

# Proofs Portfolio

## MAT 3100W: Intro to Proofs

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## 1 Introduction

(Leave this blank for now. Here's an outline of course topics for your reference.)

## 2 Mathematical concepts

### 2.1 Logic, truth tables, and DeMorgan's laws

#### 2.1.1 Logical Statements

**Definition 1.** A logical statement is a statement that can either be **true** or **false**. Logical statements must be unambiguous, meaning all rational agents with access to the same information will come to the same conclusion.

**Example 1.** “The sun rose today.” is a **true** logical statement.

*Proof.* We begin by observing that we can currently see the sun in the sky and that we could not see the sun in the sky last night. If we can not see the sun in the sky, it must be below the horizon. Because the sun follows a continuous path, and it had been below the horizon last night, it must have crossed the horizon at some point between last night and now. Thus the sun must have risen today.  $\square$

#### 2.1.2 Truth Tables

**Definition 2.** Certain logical statements' **truth value** depends on the truth of other statements. For example, “the sun rose today **and** it rained today” requires both statements to be true in order for the overall statement to be true. If the sun rose but it didn't rain, or if the sun hasn't risen but it is raining, the overall statement is false. Thus, to visualize this relationship, it is useful to have a table to lay out the possibilities.

**Example 2.**  $A =$  the sun rose today,  $B =$  it rained today.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

#### 2.1.3 DeMorgan's Laws

**Definition 3.** Logical statements and their combinations have their own form of algebra. One of the fundamental rules are DeMorgan's Laws, which state how to find the complements of conjunctions and disjunctions.

**Theorem 1.** *DeMorgan's Laws:*

$$1. \neg(A \wedge B) = \neg A \vee \neg B$$

$$2. \neg(A \vee B) = \neg A \wedge \neg B$$

## 2.2 Sets

**Definition 4.** Set: An unordered collection of unique elements.

### 2.2.1 Unions, intersections, complements, and set differences

**Definition 5.** Union: the union of two sets  $A, B$  is the set that contain elements that are in  $A$ , or in  $B$ , or both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

**Definition 6.** Intersection: the intersection of two sets  $A, B$  is the set that contains elements that are in both  $A$  and  $B$  at the same time.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

**Definition 7.** Difference: the set difference of two sets  $A, B$  is the set that contains all elements of  $A$  that are not in  $B$ . This operation is not commutative.“

$$A \setminus B = \{x \mid x \in A \wedge x \notin B\}.$$

**Definition 8.** Complement: the complement of a set  $A$  is the set of all elements that are not in  $A$ . For the complements of a set to be defined, it must be a subset of the universal set  $\mathcal{U}$ . In other words, it is the set difference between  $\mathcal{U}$  and  $A$ .

$$A^c = \mathcal{U} \setminus A.$$

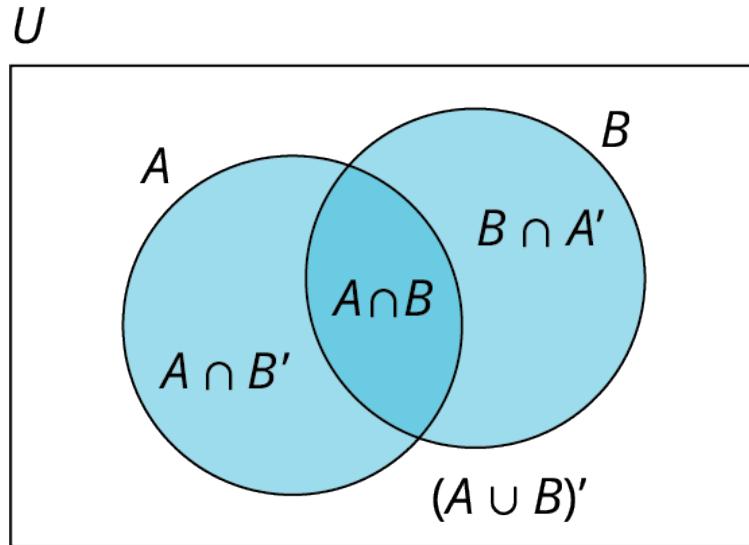
**Theorem 2.** DeMorgan’s Laws for Sets:

$$1. (A \cap B)^c = A^c \cup B^c$$

$$2. (A \cup B)^c = A^c \cap B^c$$

### 2.2.2 Venn diagrams

**Definition 9.** Venn diagrams: a visual aid for understanding sets of objects and their relationships.



## 2.3 Numbers and number systems

**Definition 10.** Number: values that symbolize quantities.

**Definition 11.** Number system: way of representing numbers. Some are more sophisticated than others.

### 2.3.1 Parity, divisibility, and modular arithmetic

**Definition 12.** Divisibility: a number  $n \in \mathbb{Z}$  is divisible by another number  $m$  if and only if  $n = k \times m$  for some integer  $k$ .

