rho}(-\alpha F_{\mu\rho}) = -\alpha g_{.\rho}F_{\mu\rho}$$

$$\Gamma^{\rho}_{00}\Gamma^{0}_{\rho\mu} = (-\Gamma_{00\rho})(-\Gamma_{\rho\mu0}) = -g_{.\rho} - (-\alpha\Sigma_{\rho\mu}) =$$

$$= -\alpha g_{.\rho}\Sigma_{\rho\mu}$$

$$\Gamma^{\rho}_{0\beta}\Gamma^{\beta}_{\rho\mu} = \left(-\Gamma_{0\beta\rho}\right)\Gamma^{\beta}_{\rho\mu} = -\alpha F_{\beta\rho}\Gamma^{\beta}_{\rho\mu}$$

$$\frac{1}{2}\Gamma^{\rho}_{0\mu}(\ln(g))_{.\rho} = -\frac{1}{2}\Gamma_{0\mu\rho}(\ln(g))_{.\rho} = -\frac{1}{2}\alpha F_{\mu\rho}(\ln(g))_{.\rho}$$

so that the final expression for $R_{0\mu}$ is

$$R_{0\mu} = -\alpha F_{\mu\rho.\rho} - \alpha g\!\!/_{.\rho}F_{\mu\rho} - \alpha g\!\!/_{.\rho}\Sigma_{\rho\mu} - \alpha F_{\beta\rho}\Gamma^{\beta}_{\rho\mu}$$

$$-\frac{1}{2}\alpha F_{\mu\rho}(\ln(g))_{.\rho}\,.$$

In the approximation I, the first term is dominant and the others are negligible, so that $R_{0\mu} \approx -\alpha F_{\mu\rho.\rho}$. To see the meaning of this term, it is useful to consider

$$\Gamma^{\rho}_{0\mu.\rho} = (g^{\rho i}\Gamma_{0\mu i})_{.\rho} = (g^{\rho\alpha}\Gamma_{0\mu\alpha} + g^{\rho 0}\Gamma_{0\mu 0})_{.\rho}$$

$$\approx \left(g^{\rho\alpha}\Gamma_{0\mu\alpha}\right)_{.\rho} = \left(g^{\rho\alpha}\alpha F_{\mu\alpha}\right)_{.\rho} = \alpha\left(F^{\rho}_{\mu}\right)_{.\rho}$$

$$= \alpha F^{\rho}_{\mu.\rho}$$

where the $g^{\rho 0}\Gamma_{0\mu 0}$ term is negligible in the approximation I. It is known that the remaining Maxwell equations can be written as $F^{\mu\rho}_{.\rho} = 4\pi J^{\mu}$. This is a compact record of

$$\frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times\vec{H} - 4\pi\vec{J}$$

$$\vec{\nabla}\vec{E} = 4\pi\rho$$

equations. Using the metric tensor of the flat space-time $\eta_{\mu\nu}$ where $1, -1, -1, -1$ are on the diagonal, and the other elements are 0, we can lower the index μ in $F^{\mu\rho}_{.\rho} = 4\pi J^{\mu}$

$$\eta_{\alpha\mu} F^{\mu\rho}_{.\rho} = \eta_{\alpha\mu} 4\pi J^{\mu} = 4\pi J_{\alpha} = (\eta_{\alpha\mu} F^{\mu\rho})_{.\rho} = (F^{\rho}_{\alpha})_{.\rho}$$

so that $F^{\rho}_{\alpha.\rho} = 4\pi J_{\alpha}$. Or in Kaluza's notation $F_{\alpha\rho.\rho} = 4\pi J_{\alpha}$. We see that $R_{0\mu} \approx -\alpha F_{\mu\rho.\rho} = -4\pi\alpha J_{\mu}$ so that $R_{0\mu}$ essentially has the meaning of the remaining Maxwell equations.

Finally, for the last relation R_{00} of the set (6) we find

$$\begin{aligned} R_{00} = \Gamma^{r}_{00.r} - {\color{red}\Gamma^{r}_{0r.0}} + \Gamma^{m}_{0r}\Gamma^{r}_{m0} - \Gamma^{m}_{00}\Gamma^{r}_{mr} = \\ = {\color{red}\Gamma^{0}_{00.0}} + \Gamma^{\rho}_{00.\rho} + \Gamma^{0}_{0r}\Gamma^{r}_{00} + \Gamma^{\rho}_{0r}\Gamma^{r}_{\rho 0} - {\color{red}\Gamma^{0}_{00}}\Gamma^{r}_{0r} \\ - \Gamma^{\rho}_{00}\Gamma^{r}_{\rho r} = \\ = \Gamma^{\rho}_{00.\rho} + \Gamma^{0}_{0r}\Gamma^{r}_{00} + \Gamma^{\rho}_{0r}\Gamma^{r}_{\rho 0} - \Gamma^{\rho}_{00}\Gamma^{r}_{\rho r} \end{aligned}$$

