A spatial model for rare binary events

September 28, 2015

1 Introduction

The goal in binary regression is to relate a latent variable to a response using a link function. Two common examples of binary regression include logistic regression and probit regression. The link functions for logistic and probit regression are symmetric, so they may not be well-suited for asymmetric data. An asymmetric alternative to these link functions is the complementary log-log (cloglog) link function. More recently, Wang and Dey (2010) introduced the generalized extreme value (GEV) link function for rare binary data. The GEV link function introduces a new shape parameter to the link function that controls the degree of asymmetry. The cloglog link is a special case of the GEV link function when the shape parameter is 0.

Want to make the case in this paragraph that spatial logistic and probit models are not appropriate because asymptotic dependence is 0. Spatial logistic and probit models are commonly presented using a hierarchical model citation. In the hierarchical framework, spatial dependence is typically modeled with an underlying latent Gaussian process, and conditioned on this process, observations are independent. However, if the latent variable is assumed to follow a GEV marginally, then a Gaussian process may not be appropriate to describe the dependence due to the fact that they do not demonstrate asymptotic dependence regardless of the strength of the dependence in the bulk of the data. As an alternative to the Gaussian process, we propose using a latent max-stable process because it allows for asymptotic dependence citation. Max-stable processes are extremely flexible, but are often challenging to work with because very few finite dimensional representations exist in more than two or three dimensions.

Paragraph outlining the structure of the paper

2 Binary regression using the GEV link

Here, we provide a brief review of the the GEV link of Wang and Dey (2010). Let $Y_i \in \{0, 1\}, i = 1, \ldots, n$ be a collection of i.i.d. binary responses. We assume that $Y_i = I(z_i > 0)$ where $I(\cdot)$ is an indicator function, $z_i = [1 - \xi \mathbf{X}_i \boldsymbol{\beta}]^{1/\xi}$ is a latent variable following a $\text{GEV}(1, 1, 1)$ distribution, $\mathbf{X}_i$ be the associated p-vector of covariates with first element equal to one for the intercept, and $\boldsymbol{\beta}$ is a p-vector of regression coefficients. Therefore, the marginal probability of an event is given by

$$\pi_i = 1 - \exp\left(-\frac{1}{z_i}\right). \tag{1}$$

Although this link was selected by Wang and Dey based on its ability to handle asymmetry, the GEV distribution is one of the primary distributions used for modeling extremes. Therefore, it should also be suitable to model phenomena that are rare.

3 Spatial dependence for binary regression

In many binary regression applications, spatial dependence is handled using a hierarchical model. Logistic and probit regression typically incorporate a hierarchical model that is based on a Gaussian process. We present this in section 3.1.

3.1 Non-rare data

In this section we present the spatial logistic and probit models.

In the case that n is large, low-rank predictive process models can be used to ease the computation.

2

3.2 Rare binary data

In this section we extend the GEV link function to allow for spatial dependence. Let $Y(\mathbf{s})$ be the observation at spatial location $\mathbf{s}$ in a spatial domain of interest $\mathcal{D} \in \mathcal{R}^2$. As before, we assume $Y(\mathbf{s}) = I[z(\mathbf{s}) > 0]$ where $z(\mathbf{s}) = [1 - \xi \mathbf{X}(\mathbf{s})\boldsymbol{\beta}]^{1/\xi}$. Although $\boldsymbol{\beta}$ and ξ could be permitted to vary across space, we assume that they are constant across $\mathcal{D}$. As with logistic and probit regression, we also account for the spatial dependence using a hierarchical framework by assuming that the $Z(\mathbf{s})$ come from a realization of a max-stable process.

