

Speed Running Calculus

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November 13, 2023

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Author's Notes

This actually isn't the first copy of my discrete math notes...I had an earlier copy that got corrupted, so now I'm just going to constantly upload to GitHub because I'm afraid of them getting wiped again 😭

These notes are based off of the textbook *Discrete Mathematics and Its Applications* by *Kenneth H. Rosen*, as well as from the lectures of *Serdar Erbatur* from the *Fall 2023* semester.

If you have any complaints, or suggestions regarding these notes, please email me at mdn220004@utdallas.edu

1 Propositional Logic

Definition Proposition

Declarative statements that are either **true** or **false**

Table 1: Examples of propositions

Statement	Proposition?
$1 + 1 = 2$	✓
What class are ya takin?	✗
I am happy	✓

Propositons can be combined to form another proposition with the use of *logical operators*

Note

Common labels for propositions in discrete mathematics include letters like $P, Q, R, S...$

Table 2: Logical Operators in Discrete Mathematics

Symbol	Meaning	Expression
\neg	negation / not	$\neg P$
\wedge	conjunction / and	$P \wedge Q$
\vee	disjunction / or	$P \vee Q$
\oplus	exclusive disjunction/ xor	$P \oplus Q$
\implies	implication / conditional	$P \implies Q$
\iff	biconditional	$P \iff Q$

Note Logical Operator Precedence

Order matters when it comes to evaluating logical operators:

1. negaiton
2. conjunction
3. disjunctions
4. conditionals

Where negation is evaluated first, and conditionals last.

Theorem More on Conditionals

Conditionals can also be expressed in three other ways:

Contrapositive: $\neg Q \implies \neg P$

Converse: $Q \implies P$

Inverse: $\neg P \implies \neg Q$

Example

Conditional: If it is raining, then I am not going to town

Contrapositive: If I go to town, then it is not raining

Converse: If I do not go to town, then it is raining

Inverse: If it is not raining, then I am going to town

Definition Truth Table

They are used as a way of seeing all possible values of a proposition

Table 3: Truth Table of $P \implies Q$

P	Q	$P \implies Q$
F	F	T
F	T	T
T	F	F
T	T	T

1.1 Propositional Equivalencies

Definition Equivalencies

A proposition is a...

- **Tautology** if it is **true** in *every case*
- **Contradiction** / **Fallacy** if it is **false** in *every case*
- **Contingency** if *neither* is the case

Table 4: Example of Tautology and Contradiction

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

Definition Logical Equivalency (\equiv)

Compound propositions P and Q are *logically equivalent* if $P \iff Q$ is a *tautology*. This is expressed as

$$P \equiv Q$$

Below is a table of useful logical equivalencies

Table 5: Logical Equivalencies

Name	Equivalence
Identity	$P \wedge T \equiv P$
	$P \vee F \equiv P$
Idempotent	$P \wedge P \equiv P$
	$P \vee P \equiv P$
Domination	$P \vee T \equiv T$
	$P \wedge F \equiv F$
Negation	$P \vee \neg P \equiv T$
	$P \wedge \neg P \equiv F$
Double Negation	$\neg(\neg P) \equiv P$
Commutative	$P \wedge Q \equiv Q \wedge P$
	$P \vee Q \equiv Q \vee P$
Associative	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
Distributive	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
De Morgan's	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
Absorption	$P \wedge (P \vee Q) \equiv P$
	$P \vee (P \wedge Q) \equiv P$

The use of showing the equivalencies between two compound propositions is called a *conditional-disjunction equivalence*

Example Conditional-Disjunction Equivalence

$$P \implies Q \equiv \neg P \vee Q$$

This can be proven with the use of a truth table 🧐

Continued on Next Page...

Table 6: Conditional-Disjunction Equivalence Proof

P	Q	$\neg P$	$\neg P \vee Q$	$P \implies Q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

1.2 Predicates and Quantifiers

Note

So far, propositional logic can only handle *singular* subjects. It can't handle statements such as:

- All computer science students can program well
- $3x + 4 \geq 0$

Definition Predicate Logic

When you have a proposition that contains a variable. It is typically written as a *propositional function*, $P(x)$, where x is the subject for the predicate P

Example

- $F(x) = "x > 3"$
- $P(x) = "x \text{ looks beautiful}"$

Note Multi-variable Propositional Functions

Propositional functions may also contain more than just one argument:

$$P(x, y)$$

Definition Quantifiers

They define the range of which a proposition holds **true**, and can be nested to produce *nested quantifiers*

