Speed Running Discrete Mathematics

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November 30, 2023

Contents

Author's Notes

This actually isn't the first copy of my discrete math notes…I had an earlier copy that got corrupted, so now I'm just going to constantly upload to GitHub because I'm afraid of them getting wiped again 😭

These notes are based off of the textbook *Discrete Mathematics and Its Applications* by *Kenneth H. Rosen*, as well as from the lectures of *Serdar Erbatur* from the *Fall 2023* semester.

If you have any complaints, or suggestions regarding these notes, please email me at mdn220004@utdallas.edu

1 Propositional Logic

Definition **Proposition**

Declarative statements that are either `true` or `false`

Table 1: Examples of propositions

Statement	**Proposition?**
$1 + 1 = 2$	✔️
What class are ya takin?	⛔
I am happy	✔️

Propositions can be combined to form another proposition with the use of *logical operators*

Note

Common labels for propositions in discrete mathematics include letters like P, Q, R, S...

Table 2: Logical Operators in Discrete Mathematics

Symbol	**Meaning**	**Expression**
$\neg$	negation / not	$\neg P$
$\wedge$	conjunction / and	$P \wedge Q$
$\vee$	disjunction / or	$P \vee Q$
$\oplus$	exclusive disjunction/ xor	$P \oplus Q$
$\implies$	implication / conditional	$P \implies Q$
$\iff$	biconditional	$P \iff Q$

Note **Logical Operator Precedence**

Order matters when it comes to evaluating logical operators:

1. negation
2. conjunction
3. disjunctions
4. conditionals

Where negation is evaluated first, and conditionals last.

Theorem More on Conditionals

Conditionals can also be expressed in three other ways:

Contrapositive: $\neg Q \implies \neg P$

Converse: $Q \implies P$

Inverse: $\neg P \implies \neg Q$

Example

Conditional: If it is raining, then I am not going to town

Contrapositive: If I go to town, then it is not raining

Converse: If I do not go to town, then it is raining

Inverse: If it is not raining, then I am going to town

Definition **Truth Table**

They are used as a way of seeing all possible values of a proposition

Table 3: Truth Table of $P \implies Q$

P	Q	$P \implies Q$
F	F	T
F	T	T
T	F	F
T	T	T

1.1 Propositional Equivalencies

Definition **Equivalencies**

A proposition is a...

- **Tautology** if it is `true` in *every case*
- **Contradiction / Fallacy** if it is `false` in *every case*
- **Contingency** if *neither* is the case

Table 4: Example of Tautology and Contradiction

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

Definition **Logical Equivalency ($\equiv$)**

Compound propositions P and Q are *logically equivalent* if $P \iff Q$ is a *tautology*. This is expressed as

$$P \equiv Q$$

Below is a table of useful logical equivalencies

Table 5: Logical Equivalencies

Name	**Equivalence**
Identity	$P \wedge \mathrm{T} \equiv P$
	$P \vee \mathrm{F} \equiv P$
Idempotent	$P \wedge P \equiv P$
	$P \vee P \equiv P$
Domination	$P \vee \mathrm{T} \equiv \mathrm{T}$
	$P \wedge \mathrm{F} \equiv \mathrm{F}$
Negation	$P \vee \neg P \equiv \mathrm{T}$
	$P \wedge \neg P \equiv \mathrm{F}$
Double Negation	$\neg(\neg P) \equiv P$
Commutative	$P \wedge Q \equiv Q \wedge P$
	$P \vee Q \equiv Q \vee P$
Associative	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
Distributive	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
De Morgan's	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
Absorption	$P \wedge (P \vee Q) \equiv P$
	$P \vee (P \wedge Q) \equiv P$

The use of showing the equivalencies between two compound propositions is called a *conditional-disjunction equivalence*

Example **Conditional-Disjunction Equivalence**

$$P \implies Q \equiv \neg P \vee Q$$

This can be proven with the use of a truth table 🥰

Continued on Next Page...

Table 6: Conditional-Disjunction Equivalence Proof

P	Q	$\neg P$	$\neg P \vee Q$	$P \implies Q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

1.2 Predicates and Quantifiers

Note

So far, propositional logic can only handle *singular* subjects. It can't handle statements such as:

- All computer science students can program well
- $3x + 4 \geq 0$

Definition **Predicate Logic**

When you have a proposition that contains a variable. It is typically written as a *propositional function*, $P(x)$, where x is the subject for the predicate P

Example

- $F(x) =$ "$x > 3$"
- $P(x) =$ "x looks beautiful!"

