

Speed Running Discrete Mathematics

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Author's Notes

This actually isn't the first copy of my discrete math notes...I had an earlier copy that got corrupted, so now I'm just going to constantly upload to GitHub because I'm afraid of them getting wiped again 😭

These notes are based off of the textbook *Discrete Mathematics and Its Applications* by *Kenneth H. Rosen*, as well as from the lectures of *Serdar Erbatur* from the *Fall 2023* semester.

If you have any complaints, or suggestions regarding these notes, please email me at mdn220004@utdallas.edu

1 Propositional Logic

Definition Proposition

Declarative statements that are either **true** or **false**

Table 1: Examples of propositions

Statement	Proposition?
$1 + 1 = 2$	✓
What class are ya takin?	✗
I am happy	✓

Propositions can be combined to form another proposition with the use of *logical operators*

Note

Common labels for propositions in discrete mathematics include letters like $P, Q, R, S...$

Table 2: Logical Operators in Discrete Mathematics

Symbol	Meaning	Expression
\neg	negation / not	$\neg P$
\wedge	conjunction / and	$P \wedge Q$
\vee	disjunction / or	$P \vee Q$
\oplus	exclusive disjunction/ xor	$P \oplus Q$
\implies	implication / conditional	$P \implies Q$
\iff	biconditional	$P \iff Q$

Note Logical Operator Precedence

Order matters when it comes to evaluating logical operators:

1. negation
2. conjunction
3. disjunctions
4. conditionals

Where negation is evaluated first, and conditionals last.

Theorem More on Conditionals

Conditionals can also be expressed in three other ways:

Contrapositive: $\neg Q \implies \neg P$

Converse: $Q \implies P$

Inverse: $\neg P \implies \neg Q$

Example

Conditional: If it is raining, then I am not going to town

Contrapositive: If I go to town, then it is not raining

Converse: If I do not go to town, then it is raining

Inverse: If it is not raining, then I am going to town

Definition Truth Table

They are used as a way of seeing all possible values of a proposition

Table 3: Truth Table of $P \implies Q$

P	Q	$P \implies Q$
F	F	T
F	T	T
T	F	F
T	T	T

1.1 Propositional Equivalencies

Definition Equivalencies

A proposition is a...

- **Tautology** if it is **true** in *every case*
- **Contradiction** / **Fallacy** if it is **false** in *every case*
- **Contingency** if *neither* is the case

Table 4: Example of Tautology and Contradiction

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

Definition Logical Equivalency (\equiv)

Compound propositions P and Q are *logically equivalent* if $P \iff Q$ is a *tautology*. This is expressed as

$$P \equiv Q$$

Below is a table of useful logical equivalencies

Table 5: Logical Equivalencies

Name	Equivalence
Identity	$P \wedge T \equiv P$
	$P \vee F \equiv P$
Idempotent	$P \wedge P \equiv P$
	$P \vee P \equiv P$
Domination	$P \vee T \equiv T$
	$P \wedge F \equiv F$
Negation	$P \vee \neg P \equiv T$
	$P \wedge \neg P \equiv F$
Double Negation	$\neg(\neg P) \equiv P$
Commutative	$P \wedge Q \equiv Q \wedge P$
	$P \vee Q \equiv Q \vee P$
Associative	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
Distributive	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
De Morgan's	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
Absorption	$P \wedge (P \vee Q) \equiv P$
	$P \vee (P \wedge Q) \equiv P$

The use of showing the equivalencies between two compound propositions is called a *conditional-disjunction equivalence*

Example Conditional-Disjunction Equivalence

$$P \implies Q \equiv \neg P \vee Q$$

This can be proven with the use of a truth table 🍷

Continued on Next Page...

Table 6: Conditional-Disjunction Equivalence Proof

P	Q	$\neg P$	$\neg P \vee Q$	$P \implies Q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

1.2 Predicates and Quantifiers

Note

So far, propositional logic can only handle *singular* subjects. It can't handle statements such as:

- All computer science students can program well
- $3x + 4 \geq 0$

Definition Predicate Logic

When you have a proposition that contains a variable. It is typically written as a *propositional function*, $P(x)$, where x is the subject for the predicate P

Example

- $F(x) = "x > 3"$
- $P(x) = "x \text{ looks beautiful!}"$

Note Multi-variable Propositional Functions

Propositional functions may also contain more than just one argument:

$$P(x, y)$$

Definition Quantifiers

They define the range of which a proposition holds **true**, and can be nested to produce *nested quantifiers*

