HW04

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October 3, 2025

Total Points: 21

Required Exercise 1 [2]

1. Did it.
2. Did it.

Required Exercise 2 [1]

- In class during week 4, we proved that since my birthday was on a Saturday this year (2025), it will be on a Sunday next year (2026). We did this by showing that

  $$x + 365 \equiv x + 1 \pmod 7$$

  Why does this prove the claim?

  This proves the claim because all "days of the week" are actually individual congruency classes (mod 7). Thus, when we add the no. of days equal to a year (365), and find that it is congruent to 1 (mod 7), we are saying that moving a year forward puts use in the same congruency class of days of the week as moving one day forward. Thus, if your birthday was on a Saturday this year, it will be on a Sunday next year.

- We also showed that if the minute hand of the clock says now, then in minutes, it will say by showing that

  $$19 + 52 \equiv 11 \pmod{60}$$

  This proves the claim because of similar reasoning as the prior exercise. Each second of the minute can be seen as a congruency class, and by seeing that $19 + 52 \equiv 11 \pmod{60}$, we see that the second hand will be on the 11 seconds mark.

Required Exercise 3 [3]

1. Prove that if all the coefficients of the quadratic equation

   $$ax^2 + bx + c = 0$$

   are odd integers, then the roots of the equation cannot be rational.

   Proof. We begin by seeing that the roots of a quadratic equation are

   $$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   This expression is not rational if $b^2 - 4ac$ is not square. We also see that $b^2 - 4ac$ must be odd because b^2 is odd, and $4ac$ is even, and an odd number minus an even number is always odd.

Lemma 1 Let m be an odd integer. $m^2 \equiv 1 \pmod 8$.

Since m is odd, it must have the form $2k+1$ for some integer k. Thus, all odd squares can b written in the as

$$(2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2+k) + 1.$$

Now, we see that $k^2 + k$ must always be even, because if k was odd, k^2 would also be odd, and an odd plus odd is always even, and if k was even, then k^2 is even, and an even plus even is always even. Thus, $k^2 + k$ can be written in the form $2j$ for some integer j.

By substituting, we get

$$4(k^2+k) + 1 = 4(2j) + 1 = 8j + 1.$$

And $8j + 1 \equiv 1 \pmod 8$ because $8 \mid (8j + 1 - 1)$.

Thus, for any odd square n, $n \equiv 1 \pmod 8$. □

Now, using the contrapositive of **Lemma 1**, we see that for any integer $k \not\equiv 1 \pmod 8$, k must not be an odd square.

Let $a = 2k_1+1$, $b = 2k_2+1$, and $c = 2k_3+1$. By substituting into $b^2 - 4ac$ and algebraic manipulation, we get

$$4{k_2}^2 + 4k_2 + 1 - 4(4k_1k_3 + 2k_1 + 2k_3 + 1)$$
$$4{k_2}^2 + 4k_2 + 1 - 16k_1k_3 - 8k_1 - 8k_3 - 4$$
$$4{k_2}^2 + 4k_2 - 16k_1k_3 - 8k_1 - 8k_3 - 3$$
$$4({k_2}^2 + k_2 - 4k_1k_3 - 2k_1 - 2k_3) - 3.$$

Then, notice that ${k_2}^2 + k_2$ is always even because if k_2 was odd, ${k_2}^2$ would also be odd, and an odd plus odd is always even, and if k_2 was even, then ${k_2}^2$ is even, and an even plus even is always even.

Also, notice that $4k_1k_3 - 2k_1 - 2k_3$ is always even because a it can be written in the form

$$2(2k_1k_3 - k_1 - k_3).$$

Now, we can see that the entire expression ${k_2}^2 + k_2 - 4k_1k_3 - 2k_1 - 2k_3$ must be even.

Thus, ${k_2}^2 + k_2 - 4k_1k_3 - 2k_1 - 2k_3 = 2i$ for some integer i.

