Math-2417.001 Notes

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Contents

Author's Notes

This is my very first project that I wrote in LaTeX, but I hope you enjoy the notes that I poured my blood, sweat, and tears into 😭

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These notes are based off of the textbook *Calculus 11e* by *Ron Larson* and *Bruce Edwards*, as well as the lectures of professor *Carlos Arreche.*

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If you have any complaints, or suggestions regarding these notes, please email me at mdn220004@utdallas.edu

1 Limits

1.1 Introduction to Calculus — "Mathematics of Change"

You can use calculus to study static objects by pretending they're changing

> **Example**
>
> A circle has area πr^2, but you can use the radius r to calculate the area of other polygons, such as a square, triangle, pentagon, etc...
> In other words, the *limit* of the areas inscribed polygons in a circle is πr^2

1.2 Finding Limits Graphically & Numerically

> **Theorem** **Informal Definition of Limit**
>
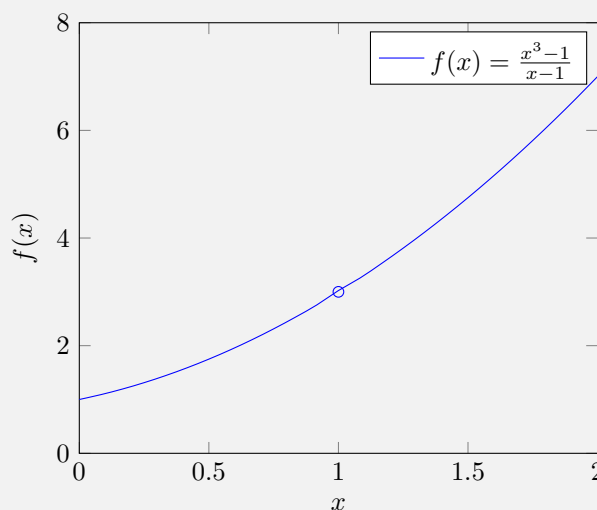> $$\lim_{x \to c} f(x) = L$$
>
> As x gets closer to c, the value of $f(x)$ becomes L

Limits also depend on what direction[1] you're coming from:

Meaning	Math Expression
From the right	$\lim_{x \to c^+} f(x)$
From the left	$\lim_{x \to c^-} f(x)$

> **Example** **Estimating Graphically and Numerically**
>
> Consider the function $f(x) = \frac{x^3 - 1}{x - 1}$:
>
>
>
> | x | 0.9 | 0.99 | 0.999 | 1 | 1.001 | 1.01 | 1.1 |
> |---|---|---|---|---|---|---|---|
> | f(x) | 2.71 | 2.97 | 2.997 | DNE | 3.003 | 3.03 | 3.31 |
>
> Notice how in both the **table and graph**, $f(x)$ looks like it's approaching $f(x) = 3$ when $x = 1$

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[1] If the limits from the left and right don't match, then the limit *doesn't exist*

1.2.1 When Limits Fail to Exist

Note **Limits Don't Exist When**

- $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

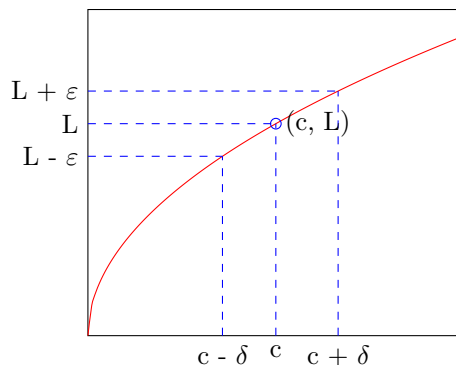- $f(x)$ is a violently oscillating function

1.2.2 A Formal Definition

Basically...

