Math Reference Notebook

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Contents

1 Notations and Definitions

1.1 Functions

Let S and T be sets. A *function* from S into T is a subset F of $S \times T$ such that for each element $x \in S$ there is exactly one element $y \in T$ such that $(x, y) \in F$. The set S is called the *domain* of the function, and the set T is called the *codomain* of the function. The subset $\{y \in T | (x, y) \in F$ for some $x \in S\}$ of the codomain is called the *image* of the function.

Functions can satisfy the following properties:

Surjective (Onto):	every image in the codomain is also in the range/image.
Injective (One-to-one):	each element of the codomain is the image of at most on
Bijectivity: (One-to-one correspondence)	both surjective and injective.

2 Logic

Implication		$P \to Q$
Negation	$\neg$	$(P \to Q)$
Inverse	$\neg$	$P \to \neg Q$
Converse		$Q \to P$
Contrapositive	$\neg$	$Q \to \neg P$
Modus ponens		Given $P \to Q$ and P, then Q.
Modus tollens		Given $P \to Q$ and Q, then P.
Distributive law		$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

3 Elementary Algebra

3.1 Sums

3.1.1 Arithmetic Sums

Arithmetic Sums $S_n = \frac{n(t_1+t_n)}{2}$

3.1.2 Geometric Sums

Geometric Sums $S_n = \frac{t_1(1-r^n)}{1-r}$

3.2 Arithmetic of the Complex Numbers

Let $i^2 = -1$. For $z = a + bi$ for $a, b \in \mathbb{R}$, we have

- $z = re^{i\theta}$ by Euler's formula
- $z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$
- $z_1 \times z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bd)i = (r_1 r_2)e^{i(\theta_1+\theta_2)}$
- $\Re(z) = a$
- $\Im(z) = b$
- $|z| = \sqrt{a^2 + b^2}$
- $\overline{z} = a - bi$
- $|\overline{z}| = |z|$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

- $z = r(\cos\theta + i\sin\theta)$ for $z \neq 0$, $|z| = r \in \mathbb{R}^+$, and θ in radians. Each value of θ is called an argument of z, and the set of all such θ is denoted $\arg z$. The principal value of $\arg z$, denoted $\operatorname{Arg} z$, is the unique value of θ such that $\pi < \theta \leq \pi$

- $\arg z_1 z_2 = \arg z_1 + \arg z_2$

The set of all complex numbers is the complex plane $\mathbb{C} = \{a + bi \,|\, a, b \in \mathbb{R},\, i^2 = -1\}$ which is isomorphic to $\mathbb{R}^2$.

4 Combinatorics

4.1 Permutations

$${}_nP_k = \frac{n!}{(n-k)!}$$

4.2 Combinations

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Factorial For $n \in \mathbb{N}$, $n! = n(n-1)! = n(n-1)(n-2) \times \cdots \times 0!$. $0! = 1$.

$b^{xy} = (b^x)^y$

Binomial Theorem For $x, y \in \mathbb{C}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + y^n$$

4.3 n-dimensional Real Space, $\mathbb{R}^n$

$\mathbb{R}^2$ is defined as the Cartesian product

$$\mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}$$

Higher dimensional real space is defined similarly:

$$\mathbb{R}^3 = \{(a, b, c) \mid a, b, c \in \mathbb{R}\}$$

4.4 Complex Numbers

4.5 Partial Fractions

Suppose we have a rational function $R(x) = \frac{p(x)}{q(x)}$. To perform a **partial fraction decomposition**, we perform the following:

1. If $\deg p \geq \deg q$, we perform polynomial division to get $\deg p < degq$.
2. We factor $q(x)$ into its linear and irreducible quadratic terms.[1]
3. Each factor of $q(x)$ of the form $(ax - b^k)$ contributes the following to the PFD:
$$\frac{A_1}{(ax-b)} + \frac{A_2}{(ax-b)^2} + \cdots + \frac{A_k}{(ax-b)^k}$$
4. Each factor of $q(x)$ of the form $(ax^2 + bx + c)^k$ contributes the following to the PFD:
$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_k x + B_k}{(ax^2 + bx + c)^k}$$
5. Once we have the correct form of the partial fraction decomposition, we recombine over the common denominator to evaluate the constants. Clearing the denominators, plugging in the roots of the various polynomial terms, and then evaluating the resulting simplified expression is one method to determine the unknown constants.

[1]"Irreducible quadratic" means that the term does not contain any real linear factors. $x^2 + 1$ is irreducible, but $x^2 - 1$ is not because it can be factored as $(x-1)(x+1)$.

