MATH-2417.001 Notes

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Contents

Author's Notes

This is my very first project that I wrote in LaTeX, but I hope you enjoy the notes that I poured my blood, sweat, and tears into 😭

These notes are based off of the textbook *Calculus 11e* by *Ron Larson* and *Bruce Edwards*, as well as the lectures of professor *Carlos Arreche* from the *Fall 2023* semester.

If you have any complaints, or suggestions regarding these notes, please email me at mdn220004@utdallas.edu

1 Limits

1.1 Introduction to Calculus — ”Mathematics of Change”

You can use calculus to study static objects by pretending they’re changing

Example

A circle has area πr^2, but you can use the radius r to calculate the area of other polygons, such as a square, triangle, pentagon, etc...
In other words, the *limit* of the areas inscribed polygons in a circle is πr^2

1.2 Finding Limits Graphically & Numerically

Theorem Informal Definition of Limit

$$\lim_{x \to c} f(x) = L$$

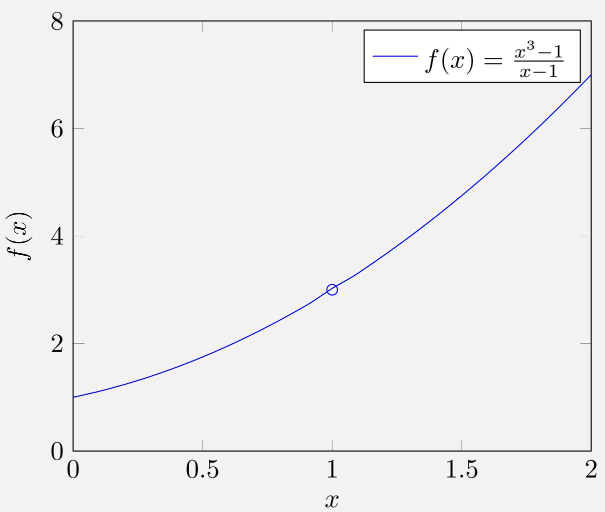
As x gets closer to c, the value of $f(x)$ becomes L

Limits also depend on what direction[1] you’re coming from:

Meaning	Math Expression
From the right	$\lim_{x \to c^+} f(x)$
From the left	$\lim_{x \to c^-} f(x)$

Example Estimating Graphically and Numerically

Consider the function $f(x) = \frac{x^3-1}{x-1}$:



x	0.9	0.99	0.999	1	1.001	1.01	1.1
f(x)	2.71	2.97	2.997	DNE	3.003	3.03	3.31

Notice how in both the **table and graph**, $f(x)$ looks like it’s approaching $f(x) = 3$ when $x = 1$

[1] If the limits from the left and right don’t match, then the limit *doesn’t exist*

1.2.1 When Limits Fail to Exist

Note **Limits Don't Exist When**

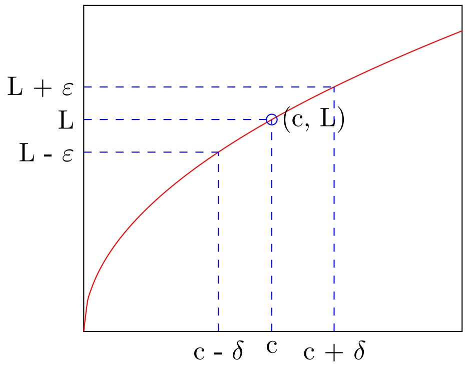
- $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$
- $f(x)$ is a violently oscillating function

1.2.2 A Formal Definition

Basically...

Theorem **Limits via.** $\varepsilon - \delta$

The statement $\lim_{x\to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$



Refer back to finding limits numerically (example 1.2):

- $\pm\varepsilon$ would be the values on the row $f(x)$
- $\pm\delta$ would be the values on the row x

Example **Proving Limits via.** $\varepsilon - \delta$ **Definition**

Consider $f(x) = 10x - 6$, prove that $\lim_{x\to 3} f(x) = 24$ using the $\epsilon - \delta$ definition:

The first thing we would have to do is to find δ:

$$\begin{aligned} |(10x - 6) - (24)| &< \varepsilon \\ |10x - 30| &< \varepsilon \\ 10|x - 3| &< \varepsilon \\ |x - 3| &< \frac{\varepsilon}{10} \end{aligned}$$

$$0 < |x - (3)| < \delta$$

Notice how $\delta = \frac{\epsilon}{10}$. This guarantees that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$

1.3 Evaluating Limits Analytically

1.3.1 Properties of Limits

$\mathbb{R}$ =all real numbers
$\mathbb{N}$ =all natural numbers

Theorem **Basic Limit Properties**

Let $\{b, c\} = \mathbb{R}$, $n = \mathbb{N}$:

1. $\lim_{x\to c} x = c$
2. $\lim_{x\to c} b = b$
3. $\lim_{x\to c} x^n = c^n$

Example **Evaluating Basic Limits**

$$\lim_{x\to 2} 5 = 5$$
$$\lim_{x\to 4} x = 4$$
$$\lim_{x\to 5} x^2 = 25$$

$f(g(x))$ may also be written as $f \circ g$

Theorem **Limit Properties**

Let $\{b, c\} = \mathbb{R}$, $n = \mathbb{N}$, and f and g are functions with limits:

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. $\lim_{x\to c} [b \cdot f(x)] = b \cdot L$
2. $\lim_{x\to c} [b \pm f(x)] = b \pm L$
3. $\lim_{x\to c} [f(x) \cdot g(x)] = L \cdot K$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}, K \neq 0$
5. $\lim_{x\to c} [f(x)]^n = L^n$
6. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$
7. $\lim_{x\to c} f\big(g(x)\big) = f\big(\lim_{x\to c} g(x)\big) = f(K)$

Example **Limit of Polynomial**

Evaluate $\lim_{x\to 5}[3x^3 + 4]$:

$$\begin{aligned}\lim_{x\to 5} [3x^3 + 4] &= \lim_{x\to 5} 3x^3 + \lim_{x\to 5} 4 \\ &= 3 \cdot \lim_{x\to 5} x^3 + \lim_{x\to 5} 4 \\ &= 3 \cdot (5)^3 + 4 \\ &= 379\end{aligned}$$

Theorem **Limits of Polynomial/Rationals**

Let $c = \mathbb{R}$ and p be a polynomial function:

$$\lim_{x\to c} p(x) = p(c)$$

Let r be a rational function $r(x) = \frac{p(x)}{q(x)}$ and $c = \mathbb{R}$ such that $q(c) \neq 0$:

$$\lim_{x\to c} r(x) = r(c) = \frac{p(r)}{q(r)}$$

1.3.2 Squeeze Theorem

Open/closed interval continuity will be discussed later in section 1.4.1.

Theorem **The Squeeze Theorem**

Suppose $h(x) \leq f(x) \leq g(x)$ for all x in an open interval *except* when $x = c$

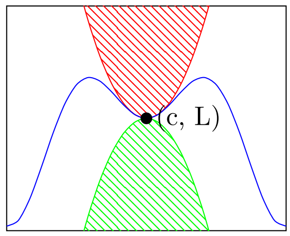
Also suppose that $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$ i.e. they share the same limit.

This would mean that $\lim_{x \to c} f(x) = L$

Theorem **Trigonometric Limits**

1. $\lim_{x \to 0} \frac{\sin x}{x}$
2. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 1$

Squeeze Theorem



1.4 Continuity

Definition **Definition of Continuity**

A function f is continuous if:

1. $f(c)$ exists
2. $\lim_{x \to c} f(x)$ exists
3. $f(c) = \lim_{x \to c} f(c)$

∴ means ”therefore”

Example **Determine Continuity**

Determine if the function is continuous at $x = 2$

$$f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{for } x \neq 2 \\ 1 & \text{for } x = 2 \end{cases}$$

We should probably start by plugging in $x = 2$ for the case that $x \neq 2$ to see if it equals the case where $x = 2$:

$$\frac{x^2 - x - 2}{x - 2} = \frac{(2)^2 - (2) - 2}{(2) - 2} = \frac{0}{0}$$

That’s so *uncool*. Let’s factor it out and try again!

$$\begin{aligned} \frac{x^2 - x - 2}{x - 2} = \frac{(x - 2)(x + 1)}{x - 2} &= x + 1 \\ &= (2) + 1 \\ &= 3 \end{aligned}$$

Now, we compare both that with the case where $x = 2$ to see if they are the same: $3 \neq 2$
$\therefore f(x)$ is *not continuous* at $x = 2$!

Note **Continuous Functions**

The following functions are always continuous *everywhere they're defined*:

- polynomial functions
- rational functions
- radical functions
- trigonometric functions

1.4.1 Open and Closed Intervals

A function is continuous on an **open interval** (a, b) if $f(x) = c$ for each c in (a, b)
A function is continuous on a **closed interval** $[a, b]$ if:

- $f(x)$ is continuous on (a, b)
- $\lim_{x \to b^-} f(x) = f(b)$
- $\lim_{x \to a^+} f(x) = f(a)$

Theorem **Intermediate Value Theorem**

If $f(x)$ is continuous on $[a, b], a \neq b$, and k is any number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that:

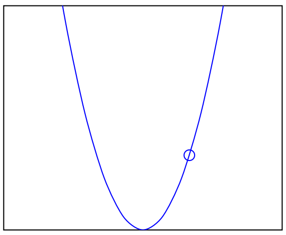
$$f(c) = k$$

1.4.2 Discontinuities

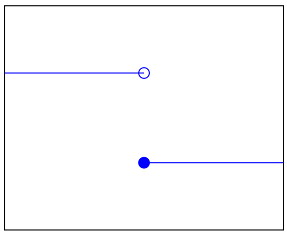
There are two cases where discontinuities happen:

Another example of removable discontinuity is in example 1.2

Removable Discontinuity



Non-Removable Discontinuity



1.4.3 Asymptotes

Definition **Asymptotes**

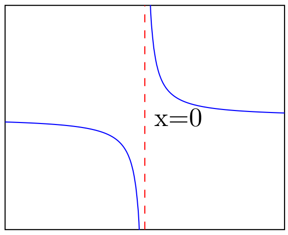
Vertical asymptote are when:

$$\lim_{x \to c^{\pm}} f(x) = \pm\infty$$

Horizontal asymptote is:

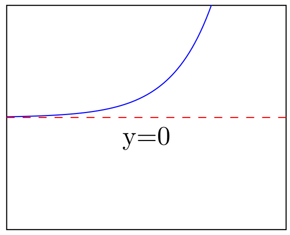
$$\lim_{x \to \pm\infty} f(x) = L$$

Vertical Asymptote



$f(x) = \frac{1}{x}$

Horizontal Asymptote



$f(x) = 2^x$

Note

We can infer something from vertical asymptotes from this graph: As the denominator becomes closer to zero, and it's a positive number, then the $f(x)$ will approach ∞. If the denominator approaches zero and is negative, then $f(x)$ will approach $-\infty$

Example **Curveball Asymptote**

Find the asymptotes:

$$f(x) = \frac{\sqrt{(x-1)(x-3)}}{(x-2)(x-4)}$$

You'd *think* that the asymptotes are $x = \{2, 4\}$, but you must consider the *domain* at which $f(x)$ exists.
Because this is a square root function, $(x-1)(x-3)$ *cannot be a negative number*. Plugging in $x = 2$ would result in a negative square root.

Moral of the story: double-check your answers!

Example **Curveball Trigonometry**

Find the asymptotes:

$$f(x) = \frac{\sin(x)}{x^3 - x}$$

Let's start by factoring out the denominator:

$$f(x) = \frac{\sin(x)}{x(x-1)(x+1)}$$

You'd *think* that the asymptotes are $x = \{-1, 0, 1\}$. However, $\lim_{x \to 0} f(x) = 1$ — $f(x)$ can also be re-written as this: $\lim_{x \to 0}[\frac{\sin x}{x}] \cdot \lim_{x \to 0}[\frac{1}{(x-1)^2}] = -1$

2 Differentiation

Derivatives are essentially the *slope* of the function at a certain point
They also *cannot exist* where the limit doesn't exist at the function

This comes in handy when solving physics problems!

Note **Derivatives and Rate of Change**

To elaborate more on this, the derivative of a function $f(x)$ at point $x = c$ is the instantaneous rate of change:

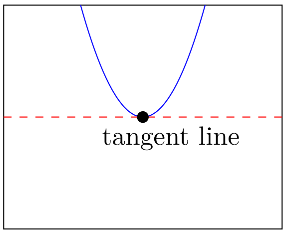
$$f'(x)\Big|_{x=c}$$

The average rate of change of the function on the interval $[a, b]$ is written as such:

$$\frac{f(b) - f(a)}{b - a}$$

2.1 Derivatives and Tangent Lines

Some mathematicians were trying to find out how to draw a line that intersects a function at *only one point*:



However, it takes *two points* to draw a line, so they were confuzzled. You can just Google up the rest of the lore behind the definition of a limit, but it boils down to this:

Theorem **Derivative via. Limits**

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(c) \text{ and } \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(c)$$

Note **Alternative Ways of Writing a Derivative**

There are other ways that mathematicians defined derivative:

- $f'(x)$
- $\frac{d}{dx} f(x)$
- $\frac{dy}{dx}$
- $Dx(y)$

This isn't essential to know, but it's pretty useful to see how other mathematicians may express derivatives

Theorem **Continuity of Derivatives**

If function $f(x)$ is differentiable at $x = c$, then it is *also continuous* at $x = c$

Example **Finding the Tangent Line**

Find the tangent lines to $f(x) = x^2 + 1$ at $(-2, 5)$:

Let's start by finding the derivative of the function at $x = -2$:

$$\begin{aligned} f'(-2) &= \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \to 0} \frac{\left((-2+h)^2 + 1\right) - (5)}{h} \\ &= \lim_{h \to 0} \frac{h^2 - 4h}{h} \\ &= \lim_{h \to 0} -4 + h = -4 + (0) \\ &= -4 \end{aligned}$$