Theorem **Limits via.** $\varepsilon - \delta$

The statement $\lim_{x \to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

Refer back to finding limits numerically (example 1.2):

- $\pm\varepsilon$ would be the values on the row $f(x)$

- $\pm\delta$ would be the values on the row x

Example **Proving Limits via.** $\varepsilon - \delta$ **Definition**

Consider $f(x) = 10x - 6$, prove that $\lim_{x \to 3} f(x) = 24$ using the $\epsilon - \delta$ definition:

———

The first thing we would have to do is to find δ:

$$|(10x - 6) - (24)| < \varepsilon$$
$$|10x - 30| < \varepsilon$$
$$10|x - 3| < \varepsilon$$
$$|x - 3| < \frac{\varepsilon}{10}$$

$$0 < |x - (3)| < \delta$$

Notice how $\delta = \frac{\epsilon}{10}$. This guarantees that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$

1.3 Evaluating Limits Analytically

1.3.1 Properties of Limits

Theorem Basic Limit Properties

Let $\{b, c\} = \mathbb{R}$, $n = \mathbb{N}$:

1. $\lim_{x \to c} x = c$

2. $\lim_{x \to c} b = b$

3. $\lim_{x \to c} x^n = c^n$

Example Evaluating Basic Limits

$$\lim_{x \to 2} 5 = 5$$

$$\lim_{x \to 4} x = 4$$

$$\lim_{x \to 5} x^2 = 25$$

$f(g(x))$ may also be written as $f \circ g$

Theorem Limit Properties

Let $\{b, c\} = \mathbb{R}$, $n = \mathbb{N}$, and f and g are functions with limits:

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. $\lim_{x \to c} [b \cdot f(x)] = b \cdot L$

2. $\lim_{x \to c} [b \pm f(x)] = b \pm L$

3. $\lim_{x \to c} [f(x) \cdot g(x)] = L \cdot K$

4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}, K \neq 0$

5. $\lim_{x \to c} [f(x)]^n = L^n$

6. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$

7. $\lim_{x \to c} f\big(g(x)\big) = f\big(\lim_{x \to c} g(x)\big) = f(K)$

Example Limit of Polynomial

Evaluate $\lim_{x \to 5}[3x^3 + 4]$:

$$\lim_{x \to 5} [3x^3 + 4] = \lim_{x \to 5} 3x^3 + \lim_{x \to 5} 4$$
$$= 3 \cdot \lim_{x \to 5} x^3 + \lim_{x \to 5} 4$$
$$= 3 \cdot (5)^3 + 4$$
$$= 379$$

Theorem Limits of Polynomial/Rationals

Let $c = \mathbb{R}$ and p be a polynomial function:

$$\lim_{x \to c} p(x) = p(c)$$

Let r be a rational function $r(x) = \frac{p(x)}{q(x)}$ and $c = \mathbb{R}$ such that $q(c) \neq 0$:

$$\lim_{x \to c} r(x) = r(c) = \frac{p(r)}{q(r)}$$

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1.3.2 Squeeze Theorem

> **Theorem** The Squeeze Theorem
>
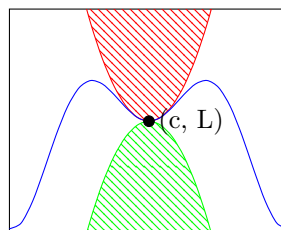> Suppose $h(x) \leq f(x) \leq g(x)$ for all x in an open interval *except* when $x = c$
>
> ———
>
> Also suppose that $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$ i.e. they share the same limit.
>
> ———
>
> This would mean that $\lim_{x \to c} f(x) = L$

> **Theorem** Trigonomic Limits
>
> 1. $\lim_{x \to 0} \frac{\sin x}{x}$
> 2. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 1$

Squeeze Theorem

1.4 Continuity

> **Definition** Definition of Continuity
>
> A function f is continuous if:
>
> 1. $f(c)$ exists
> 2. $\lim_{x \to c} f(x)$ exists
> 3. $f(c) = \lim_{x \to c} f(c)$

> **Example** Determine Continuity
>
> Determine if the function is continuous at $x = 2$
>
> $$f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{for } x \neq 2 \\ 1 & \text{for } x = 2 \end{cases}$$
>
> ———
>
> We should probably start by plugging in $x = 2$ for the case that $x \neq 2$ to see if it equals the case where $x = 2$:
>
> $$\frac{x^2 - x - 2}{x - 2} = \frac{(2)^2 - (2) - 2}{(2) - 2} = \frac{0}{0}$$
>
> That's so *uncool*. Let's factor it out and try again!
>
> $$\frac{x^2 - x - 2}{x - 2} = \frac{(x - 2)(x + 1)}{x - 2} = x + 1$$
> $$= (2) + 1$$
> $$= 3$$
>
> Now, we compare both that with the case where $x = 2$ to see if they are the same: $3 \neq 2$
> $\therefore f(x)$ is *not continuous* at $x = 2$!