4.6 Polynomials

By the **fundamental theorem of algebra**, any nonconstant polynomial of degree n is guaranteed to have n roots (though these roots need not be unique). There exist equations to solve polynomials up to degree 4 by radicals; polynomials of degree 5 or higher cannot be solved by radicals. The **Quadratic Equation** is as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

General formulas for cubic and quintic equations exist, but these formulas are exceptionally difficult to evaluate by hand. Several identities for quadratics and cubics are below:

$$\begin{aligned}
a^2 + 2ab + b^2 &= (a+b)^2 \\
a^2 - 2ab + b^2 &= (a-b)^2 \\
a^2 - b^2 &= (a+b)(a-b) \\
a^3 + b^3 &= (a+b)(a^2 - ab + b^2) \\
&= (a+b)(a-b)^2 \\
a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\
&= (a-b)(a+b)^2
\end{aligned}$$

4.7 Logarithms

Logarithms

Where x, y are positive reals, c, d are real, and b is a positive real not equal to 1:

$\log_b(xy)$	$=$	$\log_b(x) + \log_b y$	because	$b^c \cdot b^d = b^{c+d}$
$\log_b\left(\frac{x}{y}\right)$	$=$	$\log_b(x) - \log_b y$	because	$\frac{b^c}{b^d} = b^{c-d}$
$\log_b\left(x^d\right)$	$=$	$d\log_b(x)$	because	$(b^c)^d = b^{cd}$
$\log_b\left(\sqrt[y]{x}\right)$	$=$	$\frac{\log_b(x)}{y}$	because	$\sqrt[x]{y} = x^{\frac{1}{y}}$
$x^{\log_b(y)}$	$=$	$y^{\log_b(x)}$		
$\log_b(a)$	$=$	$\frac{\log_x(a)}{\log_x(b)}$		
$\log_b \frac{1}{y}$	$=$	$\log_b(y^{-1}) = -\log_b(y)$		

5 Trigonometry

5.1 Definitions

$f(x)$	Range	Period	L. Minima	L. Maxima	Zeroes
$\sin x$	$[-1,1]$	2π	$\frac{3\pi}{2}+2\pi n$	$\frac{\pi}{2}+2\pi n$	πn
$\cos x$	$[-1,1]$	2π	$\pi+2\pi n$	$2\pi n$	$\frac{\pi}{2}+\pi n$
$\tan x$	$\mathbb{R}$	π			$\frac{\pi}{2}+\pi n$
$\csc x$	$[-\infty,-\frac{\pi}{2}]\cup[\frac{\pi}{2},\infty]$	2π	$\frac{\pi}{2}+2\pi n$	$\frac{3\pi}{2}+2\pi n$	
$\sec x$	$[-\infty,-\frac{\pi}{2}]\cup[\frac{\pi}{2},\infty]$	2π	$2\pi n$	$\pi+2\pi n$	
$\cot x$	$\mathbb{R}$	π			$\frac{\pi}{2}+\pi n$

By Taylor series, $\sin x$ and $\cos x$ can be related to $\exp(ix)=e^{ix}$ to derive **Euler's Formula**:

$$e^{ix}=\cos(x)+i\sin(x)$$

So for real x, $\cos x=\Re(e^{ix})$ and $\sin x=\Im(e^{ix})$.

$$\begin{aligned}
\sin x &= \tfrac{\text{opp}}{\text{hyp}} = \tfrac{e^{ix}-e^{-ix}}{2i} = \textstyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x-\tfrac{x^3}{3!}+\tfrac{x^5}{5!}-\cdots\\
\cos x &= \tfrac{\text{adj}}{\text{hyp}} = \tfrac{e^{ix}+e^{-ix}}{2} = \textstyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1-\tfrac{x^2}{2!}+\tfrac{x^4}{4!}-\cdots\\
\tan x &= \tfrac{\text{opp}}{\text{adj}} = \tfrac{\sin x}{\cos x}
\end{aligned}$$

5.2 Euler's Formula

$e^{ix}=\cos x+i\sin x$

5.3 de Moivre's Formula

$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$
for $n\in\mathbb{Z}$

5.4 Pythagorean Identities

$$\sin^2 x+\cos^2 x=1$$
$$\sec^2 x-\tan^2 x=1$$
$$\csc^2 x-\cot^2 x=1$$

5.5 Sum Identities

$$
\begin{aligned}
\sin x + y =& \sin x \cos y + \cos x \sin y \\
\cos x + y =& \cos x \cos y - \sin x \sin y \\
\tan x + y =& \frac{\tan x + \tan y}{1 - \tan x \tan y}
\end{aligned}
$$

5.6 Difference Identities

$$
\begin{aligned}
\sin x - y =& \sin x \cos y - \cos x \sin y \\
\cos x - y =& \cos x \cos y + \sin x \sin y \\
\tan x - y =& \frac{\tan x - \tan y}{1 + \tan x \tan y}
\end{aligned}
$$

5.7 Double Angle Identities

$$
\begin{aligned}
\sin 2x =& 2 \sin x \cos x \\
\cos 2x =& \cos^2 x \sin^2 x = 2\cos^2 x - 1 \\
\tan 2x =& \frac{2 \tan x}{1 - \tan^2 x}
\end{aligned}
$$

5.8 Half Angle Identities

$$
\begin{aligned}
\sin^2 x =& \frac{1 - \cos 2x}{2} \\
\cos^2 x =& \frac{1 + \cos 2x}{2} \\
\tan \frac{x}{2} =& \frac{\sin x}{1 + \cos x}
\end{aligned}
$$

