HW09

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Required Exercise 1 [4]

Included below.

Required Exercise 2 [3]

Proposition 1. *A function $f : A \to B$ is a bijection if and only if there exists a function $G : B \to A$ such that $g(f(a)) = a$ for all $a \in A$ and $f(g(b)) = b$ for all $b \in B$.*

1. Prove the "forward direction" of the proposition by assuming that $f : A \to B$ is surjective and injective, and concluding that there exists an inverse function.

   Proof. Suppose that $f : A \to B$ is a bijection. Thus, for every unique $b \in B$ there exists a unique $a \in A$ such that $f(a) = b$.

   Therefore, there must exist the inverse function $f^{-1} : B \to A$ such that for every value $b \in B$, $f^{-1}(b)$ is a unique value $a \in A$. $\square$

2. Prove the "backward direction" of the proposition by assuming that $f^{-1} : B \to A$ is an inverse function of f, and concluding that f is surjective and injective.

   Proof. First, we see that because $f-1$ exists, f itself must be a valid fumction. f^{-1} maps every value $b \in B$ to a unique value $a \in A$. So, f must be injective because the domain of f^{-1} is B. Also, because f^{-1} is a valid function, no two values $a \in A$ map to the same value $b \in B$. Thus, f is injective. $\square$

3. (a) Give an example of sets A and B together with a pair of functions $f : A \to B$ and $g : B \to A$ where

      i. $g(f(a)) = a$ for all $a \in A$,
      ii. there exists $b \in B$ such that $f(g(b) \neq b$, and
      iii. f is not a bijection.

      Let sets $A = \mathbb{N}$ and $B = \mathbb{R}$. Let $f : \mathbb{N} \to \mathbb{R}$, where $f(n) = n$. Let $g : \mathbb{R} \to \mathbb{N}$, where $g(r) = \lfloor r \rfloor$.

      We see that for all $a \in A$, $g(f(a)) = a$. However, for any non-integer value $b \in B$, $f(g(b)) \neq b$. Thus, f is not a bijection.

   (b) Let sets $A = \mathbb{R}$ and $B = \{r \in \mathbb{R} : 0 \leq r \leq 1\}$. For $f : A \to B$, $f(x) = \sin(x)$. and $g : B \to A$, $g(x) = \sin^{-1}(x)$.

      We see that for $a = 4\pi \in A$, $g(f(a)) = 0 \neq 4\pi$. However, $f(g(b)) = b$ for all $b \in B$.

Required Exercise 3 [3]

1. Give an example of a bijection $h : \mathbb{N}_{>0} \to \mathbb{N}_{\geq 0}$ Note that the domain and the codomain are different, and explain why the map $h(x) = x$ is not a bijection.

   $h(x) = x - 1$ is a valid bijection. $h(x) = x$ is not a bijection because it is not surjective. The value 0 is not reached.

2. Consider the function $f : \mathbb{N}_{>0} \to \mathbb{N}_{\geq 0}$ where

$$f(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even} \\ -\dfrac{(n+1)}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

   (a) Prove that f is a bijection by proving that it is both injective and surjective.

   Proof. To see that f is both injective and surjective, we must first see that the individual "peices" of f are each injective and surjective, and that these "pieces" have mutually exclusive codomains.
   First, we see that for even n, $f(n) = n/2$. This is itself a valid bijection between the set A of even natural numbers including 0 and the set B of all natural numbers including 0.
   Then, we see that for odd n, $f(n) = -\frac{n+1}{2}$. This is a valid bijection between the set C of all odd natural numbers and the set D of all negative integers.
   Notice that the range of both "pieces" is equal to their codomains.
   Now, we see that $B \cap D = \emptyset$, $A \cap C = \emptyset$, and $A \cup B = \mathbb{N}_{\geq 0}$. So, f must be injective.
   Also, because $B \cup D = \mathbb{Z}$, f must be surjective.
   Therefore, f is a bijection. □

   (b) Prove that f is a bijection by describing a (piecewise-defined) function for the inverse map $g : \mathbb{Z} \to \mathbb{N}_{\geq 0}$, and checking that $g \circ f : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ and $f \circ g : \mathbb{Z} \to \mathbb{Z}$ are both the identity function on their respective domains.