1.4.1 Open and Closed Intervals

A function is continuous on an **open interval** (a, b) if $f(x) = c$ for each c in (a, b)

A function is continuous on a **closed interval** $[a, b]$ if:

- $f(x)$ is continuous on (a, b)

- $\lim_{x \to b^-} f(x) = f(b)$

- $\lim_{x \to a^+} f(x) = f(a)$

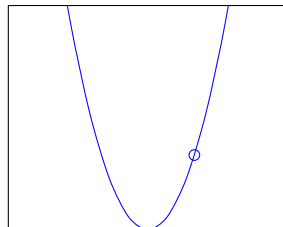> **Theorem** **Intermediate Value Theorem**
>
> If $f(x)$ is continuous on $[a, b]$, $a \neq b$, and k is any number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that:
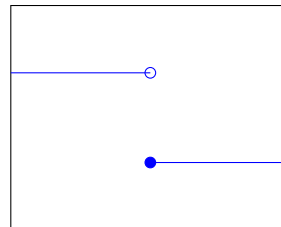> $$f(c) = k$$

1.4.2 Discontinuities

There are two cases where discontinuities happen:

Another example of removable discontinuity is in example 1.2

Removable Discontinuity
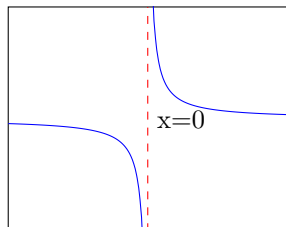
Non-Removable Discontinuity

1.4.3 Asymptotes

Definition **Asymptotes**

Vertical asymptote are when: Horizontal asymptote is:

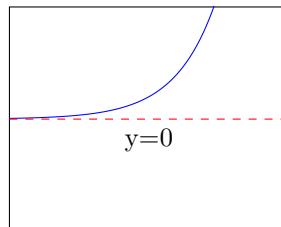$$\lim_{x \to c} f(x) = \pm\infty \qquad\qquad \lim_{x \to \pm\infty} f(x) = L$$

Vertical Asymptote

x=0

$$f(x) = \tfrac{1}{x}$$

Horizontal Asymptote

y=0

$$f(x) = 2^x$$

Note

We can infer something from vertical asymptotes from this graph: As the denominator becomes closer to zero, and it's a positive number, then the $f(x)$ will approach ∞. If the denominator approaches zero and is negative, then $f(x)$ will approach $-\infty$

Example **Curveball Asymptote**

Find the asymptotes:

$$f(x) = \frac{\sqrt{(x-1)(x-3)}}{(x-2)(x-4)}$$

———

You'd *think* that the asymptotes are $x = \{2, 4\}$, but you must consider the *domain* at which $f(x)$ exists.
Because this is a square root function, $(x-1)(x-3)$ *cannot be a negative number*. Plugging in $x = 2$ would result in a negative square root.

Example **Curveball Trigonometry**

Find the asymptotes:

$$f(x) = \frac{\sin(x)}{x^3 - x}$$

———

Let's start by factoring out the denominator:

$$f(x) = \frac{\sin(x)}{x(x-1)(x+1)}$$

You'd *think* that the asymptotes are $x = \{-1, 0, 1\}$. However, $\lim_{x \to 0} f(x) = 1$ — $f(x)$ can also be re-written as this:
$\lim_{x \to 0}\left[\frac{\sin x}{x}\right] \cdot \lim_{x \to 0}\left[\frac{1}{(x-1)^2}\right] = -1$
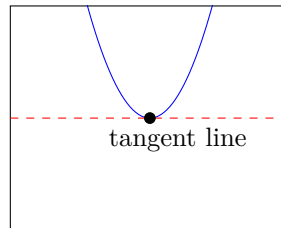
Moral of the story: double-check your answers!