where the colored terms disappear due to the cylinder condition. Now we have to calculate the remaining terms individually in the usual way. For $\Gamma^{\rho}_{00.\rho}$ we obtain

$$\Gamma^{\rho}_{00.\rho} = (-\Gamma_{00\rho})_{.\rho} = -g_{.\rho\rho} \, .$$

The remaining terms are

$$\Gamma^0_{0r}\Gamma^r_{00} = \Gamma^0_{00}\Gamma^0_{00} + \Gamma^0_{0\rho}\Gamma^\rho_{00} = (-\Gamma_{0\rho 0})(-\Gamma_{00\rho}) =$$

$$= (g_{.\rho})(-g_{.\rho}) = -(g_{.\rho})^2$$

$$\Gamma^\rho_{0r}\Gamma^r_{\rho 0} = \Gamma^\rho_{00}\Gamma^0_{\rho 0} + \Gamma^\rho_{0\beta}\Gamma^\beta_{\rho 0} =$$

$$= (-\Gamma_{00\rho})(-\Gamma_{\rho 00}) + (-\Gamma_{0\beta\rho})(-\Gamma_{\rho 0\beta}) =$$

$$= (-g_{.\rho})(g_{.\rho}) + (\alpha F_{\beta\rho})(\alpha F_{\rho\beta}) =$$

$$= -(g_{.\rho})^2 + \alpha^2 F_{\beta\rho}F_{\rho\beta}$$

$$\Gamma^\rho_{00}\Gamma^r_{\rho r} = (-\Gamma_{00\rho})\left(-\frac{1}{2}(\ln(g))_{.\rho}\right) = g_{.\rho}\frac{1}{2}(\ln(g))_{.\rho} =$$

$$= \frac{1}{2}g_{.\rho}(\ln(g))_{.\rho}$$

so that for R_{00} we get

$$R_{00} = -g_{.\rho\rho} - (g_{.\rho})^2 - (g_{.\rho})^2 + \alpha^2 F_{\beta\rho}F_{\rho\beta} - \frac{1}{2}g_{.\rho}(\ln(g))_{.\rho} =$$

$$= -g_{.\rho\rho} - 2(g_{.\rho})^2 + \alpha^2 F_{\beta\rho}F_{\rho\beta} - \frac{1}{2}g_{.\rho}(\ln(g))_{.\rho}\,,$$

which in the approximation I is reduced to $R_{00} \approx -g_{.\rho\rho} = -\Box g$ since the other members are negligible compared with $g_{.\rho\rho}$. With this we have completed the derivations of the relations (6) on page 17.

g. Energy tensor of matter

It is useful first to refresh our memory with respect to the concepts related to the energy tensor of matter. We will follow Dirac in this on the basis of his book [7] *General theory of relativity*. In the general theory of relativity we assume that we have a distribution of matter whose velocity varies continuously from one point to a neighboring one [7, Chapter 25]. If z^μ denotes the coordinates of one element of matter, we can introduce a vector of velocity $v^\mu = \frac{dz^\mu}{ds}$ which will be a continuous coordinate function, like a field function. Using $ds^2 = g_{\mu\nu} dz^\mu dz^\nu$ we find that the velocity vector has the following properties

$$1 = g_{\mu\nu} \frac{dz^\mu}{ds} \frac{dz^\nu}{ds} = g_{\mu\nu} v^\mu v^\nu$$

$$0 = (g_{\mu\nu} v^\mu v^\nu)_{:\sigma} = g_{\mu\nu} \left(v^\mu_{:\sigma} v^\nu + v^\mu v^\nu_{:\sigma} \right) =$$

$$= g_{\mu\nu} v^\mu_{:\sigma} v^\nu + g_{\nu\mu} v^\nu v^\mu_{:\sigma} = 2 g_{\mu\nu} v^\mu_{:\sigma} v^\nu$$

where we used the symmetry of the metric tensor, the exchange of the summation indexes ($\mu \leftrightarrow \nu$) in the second term, and the fact that the covariant derivation of the metric tensor vanishes ($g_{\mu\nu:\sigma} = 0$). Obviously, $v^\mu_{:\sigma} v_\mu = 0$.

In an analogy to electromagnetism where J^{μ} determines the density and flow of electricity, we can introduce a scalar field ρ so that the vector field ρv^{μ} determines the density and flow of matter. The law of conservation of electricity is undisturbed by the curvature of the space $J^{\mu}_{:\mu} = 0$. According to this analogy, the law of conservation of matter is $(\rho v^{\mu})_{:\mu} = 0$. It is now convenient to introduce a symmetric tensor $T^{\mu\nu} = \rho v^{\mu} v^{\nu}$. This is exactly the energy tensor of matter that also satisfies a similar conservation law $T^{\mu\nu}_{:\nu} = 0$.

