HW06

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October 19, 2025

Required Exercise 1 [4]

Done.

Required Exercise 2 [3]

Let $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-1}$ for $n \geq 3$. Use induction to prove that F_n is even if and only if $3 \mid n$.

Proof. First, we must see that the sum of two integers a and b is even if and only if a and b have the same parity. In other words, if a and b are both even or odd, then $a + b$ is even. But if a is even and b is odd or vice versa, then $a + b$ is odd.

We can procede with induction. We use the following as our base cases:

1. $F_1 = 1$
2. $F_2 = 1$
3. $F_3 = 1$

Then, we assume that F_n is even if and only if $3 \mid n$ for all $n \leq k$, where $3 \mid k$.
Thus, F_k is even, and F_{k-1} and F_{k-2} are both odd.
Then, we see that F_{k+1} must be odd, since $F_{k+1} = F_k + F_{k-1}$ and F_k is even while F_{k-1} is odd.
Similarly, we see that F_{k+2} must also be odd, since F_{k+1} is odd and F_k is even.
We also notice that $k + 1$ and $k + 2$ must not be divisible by 3 because

$$\begin{aligned} k &\equiv 0 \pmod 3 \\ k + 1 &\equiv 1 \pmod 3 \\ k + 2 &\equiv 2 \pmod 3. \end{aligned}$$

Then, we see that F_{k+3} must be even because F_{k+2} and F_{k+1} are both odd. Also, $k + 3 \equiv 0 \pmod 3$.
Therefore, F_n is even if and only if $3 \mid n$. □

Required Exercise 3 [3]

1. Go to the proofs portfolio instructions, set a timer for 5 minutes, and read the document until you finish or the timer goes off, and write any questions you have here.
   - How far from the topics covered in class can we stray for our discussion? Specfically, I would like to potentially explore Turing Completeness or P vs. NP for my discussions.
   - How many revisions should we do?
2. Done.
3. Done.
4. Done.

Choice Exercise 8 [8]

1. [1] Write the negation of the statement "for all real numbers $\epsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $|x - a| < \delta$, $|f(x) - f(a)| < \epsilon$."

   There exists $\epsilon > 0$, where for all $\delta > 0$, there exists x such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$.

2. [3] Show that "for all real numbers $\epsilon > 0$ there exists a positive number $\delta \in \mathbb{R}$ such that for all x satisfying $|x - a| < \delta$, $|f(x) - f(a)| \geq \epsilon$" is *not* the negation of this statement. (The way to do this is to write the correct negation, and give an example showing that it is not logically equivalent to the above statement.)

   First, we formalize the statements as open statements that take in a function $f : \mathbb{R} \to \mathbb{R}$, and $a \in \mathbb{R}$.

   So, our original statement becomes $P(f, a) = \{$for all real numbers $\epsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $|x - a| < \delta$, $|f(x) - f(a)| < \epsilon\}$. If we want to be fancy, we say

   $$P(f, a) = \forall \epsilon > 0, \exists \delta > 0, \forall x, (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon).$$

   Thus, $\sim P(f, a) = \{$ there exists $\epsilon > 0$, where for all $\delta > 0$, there exists x such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$ $\}$.

   $$\begin{aligned}
   \sim P(f, a) &= \sim (\forall \epsilon > 0, \exists \delta > 0, \forall x, (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)) \\
   \sim P(f, a) &= \exists \epsilon > 0, \sim (\exists \delta > 0, \forall x, (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)) \\
   \sim P(f, a) &= \exists \epsilon > 0, \forall \delta > 0, \sim (\forall x, (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)) \\
   \sim P(f, a) &= \exists \epsilon > 0, \forall \delta > 0, \exists x, \sim (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon) \\
   \sim P(f, a) &= \exists \epsilon > 0, \forall \delta > 0, \exists x, (|x - a| < \delta \wedge |f(x) - f(a)| \geq \epsilon)
   \end{aligned}$$

   Similarly, our "wrongly negated" statement becomes $Q(f, a) = \{$ for all real numbers $\epsilon > 0$ there exists a positive number $\delta \in \mathbb{R}$ such that for all x satisfying $|x - a| < \delta$, $|f(x) - f(a)| \geq \epsilon$ $\}$, or more formally:

   $$Q(f, a) = \forall \epsilon > 0, \exists \delta > 0, \forall x, (|x - a| < \delta \Rightarrow |f(x) - f(a)| \geq \epsilon)$$

   An example of when $P \neq Q$ would be when $f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$, and $a = 0$.