Table 7: Quantifiers in Discrete Mathematics

Quantifier	Expression	In English
Universal Quantifier	$\forall x.P(x)$	$P(x)$ is true for every x in its domain
	$\exists x.P(x)$	There exists x where $P(x)$ is true
Existential Quantifier	$\nexists x.P(x)$	There does not exist x where $P(x)$ is true
	$\exists! x.P(x)$	There exists <i>only one</i> x where $P(x)$ is true

$\forall x.P(x)$ may also be written as $\forall x P(x)$, and the same holds for the other quantifiers

Example Nested Quantifiers

$$\forall x \exists y. (x + y = 0)$$

Translates to: “For *every* x there *exists* a y such that $x + y = 0$ ”

1.3 Inference Rules and Proofs**Definition Argument**

A sequence of statements that have a conclusion

Definition Premise

The preceding statements of a mathematical argument that lead to a conclusion

Definition Valid

The conclusion, the final statement of the argument, must follow from its premises
(i.e. premises \implies conclusion)

Definition Fallacy

Incorrect reasoning in discrete mathematics that leads to an invalid argument

Example Argumentative Form

Arguments may be written as this: $((P \implies Q) \wedge P) \implies Q$, or in argumentative form:

$$\begin{array}{c} P \implies Q \\ P \\ \hline \therefore Q \end{array}$$

The next page contains a table of inference rules

Table 8: Rules of Inference

Rule	Expression	Tautology
Modus ponens	P	
	$\frac{P \implies Q}{P \implies Q}$	$(P \wedge (P \implies Q)) \implies Q$
	$\therefore Q$	
Modus tollens	$\neg Q$	
	$\frac{P \implies Q}{P \implies Q}$	$(\neg Q \wedge (P \implies Q)) \implies \neg P$
	$\therefore \neg P$	
Hypothetical syllogism	$P \implies Q$	
	$\frac{Q \implies R}{Q \implies R}$	$((P \implies Q) \wedge (Q \implies R)) \implies (P \implies R)$
	$\therefore P \implies R$	
Disjunctive syllogism	$P \vee Q$	
	$\frac{\neg P}{\neg P}$	$((P \vee Q) \wedge \neg P) \implies Q$
	$\therefore Q$	
Addition	P	
	$\frac{P}{P \vee Q}$	$P \implies (P \vee Q)$
	$\therefore P \vee Q$	
Simplification	$P \wedge Q$	
	$\frac{P \wedge Q}{P}$	$(P \wedge Q) \implies P$
	$\therefore P$	
Conjunction	P	
	$\frac{Q}{Q}$	$((P) \wedge (Q)) \implies P \wedge Q$
	$\therefore P \wedge Q$	
Resolution	$P \vee Q$	
	$\frac{\neg P \vee R}{\neg P \vee R}$	$((P \vee Q) \wedge (\neg P \vee R)) \implies (Q \vee R)$
	$\therefore Q \vee R$	

Definition Proof

Valid arguments that establish the **truth** of mathematical statements

Table 9: Rules of Inference for Quantified Statements

Rule	Expression
Universal Instantiation	$\frac{\forall x.P(x)}{\therefore P(c)}$
Universal Generalization	$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x.P(x)}$
Existential Instantiation	$\frac{\exists x.P(x)}{\therefore P(c) \text{ for some element } c}$
Existential Generalization	$\frac{P(c) \text{ for an arbitrary } c}{\therefore \exists x.P(x)}$

Definition Theorem

A statement or claim that can be proven using:

- definitions
- other theorems
- axioms
- inference rules

They are also referred to as “Lemma”, “Proposition”, or “Corollary”

There are many different ways of proving a theorem. Let’s assume a conditional statement $P \implies Q$...

Definition Direct Proof

When the first step is the assumption that P is **true** and the following steps that lead up to Q is also **true**

Example Proof by Contraposition

$P \implies Q$ can be proved **true** if $\neg Q \implies \neg P$, its contraposition, can also be proved. This is because a contraposition and a conditional proposition are tautologies

Definition Indirect Proof

Proofs that do not start with the premises and end with conclusion (the opposite of a direct proof). Ways of doing direct proofs involve *proof by contraposition*, *proof by contradiction*, and much more...

