Speed Running Calculus

Minh Nguyen

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Contents

Author's Notes

This actually isn't the first copy of my discrete math notes...I had an earlier copy that got corrupted, so now I'm just going to constantly upload to GitHub because I'm afraid of them getting wiped again 😭

These notes are based off of the textbook *Discrete Mathematics and Its Applications* by *Kenneth H. Rosen*, as well as from the lectures of *Serdar Erbatur* from the *Fall 2023* semester.

If you have any complaints, or suggestions regarding these notes, please email me at mdn220004@utdallas.edu

1 Propositional Logic

Definition **Proposition**

Declarative statements that are either `true` or `false`

Table 1: Examples of propositions

Statement	Proposition?
$1 + 1 = 2$	✔️
What class are ya takin?	⛔
I am happy	✔️

Propositons can be combined to form another proposition with the use of *logical operators*

Note

Common labels for propositions in discrete mathematics include letters like P, Q, R, S...

Table 2: Logical Operators in Discrete Mathematics

Symbol	Meaning	Expression
$\neg$	negation / not	$\neg P$
$\wedge$	conjunction / and	$P \wedge Q$
$\vee$	disjunction / or	$P \vee Q$
$\oplus$	exclusive disjunction/ xor	$P \oplus Q$
$\implies$	implication / conditional	$P \implies Q$
$\iff$	biconditional	$P \iff Q$

Note **Logical Operator Precedence**

Order matters when it comes to evaluating logical operators:

1. negaiton
2. conjunction
3. disjunctions
4. conditionals

Where negation is evaluated first, and conditionals last.

Theorem More on Conditionals

Conditionals can also be expressed in three other ways:

Contrapositive: $\neg Q \implies \neg P$

Converse: $Q \implies P$

Inverse: $\neg P \implies \neg Q$

Example

Conditional: If it is raining, then I am not going to town

Contrapositive: If I go to town, then it is not raining

Converse: If I do not go to town, then it is raining

Inverse: If it is not raining, then I am going to town

Definition **Truth Table**

They are used as a way of seeing all possible values of a proposition

Table 3: Truth Table of $P \implies Q$

P	Q	$P \implies Q$
F	F	T
F	T	T
T	F	F
T	T	T

1.1 Propositional Equivalencies

Definition **Equivalencies**

A proposition is a...

- **Tautology** if it is `true` in *every case*
- **Contradiction / Fallacy** if it is `false` in *every case*
- **Contingency** if *neither* is the case

Table 4: Example of Tautology and Contradiction

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

Definition **Logical Equivalency ($\equiv$)**

Compound propositions P and Q are *logically equivalent* if $P \iff Q$ is a *tautology*. This is expressed as

$$P \equiv Q$$

Below is a table of useful logical equivalencies

Table 5: Logical Equivalencies

Name	**Equivalence**
Identity	$P \wedge \mathrm{T} \equiv P$
	$P \vee \mathrm{F} \equiv P$
Idempotent	$P \wedge P \equiv P$
	$P \vee P \equiv P$
Domination	$P \vee \mathrm{T} \equiv \mathrm{T}$
	$P \wedge \mathrm{F} \equiv \mathrm{F}$
Negation	$P \vee \neg P \equiv \mathrm{T}$
	$P \wedge \neg P \equiv \mathrm{F}$
Double Negation	$\neg(\neg P) \equiv P$
Commutative	$P \wedge Q \equiv Q \wedge P$
	$P \vee Q \equiv Q \vee P$
Associative	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
Distributive	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
De Morgan's	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
Absorption	$P \wedge (P \vee Q) \equiv P$
	$P \vee (P \wedge Q) \equiv P$

The use of showing the equivalencies between two compound propositions is called a *conditional-disjunction equivalence*

Example **Conditional-Disjunction Equivalence**

$$P \implies Q \equiv \neg P \vee Q$$

This can be proven with the use of a truth table 🥰

Continued on Next Page...

Table 6: Conditional-Disjunction Equivalence Proof

P	Q	$\neg P$	$\neg P \vee Q$	$P \implies Q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

1.2 Predicates and Quantifiers

Note

So far, propositional logic can only handle *singular* subjects. It can't handle statements such as:

- All computer science students can program well
- $3x + 4 \geq 0$

Definition **Predicate Logic**

When you have a proposition that contains a variable. It is typically written as a *propositional function*, $P(x)$, where x is the subject for the predicate P

Example

- $F(x) =$ "$x > 3$"
- $P(x) =$ "x looks beautiful!"

