Speed Running Calculus

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Contents

Author's Notes

This is my very first project that I wrote in LaTeX, but I hope you enjoy the notes that I poured my blood, sweat, and tears into 😭

These notes are based off of the textbook *Calculus 11e* by *Ron Larson* and *Bruce Edwards*, as well as the lectures of professor *Carlos Arreche* from the *Fall 2023* semester.

Practice problems are also found within the notes. The answers for the practice problems are found at the end of the notes. Please try them yourself so that you can get a hang of the subject!

If you have any complaints, or suggestions regarding these notes, please email me at mdn220004@utdallas.edu

Thank you so much to Jing Guo and Aaron Kelly for contributing to some of the practice problems as well as corrections for these notes. You have my sincere gratitude!

Class Specific Information

The first exam will cover up to section 3.3 on September 29, 2023. I recommend that you also do the exam review on top of the practice problems in these notes if you so choose. Good luck!

1 Limits

1.1 Finding Limits Graphically & Numerically

Theorem **Informal Definition of Limit**

$$\lim_{x \to c} f(x) = L$$

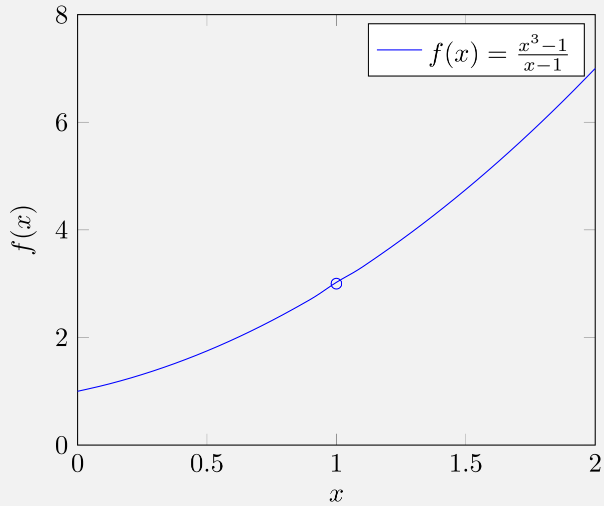
As x gets closer to c, the value of $f(x)$ becomes L

Limits also depend on what direction[1] you're coming from:

Meaning	Math Expression
From the right	$\lim_{x \to c^+} f(x)$
From the left	$\lim_{x \to c^-} f(x)$

Example **Estimating Graphically and Numerically**

Consider the function $f(x) = \frac{x^3-1}{x-1}$:



x	0.9	0.99	0.999	1	1.001	1.01	1.1
f(x)	2.71	2.97	2.997	DNE	3.003	3.03	3.31

Notice how in both the **table and graph**, $f(x)$ looks like it's approaching $f(x) = 3$ when $x = 1$

1.1.1 When Limits Fail to Exist

Note **Limits Don't Exist When**

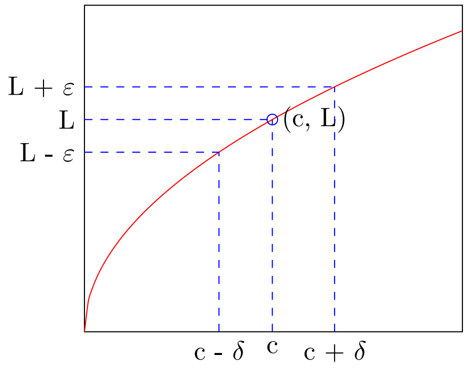
- $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$
- $f(x)$ is a violently oscillating function

[1] If the limits from the left and right don't match, then the limit *doesn't exist*

1.1.2 A Formal Definition

Theorem **Limits via.** $\varepsilon - \delta$

The statement $\lim_{x \to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$



Refer back to finding limits numerically (example 1.1):

- $\pm\varepsilon$ would be the values on the row $f(x)$
- $\pm\delta$ would be the values on the row x

1|You Try **Proving Limits via.** $\varepsilon - \delta$ **Definition**

Consider $f(x) = 10x - 6$, prove that $\lim\limits_{x \to 3} f(x) = 24$ using the $\epsilon - \delta$ definition

1.2 Evaluating Limits Analytically

1.2.1 Properties of Limits

$\mathbb{R}$ =all real numbers
$\mathbb{N}$ =all natural numbers

Theorem **Basic Limit Properties**

Let $\{b, c\} = \mathbb{R}$, $n = \mathbb{N}$:

1. $\lim_{x \to c} x = c$
2. $\lim_{x \to c} b = b$
3. $\lim_{x \to c} x^n = c^n$

Example **Evaluating Basic Limits**

$$\lim_{x \to 2} 5 = 5$$
$$\lim_{x \to 4} x = 4$$
$$\lim_{x \to 5} x^2 = 25$$

$f(g(x))$ may also be written as $f \circ g$

Theorem **Limit Properties**

Let $\{b, c\} = \mathbb{R}$, $n = \mathbb{N}$, and f and g are functions with limits:

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. $\lim_{x \to c}[b \cdot f(x)] = b \cdot L$
2. $\lim_{x \to c}[b \pm f(x)] = b \pm L$
3. $\lim_{x \to c}[f(x) \cdot g(x)] = L \cdot K$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}, K \neq 0$
5. $\lim_{x \to c}[f(x)]^n = L^n$
6. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$
7. $\lim_{x \to c} f\big(g(x)\big) = f\big(\lim_{x \to c} g(x)\big) = f(K)$

Example **Limit of Polynomial**

$$\begin{aligned} \lim_{x \to 5}[3x^3 + 4] &= \lim_{x \to 5} 3x^3 + \lim_{x \to 5} 4 \\ &= 3 \cdot \lim_{x \to 5} x^3 + \lim_{x \to 5} 4 \\ &= 3 \cdot (5)^3 + 4 = 379 \end{aligned}$$

Theorem **Limits of Polynomial/Rationals**

Let $c = \mathbb{R}$ and p be a polynomial function:

$$\lim_{x \to c} p(x) = p(c)$$

Let r be a rational function $r(x) = \frac{p(x)}{q(x)}$ and $c = \mathbb{R}$ such that $q(c) \neq 0$:

$$\lim_{x \to c} r(x) = r(c) = \frac{p(r)}{q(r)}$$

1.2.2 Squeeze Theorem

Open/closed interval continuity will be discussed later in section 1.3.1.

Theorem **The Squeeze Theorem**

Suppose $h(x) \leq f(x) \leq g(x)$ for all x in an open interval *except* when $x = c$

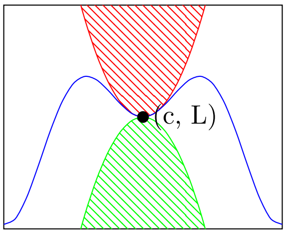
Also suppose that $\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$ i.e. they share the same limit.

