MATH-2417.001 Notes

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Contents

Author's Notes

This is my very first project that I wrote in LaTeX, but I hope you enjoy the notes that I poured my blood, sweat, and tears into 😭

These notes are based off of the textbook *Calculus 11e* by *Ron Larson* and *Bruce Edwards*, as well as the lectures of professor *Carlos Arreche* from the *Fall 2023* semester.

Practice problems are also found within the notes. The answers for the practice problems are found at the end of the notes. Please try them yourself so that you can get a hang of the subject!

If you have any complaints, or suggestions regarding these notes, please email me at
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1 Limits

1.1 Finding Limits Graphically & Numerically

Theorem **Informal Definition of Limit**

$$\lim_{x \to c} f(x) = L$$

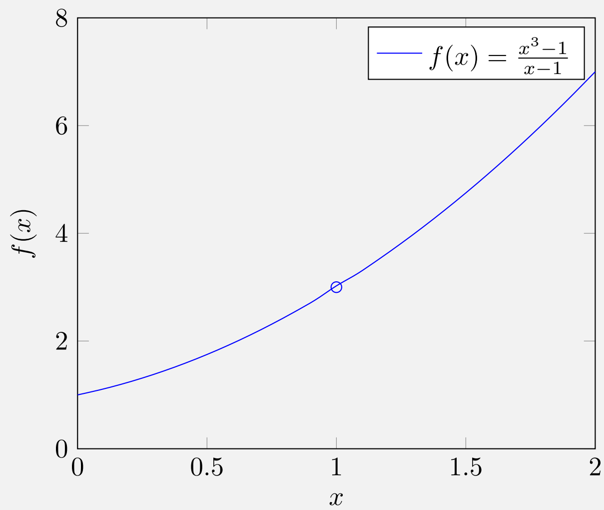
As x gets closer to c, the value of $f(x)$ becomes L

Limits also depend on what direction[1] you're coming from:

Meaning	Math Expression
From the right	$\lim_{x \to c^+} f(x)$
From the left	$\lim_{x \to c^-} f(x)$

Example **Estimating Graphically and Numerically**

Consider the function $f(x) = \frac{x^3-1}{x-1}$:



x	0.9	0.99	0.999	1	1.001	1.01	1.1
f(x)	2.71	2.97	2.997	DNE	3.003	3.03	3.31

Notice how in both the **table and graph**, $f(x)$ looks like it's approaching $f(x) = 3$ when $x = 1$

[1] If the limits from the left and right don't match, then the limit *doesn't exist*

1.1.1 When Limits Fail to Exist

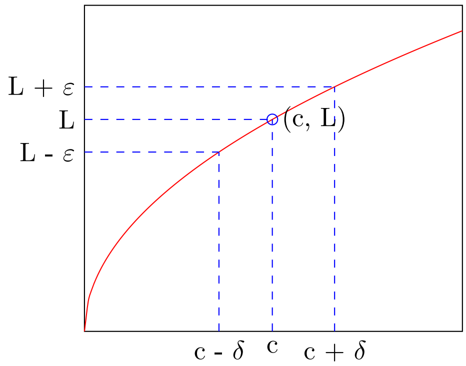
Note **Limits Don't Exist When**

- $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$
- $f(x)$ is a violently oscillating function

1.1.2 A Formal Definition

Theorem **Limits via. $\varepsilon - \delta$**

The statement $\lim_{x\to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$



Refer back to finding limits numerically (example 1.1):

- $\pm\varepsilon$ would be the values on the row $f(x)$
- $\pm\delta$ would be the values on the row x

1|You Try **Proving Limits via. $\varepsilon - \delta$ Definition**

Consider $f(x) = 10x - 6$, prove that $\lim_{x\to 3} f(x) = 24$ using the $\epsilon - \delta$ definition

1.2 Evaluating Limits Analytically

1.2.1 Properties of Limits

$\mathbb{R}$ =all real numbers
$\mathbb{N}$ =all natural numbers

Theorem **Basic Limit Properties**

Let $\{b, c\} = \mathbb{R}$, $n = \mathbb{N}$:

1. $\lim_{x\to c} x = c$
2. $\lim_{x\to c} b = b$
3. $\lim_{x\to c} x^n = c^n$

Example **Evaluating Basic Limits**

$$\lim_{x\to 2} 5 = 5$$
$$\lim_{x\to 4} x = 4$$
$$\lim_{x\to 5} x^2 = 25$$

$f(g(x))$ may also be written as $f \circ g$

Theorem **Limit Properties**

Let $\{b, c\} = \mathbb{R}$, $n = \mathbb{N}$, and f and g are functions with limits:

