Proofs Portfolio

MAT 3100W: Intro to Proofs

Sam Ly

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1 Introduction

(Leave this blank for now. Here's an outline of course topics for your reference.)

2 Mathematical concepts

2.1 Logic, truth tables, and DeMorgan's laws

2.1.1 Logical Statements

Definition 1. A logical statement is a statement that can either be **true** or **false**. Logical statements must be unambiguous, meaning all ratioal agents with access to the same information will come to the same conclusion.

Example 1. "The sun rose today." is a **true** logical statement.

Proof. We begin by observing that the we can currently see the sun in the sky and that we could not see the sun in the sky last night. If we can not see the sun in the sky, it must be below the horizon. Because the sun follows a continuous path, and it had been below the horizon last night, it must have crossed the horizon at some point between last night and now. Thus the sun must have risen today. □

Definition 2. Logical Connectives:

Disjunction: the disjunction of two statements P and Q denoted $P \vee Q$ is true when either P or Q or both P and Q are true.

Conjunction: the conjunction of two statements P and Q denoted $P \wedge Q$ is true when both P and Q are true.

Negation: the negation of a statement P denoted $\neg P$ is true when P is false.

2.1.2 Truth Tables

Definition 3. Certain logical statements' **truth value** depends on the truth of other statements. For example, "the sun rose today **and** it rained today" requires both statements to be true in order for the overall statements to be true. If the sun rose but it didn't rain, or if the sun hasn't risen but it is raining, the overall statement is false. Thus, to visualize this relationship, it is useful to have a table to lay out the possibilities.

Example 2. $A =$ the sun rose today, $B =$ it rained today.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

2.1.3 DeMorgan's Laws

Definition 4. Logical statements and their combinations have their own form of algebra. One of the fundamental rules are DeMorgan's Laws, which state how to find the complements of conjunctions and disjunctions.

Theorem 1. *DeMorgan's Laws:*

1. $\neg(A \wedge B) = \neg A \vee \neg B$
2. $\neg(A \vee B) = \neg A \wedge \neg B$

2.2 Sets

Definition 5. Set: An unordered collection of unique elements.

2.2.1 Unions, intersections, complements, and set differences

Definition 6. Set operations:

Union: the union of two sets A, B is the set that contain elements that are in A, or in B, or both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

Intersection: the intersection of two sets A, B is the set that contains elements that are in both A and B at the same time.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

Difference: the set difference of two sets A, B is the set that contains all elements of A that are not in B. This operation is not commutative.‘’

$$A \setminus B = \{x \mid x \in A \wedge x \notin B\}.$$

Complement: the complement of a set A is the set of all elements that are not in A. For the complements of a set to be defined, it must be a subset of the unversal set $\mathcal{U}$. In other words, it is the set difference between $\mathcal{U}$ and A.

$$A^c = \mathcal{U} \setminus A.$$

Theorem 2. *DeMorgan's Laws for Sets:*

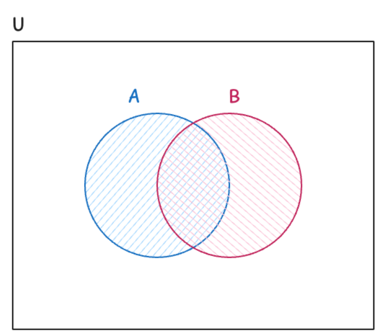
1. $(A \cap B)^c = A^c \cup B^c$
2. $(A \cup B)^c = A^c \cap B^c$

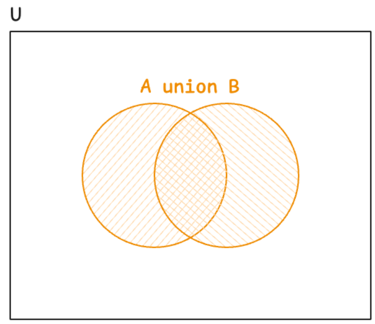
Proposition 1. *For all integers $n \geq 2$:*

1. $(A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$
2. $(A_1 \cap A_2 \cap ... \cap A_n)^c = A_1^c \cup A_2^c \cup ... \cup A_n^c$

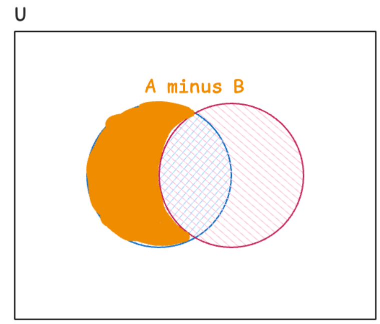
2.2.2 Venn diagrams

Definition 7. Venn diagrams: a visual aid for understanding sets and set operations.

