

**Web-based Supplementary Materials for A Space-time Skew- t Model for Threshold
Exceedances by Morris, Reich, Thibaud, and Cooley**

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Web Appendix A. MCMC details

The MCMC sampling for the model ?? is done using R (<http://www.r-project.org>). Whenever possible, we select conjugate priors (see Appendix Web Appendix B); however, for some of the parameters, no conjugate prior distributions exist. For these parameters, we use a random walk Metropolis-Hastings update step. In each Metropolis-Hastings update, we tune the algorithm during the burn-in period to give acceptance rates near 0.40.

Spatial knot locations

For each day, we update the spatial knot locations, $\mathbf{w}_1, \dots, \mathbf{w}_K$, using a Metropolis-Hastings block update. Because the spatial domain is bounded, we generate candidate knots using the transformed knots $\mathbf{w}_1^*, \dots, \mathbf{w}_K^*$ (see section ??) and a random walk bivariate Gaussian candidate distribution

$$\mathbf{w}_k^{*(c)} \sim N(\mathbf{w}_k^{*(r-1)}, s^2 I_2)$$

where $\mathbf{w}_k^{*(r-1)}$ is the location for the transformed knot at MCMC iteration $r - 1$, s is a tuning parameter, and I_2 is an identity matrix. After candidates have been generated for all K knots, the acceptance ratio is

$$R = \left\{ \frac{l[Y_t(\mathbf{s}|\mathbf{w}_1^{(c)}, \dots, \mathbf{w}_K^{(c)}, \dots)]}{l[Y_t(\mathbf{s}|\mathbf{w}_1^{(r-1)}, \dots, \mathbf{w}_K^{(r-1)}, \dots)]} \right\} \times \left\{ \frac{\prod_{k=1}^K \phi(\mathbf{w}_k^{(c)})}{\prod_{k=1}^K \phi(\mathbf{w}_k^{(r-1)})} \right\} \times \left\{ \frac{\prod_{k=1}^K p(\mathbf{w}_k^{*(c)})}{\prod_{k=1}^K p(\mathbf{w}_k^{*(r-1)})} \right\}$$

where l is the likelihood given in (??), and $p(\cdot)$ is the prior either taken from the time series given in (??) or assumed to be uniform over \mathcal{D} . The candidate knots are accepted with probability $\min\{R, 1\}$.

Spatial random effects

If there is no temporal dependence amongst the observations, we use a Gibbs update for z_{tk} , and the posterior distribution is given in Web Appendix B. If there is temporal dependence amongst the observations, then we update z_{tk} using a Metropolis-Hastings update. Because this model uses $|z_{tk}|$, we generate candidate random effects using the z_{tk}^* (see Section ??) and a random walk

Gaussian candidate distribution

$$z_{tk}^{*(c)} \sim N(z_{tk}^{*(r-1)}, s^2)$$

where $z_{tk}^{*(r-1)}$ is the value at MCMC iteration $r - 1$, and s is a tuning parameter. The acceptance ratio is

$$R = \left\{ \frac{l[Y_t(\mathbf{s})|z_{tk}^{(c)}, \dots]}{l[Y_t(\mathbf{s})|z_{tk}^{(r-1)}]} \right\} \times \left\{ \frac{p[z_{tk}^{(c)}]}{p[z_{tk}^{(r-1)}]} \right\}$$

where $p[\cdot]$ is the prior taken from the time series given in Section ?? . The candidate is accepted with probability $\min\{R, 1\}$.

