

# Speed Running Discrete Mathematics

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## Author's Notes

This actually isn't the first copy of my discrete math notes...I had an earlier copy that got corrupted, so now I'm just going to constantly upload to GitHub because I'm afraid of them getting wiped again 😭

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These notes are based off of the textbook *Discrete Mathematics and Its Applications* by *Kenneth H. Rosen*, as well as from the lectures of *Serdar Erbatur* from the *Fall 2023* semester.

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If you have any complaints, or suggestions regarding these notes, please email me at [mdn220004@utdallas.edu](mailto:mdn220004@utdallas.edu)

# 1 Propositional Logic

## Definition Proposition

Declarative statements that are either **true** or **false**

Table 1: Examples of propositions

Statement	Proposition?
$1 + 1 = 2$	✓
What class are ya takin?	✗
I am happy	✓

Propositions can be combined to form another proposition with the use of *logical operators*

## Note

Common labels for propositions in discrete mathematics include letters like  $P, Q, R, S \dots$

Table 2: Logical Operators in Discrete Mathematics

Symbol	Meaning	Expression
$\neg$	negation / not	$\neg P$
$\wedge$	conjunction / and	$P \wedge Q$
$\vee$	disjunction / or	$P \vee Q$
$\oplus$	exclusive disjunction/ xor	$P \oplus Q$
$\implies$	implication / conditional	$P \implies Q$
$\iff$	biconditional	$P \iff Q$

## Note Logical Operator Precedence

Order matters when it comes to evaluating logical operators:

1. negation
2. conjunction
3. disjunctions
4. conditionals

Where negation is evaluated first, and conditionals last.

**Theorem** More on Conditionals

Conditionals can also be expressed in three other ways:

**Contrapositive:**  $\neg Q \implies \neg P$

**Converse:**  $Q \implies P$

**Inverse:**  $\neg P \implies \neg Q$

**Example**

**Conditional:** If it is raining, then I am not going to town

**Contrapositive:** If I go to town, then it is not raining

**Converse:** If I do not go to town, then it is raining

**Inverse:** If it is not raining, then I am going to town

**Definition** Truth Table

They are used as a way of seeing all possible values of a proposition

Table 3: Truth Table of  $P \implies Q$

$P$	$Q$	$P \implies Q$
F	F	T
F	T	T
T	F	F
T	T	T

## 1.1 Propositional Equivalencies

**Definition** Equivalencies

A proposition is a...

- **Tautology** if it is **true** in *every case*
- **Contradiction** / **Fallacy** if it is **false** in *every case*
- **Contingency** if *neither* is the case

Table 4: Example of Tautology and Contradiction

$P$	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

**Definition** Logical Equivalency ( $\equiv$ )

Compound propositions  $P$  and  $Q$  are *logically equivalent* if  $P \iff Q$  is a *tautology*. This is expressed as

$$P \equiv Q$$

Below is a table of useful logical equivalencies

Table 5: Logical Equivalencies

Name	Equivalence
Identity	$P \wedge T \equiv P$
	$P \vee F \equiv P$
Idempotent	$P \wedge P \equiv P$
	$P \vee P \equiv P$
Domination	$P \vee T \equiv T$
	$P \wedge F \equiv F$
Negation	$P \vee \neg P \equiv T$
	$P \wedge \neg P \equiv F$
Double Negation	$\neg(\neg P) \equiv P$
Commutative	$P \wedge Q \equiv Q \wedge P$
	$P \vee Q \equiv Q \vee P$
Associative	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
Distributive	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
De Morgan's	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
Absorption	$P \wedge (P \vee Q) \equiv P$
	$P \vee (P \wedge Q) \equiv P$

The use of showing the equivalencies between two compound propositions is called a *conditional-disjunction equivalence*

**Example** Conditional-Disjunction Equivalence

$$P \implies Q \equiv \neg P \vee Q$$

This can be proven with the use of a truth table 🍕

Continued on Next Page...