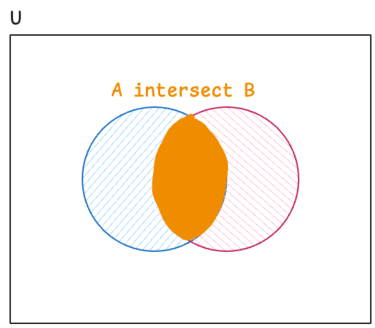




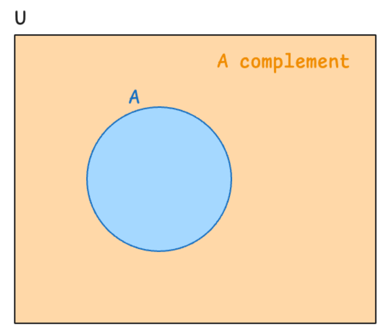
$A \cup B$



$A \setminus B$



$A \cap B$



A^c

2.3 Numbers and number systems

Definition 8. Number: values that symbolize quantities.

Definition 9. Number system: way of representing numbers. Some are more sophisticated than others.

2.3.1 Parity, divisibility, and modular arithmetic

Definition 10. Divisibility: a number $n \in \mathbb{Z}$ is divisible by another number m if and only if $n = k \times m$ for some integer k.

Definition 11. Parity: the property of a number being even or odd. The number is even if it is divisible by two, and odd otherwise.

Definition 12. Modular arithmetic: a number system that groups numbers into equivalence classes based on their remainder when divided by a specific integer. More formally, for integers n, r, and m, we say n is **congruent** to r modulo m if $(n - r)$ is divisible by m.

$$n \equiv r \pmod{m} \Leftrightarrow m \mid (n - r)$$

For example, $5 \equiv 11 \pmod 3$ since they both have a remainder 2 when divided by 3, and because $11 - 5 = 6$ is divisible by 3.

Standard arithmetic operations $+, -,$ and $\times$ are well-defined under modular arithmetic. However, $\div$ is not always well defined. These operations work the same way as they do in standard arithmetic. Notice that the parity of a number is equivalent to its divisability by 2, and a number's divisibility by $m \in \mathbb{N} > 0$ is a equivalent to it being congruent to 0 modulo m.

Proposition 2. *If $a \equiv b \pmod m$, then $b \equiv a \pmod m$.*

Proposition 3. *If $a \equiv a' \pmod m$ and $b \equiv b' \pmod m$, then:*

1. $a + b \equiv a' + b' \pmod m$
2. $a - b \equiv a' - b' \pmod m$
3. $a \times b \equiv a' \times b' \pmod m$

2.3.2 Rational and irrational numbers

Definition 13. Rational numbers $\mathbb{Q}$: the set of numbers that can be expressed as a ratio of two integers.

Definition 14. Irrational numbers: the set of numbers that can't be expressed as a ratio of two integers.

Proposition 4. $\sqrt{2}$ *is irrational.*

2.3.3 Real numbers, absolute value, and inequalities

Definition 15. Real numbers $\mathbb{R}$: the set of all numbers on our number line.

2.3.4 Combinatorics: combinations, permutations, and factorials.

Definition 16. Combinations $C(n, r)$: the cardinality of the set of all subsets of a specific cardinality.

Definition 17. Permutations $P(n, r)$: the cardinality of the set of all orderings of a specific length.

Definition 18. Factorial: the product of natural numbers before it down to zero.

$$5! = 5 \times 4 \times 3 \times 2 \times 1.$$

2.3.5 Countable sets

Definition 19. Countable set: a set that is either finite, or that has the same cardinality as natural numbers $\mathbb{N}$. The second case is called **countably infinite**.

