Algorithmic Game Theory BIT - 2023 Spring

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§ Mechanism Design §

Problem 1: Favourite Result in Class

Nash's Theorem holds a special place in my heart. It is a fundamental result in the field of game theory. It states that every finite game has at least one Nash equilibrium, in which no player can unilaterally improve their outcome by changing their strategy while other players keep their strategies unchanged. The theorem applies to all finite games, which makes it incredibly broad and universally applicable. It has had a tremendous impact on economics and other social sciences by providing a mathematical basis for predicting the behavior of strategic interactions. Above all, it ignites my curiosity and motivates me to develop deeper into the study of mechanism design.

Problem 2: Set Function

1. First, we note that $S \subseteq T$ implies $S + j \subseteq T + j$. Then, by the submodularity property, we have

$$v(S) + v(T + j) \geq v(S \cap T + j) + v(S \cup T + j) = v(S) + v(T + j) \tag{2.1}$$

and

$$v(T) + v(S + j) \geq v(T \cap S + j) + v(T \cup S + j) = v(T) + v(S + j) \tag{2.2}$$

Subtracting the two inequalities, we get

$$v(T + j) - v(T) \leq v(S + j) - v(S) \tag{2.3}$$

Dividing both sides by $v(T)$ and $v(S)$ respectively, we obtain

$$\frac{v(T+j)}{v(T)} \leq \frac{v(S+j)}{v(S)} \tag{2.4}$$

which is what we wanted to prove.

2. To prove this inequality, we can use a proof by contradiction. Assume the opposite, that there exists a set T such that

$$v(T) < \sum_{k \in T} [v(T) - v(T-k)] \tag{2.5}$$

Let's take an element k in T. Then, by the submodularity of v, we have

$$v(T) + v(T-k) \geq v(T \cap (T-k)) + v(T \cup (T-k)) = v(T-k) + v(T) \tag{2.6}$$

which simplifies to

$$v(T) \geq v(T-k) \tag{2.7}$$

Adding up these inequalities for all $k \in T$, we get

$$|T| \cdot v(T) \geq \sum_{k \in T} v(T-k) \tag{2.8}$$

which contradicts our assumption that $v(T) < \sum_{k \in T}[v(T) - v(T-k)]$. Therefore, our original inequality must hold:

$$v(T) \geq \sum_{k \in T} [v(T) - v(T-k)] \tag{2.9}$$

This completes the proof.

Problem 3: EFX

1. To show that EFX always exists for identical valuations, consider a situation where there are n agents and m goods, and each agent values each good equally. Without loss of generality, let's assume that each good has a value of 1 for each agent.

We can allocate the goods equally among the agents (or as equally as possible if m is not divisible by n). This allocation clearly fulfills the EFX condition:

- No agent envies another agent's bundle, because each bundle has the same total value (or at most a difference of 1 if m is not divisible by n).
- Even if another agent were to discard any single item, the value of their bundle would decrease. Therefore, no agent would prefer this reduced bundle to their own.

Hence, in this setting, an EFX allocation always exists.

2. Given two agents, A and B, and a set of goods M with values $v_A(m)$ and $v_B(m)$ for each good $m \in M$ according to agents A and B respectively.

For agent A, order the goods in non-increasing order of the values, i.e., $v_A(m_1) \geq v_A(m_2) \geq ... \geq v_A(m_k)$ for all goods $m_i \in M$. Divide the goods between the agents such that agent A gets goods $m_1, m_2, ..., m_i$ and agent B gets goods $m_{i+1}, m_{i+2}, ..., m_k$, where i is chosen such that the total value of the goods for agent A is as close as possible to half of the total value of all goods.

Similarly for agent B, order the goods in non-increasing order of the values and divide the goods such that the total value of goods for agent B is as close as possible to half of the total value of all goods.

Since each agent divides the goods according to their own valuations, each agent believes they are getting at least half of the total value of all goods, hence there is no envy. Since each good is given to the agent who values it more, removing any good from the other agent's bundle would not make it more attractive, hence it is also EFX.

Hence, an EFX allocation always exists when $n = 2$.

Problem 4: Valuation and XOS Function

To prove that any monotone and normalized submodular function can be written as an XOS function, we will use the following lemma:

Lemma: For any submodular function v, there exists a set of additive valuations $\{a^1, a^2, \ldots, a^m\}$ such that for any set $S \subseteq [m]$, we have $v(S) = \max_{i\in[m]} \sum_{j\in S} a^i(j)$.

Proof. We will construct the additive valuations $a^1, a^2, \ldots, a^m$ iteratively. For each $i \in [m]$, define the additive valuation a^i as follows:

$$a^i(i) = v(i). \tag{4.1}$$

$$\text{For each } j \neq i, a^i(j) = v(\{i, j\}) - v(i). \tag{4.2}$$

Now, we will show that for any set $S \subseteq [m]$, we have $v(S) = \max_{i\in[m]} \sum_{j\in S} a^i(j)$. Consider any set $S \subseteq [m]$. We have:

$$\begin{aligned} v(S) &= \sum_{i\in S}[v(S\cup\{i\}) - v(S - \{i\})] \quad \text{(using the submodularity property)} \\ &\le \sum_{i\in S} \max_{j\in[m]} a^j(i) \quad \text{(by the definition of the additive valuations } a^i \text{)} \\ &\le \max_{j\in[m]} \sum_{i\in S} a^j(i) \quad \text{(by rearranging the maximum and the summation)} \end{aligned} \tag{4.3}$$

Now, we will show that $v(S) \ge \max_{j\in[m]} \sum_{i\in S} a^j(i)$. For any $j \in [m]$. We have:

$$\begin{aligned} \sum_{i\in S} a^j(i) &= a^j(j) + \sum_{i\in S, i\neq j} a^j(i) \\ &= v(j) + \sum_{i\in S, i\neq j} (v(\{i,j\}) - v(j)) \\ &\le v(S) \quad \text{(by the monotonicity property)} \end{aligned} \tag{4.4}$$

Therefore,

$$v(S) \geq \max_{j \in [m]} \sum_{i \in S} a^j(i). \tag{4.5}$$

Combining the two inequalities, we have

$$v(S) = \max_{i \in [m]} \sum_{j \in S} a^i(j) \tag{4.6}$$

for any set $S \subseteq [m]$. This completes the proof of the lemma.

Now, we can use the lemma to show that any monotone and normalized submodular function can be written as an XOS function. Given a submodular function v, by the lemma, we know that there exists a set of additive valuations such that for any set $S \subseteq [m]$, we have

$$v(S) = \max_{i \in [m]} \sum_{j \in S} a^i(j) \tag{4.7}$$

We define the XOS function v^* as follows: for any set $S \subseteq [m]$,

$$v^*(S) = \max_{i \in [m]} \sum_{j \in S} a^i(j) \tag{4.8}$$

By construction, we have for all sets $S \subseteq [m]$ that $v^*(S) = v(S)$. Moreover, the additive valuations satisfy the definition of an XOS function. Thus, any monotone and normalized submodular function can be written as an XOS function.