Fourier Series and PDEs Notes

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1 Review of 2341 and Preliminaries

Lecture 1

We can write an ordinary differential equation to find the temperature at any point on a rod of length L; $T(x)$ could represent the temperature, where x is length. However, we know that the temperature at each point also depends on time, so we have to expand our model to cover two variables.

We need partial derivatives. For our temperature function $T = f(x,t)$, we'll be concerned with $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial t}$.

We have **heat equation in one dimension**,

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

which can also be written as $T_t = T_{xx}$. This equation has *no general solution.* However, we can solve it using Fourier series.

Wave equation in one dimension measures displacement $u(x,t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Maxwell's equation, which is used in communication.

Navier-Stokes equation which is used in weather forecasting.

In ordinary differential equations, a *general solution always exists.* In PDEs, there are *no general solutions.*

1.1 Review of MATH 2341

First order separable differential equations take the form

$$\frac{dy}{dx} = f(x)g(y)$$

with solution

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

yielding $G(y) = F(x) + C$ which is the implicit general solution, and can typically be rearranged to isolate y and yield an explicit general solution.

Example: $y' = xy$ which yields the explicit solution $|y| = e^{\frac{x^2}{2}+C}$, typically written as $Ce^{\frac{x^2}{2}}$, so y equals either $Ce^{\frac{x^2}{2}}$ or $-Ce^{\frac{x^2}{2}}$.

We also have **first-order linear differential equations** (some of which are separable!). $y' + p(x)y = q(x)$. We find the integrating factor $I(x) = e^{\int p(x)dx}$. Multiplying both sides by $I(x)$ yields $I(x)y + p(x)I(x)y = I(x)q(x)$; the left side resembles the product rule of $(I(x)y)'$. So we integrate $I(x)y = \int I(x)q(x)dx + C$.

Example: $y' - \frac{1}{x}y = 2x + 1$ for $x > 0$. We set $p(x) = -\frac{1}{x}$ and get $I(x) = e^{-\ln(x)} = \frac{1}{x}$. We then have $\frac{1}{x}y' - \frac{1}{x^2}y = \left(\frac{1}{x}y\right)' = 2 + \frac{1}{x}$. Integrating yields $\frac{1}{x}y = 2x + \ln(x) + C$ so $y = 2x^2 + x\ln(x) + Cx$.

Second order constant-coefficient differential equations take the form $ay'' + by' + cy = 0$ for $a \neq 0$. The solution is $y = e^{rx} \to y' = re^{rx} \to\to y'' = r^2e^{rx}$ where each r is the roots of the characteristic equation (polynomial) $at^2 + bt + c = 0$.

- If $b^2 - 4ac > 0$, $r_1 \neq r_2$ and both are real. Our solution is $y = C_1e^{r_1x} + C_2e^{r_2x}$, which we call a **superposition**
- If $b^2 = 4ac$ then $r_1 = r_2$ and our solution is $y = C_1e^{r_1x} + xe^{r_1x}$
- Finally, if $b^2 - 4ac < 0$, both r_1 and r_2 are complex. Setting $q = 4ac - b^2$, our solution take the form

$$y(x) = e^{\beta x}\left[c_1\cos(qx) + c_2\sin(qx)\right]$$

If $p > 0$, the sine wave's amplitude increases; if $p < 0$, it decreases.

We also should summarize some basic facts about the trigonometric functions:

- $\sin(x)$ has period 2π radians with zeroes located at $n\pi$ for $n \in \mathbb{Z}$.
- In general, $\sin(\beta x)$ and $\cos(\beta x)$ have period $\frac{2\pi}{\beta}$.
- For $L > 0$ and real, if we have $\sin(\frac{n\pi x}{L})$ and $\cos(\frac{n\pi x}{L})$, the period is $\frac{2\pi}{\frac{n\pi}{L}} = \frac{2L}{n}$ so $2L$ is the common period for any function of this form.

If we want to compute

$$I = \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

integrating by parts[1] with

$$\begin{aligned}
u &= \cos\left(\frac{n\pi x}{L}\right)\\
dv &= \cos\left(\frac{m\pi x}{L}\right)\\
du &= -\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)\\
v &= \int \cos\left(\frac{m\pi x}{L}\right)dx = \frac{L}{m\pi}\sin\left(\frac{m\pi x}{L}\right)
\end{aligned}$$

$$\begin{aligned}
I &= uv\Big|_{-L}^{L} - \frac{L}{m\pi}\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\frac{-n\pi}{L}\sin\left(\frac{m\pi x}{L}\right)dx\\
&= \frac{n}{m}\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{n}{m}\int_{-L}^{L}udv\\
&= \frac{n^2}{m^2}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \frac{n^2}{m^2}I
\end{aligned}$$

So $I = \frac{n^2}{m^2}I$ implying $(m^2 - n^2)I = 0$. Either $m = n$ or $I = 0$, so

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = 0$$

for $n = 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots$, and $m \neq n$

2 Fourier Series

$$f(x) \sim \frac{a_0}{2}\sum_{n=1}^{\infty}$$

3 Sturm-Liouville Problems

3.1 Four Standard Cases

1.