Note **Multi-variable Propositional Functions**

Propositional functions may also contain more than just one argument:

$$P(x, y)$$

Definition **Quantifiers**

They define the range of which a proposition holds `true`, and can be nested to produce *nested quantifiers*

Table 7: Quantifiers in Discrete Mathematics

Quantifier	Expression	In English
Universal Quantifier	$\forall x.P(x)$	$P(x)$ is `true` for every x in its domain
Existential Quantifier	$\exists x.P(x)$	There exists x where $P(x)$ is `true`
	$\nexists x.P(x)$	There does not exist x where $P(x)$ is `true`
	$\exists! x.P(x)$	There exists *only one* x where $P(x)$ is `true`

$\forall x.P(x)$ may also be written as $\forall x P(x)$, and the same holds for the other quantifiers

Example **Nested Quantifiers**

$$\forall x \exists y.(x + y = 0)$$

Translates to: "For *every* x *there exists* a y such that $x + y = 0$"

1.3 Inference Rules and Proofs

Definition **Argument**

A sequence of statements that have a conclusion

Definition **Valid**

The conclusion, the final statement of the argument, must follow from its premises
(i.e. premises $\implies$ conclusion)

Definition **Premise**

The preceding statements of a mathematical argument that lead to a conclusion

Definition **Fallacy**

Incorrect reasoning in discrete mathematics that leads to an invalid argument

Example **Argumentative Form**

Arguments may be written as this: $\big((P \implies Q) \land P\big) \implies Q$, or in argumentative form:

$$\begin{array}{l} P \implies Q \\ \underline{P \qquad\qquad} \\ \therefore Q \end{array}$$

The next page contains a table of inference rules

Table 8: Rules of Inference

Rule	Expression	Tautology
Modus ponens	P $\underline{P \implies Q}$ $\therefore\ Q$	$\big(P \land (P \implies Q)\big) \implies Q$
Modus tollens	$\neg Q$ $\underline{P \implies Q}$ $\therefore\ \neg P$	$\big(\neg Q \land (P \implies Q)\big) \implies \neg P$
Hypothetical syllogism	$P \implies Q$ $\underline{Q \implies R}$ $\therefore\ P \implies R$	$\big((P \implies Q) \land (Q \implies R)\big) \implies (P \implies R)$
Disjunctive syllogism	$P \lor Q$ $\underline{\neg P}$ $\therefore\ Q$	$\big((P \lor Q) \land \neg P\big) \implies Q$
Addition	$\underline{P}$ $\therefore\ P \lor Q$	$P \implies (P \lor Q)$
Simplification	$\underline{P \land Q}$ $\therefore\ P$	$(P \land Q) \implies P$
Conjunction	P $\underline{Q}$ $\therefore\ P \land Q$	$\big((P) \land (Q)\big) \implies P \land Q$
Resolution	$P \lor Q$ $\underline{\neg P \lor R}$ $\therefore\ Q \lor R$	$\big((P \lor Q) \land (\neg P \lor R)\big) \implies (Q \lor R)$

Definition **Proof**

Valid arguments that establish the `truth` of mathematical statements

Table 9: Rules of Inference for Quantified Statements

Rule	Expression
Universal Instantiation	$\begin{array}{rl} & \forall x.P(x) \\ \hline \therefore & P(c) \end{array}$
Universal Generalization	$\begin{array}{rl} & P(c) \text{ for an arbitrary } c \\ \hline \therefore & \forall x.P(x) \end{array}$
Existential Instantiation	$\begin{array}{rl} & \exists x.P(x) \\ \hline \therefore & P(c) \text{ for some element } c \end{array}$
Existential Generalization	$\begin{array}{rl} & P(c) \text{ for an arbitrary } c \\ \hline \therefore & \forall x.P(x) \end{array}$

Definition **Theorem**

A statement or claim that can be proven using:

- definitions
- other theorems
- axioms
- inference rules

They are also referred to as "Lemma", "Proposition", or "Corollary"

There are many different ways of proving a theorem. Let's assume a conditional statement $P \implies Q$...