Table 6: Conditional-Disjunction Equivalence Proof

$P$	$Q$	$\neg P$	$\neg P \vee Q$	$P \implies Q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

## 1.2 Predicates and Quantifiers

### Note

So far, propositional logic can only handle *singular* subjects. It can't handle statements such as:

- All computer science students can program well
- $3x + 4 \geq 0$

### Definition Predicate Logic

When you have a proposition that contains a variable. It is typically written as a *propositional function*,  $P(x)$ , where  $x$  is the subject for the predicate  $P$

### Example

- $F(x) = "x > 3"$
- $P(x) = "x \text{ looks beautiful!}"$

### Note Multi-variable Propositional Functions

Propositional functions may also contain more than just one argument:

$$P(x, y)$$

### Definition Quantifiers

They define the range of which a proposition holds **true**, and can be nested to produce *nested quantifiers*

Table 7: Quantifiers in Discrete Mathematics

Quantifier	Expression	In English
Universal Quantifier	$\forall x.P(x)$	$P(x)$ is <b>true</b> for every $x$ in its domain
	$\exists x.P(x)$	There exists $x$ where $P(x)$ is <b>true</b>
Existential Quantifier	$\nexists x.P(x)$	There does not exist $x$ where $P(x)$ is <b>true</b>
	$\exists! x.P(x)$	There exists <i>only one</i> $x$ where $P(x)$ is <b>true</b>

$\forall x.P(x)$  may also be written as  $\forall x P(x)$ , and the same holds for the other quantifiers

**Example Nested Quantifiers**

$$\forall x \exists y. (x + y = 0)$$

Translates to: “For *every*  $x$  there *exists* a  $y$  such that  $x + y = 0$ ”

**1.3 Inference Rules and Proofs****Definition Argument**

A sequence of statements that have a conclusion

**Definition Premise**

The preceding statements of a mathematical argument that lead to a conclusion

**Definition Valid**

The conclusion, the final statement of the argument, must follow from its premises  
(i.e. premises  $\implies$  conclusion)

**Definition Fallacy**

Incorrect reasoning in discrete mathematics that leads to an invalid argument

**Example Argumentative Form**

Arguments may be written as this:  $((P \implies Q) \wedge P) \implies Q$ , or in argumentative form:

$$\begin{array}{c} P \implies Q \\ P \\ \hline \therefore Q \end{array}$$

The next page contains a table of inference rules

Table 8: Rules of Inference

Rule	Expression	Tautology
Modus ponens	$\frac{P}{P \Rightarrow Q}$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
	$\therefore Q$	
Modus tollens	$\frac{\neg Q}{P \Rightarrow Q}$	$(\neg Q \wedge (P \Rightarrow Q)) \Rightarrow \neg P$
	$\therefore \neg P$	
Hypothetical syllogism	$\frac{P \Rightarrow Q}{Q \Rightarrow R}$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
	$\therefore P \Rightarrow R$	
Disjunctive syllogism	$\frac{P \vee Q}{\neg P}$	$((P \vee Q) \wedge \neg P) \Rightarrow Q$
	$\therefore Q$	
Addition	$\frac{P}{P \vee Q}$	$P \Rightarrow (P \vee Q)$
	$\therefore P \vee Q$	
Simplification	$\frac{P \wedge Q}{P}$	$(P \wedge Q) \Rightarrow P$
	$\therefore P$	
Conjunction	$\frac{P}{Q}$	$((P) \wedge (Q)) \Rightarrow P \wedge Q$
	$\therefore P \wedge Q$	
Resolution	$\frac{P \vee Q}{\neg P \vee R}$	$((P \vee Q) \wedge (\neg P \vee R)) \Rightarrow (Q \vee R)$
	$\therefore Q \vee R$	

**Definition Proof**

Valid arguments that establish the **truth** of mathematical statements



Table 9: Rules of Inference for Quantified Statements

Rule	Expression
Universal Instantiation	$\frac{\forall x.P(x)}{\therefore P(c)}$
Universal Generalization	$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x.P(x)}$
Existential Instantiation	$\frac{\exists x.P(x)}{\therefore P(c) \text{ for some element } c}$
Existential Generalization	$\frac{P(c) \text{ for an arbitrary } c}{\therefore \exists x.P(x)}$

### Definition Theorem

A statement or claim that can be proven using:

- definitions
- other theorems
- axioms
- inference rules

They are also referred to as “Lemma”, “Proposition”, or “Corollary”

There are many different ways of proving a theorem. Let’s assume a conditional statement  $P \implies Q$ ...