   Under these conditions, $\sim P(f, a)$ is a true by because we can pick $\epsilon = 0.5$. Then, we see that for all δ, there will always exist a value $0 < x < \delta$. Namely, $x = \frac{\delta}{2}$. Thus, $|x - a| < \delta$ is true, since $a = 0$. We also see that $f(x) = 1$, since $x > 0$, and $f(a) = 0$ since $a \leq 0$. Thus, $|f(x) - f(a)| = 1 \geq \epsilon$.

   However, $Q(f, a)$ is false because we can pick $\epsilon = 5$. This makes $|f(x) - f(a) \geq \epsilon$ false. However, there can not exist any δ such that $|x - a| < \delta$ for any x. In other words, for any δ, there will always be a value $0 < x < \delta$. This makes our final implication false.

   Because these two statements are not logically equivalent, "for all real numbers $\epsilon > 0$ there exists a positive number $\delta \in \mathbb{R}$ such that for all x satisfying $|x - a| < \delta$, $|f(x) - f(a)| \geq \epsilon$" is not the correct negation.

3. [4] We want to prove something that we know: the function $f(x) = 2x + 1$ is continuous. Let $f(x) = 2x + 1$ and prove that $\lim_{x \to a} f(x) = f(a)$ for all $a \in \mathbb{R}$.

   We begin by substituting in f into $|f(x) - f(a) \geq \epsilon$ to get

   $$\begin{aligned}
   |2x + 1 - (2a + 1)| = |2x - 2a| &< \epsilon \\
   2|x - a| &< \epsilon \\
   |x - a| &< \epsilon/2.
   \end{aligned}$$

We no notice that for any value ϵ, we can pick $\delta = \epsilon/2$, and get $|x - a| \Rightarrow |x - a|$. This value is true for all x, and for all a.

Therefore, $f(x) = 2x + 1$ is continuous.

Proofs Portfolio

MAT 3100W: Intro to Proofs

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October 19, 2025

1 Introduction

(Leave this blank for now. Here's an outline of course topics for your reference.)

2 Mathematical concepts

3 Proof techniques

3.1 Direct Proofs

Suppose $a \equiv a' \pmod{m}$ and $b \equiv b' \pmod{m}$. Prove the following:

1. $a + b \equiv a' + b' \pmod{m}$.

   Proof. We begin with by defining $a \equiv a' \pmod{m}$ as $m \mid (a - a')$. Similarly, $m \mid (b - b')$.
   Following from these definitions, we write:

   $$a - a' = m \times k_1 \tag{1}$$

   $$b - b' = m \times k_2 \tag{2}$$

   We can add equations 1 and 2 together to get $a + b - a' - b' = m \times k_1 + m \times k_2$.
   With some factoring, we get $(a + b) - (a' + b') = m(k_1 + k_2)$.
   By definition, we find that $m \mid (a + b) - (a' + b')$, and thus $a + b \equiv a' + b' \pmod{m}$. □

2. $a - b \equiv a' - b' \pmod{m}$.

   Proof. Following from Proof 1, we can instead subtract equation 1 and 2 to get $a - b - a' + b' = m \times k_1 - m \times k_2$.
   With some factoring, we get $(a - b) - (a' - b') = m(k_1 - k_2)$.
   By definition, we find that $m \mid (a - b) - (a' - b')$, and thus $a - b \equiv a' - b' \pmod{m}$. □

3. $a \times b \equiv a' \times b' \pmod{m}$.

   Proof. Following from equation 1, we get

   $$a = a' + m \times k_1. \tag{3}$$

   Similarly, from equation 2, we get

   $$b = b' + m \times k_2. \tag{4}$$

By multiplying equations 3 and 4, we get $a \times b = (a' + m \times k_1)(b' + m \times k_2)$.

From now on, I will omit the $\times$ symbol.

By distributing, we get

$$ab = a'b' + a'mk_2 + b'mk_1 + m^2k_1k_2.$$

We can factor out m to find

$$ab = a'b' + m(a'k_2 + b'k_1 + mk_1k_2).$$

We can subtract $a'b'$ from both sides to find

$$ab - a'b' = m(a'k_2 + b'k_1 + mk_1k_2).$$

By definition, we see that $m \mid (ab - a'b')$, and, by extension, $ab \equiv a'b' \pmod m$.

□

3.2 Proof by Induction

As an example of Proof by Induction, we will prove the following.

