1 Real Numbers

1.1 Construction of $\mathbb{R}$: Sequences of Rational Numbers

1.2 Construction of $\mathbb{R}$: Dedekind Cuts

Observe that $a = \sup\{r \in \mathbb{Q} \,|\, r < a\}$ for each $a \in \mathbb{R}$

Subsets α of $\mathbb{Q}$ having the form $\{r \in \mathbb{Q} \,|\, r < a\}$ satisfy these properties: $\alpha \neq \mathbb{Q}$ and α is not empty (α is a strict nonempty subset of $\mathbb{Q}$) If $r \in \alpha$, $s \in \mathbb{Q}$ and $s < r$, then $s \in \alpha$ α contains no largest rational

Every subset α of $\mathbb{Q}$ that satisfies these three properties has the form $\{r \in \mathbb{Q} : r < a\}$ for some $a \in \mathbb{R}$, and $a = \sup \alpha$. Subsets α of $\mathbb{Q}$ satisfying these properties are called *Dedekind cuts*.

$\mathbb{R}$ is defined as the space of all Dedekind cuts.

Each rational q corresponds to the Dedekind cut $q^* = \{r \in \mathbb{Q} : r < q\}$; by extension, $\mathbb{Q}^* = \{s* : s \in \mathbb{Q}\}$ ($s*$ is a Dedekind cut) shows that $\mathbb{Q} \subseteq \mathbb{R}$.

The set $\mathbb{R}$, as just defined, is given an order structure as follows: if α and β are Dedekind cuts, then we write $\alpha \leq \beta$ to mean $\alpha \subseteq \beta$. All three properties of Dedekind cuts hold for this ordering.

Example: let α be the set of all real numbers x such that $x^2 \leq 2$ and let β is the set of real numbers such that $y > 2$; note that α has no largest rational member and β has no least rational member, so the cut defines the number $\sqrt{2}$.

Addition in $\mathbb{R}$ is defined as follows: if α and β are Dedekind cuts, then

$$\alpha + \beta = \{r_1 + r_2 : r_1 \in \alpha \wedge r_2 \in \beta\}$$

which is also a Dedekind cut.

1.3 Supremum and Infimum

Let S be a nonempty subset of $\mathbb{R}$. If there is some $s_0 \in S$ such that $s \leq s_0$ for all $s \in S$ then s_0 is the *maximum* of S, denoted $s_0 = maxS$.

If there is some $s_0 \in S$ such that $s \geq s_0$ for all $s \in S$ then s_0 is the *manimum* of S, denoted $s_0 = minS$.

Every finite nonempty subset of $\mathbb{R}$ has a maximum and a minimum.

Let S be a nonempty subset of $\mathbb{R}$.

If S is bounded above and S has a least upper bound, then we call it the *supremum of* S and we denote it by $\sup S$.

If S is bounded below and S has a greatest upper bound, then we call it the *infimum of* S and we denote it by $\inf S$.

The *completeness axiom* is defined for $\mathbb{R}$. Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. Equivalently, $\sup S$ exists and is a real number.

$\sup S = +\infty$ if S is not bounded above.

$\inf S = -\infty$ if S is not bounded below.

1.4 The Completeness Property

1.5 The Intermediate Value Property

1.6 Denseness of $\mathbb{Q}$ in $\mathbb{R}$

If $a, b \in \mathbb{R}$ and $a < b$, $\exists r \in Q$ such that $a < r < b$

Proof: we need to show that $a < \frac{m}{n} < b$ for $m, n \in \mathbb{Z}$, $n > 0$, so we need $an < m < bn$. Since $b - a > 0$, the Archimedean property shows that there exists an $n \in \mathbb{N}$ such that $n(b - a) > 1$ hence $bn - an > 1$. So, there's an integer m such that $an < m < bn$. To prove that m exists, we argue that by the Archimedean property $\exists k > max|an|, |bn|$ so that $-k < an < bn < k$. Then the sets $K = j \in \mathbb{Z} : -k \leq j \leq -k$ and $j \in K : an < j$ are finite (and nonempty, since they both contain k). Let $m = minj \in K : an < j$. Then $-k < an < m$. Since $m > -k$, we have $m - 1 \in K$, so the inequality $an < m - 1$ is false by our choice of m. Thus $m - 1 \leq an$ and, using $bn - an > 1$, we have $m \leq an + 1 < bn$. Thus $an < m < bn$ holds.