Kaluza did generalize this definition of the energy tensor of matter to 5 dimensions: $T^{ik} = \mu_0 u^i u^k$. In the approximation I, we get

$$T_{ik} = g_{im} g_{kn} T^{mn} \approx (-\delta_{im})(-\delta_{kn}) T^{mn} = T^{ik}$$

so that we have $T_{ik} = T^{ik} = \mu_0 u^i u^k$.

h. The electromagnetic field does not "bend" space-time

We will now show that the electromagnetic field does not contribute to the curvature of space-time. We start from Einstein's equations of the four dimensional space and time in the presence of matter and electromagnetic field [7, Chapter 29]

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu} - 8\pi E^{\mu\nu} .$$

The right side has two terms that come from the energy tensor of matter $T^{\mu\nu}$ and from the Maxwell stress-energy tensor $E^{\mu\nu}$

$$4\pi E^{\mu\nu} = -F^{\mu}_{\rho} F^{\nu\rho} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

where $F^{\alpha\beta}$ is the usual tensor of the electromagnetic field.

If in Einstein's equations we lower the index ν by $g_{\mu\nu}$, we get

$$R^{\mu}_{\mu} - \frac{1}{2} g^{\mu}_{\mu} R = \kappa T^{\mu}_{\mu} - 8\pi E^{\mu}_{\mu},$$

which is further equal to $R - 2R = \kappa T - 8\pi E^{\mu}_{\mu}$ since $R = R^{\mu}_{\mu}$, $g^{\mu}_{\mu} = 4$ and $T = T^{\mu}_{\mu}$. This means that $R = -\kappa T + 8\pi E^{\mu}_{\mu}$. We see that the contribution of the electromagnetic field to the curvature of space-time is equal to $8\pi E^{\mu}_{\mu}$.

To see for sure what this contribution is, we lower the index ν in $E^{\mu\nu}$ by $g_{\mu\nu}$ in the expression for the Maxwell stress-energy tensor

$$4\pi E^{\mu}_{\mu} = -F^{\mu}_{\rho} g_{\mu\nu} F^{\nu\rho} + \frac{1}{4} g^{\mu}_{\mu} F_{\alpha\beta} F^{\alpha\beta} = -F_{\nu\rho} F^{\nu\rho} + F_{\alpha\beta} F^{\alpha\beta} \,.$$

In the first term on the right we lowered the μ index and got $F_{\nu\rho}$. If we replace the summation indexes ν, ρ with α, β we find that $4\pi E^{\mu}_{\mu} = -F_{\alpha\beta} F^{\alpha\beta} + F_{\alpha\beta} F^{\alpha\beta} = 0$ i.e. $E^{\mu}_{\mu} = 0$. This shows that the electromagnetic field does not contribute to the curvature of the four-dimensional space and time.

i. Lowering the indexes of the curvature tensor

In this and the following two parts of the Appendix we follow Dirac again [7] based on his book *General Theory of Relativity*. We proceed from the definition of the curvature tensor [7, Chapter 11]

$$R^{\beta}_{\nu\rho\sigma} = \Gamma^{\beta}_{\nu\sigma.\rho} - \Gamma^{\beta}_{\nu\rho.\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\beta}_{\alpha\sigma} =$$

$$= \Gamma^{\beta}_{\nu\sigma.\rho} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\alpha\rho} - < \rho\sigma >$$

where the symbol $< \rho\sigma >$ indicates the previous two terms in which the ρ and σ indices replaced the places.

We will lower the index β to be the first suffix

$$R_{\mu\nu\rho\sigma} = g_{\mu\beta}R^{\beta}_{\nu\rho\sigma} = g_{\mu\beta}\Gamma^{\beta}_{\nu\sigma.\rho} + \Gamma^{\alpha}_{\nu\sigma}g_{\mu\beta}\Gamma^{\beta}_{\alpha\rho} - g_{\mu\beta} < \rho\sigma >,$$

which is further equal to

$$R_{\mu\nu\rho\sigma} = g_{\mu\beta}\Gamma^{\beta}_{\nu\sigma.\rho} + \Gamma^{\alpha}_{\nu\sigma}\Gamma_{\mu\alpha\rho} - g_{\mu\beta} < \rho\sigma >.$$

It is useful now to go from

$$(g_{\mu\beta}\Gamma^{\beta}_{\nu\sigma})_{.\rho} = \Gamma_{\mu\nu\sigma.\rho} = g_{\mu\beta.\rho}\Gamma^{\beta}_{\nu\sigma} + g_{\mu\beta}\Gamma^{\beta}_{\nu\sigma.\rho}$$

so that

$$g_{\mu\beta}\Gamma^{\beta}_{\nu\sigma.\rho} = \Gamma_{\mu\nu\sigma.\rho} - g_{\mu\beta.\rho}\Gamma^{\beta}_{\nu\sigma}.$$

With this, the term for $R_{\mu\nu\rho\sigma}$ becomes

$$R_{\mu\nu\rho\sigma} = \Gamma_{\mu\nu\sigma.\rho} - g_{\mu\beta.\rho}\Gamma^{\beta}_{\nu\sigma} + \Gamma^{\beta}_{\nu\sigma}\Gamma_{\mu\beta\rho} - < \rho\sigma >$$

where in the $\Gamma^{\alpha}_{\nu\sigma}\Gamma_{\mu\alpha\rho}$ term we replaced the summation index α with β.

