Differential Equations Notes

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1 Convolutions

If f and g are piecewise continuous functions for $t \geq 0$, then the convolution is the function defined over $t \geq 0$ by

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

If we substitute $u = t - \tau$, then $du = -d\tau$ and

$$\int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t g(t-u)f(u)du$$

so convolution is commutative; it's also easy to check that convolution is associative and distributes over addition.

2 The Laplace Transform

$\mathscr{L}[f(t) \star g(t)] = F(s)G(s)$ for $s > c$

3 Systems of Ordinary Differential Equations with Constant Coefficients

Systems of differential equations in which knowledge of one function is required to find other functions (and vice versa) are called **coupled**.

3.1 Systems of First-Order Equations

An arbitrary linear ordinary differential equation (or a system of linear ordinary differential equations) can be converted into a first order system of linear differential equations by adding variables for all but the highest order derivatives. We illustrate this in the case of a second-order linear differential equation: consider

$$a(t)y'' + b(t)y' + c(t)y = f(t)$$

with $y(t_1) = r_1$ and $y'(t_2) = r_2$. Substitute

$$\begin{aligned} x_1(t) &= y \\ x_2(t) &= y' \end{aligned}$$

Differentiating both functions, we get

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -\frac{b(t)}{a(t)}y' - \frac{c(t)}{a(t)}y + f(t) = -\frac{b(t)}{a(t)}x_2 - \frac{c(t)}{a(t)}x_1 + f(t) \end{aligned}$$

with initial conditions given by

$$\begin{aligned} y_1(t_1) &= r_1 \\ y_2(t_2) &= r_2 \end{aligned}$$

We can write a system in of linear first-order ODEs

$$x_1' = a(t)x_1 + b(t)x_2 + f_1(t)$$
$$x_2' = c(t)x_1 + d(t)x_2 + f_2(t)$$

in **matrix form** as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

or more compactly as

$$\vec{x}\,' = A(t)\vec{x} + \vec{f}(t)$$

This algorithm can be extended to decompose linear differential equations of arbitrary degree into systems of linear ordinary differential equations.
We say a system of this form is **homogeneous** if $\vec{f}(t) = 0$, or **non-homogeneous** if $\vec{f}(t) \neq 0$.

3.2 Existence and Uniqueness of Solutions to Systems

If a solution $x(t)$ exists, then it defines a curve in n-dimensional space starting at the point x_0. Let f_x denote $\frac{\partial f_i}{\partial x_j}$ for $i, j = 1, \ldots, n$.
Theorem Suppose t_0 is a fixed point on an interval $I = (a, b)$ and x_0 is a fixed point in $\mathbb{R}^n$ such that $f(x, t)$ and $f_x(x, t)$ are continuous for $a < t < b$ and $|x - x_0| < R$. Then there is a unique solution $x(t)$ of the above system of equations defined for t for $|t - t_0| < \epsilon$ for $\epsilon > 0$.
Remark Since the dependence of f on x might be nonlinear, the solution $x(t)$ need not be defined on all of I. When the system *is* linear, we can guarantee that such a solution is defined on all of I:
Theorem If $A(t)$ and $f(t)$ are continuous on an interval I containing t_0, then the first-order linear system $x' = A(t)x + f(t)$ with initial condition $x(t_0) = b$ has a unique solution x defined on I.
Remark Note that linearity implies that existence and uniqueness hold on *all* of the interval I, not just near t_0.

3.3 Autonomous Systems and Geometric Analysis

Like first-order equations, first-order systems are called **autonomous** if the function f above does not depend on t:

$$\frac{dx}{dt} = f(x)$$

Notice that f is a vector field in $\mathbb{R}^n$. Moreover, a solution $x(t)$ of that system is a curve in $\mathbb{R}^n$ whose tangent vector at each point is given by the vector field at that point. By sketching the vector field and a few sample solution curves, we obtain a geometric insight into the behavior of systems of solutions of first-order autonomous systems. When $n = 2$, such pictures are called **phase plane portraits**.
Like first order autonomous equations, first-order autonomous systems admit equilibrium solutions and stability analysis. Values of x_0 for which $f(x_0) = 0$ (i.e. solutions for which the vector field vanishes) are called **critical points**, and they correspond to **equilibrium solutions** if we let $x(t) = x_0$. The notion of **stability** for an equilibrium solution of a first-order system is similar to the case of a singe equation discussed earlier, but is sl
An equilibrium solution x_0 for a the system above is **stable** if all "nearby" solutions remain "nearby." This is certainly true if all nearby solutions approach x_0 (all nearby vectors in the

