(Gneiting and Ranjan, 2011)
p. 415 (Table 4) gives suggested threshold weighting for Brier and Quantile scores

1 Continuous Ranked Probability Scores (CRPS)

- For each of the CRPS versions, they still integrate over all q and τ
- For these to be strictly proper, we need a finite first moment ($\xi < 1$)

1.1 Brier scores

The Brier probability scores is given by

$$\text{PS}[F(q), I(y \leq q)] = [F(q) - I(y \leq q)]^2 \tag{1}$$

where q is a quantile of interest.

The CRPS with Brier score is

$$S(f, y) = \int_{-\infty}^{\infty} \text{PS}[F(q), I(y \leq q)] u(q) \text{d}q \tag{2}$$

where $u(q)$ is a weighting function (see 2).

Discretized (integral w.r.t. a discrete Stieltjes measure)

$$S(f, y) = \frac{q_u - q_\ell}{I - 1} \sum_{i=1}^{I} u(q_i) \text{PS}[F(q), I(y \leq q)] \tag{3}$$

where (q_ℓ, q_u) is the range of interest

1.2 Quantile scores

The quantile score is given by

$$\text{QS}_\tau[q(\tau), y] = 2\{I[y \leq q(\tau)]\}[q(\tau) - y] \tag{4}$$

where $q(\tau)$ is the τth quantile of the density.

The CRPS with quantile score is

$$S(f, y) = \int_0^1 \text{QS}_\tau(q(\tau), y) v(\tau) \text{d}\tau \tag{5}$$

where $v(\tau)$ is a weighting function (see 2).

Discretized (integral w.r.t. a discrete Stieltjes measure)

$$S(f, y) = \frac{1}{J - 1} \sum_{j=1}^{J-1} v(\tau_j) \text{QS}_\tau[F^{-1}(\tau), y] \tag{6}$$

where $\tau_j = \frac{j}{J}$

2 Suggested threshold weighting functions

This comes from p. 415 - Table 4

	Brier Scores	Quantile Scores	
			(7)
Tails:	$u(q) = 1 - \phi_{a,b}(q)/\phi_{a,b}(a)$	$v(\tau) = (2\tau - 1)^2$	(8)
Right Tail:	$u(q) = \Phi_{a,b}(q)$	$v(\tau) = \tau^2$	(9)
Left Tail:	$u(q) = 1 - \Phi_{a,b}$	$v(\tau) = (1 - \tau)^2$	(10)

where $\phi_{a,b}$ and $\Phi_{a,b}$ are the pdf and cdf of a $N(a, b)$ distribution, and a and b are parameters to fine-tune the weighting function.

References

Gneiting, T. and Ranjan, R. (2011) Comparing Density Forecasts Using Threshold- and Quantile-Weighted Scoring Rules. *Journal of Business & Economic Statistics*, **29**, 411–422.