Example Proof by Contradiction

To prove P , you must assume $\neg P$ and derive that $\neg P$ is **false**. If $\neg P$ is **false**, then that must mean that $\neg\neg P$, or P , must be **true**

2 Set Theory

Definition set

A collection of *unique* objects or elements

- $a \in A$ to state that a is *contained* in the set A
- $a \notin A$ to indicate that a is *not* contained in A

Note Defining Sets

Roster method

$$A = \{a, b, c, d\}$$

$$B = \{4, b, c, a\}$$

$$C = \{1, 2, \{3.0, 3.5\}, 4, 5\}$$

$$D = \{1, 2, C\}$$

Set Builder Notation

$$E = \{x \in \mathbb{N} \mid x = 2k \text{ for some } k \in \mathbb{N}\}$$

$$F = \{\alpha \mid P(\alpha) \text{ is true}\}$$

Table 10: Important Sets

Set	Expansion	Description
\mathbb{N}	$\{0, 1, 2, 3, \dots\}$	Natural numbers
\mathbb{Z}	$\{\dots, -1, 0, 1, \dots\}$	Integers
\mathbb{Z}^+	$\{1, 2, 3, \dots\}$	Positive integers
\mathbb{Q}	$\{\frac{P}{Q} \mid P \in \mathbb{Z}, Q \in \mathbb{Z}, Q \neq 0\}$	Rational numbers
\mathbb{R}		Real numbers
\mathbb{R}^+	$\{x \mid x > 0\}$	Positive real numbers
\mathbb{C}		Complex numbers
\emptyset		Empty set

2.1 Subsets

Theorem Set Equality

Given two sets A and B :

$$A \equiv B \iff \forall x.(x \in A \iff x \in B)$$

is true

Definition Subset (\subseteq)

A is a subset of B if *all elements of* A are also contained in B :

$$A \subseteq B \iff \forall x.(x \in A \implies x \in B)$$

Example Subsets

Let $A = \{2, 3\}, B = \{a, b, c, 2, 3\}$

- $A \subseteq B$ true
- $\emptyset \subseteq A$ true
- $A \subseteq A$ true
- $B \subseteq A$ false

This notation is seen in Section 2.2

Definition Power Set

The set of all subsets of the set

$$|A| = n, |P(A)| = 2^n$$

Example

Let $S = \{1, 2\}$:

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

Definition Proper Subset (\subset)

It's a subset where $A \neq B$:

$$A \subset B \iff \forall x.(x \in A \implies x \in B) \wedge A \neq B$$

2.2 Cardinality

Definition Cardinality

It's the number of unique elements in a set. It's denoted by $|A|$

Example

$\{\{1, 2\}, 3\}$	2 elements
$\{1, 2, 3\} = \{1, 2, 3, 3\}$	3 elements
$\{\emptyset\}$	1 element
$\{\}$	0 elements
$\{\emptyset\}$	1 element

Definition Intervals

Sets of numbers *between* two numbers a and b if $a, b \in \mathbb{R} \wedge a \leq b$

- $[a, b] = \{x | a \leq x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $(a, b) = \{x | a < x < b\}$

Definition Ordered Tuples

$$(a_0, a_1, a_2, \dots, a_n)$$

An *ordered collection* of unique elements

$$(a, b, c) \neq (b, c, a)$$

Theorem Cartesian Products

Let A and B be sets:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Example Cartesian Product

Let $A = \{1, 2\}$, $B = \{a, b, c\}$:

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

2.3 Set Operations

Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$:

Table 11: Set Operations

Expression	Meaning	Result
$A \cup B$	A union B	$\{1, 2, 3, 4, 5\}$
$A \cap B$	A intersection B	$\{2, 3\}$
\bar{A}	complement of A	$\{x \notin A\}$

Definition Membership Table

It's very similar to a truth table, where **true** means that an element exists within the set, and **false** means that the element isn't.

Table 12: Membership Table Example

A	B	$A \cup B$	$A \cap B$
false	false	false	false
false	true	true	false
true	false	true	false
true	true	true	true

2.4 Functions

Definition Function

Function f is denoted from set A to set B :

$$f : A \rightarrow B$$

representing a relation that assigns each element of A to *exactly one* element of B

Example

Let $A = \{1, 2\}, B = \{a, b\}$

$$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$$

These are functions:

- $f = \{(1, a), (2, b)\}$
- $f = \{(1, a), (2, a)\}$

This isn't:

- $f = \{(1, a), (1, b)\}$

Theorem Equality of Functions

Two functions f and g are equal if:

- They share the same domains / co-domains
- They assign the same element from the domain to the same element

Let $f : A \rightarrow B, g : C \rightarrow D$:

- $A \neq C \vee B \neq D \implies f \neq g$
- $A = C \vee B = D \implies f = g$

Definition One-to-One Functions (Injective)

A function $f : A \rightarrow B$ where all values A must correspond to *one and only one* element in B :

$$\forall a \in A, \forall b \in B. (f(a) \neq f(b))$$

Definition Onto Functions (Subjective)