Definition **Direct Proof**

When the first step is the assumption that P is `true` and the following steps that lead up to Q is also `true`

Definition **Indirect Proof**

Proofs that do not start with the premises and end with conclusion (the opposite of a direct proof). Ways of doing direct proofs involve *proof by contraposition*, *proof by contradiction*, and much more...

Example **Proof by Contraposition**

$P \implies Q$ can be proved `true` if $\neg Q \implies P$, its contraposition, can also be proved. This is because a contraposition and a conditional proposition are tautologies

Example **Proof by Contradiciton**

To prove P, you must assume $\neg P$ and derive that $\neg P$ is `false`. If $\neg P$ is `false`, then that must mean that $\neg\neg P$, or P, must be `true`

2 Set Theory

Definition **set**

A collection of *unique* objects or elements

- $a \in A$ to state that a is *contained* in the set A
- $a \notin A$ to indicate that a *is not* contained in A

Note **Defining Sets**

Roster method

$$\begin{aligned} A &= \{a, b, c, d\} \\ B &= \{4, b, c, a\} \\ C &= \left\{1, 2, \{3.0, 3.5\}, 4, 5\right\} \\ D &= \{1, 2, C\} \end{aligned}$$

Set Builder Notation

$$E = \{x \in \mathbb{N} \big| x = 2k \text{ for some } k \in \mathbb{N}\}$$
$$F = \{\alpha \big| P(\alpha) \text{ is } \texttt{true}\}$$

Table 10: Important Sets

Set	**Expansion**	**Description**
$\mathbb{N}$	$\{0, 1, 2, 3, \cdots\}$	Natural numbers
$\mathbb{Z}$	$\{\cdots, -1, 0, 1, \cdots\}$	Integers
$\mathbb{Z}^+$	$\{1, 2, 3, \cdots\}$	Positive integers
$\mathbb{Q}$	$\{\frac{P}{Q} \big| P \in \mathbb{Z}, Q \in \mathbb{Z}, Q \neq 0\}$	Rational numbers
$\mathbb{R}$		Real numbers
$\mathbb{R}^+$	$\{x \big| x > 0\}$	Positive real numbers
$\mathbb{C}$		Complex numbers
$\varnothing$		Empty set

2.1 Subsets

Theorem **Set Equality**

Given two sets A and B:

$$A \equiv B \iff \forall x.(x \in A \iff x \in B)$$

is `true`

Definition **Subset ($\subseteq$)**

A is a subset of B if *all elements of* A are also contained in B:

$$A \subseteq B \iff \forall x.(x \in A \implies x \in B)$$

Example **Subsets**

Let $A = \{2, 3\}$, $B = \{a, b, c, 2, 3\}$

- $A \subseteq B$ `true`
- $A \subseteq A$ `true`
- $\varnothing \subseteq A$ `true`
- $B \subseteq A$ `false`

This notation is seen in Section 2.2

Definition **Power Set**

The set of all subsets of the set

$$|A| = n, |P(A)| = 2^n$$

Example

Let $S = \{1, 2\}$:

$$P(S) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$$

Definition **Proper Subset ($\subset$)**

It's a subset where $A \neq B$:

$$A \subset B \iff \forall x.(x \in A \implies x \in B) \land A \neq B$$

2.2 Cardinality

Definition **Cardinality**

It's the number of unique elements in a set. It's denoted by $|A|$

Example

$\{\{1, 2\}, 3\}$	2 elements
$\{1, 2, 3\} = \{1, 2, 3, 3\}$	3 elements
$\{\varnothing\}$	1 element
$\{\}$	0 elements
$\{\varnothing\}$	1 element

Definition **Intervals**

Sets of numbers *between* two numbers a and b if $a, b \in \mathbb{R} \land a \leq b$

- $[a, b] = x|a \leq x \leq b$
- $[a, b) = x|a \leq x < b$
- $(a, b] = x|a < x \leq b$
- $(a, b) = x|a < x < b$

Definition **Ordered Tuples**

$$(a_0, a_1, a_2, \cdots, a_n)$$

An *ordered collection* of unique elements

$$(a, b, c) \neq (b, c, a)$$

Theorem **Cartesian Products**

Let A and B be sets:

$$A \times B = \{(a, b) \big| a \in A, b \in B\}$$

Example **Cartesian Product**

Let $A = \{1, 2\}, B = \{a, b, c\}$:

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

2.3 Set Operations

Let $A = \{1, 2, 3\}, B = \{2, 3, 4, 5\}$:

Table 11: Set Operations

Expression	Meaning	Result
$A \cup B$	A union B	$\{1, 2, 3, 4, 5\}$
$A \cap B$	A intersection B	$\{2, 3\}$
$\bar{A}$	A complement B	$\{x \notin A\}$

Definition **Membership Table**

It's very similar to a truth table, where `true` means that an element exists within the set, and `false` means that the element isn't.