Finally, can substitute our new expression to see

$$4({k_2}^2 + k_2 - 4k_1k_3 - 2k_1 - 2k_3) - 3 = 4(2i) - 3 = 8i - 3.$$

For any integer i, $8i - 3 \not\equiv 1 \pmod 8$. Since $b^2 - 4ac \not\equiv 1 \pmod 8$ and $b^2 - 4ac$ is odd, $b^2 - 4ac$ can not be square. Thus, the roots of the quadratic equation $ax^2 + bx + c$ cannot be rational.

□

Required Exercise 4 [4]

1. Solve the following three questions about set complements.

   - Suppose that $\mathcal{O}$ is the set of odd integers. If $\mathcal{U} = \mathbb{Z}$ (the set of integers), what is $\mathcal{O}^c$?

     All integers can be classified as either even or odd. All integers that are even are not odd, and all the integers that are odd are not even. Thus, $\mathcal{O}^c$ is the set of all integers that are not odd, which is just the set of all even integers $\mathcal{E} = \{2n : n \in \mathbb{Z}\}$.

   - Suppose that $A = \{1, 2, 3, 5, 8\}$ and the universal set is $\mathcal{U} = 1, 2, ..., 10$. What is A^c?

     A^c is the set of numbers in $\mathcal{U}$ that are not in A. Thus, $A^c = 4, 6, 7, 9$.

- Suppose that $B = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 > 1\}$ and $\mathcal{U} = \mathbb{R} \times \mathbb{R}$. What is B^c?

  The set B is the set of all points that lie strictly outside the unit circle on the Cartesian plane, thus the complement of this set B^c is the set of all points inside or on the unit circle. In other words, $B^c = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

2. Now prove De Morgan's laws. In both cases, assume that A and B are both subsets of the universal set $\mathcal{U}$.

   I will be using $\subset$ to be "proper subset".

   - $(A \cup B)^c = A^c \cap B^c$

     Proof. We begin by seeing that for any two sets X an Y, if $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.
     Suppose $x \in (A \cup B)^c$, so $x \notin A$, and thus $x \in A^c$. Similarly, $x \notin B$, and thus $x \in B^c$.
     Therefore, for every $x \in (A \cup B)^c$, $x \in A^c \cap B^c$. So, $(A \cup B)^c \subseteq A^c \cap B^c$.
     Now, suppose $x \in A^c \cap B^c$, so $x \notin A$, and thus $x \in A^c$. Similarly, $x \notin B$, and thus $x \in B^c$.
     Therefore, for every $x \in A^c \cap B^c$, $x \in (A \cup B)^c$. So, $A^c \cap B^c \subseteq (A \cup B)^c$.
     Because both $A^c \cap B^c \subseteq (A \cup B)^c$ and $(A \cup B)^c \subseteq A^c \cap B^c$, $(A \cup B)^c = A^c \cap B^c$. □

   - $(A \cap B)^c = A^c \cup B^c$

     Proof. We begin by seeing that for any two sets X an Y, if $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.
     Suppose $x \in (A \cap B)^c$, so x can't be in both A and B at the same time. Thus, $x \in A^c \cup B^c$.
     Therefore, for every $x \in (A \cap B)^c$, $x \in A^c \cup B^c$. So, $(A \cap B)^c \subseteq A^c \cup B^c$.
     Now, suppose $x \in A^c \cup B^c$, so $x \in (\mathcal{U} \setminus A) \cup (\mathcal{U} \setminus B)$. If we let x be an element of B, we see that this must come from the complement of A. Similarly, if we let $x \in A$, we see that $x \in B^c$. In other words, x can't be in A and B at the same time.
     Therefore, for every $x \in A^c \cup B^c$, $x \in (A \cap B)^c$. So, $A^c \cup B^c \subseteq (A \cap B)^c$.
     Because both $A^c \cup B^c \subseteq (A \cap B)^c$ and $(A \cap B)^c \subseteq A^c \cup B^c$, $(A \cap B)^c = A^c \cup B^c$. □