1.7 Denseness of $\mathbb{I}$ in $\mathbb{R}$

1.8 Algebraic Numbers

1.9 Rational Zeroes Theorem

Corollary

1.10 The Triangle Inequality

$$|a + b| \leq |a| + |b|$$

1.11 Archimedean Property

If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$.

2 Limits of Sequences

Sequences can be denoted $(s_n)_{n=m}^{\infty}$ or just $(s_n)_{n\in\mathbb{N}}$ if $m = 1$.

For example, the sequence given by $a_n = (-1)^n$ for $n \geq 0$ is $(1, -1, 1, -1, ...)$. This is also a function $a : \mathbb{N} \to \mathbb{N}$ whose domain is $\{0, 1, 2, ...\}$ and whose codomain is $\{1, -1\}$

A sequence (s_n) of real numbers converges to a real number L provided that for each $\epsilon > 0$ there exists a number N such that $n > N$ implies $|s_n - L| < \epsilon$, in which case we write $\lim_{n\to\infty} s_n = L$, $\lim s_n = L$, or $s_n \to L$.

In other words, for all $n > N$, the value s_n is within ϵ of L.

2.1 Limit Theorems for Sequences

Convergent sequences are bounded.

Proof: Let (s_n) be a convergent sequence with $L = \lim(s_n)$. We then know $n > N$ implies $|s_n - L| < 1$ for $\epsilon = 1$. From the triangle inequality we see $n > N$ implies $|s_n| < |L| + 1$. Define $M = max\{|L| + 1, |s_1|, |s_2|, \ldots, |s_\delta|\}$. The we have $|s_n| \leq M$ for all $n \in \mathbb{N}$, so (s_n) is a bounded sequence. (The choice $\epsilon = 1$ is arbitrary.)

$$\lim(ks_n) = k \lim(s_n)$$

Proof: Assume $k \neq 0$. Since $\lim s_n = L$, there exists N such that $n > N$ implies $|s_n - L| < \frac{\epsilon}{|k|}$, then $n > N$ implies $|ks_n - kL| < \epsilon$.

$$\lim(s_n + t_n) = \lim s_n + \lim t_n$$

Proof: Let $\epsilon > 0$ ...

$$\lim(s_n t_n) = \lim(s_n) \lim(t_n)$$

Proof: ...

If (s_n) converges to L, $s_n \neq 0$ for all n, and $L \neq 0$, then $(\frac{1}{s_n}$ converges to $\frac{1}{L}$.

Proof: ...

Suppose (s_n) converges to L_1 and (t_n) converges to L_2. If $L_1 \neq 0$ and $s_n \neq 0$ for all n, then $\frac{t_n}{s_n}$ converges to $\frac{L_2}{L_1}$.

If for each $M > 0$ there is a number N such that $n > N$ implies $s_n > M$, then we write $\lim s_n = +\infty$ and say that the sequence diverges to $+\infty$. Similarly, if for each $M > 0$ there is a number N such that $n > N$ implies $s_n < M$, then we write $\lim s_n = -\infty$ and say that the sequence diverges to $-\infty$.

We say s_n *has a limit* or *the limit exists* provided (s_n) converges (to a real number) or diverges to $\pm\infty$.

If (s_n) and (t_n) are sequences with $\lim(s_n) = +\infty$ and $\lim t_n > 0$, then $\lim(s_n t_n) = +\infty$. Proof:...

For a sequence (s_n) of positive real numbers, $\lim s_n = +\infty$ if and only if $\lim \frac{1}{s_n} = 0$. Proof: ...

2.2 Monotone Sequences

A sequence (s_n) of real numbers is called increasing if $s_n \leq s_{n+1}$ for all n (additionally, $s_n \leq s_m$ whenever $n < m$), and (s_n) is called decreasing if $s_n \geq s_{n+1}$ for all n. If a sequence satisfies one of these two definitions, then that sequence is called *monotone* or *monotonic.*

Bounded monotone series converge.

Proof:...

If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.

If (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.

Proof:...

If (s_n) is a monotone sequence, then the sequence either converges to a real number or diverges to $\pm\infty$.

