Spatiotemporal Modeling of Extreme Events

Samuel Morris and Brian Reich

North Carolina State University

Motivation

- Average behavior is important to understand, but it does not paint the whole picture.
  - e.g. When constructing river levees, engineers need to be able to estimate a 100-year or 1000-year flood levels.
- In geostatistical analysis, kriging uses spatial correlation to help inform prediction at unknown locations.
- Want to explore computationally easy methods that are available in higher dimensions

Standard non-spatial analysis

- Block maxima:
  - Uses yearly maxima
  - Discards many observations
  - Models are fit using the generalized extreme value distribution
- Generalized extreme value distribution (GEV):

$$\Pr(Y_j < y) = G_j(y) = \exp\left\{ - \left[\left(1 + \xi_j \frac{y - \mu_j}{\sigma_j} \right)_+^{-1/\xi_j} \right] \right\}$$

Standard non-spatial analysis

- Peaks-over-threshold:
  - Incorporates more data than block maxima
  - Select a threshold, T, and fit data above the threshold using the generalized Pareto distribution
  - Autocorrelation may be an issue between observations (e.g. flood levels don't dissipate overnight)
- Generalized Pareto distribution (GPD):

$$\Pr(Y_j > y | Y_j > T) = F_j(y) = \left(1 + \xi_j \frac{y - T}{\sigma_j}\right)_+^{-1/\xi_j}$$

Multivariate analysis

- Multivariate max-stable and GPD models have nice features, but they are
  - computationally hard to work with
  - joint distribution only available in low dimension
- Pairwise likelihood approach (Huser and Davison, 2014)

Model objectives

- Our objective is to build a model that
  - has a flexible tail
  - has asymptotic spatial dependence
  - computation on the order of Gaussian models for large space-time datasets

Thresholding data

- We threshold the observed data at a high threshold T.
- Thresholded data:

$$Y_t^*(\mathbf{s}) = \begin{cases} Y_t(\mathbf{s}) & Y_t(\mathbf{s}) > T \\ T & Y_t(\mathbf{s}) \le T \end{cases}$$

- Allows tails of the distribution to speak for themselves.

Spatial skew-t distribution

- Assume observed data $Y_t(\mathbf{s})$ come from a skew-t (Zhang and El-Shaarawi, 2012)

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \alpha z_t + v_t(\mathbf{s})$$

where

- $\alpha \in \mathcal{R}$ controls the skewness
- $z_t \overset{iid}{\sim} N_{(0,\infty)}(0, \sigma_t^2)$ is a random effect
- $v_t(\mathbf{s})$ is a Gaussian process with variance σ_t^2 and Matérn correlation
- $\sigma_t^2 \overset{iid}{\sim} \text{IG}(a, b)$

Spatial skew-t distribution

- Conditioned on z_t and σ_t^2, $Y_t(\mathbf{s})$ is Gaussian
- Can use standard geostatistical methods to fit this model.
- Predictions can be made through kriging.
- Marginalizing over z_t and σ_t^2 (via MCMC),

$$Y_t(\mathbf{s}) \sim \text{skew-t}(\mu, \Sigma^*, \alpha, \text{df} = 2a)$$

where

  - μ is the location
  - a, b are the IG parameters for σ_t^2
  - $\Sigma^* = \frac{b}{a}\Sigma$ is a scale matrix, and Σ is a Matérn covariance matrix
  - $\alpha \in \mathcal{R}$ controls the skewness

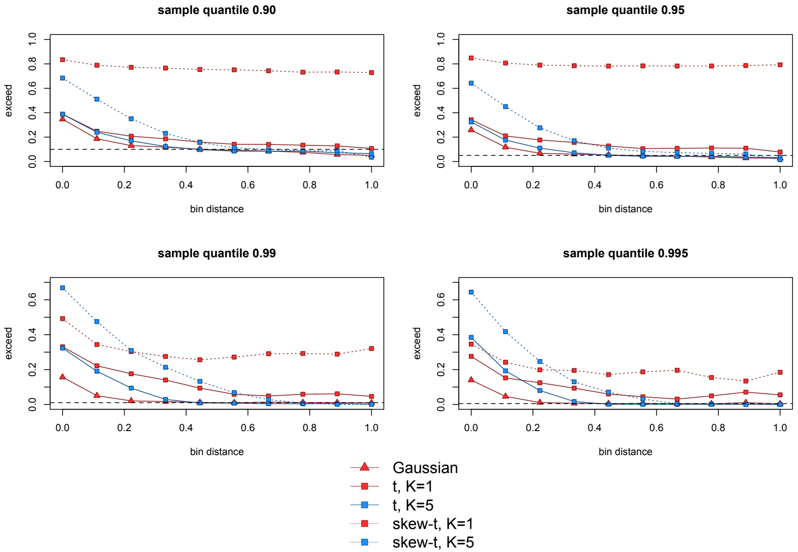
Long-range dependence

- The χ coefficient is a measure of extremal spatial correlation

$$\chi(\mathbf{h}) = \Pr(Y_t(\mathbf{s}) > c \mid Y_t(\mathbf{s}+\mathbf{h}) > c)$$

- This value shows asymptotic dependence that does not approach 0 as $\mathbf{h} \to \infty$ (Padoan, 2011)
- Deal with this through a daily random partition.

Simulated χ plots



Random daily partition

- Daily random partition allows z_t and σ_t^2 to vary by site.