From the definition of the Christoffel Symbol of the First Kind

$$\Gamma_{\mu\nu\sigma} = \frac{1}{2}\left(g_{\mu\nu.\sigma} + g_{\mu\sigma.\nu} - g_{\nu\sigma.\mu}\right),$$

we can easily see that

$$\Gamma_{\mu\nu\sigma} + \Gamma_{\nu\mu\sigma} = \frac{1}{2}\left(g_{\mu\nu.\sigma} + g_{\mu\sigma.\nu} - g_{\nu\sigma.\mu}\right) +$$

$$+ \frac{1}{2}\left(g_{\nu\mu.\sigma} + g_{\nu\sigma.\mu} - g_{\mu\sigma.\nu}\right) = g_{\mu\nu.\sigma}.$$

Using this relation we find that

$$-g_{\mu\beta.\rho}\Gamma^{\beta}_{\nu\sigma} + \Gamma^{\beta}_{\nu\sigma}\Gamma_{\mu\beta\rho} = \Gamma^{\beta}_{\nu\sigma}\left(\Gamma_{\mu\beta\rho} - g_{\mu\beta.\rho}\right) = -\Gamma^{\beta}_{\nu\sigma}\Gamma_{\beta\mu\rho}$$

so that

$$R_{\mu\nu\rho\sigma} = \Gamma_{\mu\nu\sigma.\rho} - \Gamma_{\beta\mu\rho}\Gamma^{\beta}_{\nu\sigma} - < \rho\sigma >$$

or in the explicit form

$$R_{\mu\nu\rho\sigma} = \Gamma_{\mu\nu\sigma.\rho} - \Gamma_{\beta\mu\rho}\Gamma^{\beta}_{\nu\sigma} - \Gamma_{\mu\nu\rho.\sigma} + \Gamma_{\beta\mu\sigma}\Gamma^{\beta}_{\nu\rho}.$$

Since

$$\Gamma_{\mu\nu\sigma.\rho} - \Gamma_{\mu\nu\rho.\sigma} = \frac{1}{2}\left(g_{\mu\nu.\sigma} + g_{\sigma\mu.\nu} - g_{\upsilon\sigma.\mu}\right)_{.\rho}$$

$$-\frac{1}{2}\left(g_{\mu\nu.\rho} + g_{\rho\mu.\nu} - g_{\upsilon\rho.\mu}\right)_{.\sigma} =$$

$$= \frac{1}{2}\left(g_{\mu\nu.\sigma\rho} + g_{\sigma\mu.\nu\rho} - g_{\nu\sigma.\mu\rho} - g_{\mu\nu.\rho\sigma} - g_{\rho\mu.\nu\sigma} + g_{\nu\rho.\mu\sigma}\right) =$$

$$\frac{1}{2}\left(g_{\sigma\mu.\nu\rho} - g_{\nu\sigma.\mu\rho} - g_{\rho\mu.\nu\sigma} + g_{\nu\rho.\mu\sigma}\right),$$

it is finally

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma.\nu\rho} - g_{\nu\sigma.\mu\rho} - g_{\mu\rho.\nu\sigma} + g_{\nu\rho.\mu\sigma}\right) +$$

$$+\Gamma_{\beta\mu\sigma}\Gamma^{\beta}_{\nu\rho} - \Gamma_{\beta\mu\rho}\Gamma^{\beta}_{\nu\sigma}$$

where in the first and third term we used the symmetry of the metric tensor in the indexes $\sigma\mu$ and $\rho\mu$.

j. Newton's approximation

In this part of the Appendix [7, Chapter 16], our goal is to show that Einstein's law of gravity becomes Newton's law of gravity when the gravitational field is weak and static. Therefore, we consider a static gravitational field and a static coordinate system. All references to this gravitational field are based on this static coordinate system. The components of the metric tensor $g_{\mu\nu}$ are constant in time, $g_{\mu\nu,0} = 0$. In this and in the last part of the Appendix, the usual derivative is denoted now, as Dirac did, with coma (,) instead of the point (.) used by Kaluza. Next we have that

$$g_{m0} = 0, m = 1, 2, 3.$$

Namely, if we write out the expression for the invariant distance ds between a point x^μ and a neighboring point $x^\mu + dx^\mu$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

we find

$$ds^2 = g_{00}(dx^0)^2 + 2g_{m0} dx^0 dx^m + g_{mn} dx^m dx^n.$$

Our everyday metric is such that there is no mixing of the time component and the space components so we have $g_{m0} = 0$. This further leads to $g^{m0} = 0$ and $g^{00} = (g_{00})^{-1}$. This is easy to show if we remember the relation $g^{\alpha}_{\nu} = g^{\alpha\mu} g_{\mu\nu}$. For $\alpha = \nu = 0$, we find