Lemma 1. *The cartesian product of two countable sets will always be countable.*

Proposition 5. $\mathbb{Q}$ *is countably infinite.*

2.3.6 Uncountable sets

Definition 20. Uncountable set: a set that is infinite and there does not exists a bijection from it to the natural numbers.

Proposition 6. $\mathbb{R}$ *is not countably infinite.*

2.4 Relations and functions

2.4.1 Relations and equivalence relations

Definition 21. Relation R: a set of ordered pairs that represents if a two element $a, b \in S$ are related. $a \text{and} b$ are related if and only if $(a, b) \in R$.

Definition 22. Properties of Relations:

Reflexive: a relation R on set S is reflexive if and only if for every element $s \in S$, sRs.

Symmetric: a relation R on set S is symmetric if and only if for every pair of elements $s_1, s_2 \in S$, s_1Rs_2 implies s_2Rs_1.

Transitive: a relation R on set S is transitive if and only if for every trio of elements $s_1, s_2, s_3 \in S$, s_1Rs_2 and s_2R_s3 implies s_1Rs_3.

Definition 23. Equivalence relations: a special type of relation on a set that satisfies the properties of being symmetric, reflexive, and transitive.

Theorem 3. *Congruence under modular arithmetic is an equivalence relation.*

2.4.2 Functions

Definition 24. Function: a mapping from a set called the domain to elements in a set called the codomain.

2.4.3 Injections (one-to-one), surjections (onto), and bijections

Definition 25. Types of mappings:

Injection: a function $f : A \to B$ is injective if and only if every distinct element $a \in A$ maps to a distinct element $f(a) \in B$. In other words, there does not exist a pair of elements $a, a' \in A$ where $a \neq a'$ such that $f(a) = f(a')$.

Surjection: a function $f : A \to B$ is surjective if and only if for every element $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Bijection: a function $f : A \to B$ is a bijection if and only if it is both injective and surjective.

Lemma 2. *If a bijection $f : A \to B$ exists, then $|A| = |B|$.*

Lemma 3. *If $|A| = |B|$, then a bijection $f : A \to B$ exists.*

Theorem 4. *A bijection $f : A \to B$ exists if and only if $|A| = |B|$.*

3 Proof techniques

3.1 Direct Proofs

Definition 26. Direct proof: using fundamental rules of logic to prove a statement. The fundamental rules of logic are taken for granted as **axioms**.

Example 3. Proposition 2 can be proven directly from definitions.

Proof. Assume that $a \equiv b \pmod{m}$. This means that $m \mid (a - b)$. Thus, $a - b = km$ for some $k \in \mathbb{Z}$.
If we multiply both sides by -1, we get $b - a = -km$.
Thus, by definition, $m \mid (b - a)$ and $b \equiv a \pmod{m}$. □

Using the properties of modular arithmetic in definition 12, prove proposition 3
Given $a \equiv a' \pmod{m}$ and $b \equiv b' \pmod{m}$:

1. $a + b \equiv a' + b' \pmod{m}$.

   Proof. We begin with by defining $a \equiv a' \pmod{m}$ as $m \mid (a - a')$. Similarly, $m \mid (b - b')$.

   Following from these definitions, we write:

$$a - a' = m \times k_1 \tag{1}$$

$$b - b' = m \times k_2 \tag{2}$$

   We can add equations eq. (1) and eq. (2) together to get $a + b - a' - b' = m \times k_1 + m \times k_2$.

   With some factoring, we get $(a + b) - (a' + b') = m(k_1 + k_2)$.

   By definition, we find that $m \mid (a + b) - (a' + b')$, and thus $a + b \equiv a' + b' \pmod{m}$. □

2. $a - b \equiv a' - b' \pmod{m}$.

   Proof. Following from Proof 1, we can instead subtract equation eq. (1) and eq. (2) to get $a - b - a' + b' = m \times k_1 - m \times k_2$.

   With some factoring, we get $(a - b) - (a' - b') = m(k_1 - k_2)$.

   By definition, we find that $m \mid (a - b) - (a' - b')$, and thus $a - b \equiv a' - b' \pmod{m}$. □

3. $a \times b \equiv a' \times b' \pmod{m}$.