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. $\lim_{x \to c}[b \cdot f(x)] = b \cdot L$
2. $\lim_{x \to c}[b \pm f(x)] = b \pm L$
3. $\lim_{x \to c}[f(x) \cdot g(x)] = L \cdot K$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}, K \neq 0$
5. $\lim_{x \to c}[f(x)]^n = L^n$
6. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$
7. $\lim_{x \to c} f\big(g(x)\big) = f\big(\lim_{x \to c} g(x)\big) = f(K)$

Example **Limit of Polynomial**

$$\begin{aligned} \lim_{x \to 5}[3x^3 + 4] &= \lim_{x \to 5} 3x^3 + \lim_{x \to 5} 4 \\ &= 3 \cdot \lim_{x \to 5} x^3 + \lim_{x \to 5} 4 \\ &= 3 \cdot (5)^3 + 4 = 379 \end{aligned}$$

Theorem **Limits of Polynomial/Rationals**

Let $c = \mathbb{R}$ and p be a polynomial function:

$$\lim_{x \to c} p(x) = p(c)$$

Let r be a rational function $r(x) = \frac{p(x)}{q(x)}$ and $c = \mathbb{R}$ such that $q(c) \neq 0$:

$$\lim_{x \to c} r(x) = r(c) = \frac{p(r)}{q(r)}$$

1.2.2 Squeeze Theorem

Open/closed interval continuity will be discussed later in section 1.3.1.

Theorem **The Squeeze Theorem**

Suppose $h(x) \leq f(x) \leq g(x)$ for all x in an open interval *except* when $x = c$

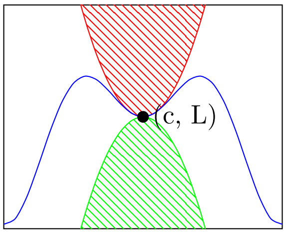
Also suppose that $\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$ i.e. they share the same limit.

This would mean that $\lim_{x\to c} f(x) = L$

Theorem **Trigonometric Limits**

1. $\lim_{x\to 0} \frac{\sin x}{x}$
2. $\lim_{x\to 0} \frac{1-\cos x}{x} = 1$

Squeeze Theorem



1|You Try **Squeeze Theorem**

Show that $\lim_{x\to 0}(\cos(\frac{2\pi}{3x}) \cdot \sqrt{x^3 + x^2}) = 0$ using the Squeeze Theorem

2|You Try **Trigonometric Limit**

Evaluate:

$$\lim_{x\to 0} \frac{\tan(3x)}{4x + \sin(2x)}$$

Hint: Break it down to the trigonometric functions you know how to take a limit of!

1.3 Continuity

Definition **Definition of Continuity**

A function f is continuous if:

1. $f(c)$ exists
2. $\lim_{x\to c} f(x)$ exists
3. $f(c) = \lim_{x\to c} f(c)$

1|You Try **Determine Continuity**

Determine if the function is continuous at $x = 2$

Note **Continuous Functions**

The following functions are always continuous *everywhere they're defined*:

- polynomial functions
- rational functions
- radical functions
- trigonometric functions

1.3.1 Open and Closed Intervals

A function is continuous on an **open interval** (a, b) if $f(x) = c$ for each c in (a, b)
A function is continuous on a **closed interval** $[a, b]$ if:

- $f(x)$ is continuous on (a, b)
- $\lim_{x \to b^-} f(x) = f(b)$
- $\lim_{x \to a^+} f(x) = f(a)$

Theorem **Intermediate Value Theorem**

If $f(x)$ is continuous on $[a, b], a \neq b$, and k is any number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that:

$$f(c) = k$$

1|You Try **Intermediate Value Thorem**

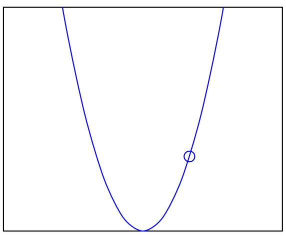
Use the Intermediate Value Theorem to prove that $x^3 + x^2 = 1$ has at least one solution on the interval $(-1, 2)$

1.3.2 Discontinuities

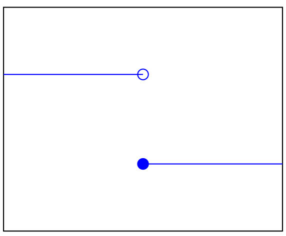
There are two cases where discontinuities happen:

Another example of removable discontinuity is in example 1.1

Removable Discontinuity



Non-Removable Discontinuity



1.3.3 Asymptotes

Definition **Asymptotes**

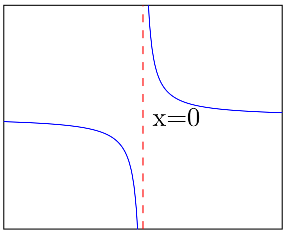
Vertical asymptote are when:

$$\lim_{x \to c^{\pm}} f(x) = \pm\infty$$

Horizontal asymptote is:

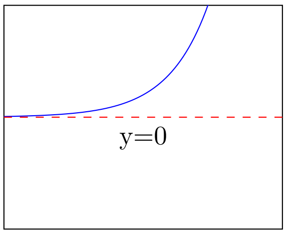
$$\lim_{x \to \pm\infty} f(x) = L$$

Vertical Asymptote



$f(x) = \frac{1}{x}$

Horizontal Asymptote



$f(x) = 2^x$

Note

We can infer something from vertical asymptotes from this graph: As the denominator becomes closer to zero, and it's a positive number, then the $f(x)$ will approach ∞. If the denominator approaches zero and is negative, then $f(x)$ will approach $-\infty$

1|You Try **Curveball Asymptote**

Find the asymptotes:

$$f(x) = \frac{\sqrt{(x-1)(x-3)}}{(x-2)(x-4)}$$

2|You Try **Curveball Trigonometry**

Find the asymptotes:

$$f(x) = \frac{\sin(x)}{x^3 - x}$$

2 Differentiation

Derivatives are essentially the *slope* of the function at a certain point
They also *cannot exist* where the limit doesn't exist at the function

This comes in handy when solving physics problems!

Note **Derivatives and Rate of Change**

To elaborate more on this, the derivative of a function $f(x)$ at point $x = c$ is the instantaneous rate of change:

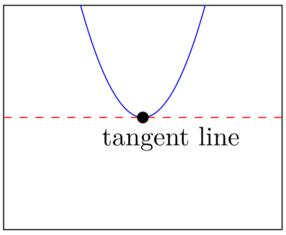
$$f'(x)\Big|_{x=c}$$

The average rate of change of the function on the interval $[a, b]$ is written as such:

$$\frac{f(b) - f(a)}{b - a}$$

2.1 Derivatives and Tangent Lines

Some mathematicians were trying to find out how to draw a line that intersects a function at *only one point*:



However, it takes *two points* to draw a line, so they were confuzzled.
You can just Google up the rest of the lore behind the definition of a limit, but it boils down to this:

Theorem **Derivative via. Limits**

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(c) \text{ and } \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(c)$$

Note **Alternative Ways of Writing a Derivative**

There are other ways that mathematicians defined derivative:

- $f'(x)$
- $\frac{dy}{dx}$
- $\frac{d}{dx}f(x)$
- $Dx(y)$

This isn't essential to know, but it's pretty useful to see how other mathematicians may express derivatives

Theorem **Continuity of Derivatives**

If function $f(x)$ is differentiable at $x = c$, then it is *also continuous* at $x = c$

1|You Try **Finding the Tangent Line**

Find the tangent lines to $f(x) = x^2 + 1$ at $(-2, 5)$

2.2 Rules of Derivatives

Finding derivatives with the limit definition can get pretty exhausting, so mathematicians came up with a lot of shortcuts to evaluate derivatives much quicker. This is a pretty lengthy section, because of the many practice example problems I'll be putting in this section, but here are the rules in short:

Theorem Derivative Rules

$$\begin{aligned} \text{Let: } f(x) &= \text{a function} \\ g(x) &= \text{another function} \\ c &= \text{a constant} \end{aligned}$$

1.	Constant	$\frac{d}{dx}c = 0$
2.	Power	$\frac{d}{dx}x^n = nx^{n-1}$
3.	Sum and Difference	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
4.	Product	$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
5.	Quotient	$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}$
6.	Trigonometry	$\frac{d}{dx}\sin(x) = \cos(x)$
		$\frac{d}{dx}\cos(x) = -\sin(x)$
		$\frac{d}{dx}\tan(x) = \sec^2(x)$
		$\frac{d}{dx}\cot(x) = -\csc^2(x)$
		$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
		$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$
7.	Chain	$\frac{d}{dx}f(g(x)) = g'(x) \cdot f'(g(x))$

You don't have to simplify completely on open-ended questions!