vector field point to x_0), in which case such an equilibrium is called a **sink**. It's also possible for solutions to remain nearby by orbiting around the equilibrium; such an equilibrium is called a **center**.
An equilibrium solution x_0 for the system is called **unstable** if at least some nearby solutions move away from x_0. This is certainly true if all nearby solutions move away from the equilibrium; such an equilibrium is called a **source**. However, for systems it could be that some solutions approach x_0 while others move away; such an equilibrium is called a **saddle**.
In applications where initial conditions can only be approximated, an unstable equilibrium is of limited significance since a very small perturbation could yield an extremely different outcome. On the other hand, stable equilibria represent very important states of a physical system.

3.4 Theory of First-Order Linear Systems

As was the case for second-order equations, the solution of the nonhomogeneous system $x' = A(t)x + f(t)$ is closely connected with the associated homogeneous system $x' = A(t)x$. If c_1, c_2 are constants and x_1, x_2 are both solutions of the homogeneous system, then by linearity $c_1x_1 + c_2x_2$ is also a solution, so the homogeneous solution set forms a vector space.
In fact, if the solutions $x_1, \ldots, x_n$ are linearly independent, then we want to show that *every* solution is of the form $x(t) = c_1x_1(t) + \cdots + c_nx_n(t)$ for some choice of constants $c_1, \ldots, c_n$. As in previous sections, the linear dependence of functions is determined by the **Wronskian**.

3.4.1 The Wronskian

Definition. Let $\vec{x}_1, \ldots, \vec{x}_n$ be n-dimensional vector functions defined on an interval I. Define

$$X = \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}$$

to be the $n \times n$ matrix where the i^{th} column is the i^{th} vector. The **Wronskian** W of $\vec{x}_1, \ldots, \vec{x}_n$ is the determinant

$$W = \det(X)$$

of X.
Theorem Let $\vec{x}_1, \ldots, \vec{x}_n$ be n-dimensional vector-valued functions on an interval I. Then if $\vec{x}_1, \ldots, \vec{x}_n$ are all solutions of the homogeneous system of first-order equations and $W[x_1, \ldots, x_n](t_0) = 0$ for some t_0 in I, then $x_1, \ldots, x_n$ are linearly dependent on I.
Theorem If $A(t)$ is continuous on an interval I and $x_1, \ldots, x_n$ are linearly independent solutions of the homogeneous system of first-order equations, then every solution of the homogeneous linear system is of the form $x(t) = c_1x_1(t) + \cdots + c_nx_n(t)$.

3.4.2 Structure of Solutions

Theorem (Principle of Superposition for Solutions) Let $\vec{x}' = A(t)\vec{x} + \vec{f}(t)$. If $A(t)$ and $f(t)$ are continuous on an interval I and x_p is a particular solution of the nonhomogeneous linear system of first-order equations, then every solution of that system is of the form

$$x(t) = x_p(t) + x_h(t)$$

where x_h is the general solution of the homogeneous system and x_p is the particular solution of the system.

3.5 Finding Solutions to Homogeneous Linear Systems

In this section we investigate the solution to $\vec{x}\,' = A(t)\vec{x} + \vec{f}(t)$ when $A(t) = A$ is a constant matrix and $\vec{f}(t)$ is identically 0. This yields the homogeneous system of linear equations

$$\vec{x}\,' = A\vec{x}$$

and we can find the general solution x_h of this homogeneous system via the **eigenvalue method**. First, notice that this system is autonomous, and the trivial solution $x(t) = 0$ is an equilibrium. If A is invertible, then 0 is the *only* equilibrium solution.
Let $\vec{x}\,' = A\vec{x}$ where A is an $n \times n$ matrix. Focusing on the $n = 1$ case, we get $x' = ax$ which has the solution $x(t) = ce^{at}$. Let's use this as a starting point for general n. Guess $\vec{x} = e^{\lambda t}\vec{v}$; plugging this into $\vec{x}\,' = A\vec{x}$, we get