Table 7: Quantifiers in Discrete Mathematics

Quantifier	Expression	In English
Universal Quantifier	$\forall x.P(x)$	$P(x)$ is true for every x in its domain
	$\exists x.P(x)$	There exists x where $P(x)$ is true
Existential Quantifier	$\nexists x.P(x)$	There does not exist x where $P(x)$ is true
	$\exists! x.P(x)$	There exists <i>only one</i> x where $P(x)$ is true

$\forall x.P(x)$ may also be written as $\forall x P(x)$, and the same holds for the other quantifiers

Example Nested Quantifiers

$$\forall x \exists y. (x + y = 0)$$

Translates to: “For *every* x there *exists* a y such that $x + y = 0$ ”

1.3 Inference Rules and Proofs**Definition Argument**

A sequence of statements that have a conclusion

Definition Premise

The preceding statements of a mathematical argument that lead to a conclusion

Definition Valid

The conclusion, the final statement of the argument, must follow from its premises
(i.e. premises \implies conclusion)

Definition Fallacy

Incorrect reasoning in discrete mathematics that leads to an invalid argument

Example Argumentative Form

Arguments may be written as this: $((P \implies Q) \wedge P) \implies Q$, or in argumentative form:

$$\begin{array}{c} P \implies Q \\ P \\ \hline \therefore Q \end{array}$$

The next page contains a table of inference rules

Table 8: Rules of Inference

Rule	Expression	Tautology
	P	
Modus ponens	$\frac{P \implies Q}{P \implies Q}$	$(P \wedge (P \implies Q)) \implies Q$
	$\therefore Q$	
	$\neg Q$	
Modus tollens	$\frac{P \implies Q}{P \implies Q}$	$(\neg Q \wedge (P \implies Q)) \implies \neg P$
	$\therefore \neg P$	
	$P \implies Q$	
Hypothetical syllogism	$\frac{Q \implies R}{Q \implies R}$	$((P \implies Q) \wedge (Q \implies R)) \implies (P \implies R)$
	$\therefore P \implies R$	
	$P \vee Q$	
Disjunctive syllogism	$\frac{\neg P}{\neg P}$	$((P \vee Q) \wedge \neg P) \implies Q$
	$\therefore Q$	
	P	
Addition	$\frac{P}{P \vee Q}$	$P \implies (P \vee Q)$
	$\therefore P \vee Q$	
	$P \wedge Q$	
Simplification	$\frac{P \wedge Q}{P \wedge Q}$	$(P \wedge Q) \implies P$
	$\therefore P$	
	P	
Conjunction	$\frac{Q}{Q}$	$((P) \wedge (Q)) \implies P \wedge Q$
	$\therefore P \wedge Q$	
	$P \vee Q$	
Resolution	$\frac{\neg P \vee R}{\neg P \vee R}$	$((P \vee Q) \wedge (\neg P \vee R)) \implies (Q \vee R)$
	$\therefore Q \vee R$	

Table 9: Rules of Inference for Quantified Statements

Rule	Expression
Universal Instantiation	$\frac{\forall x.P(x)}{\forall x.P(x)}$
	$\therefore P(c)$
Universal Generalization	$\frac{P(c) \text{ for an arbitrary } c}{P(c) \text{ for an arbitrary } c}$
	$\therefore \forall x.P(x)$
Existential Instantiation	$\frac{\exists x.P(x)}{\exists x.P(x)}$
	$\therefore P(c) \text{ for some element } c$
Existential Generalization	$\frac{P(c) \text{ for an arbitrary } c}{P(c) \text{ for an arbitrary } c}$
	$\therefore \forall x.P(x)$

Definition Proof

Valid arguments that establish the **truth** of mathematical statements

Definition Theorem

A statement or claim that can be proven using:

- definitions
- other theorems
- axioms
- inference rules

They are also referred to as “Lemma”, “Proposition”, or “Corollary”

There's many different ways of proving a theorem. Let's assume a conditional statement $P \implies Q$...

Definition Direct Proof

When the first step is the assumption that P is **true** and the following steps that lead up to Q is also **true**

Definition Indirect Proof

Proofs that do not start with the premises and end with conclusion (the opposite of a direct proof). Ways of doing direct proofs involve *proof by contraposition*, *proof by contradiction*, and much more...