Example Power Rule

1. $\dfrac{d}{dx}(3x^2 + 4x) = 6x + 4$
2. $\dfrac{d}{dx}\sqrt{x} = \dfrac{d}{dx}x^{\frac{1}{2}} = \dfrac{1}{2}x^{-\frac{1}{2}}$
3. $\dfrac{d}{dx}\sqrt[3]{x^2} = \dfrac{d}{dx}x^{\frac{2}{3}} = x^{-\frac{1}{3}}$

Example Product and Quotient Rule

1. $\dfrac{d}{dx}(2x+3)(3x^2+1) = (2x+3)(6x) + (3x^2+1)(2)$
2. $\dfrac{d}{dx}\dfrac{x^2+3}{2x-1} = \dfrac{(2x-1)(2x) - (x^2+3)(2)}{(2x-1)^2}$

Example **Chain Rule**

$$\frac{d}{dx}\sin(x^2+1)$$

$$u = x^2+1, \qquad v = \sin(u)$$
$$u' = 2x, \qquad v' = \cos(u)$$
$$v' = \cos(x^2+1)$$

$$= u' \cdot v'$$
$$= 2x \cdot \cos(x^2+1)$$

u-substitution is a good method for solving chain rule problems:

Example $\sin$ **and** $\cos$ **Rule**

1. $\frac{d}{dx}\big(2\sin(\theta) + 3\cos(\theta) - \frac{4}{\theta}\big) = 2\cos(\theta) - 3\sin(\theta) + \frac{4}{\theta^2}$
2. $\frac{d}{dx}\big(x^2 + 2\cos(x)\big) = 2x - 2\sin(x)$
3. $\frac{d}{d\theta}\big(\tan(\theta)\big) = \frac{d}{d\theta}\big(\frac{\sin(\theta)}{\cos(\theta)}\big) = \frac{\cos^2(\theta)+\sin^2(\theta)}{\cos^2(\theta)} = \sec^2(\theta)$

Remember: $\sin^2(\theta) + \cos^2(\theta) = 1$

1|You Try **Find Horizontal Tangents**

Find the horizontal tangents of $f(x) = x^4 - 2x^2 + 3$

2|You Try **Find the Derivative**

Find the derivative of $f(x) = \big((x^2+3)^5 + x\big)^2$

2.3 Higher Order Derivatives

You can take a derivative of derivative:

$$\frac{d}{dx}\left[\frac{d}{dx}f(x)\right] = \frac{d^2}{dx^2}\left[f(x)\right] = f''(x)$$

Example **Finding the Second Derivative**

Finding $f''(x)$ when $f(x) = \frac{2x}{x+1}$:

$$\begin{aligned} f'(x) &= \frac{2 \cdot (x+1) - (2x) \cdot 1}{(x+1)^2} \\ &= \frac{2}{(x+1)^2} \end{aligned}$$

$$f''(x) = -\frac{4(x+1)}{(x+4)^4}$$

You can do this for however many times you want to!

1|You Try **Velocity and Acceleration**

A ball is thrown up into the air, and its position can be modeled into of feet as a function of time: $f(t) = -5(t-2)^2 + 20$. Find the acceleration of the ball just as it's about to hit the ground again.

Note **Taking Beyond the Third Derivative**

Taking $f'''(x)$ and beyond, it'll start to be tedious writing all of those apostrophes. Mathematicians often write it like this:

$$\frac{d^n}{dx^n}\left[f(x)\right] = f^{(n)}(x)$$

2.4 Implicit Differentiation

So far, we've been using *explicit differentiation*, which is when y can be isolated on one side of the equation where y is a function of x.
What if that's not possible?
That's where *implicit differentiation comes through.* Here are the steps on how to do that:

Note **Steps for Implicit Differentiation**

1. For all terms of x and y, derive in respect to x: $(\frac{d}{dx})$
2. Isolate all terms with $\frac{dy}{dx}$
3. Factor out $\frac{dy}{dx}$ where the terms with $\frac{dy}{dx}$ are isolated
4. Solve for $\frac{dy}{dx}$

It's also important to understand that derivative rules don't have to just apply to terms of x. The example on the next page shows an implicit differentiation using the product rule on x *and* y:

Example **Implicit Differentiation with Product Rule**

Let's follow those steps to find the derivative of

$$x^2y + y^2x = -2$$

Begin by deriving all terms in respect to x:

$$\begin{aligned} x^2y + y^2x &= -2 \\ \frac{d}{dx}x^2y + \frac{d}{dx}y^2x &= \frac{d}{dx} - 2 \\ (x^2 \cdot 1\frac{dy}{dx} + y \cdot 2x\frac{dx}{dx}) + (y^2 \cdot 1\frac{dx}{dx} + x \cdot 2y\frac{dy}{dx}) &= 0 \\ x^2\frac{dy}{dx} + 2xy + y^2 + 2yx\frac{dy}{dx} &= 0 \end{aligned}$$

Next, isolate terms with $\frac{dy}{dx}$:

$$x^2\frac{dy}{dx} + 2yx\frac{dy}{dx} = -2xy - y^2$$

Factor out $\frac{dy}{dx}$

$$(x^2 + 2yx)\frac{dy}{dx} = -2xy - y^2$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2yx}$$