### Definition Direct Proof

When the first step is the assumption that  $P$  is **true** and the following steps that lead up to  $Q$  is also **true**

### Example Proof by Contraposition

$P \implies Q$  can be proved **true** if  $\neg Q \implies \neg P$ , its contraposition, can also be proved. This is because a contraposition and a conditional proposition are tautologies

### Definition Indirect Proof

Proofs that do not start with the premises and end with conclusion (the opposite of a direct proof). Ways of doing direct proofs involve *proof by contraposition*, *proof by contradiction*, and much more...

### Example Proof by Contradiction

To prove  $P$ , you must assume  $\neg P$  and derive that  $\neg P$  is **false**. If  $\neg P$  is **false**, then that must mean that  $\neg\neg P$ , or  $P$ , must be **true**

## 2 Set Theory

### Definition set

A collection of *unique* objects or elements

- $a \in A$  to state that  $a$  is *contained* in the set  $A$
- $a \notin A$  to indicate that  $a$  is *not* contained in  $A$

### Note Defining Sets

Roster method

$$A = \{a, b, c, d\}$$

$$B = \{4, b, c, a\}$$

$$C = \{1, 2, \{3.0, 3.5\}, 4, 5\}$$

$$D = \{1, 2, C\}$$

Set Builder Notation

$$E = \{x \in \mathbb{N} \mid x = 2k \text{ for some } k \in \mathbb{N}\}$$

$$F = \{\alpha \mid P(\alpha) \text{ is true}\}$$

Table 10: Important Sets

Set	Expansion	Description
$\mathbb{N}$	$\{0, 1, 2, 3, \dots\}$	Natural numbers
$\mathbb{Z}$	$\{\dots, -1, 0, 1, \dots\}$	Integers
$\mathbb{Z}^+$	$\{1, 2, 3, \dots\}$	Positive integers
$\mathbb{Q}$	$\{\frac{P}{Q} \mid P \in \mathbb{Z}, Q \in \mathbb{Z}, Q \neq 0\}$	Rational numbers
$\mathbb{R}$		Real numbers
$\mathbb{R}^+$	$\{x \mid x > 0\}$	Positive real numbers
$\mathbb{C}$		Complex numbers
$\emptyset$		Empty set

## 2.1 Subsets

### Theorem Set Equality

Given two sets  $A$  and  $B$ :

$$A \equiv B \iff \forall x.(x \in A \iff x \in B)$$

is true

### Definition Subset ( $\subseteq$ )

$A$  is a subset of  $B$  if *all elements of*  $A$  are also contained in  $B$ :

$$A \subseteq B \iff \forall x.(x \in A \implies x \in B)$$

### Example Subsets

Let  $A = \{2, 3\}, B = \{a, b, c, 2, 3\}$

- $A \subseteq B$  true
- $\emptyset \subseteq A$  true
- $A \subseteq A$  true
- $B \subseteq A$  false

This notation is seen in Section 2.2

### Definition Power Set

The set of all subsets of the set

$$|A| = n, |P(A)| = 2^n$$

### Example

Let  $S = \{1, 2\}$ :

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

### Definition Proper Subset ( $\subset$ )

It's a subset where  $A \neq B$ :

$$A \subset B \iff \forall x.(x \in A \implies x \in B) \wedge A \neq B$$

## 2.2 Cardinality

### Definition Cardinality

It's the number of unique elements in a set. It's denoted by  $|A|$

### Example

$\{\{1, 2\}, 3\}$	2 elements
$\{1, 2, 3\} = \{1, 2, 3, 3\}$	3 elements
$\{\emptyset\}$	1 element
$\{\}$	0 elements
$\{\emptyset\}$	1 element

### Definition Intervals

Sets of numbers *between* two numbers  $a$  and  $b$  if  $a, b \in \mathbb{R} \wedge a \leq b$

- $[a, b] = x | a \leq x \leq b$
- $[a, b) = x | a \leq x < b$
- $(a, b] = x | a < x \leq b$
- $(a, b) = x | a < x < b$