Proof:...

2.3 $\limsup$ and $\liminf$

Let (s_n) be a sequence in $\mathbb{R}$. We define $\limsup s_n = \lim\limits_{N \to \infty} \sup\{s_n : n > N\}$ and $\liminf s_n = \lim\limits_{N \to \infty} \inf\{s_n : n > N\}$. Note that (s_n) does not need to be bounded; however, if (s_n) is not bounded above, $sup\{s_n : n > N\} = +\infty$ for all N then we say $\limsup s_n = +\infty$. If (s_n) is not bounded below, $sup\{s_n : n > N\} = -\infty$ for all N then we say $\limsup s_n = -\infty$.

2.4 Cauchy Sequences

Definition. A **Cauchy sequence** satisfies the following: for all $\epsilon > 0$, there is some N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$. In other words, the difference between the terms in the sequence approaches zero as the sequence progresses.

Lemma. Cauchy sequences are bounded.

Lemma. A sequence converges if and only if it is a Cauchy sequence.

Definition. A **subsequene** of a sequence (s_n) is a sequence of the form (t_k) where for each k there's a positive integer n_k such that $n_1 < n_2 < \cdots < n_{k-1} < n_k < \cdots$ and $t_k = s_{n_k}$. In other words, a subsequence of the sequence (s_n) can be constructed by selecting, in order, an infinite subset of terms from (s_n).

2.5 Subsequences

Theorem. If a sequence (s_n) converges, then every subsequence converges to the same limit.

Theorem. Every sequence has a monotonic subsequence.

Bolzano-Weierstrass Theorem. Every bounded subsequence has a convergent subsequence.

2.6 Topological Concepts on Metric Spaces

Definition (Metric Space) Let S be a set and let d be a function defined for all pairs of elements in S, saisfying:

1. $d(x, x) = 0$ for all $x \in S$ and $d(x, y) > 0$ for all distinct $x, y \in S$
2. $d(x, y) = d(y, x)$ for all $x, y \in S$
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$ (the triangle inequality)

Such a function is called a **distance function** or **metric** on S. A **metric space** is the pair (S, d). The same set may have more than one metric defined on it.

Definition (Convergence) A sequence (s_n) in a metric space (S, d) **converges** to s in S if $\lim_{n\to\infty} d(s_n, s) = 0$. A sequence (s_n) in S is a **Cauchy sequence** if for each $\epsilon > 0$ there exists an N such that $m, n > N$ implies $d(s_m, s_n) < \epsilon$.

Definition (Complete) A metric space is said to be **complete** if every Cauchy sequence in S converges to some element in S.

Lemma A sequence $(x^{(n)})$ in R^k converges if and only if for each $j = 1, 2, \ldots, k$, the sequence $(x_j^{(n)})$ converges in $\mathbb{R}$. A sequence $(x^{(n)})$ in R^k is a Cauchy sequence if and only if each sequence $(x_j^{(n)})$ is a Cauchy sequence in $\mathbb{R}$.

Theorem Euclidean k-space $\mathbb{R}^k$ is complete.

Bolzano-Weierstrass Theorem Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

3 Series

The infinite series $\sum_{n=m}^{\infty} a_n$ is said to converge provided the sequence (s_n) of partial sums converges to a real number S, in which case we define $\sum_{n=m}^{\infty} a_n = S$, which is equivalent to $\lim s_n = S$ or $\lim_{n\to\infty} \left(\sum_{k=m}^{n} a_k \right) = S$.

3.1 Absolute Convergence

The sum $\sum a_n$ is said to **converge absolutely** or be **absolutely convergent** if $\sum |a_n|$ converges. Absolutely convergent series converge by the comparison test.

3.2 Geometric Series

For $|r| < 1$, we have

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

If $a \neq 0$ and $|r| \geq 1$, the series diverges.

3.3 p-series

p-series are geometric series of the form $\sum_{n=0}^{\infty} \frac{1}{n^p}$. p-series converge if and only if $|p| > 1$.

3.4 Cauchy Criterion for Series

A series converges if and only if it satisfies the Cauchy criterion.

3.5 Corollary

If a series $\sum a_n$ converges, then $\lim a_n = 0$.