2 Differentiation

2.1 Derivatives and Tangent Lines

Some mathematicians were trying to find out how to draw a line that intersects a function at *only one point*:

tangent line

However, it takes *two points* to draw a line, so they were confuzzled. You can just Google up the rest of the lore behind the definition of a limit, but it boils down to this:

> **Theorem** **Derivative via. Limits**
>
> $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(c) \text{ and } \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(c)$$

> **Note** **Alternative Ways of Writing a Derivative**
>
> There are other ways that mathematicians defined derivative:
>
> - $f'(x)$
> - $\frac{dy}{dx}$
> - $\frac{d}{dx} f(x)$
> - $Dx(y)$
>
> This isn't essential to know, but it's pretty useful to see how other mathematicians may express derivatives

> **Theorem** **Continuity of Derivatives**
>
> If function $f(x)$ is differentiable at $x = c$, then it is *also continuous* at $x = c$

Finding the Tangent Line

Find the tangent lines to $f(x) = x^2 + 1$ at $(-2, 5)$:

———

Let's start by finding the derivative of the function at $x = -2$:

$$f'(-2) = \lim_{h \to 0} \frac{f(-2 + h) - f(-2)}{h}$$
$$= \lim_{h \to 0} \frac{((-2 + h)^2 + 1) - (5)}{h}$$
$$= \lim_{h \to 0} \frac{h^2 - 4h}{h}$$
$$= \lim_{h \to 0} -4 + h = -4 + (0)$$
$$= -4$$

Now we must write a point-slope equation with that derivative.

$$y - y_1 = m(x - x_1)$$
$$y - (5) = (-4)(x - (-2))$$
$$y - 5 = -4(x + 2)$$

Tangent line to $f(x) = x^2 + 1$ at $(-2, 5)$ has equation:

$$y - 5 = -4(x + 2)$$

Another way to find the derivative of the function would be to find the limit of $f(x)$ when $x = c$ and then plugging in c with any number that you want:

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$
$$= \lim_{h \to 0} \frac{((c + h)^2 + 1) - (c^2 + 1)}{h}$$
$$= \lim_{h \to 0} \frac{(c^2 + 2ch + h^2 + 1) - (c^2 + 1)}{h}$$
$$= \lim_{h \to 0} \frac{2ch + h^2}{h} = \lim_{h \to 0} \frac{h(2c + h)}{h}$$
$$= \lim_{h \to 0} 2c + h = 2c + (0)$$
$$= 2c$$

$$c = 2$$
$$f'(-2) = 2(-2) = -4$$

2.2 Rules of Derivatives

Finding derivatives with the limit definition can get pretty exhausting, so mathematicians came up with a lot of shortcuts to evaluate derivatives much quicker. This is a pretty lengthy section, because of the many practice example problems I'll be putting in this section, but here are the rules in short:

$$\text{let: } f(x) = \text{a function}$$
$$g(x) = \text{another function}$$
$$c = \text{a constant}$$

Rule	Math Expression
Constant	$\frac{d}{dx}c = 0$
Power	$\frac{d}{dx}x^n = nx^{n-1}$
Sum and Difference	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
sin	$\frac{d}{dx}\sin(x) = \cos(x)$
cos	$\frac{d}{dx}\cos(x) = -\sin(x)$
Product	$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
Quotient	$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$
Chain	$\frac{d}{dx}f(g(x)) = g'(x) \cdot f'(g(x))$

You don't have to simplify completely on open-ended questions!

Example **Power Rule**

1. $\frac{d}{dx}(3x^2 + 4x) = 6x + 4$

2. $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$

3. $\frac{d}{dx}\sqrt[3]{x^2} = \frac{d}{dx}x^{\frac{2}{3}} = x^{-\frac{1}{3}}$

Example sin/cos **Rule**

1. $\frac{d}{dx}\left(2\sin(\theta) + 3\cos(\theta) - \frac{4}{\theta}\right) = 2\cos(\theta) - 3\sin(\theta) + \frac{4}{\theta^2}$

2. $\frac{d}{dx}\left(x^2 + 2\cos(x)\right) = 2x - 2\sin(x)$

Example **Find Horizontal Tangents**

Find the horizontal tangents of $f(x) = x^4 - 2x^2 + 3$:

———

The question is essentially asking us when the slope of the functions are equal to 0

3 Cheat Sheet

> **Limits**
>
> $\lim_{x \to c} f(x) = L$

$\lim_{x \to c} f(x) = L$