0.1.2 A polynomial $f \in \mathbb{F}[x]$ has no repeated roots iff it is relatively prime to its derivative.

0.1.4

- Dirichlet convolution is associative

  Proof Let $n = ab$

$$\begin{aligned}\big(f * (g * h)\big)(n) &= \sum_{d_1(d_4)=n} f(d_1)\left[\sum_{d_2d_3=d_4} g(d_2)h(d_3)\right] \\ &= \sum_{d_1(d_2d_3)=n} f(d_1)\left[g(d_2)h(d_3)\right] \\ &= \sum_{(d_1d_2)d_3=n} \left[f(d_1)g(d_2)\right]h(d_3) \\ &= \sum_{d_5d_3=n}\left[\sum_{d_1d_2=d_5} f(d_1)g(d_2)\right]h(d_3) \\ &= \big((f*g)*h\big)(n)\end{aligned}$$

- Dirichlet convolution is commutative

  Proof

$$\sum_{ab=n} f(a)g(b) = \sum_{ba=n} f(b)g(a) = \sum_{ba=n} g(a)f(b)$$

- Dirichlet convolution has identity $I(n)$

  Proof Plugging it in:

$$(f * I)(n) = \sum_{d|n} f(d) I\left(\frac{n}{d}\right) = f(n)1\left(\frac{n}{n}\right) = f(n)$$

- If $f(1) \neq 0$, then f has an inverse under Dirichlet convolution.

  Proof If $f(1) \neq 0$, we want some g such that $(f*g)(n) = I(n)$. Assume $f(1) \neq 0$, then we construct the function $g(1) = \frac{1}{f(1)}$ if $n = 1$. For our base case, we have $(f*g)(1) = \frac{f(1)}{f(1)} = 1$, which agrees with $I(1)$. We also know that for $n > 1$,

$$(f*g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right) = I(n) = 0$$

so

$$
\begin{aligned}
f(1)g(n) + \sum_{d>1|n} f(d)g\left(\frac{n}{d}\right) &= 0 \\
g(n) + \frac{1}{f(1)} \sum_{d>1|n} f(d)g\left(\frac{n}{d}\right) &= 0 \\
g(n) &= -\frac{1}{f(1)} \sum_{d>1|n} f(d)g\left(\frac{n}{d}\right)
\end{aligned}
$$

Now assume $f(1) = 0$; then clearly for any g we have $(f * g)(1) = f(1)g(1) = 0$ so $(f * g)(n)$ necessarily disagrees with $I(n)$.