**Definition 13.** Parity: the property of a number being even or odd. The number is even if it is divisible by two, and odd otherwise.

**Definition 14.** Modular arithmetic: a number system that groups numbers into equivalence classes based on their remainder when divided by a specific integer.

More formally, for integers  $n$ ,  $r$ , and  $m$ , we say  $n$  is **congruent** to  $r$  modulo  $m$  if  $(n - r)$  is divisible by  $m$ .

$$n \equiv r \pmod{m} \Leftrightarrow m \mid (n - r)$$

For example,  $5 \equiv 11 \pmod{3}$  since they both have a remainder 2 when divided by 3, and because  $11 - 5 = 6$  is divisible by 3.

Standard arithmetic operations  $+$ ,  $-$ , and  $\times$  are well-defined under modular arithmetic. However,  $\div$  is not always well defined. These operations work the same way as they do in standard arithmetic.

Notice that the parity of a number is equivalent to its divisibility by 2, and a number's divisibility by  $m \in \mathbb{N} > 0$  is equivalent to it being congruent to 0 modulo  $m$ .

**Proposition 1.** If  $a \equiv b \pmod{m}$ , then  $b \equiv a \pmod{m}$ .

**Proposition 2.** If  $a \equiv a' \pmod{m}$  and  $b \equiv b' \pmod{m}$ , then:

1.  $a + b \equiv a' + b' \pmod{m}$
2.  $a - b \equiv a' - b' \pmod{m}$
3.  $a \times b \equiv a' \times b' \pmod{m}$

### 2.3.2 Rational and irrational numbers

**Definition 15.** Rational numbers  $\mathbb{Q}$ : the set of numbers that can be expressed as a ratio of two integers.

**Definition 16.** Irrational numbers: the set of numbers that can't be expressed as a ratio of two integers.

### 2.3.3 Real numbers, absolute value, and inequalities

**Definition 17.** Real numbers  $\mathbb{R}$ : the set of all numbers on our number line.

### 2.3.4 Combinatorics: combinations, permutations, and factorials.

**Definition 18.** Combinations  $C(n, r)$ : the cardinality of the set of all subsets of a specific cardinality.

**Definition 19.** Permutations  $P(n, r)$ : the cardinality of the set of all orderings of a specific length.

**Definition 20.** Factorial: the product of natural numbers before it down to zero.

$$5! = 5 \times 4 \times 3 \times 2 \times 1.$$

### 2.3.5 Countable sets

**Definition 21.** Countable set: a set that is either finite, or that has a bijection to the natural numbers. A set is countably infinite if it has a bijection to the natural numbers.

**Proposition 3.**  $\mathbb{Q}$  is countably infinite.

Prove this with existence and uniqueness

### 2.3.6 Uncountable sets

**Definition 22.** Uncountable set: a set that is infinite and there does not exist a bijection from it to the natural numbers.

**Proposition 4.**  $\mathbb{R}$  is not countably infinite.

Proof by contradiction.

## 2.4 Relations and functions

### 2.4.1 Relations and equivalence relations

**Definition 23.** Relation  $R$ : a set of ordered pairs that represents if two elements  $a, b \in S$  are related.  $a$  and  $b$  are related if and only if  $(a, b) \in R$ .

**Definition 24.** Reflexive: a relation  $R$  on set  $S$  is reflexive if and only if for every element  $s \in S$ ,  $sRs$ .

**Definition 25.** Symmetric: a relation  $R$  on set  $S$  is symmetric if and only if for every pair of elements  $s_1, s_2 \in S$ ,  $s_1Rs_2$  implies  $s_2Rs_1$ .

**Definition 26.** Transitive: a relation  $R$  on set  $S$  is transitive if and only if for every trio of elements  $s_1, s_2, s_3 \in S$ ,  $s_1Rs_2$  and  $s_2Rs_3$  implies  $s_1Rs_3$ .

**Definition 27.** Equivalence relations: a special type of relation on a set that satisfies the properties of being symmetric, reflexive, and transitive.

### 2.4.2 Functions

**Definition 28.** Function: a mapping from a set called the domain to elements in a set called the codomain.

### 2.4.3 Injections (one-to-one), surjections (onto), and bijections

**Definition 29.** Injection: a function  $f : A \rightarrow B$  is injective if and only if every distinct element  $a \in A$  maps to a distinct element  $f(a) \in B$ . In other words, there does not exist a pair of elements  $a, a' \in A$  where  $a \neq a'$  such that  $f(a) = f(a')$ .

**Definition 30.** Surjection: a function  $f : A \rightarrow B$  is surjective if and only if for every element  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$ .

**Definition 31.** Bijection: a function  $f : A \rightarrow B$  is a bijection if and only if it is both injective and surjective.