   Proof. Let $g : \mathbb{Z} \to \mathbb{N}_{\geq 0}$, where

$$g(n) = \begin{cases} 2n & n \geq 0 \\ -2n - 1 & n < 0 \end{cases}$$

   Now, we observe that $g \circ f$ is the identity function because for all even $n \in \mathbb{N}$, $f(n) = n/2$. Because $n \geq 0$, $f(n) \geq 0$. Thus, $g(n) = 2n$. Finally, $g(f(n)) = n$.
   Then, for all odd $n \in \mathbb{N}$, $f(n) = -\frac{n+1}{2}$. Now, $f(n) < 0$, so $g(n) = -2n - 1$. Thus,

$$\begin{aligned} g(f(n)) &= -2(-\frac{n+1}{2}) - 1 \\ &= (n+1) - 1 = n. \end{aligned}$$

   Thus, $g \circ f$ is the identity on $\mathbb{N}_{\geq 0}$.
   Also, $f \circ g$ is the identity on $\mathbb{Z}$ because for $n \geq 0$, $g(n) = 2n$ is even, and $f(n) = n/2$. Thus, $f(g(n)) = n$. Then for $n < 0$, $g(n) = -2n - 1$ and $f(n) = -\frac{n+1}{2}$. Thus,

$$\begin{aligned} f(g(n)) &= -\frac{(-2n-1)+1}{2} \\ &= -\frac{-2n}{2} = n. \end{aligned}$$

   Therefore, f is a bijection. □

3. Describe a bijection $s : \mathbb{N}_{>0} \to \mathbb{Z}$ in ters of h and f.

   Since $h : \mathbb{N}_{>0} \to \mathbb{N}_{\geq 0}$ and $f : \mathbb{N}_{\geq 0} \to \mathbb{Z}$, $s = f \circ h$.

Choice Exercise 6 [5]

In this exercise, you will prove that the relation "there exists a bijection between" is an equivalence relation on sets.

1. Prove that the relation is reflexive: for all sets A, there exists a bijection $f : A \to A$.

   Proof. For any set A, there must exist the identity function f. This function must be a bijection since it relates all elements of set A, to a unique elemeent of A. $\square$

2. Prove that the relation is symmetric: for all sets A and B such that there exists a bijection $f : A \to B$, there also exists a bijection $g : B \to A$.

   Proof. For a function f to be well defined on a domain A, it must be defined for every value $a \in A$. This means that, if f is injective, there bust exist a surjection from B to A. Because f is a bijection, f must also be injective, thus there exists some surjection from B to A.

   Also, since f is a surjection and well-defined, there must exist an injection from B to A. Therefore, there must exist a bijection from $g : B \to A$. $\square$

3. Prove that the relation is transitive: for all sets A, B, and C such that there exist bijections $f_1 : A \to B$ and $f_2 B \to C$, there also exists a bijection $f_3 : A \to C$.

   Proof. Assume that there exists bijections $f_1 : A \to B$ and $f_2 B \to C$.

   This means that every value $a \in A$ can be mapped to a unique value $b \in B$ by a the function f_1. Also, every value $b \in B$ is mapped to by a value $a \in A$. Similarly, every value $b \in B$ is mapped to a unique value $c \in C$ by the function f_2, and all values of $c \in C$ are mapped to by a value $b \in B$.

   So, this means that the composition $f_2 \circ f_1$ first maps all unique values of $a \in A$ to a unique value $b \in B$, then maps all values $b \in B$ to a unique value $c \in C$. This means that all elements $a \in A$ can be mapped to a unique element $c \in C$. Also, this mapping is surjective because all values of C are reached by f_2, and all values of B are reached by f_1.

   Therefore, this relation is transitive. $\square$