A function $f : A \rightarrow B$ where all values B must correspond to *at least an* element in A :

$$\forall b \in B, \exists a \in A. (f(a) = b)$$

Definition Bijection

A function that is *both* one-to-one *and* onto

Definition Inverse of Function

$$f^{-1}(x) = y \iff f(y) = x$$

Definition Composite Functions

$$\text{domain}(f) = \text{range}(g) \implies f \circ g(x) = f(g(x))$$

If the composite of the function exists, then:

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

Definition Floor ($\lfloor x \rfloor$)

The biggest integer $n \leq x$

Definition Ceiling ($\lceil x \rceil$)

The smallest integer $n \geq x$

3 Algorithms

Definition Algorithm

A sequence of *precise* steps used to solve computational problems

Note Input-Output

For every input instance, a correct output *must* be produced

Algorithm 1: Binary Search Algorithm

Input: $A \leftarrow \{A_1, A_2, \dots, A_n\}$ where $A_i \in \mathbb{Z}, k \in \mathbb{Z}$

Output: i where $A_i = k$, or $i = 0 \implies k \notin A$

```

1 left ← 1
2 right ← n
3 while left < right do
4   i ← ⌊(left + right) ÷ 2⌋
5   if  $A_i = k$  then
6     return i
7   else if  $A_i < k$  then
8     left ← i + 1
9   else
10    right ← i
11 return 0

```

Algorithm 2: Insertion Sort

Input: $A \leftarrow \{A_1, A_2, \dots, A_n\}$ where $n \geq 2$

Output: $A \leftarrow \{A_1, A_2, \dots, A_n\}$ where $n \geq 2 \wedge A_i \leq A_{i+1}$

```

1 for i ← 2 to n do
2   key ←  $A_i$ 
3   index ← i - 1
4   while index ≥ 1 ∧  $A_{index} > key$  do
5      $A_{index+1} \leftarrow A_{index}$ 
6     index ← index - 1
7    $A_{index+1} \leftarrow key$ 

```

3.1 Growth of Functions

This can be used to determine the growth of functions, and is also mainly used in the field of *computer science* to analyze algorithms.

Note Proving Big-O

To prove that $f(x) = O(g(x))$, you must prove the existence of a C and k that fulfill the definition:

$$\exists C, k. (\forall x > k. (|f(x)| \leq C \times |g(x)|))$$

Definition Big-O Notation

Let f and g be functions. $f(x)$ is $O(g(x))$ (read “ $f(x)$ is Big-O of $g(x)$ ”) when there are constants C and k (the witnesses) such that:

$$|f(x)| \leq C|g(x)|, x > k$$

This is called the worst-case runtime of an algorithm

Example Identify the Witnesses

Find C, k such that $f(x) = 4x^2 + 2x + 1$ is $O(x^2)$

$$\begin{aligned} g(x) &= n^2 + n^2 + n^2 \\ &= 3n^2 \end{aligned}$$

$$f(x) < g(x) \text{ when } x > 1 \therefore C = 3, k = 1$$

Definition Big-Ω

Let f and g be functions. $f(x)$ is $\Omega(g(x))$ when there are constants C and k such that:

$$|f(x)| \geq C|g(x)|$$

This is called the best-case runtime of an algorithm

Definition Big-θ

Let f and g be functions. $f(x)$ is $\theta(g(x))$ when:

$$f(x) = O(g(x)) \wedge f(x) = \Omega(g(x))$$

This is called the average-case runtime of an algorithm

4 Number Theory

Definition Division

Let $a, b \in \mathbb{Z}$. a divides b ($a|b$) if there exists integer c such that $a \times c = b$

$$a|b, a, b \in \mathbb{Z} \iff \exists c \in \mathbb{Z}. (a \times c = b)$$

Theorem Basic Division Properties

Let $a, b, c \in \mathbb{Z}, a \neq 0$:

- $(a|b) \wedge (a|c) \implies a|(b + c)$
- $a|b \implies \forall c \in \mathbb{Z}. (a|(b \times c))$
- $(a|b) \wedge (b|c) \implies a|c$

Theorem Consequence of Division

$$\forall a, b, c, m, n \in \mathbb{Z}, a \neq 0. ((a|b) \wedge (a|c) \implies (a|(mb + nc)))$$

Let a, b, c, m, n be integers where $a \neq 0$, and suppose $a|b \wedge a|c$. Then, for any integers m and n , $a|mb + nc$.