5.9 Cofunction Identities

$$
\begin{aligned}
\sin \frac{\pi}{2} - x &= \cos x \\
\cos \frac{\pi}{2} - x &= \sin x \\
\tan \frac{\pi}{2} - x &= \cot x \\
\cot \frac{\pi}{2} - x &= \tan x \\
\sec \frac{\pi}{2} - x &= \csc x \\
\csc \frac{\pi}{2} - x &= \sec x
\end{aligned}
$$

5.10 The Law of Sines and the Law of Cosines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

5.11 Conic Sections

Circle	$(x-h)^2 + (y-k)^2 = r^2$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Parabola	$x^2 + y^2 = r^2$
Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

6 Group Theory

Groups

A group is a tuple $(G, *)$ of a set of elements G and an operation $*$ which satisfies these properties:

$$\begin{aligned}
\text{Closure: } & x, y \in G \to x * y \in G \\
\text{Associativity: } & x, y, z \in G \to x * (y * z) = (x * y) * z \\
\text{Identity: } & \exists (e \in G) \forall (x \in G)(x * e = x) \\
\text{Invertibility: } & \forall (x \in G) \exists (x^{-1} \in G) \text{ such that } x * x^{-1} = e
\end{aligned}$$

We often denote $G = (G, *)$.
A group is *commutative* (or *Abelian*) if its operation $*$ also satisfies the commutative property $x, y \in G \to x * y = y * x$.

Rings

A ring R is a commutative group under some operation $+$ ("addition") and a second operation $\times$ ("multiplication") which satisfies these properties:

$$\begin{aligned}
\text{Closure: } & x, y \in R \to x \times y \in R \\
\text{Associativity: } & x, y, z \in R \to x \times (y \times z) = (x \times y) \times z \\
\text{Distributivity: } & a \times (b + c) = a \times b + a \times c
\end{aligned}$$

Most definitions of a ring include the property

$$\text{Identity: } \exists (1 \in R) \forall (x \in R)(x \times 1 = x)$$

Rings with commutative multiplication are called *commutative rings*.
Rings that have multiplicative inverses are called *division rings* or *skew fields*, because the division operation $\div$ can be defined such that $a \div b = a \times b^{-1}$.
Commutative division rings are called *fields*.
The *center* of a noncommutative ring is the subring of elements c such that $cx = xc$ for all $x \in R$.

7 Linear Algebra

7.1 Vector Spaces

Vector Spaces

A vector space is a set of vectors V over a field F satisfying the following properties for all $v, u, w \in V$ and all $s, t \in F$:

- Closure under vector addition: $v + w \in V$
- Associative vector addition: $v + (u + w) = (v + u) + w$
- Zero vector: $\vec{0} \in V$ such that $v + \vec{0} = v$, $u + \vec{0} = u$
- Inverse vectors: $\forall v \in V \exists - v \in V$ such that $v + -v = \vec{0}$
- Scalar multiplication: $s \times v \in V$
- Associative scalar multiplication: $s \times (t \times v) = (s \times t) \times v$ for $s, t \in F$ and $v \in V$
- Identity scalar: $1 \in F$ such that $1 \times v = v$, $1 \times u = u$
- Distribution of scalar multiplication over vector addition: $s \times (v + u) = s \times v + s \times u$
- Distribution of scalar multiplication over field addition: $(s + t) \times v = s \times v + t \times v$

The members of V are called *vectors* and the members of F are called *scalars*. The first four properties of vector spaces satisfy the four properties of groups.

Span

The *span* of a set S of vectors, denoted $\operatorname{span} S$, is the set of all linear combinations of those vectors. $\operatorname{span} S$ forms vector space.

Subspace

If V is a set of vectors that forms a vector space, $W \subset V$, and W also forms a vector space, then W is a *subspace*.

Basis

A *basis* for a vector space is a linearly independent set of vectors that spans that space.

7.2 Matrices

The entries of the product matrix P of an $l \times m$ matrix A and a $m \times n$ matrix B are computed by multiplying the rows of A by the columns of B. $p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$.

The matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

represents a θ-radian counterclockwise turn about the origin in $\mathbb{R}^2$. The clockwise rotation matrix is given by

$$R(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

This group of two-dimensional rotation matrices is commutative under multiplication; higher dimensional rotation matrices are not commutative.

Row Space

The *row space* of a matrix A is the span of the row vectors of A. The dimension of the row space is the number of independent rows.

Nullspace

The nullspace of a matrix A, denoted $N(A)$, is the space of all vectors v such that $Av = \vec{0}$. The nullspace is perpendicular to the row space, because for any row vector of A, the dot product of that row with v will equal 0.

Column Space

The column space of a matrix A, denoted $C(A)$, is the span of the column vectors of A. $C(A^T)$ is the row space. The column space and row space of A have the same dimension; that is to say that the number of independent rows that a matrix has is equal to the number of independent columns that the matrix has.

Left Nullspace

The left nullspace of a matrix A, denoted $N(A^T)$, is the space of all vectors w such that $Aw = \vec{0}$. The left nullspace is perpendicular to the column space, because for any column vector of A, the dot product of that row with v will equal 0.

Rank-Nullity Theorem

Let V, W be vector spaces, where V is finite dimensional. Let $T : V \to W$ be a linear transformation. Then $Rank(T) + Nullity(T) = dim(V)$.