$$1 = g^{0\mu} g_{\mu 0} = g^{00} g_{00} + g^{0m} g_{m0} = g^{00} g_{00}$$

since $g_{m0} = 0$. Then $g^{00} = (g_{00})^{-1}$. For $\alpha = m, \nu = 0$ we get

$$0 = g^{m\mu} g_{\mu 0} = g^{m0} g_{00} + g^{mn} g_{n0} = g^{m0} g_{00.}$$

Since $g_{00} \neq 0$, otherwise the everyday metric would have no time component, it follows that $g^{m0} = 0$. Finally, for $\alpha = \nu = m$ we find

$$1 = g^{m\mu} g_{\mu m} = g^{m0} g_{0m} + g^{mn} g_{nm} = g^{mn} g_{mn}$$

so that $g^{mn} = (g_{mn})^{-1}$ due to the symmetry $g_{mn} = g_{nm}$ of the metric tensor. Latin indices such as m and n take values $1, 2, 3$. For the Christoffel Symbol of the First Kind

$$\Gamma_{\mu\nu\sigma} = \frac{1}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu})$$

we have

$$\Gamma_{m0n} = \frac{1}{2}\left(g_{m0,n} + g_{mn,0} - g_{0n,m}\right) = 0,$$

which leads to the disappearance of the Christoffel Symbol of the Second Kind

$$\Gamma^{m}_{0n} = g^{m\lambda} \Gamma_{\lambda 0n} = g^{m0} \Gamma_{00n} + g^{mp} \Gamma_{p0n} = 0,$$

where $p = 1, 2, 3$.

Let us now consider a particle that moves slowly in comparison with the speed of light. Starting from

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

if we divide both sides with ds we get $1 = g_{\mu\nu} v^\mu v^\nu$ where the velocity four-vector is $v^\mu = \frac{dx^\mu}{ds}$. If we explicitly write out the right side we find that

$$1 = g_{00}(v^0)^2 + g_{0m} v^0 v^m + g_{m0} v^m v^0 + g_{mn} v^m v^n.$$

Since $g_{m0} = g_{0m} = 0$, it follows that

$$1 = g_{00}(v^0)^2 + g_{mn} v^m v^n.$$

Further, $v^m \ll v^0$, which in essence means that $dx^m \ll cdt$, (remember that $dx^0 = cdt$ where c is the speed of light), and this is our everyday experience, so that we can ignore the second term, and this leads to

$$1 \approx g_{00}(v^0)^2.$$

This means that $v^0 \approx \frac{1}{\sqrt{g_{00}}}$. The particle moves along a geodetic line which is described by the equation

$$\frac{dv^\mu}{ds} + \Gamma^\mu_{\nu\sigma} v^\nu v^\sigma = 0.$$

For $\mu = m$ we have

$$\frac{dv^m}{ds} = -\Gamma^m_{\nu\sigma} v^\nu v^\sigma = -\Gamma^m_{0\sigma} v^0 v^\sigma - \Gamma^m_{n\sigma} v^n v^\sigma =$$

$$= -\Gamma^m_{00} (v^0)^2 - \Gamma^m_{0n} v^0 v^n - \Gamma^m_{n0} v^n v^0 =$$

$$= -\Gamma^m_{np} v^n v^p.$$

Since $\Gamma^m_{0n} = \Gamma^m_{n0} = 0$, we find further that

$$\frac{dv^m}{ds} = -\Gamma^m_{00} (v^0)^2 - \Gamma^m_{np} v^n v^p.$$

Because of the low speed, if we ignore the second term in comparison with the first, we finally get

$$\frac{dv^m}{ds} \approx -\Gamma^m_{00} (v^0)^2 = -g^{m\lambda} \Gamma_{\lambda 00} (v^0)^2 =$$

$$= -g^{m0} \Gamma_{000} (v^0)^2 - g^{mn} \Gamma_{n00} (v^0)^2.$$

Since $g^{m0} = 0$, it follows that

$$\frac{dv^m}{ds} \approx -g^{mn} \Gamma_{n00} (v^0)^2.$$

We know that

$$\Gamma_{n00} = \frac{1}{2} \left(g_{n0,0} + g_{n0,0} - g_{00,n} \right) = -\frac{1}{2} g_{00,n},$$

since $g_{n0} = 0$, so that $\frac{dv^m}{ds}$ reduces to

$$\frac{dv^m}{ds} \approx \frac{1}{2} g^{mn} g_{00,n} (v^0)^2.$$

Further, v^m can be understood as a function of x^μ so that

$$dv^m = \frac{\partial v^m}{\partial x^\mu} dx^\mu.$$

If we divide both sides with ds, we find

$$\frac{dv^m}{ds} = \frac{\partial v^m}{\partial x^\mu}\frac{dx^\mu}{ds} = \frac{\partial v^m}{\partial x^0}\frac{dx^0}{ds} + \frac{\partial v^m}{\partial x^n}\frac{dx^n}{ds} =$$

$$= \frac{\partial v^m}{\partial x^0} v^0 + \frac{\partial v^m}{\partial x^n} v^n$$

which in the small-speed approximation reduces to the first term

$$\frac{dv^m}{ds} \approx \frac{\partial v^m}{\partial x^0} v^0 = \frac{dv^m}{dx^0} v^0$$

since $\frac{dv^m}{dx^0} = \frac{\partial v^m}{\partial x^\mu}\frac{dx^\mu}{dx^0} = \frac{\partial v^m}{\partial x^0}$.