Proof. Following from equation eq. (1), we get

$$a = a' + m \times k_1. \tag{3}$$

Similarly, from equation eq. (2), we get

$$b = b' + m \times k_2. \tag{4}$$

By multiplying equations eq. (3) and eq. (4), we get $a \times b = (a' + m \times k_1)(b' + m \times k_2)$.

From now on, I will omit the $\times$ *symbol.*

By distributing, we get

$$ab = a'b' + a'mk_2 + b'mk_1 + m^2k_1k_2.$$

We can factor out m to find

$$ab = a'b' + m(a'k_2 + b'k_1 + mk_1k_2).$$

We can subtract $a'b'$ from both sides to find

$$ab - a'b' = m(a'k_2 + b'k_1 + mk_1k_2).$$

By definition, we see that $m \mid (ab - a'b')$, and, by extension, $ab \equiv a'b' \pmod{m}$.

□

3.2 Transformation of conditionals

Definition 27. Transformation of conditionals: using rules of conditional logic to prove conditional statements.

For the following proofs, I will prove similar/related statements as it makes it easier to see the relationships between the transformations. We first perform a direct proof.

Example 4. Lemma 3 can be proven directly.

Proof. Suppose you have two sets A and B such that $|A| = |B| = n$. Thus, the elements of A can be enumerated by a_i for $0 < i \leq n$. Similarly, the elements of B can be enumerated by b_i for $0 < i \leq n$.

So, we can construct the function $f : A \to B$ as $f(a_i) = b_i$. f is injective because there does not exist a pair a_i, a_i' such that $f(a_i) = f(a_i')$. f is also surjective becuase for every $b_i \in B$, there exists $a_i \in A$ such that $f(a_i) = b_i$. Therefore, by definition, f is a bijection. □

3.2.1 Contrapositive proofs

Definition 28. Contrapositive: given a statement $P \Rightarrow Q$, the converse is $\neg Q \Rightarrow \neg P$ The truth value of a statement is equivalent to its contrapositive. By proving the contrapositive, you also prove the original statement.

Example 5. Lemma 3 can also be proven using its contrapositive. We find that this proof is slightly simplier.

Proof. We proceed by contrapositive by saying if no bijection $f : A \to B$ exists, then $|A| \neq |B|$.

Suppose two sets A and B such that there can not exist a bijection f. Thus, for any $f : A \to B$, f is either not injective or it is not surjective.

If f is not injective, then there exists $a, a' \in A$ such that $f(a) = f(a')$. Thus, $|A| > |B|$. However, if f is not surjective, then there exists $b \in B$ such that there does not exist $a \in A$ where $f(a) = b$. Thus, $|B| > |A|$. Therefore, in either case, $|A| \neq |B|$. □

3.2.2 Converse statements

Definition 29. Converse: given a statement $P \Rightarrow Q$, the converse is $Q \Rightarrow P$ The truth value of a statement's converse is equivalent to its **inverse**.

Example 6. Lemma 2 is the converse of lemma 3, and can be proven directly.

Proof. Since f is injective, distinct elements $a_i \in A$ will always map to different elements $b \in B$. In other words, $f(a_1) \neq f(a_2)$. Also, since f is surjective, for all elements $b \in B$, there exists an element $a \in A$ such that $f(a) = b$. Therefore, $|A| = |B|$. □

3.2.3 Inverse statements

Definition 30. Inverse: given a statement $P \Rightarrow Q$, the inverse is $\neg P \Rightarrow \neg Q$. The truth value of a statement's inverse is equivalent to its **converse**.

Example 7. We can prove lemma 2 by proving the inverse of lemma 3. We will see that this is slightly simplier than the direct proof.

Proof. Suppose we have two sets A and B such that $|A| \neq |B|$. Thus, we have two cases:

$|A| > |B|$: Since there are more elements $a \in A$ than there are $b \in B$, there must be a pair of elements $a, a' \in A$ such that $f(a) = f(a')$.

Thus, f is not injective.

$|A| < |B|$: Since there are more elements $b \in B$ than there are $a \in A$, there must exist an element $b \in B$ such that $f(a) \neq b$ for all $a \in A$.