$$(A - \lambda I)e^{\lambda t}\vec{v} = 0$$

We know that $e^{\lambda t}$ is nonzero for any λ and t, so we must have

$$(A - \lambda I)\vec{v} = 0$$

So to be a solution, $\vec{v}$ must be an eigenvector of A with associated eigenvalue λ.
Note that for all values of t, $\vec{x}(t) = e^{\lambda t}\vec{v}$ lies on the ray passing through vector $\vec{v}$, so a solution of the form $e^{\lambda t}\vec{v}$ is called a **straight-line solution**. Moreover, if $\lambda < 0$, then $\lim_{t\to\infty} \vec{x}(t) = 0$, and if $\lambda > 0$, then $\vec{x}(t)$ tends away from 0 as $t \to \infty$, so the signs of the eigenvalues determines the stability of the equilibrium solution $\vec{x}(t) = 0$.
Our algorithm is as follows: if A has n linearly independent eigenvectors $v_1, \ldots, v_n$ with associated eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ (which need not be unique) then we have n distinct solutions of $x' = Ax$ that are part of the special form

$$x_1(t) = e^{\lambda_1 t}\vec{v}_1, \ \ldots, x_n = e^{\lambda_n t}\vec{v}_n$$

Moreover, the solutions are linearly independent because

$$W[x_1, \ldots, x_n](t) = \det([e^{\lambda_1 t}\vec{v}_1 \ \cdots \ e^{\lambda_n t}\vec{v}_n]) = e^{(\lambda_1 + \cdots + \lambda_n)t} \det([\vec{v}_1 \cdots \vec{v}_n]) \neq 0$$

for any t.

3.5.1 Generating Real Solutions from Complex Eigenvalues

Suppose that the characteristic polynomial $p(\lambda)$ has a complex root $\lambda = \alpha + \beta i$ leading to a complex eigenvector $\vec{v}$. We know that $e^{\lambda t}\vec{v}$ is a solution, but if A is real we might want a real solution – particularly if we're dealing with real-world applications.
We know that complex eigenvalues and eigenvectors for real-valued matrices come in complex conjugate pairs (because they come in pairs for real polynomials), so $w(t) = e^{\lambda t}\vec{v}$ and $\overline{w}(t) = e^{\overline{\lambda} t}\overline{\vec{v}}$ are linearly independent complex-valued solutions of $x' = Ax$.
Decomposing $w(t)$ into its real and imaginary parts, we have

$$w(t) = a(t) + ib(t) \quad \overline{w}(t) = a(t) - ib(t)$$

Knowing that a linear combination of solutions is a solution, we have

$$a(t) = \frac{1}{2}\left(w(t) + \overline{w}(t)\right) \quad b(t) = \frac{1}{2i}\left(w(t) - \overline{w}(t)\right)$$

as two real-valued solutions of $x' = Ax$. Moreover, $a(t)$ and $b(t)$ are linearly independent because their span over $\mathbb{C}$ includes both $w(t)$ and $\overline{w}(t)$. We've can neatly express this as a theorem:
Theorem (Real Solutions from Complex Eigenvalues) If A is real constant $(n \times n)$ matrix with complex eigenvalue λ and eigenvector $\vec{v}$, then the real and imaginary components of $w(t) = e^{\lambda t}\vec{v}$ are linearly independent real-valued solutions of $x' = Ax$:

$$x_1(t) = \Re(w(t)) \text{ and } x_2(t) = \Im(w(t))$$

When $(\lambda, \vec{v})$ is a complex eigenpair, $w(t) = e^{\lambda t}\vec{v}$ isn't a straight-line solution because we've seen that their real-valued solutions circle the equilibrium in spirals or closed curves rather than moving in straight lines.
Theorem (Eigenbasis forms a Solution) If A is an $(n \times n)$ real constant matrix with eigenbasis $v_1, \ldots, v_n$ for $\mathbb{R}^n$, then the general solution of $x' = Ax$ is

$$x(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + c_n e^{\lambda_n t}\vec{v}_n$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the (not necessarily distinct) eigenvalues associated with $v_1, \ldots, v_n$.