Example Proof by Contraposition

$P \implies Q$ can be proved **true** if $\neg Q \implies \neg P$, its contraposition, can also be proved. This is because a contraposition and a conditional proposition are tautologies

Example Proof by Contradiction

To prove P , you must assume $\neg P$ and derive that $\neg P$ is **false**. If $\neg P$ is **false**, then that must mean that $\neg\neg P$, or P , must be **true**

2 Set Theory

Definition set

A collection of *unique* objects or elements

- $a \in A$ to state that a is *contained* in the set A
- $a \notin A$ to indicate that a is *not* contained in A

Note Defining Sets

Roster method

$$A = \{a, b, c, d\}$$

$$B = \{4, b, c, a\}$$

$$C = \{1, 2, \{3.0, 3.5\}, 4, 5\}$$

$$D = \{1, 2, C\}$$

Set Builder Notation

$$E = \{x \in \mathbb{N} \mid x = 2k \text{ for some } k \in \mathbb{N}\}$$

$$F = \{\alpha \mid P(\alpha) \text{ is true}\}$$

Table 10: Important Sets

Set	Expansion	Description
\mathbb{N}	$\{0, 1, 2, 3, \dots\}$	Natural numbers
\mathbb{Z}	$\{\dots, -1, 0, 1, \dots\}$	Integers
\mathbb{Z}^+	$\{1, 2, 3, \dots\}$	Positive integers
\mathbb{Q}	$\{\frac{P}{Q} \mid P \in \mathbb{Z}, Q \in \mathbb{Z}, Q \neq 0\}$	Rational numbers
\mathbb{R}		Real numbers
\mathbb{R}^+	$\{x \mid x > 0\}$	Positive real numbers
\mathbb{C}		Complex numbers
\emptyset		Empty set

2.1 Subsets

Theorem Set Equality

Given two sets A and B :

$$A \equiv B \iff \forall x.(x \in A \iff x \in B)$$

is true

Definition Subset (\subseteq)

A is a subset of B if *all elements of* A are also contained in B :

$$A \subseteq B \iff \forall x.(x \in A \implies x \in B)$$

Example Subsets

Let $A = \{2, 3\}, B = \{a, b, c, 2, 3\}$

- $A \subseteq B$ true
- $\emptyset \subseteq A$ true
- $A \subseteq A$ true
- $B \subseteq A$ false

This notation is seen in Section 2.2

Definition Power Set

The set of all subsets of the set

$$|A| = n, |P(A)| = 2^n$$

Example

Let $S = \{1, 2\}$:

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

Definition Proper Subset (\subset)

It's a subset where $A \neq B$:

$$A \subset B \iff \forall x.(x \in A \implies x \in B) \wedge A \neq B$$

2.2 Cardinality

Definition Cardinality

It's the number of unique elements in a set. It's denoted by $|A|$

Example

$\{\{1, 2\}, 3\}$	2 elements
$\{1, 2, 3\} = \{1, 2, 3, 3\}$	3 elements
$\{\emptyset\}$	1 element
$\{\}$	0 elements
$\{\emptyset\}$	1 element

Definition Intervals

Sets of numbers *between* two numbers a and b if $a, b \in \mathbb{R} \wedge a \leq b$

- $[a, b] = x | a \leq x \leq b$
- $[a, b) = x | a \leq x < b$
- $(a, b] = x | a < x \leq b$
- $(a, b) = x | a < x < b$

Definition Ordered Tuples

$$(a_0, a_1, a_2, \dots, a_n)$$

An *ordered collection* of unique elements

$$(a, b, c) \neq (b, c, a)$$

Theorem Cartesian Products

Let A and B be sets:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Example Cartesian Product

Let $A = \{1, 2\}$, $B = \{a, b, c\}$:

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

2.3 Set Operations

Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$:

Table 11: Set Operations

Expression	Meaning	Result
$A \cup B$	A union B	$\{1, 2, 3, 4, 5\}$
$A \cap B$	A intersection B	$\{2, 3\}$
\bar{A}	A complement B	$\{x \notin A\}$

Definition Membership Table

It's very similar to a truth table, where **true** means that an element exists within the set, and **false** means that the element isn't.

