Gradient calculations

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Consider the case when $Y_1 = 0$ and $Y_2 = 0$. Then the bivariate likelihood is given by

$$f = \exp\left\{\sum_{l=1}^{L}\left(\sum_{i=1}^{2} k_{i,l}\right)^{\alpha}\right\} \tag{1}$$

where $k_{i,l} = \left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}$, $z_i = (1 - \xi \mathbf{X}_i^T \boldsymbol{\beta})^{1/\xi}$, and $w_l(\mathbf{s}_i) = \frac{\exp\left\{\frac{-||\mathbf{s}_i - \mathbf{v}_l||^2}{2\rho^2}\right\}}{\sum_{k=1}^{L}\exp\left\{\frac{-||\mathbf{s}_i - \mathbf{v}_k||^2}{2\rho^2}\right\}}$. Let $\ell = \log(f)$ be the log likelihood function. Then these are the gradients with respect to model parameters α, $\boldsymbol{\beta}$, ρ, and ξ.

$$\frac{\partial \ell}{\partial \alpha} = -\sum_{l=1}^{L}\left\{\left(\sum_{i=1}^{2}\left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}\right)^{\alpha-1}\left[\log\left(\sum_{i=1}^{2}\left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}\right) - \sum_{i=1}^{2}\left(\left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}\log\left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}\right)\right]\right\} \tag{2}$$

$$\frac{\partial \ell}{\partial \beta_p} = -\sum_{l=1}^{L}\left\{\left(\sum_{i=1}^{2}\left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}\right)^{\alpha-1}\sum_{i=1}^{2}\left(x_{i,p}\left[\frac{w_l(\mathbf{s}_i)}{z_i^{\alpha\xi+1}}\right]^{1/\alpha}\right)\right\} \tag{3}$$

where β_p is the pth regression coefficient, and $x_{i,p}$ is the pth covariate for observation i.

$$\begin{aligned}\frac{\partial \ell}{\partial \rho} = -\sum_{l=1}^{L}\Bigg\{&\left(\sum_{i=1}^{2}\left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}\right)^{\alpha-1} \times \\ &\frac{1}{\rho^3}\sum_{i=1}^{2}\left(\left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}\left[||\mathbf{s}_i - \mathbf{v}_l||^2 - \frac{\sum_{k=1}^{L}||\mathbf{s}_i - \mathbf{v}_k||^2\exp\left\{-\frac{||\mathbf{s}_i - \mathbf{v}_k||^2}{2\rho^2}\right\}}{\sum_{k=1}^{L}\exp\left\{-\frac{||\mathbf{s}_i - \mathbf{v}_k||^2}{2\rho^2}\right\}}\right]\right)\Bigg\}\end{aligned} \tag{4}$$

$$\frac{\partial \ell}{\partial \xi} = -\sum_{l=1}^{L}\left\{\left(\sum_{i=1}^{2}\left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}\right)^{\alpha-1}\sum_{i=1}^{2}\left(\frac{1}{\xi}\left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}\left[\ln(z_i) + \frac{\mathbf{X}_i^T\boldsymbol{\beta}}{z_i^{\xi}}\right]\right)\right\} \tag{5}$$

Consider the case when $Y_1 = 1$ and $Y_2 = 0$. Then the likelihood is given by

$$f = \exp\left\{-\frac{1}{z_2}\right\} - \exp\left\{\sum_{l=1}^{L}\left(\sum_{i=1}^{2}\left[\frac{w_l(\mathbf{s}_i)}{z_i}\right]^{1/\alpha}\right)^{\alpha}\right\}. \tag{6}$$

Let $\ell = \log(f)$ be the log likelihood function. Then these are the gradients with respect to model parameters α, $\boldsymbol{\beta}$, ρ, and ξ.