A new spatial model for points above a threshold

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1 Introduction

2 Statistical model

Let $Y_t(\mathbf{s}) \in \mathcal{R}$ be the observed value at location $\mathbf{s}$ on day t. To avoid bias in estimating tail parameters, we model the thresholded data

$$\tilde{Y}_t(\mathbf{s}) = \begin{cases} Y_t(\mathbf{s}) & Y_t(\mathbf{s}) > T \\ T & Y_t(\mathbf{s}) \leq T \end{cases} \tag{1}$$

where T is a pre-specified threshold.

We first specify a model for the complete data, $Y_t(\mathbf{s})$, and then study the induced model for thresholded data, $\tilde{Y}_t(\mathbf{s})$. The full data model is given in Section 2.2 assuming a skew normal distribution with a different variance each day. Computationally, the values below the threshold are updated using standard Bayesian missing data methods as described in Section 3. The skew normal representation is from (Minozzo and Ferracuti, 2012) and is the sum of a normal and half-normal random variable.

2.1 Half-normal

Let $u = \xi + \eta|z|$ where $Z \sim N(0, 1)$. Then U follows a half-normal distribution, $U \sim \text{HN}(\xi, \eta)$ (Wiper et al., 2008), and the density is given by

$$f_U(u) = \frac{\sqrt{\pi}}{\sqrt{2\eta^2}} \exp\left(-\frac{(u-\xi)^2}{2\eta^2}\right), \quad u > \xi. \tag{2}$$

2.2 Complete data

Consider the spatial process

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + e_t(\mathbf{s}) \tag{3}$$
$$e_t(\mathbf{s}) = \delta z_t(\mathbf{s}) + v_t(\mathbf{s}) \tag{4}$$

where $z_t(\mathbf{s}) = z_{tl}$ if $s \in P_{tl}$ where $P_{t1}, \dots, P_{tL}$ form a partition, and $z_{tl} \overset{iid}{\sim} \text{HN}(0, \sigma_t^2)$, $\delta \in (-1, 1)$ controls skew, and $v_t(\mathbf{s})$ is a spatial Gaussian process with mean zero and variance $\sigma_t^2(1 - \delta^2)$. Then $Y_t(\mathbf{s})$ is skew normal within each partition (Minozzo and Ferracuti, 2012). We model this with a Bayesian hierarchical model as follows. Let $w_{t1}, \dots, w_{tL}$ be partition centers so that $P_{tl} = \{\mathbf{s}_t : l = \arg\min_k ||\mathbf{s}_t - w_{tk}||\}$. Then

$$Y_t(\mathbf{s}) \mid \Theta = X_t(\mathbf{s})\beta + \delta z_t(\mathbf{s}) + v_t(\mathbf{s}) \tag{5}$$
$$z_{tl}(\mathbf{s}) \mid \Theta \sim \text{Half-Normal}(0, \sigma_t^2) \tag{6}$$
$$v_t(\mathbf{s}) \mid \Theta \sim \text{Matérn } (0, \Sigma) \tag{7}$$
$$\sigma \sim \text{IG}(\alpha, \beta) \tag{8}$$
$$\delta \sim \text{U}(-1, 1) \tag{9}$$
$$w_{tk} \sim \text{U}(\mathcal{D}) \tag{10}$$

where $\Theta = \{z_{t1}, \dots, z_{tL}, w_{t1}, \dots, w_{tL}, \beta, \sigma_t, \delta, \rho, \nu\}$; $l = \arg\min_k ||\mathbf{s} - w_k||$; Σ_t is a Matérn covariance matrix with variance $\sigma_t^2(1-\delta^2)$, spatial range ρ and smoothness ν; and $\mathcal{D}$ is the spatial domain of interest.

3 Computation

The MCMC for this model is fairly straightforward. First, we impute values below the threshold. Then, we update Θ using random walk MH or Gibbs sampling when appropriate. Finally, we make spatial predictions. Each requires the joint distribution for the complete data given Θ. As defined in 5, the distribution of $Y_t(\mathbf{s}) \mid \Theta$ is the usual multivariate normal distribution with a Matérn spatial covariance structure.

3.1 Imputation

We can use Gibbs sampling to update $\tilde{Y}_t(\mathbf{s})$ for observations that are below T, the thresholded value. Given Θ, $Y_t(\mathbf{s})$ has truncated normal full conditional with these parameter values. So we sample $Y_t(\mathbf{s}) \sim \text{TN}_{(-\infty, T)}$

3.2 Parameter updates

To update Θ given the current value of the complete data $\mathbf{Y}_1, \dots, \mathbf{Y}_T$, we use a standard Gibbs updates for all parameters except for the knot locations which are done using a Metropolis update. See Appendix A.1 for details regarding Gibbs sampling and $|u_t(\mathbf{s})|$.