So we have $\frac{dv^m}{dx^0} \approx \frac{1}{v^0}\frac{dv^m}{ds}$ and if we plug in the expression for $\frac{dv^m}{ds}$, we get

$$\frac{dv^m}{dx^0} \approx \frac{1}{v^0}\frac{1}{2} g^{mn} g_{00,n} (v^0)^2 = \frac{1}{2} g^{mn} g_{00,n}\, v^0 \approx$$

$$\frac{1}{2} g^{mn} g_{00,n} \frac{1}{\sqrt{g_{00}}} = g^{mn} \frac{1}{2\sqrt{g_{00}}} g_{00,n} = g^{mn} (g_{00}{}^{\frac{1}{2}})_{,n}$$

where we used an explicit expression $\frac{1}{\sqrt{g_{00}}}$ for v^0. Thus we finally find that $\frac{dv^m}{dx^0} \approx g^{mn} (g_{00}{}^{\frac{1}{2}})_{,n}$. Because

$$\frac{dv^0}{dx^0} = \frac{d}{dx^0}\left(\frac{dx^0}{ds}\right) = \frac{d}{ds}\left(\frac{dx^0}{dx^0}\right) = \frac{d}{ds}(1) = 0$$

and $g^{0n} = 0$, we can extend the previous relation to all four indexes

$$\frac{dv^{\lambda}}{dx^0} \approx g^{\lambda n}(g_{00}{}^{\frac{1}{2}})_{,n}.$$

Since $g_{\mu\nu}$ is independent of x^0 we can lower the index m

$$g_{m\lambda}\frac{dv^{\lambda}}{dx^0} \approx g_{m\lambda}g^{\lambda n}(g_{00}{}^{\frac{1}{2}})_{,n} = g_m^n(g_{00}{}^{\frac{1}{2}})_{,n} = (g_{00}{}^{\frac{1}{2}})_{,m}$$

$$= \frac{d}{dx^0}\left(g_{m\lambda}v^{\lambda}\right) = \frac{dv_m}{dx^0}.$$

So $\frac{dv_m}{dx^0} \approx (g_{00}{}^{\frac{1}{2}})_{,m}$ which means that the particle moves as if it is under the influence of $g_{00}{}^{\frac{1}{2}}$ potential. It is important to note here that we did not use Einstein's law $R_{\mu\nu} = 0$ to get to this result.

Now we will use Einstein's law to find the condition for potential, so that we can compare it with Newton's condition. Einstein assumed that in the empty space

$$R_{\mu\nu} = 0$$

and exactly this equation represents his law of gravity. “Empty” here means that the matter is not present and that there are no physical fields other than the gravitational field. The gravitational field does not disturb the emptiness. Other fields disturb the emptiness.

We assume that the gravitational field is weak so that the curvature of the space is small. Then we can choose a coordinate system so that the curvature of the coordinate lines is small. Under these conditions $g_{\mu\nu}$ is approximatively constant and $g_{\mu\nu,\sigma}$ and all Christoffel Symbols are small. Then Einstein's law becomes

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \approx \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} = 0$$

if we ignore the quantities of the second order. Dirac has shown that this term can be easily calculated by performing a contraction of the term for $R_{\mu\nu\rho\sigma}$ (for a detailed derivation of this term see the preceding part of the Appendix) in which we exchanged ρ and μ indices and ignored the terms of the second order. Therefore,

$$R_{\rho\nu\mu\sigma} = \frac{1}{2}\left(g_{\rho\sigma,\nu\mu} - g_{\nu\sigma,\rho\mu} - g_{\rho\mu,\nu\sigma} + g_{\nu\mu,\rho\sigma}\right) + \Gamma_{\beta\rho\sigma}\Gamma^{\beta}_{\nu\mu} - \Gamma_{\beta\rho\mu}\Gamma^{\beta}_{\nu\sigma}$$

and

$$g^{\rho\sigma}R_{\rho\nu\mu\sigma} = R^{\sigma}_{\nu\mu\sigma} = R_{\nu\mu} = R_{\mu\nu} = 0$$

so that

$$\frac{1}{2}g^{\rho\sigma}\left(g_{\rho\sigma,\mu\nu} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\mu\nu,\rho\sigma}\right) + g^{\rho\sigma}\Gamma_{\beta\rho\sigma}\Gamma_{\nu\mu}^{\beta}$$

$$- g^{\rho\sigma}\Gamma_{\beta\rho\mu}\Gamma_{\nu\sigma}^{\beta} \approx$$

$$\approx \frac{1}{2}g^{\rho\sigma}\left(g_{\rho\sigma,\mu\nu} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\mu\nu,\rho\sigma}\right) = 0$$

which means that

$$g^{\rho\sigma}\left(g_{\rho\sigma,\mu\nu} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\mu\nu,\rho\sigma}\right) = 0.$$