Thus, f is not surjective.

Therefore, f is not a bijection. □

3.2.4 Bidirectional (”if and only if” proofs)

Definition 31. Bidirectional proof: A bidirectional proof involves proving both a conditional statement and its converse to conclude that the **antecedent** and the **consequent** are logically equivalent.

Example 8. Theorem 4 can be proven bidirectionally by proving lemma 2 and lemma 3.

Proof. We have proven both lemma 3 and lemma 2 above. Thus, theorem 4 must be true. □

3.3 Quantifiers

Feedback requested. [Are these propositions good enough?]

Definition 32. Quantifier: a logical expression that denotes whether a statement is true for all cases or for specific cases.

3.3.1 Universal quantifiers

Definition 33. For a statement with a universal quantifier to be true on set S, the statement must be true for every single $s \in S$. Universal statements can be disproven by one counterexample.

Example 9. A function $f : D \to C$ is well defined on a domain D and codomain C if $f(d) \in C$ for all $d \in D$. Let $f(x) = \sqrt{x}$. Is $f : \mathbb{R} \to \mathbb{R}$ well defined?

Proof. We find that for $x < 0$, $f(x) \notin \mathbb{R}$. We have not just found one counterexample, we have found an uncountably infinite number of counterexamples.

So, f is not well defined. □

3.3.2 Existential quantifiers

Definition 34. For a statement with an existential quantifier to be true on set S, the statement must be true for at least one $s \in S$. Existential statements can be proven by one example.

Example 10. There exists a real number $r \in \mathbb{R}$, where $\sqrt{r} \in \mathbb{Q}$.

Proof. Let $r = 4$. $\sqrt{4} = 2 = \frac{2}{1}$. Thus, $\sqrt{2} \in \mathbb{Q}$. So, there exists a real number $r \in \mathbb{R}$, where $\sqrt{r} \in \mathbb{Q}$. □

3.3.3 Multiply quantified statements

Definition 35. Multiply quantified statement: statements that include more than one quantifier. Typically, they go "for all x, there exists y, such that z." They can be reasoned about by using an "adversarial" game, where player 1 picks an x that makes it hard for player 2 to pick a y to satisfies z. If player 1 can pick an x such that player 2 can't pick a valid y to satisfy z, player 1 'wins' and the statement is false.

In another case, the statement can go "there exists x, for all y, such that z. " In this case, player 1 just needs to pick one value x, such that for all values player 2 picks for y, it satisfies z. This makes the statement true.

Example 11. For all pairs of real numbers $x, y \in \mathbb{R}$ with $x < y$, there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

Proof. We first define a useful interpretation of constructing rational numbers by dividing two integers $r = n/d$. We are essentially taking n steps of length $1/d$.

Thus, if we have a sufficiently small step size, there must be an integer number of steps for us to land on the range $x < r < y$.

We see that if we have a step size smaller than $y - x$, we must always land within the range $x < r < y$.

We can see this by imagining a worst case scenario where $\frac{n-1}{d} = x$. Thus, because $1/d < y - x$,

$$\begin{aligned} x < \frac{n-1}{d} + \frac{1}{d} < y \\ x < \frac{n}{d} < y \end{aligned}$$

Similarly, we can see the other worst case scenario where $\frac{n+1}{d} = y$. Thus, because $1/d < y - x$,

$$\begin{aligned} x < \frac{n+1}{d} - \frac{1}{d} < y \\ x < \frac{n}{d} < y \end{aligned}$$

In order to achieve a step size smaller than $y - x$, we must satisfy the condition $1/d < y - x$. In other words, $d > \frac{1}{y-x}$. Furthermore, increasing d will only decrease the step size.

Thus, for all pairs of real numbers $x, y \in \mathbb{R}$ with $x < y$, we can find $d \in \mathbb{Z}$ with $d > \frac{1}{y-x}$. Then, there must exist an integer n where $x < n/d < y$. □

3.4 Existence and uniqueness proofs

Definition 36. Existence and uniqueness proof: a proof that results in us being sure that an element exists with a given property, and that it is the only element that exhibits such property.