Theorem Division Algorithm

Let $a \in \mathbb{Z}, d \in \mathbb{Z}^+$. Then there are unique integers q and r where $0 \leq r < d$ such that $a = q \times d + r$

$$\forall a, q, r \in \mathbb{Z}, d \in \mathbb{Z}^+, 0 \leq r < d. (a = q \times d + r)$$

Definition Quotient

$$q = a \text{ div } d$$

Definition Remainder

$$r = a \text{ mod } d$$

Definition Congruency (\equiv)

Let $a, b \in \mathbb{Z}, m \in \mathbb{Z}^+$. a is congruent to $b \text{ mod } m$ if...

- $m|a - b$
- $a \text{ mod } m \equiv b \text{ mod } m$
- $k \in \mathbb{Z}. (a = b + k \times m)$

Theorem Preservation of Congruencies

Let $a, b, c, d \in \mathbb{Z}, m \in \mathbb{Z}^+$. If $a \equiv b \text{ mod } m \wedge c \equiv d \text{ mod } m$, then

- $a + c \equiv b + d \text{ (mod } m)$
- $a \times c \equiv b \times d \text{ (mod } m)$

5 Induction and Recursion

Definition Proof via. Induction

To prove that $P(n)$ is true for $n \in \mathbb{Z}^+$, we must complete these two steps:

- **Basis Step:** Verify that $P(1)$ is **true**
- **Inductive Step:** Show that $P(k) \implies P(k+1), k \in \mathbb{Z}^+$ is **true**

Example Sum to Squares

Prove that the sum of the first positive odd numbers is n^2 :

$$P(n) = (1 + 3 + 5 + \cdots + 2n - 1 = n^2)$$

—

Basis Step: $n = 1$

$$\begin{aligned} 1 &= (1)^2 \\ \therefore P(1) &= \text{true} \end{aligned}$$

Inductive Step: Prove $P(n) \implies P(n+1)$

Assume that $P(n)$ is true, and prove that $P(n+1)$ is also true:

$$1 + 3 + \cdots + 2n - 1 + 2n + 1 = n^2 + 2n + 1 = (n+1)^2$$

$1 + 3 + \cdots + 2n - 1$ is n^2 , so we can cancel out those terms. We're now left with $2n + 1 = 2n + 1$, which is a tautology!

Definition Recursion

A way to express a definition of an element in terms of itself. There's two required steps to recursively define a function:

- **Basis Step:** Specify the value of the function at 0
- **Recursive Step:** Specify a rule for finding its value at an integer from its smaller values at smaller integers

6 Counting

6.1 Counting Basics

Theorem The Product Rule

Suppose a procedure can be broken down into a sequence of multiple tasks. If there are n_1 ways to do the first task, and for each task $k \in n_1$ there are n_2 and continuing on until with the final task having n_a ways to do that task, then there are

$$n_1 \times n_2 \times \cdots \times n_a$$

ways to do that procedure.

Think of nested
for loops

Example Product Rule

An ice-cream shop has 12 different types of cones, and 32 flavors of ice-cream. What is the number of unique single-scoop ice-creams the ice-cream shop can serve?

For each ice-cream cone there are 32 unique flavors of ice-cream that can be served with it. This means that there are

$$12 \times 32 = 384$$

unique single-scoop ice-cream combinations that the shop can serve.

Think of
sequential **for**
loops

Theorem The Sum Rule

Suppose a procedure can be broken down into a sequence of multiple tasks. If a task can be done in one of n_1 ways, or one of n_2 ways, or so on and so forth until one of n_a ways, then there are

$$n_1 + n_2 + \cdots + n_a$$

ways to do the task.

Example Sum Rule

You spot three potential people to steal wallets from. The first person's wallet has 4 different types of credit / reward cards, the second person has 18 (wow that's a large wallet!), and the third only has one. However, you can only take one card in the wallet that you choose. How many possible cards do you have to steal?

$$4 + 18 + 1 = 23$$

different cards to rob 🐱

6.2 Pigeonhole Principle

Theorem The Pigeonhole Principle

If $k \in \mathbb{Z}^+$ and $k + 1$ or more objects are placed into k containers, then there's at least one container containing more than 1 object.