Proposition 1. *Let F_n be the n-th Fibonacci number, where $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Prove that $F_n \leq 1.9^n$ for all $n \geq 1$.*

Proof. We begin by verifying the relation for small n to create our base cases:

$$n = 1, F_1 = 1 \leq 1.9^1 = 1.9$$

$$n = 2, F_2 = 2 \leq 1.9^2 = 3.61.$$

We then form our inductive hypothesis by assumimg $F_n \leq 1.9^n$ for $1 \leq n \leq k$.

$$F_{k+1} \leq 1.9^{k+1}$$

$$F_k + F_{k-1} \leq 1.9^{k+1}.$$

Using our inductive hypothesis, we see that $F_k \leq 1.9^k$ and $F_{k-1} \leq 1.9^{k-1}$.
So, $F_k + F_{k-1} \leq 1.9^k + 1.9^{k-1}$.
By refactoring $1.9^k + 1.9^{k-1}$, we get:

$$1.9^k + 1.9^{k-1} = 1.9(1.9^{k-1}) + 1.9^{k-1} = 2.9(1.9^{k-1}).$$

Also, 1.9^{k+1} can be rewritten as $1.9^2(1.9^{k-1}) = 3.61(1.9^{k-1})$.
Finally, we see

$$F_{k+1} = F_k + F_{k-1} \leq 2.9(1.9^{k-1}) \leq 3.61(1.9^{k-1}) = 1.9^{k+1}$$

$$F_{k+1} \leq 1.9^{k+1}.$$

Therefore, $F_n \leq 1.9^n$ for all $n \geq 1$. □

Proposition 2. *Look up the Tower of Hanoi puzzle. Prove that given a stack of disks, you can solve the puzzle in moves.*

Proof. We begin by defining the Tower of Hanoi problem.

In this problem, we begin with a stack of n disks. The disks are ordered from largest at the bottom to smallest at the top. We are also given 3 'spots' to place our disks under one condition: that we never place a larger disk on top of a smaller disk.

Following these rules, what is the minimum number of moves required to move the entire pile to a new 'spot'?

We define the function $f : \mathbb{N} \to \mathbb{N}$ such that it maps the starting stack height n to the minimum number of moves required to move the entire pile $f(n)$.

Before immediately proving that $f(n) = 2^n - 1$, it is more intuitive to first define f as a recurrence relation, then prove that the recurrence relation is equal to $2^n - 1$.

We notice that moving the entire pile of n disks essentially requires 3 'phases':

1. Moving the top $n-1$ disks onto a single pile.

2. Moving the nth disk to another vacant spot.

3. Moving the top $n-1$ disks onto the new spot.

Thus, we know that $f(n) = f(n-1) + 1 + f(n-1) = 1 + 2f(n-1)$, where $f(1) = 1$. We can then prove $f(n) = 2^n - 1$ using induction.

We begin with our base cases:

n	$f(n)$
1	$1 = 2^1 - 1$
2	$3 = 2^2 - 1$
3	$7 = 2^3 - 1$

Now, we assume that $f(k) = 2^k - 1$ for all $1 \leq k \leq n$.

We see that

$$f(k+1) = 1 + 2f(k)$$

$$f(k+1) = 1 + 2(2^k - 1)$$

$$f(k+1) = 2^{k+1} - 1.$$

Thus, $f(n) = 2^n - 1$. □

4 Final project

5 Conclusion and reflection

Appendix

(The first section, "Course objectives and student learning outcomes" is just here for your reference.)

A Course objectives and student learning outcomes

1. Students will learn to identify the logical structure of mathematical statements and apply appropriate strategies to prove those statements.
2. Students learn methods of proof including direct and indirect proofs (contrapositive, contradiction) and induction.
3. Students learn the basic structures of mathematics, including sets, functions, equivalence relations, and the basics of counting formulas.
4. Students will be able to prove multiply quantified statements.
5. Students will be exposed to well-known proofs, like the irrationality of $\sqrt{2}$ and the uncountability of the reals.

A.1 Expanded course description

- Propositional logic, truth tables, DeMorgan's Laws
- Sets, set operations, Venn diagrams, indexed collections of sets
- Conventions of writing proofs
- Proofs
  - Direct proofs
  - Contrapositive proofs
  - Proof by cases
  - Proof by contradiction
  - Existence and Uniqueness proofs
  - Proof by Induction
- Quantifiers
  - Proving universally and existentially quantified statements
  - Disproving universally and existentially quantified statements
  - Proving and disproving multiply quantified statements
- Number systems and basic mathematical concepts
  - The natural numbers and the integers, divisibility, and modular arithmetic
  - Counting: combinations and permutations, factorials
  - Rational numbers, the irrationality of $\sqrt{2}$
  - Real numbers, absolute value, and inequalities
- Relations and functions
  - Relations, equivalence relations
  - Functions
  - Injections, surjections, bijections

- Cardinality
  - Countable and uncountable sets
  - Countability of the rational numbers, $\mathbb{Q}$
  - Uncountability of the real numbers, $\mathbb{R}$