Variance terms

When there is more than one site in a partition, then we update σ_{tk}^2 using a Metropolis-Hastings update. First, we generate a candidate for σ_{tk}^2 using an $IG(a^*/s, b^*/s)$ candidate distribution in an independence Metropolis-Hastings update where $a^* = (n_{tk} + 1)/2 + a$, $b^* = [Y_{tk}^T \Sigma_{tk}^{-1} Y_{tk} + z_{tk}^2]/2 + b$, n_{tk} is the number of sites in partition k on day t , and Y_{tk} and Σ_{tk}^{-1} are the observations and precision matrix for partition k on day t . The acceptance ratio is

$$R = \left\{ \frac{l[Y_t(\mathbf{s})|\sigma_{tk}^{2(c)}, \dots]}{l[Y_t(\mathbf{s})|\sigma_{tk}^{2(r-1)}]} \right\} \times \left\{ \frac{l[z_{tk}|\sigma_{tk}^{2(c)}, \dots]}{l[z_{tk}|\sigma_{tk}^{2(r-1)}, \dots]} \right\} \times \left\{ \frac{p[\sigma_{tk}^{2(c)}]}{p[\sigma_{tk}^{2(r-1)}]} \right\} \times \left\{ \frac{c[\sigma_{tk}^{2(r-1)}]}{c[\sigma_{tk}^{2(c)}]} \right\}$$

where $p[\cdot]$ is the prior either taken from the time series given in Section ?? or assumed to be $IG(a, b)$, and $c[\cdot]$ is the candidate distribution. The candidate is accepted with probability $\min\{R, 1\}$.

Spatial covariance parameters

We update the three spatial covariance parameters, $\log(\rho)$, $\log(\nu)$, γ , using a Metropolis-Hastings block update step. First, we generate a candidate using a random walk Gaussian candidate distribution

$$\log(\rho)^{(c)} \sim N(\log(\rho)^{(r-1)}, s^2)$$

where $\log(\rho)^{(r-1)}$ is the value at MCMC iteration $r - 1$, and s is a tuning parameter. Candidates are generated for $\log(\nu)$ and γ in a similar fashion. The acceptance ratio is

$$R = \left\{ \frac{\prod_{t=1}^T l[Y_t(\mathbf{s}) | \rho^{(c)}, \nu^{(c)}, \gamma^{(c)}, \dots]}{\prod_{t=1}^T l[Y_t(\mathbf{s}) | \rho^{(r-1)}, \nu^{(r-1)}, \gamma^{(r-1)}, \dots]} \right\} \times \left\{ \frac{p[\rho^{(c)}]}{p[\rho^{(r-1)}]} \right\} \times \left\{ \frac{p[\nu^{(c)}]}{p[\nu^{(r-1)}]} \right\} \times \left\{ \frac{p[\gamma^{(c)}]}{p[\gamma^{(r-1)}]} \right\}.$$

All three candidates are accepted with probability $\min\{R, 1\}$.

Web Appendix B. Posterior distributions

Conditional posterior of $z_{tk} \mid \dots$

If knots are independent over days, then the conditional posterior distribution of $|z_{tk}|$ is conjugate.

For simplicity, drop the subscript t , let $\tilde{z}_{tk} = |z_{tk}|$, and define

$$R(\mathbf{s}) = \begin{cases} Y(\mathbf{s}) - X(\mathbf{s})\beta & s \in P_l \\ Y(\mathbf{s}) - X(\mathbf{s})\beta - \lambda\tilde{z}(\mathbf{s}) & s \notin P_l \end{cases}$$

Let

$R_1 =$ the vector of $R(\mathbf{s})$ for $s \in P_l$

$R_2 =$ the vector of $R(\mathbf{s})$ for $s \notin P_l$

$$\Omega = \Sigma^{-1}.$$

Then

$$\begin{aligned} \pi(z_l | \dots) &\propto \exp \left\{ -\frac{1}{2} \left[\begin{pmatrix} R_1 - \lambda\tilde{z}_l \mathbf{1} \\ R_2 \end{pmatrix}^T \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \begin{pmatrix} R_1 - \lambda\tilde{z}_l \mathbf{1} \\ R_2 \end{pmatrix} + \frac{\tilde{z}_l^2}{\sigma_l^2} \right] \right\} I(z_l > 0) \\ &\propto \exp \left\{ -\frac{1}{2} [\Lambda_l \tilde{z}_l^2 - 2\mu_l \tilde{z}_l] \right\} \end{aligned}$$

where

$$\mu_l = \lambda(R_1^T \Omega_{11} + R_2^T \Omega_{21}) \mathbf{1}$$

$$\Lambda_l = \lambda^2 \mathbf{1}^T \Omega_{11} \mathbf{1} + \frac{1}{\sigma_l^2}.$$