Table 12: Membership Table Example

A	B	$A \cup B$	$A \cap B$
false	false	false	false
false	true	true	false
true	false	true	false
true	true	true	true

2.4 Functions

Definition Function

Function f is denoted from set A to set B :

$$f : A \rightarrow B$$

representing a relation that assigns each element of A to *exactly one* element of B

Example

Let $A = \{1, 2\}, B = \{a, b\}$

$$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$$

These are functions:

- $f = \{(1, a), (2, b)\}$
- $f = \{(1, a), (2, a)\}$

This isn't:

- $f = \{(1, a), (1, b)\}$

Theorem Equality of Functions

Two functions f and g are equal if:

- They share the same domains / co-domains
- They assign the same element from the domain to the same element

Let $f : A \rightarrow B, g : C \rightarrow D$:

- $A \neq C \vee B \neq D \implies f \neq g$
- $A = C \vee B = D \implies f = g$

Definition One-to-One Functions (Injective)

A function $f : A \rightarrow B$ where all values A must correspond to *one and only one* element in B :

$$\forall a \in A, \forall b \in B. (f(a) \neq f(b))$$

Definition Onto Functions (Subjective)

A function $f : A \rightarrow B$ where all values B must correspond to *at least an* element in A :

$$\forall b \in B, \exists a \in A. (f(a) = b)$$

Definition Bijection

A function that is *both* one-to-one *and* onto

Definition Inverse of Function

$$f^{-1}(x) = y \iff f(y) = x$$

Definition Composite Functions

$$\text{domain}(f) = \text{range}(g) \implies f \circ g(x) = f(g(x))$$

If the composite of the function exists, then:

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

Definition Floor ($\lfloor x \rfloor$)

The biggest integer $n \leq x$

Definition Ceiling ($\lceil x \rceil$)

The smallest integer $n \geq x$

3 Algorithms

Definition Algorithm

A sequence of *precise* steps used to solve computational problems

Note Input-Output

For every input instance, a correct output *must* be produced

Algorithm 1: Binary Search Algorithm

Input: $A \leftarrow \{A_1, A_2, \dots, A_n\}$ where $A_i \in \mathbb{Z}, k \in \mathbb{Z}$

Output: i where $A_i = k$, or $i = 0 \implies k \notin A$

```

1 left ← 1
2 right ← n
3 while left < right do
4   i ← ⌊(left + right) ÷ 2⌋
5   if  $A_i = k$  then
6     return i
7   else if  $A_i < k$  then
8     left ← i + 1
9   else
10    right ← i
11 return 0

```

Algorithm 2: Insertion Sort

Input: $A \leftarrow \{A_1, A_2, \dots, A_n\}$ where $n \geq 2$

Output: $A \leftarrow \{A_1, A_2, \dots, A_n\}$ where $n \geq 2 \wedge A_i \leq A_{i+1}$

```

1 for i ← 2 to n do
2   key ←  $A_i$ 
3   index ← i - 1
4   while index ≥ 1 ∧  $A_{index} > key$  do
5      $A_{index+1} \leftarrow A_{index}$ 
6     index ← index - 1
7    $A_{index+1} \leftarrow key$ 

```

3.1 Growth of Functions

This can be used to determine the growth of functions, and is also mainly used in the field of *computer science* to analyze algorithms.

Note Proving Big-O

To prove that $f(x) = O(g(x))$, you must prove the existence of a C and k that fulfill the definition:

$$\exists C, k. (\forall x > k. (|f(x)| \leq C \times |g(x)|))$$

Definition Big-O Notation

Let f and g be functions. $f(x)$ is $O(g(x))$ (read “ $f(x)$ is Big-O of $g(x)$ ”) when there are constants C and k (the witnesses) such that:

$$|f(x)| \leq C|g(x)|, x > k$$

This is called the worst-case runtime of an algorithm

Example

Find C, k such that $f(x) = 4x^2 + 2x + 1$ is $O(x^2)$

$$\begin{aligned} g(x) &= n^2 + n^2 + n^2 \\ &= 3n^2 \end{aligned}$$

$$f(x) < g(x) \text{ when } x > 1 \therefore C = 3, k = 1$$

Definition Big-Ω

Let f and g be functions. $f(x)$ is $\Omega(g(x))$ when there are constants C and k such that:

$$|f(x)| \geq C|g(x)|$$

This is called the best-case runtime of an algorithm

Definition Big-θ

Let f and g be functions. $f(x)$ is $\theta(g(x))$ when:

$$f(x) = O(g(x)) \wedge f(x) = \Omega(g(x))$$

This is called the average-case runtime of an algorithm