Table 12: Membership Table Example

A	B	$A \cup B$	$A \cap B$
`false`	`false`	`false`	`false`
`false`	`true`	`true`	`false`
`true`	`false`	`true`	`false`
`true`	`true`	`true`	`true`

2.4 Functions

Definition **Function**

Function f is denoted from set A to set B:

$$f : A \to B$$

representing a relation that assigns each element of A to *exactly one* element of B

Example

Let $A = \{1, 2\}, B = \{a, b\}$

$$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$$

These are functions:

- $f = \{(1, a), (2, b)\}$
- $f = \{(1, a), (2, a)\}$

This isn't:

- $f = \{(1, a), (1, b)\}$

Theorem **Equality of Functions**

Two functions f and g are equal if:

- They share the same domains / co-domains
- They assign the same element from the domain to the same element

Let $f : A \to B, g : C \to D$:

- $A \neq C \vee B \neq D \implies f \neq g$
- $A = C \vee B = D \implies f = g$

Definition **One-to-One Functions (Injective)**

A function $f : A \to B$ where all values A must correspond to *one and only one* element in B:

$$\forall a \in A, \forall b \in B.\big(f(a) \neq f(b)\big)$$

Definition **Onto Functions (Subjective)**

A function $f : A \to B$ where all values B must correspond to *at least an* element in A:

$$\forall b \in B, \exists a \in A.\big(f(a) = b\big)$$

Definition **Bijection**

A function that is *both* one-to-one *and* onto

Definition **Inverse of Function**

$$f^{-1}(x) = y \iff f(y) = x$$

Definition **Compisite Functions**

$$\text{domain}(f) = \text{range}(g) \implies f \circ g(x) = f(g(x))$$

If the composite of the function exists, then:

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

Definition **Floor** ($\lfloor x \rfloor$)

The biggest integer $n \leq x$

Definition **Ceiling** ($\lceil x \rceil$)

The smallest integer $n \geq x$

3 Algorithms

Definition **Algorithm**

A sequence of *precise* steps used to solve computational problems

Note **Input-Output**

For every input instance, a correct output *must* be produced

Algorithm 1: Binary Search Algorithm

Input: $A \leftarrow \{A_1, A_2, \ldots, A_n\}$ where $A_i \in \mathbb{Z}$, $k \in \mathbb{Z}$
Output: i where $A_i = k$, or $i = 0 \implies k \notin A$

```
left ← 1
right ← n
while left < right do
    i ← ⌊(left + right) ÷ 2⌋
    if A_i = k then
        return i
    else if A_i < k then
        left ← i + 1
    else
        right ← i
return 0
```

Algorithm 2: Insertion Sort

Input: $A \leftarrow \{A_1, A_2, \ldots, A_n\}$ where $n \geq 2$
Output: $A \leftarrow \{A_1, A_2, \ldots, A_n\}$ where $n \geq 2 \wedge A_i \leq A_{i+1}$

```
for i ← 2 to n do
    key ← A_i
    index ← i − 1
    while index ≥ 1 ∧ A_index > key do
        A_index+1 ← A_index
        index ← index − 1
    A_index+1 ← key
```

3.1 Growth of Functions

This can be used to determine the growth of functions, and is also mainly used in the field of *computer science* to analyze algorithms.