Then $\tilde{Z}_l | \dots \sim N_{(0,\infty)}(\Lambda_l^{-1}\mu_l, \Lambda_l^{-1})$

Conditional posterior of $\beta | \dots$

Let $\beta \sim N_p(0, \Lambda_0)$ where Λ_0 is a precision matrix. Then

$$\begin{aligned} \pi(\beta | \dots) &\propto \exp \left\{ -\frac{1}{2}\beta^T \Lambda_0 \beta - \frac{1}{2} \sum_{t=1}^T [\mathbf{Y}_t - X_t \beta - \lambda |z_t|]^T \Omega [\mathbf{Y}_t - X_t \beta - \lambda |z_t|] \right\} \\ &\propto \exp \left\{ -\frac{1}{2} \left[\beta^T \Lambda_\beta \beta - 2 \sum_{t=1}^T [\beta^T X_t^T \Omega (\mathbf{Y}_t - \lambda |z_t|)] \right] \right\} \\ &\propto N(\Lambda_\beta^{-1} \mu_\beta, \Lambda_\beta^{-1}) \end{aligned}$$

where

$$\begin{aligned} \mu_\beta &= \sum_{t=1}^T [X_t^T \Omega (\mathbf{Y}_t - \lambda |z_t|)] \\ \Lambda_\beta &= \Lambda_0 + \sum_{t=1}^T X_t^T \Omega X_t. \end{aligned}$$

Conditional posterior of $\sigma^2 | \dots$

In the case where $L = 1$ and temporal dependence is negligible, then σ^2 has a conjugate posterior distribution. Let $\sigma_t^2 \stackrel{iid}{\sim} \text{IG}(\alpha_0, \beta_0)$. For simplicity, drop the subscript t . Then

$$\begin{aligned} \pi(\sigma^2 | \dots) &\propto (\sigma^2)^{-\alpha_0 - 1/2 - n/2 - 1} \exp \left\{ -\frac{\beta_0}{\sigma^2} - \frac{|z|^2}{2\sigma^2} - \frac{(\mathbf{Y} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{Y} - \boldsymbol{\mu})}{2\sigma^2} \right\} \\ &\propto (\sigma^2)^{-\alpha_0 - 1/2 - n/2 - 1} \exp \left\{ -\frac{1}{\sigma^2} \left[\beta_0 + \frac{|z|^2}{2} + \frac{1}{2} (\mathbf{Y} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{Y} - \boldsymbol{\mu}) \right] \right\} \\ &\propto \text{IG}(\alpha^*, \beta^*) \end{aligned}$$

where

$$\begin{aligned} \alpha^* &= \alpha_0 + \frac{1}{2} + \frac{n}{2} \\ \beta^* &= \beta_0 + \frac{|z|^2}{2} + \frac{1}{2} (\mathbf{Y} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{Y} - \boldsymbol{\mu}). \end{aligned}$$

In the case that $L > 1$, a random walk Metropolis Hastings step will be used to update σ_{lt}^2 .

Conditional posterior of $\lambda \mid \dots$

For convergence purposes we model $\lambda = \lambda_1 \lambda_2$ where

$$\lambda_1 = \begin{cases} +1 & \text{w.p.0.5} \\ -1 & \text{w.p.0.5} \end{cases} \quad (1)$$

$$\lambda_2^2 \sim IG(\alpha_\lambda, \beta_\lambda). \quad (2)$$

$$(3)$$

Then

$$\begin{aligned} \pi(\lambda_2 \mid \dots) &\propto \lambda_2^{2(-\alpha_\lambda-1)} \exp \left\{ -\frac{\beta_\lambda}{\lambda_2^2} \right\} \prod_{t=1}^T \prod_{k=1}^K \frac{1}{\lambda_2} \exp \left\{ -\frac{z_{tk}^2}{2\lambda_2^2 \sigma_{tk}^2} \right\} \\ &\propto \lambda_2^{2(-\alpha_\lambda-kt-1)} \exp \left\{ -\frac{1}{\lambda_2^2} \left[\beta_\lambda + \frac{z^2}{2\sigma_{tk}^2} \right] \right\} \end{aligned}$$