To incorporate spatial dependence into the model, we consider the hierarchical max-stable process of Reich and Shaby (2012). For a finite collection of locations $\mathbf{s}_1, \ldots, \mathbf{s}_n$, denote the vector of observations $\mathbf{Y} = [Y(\mathbf{s}_1), \ldots, Y(\mathbf{s}_n)]^T$. The spatial dependence is determined by the joint distribution of $\mathbf{z} = [z(\mathbf{s}_1), \ldots, z(\mathbf{s}_n)]^T$,

$$G(\mathbf{z}) = \mathrm{P}[Z(\mathbf{s}_1) < z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_n) < z(\mathbf{s}_n)] = \exp\left\{ -\sum_{l=1}^{L} \left[\sum_{i=1}^{n} \left(\frac{w_l(\mathbf{s}_i)}{z(\mathbf{s}_i)} \right)^{1/\alpha} \right]^{\alpha} \right\}, \qquad (2)$$

where $w_l(\mathbf{s}_i)$ are a set of weights that determine the spatial dependence structure and are discussed further in Section 3.3, and $\alpha \in (0, 1)$ determines the strength of dependence, with α near zero giving strong dependence and $\alpha = 1$ giving joint independence. This is a special case of the multivariate GEV distribution with asymmetric Laplace dependence function (Tawn, 1990). One nice feature to this representation for the max-stable process is that the lower-dimensional marginal distributions also follow a multivariate extreme value distribution. More importantly, at a single site i, the marginal distribution gives $P[Y(\mathbf{s}_i) = 1] = 1 - \exp\left[-\frac{1}{z(\mathbf{s}_i)} \right]$ which is the same as the marginal distributions given by Wang and Dey (2010).

3

3.3 Weight functions

The weights $w_l(\mathbf{s}_i)$ in (2) should vary smoothly across space to induce spatial dependence. Many weight functions are possible, but the weights must be constrained so that $\sum_{l=1}^{L} w_l(\mathbf{s}_i) = 1$ for all $i = 1, \ldots, n$ to preserve the marginal GEV distribution. For example, Reich and Shaby (2012) take the weights to be scaled Gaussian kernels with knots $\mathbf{v}_l$, that is

$$w_l(\mathbf{s}_i) = \frac{\exp\left[-0.5\left(||\mathbf{s}_i - \mathbf{v}_l||/\rho\right)^2\right]}{\sum_{j=1}^{L} \exp\left[-0.5\left(||\mathbf{s}_i - \mathbf{v}_j||/\rho\right)^2\right]}. \tag{3}$$

The kernel bandwidth $\rho > 0$ determines the spatial range of the dependence, with large ρ giving long-range dependence and vice versa.

4 Joint distribution

The joint likelihood of Y is computationally challenging to obtain. In this paper, we combine the random effect representation with MCMC; however, when exact expressions of the likelihood are available, better methods could be used. In section 4.1, we give an exact expression in the case where there are only two spatial locations which is useful for constructing a pairwise composite likelihood and studying spatial dependence. For more than two locations, we are also able to compute the exact likelihood when the number of locations is large but the number of events is small, as might be expected for very rare events (see Appendix A.1).

4.1 Bivariate distribution

Then in a bivariate setting, the probability mass function as a function of α is

$$
P[Y(\mathbf{s}_i), Y(\mathbf{s}_j)] = \begin{cases} \exp\left\{-\sum_{l=1}^{L}\left[\left(\frac{w_l(\mathbf{S}_i)}{z(\mathbf{S}_i)}\right)^{1/\alpha} + \left(\frac{w_l(\mathbf{S}_j)}{z(\mathbf{S}_j)}\right)^{1/\alpha}\right]^{\alpha}\right\} & Y(\mathbf{s}_i) = 0, Y(\mathbf{s}_j) = 0 \\[2ex] \frac{1}{z(\mathbf{S}_i)} - \exp\left\{-\sum_{l=1}^{L}\left[\left(\frac{w_l(\mathbf{S}_i)}{z(\mathbf{S}_i)}\right)^{1/\alpha} + \left(\frac{w_l(\mathbf{S}_j)}{z(\mathbf{S}_j)}\right)^{1/\alpha}\right]^{\alpha}\right\} & Y(\mathbf{s}_i) = 1, Y(\mathbf{s}_j) = 0 \\[2ex] 1 - \exp\left\{-\frac{1}{z(\mathbf{S}_i)}\right\} - \exp\left\{-\frac{1}{z(\mathbf{S}_j)}\right\} + \exp\left\{-\sum_{l=1}^{L}\left[\left(\frac{w_l(\mathbf{S}_i)}{z(\mathbf{S}_i)}\right)^{1/\alpha} + \left(\frac{w_l(\mathbf{S}_j)}{z(\mathbf{S}_j)}\right)^{1/\alpha}\right]^{\alpha}\right\} & Y(\mathbf{s}_i) = 1, Y(\mathbf{s}_j) = 1 \end{cases}
\tag{4}
$$