2.

[1] $\int udv = uv - \int vdu$

3.

4.

4 The 1-D Homogeneous Heat Equation

$$u_t = k u_{xx}$$

4.1 Derivation

4.2 Solution

5 n-Dimensional Heat Equation

$$u_t = k(\Delta u)$$

where Δ is the Laplacian operator given by

$$(\Delta v)(x_1, ..., x_n) = \sum_{i=1}^{n} v_{x_i x_i}$$

6 The Wave Equation

6.1 1-Dimensional Case

$$u_{tt} = c^2 u_{xx} + q(x,t)$$

6.1.1 Derivation

6.1.2 Homogeneous Solution

When $q(x,t)$ is identically zero, we have the homogeneous equation

$$u_{tt} = c^2 u_{xx}$$

6.1.3 Non-homogeneous Solution

When $q(x,t)$ is not identically zero, the solution to

$$u_{tt} = c^2 u_{xx} + q(x,t)$$

is

6.2 On a Circular Membrane

6.2.1 Solution

6.3 n-Dimensional Case

$$u_{tt} =$$

6.3.1 Solution

7 The Laplace Equation

7.1 2-Dimensional Case

$$u_{xx} + u_{yy} = 0$$

7.1.1 Derivation

7.1.2 Homogeneous Solution

7.1.3 Non-homogeneous Solution

7.2 n-Dimensional Case

7.2.1 Solution

8 Other Equations

8.1 Brownian Motion

$$u_t(x,t) = au_{xx}(x,t) - bu_x(x,t)$$

8.2 Diffusion-Convection Problems

$$u_t(x,t) = ku_{xx}(x,t) - au_x(x,t)bu(x,t)$$

8.3 Black-Scholes Equation

$$V_t(S,t) + \frac{1}{2}\sigma^2 S^2 V_{SS}(S,t) + rSV_S(S,t) - rV(S,t) = 0$$

8.4 Schrödinger Equation

Evolution of a Quantum State

$$i h \psi_t(x,t) = -\frac{\hbar^2}{2m}$$

8.5 Klein-Gordon Equation

Motion of a Quantum Scalar Field

$$\psi_{tt}(x,y,z,t) = c^2 \Delta \psi(x,y,z,t) - \frac{m^2 c^4}{\hbar^2} \psi(x,y,z,t)$$

With $i^2 = -1$ and c being the speed of light. In one dimension,

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) - a u(x,t)$$

for constant $a > 0$.

8.6 Telegraph Equation

$$a u_{tt}(x,t) + b u_t(x,t) + c u(x,t) = u_{xx}(x,t)$$

8.7 Dissipative Waves

$$u_{tt}(x,t) + a u_t(x,t) + b u(x,t) = c^2 u_{xx}(x,t) - d u_x(x,t)$$

$a, b, d \geq 0$ not all 0 and $c > 0$

8.8 Transverse Vibrations of a Rod

$$u_{tt}(x,t) + c^2 u_{xxxx}(x,t) = 0$$

8.9 Helmholtz Equation

$$\Delta u(x,y,z) + k^2 u(x,y,z) = 0$$

$$\Delta u(x,y,z) - k^2 u(x,y,z) = 0$$

8.10 Steady-state Connective Heat

$$\Delta u(x, y) - au_x(x, y) - bu_y(x, y) + cu(x, y) = 0$$

8.11 Plane Problems in Continuum Mechanics

$$\Delta\Delta u(x, y) = u_{xxxx}(x, y) + 2u_{xxyy}(x, y) + u_{yyyy}(x, y) = 0$$

8.12 Euler-Tricomi Equation

Plane Transonic Flow

$$u_{xx}(x, t) = xu_{xx}(x, t)$$

where $u(x, t)$ is a function of speed.

8.13 Fisher Equation

Advance of advantageous genes in a population

$$u_t(x, t) = Du_{xx}(x, t) + ru(x, t)(1 - u(x, t))$$

8.14 Boussinesq Equation

Fluid dynamics

$$\eta_{tt}(x, t) - gh\eta_{xx}(x, t) - gh\left(\frac{2}{3h}\eta^2(x, t) + \frac{h^2}{3}\eta_{xx}(x, t)\right)_{xx} = 0$$

8.15 Navier-Stokes Fluid Flow Equations

The Navier-Stokes fluid-flow equations are the system of partial differential equations given by

$$\nabla \cdot u = 0$$
$$\rho\left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right) = -\nabla p + u\nabla^2 u + \rho F$$

which model every known fluid. The first equation in the system represents the conservation of mass, and the second represents the conservation of momentum.

Millenium Problem (Navier-Stokes Existence and Smootheness) Prove or give a counterexample to the following statement: in three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier-Stokes equations.

9 The Laplace Transform

The Laplace transform can be used to "suppress" the time variable $t > 0$ in a PDE; for one-dimensional equations, this yields an ordinary differential equation in $U(x, s)$ where s is a constant.

10 The Fourier Transform

The Fourier transform can be used to "suppress" spatial variables $-\infty < x, y, z, \cdots < \infty$ much the same way that the Laplace transform "suppresses" the time variable.

11 The Method of Green's Functions