7.3 Determinants

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

A square matrix's rows are linearly independent if and only if its columns are also linearly independent. A square matrix's rows are linearly dependent if and only if its columns are also linearly dependent.

The determinant of a triangular matrix is the product of the entries in its main diagonal. (Try computing it — every term beyond the first is guaranteed to be zero! Alternately, eliminate the non-leading entries, calculate the determinant, and then simplify.)

For an $n \times n$ diagonal matrix A raised to a power m,

$$\begin{bmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{bmatrix}^m = \begin{bmatrix} a_{1,1}^m & 0 & \cdots & 0 \\ 0 & a_{2,2}^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n}^m \end{bmatrix}$$

For invertible $n \times n$ matrices $A_0, A_1, ..., A_n$, the product $A_0 A_1 \cdots A_n$ is invertible. Additionally, $A_0^{-1} A_1^{-1} \cdots A_n^{-1} = (A_0 A_1 \cdots A_n)^{-1}$.

$det(A_1 A_2) = det(A_1) det(A_2)$

The inverse of matrix is unique; $AA^{-1} = A^{-1}A = I$.

If A, B are $n \times n$ square matrices and $AB = I$, then A and B are invertible and $A = B^{-1}$.

Properties of the Transpose

- $(A^t)^t = A$
- $(A + B)^t = A^t + B^t$
- $(rA)^t = r(A^t)$
- $(AB)^t = A^t B^t$

Invertible Matrix Theorem

Let A be an $n \times n$ square matrix, X_1 be a constant $n \times 1$ column matrix, and Z_1 be the $n \times 1$ zero matrix. The following statements are equivalent:

- A is invertible.

- The reduced row-echelon form of $A = I$.
- The unique solution to $B = AX_1$ is $X_1 = A^{-1}B$.
- The unique solution to $AX_1 = Z_1$ is $X_1 = Z^1$.

Definition: Elementary An $n \times n$ matrix E is *elementary* if it is the result of applying one of the three Gaussian row operations to the $n \times n$ identity matrix.

The matrices which result from swapping the rows of the identity matrix are called *permutation matrices*.

The matrix which results from performing a row operation on a matrix M is the same as the matrix which matrix which results from the product EM, where E is the elementary matrix constructed by using that row operation on the identity matrix.

7.4 Linear Independence

A set of vectors can be tested for linearly independence by putting the associated matrix into reduced row echelon form. If each variable $x_1, x_2, \ldots x_n$ equals zero, then the system is linearly independent. If at least one of the variables is nonzero, the system is linearly dependent.

A set of n $n \times 1$ vectors can be shown to be independent if the determinant of the associated matrix is nonzero (equivalently, if the matrix's reduced row-echelon form is the identity matrix); if the determinant is zero (the matrix's reduced row-echelon form is not the identity matrix), then the vectors are linearly dependent. (This is because the column vectors are linearly independent if an only if the row vectors are linearly independent).

7.5 Determinant

The identity matrix

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is an $n \times n$ square matrix with all diagonal entries equal to 1 and all other entries equal to 0.

7.6 Consequences of a Zero Determinant

7.7 Row Operations as Matrices

A row operation on a matrix A can be represented as left multiplication of A by an invertible matrix.

7.8 Eigenvalues and Eigenvectors

An eigenvector $\vec{v}$ of an $n \times n$ square matrix A is a nonzero $n \times 1$ column vector satisfying the equation $A\vec{v} = \lambda\vec{v}$ for some scalar $\lambda \in \mathbb{C}$ (which is that eigenvector's associated eigenvalue). In other words, an eigenvector is a vector which is only scaled by its associated eigenvalue λ (and therefore remain on its own span) when left-multiplied by the matrix A.

The equation $A\vec{v} = \lambda\vec{v}$ can be rearranged to yield $(A - \lambda I)\vec{v} = \vec{0}$. If left-multiplied by $(A - \lambda I)^{-1}$, $\vec{v} = \vec{0}$ will be the only solution, so eigenvectors only exist if $A - \lambda I$ is not invertible; in other words, when

$$det(A - \lambda I) = 0$$

The eigenvalues of A are the zeroes of this polynomial (which is the characteristic equation of A). The same eigenvalue may occur more than once; the *multiplicity* of an eigenvalue is the number of times the eigenvalue appears as a root. Each eigenvalue λ with multiplicity m has a set of up to m linearly independent eigenvectors; the span of this set of eigenvectors is called an *eigenspace*. Every vector in that eigenspace is an eigenvector for λ.

An eigenvalue with multiplicity 1 is sometimes called *simple*.

Eigenvalues can be substituted back into the eigenvalue equation $(A - \lambda I)$ to solve for the values of $x_1, x_2, \ldots x_n$ in the corresponding $n \times 1$ eigenvector.

If A is an $n \times n$ matrix with only real numbers and if $\lambda_1 = a + bi$ is an eigenvalue with eigenvector $\vec{v}^1$ Then $\lambda_2 = \overline{\lambda_1} = a - bi$ is also an eigenvalue and its eigenvector is the conjugate of $\vec{v}^1$.

The characteristic equation of an $n \times n$ matrix A will not have more than n roots, so A will have no more than n eigenvalues and no more than n eigenvectors. (These eigenvalues and eigenvectors can be duplicates).