Note **Proving Big-O**

To prove that $f(x) = O\big(g(x)\big)$, you must prove the existence of a C and k that fulfill the definition:

$$\exists C, k.\big(\forall x > k.(|f(x)| \leq C \times |g(x)|)\big)$$

Definition **Big-O Notation**

Let f and g be functions. $f(x)$ is $O\big(g(x)\big)$ (read "$f(x)$ is Big-O of $g(x)$") when there are constants C and k (the witnesses) such that:

$$|f(x)| \leq C|g(x)|, x > k$$

This is called the worst-case runtime of an algorithm

Example **Identify the Witnesses**

Find C, k such that $f(x) = 4x^2 + 2x + 1$ is $O(x^2)$

$$\begin{aligned} g(x) &= n^2 + n^2 + n^2 \\ &= 3n^2 \end{aligned}$$

$$f(x) < g(x) \text{ when } x > 1 \therefore C = 3, k = 1$$

Definition **Big-Ω**

Let f and g be functions. $f(x)$ is $\Omega\big(g(x)\big)$ when there are constants C and k such that:

$$|f(x)| \geq C|g(x)|$$

This is called the best-case runtime of an algorithm

Definition **Big-θ**

Let f and g be functions. $f(x)$ is $\theta\big(g(x)\big)$ when:

$$f(x) = O\big(g(x)\big) \land f(x) = \Omega\big(g(x)\big)$$

This is called the average-case runtime of an algorithm

4 Number Theory

Definition **Division**

Let $a, b \in \mathbb{Z}$. a divides b ($a|b$) if there exists integer c such that $a \times c = b$

$$a|b, a, b \in \mathbb{Z} \iff \exists c \in \mathbb{Z}.(a \times c = b)$$

Theorem **Basic Division Properties**

Let $a, b, c \in \mathbb{Z}, a \neq 0$:

- $(a|b) \wedge (a|c) \implies a|(b + c)$
- $a|b \implies \forall c \in \mathbb{Z}.\big(a|(b \times c)\big)$
- $(a|b) \wedge (b|c) \implies a|c$

Theorem **Consequence of Division**

$$\forall a, b, c, m, n \in \mathbb{Z}, a \neq 0.\Big((a|b) \wedge (a|c) \implies (a|(mb + nc))\Big)$$

Let a, b, c, m, n be integers where $a \neq 0$, and suppose $a|b \wedge a|c$. Then, for any integers m and n, $a|mb + nc$.

Theorem **Division Algorithm**

Let $a \in \mathbb{Z}, d \in \mathbb{Z}^+$. Then there are unique integers q and r where $0 \leq r < d$ such that $a = q \times d + r$

$$\forall a, q, r \in \mathbb{Z}, d \in \mathbb{Z}^+, 0 \leq r < d.(a = q \times d + r)$$

Definition **Quotient**

$$q = a \texttt{ div } d$$

Definition **Remainder**

$$r = a \texttt{ mod } d$$

Definition **Congruency ($\equiv$)**

Let $a, b \in \mathbb{Z}, m \in \mathbb{Z}^+$. a is congruent to b $\texttt{mod}$ m if...

- $m|a - b$
- $a \texttt{ mod } m \equiv b \texttt{ mod } m$
- $k \in \mathbb{Z}, a = b + k \times m$

Theorem **Preservation of Congruencies**

Let $a, b, c, d \in \mathbb{Z}, m \in \mathbb{Z}^+$. If $a \equiv b \texttt{ mod } m \wedge c \equiv d \texttt{ mod } m$, then

- $a + c \equiv b + d \pmod{m}$
- $a \times c \equiv b \times d \pmod{m}$

5 Induction and Recursion

Definition **Proof via. Induction**

To prove that $P(n)$ is true for $n \in \mathbb{Z}^+$, we must complete these two steps:

- **Basis Step:** Verify that $P(1)$ is `true`
- **Inductive Step:** Show that $P(k) \implies P(k+1), k \in \mathbb{Z}^+$ is `true`

Definition **Recursion**

A way to express a definition of an element in terms of itself. There's two required steps to recursively define a function:

- **Basis Step:** Specify the value of the function at 0
- **Recursive Step:** Specify a rule for finding its value at an integer from its smaller values at smaller integers

6 Counting

6.1 Counting Basics

Theorem **The Product Rule**

Think of nested `for` loops

Suppose a procedure can be broken down into a sequence of multiple tasks. If there are n_1 ways to do the first task, and for each task $k \in n_1$ there are n_2 and continuing on until with the final task having n_a ways to do that task, then there are

$$n_1 \times n_2 \times \cdots \times n_a$$

ways to do that procedure.

Example **Product Rule**

An ice-cream shop has 12 different types of cones, and 32 flavors of ice-cream. What is the number of unique single-scoop ice-creams the ice-cream shop can serve?