4 Continuity

Let $U \subseteq \mathbb{R}$ and $f : U \to \mathbb{R}$. f is continuous at $x_0 \in U$ if:

1. for every sequence (x_n) in U converging to x_0, we have $\lim_n f(x_n) = f(x_0)$
2. for every $\epsilon > 0$ there exists a $\delta > 0$ such that $x \in U$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \epsilon$

These definitions are equivalent. We can also think of δ as a function of x_0 and ϵ, and it's sometimes denoted $\delta(x_0, \epsilon)$ or similar when discussing continuity.

4.1 Limit Theorems for Continuity

5 Properties of Continuous Functions

5.1 Intermediate Value Theorem

If f is a real-valued function on an interval I, then f has the intermediate property on I: Whenever $a, b \in I$, $a < b$ and y lies between $f(a)$ and $f(b)$ (either $f(a) < y < f(b)$ or $f(b) < y < f(a)$) then there exists at least one in $x \in (a, b)$ such that $f(x) = y$.

Corollary If f is a real-valued function on an interval I, then the set $f(I) = \{f(x) \,|\, x \in I\}$ is also an interval or a single point.

Corollary If f is a one-to-one continuous function on an interval I, then f is strictly increasing or strictly decreasing.

5.2 Extreme Value Theorem

6 Uniform Continuity

Let f be a function on $U \subseteq \mathbb{R}$. Then f is **uniformly continuous** on U if

for every $\epsilon > 0$ there exists some $\delta > 0$ such that $x, y \in U$ and $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$

Note that a function cannot be uniformly continuous *only* at a point; it must be uniformly continuous on an interval. Notice also that δ not depends only on ϵ, not on x or y, so δ must satisfy the $|f(x) - f(y)| < \delta$ for any choice of x and y on U (hence the "uniformity" in uniformly continuous). As such, δ is sometimes denoted $\delta(\epsilon)$ or similar when discussing uniform continuity.

6.1 Corollary

If a function is uniformly continuous on its domain, it is continuous on its domain.

6.2 Corollary

If a function is continuous on $[s, b]$ then it is uniformly continuous on $[a, b]$.

6.3 Corollary

If a function f is uniformly continuous on U and (s_n) is a Cauchy sequence in U, then $(f(s_n))$ is a Cauchy sequence.

6.4 Corollary

A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\bar{f}$ on $[a, b]$.

7 Limits of Function

Let $U \subseteq \mathbb{R}$, f be a function defined on U, $a \in \mathbb{R} \cup \{-\infty, \infty\}$ that is the limit of some sequence in U, and let $L \in \mathbb{R} \cup \{-\infty, \infty\}$. We write $\lim_{x \to a^U} f(x) = L$ if

- for every sequence $(x_n) \in U$ with limit a, we have $\lim_{n\to\infty} f(x_n) = L$
- for every $\epsilon > 0$ there exists a $\delta > 0$ such that $x \in U$ and $|x - a| < \delta$ imply $|f(x) - L| < \epsilon$

These definitions are equivalent. $\lim_{x\to a^U} f(x)$ is read "limit as x tends to a along U of $f(x)$."

We see that a function f is continuous at $a \in U$ if and only if $\lim_{x\to a^U} f(x) = f(a)$.

For $a \in \mathbb{R}$ and a function f, we write

1. $\lim_{x\to a} f(x) = L$ provided $\lim_{x\to a} f(x) = L$ for some $U = J\backslash\{a\}$ where J is an open interval containing a. $\lim_{x\to a} f(x)$ is called the two-sided limit of f at a.
2. $\lim_{x\to a^+} f(x) = L$ provided $\lim_{x\to a^U} f(x) = L$ for some open interval $U = (a, b)$. In this case, we call L the **right-hand limit of f at a.**
3. $\lim_{x\to a^-} f(x) = L$ provided $\lim_{x\to a^U} f(x) = L$ for some open interval $U = (c, a)$. In this case, we call L the **right-hand limit of f at a.**
4. $\lim_{x\to\infty} f(x) = L$ provided $\lim_{x\to\infty^U} f(x) = L$ for some interval $U = (c, \infty)$. Likewise, we write $\lim_{x\to\infty} f(x) = L$ provided $\lim_{x\to\infty^U} f(x) = L$ for some interval $U = (c, \infty)$.