1|You Try **Tangent Lines and Circles**

Find the tangent line at $(4, -3)$ of *the circle*:

$$x^2 + y^2 = 5^2$$

2|You Try **Differentiate**

Find the derivative of this function:

$$(4x + 4y)^3 = 64x^3 + 64y^3$$

2.5 Related Rates

Solving related-rate problems are similar to solving implicit differentiation. However, in this case, related-rate problems are typically differentiated in *respect to time*. Additionally, related-rate problems provide given quantities as well as quantities to find:

Note **Guideline for Related-Rate Problems**

1. Take note of all quantities *given* and quantities *yet to find* within the problem. Create a sketch if it helps to visualize the problem.
2. Write equation with those quantities and rates that were identified in the first step.
3. Implicitly differentiate the equation *with respect to time*
4. Substitute the given values, and solve for the required rate of change

On the next page, Let's see how those steps can be used on an example problem:

Example **Related Rate with Cylindrical Glass**

The radius r of a circle is increasing at a rate of 8 centimeters per minute. Find the rate of change of the area when $r = 45$ centimeters.
The first step is to make a table of all given values and values to find:

Quantity	Value
(r) radius	45 cm
$\frac{dr}{dt}$	$8\frac{cm}{min}$
$\frac{dA}{dt}$	?

The next step is to write an equation with those given quantities: For this case, it'll be the equation of the area of a circle:

$$A = \pi r^2$$

The next step is to derivatize all terms in respect to time:

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

Finally, substitute in the given values and solve for the required rate of change:

$$\frac{dA}{dt} = 2\pi(45cm)(8\frac{cm}{min})$$
$$\frac{dA}{dt} = 720\pi\frac{cm^2}{min}$$

The rate of change of the volume of the barrel is $720\pi\frac{cm^2}{min}$.

1|You Try **Triangles Galore**

A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

3 Applications of Differentiation

3.1 Extrema on an Interval

3.2 Rolle's Theorem & Mean Value Theorem

3.3 Increasing & Decreasing Functions—First Derivative Test

3.4 Concavity—Second Derivative Test

3.5 Limits at Infinity

3.6 Curve Sketching

3.7 Optimization Problems

3.8 Newton's Method

3.9 Differentials

4 Practice Problem Answers

1.1.2—Proving Limits via. $\varepsilon - \delta$ Definition

Consider $f(x) = 10x - 6$, prove that $\lim_{x \to 3} f(x) = 24$ using the $\epsilon - \delta$ definition:

—–

The first thing we would have to do is to find δ:

$$\begin{aligned} |(10x - 6) - (24)| &< \varepsilon \\ |10x - 30| &< \varepsilon \\ 10|x - 3| &< \varepsilon \\ |x - 3| &< \frac{\varepsilon}{10} \end{aligned}$$

$$0 < |x - (3)| < \delta$$

Notice how $\delta = \frac{\epsilon}{10}$. This guarantees that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$

1.2.2—Squeeze Theorem

Show that $\lim_{x \to 0} (\cos(\frac{2\pi}{3x}) \cdot \sqrt{x^3 + x^2}) = 0$ using the Squeeze Theorem

—–

We know that $-1 \leq \cos(\frac{2\pi}{3x}) \leq 1$:

$$\begin{aligned} -\sqrt{x^3 + x^2} &\leq \cos(\frac{2\pi}{3x})\sqrt{x^3 + x^2} && \leq \sqrt{x^3 + x^2} \\ \lim_{x \to 0} -\sqrt{x^3 + x^2} &\leq \lim_{x \to 0} \cos(\frac{2\pi}{3x})\sqrt{x^3 + x^2} && \leq \lim_{x \to 0} \sqrt{x^3 + x^2} \\ -\sqrt{(0)^3 + (0)^2} &\leq \lim_{x \to 0} \cos(\frac{2\pi}{3x})\sqrt{x^3 + x^2} && \leq \sqrt{(0)^3 + (0)^2} \\ 0 &\leq \lim_{x \to 0} \cos(\frac{2\pi}{3x}) && \leq 0 \end{aligned}$$

Because of the Squeeze Theorem, $\lim_{x \to 0} \cos(\frac{2\pi}{3x}) = 0$

1.2.2—Trigonomic Limit

Evaluate:

$$\lim_{x \to 0} \frac{\tan(3x)}{4x + \sin(2x)}$$

—–

Let's start by breaking down tan into sin and cos:

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(3x)}{4x+\sin(2x)} &= \lim_{x\to 0}\frac{\sin(3x)}{\cos(3x)\cdot(4x+\sin(2x))}\\ &= \lim_{x\to 0}\frac{\sin(3x)}{4x\cos(3x)+\sin(2x)\cos(3x)}\end{aligned}$$

Note that we can further break down $\cos(3x)$ into $\cos(2x+x)$, which can be converted to the trigonomic identity:$\cos(2x)\cos(x)-\sin(2x)\sin(x)$

$$\begin{aligned}&= \lim_{x\to 0}\frac{\sin(3x)}{4x\big(\cos(2x)\cos(x)-\sin(2x)\sin(x)\big)+\sin(2x)\cos(3x)}\\ &= \lim_{x\to 0}\frac{\sin(3x)}{4x\cos(2x)\cos(x)-4x\sin(2x)\sin(x)+\sin(2x)\cos(3x)}\end{aligned}$$

Next, you multiply by a *big 1*. In this case it would be $\frac{\frac{1}{x}}{\frac{1}{x}}$:

$$\begin{aligned}&= \lim_{x\to 0}\frac{\sin(3x)}{4x\cos(2x)\cos(x)-4x\sin(2x)\sin(x)+\sin(2x)\cos(3x)}\\ &= \lim_{x\to 0}\frac{\frac{\sin(3x)}{x}}{\frac{4x\cos(2x)\cos(x)}{x}-\frac{4x\sin(2x)\sin(x)}{x}+\frac{\sin(2x)\cos(3x)}{x}}\\ &= \lim_{x\to 0}\frac{\frac{\sin(3x)}{x}}{4\cos(2x)\cos(x)-4\sin(2x)\sin(x)+\cos(3x)\cdot\frac{\sin(2x)}{x}}\\ &= \frac{\lim_{x\to 0}\frac{\sin(3x)}{x}}{\lim_{x\to 0}4\cos(2x)\cos(x)-\lim_{x\to 0}4\sin(2x)\sin(x)+\lim_{x\to 0}\cos(3x)\frac{\sin(2x)}{x}}\\ &= \frac{(3)}{4\cos\big(2(0)\big)\cos\big((0)\big)-4\sin\big(2(0)\big)\sin\big((0)\big)+\cos\big(3(0)\big)(2)}\\ &= \frac{3}{4(1)(1)-4(0)(0)+(1)\cdot 2}\\ &= \frac{3}{4+2}=\frac{3}{6}\\ &= \frac{1}{2}\end{aligned}$$

That problem was quite lengthy, but we did it 🥰

1.3—Determine Continuity

Determine if the function is continuous at $x = 2$

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{for } x \neq 2 \\ 1 & \text{for } x = 2 \end{cases}$$

——

Let's start by plugging in $x = 2$ for the case that $x \neq 2$ to see if it equals the case where $x = 2$:

$$\frac{x^2 - x - 2}{x - 2} = \frac{(2)^2 - (2) - 2}{(2) - 2} = \frac{0}{0}$$

That's so *uncool*. Let's factor it out and try again!

$$\begin{aligned} \frac{x^2 - x - 2}{x - 2} = \frac{(x-2)(x+1)}{x-2} &= x + 1 \\ &= (2) + 1 \\ &= 3 \end{aligned}$$

Now, we compare both that with the case where $x = 2$ to see if they are the same: $3 \neq 2$
$\therefore f(x)$ is *not continuous* at $x = 2$!

$\therefore$ means "therefore"

1.3.1—Intermediate Value Theorem

Use the Intermediate Value Theorem to prove that $x^3 + x^2 = 1$ has at least one solution on the interval $(-1, 2)$

——

Let's treat $x^3 + x^2$ as a function $f(x)$. We know that the function is continuous on the interval $[-1, 2]$. The next thing we have to do is to evaluate $f(-1)$ and $f(2)$ to compare if they are equal:

$$\begin{aligned} f(-1) &= (-1)^3 + (-1)^2 \\ f(2) &= (2)^3 + (2)^2 \\ f(-1) &\neq f(2) \end{aligned}$$

Now that we know that $f(-1)$ and $f(2)$ are not equal, and $f(-1) < 1 < f(2)$, we can conclude that there is a number 'c' in the interval $(-1, 2)$ such that $f(c) = 1$

1.3.3—Curveball Asymptote

Find the asymptotes:

$$f(x) = \frac{\sqrt{(x-1)(x-3)}}{(x-2)(x-4)}$$

——

You'd *think* that the asymptotes are $x = \{2, 4\}$, but you must consider the *domain* at which $f(x)$ exists.
Because this is a square root function, $(x-1)(x-3)$ *cannot be a negative number*. Plugging in $x = 2$ would result in a negative square root.