Find a good existence and uniqueness proof.

3.5 Proof by Induction

Definition 37. Proof by Induction: proof technique used to prove a statement is true for a countably infinite set of discrete elements.

When doing proofs by induction, it is useful to enumerate the cases as $n \in \mathbb{N}$. This proof begins with a **base case** (or in some scenarios **base cases**) proving that the proposition is true for some "small" cases.

Typically this means proving the proposition is true for $n = 0, 1, 2$. It is also helpful to "play" to gain intuition about our problem before proceding to the inductive step.

We then form an **inductive hypothesis** that assumes our proposition is true for cases $n \leq k$. Our **inductive step** is to prove that this assumption necessarily means that the $n = k + 1$ case must be true.

Thus, the cases n up to k implies case $n = k + 1$. So, starting at our base case, we know that $n = 0$ is true. Then we know that $n = 1$ is true. Since, cases $n = 0$ and $n = 1$ are true, case $n = 2$ is true. Then since cases $0 \leq n \leq 2$ are true, case $n = 3$ is true, and so on.

Now, it may seem that this proof is circular at first, since we are making a massive assumption in the form of our inductive hypothesis. However, our assumption is not the same as our conclusion. Our inductive hypothesis is used to prove that for any case $n = k$, its successor must be true.

Example 12. Proposition 1 can be proven with induction.

For the first case, we have

$$(A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c.$$

Proof. First, we define some useful notation for a union and intersection for a large series of sets $A_1, A_2, ..., A_3$:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup ... \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap ... \cap A_n.$$

First, we use theorem 2 (DeMorgan's Law for Sets) as the base case $n = 2$, $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$.
Then, as our inductive hypothesis, we assume that

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c, \text{ for } n \leq k.$$

Then, we see that for $n + 1$,

$$\left(\bigcup_{i=1}^{n+1} A_i\right)^c = \left(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right)^c$$
$$\left(\bigcup_{i=1}^{n+1} A_i\right)^c = \left(\bigcup_{i=1}^{n} A_i\right)^c \cap A_{n+1}^c.$$

We can use our inductive hypothesis to substitute $(\bigcup_{i=1}^{n} A_i)^c = \bigcap_{i=1}^{n} A_i^c$ to get,

$$\left(\bigcup_{i=1}^{n+1} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c \cap A_{n+1}^c.$$
$$= A_1^c \cap A_2^c \cap ... \cap A_n^c \cap A_{n+1}^c.$$

Therefore,

$$(A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$$

for any integer $n \geq 2$. □

Similarly, for the second case, we have

$$(A_1 \cap A_2 \cap ... \cap A_n)^c = A_1^c \cup A_2^c \cup ... \cup A_n^c.$$

Proof. First, we use theorem 2 (DeMorgan's Law for Sets) as the base case $n = 2$, $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$.
Then, as our inductive hypothesis, we assume that

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c, \text{ for } n \leq k.$$

Then, we see that for $n+1$,

$$\left(\bigcap_{i=1}^{n+1} A_i\right)^c = \left(\bigcap_{i=1}^{n} A_i \cap A_{n+1}\right)^c$$
$$\left(\bigcap_{i=1}^{n+1} A_i\right)^c = \left(\bigcap_{i=1}^{n} A_i\right)^c \cup A_{n+1}^c.$$

We can use our inductive hypothesis to substitute $(\bigcap_{i=1}^{n} A_i)^c = \bigcup_{i=1}^{n} A_i^c$ to get,

$$\left(\bigcap_{i=1}^{n+1} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c \cup A_{n+1}^c.$$
$$= A_1^c \cup A_2^c \cup ... \cup A_n^c \cup A_{n+1}^c.$$

Therefore,

$$(A_1 \cap A_2 \cap ... \cap A_n)^c = A_1^c \cup A_2^c \cup ... \cup A_n^c$$

for any integer $n \geq 2$. □

Example 13. Let F_n be the n-th Fibonacci number, where $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Prove that $F_n \leq 1.9^n$ for all $n \geq 1$.