**Definition** Ordered Tuples

$$(a_0, a_1, a_2, \dots, a_n)$$

An *ordered collection* of unique elements

$$(a, b, c) \neq (b, c, a)$$

**Theorem** Cartesian Products

Let  $A$  and  $B$  be sets:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

**Example** Cartesian Product

Let  $A = \{1, 2\}$ ,  $B = \{a, b, c\}$ :

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

## 2.3 Set Operations

Let  $A = \{1, 2, 3\}$ ,  $B = \{2, 3, 4, 5\}$ :

Table 11: Set Operations

Expression	Meaning	Result
$A \cup B$	$A$ union $B$	$\{1, 2, 3, 4, 5\}$
$A \cap B$	$A$ intersection $B$	$\{2, 3\}$
$\bar{A}$	$A$ complement $B$	$\{x \notin A\}$

**Definition** Membership Table

It's very similar to a truth table, where **true** means that an element exists within the set, and **false** means that the element isn't.

Table 12: Membership Table Example

$A$	$B$	$A \cup B$	$A \cap B$
false	false	false	false
false	true	true	false
true	false	true	false
true	true	true	true

## 2.4 Functions

### Definition Function

Function  $f$  is denoted from set  $A$  to set  $B$ :

$$f : A \rightarrow B$$

representing a relation that assigns each element of  $A$  to *exactly one* element of  $B$

### Example

Let  $A = \{1, 2\}, B = \{a, b\}$

$$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$$

These are functions:

- $f = \{(1, a), (2, b)\}$
- $f = \{(1, a), (2, a)\}$

This isn't:

- $f = \{(1, a), (1, b)\}$

### Theorem Equality of Functions

Two functions  $f$  and  $g$  are equal if:

- They share the same domains / co-domains
- They assign the same element from the domain to the same element

Let  $f : A \rightarrow B, g : C \rightarrow D$ :

- $A \neq C \vee B \neq D \implies f \neq g$
- $A = C \vee B = D \implies f = g$

### Definition One-to-One Functions (Injective)

A function  $f : A \rightarrow B$  where all values  $A$  must correspond to *one and only one* element in  $B$ :

$$\forall a \in A, \forall b \in B. (f(a) \neq f(b))$$

### Definition Onto Functions (Subjective)

A function  $f : A \rightarrow B$  where all values  $B$  must correspond to *at least an* element in  $A$ :

$$\forall b \in B, \exists a \in A. (f(a) = b)$$

### Definition Bijection

A function that is *both* one-to-one *and* onto

### Definition Inverse of Function

$$f^{-1}(x) = y \iff f(y) = x$$

### Definition Composite Functions

$$\text{domain}(f) = \text{range}(g) \implies f \circ g(x) = f(g(x))$$

If the composite of the function exists, then:

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

**Definition** Floor ( $\lfloor x \rfloor$ )

The biggest integer  $n \leq x$

**Definition** Ceiling ( $\lceil x \rceil$ )

The smallest integer  $n \geq x$

### 3 Algorithms

**Definition** Algorithm

A sequence of *precise* steps used to solve computational problems

**Note** Input-Output

For every input instance, a correct output *must* be produced

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**Algorithm 1:** Binary Search Algorithm
 

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**Input:**  $A \leftarrow \{A_1, A_2, \dots, A_n\}$  where  $A_i \in \mathbb{Z}, k \in \mathbb{Z}$

**Output:**  $i$  where  $A_i = k$ , or  $i = 0 \implies k \notin A$

```

1 left ← 1
2 right ← n
3 while left < right do
4   i ← ⌊(left + right) ÷ 2⌋
5   if Ai = k then
6     return i
7   else if Ai < k then
8     left ← i + 1
9   else
10    right ← i
11 return 0

```

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**Algorithm 2:** Insertion Sort
 

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**Input:**  $A \leftarrow \{A_1, A_2, \dots, A_n\}$  where  $n \geq 2$

**Output:**  $A \leftarrow \{A_1, A_2, \dots, A_n\}$  where  $n \geq 2 \wedge A_i \leq A_{i+1}$

```

1 for i ← 2 to n do
2   key ← Ai
3   index ← i - 1
4   while index ≥ 1 ∧ Aindex > key do
5     Aindex+1 ← Aindex
6     index ← index - 1
7   Aindex+1 ← key

```

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### 3.1 Growth of Functions

This can be used to determine the growth of functions, and is also mainly used in the field of *computer science* to analyze algorithms.