5 Quantifying spatial dependence

I still need to incorporate Brian's suggestions here In the literature on extremes, one common metric to describe the bivariate dependence is the χ statistic of Coles et al. (1999). The χ statistic between two observations z_1 and z_2 is given by

$$
\chi(\mathbf{s}_1, \mathbf{s}_2) = \lim_{c \to \infty} = P(Z_1 > c | Z_2 > c).
\tag{5}
$$

However, in this latent variable approach, $\lim_{c \to \infty}$ may not be the most reasonable metric because the observed data are a series of zeros and ones. Therefore, we chose the κ statistic of Cohen (1960) defined by

$$
\kappa = \frac{P(A) - P(E)}{1 - P(E)}
\tag{6}
$$

where $P(A)$ is the joint probability of agreement and $P(E)$ is the joint probability of agreement under an assumption of independence. We believe this measure of dependence to be reasonable because,

$$
\lim_{\beta_0 \to \infty} \kappa(h) = \chi(h) = 2 - \vartheta(\mathbf{s}_i, \mathbf{s}_j)
\tag{7}
$$

where β_0 is the intercept from $\mathbf{X}^T\boldsymbol{\beta}$ and $\vartheta(\mathbf{s}_i, \mathbf{s}_j) = \sum_{l=1}^{L}\left[w_l(\mathbf{s}_i)^{1/\alpha} + w_l(\mathbf{s}_j)^{1/\alpha}\right]^{\alpha}$ is the pairwise extremal coefficient given by Reich and Shaby (2012) (see Appendix A.2). In the case of complete dependence,

83    $\kappa = 1$, and in the case of complete independence, $\kappa = 0$.

6   Computation

85   For small K we can evaluate the likelihood directly. In the random effects model, the expression for the

86   joint density conditional on θ is

$$P(Y_1 = y_1, \ldots, Y_n = y_n) = \prod_{i=1}^{n} \left[\exp \left\{ \sum_{l=1}^{L} A_l \left(\frac{w_l(\mathbf{s}_i)}{z_i} \right)^{1/\alpha} \right\} \right]^{1-Y_i} \left[1 - \exp \left\{ \sum_{l=1}^{L} A_l \left(\frac{w_l(\mathbf{s}_i)}{z_i} \right)^{1/\alpha} \right\} \right]^{Y_i}.$$

(8)

7   Simulation study

88   For our simulation study, we generate $n_m = 100$ datasets under 4 different settings to explore the impact of

89   rareness of observations, sample size, and knot spacing. We consider two degrees of rareness $\pi = 0.01, 0.05$

90   and two sample sizes $n_s = 1000, 2000$. For the different knot spacings, we use knots in $[0, 1] \times [0, 1]$ on

91   a 21×21 grid and 31×31 grid. For each dataset, we fit the model using three different methods, spatial

92   logistic regression, spatial probit regression, and the proposed spatial GEV method. In each case, we fit

93   the model using Bayesian methods with proper, but fairly uninformative priors. For each method, we fit

94   the model using 75% of the observations as a training set, and the remaining observations are used as a

95   validation set to assess the model's predictive power.

8   Data analysis

97   For the data analysis, we consider data from the eBirds dataset, a citizen-based observation network of bird

98   sitings in the United States (Sullivan et al., 2009). The data are publicly available from `http://ebird.org`.