Theorem Pigeonhole Consequence I

A function f from a set of $k + 1$ elements to a set of k elements will *never be* one-to-one

Example Pigeonhole and Ice-cream

For a total count of 23 scoops of ice-cream to be present in 22 cones, there must be at least one cone with two ice-cream scoops

Theorem Generalized Pigeon-hole Principle

If N objects are placed into k containers, then there must be at least one box containing at least N/k objects

$$\left\lceil \frac{N}{k} \right\rceil \geq m$$

Example Pigeons and Cards

From a standard deck of 52 playing cards, how many need to be drawn for there to be at least 5 cards in the same suite?

$$\left\lceil \frac{N}{4} \geq 5 \right\rceil = 21$$

Theorem Pigeonhole Consequence II

Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or decreasing

6.3 Permutations and Combinations

Definition Permutation

A set of distinct objects in an ordered arrangement. An ordered arrangement of n elements of a set is called an n -permutation

Example $S = \{\text{🍎}, \text{🍌}, \text{🐭}\}$

- $\{\text{🍌}, \text{🐭}, \text{🍎}\}$ is a permutation
- $\{\text{🐭}, \text{🍎}\}$ is a 2-permutation

Theorem Number of R-Permutations

If $n \in \mathbb{Z}^+, r \in \mathbb{Z}$ where $1 \leq r \leq n$, then there are

$$P(n, r) = n \times (n - 1) \times (n - 1) \dots \times (n - r + 1)$$

r -permutations with a set of n distinct elements

Note Simplification of the theorem

The theorem can be simplified (and more commonly used) like this:

$$\forall n, r \in \mathbb{Z}. \left(P(n, r) = \frac{n!}{(n - r)!}, 0 \leq r \leq n \right)$$

Definition Combination

An unordered subset with r amount of elements from a set

Example $S = \{\text{🦊}, \text{🍌}, \text{🍌}\}$

- $\{\text{🦊}, \text{🍌}\}$ is a combination
- $\{\text{🦊}, \text{🍌}\}$ is a combination as well

Note Combinations vs. Permutations

$$S = \{\text{🐔}, \text{🐓}, \text{🐓}\}$$

$$\{\text{🐔}, \text{🐓}\}$$

$$\{\text{🐓}, \text{🐔}\}$$

That would be 2 permutations, but *only one* combination

Theorem Number of combinations

$$\forall n, r \in \mathbb{Z}. \left(C(n, r) = C(n, n-r) = \frac{n!}{r!(n-r)!}, 0 \leq r \leq n \right)$$

factorials make everything big!

Example Gold, Silver, Bronze

How many unique ways are there to select a first, second, and third place winners in a contest with 15 contestants?

$$P(15, 3) = \frac{(15)!}{((15) - (3))!} = 2730$$

different ways!

7 Discrete Probability

Definition Finite Probability

If S is a non-empty sample space (set of possible outcomes) of equally likely outcomes, and E is an event (subset of sample space), the probability of E is

$$p(E) = \frac{|E|}{|S|}$$

its range is between 0 and 1, inclusive

Example Diamond Rings and Seventeen

There are two pouches in front of you. In once pouch, there are ten diamond rings, but only four of them are ones that your girlfriend likes. In another, there are a set of 13 Seventeen photocards—one for each member in the group. Only one of them is Minghao. What’s the probability of getting a diamond ring that your girlfriend likes and a Seventeen photocard?

—

The probability of getting a diamond ring that your girlfriend likes is

$$p(E_1) = \frac{4}{10} = \frac{2}{5}$$

While the probability of getting a Minghao photocard is

$$p(E_2) = \frac{1}{13}$$

To get the combined probability, remember the Product Rule

$$p(E) = \frac{2}{5} \times \frac{1}{13} = \frac{2}{65}$$

Theorem Complementary Probability

Let E be an event in a sample space S . The probability of the event $\bar{E} = S - E$, the complementary event of E is

$$p(\bar{E}) = 1 - p(E)$$

Theorem Union of Two Events

The union of two events E_1 and E_2 in a sample space S is

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Example Monty Hall Three-Door Puzzle

You’re in a game show, and the host invites you to open one of three doors that are in front of you. Behind only one of those doors is a scholarship that will pay for the rest of your education at UTD. After picking one of those doors, the game show host opens a door that a prize isn’t behind, and asks if you want to switch again. What should you do?

—

The odds that one of the doors has the prize is $p(E) = \frac{1}{3}$. However, the game show host opening one of the doors without the prize in it raises the odds to the other door that you didn’t pick to $p(E) = \frac{2}{3}$. This is because the game show host is taking the set of 2 doors that you didn’t pick and keeping only the best door for you.

Basically, the door you initially picked is just normal and plain, but the other one survived the gauntlet in a battle against all the other doors and won.