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \alpha z_t(\mathbf{s}) + \sigma(\mathbf{s})v_t(\mathbf{s})$$

- Consider a set of daily knots $\{w_{t1}, \ldots, w_{tK}\}$ that define a daily partition $P_{t1}, \ldots, P_{tK}$ such that

$$P_{tk} = \{s : k = \arg\min_{\ell} ||\mathbf{s} - w_{t\ell}||\}$$

- For $\mathbf{s} \in P_{tk}$

$$z_t(\mathbf{s}) = z_{tk}$$
$$\sigma_t^2(\mathbf{s}) = \sigma_{tk}^2$$

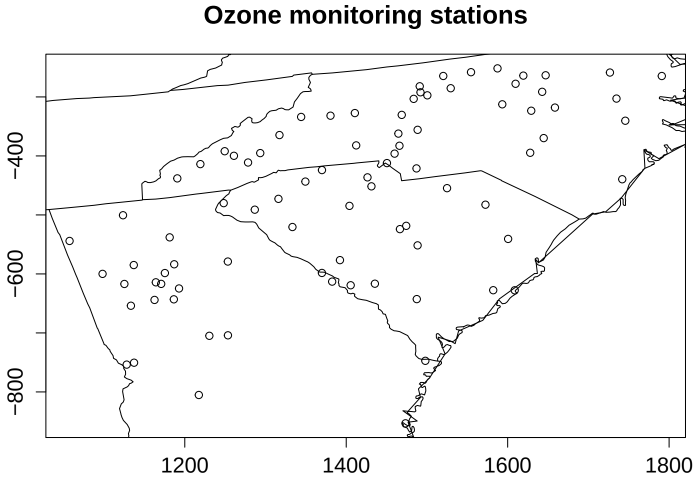
- Within each partition $Y_t(\mathbf{s})$ has the same MVT distribution as before.

MCMC details

- Three main steps:
  1. Impute missing observations and data below T
  2. Update parameters with standard random walk Metropolis Hastings or Gibbs sampling
  3. Make spatial predictions
- Priors are selected to be conjugate when possible.

Data analysis

- Ozone analysis at 85 sites in NC, SC, and GA for 92 days



Model comparisons

- 9 different analysis methods incorporating
  - Gaussian vs t vs skew-t marginal distribution
  - $K = 1$ partition vs $K = 5$ partitions
  - No thresholding vs thresholded
    - Thresholded data at $T = 0.90$ sample quantile
- All methods use a Matérn or exponential covariance ($\nu = 0.5$)
- Compare quantile and Brier scores using 5-fold cross validation (Gneiting and Raftery, 2007)
- Mean function modeled using a first-order spatial trend

Quantile score

- The quantile score for the τth quantile is

$$2\{I[y < \widehat{q}(\tau)] - \tau\}(\widehat{q} - y)$$

where:

  - y is a test set value
  - $\widehat{q}(\tau)$ is the estimated τth quantile

Brier score

- Brier score for predicting exceedance of threshold c

$$[e(c) - P(c)]^2$$

where

  - y is a test set value
  - $e(c) = I[y > c]$
  - $P(c)$ is the predicted probability of exceeding c

Five-fold cross-validation results

			Quantile				
Marginal	K	T	0.900	0.950	0.990	0.995	0.999
Gaussian	1	0	16.38	15.76	14.52	14.08	13.22
t	1	0	16.15	15.51	14.00	13.43	12.32
t	5	0	13.61	12.66	10.96	10.40	9.34
skew t	1	0	9.24	7.27	4.13	3.27	1.96
skew t	5	0	15.81	14.46	11.57	10.57	8.60
t	1	0.9	5.52	3.58	1.77	1.47	1.10
t	5	0.9	5.98	4.27	2.41	2.03	1.49
skew t	1	0.9	**4.91**	**3.16**	**1.45**	**1.16**	**0.82**
skew t	3	0.9	5.58	3.78	1.93	1.58	1.11

- Brier score results are similar.

Simulation study settings

- Data generated from 6 different settings.
  1. Gaussian
  2. t-1 with 4 degrees of freedom
  3. t-5 with 4 degrees of freedom
  4. skew t-1 with 4 degrees of freedom ($\alpha = 3$)
  5. skew t-5 with 4 degrees of freedom ($\alpha = 3$)
  6. Half-Gaussian, Half t-1 (spatial range = 0.4)
- Spatial settings
  - $\mathbf{s} \in [0, 1] \times [0, 1]$
  - Exponential covariance with range: 0.1
  - Ratio of spatial to nugget error: 0.9

Simulation study methods

- 5 different analysis methods
  1. Gaussian
  2. Skew t-1 ($T = 0$)
  3. Skew t-1 ($T = 0.9$)
  4. Skew t-5 ($T = 0$)
  5. Skew t-5 ($T = 0.9$)
- All methods use a Matérn covariance structure except for method 5 which uses an exponential covariance ($\nu = 0.5$)

Simulation study results

- Results are similar to the results from the data analysis
- Biggest gains come from thresholding.
- Using skew models give additional gain, but small relative to gain for thresholding.

Future work

- Comparison with extreme value analysis methods
- Including time in the model
  - AR(1): $Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \phi Y_{t-1}(\mathbf{s}) + \alpha z_t(\mathbf{s}) + v_t(\mathbf{s})$

Questions

- Any questions?
- Thank you for your attention.

References

- Huser, R. and Davison, A. C. (2014) Space-time modelling of extreme events. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **76**, 439–461.
- Padoan, S. A. (2011) Multivariate extreme models based on underlying skew-t and skew-normal distributions. *Journal of Multivariate Analysis*, **102**, 977–991.
- Zhang, H. and El-Shaarawi, A. (2010) On spatial skew-Gaussian processes and applications. *Environmetrics*, **21**, 33–47.