7.9 Matrix Diagonalization

The relationship between an $n \times n$ matrix A, its eigenvalues, and its eigenvectors can be written:

$$\begin{aligned} A\vec{v}_1 &= \lambda_1\vec{v}_1 \\ A\vec{v}_2 &= \lambda_2\vec{v}_2 \\ &\vdots \\ A\vec{v}_n &= \lambda_n\vec{v}_n \end{aligned}$$

or equivalently in matrix form

$$A\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix} = \begin{bmatrix}\lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n\end{bmatrix} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}$$

Letting P be the matrix of eigenvalues and D be the matrix of eigenvectors, we can rewrite the above equation as

$$A = PDP^{-1}$$

By raising both sides of this equation to some power arbitrary power m and canceling values of P and P^{-1}, we see

$$A^m = PD^mP^{-1}$$

It should be noted that there are other possible values of P and D. However, this algorithm ensures that D is diagonal, so we can exploit the fact that

$$D^m = \begin{bmatrix} \lambda_1^m & 0 & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^m \end{bmatrix}$$

to find values for $\sin D$, e^D, and other functions that can be expressed via power series (and therefore calculate $\sin A$, e^A, etc).

7.10 Dominant Eigenvectors

An eigenvector is said to be *dominant* if its eigenvalue is greater than the eigenvalues of the other eigenvectors.

Consider a diagonalizable matrix A with real-valued eigenvectors. Repeatedly multiplying a vector v by A will tend to "pull" the product vectors towards A's dominant eigenvector. This happens because the product vector can be expressed as a linear combination of A's eigenvectors, and the dominant eigenvector will be increased the most (and therefore will grow faster and hold more weight in the final linear combination than the other eigenvectors).

8 Calculus on $\mathbb{R}$

8.1 Neighborhoods and Limit Points

For all $a \in \mathbb{C}$, the ε-**neighborhood** of a is the set $V_\varepsilon(a) = \{x \in \mathbb{C} \mid \mathrm{dist}(a, x) < \varepsilon\}$. When restricting our attention to only the reals, the equivalent notion is the interval $V_\varepsilon(a) = (a - \epsilon, a + \epsilon)$.

8.2 (ε, N) Definition of a Limit for Sequences

A sequence (s_n) of real numbers is said to *converge* to a real number s if for each $\epsilon > 0$ there exists a number $N \in \mathbb{N}$ such that $n > \delta$ implies $|s_n - s| < \epsilon$. In such a case, we say that s is the **limit** of (s_n), or that (s_n) *converges* to s. If the above does not hold for *any* s, we say that the limit does not exist and (s_n) diverges.

This condition means that a sequence will eventually get *arbitrarily* (within ε) close to the limit s and stay there, granted we go out far enough (go to at least the N^{th} entry in the sequence).

8.3 $(\varepsilon,\ \delta)$ Definition of a Limit for Functions

Let f be a real-valued function defined on a subset D of $\mathbb{R}$, c be a limit point of D, and L be a real number. We say that $\lim_{x \to c} = L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$. In such a case, we say that L is the **limit** of f as x approaches c. If the above does not hold for *any* L, we say that the limit of f as it approaches c does not exist.

Essentially, this says that if we want to put some arbitrary bound on the input space (distance from c) – say, δ – then we can always find a corresponding bound on the input space (distance from L), ε.

Some authors write $\delta(\varepsilon)$ because the value of ϵ typically determines the value of δ; $\varepsilon - \delta$ proofs typically establish the existence of a δ in terms of ε. Intuitively, this makes sense – if we say ε (the bound on the output space) is very small, then we'll typically require δ (the bound on the input space) to be very small as well.

8.3.1 Continuity

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at the point $c \in \text{dom}(f)$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all $x \in D$, if $0 < |x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. More compactly, if

$$\lim_{x \to c} f(x) = f(c)$$

In other words function is continuous if changing the input by some bounded amount (the distance from c to x is less than δ) means that we can put a related bound on how much the output changes (it won't increase or decrease by more than ϵ). Essentially, changing the input to a function produces *proportionate* (i.e. bounded) change in the output of that function – varying the function's input by a "small" amount won't lead to unpredictable, explosive changes.

Continuous functions look like continuous lines, even if these lines can look jagged, rough, or unwieldy.

8.4 Quotient Rule

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$$

8.5 Integration by Parts

$$\int u dv = uv - \int v du$$

Functions that get "simpler" when differentiated are good choices for u, while functions that get "simpler" when integrated are good choices for dv.