## 3 Proof techniques

### 3.1 Direct Proofs

**Definition 32.** Direct proof: using fundamental rules of logic to prove a statement. The fundamental rules of logic are taken for granted as **axioms**.

Prove proposition 1.

*Proof.* Assume that  $a \equiv b \pmod{m}$ . This means that  $m \mid (a - b)$ . Thus,  $a - b = km$  for some  $k \in \mathbb{Z}$ .

If we multiply both sides by  $-1$ , we get  $b - a = -km$ .

Thus, by definition,  $m \mid (b - a)$  and  $b \equiv a \pmod{m}$ . □

Feedback requested. [How is this formatting? Does this count as 3 proofs?]

Using the properties of modular arithmetic in definition 14, prove proposition 2

Given  $a \equiv a' \pmod{m}$  and  $b \equiv b' \pmod{m}$ :

1.  $a + b \equiv a' + b' \pmod{m}$ .

*Proof.* We begin with by defining  $a \equiv a' \pmod{m}$  as  $m \mid (a - a')$ . Similarly,  $m \mid (b - b')$ .

Following from these definitions, we write:

$$a - a' = m \times k_1 \quad (1)$$

$$b - b' = m \times k_2 \quad (2)$$

We can add equations eq. (1) and eq. (2) together to get  $a + b - a' - b' = m \times k_1 + m \times k_2$ .

With some factoring, we get  $(a + b) - (a' + b') = m(k_1 + k_2)$ .

By definition, we find that  $m \mid (a + b) - (a' + b')$ , and thus  $a + b \equiv a' + b' \pmod{m}$ .  $\square$

2.  $a - b \equiv a' - b' \pmod{m}$ .

*Proof.* Following from Proof 1, we can instead subtract equation eq. (1) and eq. (2) to get  $a - b - a' + b' = m \times k_1 - m \times k_2$ .

With some factoring, we get  $(a - b) - (a' - b') = m(k_1 - k_2)$ .

By definition, we find that  $m \mid (a - b) - (a' - b')$ , and thus  $a - b \equiv a' - b' \pmod{m}$ .  $\square$

3.  $a \times b \equiv a' \times b' \pmod{m}$ .

*Proof.* Following from equation eq. (1), we get

$$a = a' + m \times k_1. \quad (3)$$

Similarly, from equation eq. (2), we get

$$b = b' + m \times k_2. \quad (4)$$

By multiplying equations eq. (3) and eq. (4), we get  $a \times b = (a' + m \times k_1)(b' + m \times k_2)$ .

*From now on, I will omit the  $\times$  symbol.*

By distributing, we get

$$ab = a'b' + a'mk_2 + b'mk_1 + m^2k_1k_2.$$

We can factor out  $m$  to find

$$ab = a'b' + m(a'k_2 + b'k_1 + mk_1k_2).$$

We can subtract  $a'b'$  from both sides to find

$$ab - a'b' = m(a'k_2 + b'k_1 + mk_1k_2).$$

By definition, we see that  $m \mid (ab - a'b')$ , and, by extension,  $ab \equiv a'b' \pmod{m}$ .  $\square$

## 3.2 Transformation of conditionals

**Definition 33.** Transformation of conditionals: using rules of conditional logic to prove conditional statements.

### 3.2.1 Inverse statements

### 3.2.2 Converse statements

### 3.2.3 Contrapositive proofs

### 3.2.4 Bidirectional ("if and only if" proofs)

## 3.3 Quantifiers

**Definition 34.** Quantifier: a logical expression that denotes whether a statement is true for all cases or for specific cases.

### 3.3.1 Universal quantifiers

### 3.3.2 Existential quantifiers

### 3.3.3 Multiply quantified statements

## 3.4 Existence and uniqueness proofs

**Definition 35.** Existence and uniqueness proof: a proof that results in us being sure that an element exists with a given property, and that it is the only element that exhibits such property.

## 3.5 Proof by Induction

**Definition 36.** Proof by Induction: proof technique used to prove a statement is true for a countably infinite set of discrete elements.

**Proposition 5.** Let  $F_n$  be the  $n$ -th Fibonacci number, where  $F_0 = F_1 = 1$  and  $F_n = F_{n-1} + F_{n-2}$ . Prove that  $F_n \leq 1.9^n$  for all  $n \geq 1$ .

Prove generalized De-Morgan's law.

*Proof.* We proceed by induction, starting with the base cases, where  $n = 1, 2$ :

$$n = 1, F_1 = 1 \leq 1.9^1 = 1.9$$

$$n = 2, F_2 = 2 \leq 1.9^2 = 3.61.$$

We assume as an inductive hypothesis that  $F_n \leq 1.9^n$  for  $1 \leq n \leq k$ .

Using our inductive hypothesis, we see that  $F_k \leq 1.9^k$  and  $F_{k-1} \leq 1.9^{k-1}$ .