This would mean that $\lim_{x\to c} f(x) = L$

Theorem **Trigonometric Limits**

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Squeeze Theorem



1|You Try **Squeeze Theorem**

Show that $\lim_{x\to 0} (\cos(\frac{2\pi}{3x}) \cdot \sqrt{x^3 + x^2}) = 0$ using the Squeeze Theorem

2|You Try **Trigonometric Limit**

Evaluate:

$$\lim_{x\to 0} \frac{\tan(3x)}{4x + \sin(2x)}$$

Hint: Break it down to the trigonometric functions you know how to take a limit of!

1.3 Continuity

Definition **Definition of Continuity**

A function f is continuous if:

1. $f(c)$ exists
2. $\lim_{x\to c} f(x)$ exists
3. $f(c) = \lim_{x\to c} f(c)$

1|You Try **Determine Continuity**

Determine if the function is continuous at $x = 2$

Note **Continuous Functions**

The following functions are always continuous *everywhere they're defined*:

- polynomial functions
- rational functions
- radical functions
- trigonomic functions

1.3.1 Open and Closed Intervals

A function is continuous on an **open interval** (a, b) if $f(x) = c$ for each c in (a, b)
A function is continuous on a **closed interval** $[a, b]$ if:

- $f(x)$ is continuous on (a, b)
- $\lim_{x\to b^-} f(x) = f(b)$
- $\lim_{x\to a^+} f(x) = f(a)$

Theorem **Intermediate Value Theorem**

If $f(x)$ is continuous on $[a, b], a \neq b$, and k is any number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that:

$$f(c) = k$$

1|You Try **Intermediate Value Thorem**

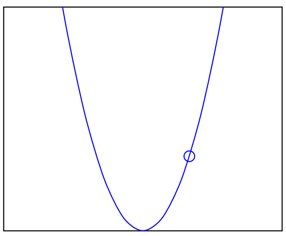
Use the Intermediate Value Theorem to prove that $x^3 + x^2 = 1$ has at least one solution on the interval $(-1, 2)$

1.3.2 Discontinuities

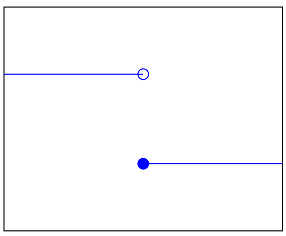
There are two cases where discontinuities happen:

Another example of removable discontinuity is in example 1.1

Removable Discontinuity



Non-Removable Discontinuity



1.3.3 Asymptotes

Definition **Asymptotes**

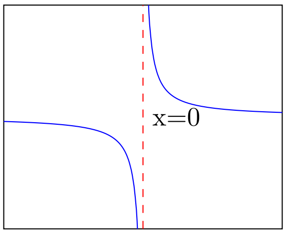
Vertical asymptotes are when:

$$\lim_{x \to c^{\pm}} f(x) = \pm\infty$$

Horizontal asymptotes are when:

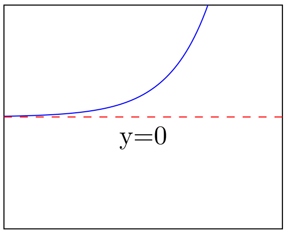
$$\lim_{x \to \pm\infty} f(x) = L$$

Vertical Asymptote



$f(x) = \frac{1}{x}$

Horizontal Asymptote



$f(x) = 2^x$

Note

We can infer something from vertical asymptotes from this graph: As the denominator becomes closer to zero, and it's a positive number, then the $f(x)$ will approach ∞. If the denominator approaches zero and is negative, then $f(x)$ will approach $-\infty$

1|You Try **Curveball Asymptote**

Find the asymptotes:

$$f(x) = \frac{\sqrt{(x-1)(x-3)}}{(x-2)(x-4)}$$

2|You Try **Curveball Trigonometry**

Find the asymptotes:

$$f(x) = \frac{\sin(x)}{x^3 - x}$$

2 Differentiation

Derivatives are essentially the *slope* of the function at a certain point
They also *cannot exist* where the limit doesn't exist at the function

This comes in handy when solving physics problems!

Note **Derivatives and Rate of Change**

To elaborate more on this, the derivative of a function $f(x)$ at point $x = c$ is the instantaneous rate of change:

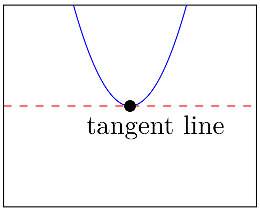
$$f'(x)\Big|_{x=c}$$

The average rate of change of the function on the interval $[a, b]$ is written as such:

$$\frac{f(b) - f(a)}{b - a}$$

2.1 Derivatives and Tangent Lines

Some mathematicians were trying to find out how to draw a line that intersects a function at *only one point*:



However, it takes *two points* to draw a line, so they were confuzzled.
You can just Google up the rest of the lore behind the definition of a limit, but it boils down to this:

Theorem **Derivative via. Limits**

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(c) \text{ and } \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(c)$$

Note **Alternative Ways of Writing a Derivative**

There are other ways that mathematicians defined derivative:

- $f'(x)$
- $\frac{dy}{dx}$
- $\frac{d}{dx}f(x)$
- $Dx(y)$

This isn't essential to know, but it's pretty useful to see how other mathematicians may express derivatives

Theorem **Continuity of Derivatives**

If function $f(x)$ is differentiable at $x = c$, then it is *also continuous* at $x = c$

1|You Try **Finding the Tangent Line**

Find the tangent lines to $f(x) = x^2 + 1$ at $(-2, 5)$

2.2 Rules of Derivatives

Finding derivatives with the limit definition can get pretty exhausting, so mathematicians came up with a lot of shortcuts to evaluate derivatives much quicker. Here are the rules in short:

Theorem **Derivative Rules**

$$\begin{aligned} \text{Let: } f(x) &= \text{a function} \\ g(x) &= \text{another function} \\ c &= \text{a constant} \end{aligned}$$

1.	Constant	$\frac{d}{dx}c = 0$
2.	Power	$\frac{d}{dx}x^n = nx^{n-1}$
3.	Sum and Difference	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
4.	Product	$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
5.	Quotient	$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\big(g(x)\big)^2}$
6.	Trigonometry	$\frac{d}{dx}\sin(x) = \cos(x)$
		$\frac{d}{dx}\cos(x) = -\sin(x)$
		$\frac{d}{dx}\tan(x) = \sec^2(x)$
		$\frac{d}{dx}\cot(x) = -\csc^2(x)$
		$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
		$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$
7.	Chain	$\frac{d}{dx}f(g(x)) = g'(x)\cdot f'(g(x))$

You don't have to simplify completely on open-ended questions!