8 Relations

Definition Binary Relation

Given sets A, B , it's a subset of $A \times B$

Example

$$\begin{aligned} R_1 &= \{(a, b) | a, b \in \mathbb{Z} \wedge a \leq b\} \subseteq \mathbb{Z} \times \mathbb{Z} \\ R_2 &= \{(a, b) | a, b \in \mathbb{Z} \wedge a < b\} \subseteq \mathbb{Z} \times \mathbb{Z} \\ R_3 &= \{(a, b) | a, b \in \mathbb{Z} \wedge a = b\} \subseteq \mathbb{Z} \times \mathbb{Z} \\ R_4 &= \{(a, b) | a, b \in \mathbb{Z} \wedge a = b + \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z} \\ R_5 &= \{(a, b) | a, b \in \mathbb{Z} \wedge a|b\} \subseteq \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

Definition Reflexivity

A relation R on a set A is *reflexive* if and only if:

$$\forall x. (x \in A \implies (x, x) \in R)$$

Example $A = \{\text{👤}, \text{👤}\}$

This relation is reflexive:

$$R = \{(\text{👤}, \text{👤}), (\text{👤}, \text{👤}), (\text{👤}, \text{👤})\}$$

However, this one isn't:

$$R = \{(\text{👤}, \text{👤}), (\text{👤}, \text{👤}), (\text{👤}, \text{👤})\}$$

Example $A = \{\text{🍎}, \text{🍌}, \text{🍌}, \text{🍌}\}$

This one is symmetric:

$$R = \{(\text{🍌}, \text{🍌}), (\text{🍌}, \text{🍌}), (\text{🍌}, \text{🍌})\}$$

This one isn't, however:

$$R = \{(\text{🍌}, \text{🍌}), (\text{🍌}, \text{🍌}), (\text{🍌}, \text{🍌})\}$$

Definition Symmetry

Relation R on a set A is symmetric if and only if

$$\forall x \forall y. ((x, y) \in R \implies (y, x) \in R)$$

Definition Anti-Symmetry

Relation R on a set A is antisymmetric if and only if

$$\forall x \forall y. ((x, y) \in R \wedge (y, x) \in R \implies x = y)$$

Example

\leq	Not symmetric: $3 \leq 5, 5 \not\leq 3$
	Antisymmetric: $a \leq b, b \leq a \implies a = b$
$>$	Not symmetric: $5 > 3, 3 \not> 5$
	Antisymmetric: $a > b, b > a \implies a = b$
$=$	Symmetric
	Antisymmetric: $a = b, b = a \implies a = b$
$\{(a, b) a, b \in \mathbb{Z} \wedge a = -b\}$	Symmetric: $(a, -a) \in R \implies (-a, a) \in R$
	Not antisymmetric: $(a, -a) \wedge (-a, a) \in R, a \neq -a$

Definition Transitive

A relation R on set A such that

$$\forall x \forall y \forall z. ((x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R)$$

Example

$$A = \{a, b, c\}$$

$$R = \{(a, b), (a, a), (b, b), (b, c), (c, b)\}$$

It's nothing, nada, it's...disappointing

Note Equivalence Relation

A relation that is reflexive, symmetric, and transitive

9 Graphs

Definition Graph

Denoted by $G = (V, E)$, consists of a non-empty set of *vertices* (or *nodes*) V and a set of edges E

What's a graph without its nodes?

Definition Adjacent Vertices (Undirected)

When the endpoints of an edge e are vertices a and b , the vertices are considered to be *adjacent* (or *neighbors*) with one another.

That edge is called an *incident edge*, *connecting* vertices a and b

Definition Adjacent Vertices (Directed)

When (a, b) is an edge of a directed graph G , a is *adjacent to* b , and a is *adjacent from* b .

Vertex a is called the *initial vertex*, and b is called the *terminal* or *end vertex* of (u, v)

Definition Neighborhood ($N(v)$)

"The set of all neighbors of u " of a graph $G = (V, E)$, denoted by $(N(v))$

If A is a subset of V , $N(A)$ is the set of vertices in G that are adjacent to at least one vertex in A :

$$V = \bigcup_{v \in A} N(v)$$

Definition Degree (Undirected Graph) ($\deg(v)$)

The number of edges incident with vertex v . A loop contributes *twice* to the count

Definition Degree (Directed Graph) ($\deg^+(v)$)

$\deg^-(v)$ number of edges pointing into v

$\deg^+(v)$ number of edges pointing out from v

Example Neighborhoods and Degrees

Figure 1: Neighborhoods Example

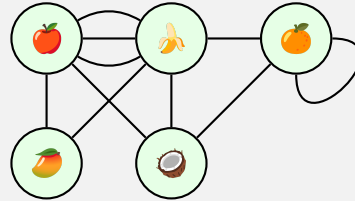


Table 13: Table of Neighbors and Degrees

Vertex	Degree	Neighborhood
	2	{,
	3	{, ,
	4	{, , ,
	5	{, , ,
	6	{, , ,

Note Sum of Degrees

When adding the degrees of all vertices in graph $G = (V, E)$, you'll find that each edge contributes *two* to the sum of degrees, since each edge is incident with exactly two vertices