#### Note Proving Big-O

To prove that  $f(x) = O(g(x))$ , you must prove the existence of a  $C$  and  $k$  that fulfill the definition:

$$\exists C, k. (\forall x > k. (|f(x)| \leq C \times |g(x)|))$$

#### Definition Big-O Notation

Let  $f$  and  $g$  be functions.  $f(x)$  is  $O(g(x))$  (read “ $f(x)$  is Big-O of  $g(x)$ ”) when there are constants  $C$  and  $k$  (the witnesses) such that:

$$|f(x)| \leq C|g(x)|, x > k$$

This is called the worst-case runtime of an algorithm

#### Example Identify the Witnesses

Find  $C, k$  such that  $f(x) = 4x^2 + 2x + 1$  is  $O(x^2)$

$$\begin{aligned} g(x) &= n^2 + n^2 + n^2 \\ &= 3n^2 \end{aligned}$$

$$f(x) < g(x) \text{ when } x > 1 \therefore C = 3, k = 1$$

#### Definition Big-Ω

Let  $f$  and  $g$  be functions.  $f(x)$  is  $\Omega(g(x))$  when there are constants  $C$  and  $k$  such that:

$$|f(x)| \geq C|g(x)|$$

This is called the best-case runtime of an algorithm

#### Definition Big-θ

Let  $f$  and  $g$  be functions.  $f(x)$  is  $\theta(g(x))$  when:

$$f(x) = O(g(x)) \wedge f(x) = \Omega(g(x))$$

This is called the average-case runtime of an algorithm

## 4 Number Theory

### Definition Division

Let  $a, b \in \mathbb{Z}$ .  $a$  divides  $b$  ( $a|b$ ) if there exists integer  $c$  such that  $a \times c = b$

$$a|b, a, b \in \mathbb{Z} \iff \exists c \in \mathbb{Z}. (a \times c = b)$$

### Theorem Basic Division Properties

Let  $a, b, c \in \mathbb{Z}, a \neq 0$ :

- $(a|b) \wedge (a|c) \implies a|(b + c)$
- $a|b \implies \forall c \in \mathbb{Z}. (a|(b \times c))$
- $(a|b) \wedge (b|c) \implies a|c$

### Theorem Consequence of Division

$$\forall a, b, c, m, n \in \mathbb{Z}, a \neq 0. ((a|b) \wedge (a|c) \implies (a|(mb + nc)))$$

Let  $a, b, c, m, n$  be integers where  $a \neq 0$ , and suppose  $a|b \wedge a|c$ . Then, for any integers  $m$  and  $n$ ,  $a|mb + nc$ .

### Theorem Division Algorithm

Let  $a \in \mathbb{Z}, d \in \mathbb{Z}^+$ . Then there are unique integers  $q$  and  $r$  where  $0 \leq r < d$  such that  $a = q \times d + r$

$$\forall a, q, r \in \mathbb{Z}, d \in \mathbb{Z}^+, 0 \leq r < d. (a = q \times d + r)$$

### Definition Quotient

$$q = a \text{ div } d$$

### Definition Remainder

$$r = a \text{ mod } d$$

### Definition Congruency ( $\equiv$ )

Let  $a, b \in \mathbb{Z}, m \in \mathbb{Z}^+$ .  $a$  is congruent to  $b \text{ mod } m$  if...