Example **Power Rule**

1. $\dfrac{d}{dx}(3x^2 + 4x) = 6x + 4$
2. $\dfrac{d}{dx}\sqrt{x} = \dfrac{d}{dx}x^{\frac{1}{2}} = \dfrac{1}{2}x^{-\frac{1}{2}}$
3. $\dfrac{d}{dx}\sqrt[3]{x^2} = \dfrac{d}{dx}x^{\frac{2}{3}} = x^{-\frac{1}{3}}$

Example **Product and Quotient Rule**

1. $\dfrac{d}{dx}(2x+3)(3x^2+1) = (2x+3)(6x) + (3x^2+1)(2)$
2. $\dfrac{d}{dx}\dfrac{x^2+3}{2x-1} = \dfrac{(2x-1)(2x) - (x^2+3)(2)}{(2x-1)^2}$

u-substitution is a good method for solving chain rule problems:

Example **Chain Rule**

$$\frac{d}{dx}\sin(x^2+1)$$

$$u = x^2+1, \qquad v = \sin(u)$$
$$u' = 2x, \qquad v' = \cos(u)$$
$$v' = \cos(x^2+1)$$

$$= u' \cdot v'$$
$$= 2x \cdot \cos(x^2+1)$$

Remember: $\sin^2(\theta) + \cos^2(\theta) = 1$

Example $\sin$ **and** $\cos$ **Rule**

1. $\frac{d}{dx}\big(2\sin(\theta) + 3\cos(\theta) - \frac{4}{\theta}\big) = 2\cos(\theta) - 3\sin(\theta) + \frac{4}{\theta^2}$
2. $\frac{d}{dx}\big(x^2 + 2\cos(x)\big) = 2x - 2\sin(x)$
3. $\frac{d}{d\theta}\big(\tan(\theta)\big) = \frac{d}{d\theta}\big(\frac{\sin(\theta)}{\cos(\theta)}\big) = \frac{\cos^2(\theta)+\sin^2(\theta)}{\cos^2(\theta)} = \sec^2(\theta)$

1|You Try **Find Horizontal Tangents**

Find the horizontal tangents of $f(x) = x^4 - 2x^2 + 3$

2|You Try **Find the Derivative**

Find the derivative of $f(x) = \big((x^2+3)^5 + x\big)^2$

2.3 Higher Order Derivatives

You can take a derivative of derivative:

$$\frac{d}{dx}\Big[\frac{d}{dx}f(x)\Big] = \frac{d^2}{dx^2}\big[f(x)\big] = f''(x)$$

Example **Finding the Second Derivative**

Finding $f''(x)$ when $f(x) = \frac{2x}{x+1}$:

$$f'(x) = \frac{2 \cdot (x+1) - (2x) \cdot 1}{(x+1)^2}$$
$$= \frac{2}{(x+1)^2}$$

$$f''(x) = -\frac{4(x+1)}{(x+4)^4}$$

1|You Try **Velocity and Acceleration**

A ball is thrown up into the air, and its position can be modeled into of feet as a function of time: $f(t) = -5(t-2)^2 + 20$. Find the acceleration of the ball just as it's about to hit the ground again.

Note **Taking Beyond the Third Derivative**

Taking $f'''(x)$ and beyond, it'll start to be tedious writing all of those apostrophes. Mathematicians often write it like this:

$$\frac{d^n}{dx^n}\big[f(x)\big] = f^{(n)}(x)$$

2.4 Implicit Differentiation

So far, we've been using *explicit differentiation*, which is when y can be isolated on one side of the equation where y is a function of x.
What if that's not possible?
That's where *implicit differentiation comes through*. Here are the steps on how to do that:

Note **Steps for Implicit Differentiation**

1. For all terms of x and y, derive in respect to x: $(\frac{d}{dx})$
2. Isolate all terms with $\frac{dy}{dx}$
3. Factor out $\frac{dy}{dx}$ where the terms with $\frac{dy}{dx}$ are isolated
4. Solve for $\frac{dy}{dx}$

It's also important to understand that derivative rules don't have to just apply to terms of x. The example on the next page shows an implicit differentiation using the product rule on x *and* y:

$\frac{dx}{dx} = 1$. I'm only showing it in this example $\frac{dx}{dx}$

Example **Implicit Differentiation with Product Rule**

Let's follow those steps to find the derivative of $x^2y + y^2x = -2$:
Begin by deriving all terms in respect to x:

$$\begin{aligned} x^2y + y^2x &= -2 \\ \frac{d}{dx}x^2y + \frac{d}{dx}y^2x &= \frac{d}{dx} - 2 \\ (x^2 \cdot 1\frac{dy}{dx} + y \cdot 2x\frac{dx}{dx}) + (y^2 \cdot 1\frac{dx}{dx} + x \cdot 2y\frac{dy}{dx}) &= 0 \\ x^2\frac{dy}{dx} + 2xy + y^2 + 2yx\frac{dy}{dx} &= 0 \end{aligned}$$

Next, isolate terms with $\frac{dy}{dx}$:

$$x^2\frac{dy}{dx} + 2yx\frac{dy}{dx} = -2xy - y^2$$

Factor out $\frac{dy}{dx}$

$$(x^2 + 2yx)\frac{dy}{dx} = -2xy - y^2$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2yx}$$

1|You Try **Tangent Lines and Circles**

Find the tangent line at $(4, -3)$ of *the circle*:

$$x^2 + y^2 = 5^2$$

2|You Try **Differentiate**

Find the derivative of this function:

$$(4x + 4y)^3 = 64x^3 + 64y^3$$

2.5 Related Rates

Solving related-rate problems are similar to solving implicit differentiation. However, in this case, related-rate problems are typically differentiated in *respect to time*. Additionally, related-rate problems provide given quantities as well as quantities to find:

Note **Guideline for Related-Rate Problems**

1. Take note of all quantities *given* and quantities *yet to find* within the problem. Create a sketch if it helps to visualize the problem.
2. Write equation with those quantities and rates that were identified in the first step.
3. Implicitly differentiate the equation *with respect to time*
4. Substitute the given values, and solve for the required rate of change

Example **Related Rate with Cylindrical Glass**

The radius r of a circle is increasing at a rate of 8 centimeters per minute. Find the rate of change of the area when $r = 45$ centimeters.
The first step is to make a table of all given values and values to find:

Quantity	Value
(r) radius	45 cm
$\frac{dr}{dt}$	$8\frac{cm}{min}$
$\frac{dA}{dt}$	?

The next step is to write an equation with those given quantities: For this case, it'll be the equation of the area of a circle:

$$A = \pi r^2$$

The next step is to derivatize all terms in respect to time:

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

Finally, substitute in the given values and solve for the required rate of change:

$$\frac{dA}{dt} = 2\pi(45cm)(8\frac{cm}{min})$$
$$\frac{dA}{dt} = 720\pi\frac{cm^2}{min}$$

The rate of change of the volume of the barrel is $720\pi\frac{cm^2}{min}$.

1|You Try **Triangles Galore**

A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second. Find the rate at which the angle between the wall of the house and the ladder are changing when the base of the ladder is 7 feet from the wall.