99   We use data from 2002, and focus specifically on cattle egrets and vesper sparrows.

9   Conclusions

Acknowledgments

A   Appendices

A.1   Derivation of the likelihood

104   We use the hierarchical max-stable spatial model given by Reich and Shaby (2012). If at each margin, $Z_i \sim$

105    $\text{GEV}(1, 1, 1)$, then $Z_i | \theta_i \overset{indep}{\sim} \text{GEV}(\theta, \alpha\theta, \alpha)$. We reorder the data such that $Y_1 = \ldots = Y_K = 1$, and

106    $Y_{K+1} = \ldots = Y_n = 0$. Then the joint likelihood conditional on the random effect θ is

$$
\begin{aligned}
P(Y_1 = y_1, \ldots, Y_n = y_n) &= \prod_{i \leq K} \left\{ 1 - \exp\left[-\left(\frac{\theta_i}{z_i}\right)^{1/\alpha} \right] \right\} \prod_{i > K} \exp\left[-\left(\frac{\theta_i}{z_i}\right)^{1/\alpha} \right] \\
&= \exp\left[-\sum_{i=K+1}^{n} \left(\frac{\theta_i}{z_i}\right)^{1/\alpha} \right] - \exp\left[-\sum_{i=K+1}^{n} \left(\frac{\theta_i}{z_i}\right)^{1/\alpha} \right] \sum_{i=1}^{K} \exp\left[-\left(\frac{\theta_i}{z_i}\right)^{1/\alpha} \right] \\
&\quad + \exp\left[-\sum_{i=K+1}^{n} \left(\frac{\theta_i}{z_i}\right)^{1/\alpha} \right] \sum_{1 < i < j \leq K} \left\{ \exp\left[-\left(\frac{\theta_i}{z_i}\right)^{1/\alpha} - \left(\frac{\theta_j}{z_j}\right)^{1/\alpha} \right] \right\} \\
&\quad + \cdots + (-1)^K \exp\left[-\sum_{i=1}^{n} \left(\frac{\theta_i}{z_i}\right)^{1/\alpha} \right]
\end{aligned}
\tag{9}
$$

107   Finally marginalizing over the random effect, we obtain

$$P(Y_1 = y_1, \ldots, Y_n = y_n) = \int G(\mathbf{z}|\mathbf{A})p(\mathbf{A}|\alpha)d\mathbf{A}.$$

$$= \int \exp\left[-\sum_{i=K+1}^{n}\left(\frac{\theta_i}{z_i}\right)^{1/\alpha}\right] - \exp\left[-\sum_{i=K+1}^{n}\left(\frac{\theta_i}{z_i}\right)^{1/\alpha}\right]\sum_{i=1}^{K}\exp\left[-\left(\frac{\theta_i}{z_i}\right)^{1/\alpha}\right]$$

$$+ \exp\left[-\sum_{i=K+1}^{n}\left(\frac{\theta_i}{z_i}\right)^{1/\alpha}\right]\sum_{1<i<j\leq K}\left\{\exp\left[-\left(\frac{\theta_i}{z_i}\right)^{1/\alpha} - \left(\frac{\theta_j}{z_j}\right)^{1/\alpha}\right]\right\}$$

$$+ \cdots + (-1)^K \exp\left[-\sum_{i=1}^{n}\left(\frac{\theta_i}{z_i}\right)^{1/\alpha}\right]p(\mathbf{A}|\alpha)d\mathbf{A}. \tag{10}$$

Consider the first term in the summation,

$$\int \exp\left\{-\sum_{i=K+1}^{n}\left(\frac{\theta_i}{z_i}\right)^{1/\alpha}\right\}p(\mathbf{A}|\alpha)d\mathbf{A} = \int \exp\left\{-\sum_{i=K+1}^{n}\left(\frac{\left[\sum_{l=1}^{L} A_l w_l(\mathbf{s}_i)^{1/\alpha}\right]^\alpha}{z_i}\right]^{1/\alpha}\right\}p(\mathbf{A}|\alpha)d\mathbf{A}$$

$$= \int \exp\left\{-\sum_{i=K+1}^{n}\sum_{l=1}^{L}A_l\left(\frac{w_l(\mathbf{s}_i)}{z_i}\right)^{1/\alpha}\right\}p(\mathbf{A}|\alpha)d\mathbf{A}$$

$$= \exp\left\{-\sum_{l=1}^{L}\left[\sum_{i=K+1}^{n}\left(\frac{w_l(\mathbf{s}_i)}{z_i}\right)^{1/\alpha}\right]^\alpha\right\}. \tag{11}$$