1.3.3—Curveball Trigonometry

Find the asymptotes:

$$f(x) = \frac{\sin(x)}{x^3 - x}$$

—

Let's start by factoring out the denominator:

$$f(x) = \frac{\sin(x)}{x(x-1)(x+1)}$$

You'd *think* that the asymptotes are $x = \{-1, 0, 1\}$. However, $\lim_{x\to 0} f(x) = 1$ — $f(x)$ can also be re-written as this: $\lim_{x\to 0}[\frac{\sin x}{x}] \cdot \lim_{x\to 0}[\frac{1}{(x-1)^2}] = -1$

Moral of the story: double-check your answers!

2.1—Finding the Tangent Line

Find the tangent lines to $f(x) = x^2 + 1$ at $(-2, 5)$:

—

Let's start by finding the derivative of the function at $x = -2$:

$$\begin{aligned} f'(-2) &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h\to 0} \frac{\big((-2+h)^2 + 1\big) - (5)}{h} \\ &= \lim_{h\to 0} \frac{h^2 - 4h}{h} \\ &= \lim_{h\to 0} -4 + h = -4 + (0) \\ &= -4 \end{aligned}$$

Now we must write a point-slope equation with that derivative.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (5) &= (-4)(x - (-2)) \\ y - 5 &= -4(x + 2) \end{aligned}$$

Tangent line to $f(x) = x^2 + 1$ at $(-2, 5)$ has equation:

$$y - 5 = -4(x + 2)$$

Note **Alternative Route**

Another way to solve this would be to find the limit of $f(x)$ when $x = c$ and then plugging in c with any number that you want:

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h\to 0} \frac{\big((c+h)^2 + 1\big) - (c^2+1)}{h} \\ &= \lim_{h\to 0} \frac{(c^2 + 2ch + h^2 + 1) - (c^2+1)}{h} \\ &= \lim_{h\to 0} \frac{2ch + h^2}{h} = \lim_{h\to 0} \frac{h(2c+h)}{h} \\ &= \lim_{h\to 0} 2c + h = 2c + (0) \\ &= 2c \end{aligned}$$

$$\begin{aligned} c &= 2 \\ f'(-2) &= 2(-2) = -4 \end{aligned}$$

2.2—Find Horizontal Tangents

Find the horizontal tangents of $f(x) = x^4 - 2x^2 + 3$:

The question is essentially asking us to find the tangent lines whose slopes are equal to 0: $f'(x) = 0$

$$\begin{aligned} \frac{d}{dy} f(x) &= 0 \\ \frac{d}{dy}(x^4 - 2x^2 + 3) &= 0 \\ 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \end{aligned}$$

$$x = \{0, \pm 1\}$$

Now we plug in these values into $f(x)$ to get the tangent lines:

$$\begin{aligned} f(0) &= (0)^4 - 2(x)^2 + 3 && = 3 \\ f(-1) &= (-1)^4 - 2(-1)^2 + 3 && = 2 \\ f(1) &= (1)^4 - 2(1)^2 + 3 && = 2 \end{aligned}$$

So the horizontal tangent lines are at $y = 3$ and $y = 2$

2.2—Find the Derivative

Find the derivative of $f(x) = \big((x^2+3)^5 + x\big)^2$

—

Let's apply the chain rule with u-substitution to this problem:

$$\begin{aligned} u &= x^2+3 & v &= u^5 + x & w &= v^2 \\ u' &= 2x & v' &= 5u^4 + x & w' &= 2v \\ & & v' &= 5(x^2+3)^4 + x & w' &= 2\big((x^2+3)^5 + x\big) \end{aligned}$$

Now we multiply each of those terms to get the answer:

$$f'(x) = 2x \cdot \big(5(x^2+3)^4 + x\big) \cdot 2\big((x^2+3)^5 + x\big)$$

2.3—Velocity and Acceleration

A ball is thrown up into the air, and its position can be modeled into of feet as a function of time: $p(t) = -5(t-2)^2 + 20$. Find the velocity and acceleration of the ball just as it's about to hit the ground again.