3 Applications of Differentiation

3.1 Extrema on an Interval

Absolute maximum/minimum is also known as the *global* maximum/minimum

Definition **Extrema**

Let function f be defined on an interval I containing c:

- $f(c)$ is an *absolute minimum of f on I* when $f(c) \leq f(x)$ for x in I
- $f(c)$ is an *absolute maximum of f on I* when $f(c) \leq f(x)$ for x in I

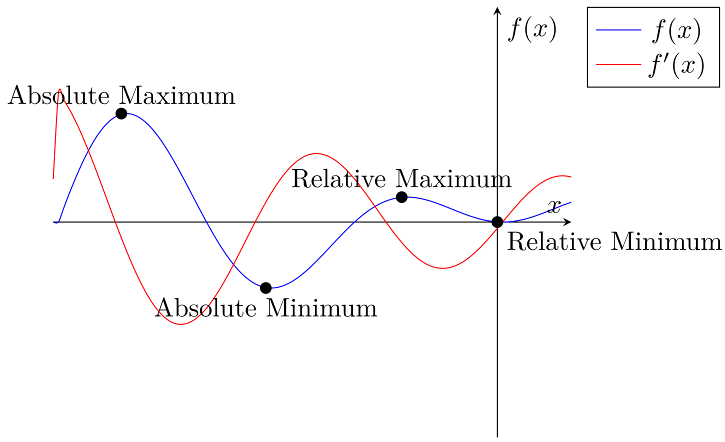
There can be multiple minimums or maximums on a function interval, but there will only be one absolute minimum or maximum.

Note **Minimums and Maximums**

Minimums and maximums occur when the slope of a function changes from positive to negative, or vice versa.

- Minimums occur when the slope before $f(c) < 0$ and the slope after $f(c) > 0$
- Maximums occur when the slope before $f(c) > 0$ and the slope after $f(c) < 0$

Below is a graph to help you graphically identify these terms:



Minimums and maximums can also exist on the endpoints of a function's interval.

Definition **Critical Value**

If $f(x)$ is defined at $x = c$, then c is a *critical number* if either:

- $f'(c) = 0$
- $f'(c) =$ DNE

However! c can't be a critical point if $f(c)$ itself is not defined.

Critical values are also known as *critical points*. The terms will be used interchangeably throughout the textbook.

Note **Finding Absolute Extrema**

To find the absolute extrema of $f(x)$ at the interval $[a, b]$

1. Find all critical points
2. Evaluate $f(x)$ at all critical points *and* at the endpoints a and b
3. The largest and smallest of those values are the absolute maximum and minimum of $f(x)$ in the interval $[a, b]$

Make sure when doing these problems that the critical values you find are *still on the interval* of the function

Example **Finding Absolute Extrema**

To find the absolute extrema of $f(x) = 3x^4 - 16x^3 + 18x^2$ on the interval $[-1, 4]$, we must first find all of its critical points:

$$\begin{aligned}
f'(x) &= 12x^3 - 48x^2 + 36x \\
0 &= 12x^3 - 48x^2 + 36x \\
0 &= 12x(x^2 - 4x + 3) \\
0 &= 12x(x-3)(x-1) \\
x &= \{0, 1, 3\}
\end{aligned}$$

Now the next step is to evaluate $f(x)$ at those critical points and endpoints:

critical values

$$\begin{aligned}
&f(0) && = 0 \\
&f(1) = 3(1)^4 - 16(1)^3 + 18(1)^2 && = 5 \\
&f(3) = 3(3)^4 - 16(3)^3 + 18(3)^2 && = -27
\end{aligned}$$

endpoints

$$\begin{aligned}
&f(-1) = 3(-1)^4 - 16(-1)^3 + 18(-1)^2 && = 37 \\
&f(4) = 3(4)^4 - 16(4)^3 + 18(4)^2 && = 32
\end{aligned}$$

The absolute minimum of $f(x)$ is (3, -27), and the absolute maximum of $f(x)$ is (-1, 37)

1|You Try **Finding Absolute Extrema**

Find the absolute extrema of $f(\theta) = \sin(\theta) - cos^2(\theta)$ on the interval $[0, 2\pi]$

2|You Try **Critical Values of a Trigonometric Function**

Find all θ where $f(\theta) = 2\cos\theta + \sin^2\theta$ is a critical value.

3.2 Rolle's Theorem & Mean Value Theorem

Theorem **Mean Value Theorem**

If $f(x)$ is:

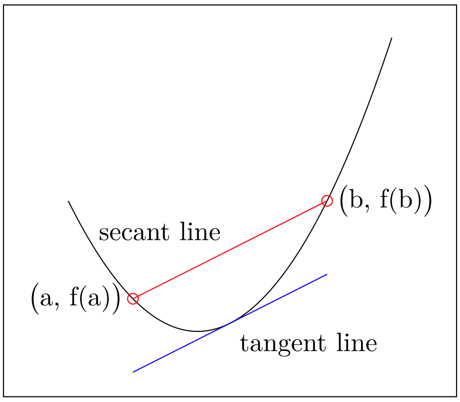
- continuous on $[a, b]$
- differentiable on (a, b)

Then there exists c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

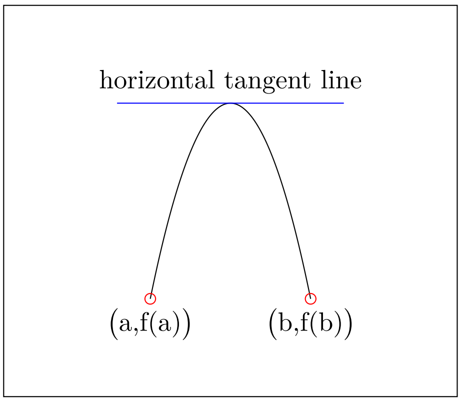
In layman's terms, this means that there will be a point c such that the function's instantaneous rate of change *is the same as* its average rate of change

Mean Value Theorem



Notice that the slope of the tangent line at $x = c$ is the same as the slope of the secant line from $(a, f(a)) \to (b, f(b))$

Rolle's Theorem



Theorem **Rolle's Theorem**

If $f(x)$ is:

- continuous on $[a, b]$
- differentiable on (a, b)
- $f(a) = f(b)$

Then there must be at least one number c such that $f'(c) = 0$

1|You Try **Inequalities**

Use the *Mean Value Theorem* to verify the following inequality:

$$\sqrt{b} - \sqrt{a} < \frac{b - a}{2\sqrt{b}}, b - a > 0$$

Hint: Divide both sides by $b - a$

2|You Try **MVT with $f(x)$**

The *Mean Value Theorem* applies to $f(x) = x^{\frac{2}{3}}$ on the interval $[-1, 8]$. If so, find all c that apply. If the theorem cannot be applied, explain why.

3|You Try **IVT & Rolle's Theorem**

Use the *Intermediate Value Theorem* and *Rolle's Theorem* to prove that the following equation has exactly one real solution:

$$6x^7 + 2x^5 + 3x + 6 = 0$$

3.3 Increasing & Decreasing Functions—First Derivative Test

Theorem **Decreasing and Increasing Functions**

Let f be a continuous function on the closed interval $[a, b]$, and differentiable on the open interval (a, b):

Condition	Meaning
$f'(x) > 0$ for x in (a, b)	f is increasing on $[a, b]$
$f'(x) < 0$ for x in (a, b)	f is decreasing on $[a, b]$
$f'(x) = 0$ for x in (a, b)	f is constant on $[a, b]$

Theorem **The First Derivative Test**

This is used to tell if a *critical point* x is a relative maximum, relative minimum, or neither. If $f'(x)$ is changing from...