8.6 Trigonometric Substitution

$a^2 - x^2$	$x = a \sin\theta$
$a^2 + x^2$	$x = a \tan\theta$
$x^2 - a^2$	$x = a \sec\theta$

8.7 Trigonometric Integrals

$\int \sin^m x \cos^n x$ m or n odd? $\rightarrow$ split off a factor of that function. If both functions are even $\rightarrow$ half-angle identities

$$\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$$
$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$$
$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \frac{n-2}{n-1} \int \tan^{n-2} x dx$$
$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx$$

$$\int \tan^m x \sec^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$$

n even? $\rightarrow$ split off $\sec^2 x = 1 + \tan^2 x$

m odd? $\rightarrow$ split off $\sec x \tan x$, rewrite using $\tan^2 x = \sec^2 x - 1$

Neither? $\rightarrow$ rewrite $1 + \tan^2 x = \sec^2 x$ and use redux form

8.8 Table of Integrals

$\int f(x)dx$	$F(x)$
$\int \frac{1}{x}dx$	$\ln\lvert x\rvert + C$
$\int a^x dx$	$\frac{a^x}{\ln a} + C$
$\int \cosh x dx$	$\sinh x + C$
$\int \sinh x dx$	$\cosh x + C$
$\int \operatorname{sech}^2 x dx$	$\tanh x + C$
$\int -\coth x \operatorname{csch} x dx$	$\operatorname{csch} x + C$
$\int -\operatorname{sech} x \tanh x dx$	$\operatorname{sech} x + C$
$\int -\operatorname{csch}^2 x dx$	$\coth x + C$
$\int \frac{1}{\sqrt{1-x^2}}dx$	$\arcsin x + C$
$\int -\frac{1}{\sqrt{1-x^2}}dx$	$\arccos x + C$
$\int \frac{1}{1+x^2}dx$	$\arctan x + C$
$\int -\frac{1}{x\sqrt{x^2-1}}dx$	$\operatorname{arccsc} x + C$
$\int \frac{1}{x\sqrt{x^2-1}}dx$	$\operatorname{arcsec} x + C$
$\int -\frac{1}{1+x^2}dx$	$\operatorname{arccot} x + C$
$\int \frac{1}{\sqrt{1+x^2}}$	$\operatorname{arcsinh} x + C$
$\int \frac{1}{\sqrt{x^2-1}}dx$	$\operatorname{arccosh} x + C$
$\int \frac{1}{1-x^2}dx$	$\operatorname{arctanh} x + C$
$\int -\frac{1}{\lvert x\rvert\sqrt{1-x^2}}dx$	$\operatorname{arccsch} x + C$
$\int -\frac{1}{x\sqrt{1-x^2}}dx$	$\operatorname{arcsech} x + C$
$\int \frac{1}{1-x^2}dx$	$\operatorname{arccoth} x + C$
$\int \frac{1}{\sqrt{a^2-x^2}}dx$	$\arcsin \frac{x}{a} + C$
$\int \frac{1}{a^2+x^2}dx$	$\frac{1}{a}\arctan \frac{x}{a} + C$

8.9 Arc Length

The length of a curve from a to b is given by $\int_a^b \sqrt{1+f'(x)^2}dx$.

8.10 Volumes of Solids of Revolution

	about x-axis	about y-axis	about x-axis (two curves)	about y-axis
Disk/Washer Method	$\pi\int_a^b f(x)^2dx$	$\pi\int_a^b h(y)^2dy$	$\pi\int_a^b \big(f(x)^2-g(x)^2\big)dx$	$\pi\int_a^b \big(h(y)^2$
Shell Method	$2\pi\int_a^b yh(y)dy$	$2\pi\int_a^b xf(x)dx$	$2\pi\int_a^b y\big(h(y)-j(y)\big)dy$	$2\pi\int_a^b x\big(f(x$

where $h(y)=f^{-1}(x)$, $j(y)=g^{-1}(x)$

8.11 Surface Areas of Solids of Revolution

For rotations about the x-axis: $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$

For rotations about the y-axis: $\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$

8.12 Convergence Tests

Comparison Test:

Ratio Test:

Root Test:

Integral Test:

8.13 Values of Infinite Sums

We assign values to infinite sums if the sequence of partial sums converges.

$\sum_{n=1}^{\infty} ar^k = \frac{a}{1-r}$ for $|r| < 1$

9 Calculus on $\mathbb{C}$

When we speak about regions in the complex plane, we'll reference the ϵ neighborhood

$$|z - z_0| < \epsilon$$

of a given point z_0. It consists of all the points lying inside but not on the circle centered at z_0 with radius ϵ. Sometimes we also speak of a deleted neighborhood,

$$0 < |z - z_0| < \epsilon$$

containing all points withing the ϵ neighborhood except z_0 itself.

A point z_0 is said to be an interior point of a set S whenever there is some neighborhood of z_0 that contains only points of S; it is called an exterior point of S when there exists a neighborhood of it containing no points in S. If z_0 is neither of these, it is a boundary point of S. A set is open if it does not contain any of its boundary points, and a set is closed if it contains all of its boundary points, and the closure of a set S is the closed set consisting of all points in S together with the boundary of S. $|z| < 1$ is open, and $|z| \leq 1$ is its closure. The punctured disk $0 <\leq |z| \leq 1$ is neither open nor closed. $\mathbb{C}$ on the other hand is both open and closed because it has no boundary points.

An open set is called connected if each pair of points z_1 and z_2 in it can be joined by a polygonal line consisting of a finite number of line segments, joined end to end, that lies entirely on S. A nonempty open set that is connected is called a domain. Any neighborhood is a domain. A domain with some, none, or all of its boundary points is usually referred to as a region. A set S is bounded if every point in S lies inside some circle $|z| = R$; otherwise, it is unbounded.