For each ice-cream cone there are 32 unique flavors of ice-cream that can be served with it. This means that there are

$$12 \times 32 = 384$$

unique single-scoop ice-creams combinations that the shop can serve.

Think of sequental `for` loops

Theorem The Sum Rule

Suppose a procedure can be broken down into a sequence of multiple tasks. If a task can be done in one of n_1 ways, or one of n_2 ways, or so on and so forth until one of n_a ways, then there are

$$n_1 + n_2 + \cdots + n_a$$

wasy to do the task.

Example Sum Rule

You spot three potential people to steal wallets from. The first person's wallet has 4 different types of credit / reward cards, the second person has 18 (wow that's a large wallet!), and the third only has one. However, you can only take one card in the wallet that you choose. How many possible cards do you have to steal?

$$4 + 18 + 1 = 23$$

different cards to rob 😈

6.2 Pigeonhole Principle

Theorem The Pigeonhole Principle

If $k \in \mathbb{Z}^+$ and $k+1$ or more objects are placed into k containers, then there's at least one container containing more than 1 object.

Theorem Pigeonhole Consequence I

A function f from a set of $k+1$ elements to a set of k elements will *never be* one-to-one

Example Pigeonhole and Ice-cream I

For a total count of 23 scoops of ice-cream to be present in 22 cones, there must be at least one cone with two ice-cream scoops

Theorem Generalized Pigeonhole Principle

If N objects are placed into k containers, then there must be at least be one box containing at least N/k objects

$$\left\lceil \frac{N}{k} \right\rceil \geq m$$

Example Pigeons and Cards

From a standard deck of 52 playing cards, how many need to be drawn for there to be at least 5 cards in the same suite?

$$\left\lceil \frac{N}{4} \geq 5 \right\rceil = 21$$

Theorem **Pigeonhole Consequence II**

Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or decreasing

6.3 Permutations and Combinations

Definition **Permutation**

A set of distinct objects in an ordered arragement. An ordered arrangement of n elements of a set is called an n-*permutation*

Example $S =$ **{🍎, 🍌, 🦈}**

- {🍌, 🦈, 🍎} is a permutation
- {🦈, 🍎} is a 2-permutation

Theorem **Number of R-Permutations**

If $n \in \mathbb{Z}^+, r \in \mathbb{Z}$ where $1 \leq r \leq n$, then there are

$$P(n, r) = n \times (n - 1) \times (n - 1) \ldots \times (n - r + 1)$$

r-permutations with a set of n distinct elements

Note **Simplification of the theorem**

The theorem can be simplified (and more commonly used) like this:

$$\forall n, r \in \mathbb{Z}. \left(P(n, r) = \frac{n!}{(n - r)!}, 0 \leq r \leq n \right)$$

Definition **Combination**

An unordered subset with r amoutn of elements from a set

Example $S =$ **{🦐, 🍳, 🍚}**

- {🦐, 🍚} is a combination
- {🦐, 🍳} is a combination as well

Note **Combinations vs. Permutations**

$S = \{$🐄, 🐔, 🦃$\}$

{🐄, 🦃} {🦃, 🐄}

That would be 2 permutations, but *only one* combination

Theorem **Number of combinations**

$$\forall n, r \in \mathbb{Z}. \left(C(n, r) = C(n, n - r) = \frac{n!}{r!\,(n - r)!}, 0 \leq r \leq n \right)$$

factorials make everything big!

Example **Permutations**

How many unique ways are there to select a first, second, and third place winners in a contest with 15 contestants?

$$P(15,3) = \frac{(15)!}{((15)-(3))!} = 2730$$

different ways!