3.5.2 Multiple Eigenvalues and Generalized Eigenvectors

Recall that a real eigenvalue may have algebraic multiplicity m_a greater than its geometric multiplicity m_g. In this case, such an eigenvalue is called **defective** and its **defect** is $d = m_a - m_g$. When a matrix has a defective eigenvalue, it fails to have an eigenbasis, so we can't leverage the theorem above.
To find a general solution to a homogeneous linear system of ordinary differential equations, we first define the **generalized eigenvector**:
Definition (Generalized Eigenvector) If A is a square matrix and p is a positive integer, then a nonzero solution v of $(A - \lambda I)^p\vec{v} = 0$ is called a **generalized eigenvector** for the eigenvalue λ. Eigenvectors correspond to the special case $p = 1$. If the above holds for $p > 1$, then it could hold for a smaller value of p. If we let p_0 denote the smallest possible integer for which the above holds, then $v_1 = (A - \lambda)^{p_0 - 1}\vec{v} \neq 0$ and satisfies $(A - \lambda I)\vec{v}_1 = (A - \lambda)^{p_0}\vec{v} = 0$ so v_1 is an eigenvector for λ. In particular, we see that generalized eigenvectors for an eigenvalue λ only exist when an actual eigenvector exists. When the eigenvalue is defective, we can use the generalized eigenvectors to find solutions of $x' = Ax$ by generalizing $x(t) = e^{\lambda t}\vec{v}$.
In the specific case where λ is an eigenvalue with $m_a = 2$ and $m_g = 1$, then two linearly independent solutions of $x' = Ax$ are

- $x_1(t) = e^{\lambda t}\vec{v}_1$ where v_1 is an eigenvalue for λ
- $x_1(t) = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2$ where $(A - \lambda I)\vec{v}_2 = v_1$

Note that v_2 is a generalized eigenvector for λ since $(A - \lambda I)^2\vec{v}_2 = 0$ although $(A - \lambda I)\vec{v}_2 \neq 0$. Because we define $v_1 = (A - \lambda I)\vec{v}_2$, we know that $\vec{v}_1 \neq \vec{0}$ and $(A - \lambda)\vec{v}_1 = 0$, so v_1 is an eigenvector and the equation $(A - \lambda I)v_2 = v_1$ is always solvable.

3.6 Finding Solutions to Non-homogeneous Linear Systems

In this section we investigate the solution to $\vec{x}\,' = A(t)\vec{x} + \vec{f}(t)$ when $A(t) = A$ is a constant matrix but $\vec{f}(t)$ is nonzero and its components are functions of t. This non-homogeneous system of linear equations

$$\vec{x}\,' = A\vec{x} + \vec{f}(t)$$

We know that the general solution to this system will be of the form

$$x(t) = x_p(t) + x_h(t)$$

We saw how to find x_h in the preceding section, so now we focus on finding the particular solution x_p.
The naive method for finding a solution is the **method of undetermined coefficients**. If we're given the system

$$\vec{x}\,' = A(t)\vec{x} + \vec{f}(t)$$

we can simply guess a solution based on what $f(t)$ is; the only difference being that the undetermined coefficients will be constant column vectors with undetermined entries. However, finding these undetermined coefficients is somewhat tedious. A more efficient method is the **method of variation of parameters**.

3.6.1 Method of Variation of Parameters

Let

$$X = \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}$$

to be the $n \times n$ matrix where the i^{th} column is the i^{th} solution vector to the homogeneous system $\vec{x}\,' = A\vec{x}$. By definition, $X'(t) = AX(t)$; now when confronted with

$$\vec{x}\,' = A\vec{x} + \vec{f}(t)$$

we guess a solution of the form

$$\vec{x_p}\,' = X(t)\vec{v}$$

for some undetermined vector $\vec{v}$. Plugging this guess into the nonhomogeneous system and applying the product rule, we get

$$(X(t)\vec{v})' = X'(t)\vec{v} + X(t)\vec{v}\,' = AX(t)\vec{v} + \vec{f}(t)$$

Knowing that $X'(t) = AX(t)$, we can rewrite this system in terms of X:

$$\begin{aligned} AX(t)\vec{v} + X(t)\vec{v}\,' &= AX(t)\vec{v} + \vec{f}(t) \\ X(t)\vec{v}\,' &= \vec{f}(t) \\ \vec{v}' &= X^{-1}\vec{f}(t) \end{aligned}$$

We can be sure that X^{-1} exists because the solutions that make up the column vectors of X are linearly independent. Finally,

$$\vec{v} = \int X^{-1}\vec{f}(t)\,dt$$

so we can neatly express x_p in a short formula:

$$\vec{x}_p = X \int X^{-1}\vec{f}(t)\,dt$$

3.6.2 Laplace Transform on Linear Systems of Equations

We can define the Laplace transform for matrices much the same way that we defined the derivative and integral operators, and solve systems of linear equations using Gauss-Jordan elimination.