A function f defined on a set S of complex numbers is a rule that assigns ever $z \in S$ a complex number w so $f(z) = w$. The set S is called the domain of definition of f, and does not need to be a domain as outlined above (although domains of definition typically are domains).

If for each positive real number ϵ, there is a positive real number δ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$, then we write

$$\lim_{z \to z_0} f(z) = w_0$$

This definition says that for each ϵ neighborhood $|w - w_0| < \epsilon$ of w_0, there is a deleted δ neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in that nieghborhood has an image lying in the ϵ neighborhood. Even though all points in the δ neighborhood are to be considered, their images need not fill the entire ϵ neighborhood.

The derivative of a complex function f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If this limit exists, the function is said to be differentiable at z_0. The constant, sum, product, quotient, and chain rules from real analysis all apply; the power rule

$$\frac{d}{dz} z^n = nz^{n-1}$$

holds for nonzero integers n.

Cauchy-Riemann Equations

Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy-Riemann equations

$$\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ u_y(x_0, y_0) &= -v_x(x_0, y_0) \end{aligned}$$

Furthermore, $f'(z_0) = u_x + iv_x$ where these partial derivatives are to be evaluated at (x_0, y_0). Satisfaction of the Cauchy-Riemann equations at z_0

are necessary but not sufficient to ensure that the derivative of f exists at z_0.

However, we do have (p. 66)

Analytic Functions

A function f of a complex variable z is **analytic** (also called **regular** or **holomorphic**) on an open set S if it has a derivative everywhere in S. It is analytic at a point z_0 if it is analytic in some neighborhood of z_0. An **entire function** is a function that is analytic at each point in the entire complex plane.

Let f and g be analytic functions. Then the following gold

Theorem (Properties of Analytic Functions) Let $f = u + iv$ and g be analytic on a domain D. Then the following hold:

- $f(g(z))$, $f(z) + g(z)$, and $f(z)g(z)$ are analytic. $f(z)/g(z)$ is analytic wherever $g(z) \neq 0$
- All derivatives of f exist and are also analytic on D
- All partial derivatives of u and d exist and are continuous on D
- If f is analytic on the disk $|z - z_0| < R$, then the Taylor series of f converges to $f(z)$ for all z in the disk. Furthermore, the Taylor series converges uniformly in any closed subdisk $|z - z_0| \leq R' < R$
- If f is nonconstant, its range is an open set
- f is analytic at z_0 and $f'(z_0) \neq 0$, there is some open disk centered at z_0 on which f is one-to-one
- f is conformal at every point z_0 for which $f'(z_0) \neq 0$

If $f'(z) = 0$ everywhere on a domain D, then $f(z)$ must be constant on D.

If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic (satisfy Laplace's equation) in D.

10 Calculus on $\mathbb{R}^n$

10.1 Directional Derivatives

$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$D_{\hat{u}} f(x, y) = u_1 \times f_x + u_2 \times f_y$

10.2 Proof of Symmetry of Mixed Partials

Theorem: Equality of Mixed Partials

If the partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are continuous at (x_0, y_0), then $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$.

11 Ordinary Differential Equations

11.1 Separation of Variables

$$\frac{dx}{dt} = p(x)q(t) \to \int \frac{1}{p(x)}dx = \int q(t)dt$$

11.2 Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - T_a)$$

11.3 Linear Differential Equations

$\frac{dx}{dt} + p(t)x = q(t)$

Multiply by an integrating factor $I(t) = e^{\int p(t)dt}$

Multiply $I(t)\dfrac{dx}{dt} + I(t)p(t)x = I(t)q(t)$

Rewrite $\dfrac{dx}{dt}\Big(I(t)\Big) = I(t)q(t)$

Integrate and solve!

11.4 Second-Order Homogeneous Differential Equations

Unique roots? $\to$ Solvable by characteristic polynomial

Double root? $\to$ e^{rt} and te^{rt} are solutions

Complex roots ($r_1 = a + bi$ and $r_2 = a - bi$)? $\to$ $e^{(r_1)t}$ and $e^{(r_1)t}$ (equivalently $e^{at}(c_1 \cos bt + c_2 \sin bt)$) are solutions

Two real roots $\to$ overdamped

Two complex roots $\to$ underdamped

Real double root $\rightarrow$ critically damped

The solution set of a homogeneous linear differential equation forms a vector space.

11.5 Second-Order Non-Homogeneous Differential Equations

$ay'' + by' + c = f(t) + g(t) + \cdots$

Guess $y = k_1 f(t)$, $y = k_2 g(t)$, and so on for each function.

Solve for each constant k_1, k_2, etc.

General solution = homogeneous solutions + particular solutions.

The solution set of a non-homogeneous differential equation does not form a vector space because the zero vector is not a solution.

11.6 Resonant Cases

A resonant case occurs when one term of the homogeneous solution is equal to the particular solution for some function $f(t)$. This can be resolved by guessing $y = k_1 t f(t)$; for a critically damped resonant case, guess $y = k_1 t^2 f(t)$.

11.7 Laplace Transform

The Laplace transform is defined as

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s)$$

.