—

To find when the ball reaches the ground, we must find when $p(t) = 0$:

$$\begin{aligned} p(t) &= -5(t-2)^2 + 20 \\ 0 &= -5(t-2)^2 + 20 \\ 5(t-2)^2 &= 20 \\ (t-2)^2 &= 4 \\ t-2 &= 2 \\ t &= 4 \end{aligned}$$

Now that we have this, we need to take the derivative of $p(t)$ to get the function of the ball's instantaneous velocity $v(t)$, and the second derivative to get the instantaneous acceleration of the ball, $a(t)$:

$$
\begin{aligned}
v(4) &= p'(4) \\
&= \frac{d}{dx}\big(-5(t-2)^2+20\big)\Big|_{t=4} \\
&= \frac{d}{dx}\big(-5(t^2-4t+4)+20\big)\Big|_{t=4} \\
&= \frac{d}{dx}(-5t^2+20t+24)\Big|_{t=4} \\
&= -10t+20\Big|_{t=4} \\
&= -10(4)+20 = -20\frac{\text{ft}}{\text{s}}
\end{aligned}
$$

$$
\begin{aligned}
a(4) &= v'(4) \\
&= \frac{d}{dx}(-10t+20)\Big|_{t=4} \\
&= -10\frac{\text{ft}}{\text{s}^2}
\end{aligned}
$$

The velocity of the ball just when it hits the ground is -20 $\frac{\text{ft}}{\text{s}}$, and the acceleration of the ball as it hits the ground is -10

$$\frac{\text{ft}}{\text{s}^2}$$

2.2—Find the Derivative

Find the derivative of

$$f(x) = \big((x^2+3)^5+x\big)^2$$

This seems to be a chain rule problem, but instead of just nesting once, it's nesting twice: $f(g(h(x)))$

It's best to use u-substitution for this problem:

$$
\begin{aligned}
u &= x^2+3 & v &= u^5+x & w &= v^2 \\
u' &= 2x & v' &= 5u^4+1 & w' &= 2v \\
& & v' &= 5(x^2+3)^4+1 & w' &= 2(u^5+x) \\
& & & & w' &= 2\big((x^2+3)^5+x\big)
\end{aligned}
$$

Now we multiply each of those bottom terms together to reach the answer:

Don't simplify *unless needed*

$$f'(x) = 2x \cdot \big(5(x^2+3)^4+1\cdot 2\big)\big((x^2+3)^5+x\big)$$

2.4—Tangent Lines and Circles

Find the tangent line at $(4, -3)$ of *the circle*

$$x^2 + y^2 = 5^2$$

—

Start by taking the derivative of all terms in *respect to* x:

$$\begin{aligned}\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(5^2)\\ 2x\frac{dx}{dx} + 2y\frac{dy}{dx} &= 0\frac{dx}{dx}\\ 2x + 2y\frac{dy}{dx} &= 0\end{aligned}$$

Now isolate terms with $\frac{dy}{dx}$ and factor that out:

$$\begin{aligned}2x + 2y\frac{dy}{dx} &= 0\\ 2y\frac{dy}{dx} &= -2x\end{aligned}$$

And now get $\frac{dy}{dx}$ by itself for the equation of the derivative:

$$\begin{aligned}2y\frac{dy}{dx} &= -2x\\ \frac{dy}{dx} &= -\frac{2x}{2y}\end{aligned}$$

Now, plug in (4, -3) to get the slope for the tangent line:

$$\begin{aligned}\frac{dy}{dx}\Big|_{x=4,y=-3} &= -\frac{2x}{2y}\\ \frac{dy}{dx}\Big|_{x=4,y=-3} &= -\frac{2(4)}{2(-3)} = \frac{8}{-6} = -\frac{4}{3}\end{aligned}$$

The last step to get the equation of the tangent line lies in plugging in those values:

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - (-3) &= (-\frac{4}{3})(x - (4))\\ y + 3 &= -\frac{4}{3}(x - 4)\end{aligned}$$

2.4—Differentiate

Find the derivative of this function:

$$(4x+4y)^3 = 64x^3 + 64y^3$$

—

Though chain rule could be applied first (and there's nothing stopping you from going down that route for implicit differentiation), I believe that expanding the exponent and simplifying first would make implicit differentiation easier:

$$\begin{aligned}
(4x+4y)^3 &= 64x^3 + 64y^3 \\
64x^2 + 192x^2y + 192xy^2 + 64y^3 &= 64x^3 + 64y^3 \\
x^2 + 3x^2y + 3xy^2 + y^3 &= x^3 + y^3 \\
3x^2y + 3xy^2 &= 0 \\
x^2y + xy^2 &= 0
\end{aligned}$$

Now that it looks much more manageable, we can continue as usual:

$$\begin{aligned}
\frac{d}{dx}(x^2y + xy^2) &= \frac{d}{dx}(0) \\
\frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) &= 0 \\
x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 &= 0 \\
x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} &= -2xy - y^2 \\
(x^2+2xy)\frac{dy}{dx} &= -(2xy + y^2) \\
\frac{dy}{dx} &= -\frac{2xy+y^2}{x^2+2xy} \\
\frac{dy}{dx} &= -\frac{y(2x+y)}{x(x+2y)}
\end{aligned}$$

It was lengthy, but we did it!

2.5—Triangles Galore