Proof. We proceed by induction, starting with the base cases, where $n = 1, 2$:

$$n = 1, F_1 = 1 \leq 1.9^1 = 1.9$$

$$n = 2, F_2 = 2 \leq 1.9^2 = 3.61.$$

We assume as an inductive hypothesis that $F_n \leq 1.9^n$ for $1 \leq n \leq k$.
Using our inductive hypothesis, we see that $F_k \leq 1.9^k$ and $F_{k-1} \leq 1.9^{k-1}$.
So, $F_k + F_{k-1} \leq 1.9^k + 1.9^{k-1}$.
By refactoring $1.9^k + 1.9^{k-1}$, we get:

$$1.9^k + 1.9^{k-1} = 1.9(1.9^{k-1}) + 1.9^{k-1} = 2.9(1.9^{k-1}).$$

Also, 1.9^{k+1} can be rewritten as $1.9^2(1.9^{k-1}) = 3.61(1.9^{k-1})$.
Finally, we see

$$F_{k+1} = F_k + F_{k-1} \leq 2.9(1.9^{k-1}) \leq 3.61(1.9^{k-1}) = 1.9^{k+1}$$
$$F_{k+1} \leq 1.9^{k+1}.$$

Thus, we conclude that by the principle of mathematical induction, $F_n \leq 1.9^n$ for all $n \geq 1$. □

Example 14. Look up the Tower of Hanoi puzzle. Prove that given a stack of disks, you can solve the puzzle in moves.

Proof. We begin by defining the Tower of Hanoi problem.

In this problem, we begin with a stack of n disks. The disks are ordered from largest at the bottom to smallest at the top. We are also given 3 'spots' to place our disks under one condition: that we never place a larger disk on top of a smaller disk.

Following these rules, what is the minimum number of moves required to move the entire pile to a new 'spot'?

We define the function $f : \mathbb{N} \to \mathbb{N}$ such that it maps the starting stack height n to the minimum number of moves required to move the entire pile $f(n)$.

Before immediately proving that $f(n) = 2^n - 1$, it is more intuitive to first define f as a recurrence relation, then prove that the recurrence relation is equal to $2^n - 1$.

We notice that moving the entire pile of n disks essentially requires 3 'phases':

1. Moving the top $n - 1$ disks onto a single pile.
2. Moving the nth disk to another vacant spot.
3. Moving the top $n - 1$ disks onto the new spot.

Thus, we know that $f(n) = f(n-1) + 1 + f(n-1) = 1 + 2f(n-1)$, where $f(1) = 1$. We can then prove $f(n) = 2^n - 1$ using induction.

We begin with our base cases:

n	$f(n)$
1	$1 = 2^1 - 1$
2	$3 = 2^2 - 1$
3	$7 = 2^3 - 1$

Now, we assume that $f(k) = 2^k - 1$ for all $1 \leq k \leq n$.

We see that

$$f(k+1) = 1 + 2f(k)$$
$$f(k+1) = 1 + 2(2^k - 1)$$
$$f(k+1) = 2^{k+1} - 1.$$

Thus, $f(n) = 2^n - 1$. □

3.6 Proof by Contradiction

Definition 38. Proof by contradiction: proof technique that assumes the opposite of our proposition, then showing that this leads to an absurd conclusion, ie. a contradiction. Used as a "last resort" proof technique.

This proof works because all statements in our mathematical universe can either be true or false. Thus, by assuming the opposite of our proposition, and showing that the opposite of our proposition **can not** be true, our original proposition then **must** be true. We show that the opposite of our proposition can't be true through finding a contradiction.

This proof technique is used as a last resort since it doesn't necessarily tell you much abou why/how our proposition is true, just that it must be true.

Example 15. Proposition 4 can be proven by contradiction.

We proceed by contradiction by assuing that $\sqrt{2} \in \mathbb{Q}$. So, we can write

$$\sqrt{2} = \frac{a}{b}$$

for $a, b \in \mathbb{N}$, and such that $\frac{a}{b}$ is in lowest form.