Now we must write a point-slope equation with that derivative.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (5) &= (-4)(x - (-2)) \\ y - 5 &= -4(x + 2) \end{aligned}$$

Tangent line to $f(x) = x^2 + 1$ at $(-2, 5)$ has equation:

$$y - 5 = -4(x + 2)$$

Another way to find the derivative of the function would be to find the limit of $f(x)$ when $x = c$ and then plugging in c with any number that you want:

$$\begin{aligned} f'(c) &= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h \to 0} \frac{\left((c+h)^2 + 1\right) - (c^2 + 1)}{h} \\ &= \lim_{h \to 0} \frac{(c^2 + 2ch + h^2 + 1) - (c^2 + 1)}{h} \\ &= \lim_{h \to 0} \frac{2ch + h^2}{h} = \lim_{h \to 0} \frac{h(2c + h)}{h} \\ &= \lim_{h \to 0} 2c + h = 2c + (0) \\ &= 2c \end{aligned}$$

$$\begin{aligned} c &= 2 \\ f'(-2) &= 2(-2) = -4 \end{aligned}$$

2.2 Rules of Derivatives

Finding derivatives with the limit definition can get pretty exhausting, so mathematicians came up with a lot of shortcuts to evaluate derivatives much quicker. This is a pretty lengthy section, because of the many practice example problems I'll be putting in this section, but here are the rules in short:

$$\begin{aligned} \text{let: } f(x) &= \text{a function} \\ g(x) &= \text{another function} \\ c &= \text{a constant} \end{aligned}$$

Rule	Math Expression
Constant	$\frac{d}{dx}c = 0$
Power	$\frac{d}{dx}x^n = nx^{n-1}$
Sum and Difference	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
Product	$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
Quotient	$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$
Trigonometry	$\frac{d}{dx}\sin(x) = \cos(x)$ $\frac{d}{dx}\cos(x) = -\sin(x)$ $\frac{d}{dx}\tan(x) = \sec^2(x)$ $\frac{d}{dx}\cot(x) = -\csc^2(x)$ $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$ $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$
Chain	$\frac{d}{dx}f(g(x)) = g'(x) \cdot f'(g(x))$

You don't have to simplify completely on open-ended questions!

Example **Power Rule**

1. $\dfrac{d}{dx}(3x^2 + 4x) = 6x + 4$
2. $\dfrac{d}{dx}\sqrt{x} = \dfrac{d}{dx}x^{\frac{1}{2}} = \dfrac{1}{2}x^{-\frac{1}{2}}$
3. $\dfrac{d}{dx}\sqrt[3]{x^2} = \dfrac{d}{dx}x^{\frac{2}{3}} = x^{-\frac{1}{3}}$

Example **Product and Quotient Rule**

1. $\dfrac{d}{dx}(2x + 3)(3x^2 + 1) = (2x + 3)(6x) + (3x^2 + 1)(2)$
2. $\dfrac{d}{dx}\dfrac{x^2 + 3}{2x - 1} = \dfrac{(2x - 1)(2x) - (x^2 + 3)(2)}{(2x - 1)^2}$

Remember: $\sin^2(\theta) + \cos^2(\theta) = 1$

Example $\sin$ **and** $\cos$ **Rule**

1. $\frac{d}{dx}\big(2\sin(\theta) + 3\cos(\theta) - \frac{4}{\theta}\big) = 2\cos(\theta) - 3\sin(\theta) + \frac{4}{\theta^2}$
2. $\frac{d}{dx}\big(x^2 + 2\cos(x)\big) = 2x - 2\sin(x)$
3. $\frac{d}{d\theta}\big(\tan(\theta)\big) = \frac{d}{d\theta}\big(\frac{\sin(\theta)}{\cos(\theta)}\big) = \frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = \sec^2(\theta)$

Example **Find Horizontal Tangents**

Find the horizontal tangents of $f(x) = x^4 - 2x^2 + 3$:

The question is essentially asking us to find the tangent lines whose slopes are equal to 0: $f'(x) = 0$

$$\begin{aligned}
\frac{d}{dy}f(x) &= 0 \\
\frac{d}{dy}(x^4 - 2x^2 + 3) &= 0 \\
4x^3 - 4x &= 0 \\
4x(x^2 - 1) &= 0
\end{aligned}$$

$$x = \{0, \pm 1\}$$

Now we plug in these values into $f(x)$ to get the tangent lines:

$$\begin{aligned}
f(0) = (0)^4 - 2(x)^2 + 3 &= 3 \\
f(-1) = (-1)^4 - 2(-1)^2 + 3 &= 2 \\
f(1) = (1)^4 - 2(1)^2 + 3 &= 2
\end{aligned}$$

So the horizontal tangent lines are at $y = 3$ and $y = 2$

3 Cheat Sheet

Limits

$\lim_{x \to c} f(x) = L$