In the case $\mu = \nu = 0$, only the last term remains

$$g^{\rho\sigma}\left(g_{\rho\sigma,00} - g_{0\sigma,0\rho} - g_{0\rho,0\sigma} + g_{00,\rho\sigma}\right) = g^{\rho\sigma}g_{00,\rho\sigma} = 0$$

since $g_{\mu\nu}$ is independent of x^0. For this term, we find further

$$g^{0\sigma}g_{00,0\sigma} + g^{m\sigma}g_{00,m\sigma} = g^{m0}g_{00,m0} + g^{mn}g_{00,mn} =$$

$$= g^{mn}g_{00,mn} = 0.$$

To see what $g^{mn}g_{00,mn} = 0$ means, we need to remember that the laws of physics must be valid in all coordinate systems. This means that they must be expressed as tensor equations. When they involve a field derivation, this derivation must be a covariant derivation. Field equations in physics must be written so that each usual derivation is replaced by a covariant derivation. For example, d'Alembert equation $\Box V = 0$ for the scalar V becomes in the covariant form

$$g^{\mu\nu} V_{:\mu:\nu} = 0.$$

Since the covariant derivation of a scalar is equal to the usual derivation $V_{:\mu} = V_{,\mu}$ applying the formula for the covariant derivation of the vector

$$A_{\mu:\nu} = A_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu} A_{\alpha}$$

we find that

$$g^{\mu\nu} V_{:\mu:\nu} = g^{\mu\nu} (V_{,\mu})_{:\nu} = g^{\mu\nu} \left(V_{,\mu\nu} - \Gamma^{\alpha}_{\mu\nu} V_{,\alpha} \right) = 0.$$

In the weak field approximation, this equation is further reduced to

$$g^{\mu\nu} V_{,\mu\nu} = 0.$$

For the static field, this is further reduced to the Laplace equation

$$g^{\mu\nu} V_{,\mu\nu} = g^{0\nu} V_{,0\nu} + g^{m\nu} V_{,m\nu} = g^{m0} V_{,m0} + g^{mn} V_{,mn} =$$

$$= g^{mn} V_{,mn} = \nabla^2 V = 0.$$

Now we finally see that $g^{mn} g_{00,mn} = 0$ is nothing else than the Laplace equation for g_{00}. We can choose the unit of time so that g_{00} is approximately equal to 1.

This allows us to put $g_{00} = 1 + 2V$ where V is a small quantity. Precisely because V is small, $g_{00}{}^{\frac{1}{2}} \approx 1 + V$ so that V is the potential and

$$\frac{dv^m}{dx^0} = g^{mn}(g_{00}{}^{\frac{1}{2}})_{,n} = g^{mn}V_{,n}$$

which means that V satisfies the Laplace equation and can be identified with the Newton's potential $-\frac{m}{r}$ for a mass m located in the origin of the coordinate system. Since the diagonal elements of g^{mn} are approximately equal to -1, the upper equation leads to the usual Newton’s equation

$$acceleration = -grad\, V$$

so that Einstein's law of gravitation becomes Newton's law of gravitation when the gravitational field is weak and static. The static approximation is a good approximation because the speeds of the planets are small compared to the speed of light. The approximation of a weak gravitational field is a good approximation because the space is almost flat so that the metric tensor is approximatively equal to the metric tensor of the special theory of relativity

$$g_{\mu\nu} = diag(1, -1, -1, -1).$$

It is interesting now to consider some orders of magnitude. The focus will be on the exponents of base 10 and we will not worry much about distinguishing the size of things that differ in factor 2 and the like. $2V$ on the surface of the earth is 10^{-9} of the order of magnitude so that $g_{00} = 1 + 2V$ is very close to 1. But even such a slight deviation of 1 is large enough to produce important gravitational effects that we see on the surface of the earth. Taking into account that the order of magnitude of the radius of the earth is $10^9\, cm$, for $(g_{00}{}^{\frac{1}{2}})_{,n}$ we find that the order of magnitude is $10^{-18} cm^{-1}$. This is easy to show. Since $g_{00}{}^{\frac{1}{2}} \approx 1 + V$ and $V = -\frac{m}{r}$ it follows that $(g_{00}{}^{\frac{1}{2}})_{,r} = \frac{\partial V}{\partial r} = \frac{m}{r^2} = \frac{|V|}{r}$ so that the order of magnitude for $(g_{00}{}^{\frac{1}{2}})_{,r}$ is equal to $\frac{10^{-9}}{10^9 cm} = 10^{-18} cm^{-1}$. We see that the deviation from the flat space is extremely small and the order of magnitude for $\frac{dV_m}{dx_0}$ is $10^{-18} cm^{-1}$. Since $x_0 = ct$ we find further that