1. negative to positive at x, then it's a *relative minimum* at $(x, f(x))$
2. positive to negative at x, then it's a *relative minimum* at $(x, f(x))$

If $f'(x)$ isn't changing at a critical point, then it's neither a relative minimum nor maximum

1|You Try **The First Derivative Test**

Given $f(x) = 3x^5 - 4x^3$, find:

1. All open interval(s) where $f(x)$ is decreasing or increasing
2. Relative minimum and maximum points of $f(x)$

3.4 Concavity—The Second Derivative Test

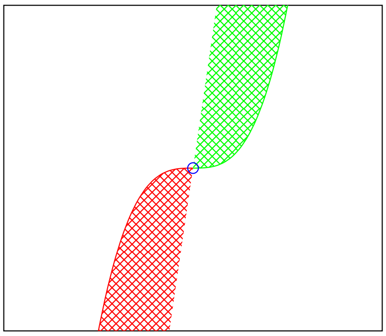
Inflection points only occur when $f'' = 0 \vee DNE$. In the case of this graph, it's at (0, 0)

Definition **Concavity**

Let f be a function that's differentiable on an open interval I:

- f is concave upward when f' is increasing
- f is concave downward when f' is decreasing

The points at which they happen are called *inflection points*



Theorem **Second Derivative Test**

Let f be a function whose $f'(c) = 0$ and its second derivative exists on an open interval cnotaining c:

- $\big(c, f(c)\big)$ is a relative minimum if $f''(c) > 0$
- $\big(c, f(c)\big)$ is a relative maximum if $f''(c) < 0$

If $f''(c) = 0$ then the test fails. Use the first derivative test instead.

An alternative way to decide whether a critical number is a relative maximum or minimum

Example **Relative Extrema with Second Derivative Test**

Find all relative extrema of

$$f(x) = 3x^4 + 8x^3 - 18x^2 + 5$$

$$\begin{aligned} f'(x) &= 12x^3 + 24x^3 - 36x \\ &= 12(x+3)(x-1) \end{aligned}$$

Critical numbers at: $x = \{-3, 0, 1\}$
Now we find the second derivative and find maximums and minimums:

$$\begin{aligned} f''(x) &= 36x^2 + 48x - 36 \\ &= 12(3x^2 + 4x - 3) \end{aligned}$$

$$\begin{aligned} f''(-3) &= 12\big(3(-3)^2 + 4(-3) - 3\big) > 0 \\ f''(0) &= 12\big(3(0)^2 + 4(0) - 3\big) < 0 \\ f''(1) &= 12\big(3(1)^2 + 4(1) - 3\big) > 0 \end{aligned}$$

The relative minimum is at $\big(-3, f(-3)\big)$, $\big(1, f(1)\big)$, and relative maximum is at $\big(0, f(0)\big)$

1|You Try

2|You Try

4 Practice Problem Answers

1.1.2—Proving Limits via. $\varepsilon - \delta$ Definition

Consider $f(x) = 10x - 6$, prove that $\lim_{x \to 3} f(x) = 24$ using the $\epsilon - \delta$ definition:

The first thing we would have to do is to find δ:

$$\begin{aligned} |(10x-6)-(24)| &< \varepsilon \\ |10x-30| &< \varepsilon \\ 10|x-3| &< \varepsilon \\ |x-3| &< \frac{\varepsilon}{10} \end{aligned}$$

$$0 < |x-(3)| < \delta$$

Notice how $\delta = \frac{\epsilon}{10}$. This guarantees that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$

1.2.2—Squeeze Theorem

Show that $\lim_{x \to 0} (\cos(\frac{2\pi}{3x}) \cdot \sqrt{x^3 + x^2}) = 0$ using the Squeeze Theorem

---

We know that $-1 \le \cos(\frac{2\pi}{3x}) \le 1$:

$$\begin{aligned} -\sqrt{x^3+x^2} \le \cos(\frac{2\pi}{3x})\sqrt{x^3+x^2} && \le \sqrt{x^3+x^2} \\ \lim_{x\to 0} -\sqrt{x^3+x^2} \le \lim_{x\to 0} \cos(\frac{2\pi}{3x})\sqrt{x^3+x^2} && \le \lim_{x\to 0} \sqrt{x^3+x^2} \\ -\sqrt{(0)^3+(0)^2} \le \lim_{x\to 0} \cos(\frac{2\pi}{3x})\sqrt{x^3+x^2} && \le \sqrt{(0)^3+(0)^2} \\ 0 \le \lim_{x\to 0} \cos(\frac{2\pi}{3x}) && \le 0 \end{aligned}$$

Because of the Squeeze Theorem, $\lim_{x \to 0} \cos(\frac{2\pi}{3x}) = 0$

1.2.2—Trigonomic Limit

Evaluate:

$$\lim_{x\to 0} \frac{\tan(3x)}{4x + \sin(2x)}$$

Let's start by breaking down tan into sin and cos:

$$\begin{aligned} &\lim_{x\to 0} \frac{\tan(3x)}{4x+\sin(2x)} \\ =&\lim_{x\to 0} \frac{\frac{\sin(3x)}{\cos(3x)}}{4x+\sin(2x)} \end{aligned}$$

Continued on Next Page...

Next, you multiply by a *big 1*. In this case it would be $\frac{\frac{1}{x}}{\frac{1}{x}}$

$$\begin{aligned}
&\lim_{x\to 0} \frac{\frac{\sin(3x)}{\cos(3x)}}{4x+\sin(2x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to 0} \Big(\frac{\frac{1}{\cos(x)} \cdot \frac{\sin(3x)}{x} \Big)}{\frac{4x+\sin(2x)}{x}} \\
&= \frac{\lim_{x\to 0} \frac{1}{\cos(x)} \cdot \frac{\sin(3x)}{x}}{\lim_{x\to 0} \left(4 + \frac{\sin(2x)}{x}\right)} \\
&= \frac{(1)(3)}{4+(2)} = \frac{3}{6} \\
&= \frac{1}{2}
\end{aligned}$$

That problem was quite tricky, but we did it 🥰

1.3—Determine Continuity

Determine if the function is continuous at $x = 2$

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{for } x \neq 2 \\ 1 & \text{for } x = 2 \end{cases}$$

Let's start by plugging in $x = 2$ for the case that $x \neq 2$ to see if it equals the case where $x = 2$:

$$\frac{x^2-x-2}{x-2} = \frac{(2)^2-(2)-2}{(2)-2} = \frac{0}{0}$$

That's so *uncool*. Let's factor it out and try again!

$$\begin{aligned}
\frac{x^2-x-2}{x-2} = \frac{(x-2)(x+1)}{x-2} &= x+1 \\
&= (2)+1 \\
&= 3
\end{aligned}$$

Now, we compare both that with the case where $x = 2$ to see if they are the same: $3 \neq 2$
$\therefore f(x)$ is *not continuous* at $x = 2$!