Note **Multi-variable Propositional Functions**

Propositional functions may also contain more than just one argument:

$$P(x, y)$$

Definition **Quantifiers**

They define the range of which a proposition holds `true`, and can be nested to produce *nested quantifiers*

$\forall x.P(x)$ may also be written as $\forall x P(x)$, and the same holds for the other quantifiers

Table 7: Quantifiers in Discrete Mathematics

Quantifier	**Expression**	**In English**
Universal Quantifier	$\forall x.P(x)$	$P(x)$ is `true` for every x in its domain
Existential Quantifier	$\exists x.P(x)$	There exists x where $P(x)$ is `true`
	$\nexists x.P(x)$	There does not exist x where $P(x)$ is `true`
	$\exists! x.P(x)$	There exists *only one* x where $P(x)$ is `true`

Example **Nested Quantifiers**

$$\forall x \exists y.(x + y = 0)$$

Translates to: "For *every* x *there exists* a y such that $x + y = 0$"

1.3 Inference Rules and Proofs

Definition **Argument**

A sequence of statements that have a conclusion

Definition **Valid**

The conclusion, the final statement of the argument, must follow from its premises
(i.e. premises $\implies$ conclusion)

Definition **Premise**

The preceeding statements of a mathematical argument that lead to a conclusion

Definition **Fallacy**

Incorrect reasoning in discrete mathematics that leads to an invalid argument

Example **Argumentative Form**

Arguments may be written as this: $\big((P \implies Q) \wedge P\big) \implies Q$, or in argumentative form:

$$\begin{array}{l} P \implies Q \\ \underline{P \qquad\qquad} \\ \therefore Q \end{array}$$

The next page contains a table of inference rules

Table 8: Rules of Inference

Rule	Expression	Tautology
Modus ponens	$\begin{array}{ll} & P \\ & \underline{P \implies Q} \\ \therefore & Q \end{array}$	$\big(P \land (P \implies Q)\big) \implies Q$
Modus tollens	$\begin{array}{ll} & \neg Q \\ & \underline{P \implies Q} \\ \therefore & \neg P \end{array}$	$\big(\neg Q \land (P \implies Q)\big) \implies \neg P$
Hypothetical syllogism	$\begin{array}{ll} & P \implies Q \\ & \underline{Q \implies R} \\ \therefore & P \implies R \end{array}$	$\big((P \implies Q) \land (Q \implies R)\big) \implies (P \implies R)$
Disjunctive syllogism	$\begin{array}{ll} & P \lor Q \\ & \underline{\neg P} \\ \therefore & Q \end{array}$	$\big((P \lor Q) \land \neg P\big) \implies Q$
Addition	$\begin{array}{ll} & \underline{P} \\ \therefore & P \lor Q \end{array}$	$P \implies (P \lor Q)$
Simplification	$\begin{array}{ll} & \underline{P \land Q} \\ \therefore & P \end{array}$	$(P \land Q) \implies P$
Conjunction	$\begin{array}{ll} & P \\ & \underline{Q} \\ \therefore & P \land Q \end{array}$	$\big((P) \land (Q)\big) \implies P \land Q$
Resolution	$\begin{array}{ll} & P \lor Q \\ & \underline{\neg P \lor R} \\ \therefore & Q \lor R \end{array}$	$\big((P \lor Q) \land (\neg P \lor R)\big) \implies (Q \lor R)$

Table 9: Rules of Inference for Quantified Statements

Rule	Expression
Universal Instantiation	$\begin{array}{ll} & \underline{\forall x.P(x)} \\ \therefore & P(c) \end{array}$
Universal Generalization	$\begin{array}{ll} & \underline{P(c) \text{ for an arbitrary } c} \\ \therefore & \forall x.P(x) \end{array}$
Existential Instantiation	$\begin{array}{ll} & \underline{\exists x.P(x) \qquad\qquad\quad} \\ \therefore & P(c) \text{ for some element } c \end{array}$
Existential Generalization	$\begin{array}{ll} & \underline{P(c) \text{ for an arbitrary } c} \\ \therefore & \forall x.P(x) \end{array}$

Definition **Proof**

Valid arguments that establish the `truth` of mathematical statements

Definition **Theorem**

A statement or claim that can be proven using:

- definitions
- other theorems
- axioms
- inference rules

They are also referred to as “Lemma”, “Proposition”, or “Corollary”

There’s many different ways of proving a theorem. Let’s assume a conditional statement $P \implies Q$...

Definition **Direct Proof**

When the first step is the assumption that P is `true` and the following steps that lead up to Q is also `true`

Definition **Indirect Proof**

Proofs that do not start with the premimses and end with conclusion (the opposite of a direct proof). Ways of doing direct proofs involve *proof by contraposition*, *proof by contradiction*, and much more...

Example **Proof by Contraposition**

$P \implies Q$ can be proved `true` if $\neg Q \implies P$, its contraposition, can also be proved. This is because a contraposition and a conditional proposition are tautologies

Example **Proof by Contradiciton**

To prove P, you must assume $\neg P$ and derive that $\neg P$ is `false`. If $\neg P$ is `false`, then that must mean that $\neg\neg P$, or P, must be `true`

2 Set Theory

2.1 Cardinality

2.2 Set Operations

2.3 Functions

2.4 Floor and Ceiling Functions

3 Algorithms