- $m|a - b$
- $a \text{ mod } m \equiv b \text{ mod } m$
- $k \in \mathbb{Z}, a = b + k \times m$

### Theorem Preservation of Congruencies

Let  $a, b, c, d \in \mathbb{Z}, m \in \mathbb{Z}^+$ . If  $a \equiv b \text{ mod } m \wedge c \equiv d \text{ mod } m$ , then

- $a + c \equiv b + d \pmod{m}$
- $a \times c \equiv b \times d \pmod{m}$



## 5 Induction and Recursion

### Definition Proof via. Induction

To prove that  $P(n)$  is true for  $n \in \mathbb{Z}^+$ , we must complete these two steps:

- **Basis Step:** Verify that  $P(1)$  is **true**
- **Inductive Step:** Show that  $P(k) \implies P(k+1), k \in \mathbb{Z}^+$  is **true**

### Definition Recursion

A way to express a definition of an element in terms of itself. There's two required steps to recursively define a function:

- **Basis Step:** Specify the value of the function at 0
- **Recursive Step:** Specify a rule for finding its value at an integer from its smaller values at smaller integers

## 6 Relations

### Definition Relation

Given sets  $A, B$ , it's a subset of  $A \times B$

### Example

$$\begin{aligned}R_1 &= \{(a, b) | a, b \in \mathbb{Z} \wedge a \leq b\} \subseteq \mathbb{Z} \times \mathbb{Z} \\R_2 &= \{(a, b) | a, b \in \mathbb{Z} \wedge a < b\} \subseteq \mathbb{Z} \times \mathbb{Z} \\R_3 &= \{(a, b) | a, b \in \mathbb{Z} \wedge a = b\} \subseteq \mathbb{Z} \times \mathbb{Z} \\R_4 &= \{(a, b) | a, b \in \mathbb{Z} \wedge a = b + \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z} \\R_5 &= \{(a, b) | a, b \in \mathbb{Z} \wedge a|b\} \subseteq \mathbb{Z} \times \mathbb{Z}\end{aligned}$$

### Definition Reflexivity

A relation  $R$  on a set  $A$  is *reflexive* if and only if:

$$\forall x. (x \in A \implies (x, x) \in R)$$

### Example

$$\begin{aligned}A &= \{1, 2, 3\} \\R &= \{(1, 1), (2, 2), (1, 3), (3, 3)\} \\R &= \{(3, 3), (1, 3), (1, 1)\} \leftarrow \text{not reflexive}\end{aligned}$$

### Definition Symmetry

Relation  $R$  on a set  $A$  is symmetric if and only if

$$\forall x \forall y. ((x, y) \in R \implies (y, x) \in R)$$

**Definition Anti-Symmetry**

Relation  $R$  on a set  $A$  is antisymmetric if and only if

$$\forall x \forall y. ((x, y) \in R \wedge (y, x) \in R \implies (y, x) \in x = y)$$

**Example**

$\leq$	Not symmetric: $3 \leq 5, 5 \not\leq 3$ Antisymmetric: $a \leq b, b \leq a \implies a = b$
$>$	Not symmetric: $5 > 3, 3 \not> 5$ Antisymmetric: $a > b, b > a \implies a = b$
$=$	Symmetric Antisymmetric: $a = b, b = a \implies a = b$
$R = \{(a, b)   a, b \in \mathbb{Z} \wedge a = -b\} \subseteq \mathbb{Z} \times \mathbb{Z}$	Symmetric: $(a, -a) \in R \implies (-a, a) \in R$ Not antisymmetric: $(a, -a) \wedge (-a, a) \in R, a \neq -a$
Let $A = \{1, 2, 3\}$ $R = \{(2, 3), (3, 2), (1, 3)\}$	Not symmetric: $(1, 3) \in R, (3, 1) \notin R$ Not antisymmetric: $(2, 3), (3, 2) \in R, 2 \neq 3$

**Definition Transitive**

A relation  $R$  on set  $A$  such that

$$\forall x \forall y \forall z. ((x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R)$$

**Example**

$$A = \{a, b, c\}$$

$$R = \{(a, b), (a, a), (b, b), (b, c), (c, b)\}$$

It's nothing, nada, it's...disappointing

**Note Equivalence Relation**

A relation that is reflexive, symmetric, and transitive

## 7 Graphs

### Definition Graph

Denoted by  $G = (V, E)$ , consists of a non-empty set of vertices (or nodes)  $V$  and a set of edges  $E$

What's a graph without its nodes?

### Definition Adjacent Vertices (Undirected)

When the endpoints of an edge  $e$  are vertices  $a$  and  $b$ , the vertices are considered to be *adjacent* (or *neighbors*) with one another.