$\therefore$ means "therefore"

1.3.1—Intermediate Value Theorem

Use the Intermediate Value Theorem to prove that $x^3 + x^2 = 1$ has at least one solution on the interval $(-1, 2)$

—

Let's treat $x^3 + x^2$ as a function $f(x)$. We know that the function is continuous on the interval $[-1, 2]$. The next thing we have to do is to evaluate $f(-1)$ and $f(2)$ to compare if they are equal:

$$\begin{aligned} f(-1) &= (-1)^3 + (-1)^2 \\ f(2) &= (2)^3 + (2)^2 \\ f(-1) &\neq f(2) \end{aligned}$$

Now that we know that $f(-1)$ and $f(2)$ are not equal, and $f(-1) < 1 < f(2)$, we can conclude that there is a number 'c' in the interval $(-1, 2)$ such that $f(c) = 1$

1.3.3—Curveball Asymptote

Find the asymptotes:

$$f(x) = \frac{\sqrt{(x-1)(x-3)}}{(x-2)(x-4)}$$

—

You'd *think* that the asymptotes are $x = \{2, 4\}$, but you must consider the *domain* at which $f(x)$ exists.
Because this is a square root function, $(x-1)(x-3)$ *cannot be a negative number.* Plugging in $x = 2$ would result in a negative square root.

1.3.3—Curveball Trigonometry

Find the asymptotes:

$$f(x) = \frac{\sin(x)}{x^3 - x}$$

—

Let's start by factoring out the denominator:

$$f(x) = \frac{\sin(x)}{x(x-1)(x+1)}$$

You'd *think* that the asymptotes are $x = \{-1, 0, 1\}$. However, $\lim_{x\to 0} f(x) = 1$ — $f(x)$ can also be re-written as this: $\lim_{x\to 0}[\frac{\sin x}{x}] \cdot \lim_{x\to 0}[\frac{1}{(x-1)^2}] = -1$

Moral of the story: double-check your answers!

2.1—Finding the Tangent Line

Find the tangent lines to $f(x) = x^2 + 1$ at $(-2, 5)$:

Let's start by finding the derivative of the function at $x = -2$:

$$\begin{aligned}
f'(-2) &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \\
&= \lim_{h\to 0} \frac{\left((-2+h)^2 + 1\right) - (5)}{h} \\
&= \lim_{h\to 0} \frac{h^2 - 4h}{h} \\
&= \lim_{h\to 0} -4 + h = -4 + (0) \\
&= -4
\end{aligned}$$

Now we must write a point-slope equation with that derivative.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - (5) &= (-4)(x - (-2)) \\
y - 5 &= -4(x + 2)
\end{aligned}$$

Tangent line to $f(x) = x^2 + 1$ at $(-2, 5)$ has equation:

$$y - 5 = -4(x + 2)$$

Note **Alternative Route**

Another way to solve this would be to find the limit of $f(x)$ when $x = c$ and then plugging in c with any number that you want:

$$\begin{aligned}
f'(c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\
&= \lim_{h\to 0} \frac{\left((c+h)^2 + 1\right) - (c^2 + 1)}{h} \\
&= \lim_{h\to 0} \frac{(c^2 + 2ch + h^2 + 1) - (c^2 + 1)}{h} \\
&= \lim_{h\to 0} \frac{2ch + h^2}{h} = \lim_{h\to 0} \frac{h(2c+h)}{h} \\
&= \lim_{h\to 0} 2c + h = 2c + (0) \\
&= 2c
\end{aligned}$$

$$\begin{aligned}
c &= 2 \\
f'(-2) &= 2(-2) = -4
\end{aligned}$$

2.2—Find Horizontal Tangents

Find the horizontal tangents of $f(x) = x^4 - 2x^2 + 3$:

The question is essentially asking us to find the tangent lines whose slopes are equal to 0: $f'(x) = 0$

$$\begin{aligned} \frac{d}{dx} f(x) &= 0 \\ \frac{d}{dx}(x^4 - 2x^2 + 3) &= 0 \\ 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \end{aligned}$$

$$x = \{0, \pm 1\}$$

Now we plug in these values into $f(x)$ to get the tangent lines:

$$\begin{aligned} f(0) &= (0)^4 - 2(x)^2 + 3 & &= 3 \\ f(-1) &= (-1)^4 - 2(-1)^2 + 3 & &= 2 \\ f(1) &= (1)^4 - 2(1)^2 + 3 & &= 2 \end{aligned}$$

So the horizontal tangent lines are at $y = 3$ and $y = 2$

2.2—Find the Derivative

Find the derivative of

$$f(x) = \big((x^2 + 3)^5 + x\big)^2$$

This seems to be a chain rule problem, but instead of just nesting once, it's nesting twice: $f(g(h(x)))$

It's best to use u-substitution for this problem:

$$\begin{aligned} u &= x^2 + 3 & v &= u^5 + x & w &= v^2 \\ u' &= 2x & v' &= 5u^4 \cdot u' + 1 & w' &= 2v \\ & & v' &= 5(x^2 + 3)^4 \cdot 2x + 1 & w' &= 2(u^5 + x) \\ & & & & w' &= 2\big((x^2 + 3)^5 + x\big) \end{aligned}$$

Now we multiply each of those bottom terms together to reach the answer:

Don't simplify *unless needed*

$$f'(x) = \big(5(x^2 + 3)^4 \cdot 2x + 1\big) \cdot 2\big((x^2 + 3)^5 + x\big)$$

2.3—Velocity and Acceleration

A ball is thrown up into the air, and its position can be modeled into of feet as a function of time: $p(t) = -5(t-2)^2 + 20$. Find the velocity and acceleration of the ball just as it's about to hit the ground again.

—

To find when the ball reaches the ground, we must find when $p(t) = 0$:

$$\begin{aligned}
p(t) &= -5(t-2)^2 + 20 \\
0 &= -5(t-2)^2 + 20 \\
5(t-2)^2 &= 20 \\
(t-2)^2 &= 4 \\
t-2 &= 2 \\
t &= 4
\end{aligned}$$

Now that we have this, we need to take the derivative of $p(t)$ to get the function of the ball's instantaneous velocity $v(t)$, and the second derivative to get the instantaneous acceleration of the ball, $a(t)$:

$$\begin{aligned}
v(4) &= p'(4) \\
&= \frac{d}{dx}\Big(-5(t-2)^2 + 20\Big)\Big|_{t=4} \\
&= \frac{d}{dx}\Big(-5(t^2 - 4t + 4) + 20\Big)\Big|_{t=4} \\
&= \frac{d}{dx}(-5t^2 + 20t + 24)\Big|_{t=4} \\
&= -10t + 20\Big|_{t=4} \\
&= -10(4) + 20 = -20\frac{\text{ft}}{\text{s}}
\end{aligned}$$

$$\begin{aligned}
a(4) &= v'(4) \\
&= \frac{d}{dx}(-10t + 20)\Big|_{t=4} \\
&= -10\frac{\text{ft}}{\text{s}^2}
\end{aligned}$$

The velocity of the ball just when it hits the ground is -20 $\dfrac{\text{ft}}{\text{s}}$, and the acceleration of the ball as it hits the ground is -10 $\frac{\text{ft}}{\text{s}^2}$