7 Relations

Definition **Relation**

Given sets A, B, it's a subset of $A \times B$

Example

$$\begin{aligned}
R_1 &= \{(a,b) | a,b \in \mathbb{Z} \wedge a \leq b\} \subseteq \mathbb{Z} \times \mathbb{Z} \\
R_2 &= \{(a,b) | a,b \in \mathbb{Z} \wedge a < b\} \subseteq \mathbb{Z} \times \mathbb{Z} \\
R_3 &= \{(a,b) | a,b \in \mathbb{Z} \wedge a = b\} \subseteq \mathbb{Z} \times \mathbb{Z} \\
R_4 &= \{(a,b) | a,b \in \mathbb{Z} \wedge a = b + \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z} \\
R_5 &= \{(a,b) | a,b \in \mathbb{Z} \wedge a | b\} \subseteq \mathbb{Z} \times \mathbb{Z}
\end{aligned}$$

Definition **Reflexivity**

A relation R on a set A is *reflexive* if and only if:

$$\forall x. \big(x \in A \implies (x,x) \in R\big)$$

Example

$$\begin{aligned}
&A = \{1,2,3\} \\
&R = \{(1,1),(2,2),(1,3),(3,3)\} \\
&R = \{(3,3),(1,3),(1,1)\} \leftarrow \text{not reflexive}
\end{aligned}$$

Definition **Symmetry**

Relation R on a set A is symmetric if and only if

$$\forall x \forall y. \big((x,y) \in R \implies (y,x) \in R\big)$$

Definition **Anti-Symmetry**

Relation R on a set A is antisymmetric if and only if

$$\forall x \forall y. \big((x,y) \in R \wedge (y,x) \in R \implies (y,x) \in x = y\big)$$

Example

$\leq$	Not symmetric: $3 \leq 5, 5 \not\leq 3$ Antisymmetric: $a \leq b, b \leq a \implies a = b$
$>$	Not symmetric: $5 > 3, 3 \not> 5$ Antisymmetric: $a > b, b > a \implies a = b$
$=$	Symmetric Antisymmetric: $a = b, b = a \implies a = b$
$R = \{(a,b) \mid a, b \in \mathbb{Z} \wedge a = -b\} \subseteq \mathbb{Z} \times \mathbb{Z}$	Symmetric: $(a, -a) \in R \implies (-a, a) \in R$ Not antisymmetric: $(a, -a) \wedge (-a, a) \in R, a \neq -a$
Let $A = \{1, 2, 3\}$ $R = \{(2,3), (3,2), (1,3)\}$	Not symmetric: $(1,3) \in R, (3,1) \notin R$ Not antisymmetric: $(2,3), (3,2) \in R, 2 \neq 3$

Definition **Transitive**

A relation R on set A such that

$$\forall x \forall y \forall z. \big((x,y) \in R \wedge (y,z) \in R \implies (x,z) \in R\big)$$

Example

$$A = \{a, b, c\}$$
$$R = \{(a,b), (a,a), (b,b), (b,c), (c,b)\}$$

It's nothing, nada, it's...disappointing

Note **Equivalence Relation**

A relation that is reflexive, symmetric, and transitive

8 Graphs

What's a graph without its nodes?

Definition **Graph**

Denoted by $G = (V, E)$, consists of a non-empty set of *vertices* (or nodes) V and a set of edges E

Definition **Adjacent Verticies (Undirected)**

When the endpoints of an edge e are vertices a and b, the verticies are considered to be *adjacent* (or *neighbors*) with one another.

That edge is called an *incident* edge, *connecting* verticies a and b

Definition **Adjacent Verticies (Directed)**

When (a, b) is an edge of a directed graph G, a is *adjacent to* b, and a is *adjacent from* b.

Vertex a is called the *initial vertex*, and b is called the *terminal* or *end vertex* of (u, v)

Definition **Neighborhood $(N(v))$**

"The set of all neighbors of u" of a graph $G = (V, E)$, denoted by $(N(v))$

If A is a subset of V, $N(A)$ is the set of verticies in G that are adjacent to at least one vertex in A:

$$V = \bigcup_{v \in A} N(v)$$

Definition **Degree (Undirected Graph) $(\texttt{deg}(v))$**

The number of edges incident with vertex v. A loop contributes *twice* to the count

Definition **Degree (Directed Graph) $(\texttt{deg}^{\pm}(v))$**

$\texttt{deg}^{-}(v)$ number of edges pointing into v
$\texttt{deg}^{+}(v)$ number of edges pointing out from v

Definition **Infinite Graphs**

If $|V| = \infty$, then the graph is referred to as an *infinite* graph, otherwise it's called a *finite* graph