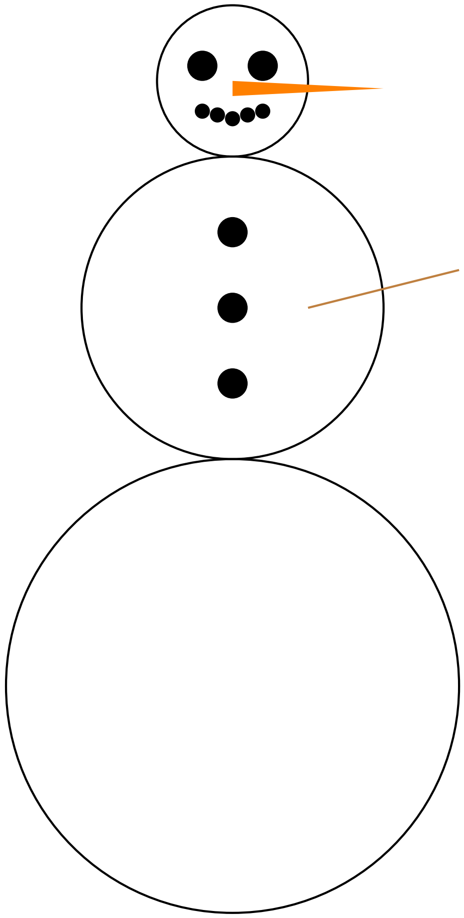
Choice Exercise 9 [4]

I really enjoyed the premise of this video and I feel like I've got a decent intuition for the problem. I understood the "mapping" of the original problem to the problem of coloring some n-dimensional cube. Because I have a CS background, graph theory problems are somewhat familiar to me, so reasoning about the coloring point of view wasn't too foreign. However, the combinitorical proof for showing only cases where the no. of squares is a power of two are even possible was a little bit confusing. I get the concrete example with a 3D cube where we literally can't evenly partition the colors, but I struggle to comprehend the generallization to higher dimensions. After watching, I wonder what if instead of having coins on the chessboard, what if we had n-sided "dice" where the position can be any value from one to n. Now, with this variant, we can either limit the ways we can manipulate the dice (only increase or decrease value by 1), or let us arbitrarily set the value.

Just writing this description, I can already sort of see that for n-sided dice where n is a power of 2 is essentially the same as our original problem in special cases, because the n-sided dice is equal to $log(n)$ squares worth of information. There seems to be more to this problem, but I'm struggling to articulate everything because I feel as though I don't have the language to convey the ideas.

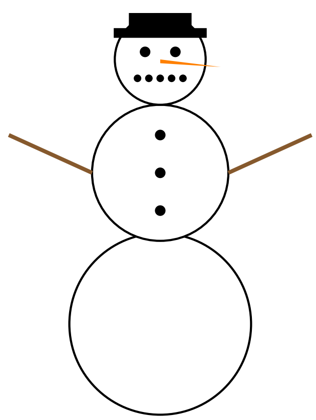
Even stranger still if we can extend the values to the Reals. I don't know if this problem even makes sense to ask.

Choice Exercise 10 [3]



My drawing

Not too shabby for an 8 minute drawing if I do say so myself.



ChatGPT's drawing

Analysis Damn I got cooked.

Choice Exercise 11 [4]

1. How do the roots that I got compare with the roots you get when you use the quadratic formula on the polynomial $2x^2 - 5x + 6$?

Using the quadratic equation, we get

$$x = \frac{5 \pm \sqrt{25 - 4(2(6))}}{2(2)} = \frac{5 \pm \sqrt{-23}}{4}.$$

This is the same as our result from completing the square.

2. Do the same process again, but where the quadratic polynomial has coefficients a, b, and c.

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\
\left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}$$

We have derived the quadratic formula.