2.4—Tangent Lines and Circles

Find the tangent line at $(4, -3)$ of *the circle*

$$x^2 + y^2 = 5^2$$

Start by taking the derivative of all terms in *respect to* x:

$$\begin{aligned}
\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(5^2) \\
2x\frac{dx}{dx} + 2y\frac{dy}{dx} &= 0\frac{dx}{dx} \\
2x + 2y\frac{dy}{dx} &= 0
\end{aligned}$$

Now isolate terms with $\frac{dy}{dx}$ and factor that out:

$$\begin{aligned}
2x + 2y\frac{dy}{dx} &= 0 \\
2y\frac{dy}{dx} &= -2x
\end{aligned}$$

And now get $\frac{dy}{dx}$ by itself for the equation of the derivative:

$$\begin{aligned}
2y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= -\frac{2x}{2y}
\end{aligned}$$

Now, plug in (4, -3) to get the slope for the tangent line:

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_{x=4,y=-3} &= -\frac{2x}{2y} \\
\left.\frac{dy}{dx}\right|_{x=4,y=-3} &= -\frac{2(4)}{2(-3)} = -\frac{8}{-6} = \frac{4}{3}
\end{aligned}$$

The last step to get the equation of the tangent line lies in plugging in those values:

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - (-3) &= (\frac{4}{3})(x - (4)) \\
y + 3 &= \frac{4}{3}(x - 4)
\end{aligned}$$

2.4—Differentiate

Find the derivative of this function:

$$(4x + 4y)^3 = 64x^3 + 64y^3$$

——

Though chain rule could be applied first (and there's nothing stopping you from going down that route for implicit differentiation), I believe that expanding the exponent and simplifying first would make implicit differentiation easier:

You could also apply chain rule to $(4x + 4y)^3$ if you so choose

$$\begin{aligned}
(4x + 4y)^3 &= 64x^3 + 64y^3 \\
64x^2 + 192x^2y + 192xy^2 + 64y^3 &= 64x^3 + 64y^3 \\
x^2 + 3x^2y + 3xy^2 + y^3 &= x^3 + y^3 \\
3x^2y + 3xy^2 &= 0 \\
x^2y + xy^2 &= 0
\end{aligned}$$

Now that it looks much more manageable, we can continue as usual:

$$\begin{aligned}
\frac{d}{dx}(x^2y + xy^2) &= \frac{d}{dx}(0) \\
\frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) &= 0 \\
x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 &= 0 \\
x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} &= -2xy - y^2 \\
(x^2 + 2xy)\frac{dy}{dx} &= -(2xy + y^2) \\
\frac{dy}{dx} &= -\frac{2xy + y^2}{x^2 + 2xy} \\
\frac{dy}{dx} &= -\frac{y(2x + y)}{x(x + 2y)}
\end{aligned}$$

It was lengthy, but we did it!

2.5—Triangles Galore

A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second. Find the rate at which the angle between the wall of the house and the ladder are changing when the base of the ladder is 7 feet from the wall.

——

Let's start by labeling what we know:

I used these variable letters, but you can use your own if you wish!

- l = length of ladder $= 25ft$
- $\frac{dl}{dt} = 0$: Since the length of the ladder is constant, it has no rate of change.
- b = distance from bottom of ladder to house wall $= 7ft$
- $\frac{db}{dt}$ = rate of change of distance from bottom of ladder to house wall $= 2\frac{ft}{sec}$

Continued on Next Page...

Let's identify what we need to find:

It's a 7-24-25 triangle, so h is actually 24 ft!

- h the height of the top of the ladder on the wall of the house to the floor
- $\frac{d\theta}{dt}$ the rate of change of the angle between the ladder and the wall of the house

An equation that relates to all of these values would be:

$$\sin\theta = \frac{b}{l}$$

Let's take the derivative of it so that we can solve for $\frac{d\theta}{dt}$:

$$\begin{aligned}
\cos\theta \cdot \frac{d\theta}{dt} &= \frac{l \cdot \frac{db}{dt} - b \cdot \frac{dl}{dt}}{l^2} \\
\cos\theta \cdot \frac{d\theta}{dt} &= \frac{l \cdot \frac{db}{dt}}{l^2} \\
\frac{d\theta}{dt} &= \frac{l \cdot \frac{db}{dt}}{l^2} \cdot \sec\theta
\end{aligned}$$

Before we plug in all of these values, let's evaluate $\sec\theta$:

$$\begin{aligned}
\theta &= \sin^{-1}(\frac{b}{l}) \\
\cos\theta &= \cos\big(\sin^{-1}(\frac{b}{l})\big) \\
\cos\theta &= \frac{h}{l} \\
\sec\theta &= \frac{l}{h}
\end{aligned}$$

Now let's continue solving for $\frac{d\theta}{dt}$:

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{l \cdot \frac{db}{dt}}{l^2} \cdot \frac{l}{h} \\
\frac{d\theta}{dt} &= \frac{l^2 \cdot \frac{db}{dt}}{l^2 \cdot h} \\
\frac{d\theta}{dt} &= \frac{\frac{db}{dt}}{h} \\
\frac{d\theta}{dt} &= \frac{(2\frac{ft}{sec})}{(24ft)} \\
\frac{d\theta}{dt} &= \frac{1}{12}\frac{rad}{sec}
\end{aligned}$$

Moral of the story: learn your trig. identities!

3.1—Finding Absolute Extrema

Find the absolute extrema of $f(\theta) = \sin(\theta) - cos^2(\theta)$ on the interval $[0, 2\pi]$

First we find its critical values:

$$\begin{aligned}
f'(\theta) &= \cos(\theta) - 2\cos(\theta)\big(-\sin(\theta)\big) \\
0 &= \cos(\theta) + 2\cos(\theta)\sin(\theta) \\
0 &= \cos(\theta) \cdot \big(1 + 2\sin(\theta)\big)
\end{aligned}$$

Continued on Next Page...

So the critical values can only happen if:

- $\cos(\theta) = 0 \implies \theta = \{\frac{\pi}{2}, \frac{3\pi}{2}\}$
- $\sin(\theta) = -\frac{1}{2} \implies \theta = \{\frac{7\pi}{6}, \frac{11\pi}{6}\}$

Let's evaluate the points at the critical values and endpoints now:

$$\begin{aligned}
f(0) &= -1 \\
f(\frac{\pi}{2}) &= 1 \\
f(\frac{7\pi}{6}) &= -\frac{5}{4} \\
f(\frac{3\pi}{2}) &= -1 \\
f(\frac{11\pi}{6}) &= -\frac{5}{4} \\
f(2\pi) &= -1
\end{aligned}$$

$$\begin{aligned}
\text{Absolute maximum:} &(\frac{\pi}{2}, 1) \\
\text{Absolute minimum:} &(\frac{7\pi}{6}, -\frac{5}{4}) \\
&(\frac{11\pi}{6}, -\frac{5}{4})
\end{aligned}$$

3.1—Critical Values of a Trigonomic Function

Find all θ where $f(\theta) = 2\cos\theta + \sin^2\theta$ is a critical value.