So,  $F_k + F_{k-1} \leq 1.9^k + 1.9^{k-1}$ .

By refactoring  $1.9^k + 1.9^{k-1}$ , we get:

$$1.9^k + 1.9^{k-1} = 1.9(1.9^{k-1}) + 1.9^{k-1} = 2.9(1.9^{k-1}).$$

Also,  $1.9^{k+1}$  can be rewritten as  $1.9^2(1.9^{k-1}) = 3.61(1.9^{k-1})$ .

Finally, we see

$$F_{k+1} = F_k + F_{k-1} \leq 2.9(1.9^{k-1}) \leq 3.61(1.9^{k-1}) = 1.9^{k+1}$$

$$F_{k+1} \leq 1.9^{k+1}.$$

Thus, we conclude that by the principle of mathematical induction,  $F_n \leq 1.9^n$  for all  $n \geq 1$ .  $\square$

**Proposition 6.** Look up the Tower of Hanoi puzzle. Prove that given a stack of disks, you can solve the puzzle in moves.

*Proof.* We begin by defining the Tower of Hanoi problem.

In this problem, we begin with a stack of  $n$  disks. The disks are ordered from largest at the bottom to smallest at the top. We are also given 3 'spots' to place our disks under one condition: that we never place a larger disk on top of a smaller disk.

Following these rules, what is the minimum number of moves required to move the entire pile to a new ‘spot’?

We define the function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that it maps the starting stack height  $n$  to the minimum number of moves required to move the entire pile  $f(n)$ .

Before immediately proving that  $f(n) = 2^n - 1$ , it is more intuitive to first define  $f$  as a recurrence relation, then prove that the recurrence relation is equal to  $2^n - 1$ .

We notice that moving the entire pile of  $n$  disks essentially requires 3 ‘phases’:

1. Moving the top  $n - 1$  disks onto a single pile.
2. Moving the  $n$ th disk to another vacant spot.
3. Moving the top  $n - 1$  disks onto the new spot.

Thus, we know that  $f(n) = f(n - 1) + 1 + f(n - 1) = 1 + 2f(n - 1)$ , where  $f(1) = 1$ . We can then prove  $f(n) = 2^n - 1$  using induction.

We begin with our base cases:

$n$	$f(n)$
1	$1 = 2^1 - 1$
2	$3 = 2^2 - 1$
3	$7 = 2^3 - 1$

Now, we assume that  $f(k) = 2^k - 1$  for all  $1 \leq k \leq n$ .

We see that

$$\begin{aligned} f(k+1) &= 1 + 2f(k) \\ f(k+1) &= 1 + 2(2^k - 1) \\ f(k+1) &= 2^{k+1} - 1. \end{aligned}$$

Thus,  $f(n) = 2^n - 1$ . □

### 3.6 Proof by Contradiction

**Definition 37.** Proof by contradiction: proof technique that assumes the opposite of our proposition, then showing that this leads to an absurd conclusion, ie. a contradiction. Used as a “last resort” proof technique.

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## 4 Final project

## 5 Conclusion and reflection

Prove the uncountability of the reals.

# Appendix

(The first section, “Course objectives and student learning outcomes” is just here for your reference.)

## A Course objectives and student learning outcomes

1. Students will learn to identify the logical structure of mathematical statements and apply appropriate strategies to prove those statements.
2. Students learn methods of proof including direct and indirect proofs (contrapositive, contradiction) and induction.
3. Students learn the basic structures of mathematics, including sets, functions, equivalence relations, and the basics of counting formulas.
4. Students will be able to prove multiply quantified statements.
5. Students will be exposed to well-known proofs, like the irrationality of  $\sqrt{2}$  and the uncountability of the reals.

### A.1 Expanded course description

- Propositional logic, truth tables, DeMorgan’s Laws
- Sets, set operations, Venn diagrams, indexed collections of sets
- Conventions of writing proofs
  - Direct proofs
  - Contrapositive proofs
  - Proof by cases
  - Proof by contradiction
  - Existence and Uniqueness proofs
  - Proof by Induction
- Quantifiers
  - Proving universally and existentially quantified statements
  - Disproving universally and existentially quantified statements
  - Proving and disproving multiply quantified statements
- Number systems and basic mathematical concepts
  - The natural numbers and the integers, divisibility, and modular arithmetic
  - Counting: combinations and permutations, factorials
  - Rational numbers, the irrationality of  $\sqrt{2}$
  - Real numbers, absolute value, and inequalities
- Relations and functions
  - Relations, equivalence relations
  - Functions
  - Injections, surjections, bijections

- Cardinality
  - Countable and uncountable sets
  - Countability of the rational numbers,  $\mathbb{Q}$
  - Uncountability of the real numbers,  $\mathbb{R}$