Then, with algebraic manipulation, we see

$$2 = \frac{a^2}{b^2}.$$

Thus, we get $a^2 = 2b^2$ and $b^2 = a^2/2$. So, a^2 is even, and by extension, a is even. Since a is even, we write $a = 2m$ for $m \in \mathbb{Z}$. Thus,

$$\begin{aligned} b^2 &= \frac{(2m)^2}{2}, \\ b^2 &= \frac{4m^2}{2}, \\ b^2 &= 2m^2, \end{aligned}$$

showing that b^2 must also be even.

This is a contradiction because we assumed that a/b was in lowest terms, however, we have just found a way to factor out a common 2. This contradiction shows that our original proposition was be false. So, $\sqrt{2}$ must be irrational.

Example 16. Proposition 6 can be proven by contradiction. This is the famous Cantor's diagonalization argument.

First we assume that $\mathbb{R}$ is countably infinite. In other words, $|\mathbb{R}| = |\mathbb{N}|$, and that there exists a bijection between $\mathbb{R}$ and $\mathbb{N}$.

Thus, we can list out all the real numbers $r_i \in \mathbb{R}$ for $i \in \mathbb{N}$ in a table as follows where the rows are distinct real numbers r_i and the columns are their decimal places:

	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	$\cdots$	
r_1	$\mathbf{a_{11}}$	a_{12}	a_{13}	a_{14}	a_{15}	$\cdots$	a_{1n}
r_2	a_{21}	$\mathbf{a_{22}}$	a_{23}	a_{24}	a_{25}	$\cdots$	a_{2n}
r_3	a_{31}	a_{32}	$\mathbf{a_{33}}$	a_{34}	a_{35}	$\cdots$	a_{3n}
r_4	a_{41}	a_{42}	a_{43}	$\mathbf{a_{44}}$	a_{45}	$\cdots$	a_{4n}
r_5	a_{51}	a_{52}	a_{53}	a_{54}	$\mathbf{a_{55}}$	$\cdots$	a_{5n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
r_n	a_{n1}	a_{n2}	a_{n3}	a_{n4}	a_{n5}	$\cdots$	$\mathbf{a_{nn}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	

Then, we see the bolded diagonal. With this diagonal, for each a_{jj} for $j \in \mathbb{N}$, we can construct an **entirely new** real number just be changing a_{jj}. In the original arument, Cantor states to add 1 if $a_{jj} < 9$ and to subtract 1 if $a_{jj} = 9$. However a more general way of saying this is to just pick a new integer $0 \leq a'_{jj} \leq 9$ that is not equal to a_{jj}.

So, we have now constructed a new real number, contradicting our assumption. Thus, we see that $\mathbb{R}$ must not be countably infinite.

4 Final project

5 Conclusion and reflection

Appendix

(The first section, "Course objectives and student learning outcomes" is just here for your reference.)

A Course objectives and student learning outcomes

1. Students will learn to identify the logical structure of mathematical statements and apply appropriate strategies to prove those statements.
2. Students learn methods of proof including direct and indirect proofs (contrapositive, contradiction) and induction.
3. Students learn the basic structures of mathematics, including sets, functions, equivalence relations, and the basics of counting formulas.
4. Students will be able to prove multiply quantified statements.
5. Students will be exposed to well-known proofs, like the irrationality of $\sqrt{2}$ and the uncountability of the reals.

A.1 Expanded course description

- Propositional logic, truth tables, DeMorgan's Laws
- Sets, set operations, Venn diagrams, indexed collections of sets
- Conventions of writing proofs
- Proofs
  - Direct proofs
  - Contrapositive proofs
  - Proof by cases
  - Proof by contradiction
  - Existence and Uniqueness proofs
  - Proof by Induction
- Quantifiers
  - Proving universally and existentially quantified statements
  - Disproving universally and existentially quantified statements
  - Proving and disproving multiply quantified statements
- Number systems and basic mathematical concepts
  - The natural numbers and the integers, divisibility, and modular arithmetic
  - Counting: combinations and permutations, factorials
  - Rational numbers, the irrationality of $\sqrt{2}$
  - Real numbers, absolute value, and inequalities
- Relations and functions
  - Relations, equivalence relations
  - Functions
  - Injections, surjections, bijections

- Cardinality
  - Countable and uncountable sets
  - Countability of the rational numbers, $\mathbb{Q}$
  - Uncountability of the real numbers, $\mathbb{R}$