Let's take the derivative of $f(x)$:

Chain rule was used to find $\frac{d}{dx}(\sin^2\theta)$

$$\begin{aligned}
f'(x) &= -2\sin\theta + (\cos\theta \cdot 2\sin\theta) \\
f'(x) &= -2\sin\theta + 2\sin\theta\cos\theta \\
f'(x) &= -2\sin\theta(1 - \cos\theta)
\end{aligned}$$

So the critical values can only happen when:

- $\sin\theta = 0 \implies \theta = 2k\pi, k \in \mathbb{Z}$
- $\cos\theta = 1 \implies \theta = 2k\pi, k \in \mathbb{Z}$

$\therefore$ Critical values of $f(x) = 2k\pi, k \in \mathbb{Z}$

3.2—Inequalities

Use the *Mean Value Theorem* to verify the following inequality:

$$\sqrt{b}-\sqrt{a}<\frac{b-a}{2\sqrt{b}}, b-a>0$$

Hint: Divide both sides by $b-a$

—

Let's follow the hint:

$$\frac{\sqrt{a}-\sqrt{b}}{b-a}<\frac{1}{2\sqrt{b}}$$

The left side resembles the formula for average rate of change:

$$\frac{f(b)-f(a)}{b-a}, b-a>0$$

From this, we can infer that $f(x)=\sqrt{x}$. Recall the *Mean Value Theorem*—it only applies to $f(x)$ if $f(x)$ is continuous on $[a,b]$ and is differentiable on (a,b). Both of these requirements are fulfilled by the function, as polynomial functions are *continuous and differentiable.*

Because the theorem applies to $f(x)$, we know that there is a value c where:

$\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}$

- $a<c<b$
- $f'(c)=\frac{f(b)-f(a)}{b-a}$

From this we can infer:

$$\frac{\sqrt{b}-\sqrt{a}}{b-a}=\frac{1}{2\sqrt{c}}<\frac{1}{2\sqrt{a}}$$

$$\frac{\sqrt{b}-\sqrt{a}}{b-a}<\frac{1}{2\sqrt{a}}$$

$$\therefore \sqrt{b}-\sqrt{a}<\frac{b-a}{2\sqrt{a}}$$

This problem was quite the verbage wasn't it! Moral of the story: sometimes math isn't just all about numbers. And when it isn't, don't give up!

3.2—MVT with $f(x)$

The *Mean Value Theorem* applies to $f(x) = x^{\frac{2}{3}}$ on the interval $[-1,8]$. If so, find all c that apply. If the theorem cannot be applied, explain why.

—

Let's prove that the theorem is valid for $f(x)$: because the function is a polynomial function, it is both continuous and differentiable. This means that the theorem applies to $f(x)$ Let's find the average rate of change with the given interval:

This is a little redundant, but some questions still ask for this step

$$\frac{f(8)-f(-1)}{(8)-(-1)}=\frac{(8)^{\frac{2}{3}}-(-1)^{\frac{2}{3}}}{9}=\frac{4+1}{9}=\frac{5}{9}$$

Continued on Next Page...

Now we need to find where $f'(c) = \frac{5}{9}$:

$$\begin{aligned}
f'(c) &= \frac{5}{9} \\
\frac{d}{dc}(c^{\frac{2}{3}}) &= \frac{5}{9} \\
\frac{2}{3c^{\frac{1}{3}}} &= \frac{5}{9} \\
\frac{1}{c^{\frac{1}{3}}} &= \frac{5}{9} \cdot \frac{3}{2} = \frac{15}{18} = \frac{5}{6} \\
\frac{1}{c^{\frac{1}{3}}} &= \frac{5}{6} \\
c^{\frac{1}{3}} &= \frac{6}{5} \\
c &= (\frac{6}{5})^3 = \frac{216}{125} \\
c &= \frac{216}{125}
\end{aligned}$$

3.2—IVT & Rolle's Theorem

Use the *Intermediate Value Theorem* and *Rolle's Theorem* to prove that the following equation has exactly one real solution:

$$6x^7 + 2x^5 + 3x + 6 = 0$$

———

Let's use $f(x) = 6x^7 + 2x^5 + 3x + 6 = 0$ to prove that the *Intermediate Value Theorem* applies to $f(x)$. Because $f(x)$ is a polynomial function, it is continuous, and therefore the theorem applies.

Let's pick two arbitrary numbers, -2 and 2, to be a and b to check if 0 exists across that interval:

$$f(-2) = 6(-2)^7 + 2(-2)^5 + 3(-2) + 6 < 0$$
$$f(2) = 6(2)^7 + 2(2)^5 + 3(2) + 6 > 0$$

$f(-2) < 0 < f(2) \implies$ at least a number c in the interval $(-2, 2)$ such that:

$$f(c) = 0 = 6c^7 + 2c^5 + 3c + 6$$

To prove that there is only one real solution to $f(x)$, we can use a *proof of contradiction*[2] with *Rolle's Theorem* to prove that another solution cannot be possible:

Suppose there is at least two solutions, l and m where $l \neq m \wedge f(l) = f(m) = 0$ $\wedge$ means "and"

Rolle's Theorem implies that there is some d between l and m where $l < d < m$ where $f'(d) = 0$.

We can find the derivative of $f(x)$ to be:

$$f'(x) = 42x^6 + 10x^4 + 3 = 0$$

Continued on Next Page...

[2]Proof by contradiction means to assume that the inverse of the premise, and prove that it's a contradiction. In this problem's case, the premise is that there are more than one solution to $f(x)$

We know that $x \in \mathbb{R}$ that's raised to an even power will *always* be greater than or equal to 0. This implies that $\forall x \in \mathbb{R}, f'(x) \geq 3$

The expression translates to: "for all x in the set of real numbers, $f'(x) \geq 3$"

Because we proved that there does not exist an x such that $f'(x) = 0$, there's a contradiction.
$\therefore$ our assumption is *incorrect*, and so *there must be* only one solution to $f(x) = 0$

3.3—The First Derivative Test

Given $f(x) = 4x^5 - 2x^3$, find:

1. All open interval(s) where $f(x)$ is decreasing or increasing
2. Relative minimum and maximum points of $f(x)$

Let's find the critical values of $f(x)$:

$$\begin{aligned} f'(x) &= 12x^4 - 6x^2 \\ f'(x) &= 6x^2(2x^2 - 1) \\ 0 &= 6x^2(2x^2 - 1) \end{aligned}$$

$$x = \{0, \pm\frac{1}{\sqrt{2}}\}$$

Let's make a table to find how the slope of $f(x)$ behaves outside the critical values:

-1	$-\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1
>0	0	<0	0	<0	0	>0

By plugging those values into $f'(x)$ we can conclude the intervals at which $f(x)$ is increasing and decreasing:

Decreasing: $(-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)$

Increasing: $(-\frac{1}{\sqrt{2}}, 0) \cup (0, \frac{1}{\sqrt{2}})$

3.4—

3.4—