The Laplace transform is useful for solving differential equations because $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. By performing the Laplace transform, f and its derivatives can be put in terms of the single function $F(s)$. Solving this function and performing the inverse Laplace transform will yield the solution for the differential equation.

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)} - f^{(n-1)}(0)$$

$$\frac{d}{dt} F(s) = -\mathscr{L}\{t f'(t)\}$$

11.8 Shifting Theorems

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathscr{L}\{\delta(t-a) f(t-a)\} = e^{-as} F(s)$$

11.9 Table of Laplace Transforms

$f(x)$	$F(s)$	Restriction
t^n	$\frac{n!}{s^{n+1}}$	
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \|a\|$

12 Partial Differential Equations

12.1 Fourier Series

13 Probability

14 Proofs of Notable Theorems

14.1 The Binomial Theorem

14.2 The Fundamental Theorem of Arithmetic

14.3 The Fundamental Theorem of Algebra

Every polynomial in $\mathbb{C}[x]$ has a root in $\mathbb{C}$. Equivalently, $\mathbb{C}$ is its own algebraic closure.

14.4 The Chain Rule

14.5 L'Hôpital's Rule

14.6 Existence and Uniqueness of Solutions to Ordinary Differential Equations

14.7 The Cauchy-Riemann Equations

14.8 Clairaut's Theorem

14.9 Green's Theorem

14.10 Stokes' Theorem

14.11 Lagrange's Theorem

14.12 Abel-Ruffini

Insolubility of Quintics by Radicals

14.13 Rank-Nullity Theorem

15 Derivations

15.1 The Laplace Transform

Consider a power series solution $A(x) = \sum_{n=0}^{\infty} a_n x^n$ to a differential equation.

Let a_n be a discrete function $a(n) = a_n$ for $n \geq 0$. (For example, where $a(n) = \frac{1}{n!}$, $A(x) = e^x$.)

To make $A(x)$ continuous, we let $n = 0, 1, 2, \ldots \to 0 \leq t < \infty$ for $t \in \mathbb{R}$, $a(n) \to f(t)$, and $A(x) = \displaystyle\int_0^\infty f(t)x^t dt$

For convenience, we change $x^t = e^{(\ln x)t}$

This integral typically fails to converge for $x > 1$ and has imaginary values for $x < 0$, so we limit $0 < x < 1$; then $\ln x < 0$.

We then let $-s := \ln x$, so $A(x) = A(e^{-s})$ and we see that

$$A(x) = \int_0^\infty f(t)e^{(\ln x)t}dt \to A(e^{-s}) = \int_0^\infty f(t)e^{-st}dt = \mathscr{L}\{f(t)\} = F(s)$$

15.2 The Fourier Transform

15.3 The Fast Fourier Transform

15.4 Undecidability of HALT

15.5 Undecidability of Peano Arithmetic

15.6 Baire Category Theorem

16 Other Notable Objects and Functions

16.1 The Quaternions $\mathbb{H}$

$\mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$ where

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

and

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

16.2 Everywhere-Continuous, Nowhere Differentiable Functions

16.3 Differentiation as a Linear Map

Differentiation of an n-degree polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ can be expressed by left multiplication of the associated polynomial vector by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 4a_4 \\ \vdots \\ na_n \\ 0 \end{pmatrix} = \frac{d}{dx} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}$$

Differentiation of polynomials of an arbitrary degree can be expressed as left multiplication by the infinite-dimensional matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

16.4 Functions with Non-elementary Antiderivatives

A non-elementary antiderivative is an antiderivative with an infinite number of terms. The following elementary functions have non-elementary antiderivatives:

$e^{-\frac{x^2}{2}}$

$\frac{\sin x}{x}$

$\frac{1}{\ln x}$

16.5 Sigmoid Function

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

16.6 Pi and Gamma Functions

$$\Pi(z) = \Gamma(z+1) = z\Gamma(z) = \int_0^\infty e^{-t} t^z dt = z!$$

16.7 Lambert W Function

$W(x) = f^{-1}(x)$ where $f(x) = xe^x$ on the interval $[-1, \infty)$

$D : [\frac{-1}{e}, \infty), R : [-1, \infty)$

$W(xe^x) = x$ and $W(x)e^{W(x)} = x$

$$\text{Exponential Function: } \exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\text{Sine Derivative Identity: } \frac{d^n y}{dx^n} \sin x = \sin \frac{n\pi}{2} - x$$

16.8 Lagrange Polynomial

Given a set of points

where every value of x_n is unique, the Lagrange polynomial will generate a polynomial that satisfies all of the given points.

16.9 Newton Polynomial

16.10 Euler's Number

$$e = \exp 1 = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

$$e^x = \exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\int_1^e \frac{1}{x} dx = 1$$

ce^x is the unique solution to the differential equation $y' = y$

$$\frac{d}{dx}(log_e x) = \frac{1}{x}$$

16.11 Extension of the Natural Log to the Negative Numbers

$\ln -x = ln{-1} \cdot x = \ln -1 + \ln x$

$e^{i(\pi+2\pi m)} = -1 \therefore \ln -1 = \pi + 2\pi m,\ m \in \mathbb{Z}$

$\therefore \ln -x = \ln x \cdot \pi + 2\pi m,\ m \in \mathbb{Z}$