The remaining terms in equation (10) are straightforward to obtain, and after integrating out the random effect, the joint density for $K = 0, 1, 2$ is given by

$$P(Y_1 = y_1, \ldots, Y_n = y_n) = \begin{cases} G(\mathbf{z}) & K = 0 \\ G(\mathbf{z}_{(1)}) - G(\mathbf{z}) & K = 1 \\ G(\mathbf{z}_{(12)}) - G(\mathbf{z}_{(1)}) - G(\mathbf{z}_{(2)}) + G(\mathbf{z}) & K = 2 \end{cases} \tag{12}$$

where

$$G[\mathbf{z}_{(1)}] = P[Z(\mathbf{s}_2) < z(\mathbf{s}_2), \ldots, Z(\mathbf{s}_n) < z(\mathbf{s}_n)]$$

$$G[\mathbf{z}_{(2)}] = P[Z(\mathbf{s}_1) < z(\mathbf{s}_1), Z(\mathbf{s}_3) < z(\mathbf{s}_3), \ldots, Z(\mathbf{s}_n) < z(\mathbf{s}_n)]$$

$$G[\mathbf{z}_{(12)}] = P[Z(\mathbf{s}_3) < z(\mathbf{s}_3), \ldots, Z(\mathbf{s}_n) < z(\mathbf{s}_n)].$$

Similar expressions can be derived for all K, but become cumbersome for large K.

A.2 Derivation of the χ statistic

$$
\begin{aligned}
\chi &= \lim_{p \to 0} \mathrm{P}(Y_i = 1 | Y_j = 1) \\
&= \lim_{p \to \infty} \frac{p + p - \left(1 - \exp\left\{-\sum_{l=1}^{L}\left[(-\log(1-p)w_l(\mathbf{s}_i))^{1/\alpha} + (-\log(1-p)w_l(\mathbf{s}_j))^{1/\alpha}\right]^\alpha\right\}\right)}{p} \\
&= \lim_{p \to 0} \frac{2p - \left(1 - \exp\left\{\log(1-p)\sum_{l=1}^{L}\left[w_l(\mathbf{s}_i)^{1/\alpha} + w_l(\mathbf{s}_j)^{1/\alpha}\right]^\alpha\right\}\right)}{p} \\
&= \lim_{p \to 0} \frac{2p - \left(1 - (1-p)^{\sum_{l=1}^{L}\left[(w_l(\mathbf{s}_i))^{1/\alpha} + (w_l(\mathbf{s}_j))^{1/\alpha}\right]^\alpha}\right)}{p} \\
&= \lim_{p \to 0} 2 - \sum_{l=1}^{L}\left[w_l(\mathbf{s}_i)^{1/\alpha} + w_l(\mathbf{s}_j)^{1/\alpha}\right]^\alpha (1-p)^{-1+\sum_{l=1}^{L}\left[w_l(\mathbf{s}_i)^{1/\alpha} + w_l(\mathbf{s}_j)^{1/\alpha}\right]^\alpha} \\
&= 2 - \sum_{l=1}^{L}\left[w_l(\mathbf{s}_i)^{1/\alpha} + w_l(\mathbf{s}_j)^{1/\alpha}\right]^\alpha.
\end{aligned} \tag{13}
$$

References

Cohen, J. (1960) A Coefficient of Agreement for Nominal Scales. *Educational and Psychological Measurement*, **20**, 37–46.

117 Coles, S., Heffernan, J. and Tawn, J. (1999) Dependence Measures for Extreme Value Analyses. *Extremes*,
118   **2**, 339–365.

119 Reich, B. J. and Shaby, B. A. (2012) A hierarchical max-stable spatial model for extreme precipitation. *The*
120   *Annals of Applied Statistics*, **6**, 1430–1451.

121 Sullivan, B. L., Wood, C. L., Iliff, M. J., Bonney, R. E., Fink, D. and Kelling, S. (2009) eBird: A citizen-
122   based bird observation network in the biological sciences. *Biological Conservation*, **142**, 2282–2292.

123 Tawn, J. A. (1990) Modelling multivariate extreme value distributions. *Biometrika*, **77**, 245–253.

124 Wang, X. and Dey, D. K. (2010) Generalized extreme value regression for binary response data: An appli-
125   cation to B2B electronic payments system adoption. *The Annals of Applied Statistics*, **4**, 2000–2023.