HomeWork2

Problem 1. (2pt) Pick your favourite result in our class and state your reason.

My favourite result is EFX which means Envy Freeness up to any Good. Exactly what it is that A is EFX if for all i,j we always have vi(Ai)≥vi(Aj\{g}) for all g ∈Aj. From this viewpoint, we don’t need to allocate rigorously to make envy-freeness. What you need to do to reduce jealousy is to hide the most valuable good for the other one. I think it is an interesting and meaningful thing and we can also study it from different fields to maximize envy free.

Problem2.

1.

We also know some inequation such as v(S)≤v(T), v(S+j)=v({j}∪S).

So let us use the inequation condition in the title and then we can have

v(S+j)+v(T)≥v({j}∪S∪T)+v((S∪{j})∩T).

Forward, it can be v(S+j)+v(T)≥v({j}∪S∪T)+v((S∪{j})∩T)=v(T+j)+v(S).

After exchanging the values of the two sides of the inequality, we can have

v(S+j)-v(S)≥v(T+j)-v(T).

Since v(T)>v(S)>0, we can deduce from this that . That is all.

2.

v(T) ≥. Let’s use this and the inequation condition in the title .Then we can have an inequation, v(k)+v(T\{k})≥v({k}∪T\{k})+v(T\{k}∩{k}).

Easily we know v(T\{k}∩{k})= and v({k}∪T\{k})=v(T).

So we now have the simplified inequation v(k)+v(T\{k})≥v({k}∪T\{k})+v(T\{k}∩{k})=v(T), which it can be also written by v(T)-v(T\{k})≤v(k).

Forward, we can deduce from this and sum the all k from T that we will have this in equation

That is all.

Problem 3

1.

Set up:

* + n agents, m goods,
  + same value for every agent . I assume that the value is x.

Let us define the allocation: distribute as evenly as possible or one for each person in turn until the distribution is complete. From this, we can deduce that everyone has equal number or one more or one less than others.

So we can infer that no one envies another one because everyone’s value of his goods is equal. Though someone may have one more than this guy, this guy will not envy someone who has one more than his, if someone hides the extra good. What we define satisfies the definition of the EFX and now we proof that EFX always exists for identical valuations.

2.

Given two agents, A and B, and a set of goods M with values and for each good m ∈ M according to agents A and B respectively.

For agent A, order the goods in non-increasing order of the values, i.e., ≥≥ ... ≥ for all goods mi ∈ M. Divide the goods between the agents such that agent A gets goods m1, m2, ..., mi and agent B gets goods mi+1, mi+2, ..., mk, where i is chosen such that the total value of the goods for agent A is as close as possible to half of the total value of all goods.

Similarly for agent B, order the goods in non-increasing order of the values and divide the goods such that the total value of goods for agent B is as close as possible to half of the total value of all goods.

Since each agent divides the goods according to their own valuations, each agent believes they are getting at least half of the total value of all goods, hence there is no envy.

Since each good is given to the agent who values it more, removing any good from the other agent’s bundle would not make it more attractive, hence it is also EFX.

Hence, an EFX allocation always exists when n = 2.

Thanks to Sun JinChuan’s answer.

Problem4.

We will construct the additive valuations , , . . . , iteratively. For each i ∈ [m], define the additive valuation as follows:(i) = v(i). (4.1)

For each j≠i, (j) = v({i, j}) − v(i). (4.2)

Now, we will show that for any set S ⊆ [m], we have v(S) = (j).Let us think about any set S ⊆ [m]. We have:

v(S) = (using the submodularity property)≤(by the definition of the additive valuations ) (4.3)≤ (by rearranging the maximum and the summation)

Now, we will show that v(S) ≥ . For any j ∈ [m]. We have: = v(j) + (4.4)≤ v(S) (by the monotonicity property)

So we can have the inequation v(S)≥.

Compared with v(S)≤, obviously

v(S)=

Now, given a submodular function v, by the lemma, we know that there exists a set

of additive valuations such that for any set S and we have the equation:

v(S)=

Furthermore, we define the XOS function v\* as follows:

v\*(S)=

By construction, we have for all sets S ⊆ [m] that v\*(S) = v(S). Moreover, the additive valuations satisfy the definition of an XOS function. Thus, any monotone and normalized submodular function can be written as an XOS function.

By the way, thanks to Sun JinChuan’s answer.