3.3 Spatial prediction

Given $\mathbf{Y}_t$ the usual Kriging equations give the predictive distribution for $Y_t(\mathbf{s}^*)$ at prediction location $(\mathbf{s}^*)$

4 Data analysis

5 Conclusions

Acknowledgments

Appendix A.1: Posterior distributions

Conditional posterior of $U_{tl} \mid \dots$

For a single day, consider $Y(\mathbf{s})$ as given by (5) with two partitions. Then conditioned on the observations in partition 2,

$$Y_1 \mid Y_2 \sim \text{N}_{n_1}(\overline{\mu}, \overline{\Sigma}) \tag{11}$$

where $\overline{\mu} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y_2 - \mu_2)$, and $\overline{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Let $U_l \overset{iid}{\sim} \mathrm{HN}(0, \eta_0), l = 1, 2$ where $\eta_0 = a^2\sigma^2\delta^2$. Then conditional posterior of $U_1 \mid \ldots$ is

$$\begin{aligned}
\pi(U_1 \mid \mathbf{Y}_1) &\propto \exp\left\{-\frac{u^2}{2a^2\sigma^2\delta^2} - \frac{1}{\sigma^2(1-\delta^2)}\left[\mathbf{Y}_1 - \overline{\mu}\right]^T \overline{\Sigma}^{-1}\left[\mathbf{Y}_1 - \overline{\mu}\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{1}{a^2\delta^2} + \frac{\mathbf{1}^T\overline{\Sigma}^{-1}\mathbf{1}}{(1-\delta^2)}\right]u_1^2 - 2u_1\mathbf{1}^T\overline{\Sigma}^{-1}[\mathbf{Y}_1 - X_1\beta - \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{Y}_2 - \mu_2)]\right\} \\
&\propto \exp\left\{-\frac{(u_1 - \xi^*)^2}{2\eta^*}\right\}
\end{aligned}$$

where

$$\begin{aligned}
\xi^* &= \frac{\mathbf{1}^T\overline{\Sigma}^{-1}\left[\mathbf{Y}_1 - X_1\beta - \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{Y}_2 - \mu_2)\right]}{\eta^*} \\
\eta^* &= \left\{\frac{1}{\sigma^2}\left[\frac{1}{a^2\delta^2} + \frac{\mathbf{1}^T\overline{\Sigma}^{-1}\mathbf{1}}{(1-\delta^2)}\right]\right\}^{-1}
\end{aligned}$$

Conditional posterior of $\beta \mid \ldots$

Let $\beta \sim \mathrm{N}_p(0, \Lambda_0)$ where Λ_0 is a precision matrix. Then

$$\begin{aligned}
\pi(\beta \mid \ldots) &\propto \exp\left\{-\frac{1}{2}\beta^T\Lambda_0\beta - \sum_{t=1}^{T}\frac{1}{2}[\mathbf{Y}_t(\mathbf{s}) - X_t(\mathbf{s})\beta - \sigma\delta|u_t|]^T\Sigma^{-1}[\mathbf{Y}_t(\mathbf{s}) - X_t(\mathbf{s})\beta - u_t^*]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\beta^T\Lambda_p\beta - \sum_{t=1}^{T}2[\beta^T X_t(\mathbf{s})\Sigma^{-1}(\mathbf{Y}_t(\mathbf{s}) + u_t^*)]\right]\right\} \\
&\propto \mathrm{N}_p(\mu_p, \Lambda_p)
\end{aligned}$$

where

$$\begin{aligned}
\mu_p &= \Lambda_p^{-1}\left[X_t(\mathbf{s})^T\Sigma^{-1}(\mathbf{Y}_t(\mathbf{s}) + u_t^*)\right] \\
\Lambda_p &= \left(\Lambda_0 + \sum_{t=1}^{T}X_t(\mathbf{s})^T\Sigma^{-1}X_t(\mathbf{s})\right)
\end{aligned}$$

and Λ_p is a precision matrix.

Conditional posterior of $\sigma^2 \mid \ldots$

Let $\sigma_t^2 \overset{iid}{\sim} \mathrm{IG}(\alpha, \beta)$. Then for a given day,

$$\begin{aligned}
\pi(\sigma_t^2 \mid \ldots) &\propto (\sigma_t^2)^{-\alpha-1}\exp\left\{-\frac{\beta}{\sigma_t^2}\right\} - (\sigma_t^2)^{-L/2}\exp\left\{-\sum_{l=1}^{L}\frac{u_{tl}^2}{2a^2\sigma_t^2\delta^2}\right\}(\sigma_t^2)^{-n/2}\exp\left\{-\frac{[\mathbf{Y}_t - \mu]^T\Sigma^{-1}[\mathbf{Y}_t - \mu]}{2\sigma_t^2(1-\delta^2)}\right\} \\
&\propto (\sigma_t^2)^{-\alpha-L/2-n/2-1}\exp\left\{-\frac{1}{\sigma^2}\left[\beta + \sum_{l=1}^{L}\frac{u_{tl}^2}{2a^2\delta^2} + \frac{(\mathbf{Y}_t - \mu)^T\Sigma^{-1}(\mathbf{Y}_t - \mu)}{2(1-\delta^2)}\right]\right\} \\
&\propto \mathrm{IG}(\alpha^*, \beta^*)
\end{aligned}$$

where

$$\alpha^* = \alpha + L/2 + n/2$$
$$\beta^* = \beta + \sum_{l=1}^{L} \frac{u_{tl}^2}{2a^2\delta^2} + \frac{(\mathbf{Y}_t - \mu)^T \Sigma^{-1} (\mathbf{Y}_t - \mu)}{2(1-\delta^2)}$$

and L is the number of partitions.

Appendix A.2: MCMC Details

Priors

References

Minozzo, M. and Ferracuti, L. (2012) On the existence of some skew-normal stationary processes. *Chilean Journal of Statistics (ChJS)*, **3**, 157–170.

Wiper, M. P., Girón, F. J. and Pewsey, A. (2008) Objective Bayesian Inference for the Half-Normal and Half- t Distributions. *Communications in Statistics - Theory and Methods*, **37**, 3165–3185.