$$\frac{dV_m}{dx_0} = \frac{d^2 x_m}{d{x_0}^2} = \frac{1}{c^2}\frac{d^2 x_m}{dt^2} = 10^{-18} cm^{-1}$$

so that for acceleration on the surface of the earth due to gravity, we get a known result

$$\frac{d^2x_m}{dt^2} = c^2 \times 10^{-18}cm^{-1} = (3 \times 10^{10}\frac{cm}{s})^2 \times 10^{-18}cm^{-1}$$

$$= 9 \times 10^{20}\frac{cm^2}{s^2} \times 10^{-18}cm^{-1} = 9 \times 10^2\frac{cm^2}{s^2}$$

$$\approx 10^3\frac{cm}{s^2} = 10\frac{m}{s^2}$$

which is quite noticeable, even though the deviation from the flat space is too small to be directly felt.

k. Derivation of the Poisson equation for gravitational potential

We begin again from Einstein’s equation in Dirac's notation [7, Chapter 25]

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = k\rho v^{\mu} v^{\nu}.$$

If we perform the left and right side contraction by $g_{\mu\nu}$,

$$g_{\mu\nu} R^{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\mu\nu} R = k\rho g_{\mu\nu} v^{\mu} v^{\nu},$$

we find $R_{\nu}^{\nu} - \frac{1}{2} g_{\nu}^{\nu} R = k\rho$, since in the section g of the Appendix we have shown that $g_{\mu\nu} v^{\mu} v^{\nu} = 1$. This equation is further reduced to

$$R - \frac{1}{2} 4R = k\rho = R - 2R = -R$$

i.e. $R = -k\rho$. If this expression is placed back into Einstein's equation we get

$$R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} k\rho = k\rho v^{\mu} v^{\nu}$$

which means that $R^{\mu\nu} = k\rho \left(v^{\mu} v^{\nu} - \frac{1}{2} g^{\mu\nu} \right)$ or if we lower the indexes

$$R_{\mu\nu} = k\rho\left(v_\mu v_\nu - \frac{1}{2}g_{\mu\nu}\right).$$

As we have shown that in the approximation of a weak gravitational field (the previous part of the Appendix)

$$R_{\mu\nu} \approx \frac{1}{2}g^{\rho\sigma}\left(g_{\rho\sigma,\mu\nu} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\mu\nu,\rho\sigma}\right),$$

this means that

$$\frac{1}{2}g^{\rho\sigma}\left(g_{\rho\sigma,\mu\nu} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\mu\nu,\rho\sigma}\right) \approx k\rho\left(v_\mu v_\nu - \frac{1}{2}g_{\mu\nu}\right).$$

In Newton's approximation we have $v_0 \approx 1$ and $v_m \approx 0$ in the case of a static field and a static distribution of matter. We recall that in the part j of the Appendix we have shown that $1 \approx g_{00}(v^0)^2$ and $g_{00} \approx 1 + 2V$ where V is small, which means that $g_{00} \approx 1$. For $\mu = \nu = 0$ we find

$$\frac{1}{2}g^{\rho\sigma}\left(g_{\rho\sigma,00} - g_{0\sigma,0\rho} - g_{0\rho,0\sigma} + g_{00,\rho\sigma}\right) \approx k\rho\left(v_0^2 - \frac{1}{2}g_{00}\right)$$

$$\approx k\rho\left(1 - \frac{1}{2}\right) = \frac{1}{2}k\rho \approx \frac{1}{2}g^{\rho\sigma}g_{00,\rho\sigma}.$$

Since

$$g^{\rho\sigma}g_{00,\rho\sigma} = g^{0\sigma}g_{00,0\sigma} + g^{m\sigma}g_{00,m\sigma} = g^{m0}g_{00,m0} + g^{mn}g_{00,mn}$$

$$= g^{mn}g_{00,mn}$$

we find that $\frac{1}{2}k\rho \approx \frac{1}{2}g^{mn}g_{00,mn}$. In the approximation of a weak gravitational field, the space is almost flat so that the metric tensor is approximatively equal to the metric tensor of the special theory of relativity

$$g_{\mu\nu} = diag(1, -1, -1, -1)$$

and

$$g^{\mu\nu} = diag(1, -1, -1, -1)$$

which leads to $-\frac{1}{2}\nabla^2 g_{00} \approx \frac{1}{2}k\rho$. Since $g_{00} \approx 1 + 2V$ we find further that

$$-\frac{1}{2}\nabla^2(1 + 2V) = -\nabla^2 V \approx \frac{1}{2}k\rho$$

i.e. $\nabla^2 V \approx -\frac{1}{2}k\rho$. The Poisson equation $\nabla^2 V \approx 4\pi\rho$ is obtained for $k = -8\pi$.

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The picture of the Fountain of the Lions on the cover page is downloaded from this website:

https://commons.wikimedia.org/wiki/File:Alhambra_Dec_2004_5.jpg

An English translation of the Poem of the Fountain of the Lions can be found on this website:

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The wisdom from the *stećak* is from this wonderful collection of Bosnian souls:

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