That edge is called an *incident edge*, *connecting* vertices  $a$  and  $b$

### Definition Adjacent Vertices (Directed)

When  $(a, b)$  is an edge of a directed graph  $G$ ,  $a$  is *adjacent to*  $b$ , and  $a$  is *adjacent from*  $b$ .

Vertex  $a$  is called the *initial vertex*, and  $b$  is called the *terminal* or *end vertex* of  $(u, v)$

### Definition Neighborhood ( $N(v)$ )

"The set of all neighbors of  $u$ " of a graph  $G = (V, E)$ , denoted by  $(N(v))$

If  $A$  is a subset of  $V$ ,  $N(A)$  is the set of vertices in  $G$  that are adjacent to at least one vertex in  $A$ :

$$V = \bigcup_{v \in A} N(v)$$

### Definition Degree (Undirected Graph) ( $\deg(v)$ )

The number of edges incident with it. A loop contributes *twice* to the degree of that vertex

### Definition Degree (Directed Graph) ( $\deg^{\pm}(v)$ )

The number of edges incident with it. A loop contributes *twice* to the degree of that vertex

### Definition Infinite Graphs

If  $|V| = \infty$ , then the graph is referred to as an *infinite* graph, otherwise it's called a *finite* graph

### Example Neighborhoods and Degrees

Figure 1: Neighborhoods Example

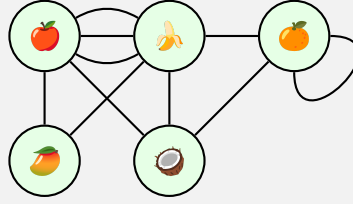


Table 13: Table of Neighbors and Degrees

Vertex	Degree	Neighborhood
	2	{, }
	3	{, , }
	4	{, , }
	5	{, , }
	6	{, , , }

### Note Sum of Degrees

When adding the degrees of all vertices in graph  $G = (V, E)$ , you'll find that each edge contributes *two* to the sum of degrees, since each edge is incident with exactly two vertices

### Theorem Handshake Theorem

If  $G = (V, E)$  is an undirected graph with  $|E| = m$  edges, then

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v) = 2 \times m$$

This applies even if multiple edges and loops are present

### Example Handshake

We can see this theorem in effect if we look back at Figure 1:

$$\begin{aligned}
 2 \times m &= \sum_{v \in V} \deg(v) \\
 &= 2 + 3 + 4 + 5 + 6 \\
 2 \times m &= 20 \\
 m &= 10
 \end{aligned}$$

**Theorem** Consequences of Handshaking

In an undirected graph, the sum of degree of nodes with odd degrees is *even*.

**Theorem** Directed Handshake

Let  $G = (V, E)$  be a directed graph:

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

Table 14: Graph Terminology

Type	Direction	Edges	Loops
Simple Graph	Undirected	Singular	No
Multigraph	Undirected	Multiple	No
Pseudograph	Undirected	Multiple	Yes
Simple Directed Graph	Directed	Singular	No
Directed Multigraph	Directed	Multiple	Yes
Mixed Graph	Both	Multiple	Yes

Figure 2: Simple Graph

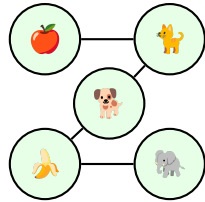


Figure 3: Multigraph

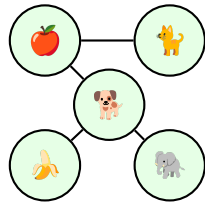


Figure 4: Pseudograph

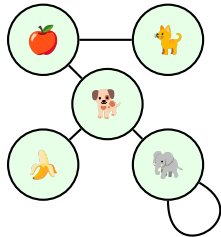


Figure 5: Simple Directed Graph

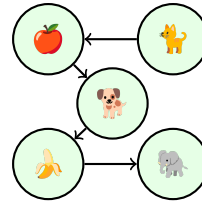


Figure 6: Directed Multigraph

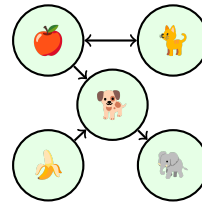


Figure 7: Mixed Graph