Note that f need not be defined at a, and that $\lim_{x\to a} f(x)$ need not equal $f(a)$ even if $f(a)$ is defined. These limits are also unique (they don't depend on U).

Theorem.

Corollary Let f be a function defined on $J\ \{a\}$ for some open interval J containing a, and let L be a real number. Then $\lim_{x\to a} f(x) = L$ if and only if

$$\text{for each } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } 0 < |x - a| < \delta \text{ implies } |f(x) - L| < \epsilon.$$

Corollary Let f be a function defined on $J\ \{a\}$ for some open interval J containing a. Then $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and are equal, in which case all three limits are equal.

7.1 Limit Theorems for Functions

8 Differentiation

9 Theorem.

If a function f is differentiable at a point a, then f is continuous at a.

9.1 Proof of the Product Rule

9.2 Proof of the Quotient Rule

9.3 Proof of the Chain Rule

9.4 The Mean Value Theorem

To prove the Mean Value Theorem, we must first establish several other theorems:

Theorem. If f is defined on an open interval containing x_0, is differentiable at $x = 0$, and f assumes a local maximum or minimum at x_0, then $f'(x_0) = 0$.

Rolle's Theorem. Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$, then there exists at least one $x \in (a, b)$ such that $f'(x) = 0$.

The Mean Value Theorem

Let f and g be continuous functions on $[a, b]$ that are differentiable on (a, b). Then there exists at least one $x \in (a, b)$ such that

$$f'(x)[g(b) - g(a)] = g'(x)[f(b) - f(a)]$$

When $g(x) = x$, the equality reduces to $f'(x)[b - a] = f(b) - f(a)$; if $f(a) = f(b)$ as well, then the equality further reduces to $f'(x) = 0$ (this special case is called **Rolle's Theorem**).

Corollary 29.5 - include discussion too!

Corollary 29.7

Intermediate Value Theorem for Derivatives Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$ and c lies between $f'(x_1)$ and $f'(x_2)$, then there exists at least one $x \in (x_1, x_2)$ such that $f'(x) = c$.

9.5 L'Hôpital's Rule

Sometimes written L'Hospital's rule,[1] this theorem states that if f and g are differentiable functions for which the following limits exist:

$$\lim_{x \to s} \frac{f'(x)}{g'(x)} = L$$

then if

$$\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$$

or if

$$\lim_{x \to s} |g(x)| = \infty$$

then

$$\lim_{x \to s} \frac{f(x)}{g(x)} = L$$

where s represents any of a, a^+, a^-, ∞, or $-\infty$ for $a \in \mathbb{R}$.

10 Taylor's Theorem

11 Integration

11.1 The Darboux Integral

Let f be a bounded function over a closed interval $[a, b]$ and $a < b$. For $U \subseteq [a, b]$ we adopt the notation

$$M(f, S) = \sup\{f(x) \,|\, x \in U\}$$

$$m(f, S) = \inf\{f(x) \,|\, x \in S\}$$

$$U(f, P) = \sum_{k=1}^{n} M\big(f, [t_{k-1}, t_k]\big) \cdot \big(t_k - t_{k-1}\big)$$

[1] In the 17^{th} and 18^{th} centuries, the name was commonly spelled L'Hospital (including by L'Hospital himself). However, French spellings have since changed, and the former spelling is mostly used by languages that do not typically use the circumflex.

$$L(f,P) = \sum_{k=1}^{n} m\big(f,[t_{k-1},t_k]\big)\big) \cdot \big(t_k - t_{k-1}\big)$$

$$U(f) = \inf\{U(f,P) \mid P \text{ a partition of } S\}$$

$$L(f) = \sup\{L(f,P) \mid P \text{ a partition of } S\}$$

Theorem. A bounded function f on $[a,b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

Definition. The Riemann Integral

Theorem. A bounded function on $[a,b]$ is Riemann integrable if and only if it is Darboux integrable, in which case the values of the integrals agree.

Theorem. Every monotonic function on $[a,b]$ is integrable.

Theorem. Every continuous function on $[a,b]$ is integrable.

12 The Fundamental Theorem of Calculus

13 Improper Integrals

14 Continuous Nowhere-Differentiable Functions