Example **Neighborhoods and Degrees**

Figure 1: Neighborhoods Example

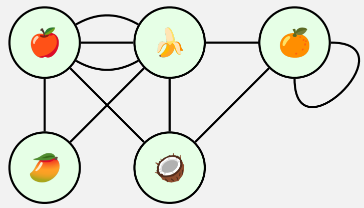

Table 13: Table of Neighbors and Degrees

Vertex	Degree	Neighborhood
🥭	2	{🍎,🍌}
🥥	3	{🍎,🍌,🍊}
🍊	4	{🍊,🍌,🥥}
🍎	5	{🥭,🍌,🥥}
🍌	6	{🍎,🥭,🥥,🍊}

Note **Sum of Degrees**

When adding the degrees of all verticies in graph $G = (V, E)$, you'll find that each edge contributes *two* to the sum of degrees, since each edge is incident with exactly two verticies

Theorem **Handshake Theorem**

If $G = (v, E)$ is an undirected graph with $|E| = m$ edges, then

$$\sum_{v \in V} \texttt{deg}(v) = \sum_{v \in V_1} \texttt{deg}(v) + \sum_{v \in V_2} \texttt{deg}(v) = 2 \times m$$

This applies even if multiple edges and loops are present

Example **Handshake**

We can see this theorem in effect if we look back at Figure 1:

$$\begin{aligned} 2 \times m =& \sum_{v \in V} \texttt{deg}(v) \\ =& 2 + 3 + 4 + 5 + 6 \\ 2 \times m =& 20 \\ m =& 10 \end{aligned}$$

Theorem **Consequences of Handshaking**

In an undirected graph, the sum of degree of nodes with odd degrees is *even*.

Theorem **Directed Handshake**

Let $G = (V, E)$ be a directed graph:

$$\sum_{v \in V} \texttt{deg}^{-}(v) = \sum_{v \in V} \texttt{deg}^{+}(v) = |E|$$

Table 14: Graph Terminology

Type	**Direction**	**Edges**	**Loops**
Simple Graph	Undirected	Singular	No
Multigraph	Undirected	Multiple	No
Pseudograph	Undirected	Multiple	Yes
Simple Directed Graph	Directed	Singular	No
Directed Multigraph	Directed	Multiple	Yes
Mixed Graph	Both	Multiple	Yes

Figure 2: Simple Graph

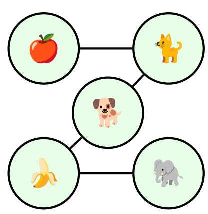

Figure 3: Multigraph

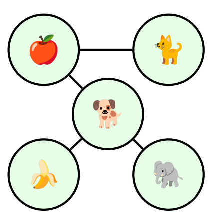

Figure 4: Pseudograph

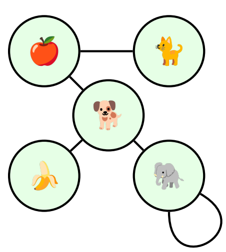

Figure 5: Simple Directed Graph

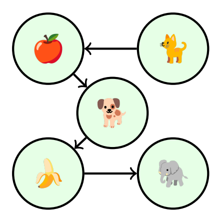

Figure 6: Directed Multigraph

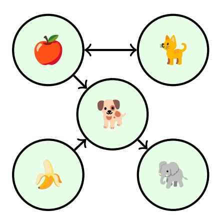

Figure 7: Mixed Graph

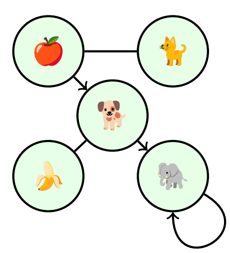

Table 15: Special Graphs

Graph	Definition
Complete Graph	For n nodes, there exists *exactly one* edge between any pair of nodes
Cycles Graph	For $n \geq 3$, the graph C_n consists of path: $(v_1, v_2), (v_2, v_2) \ldots, (v_{n-1}, v_n), (v_n, 1)$
Bipartite Graph	A simple graph $G = (V, E)$ such that V can be partitioned into two disjoint subsets V_1 and V_2
Complete Bipartite Graph	A bipartite graph where $(u, v), u \in V_1, v \in V_2$
Subgraph	Let $G = (V, E)$. $H = (A, B)$ is a subgraph of G if $A \subseteq V \wedge B \subseteq E$

Figure 8: Complete Graph

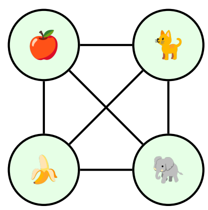

Figure 9: Cycles Graph