Theorem Handshake Theorem

If $G = (V, E)$ is an undirected graph with $|E| = m$ edges, then

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v) = 2 \times m$$

This applies even if multiple edges and loops are present

Example Handshake

We can see this theorem in effect if we look back at Figure 1:

$$\begin{aligned}2 \times m &= \sum_{v \in V} \deg(v) \\ &= 2 + 3 + 4 + 5 + 6 \\ 2 \times m &= 20 \\ m &= 10\end{aligned}$$

Theorem Consequences of Handshaking

In an undirected graph, the sum of degree of nodes with odd degrees is *even*.

Theorem Directed Handshake

Let $G = (V, E)$ be a directed graph:

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

Table 14: Graph Terminology

Type	Direction	Edges	Loops
Simple Graph	Undirected	Singular	No
Multigraph	Undirected	Multiple	No
Pseudograph	Undirected	Multiple	Yes
Simple Directed Graph	Directed	Singular	No
Directed Multigraph	Directed	Multiple	Yes
Mixed Graph	Both	Multiple	Yes

Figure 2: Simple

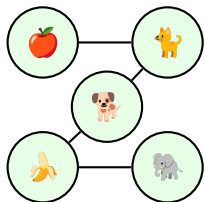


Figure 4: Pseudograph

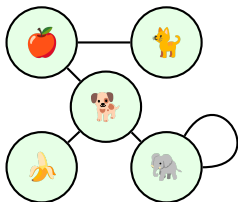


Figure 6: Dir. Multigraph

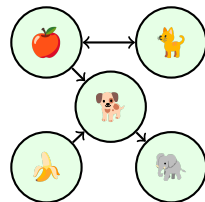


Figure 3: Multigraph

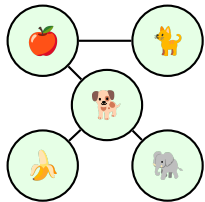


Figure 5: Simple Directed

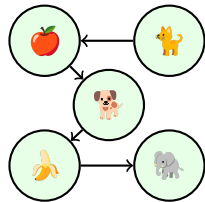
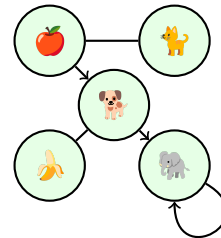


Figure 7: Mixed Graph



Definition Infinite Graphs

If $|V| = \infty$, then the graph is referred to as an *infinite* graph, otherwise it's called a *finite* graph

Table 15: Special Graphs

Graph	Definition
Complete Graph	For n nodes, there exists <i>exactly one</i> edge between any pair of nodes
Cyclical Graph	For $n \geq 3$, the graph C_n consists of path: $(v_1, v_2), (v_2, v_3) \dots, (v_{n-1}, v_n), (v_n, v_1)$
Bipartite Graph	A simple graph $G = (V, E)$ such that V can be partitioned into two disjoint subsets V_1 and V_2
Complete Bipartite Graph	A bipartite graph where $(u, v), u \in V_1, v \in V_2$
Subgraph	Let $G = (V, E)$. $H = (A, B)$ is a subgraph of G if $A \subseteq V \wedge B \subseteq E$

Figure 8: Complete Graph

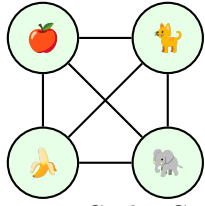


Figure 9: Cycles Graph

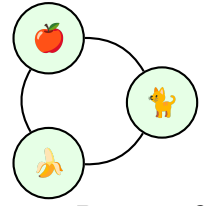


Figure 10: Bipartite Graph

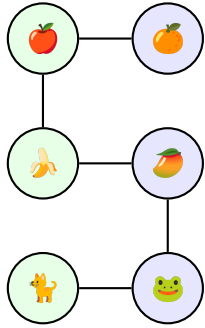


Figure 11: Complete Bipartite Graph

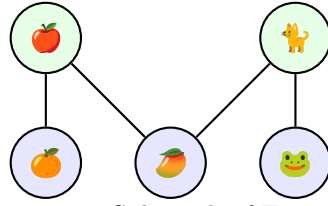


Figure 12: Subgraph of Figure 8