Then $\lambda_2 \mid \dots \sim IG \left(\alpha_\lambda + kt, \beta_\lambda + \frac{z^2}{2\sigma_{tk}^2} \right)$

Web Appendix C. Proof that $\lim_{h \rightarrow \infty} \pi(h) = 0$

Let c be the midpoint of \mathbf{s}_1 and \mathbf{s}_2 . Define A as the circle centered at c with radius $h/2$ where $h = \|\mathbf{s}_1 - \mathbf{s}_2\|$ is the distance between sites \mathbf{s}_1 and \mathbf{s}_2 . Consider a homogeneous spatial Poisson process with intensity μ . Define over A with intensity given by

$$\mu(A) = \lambda_{PP}|A| = \lambda_{PP}\pi \left(\frac{h}{2} \right)^2 = \lambda_{PP}^* h^2.$$

Consider a partition of A as the circle with center and radius $h/2$. Then into four regions, B_1, B_2, R_1, R_2 as seen in Web Figure 1.

[Figure 1 about here.]

Let N_j be the number of knots in $B_j, j = 1, 2$. Then

$$P(\mathbf{s}_1 \in P_i, \mathbf{s}_2 \in P_{j \neq i}) \geq P(N_1 > 0, N_2 > 0) \quad (4)$$

since knots in both B_1 and B_2 is sufficient, but not necessary, to ensure that \mathbf{s}_1 and \mathbf{s}_2 are in different partitions almost surely if two or more points are in A . Let $N(A)$ be the number of points in A , and let

$$\mu(A) = \mu|A| = \mu\pi \left(\frac{h}{2}\right)^2 = \lambda h^2.$$

Then

$$\begin{aligned} P[N(A) \geq 2] &= 1 - P[N(A) = 0] - P[N(A) = 1] \\ &= 1 - \exp\{-\lambda h^2\} - \lambda h^2 \exp\{-\lambda h^2\} \\ &= 1 - (1 + \lambda h^2) \exp\{-\lambda h^2\} \end{aligned}$$

partition sets. By definition of a Poisson process, N_1 and N_2 are independent and thus $P(N_1 > 0, N_2 > 0) = P(N_1 > 0)P(N_2 > 0)$ and the intensity measure over B_1 is given by

$$\begin{aligned} \mu(B_1) &= \lambda_{PP}|B_1| = \lambda_{PP} \frac{h^2}{4} \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \\ &= \lambda_{PPB1}^* h^2. \end{aligned} \tag{5}$$

So,

$$P(\mathbf{s}_1 \in P_i, \mathbf{s}_2 \in P_{j \neq i}) \geq P(N_1 > 0)^2 = [1 - P(N_1 = 0)]^2 = [1 - \exp(-\lambda_{PPB1}^* h^2)]^2 \tag{6}$$

which goes to one as $h \rightarrow \infty$, as h goes to infinity.

Web Appendix D. Skew- t distribution

Univariate skew- t distribution

We say that Y follows a univariate extended skew- t distribution with location $\xi \in \mathcal{R}$, scale $\omega > 0$, skew parameter $\alpha \in \mathcal{R}$, and degrees of freedom ν if has distribution function

$$f_{\text{EST}}(y) = 2f_T(z; \nu)F_T \left[\alpha z \sqrt{\frac{\nu+1}{\nu+z^2}}; \nu+1 \right] \tag{7}$$

where $f_T(t; \nu)$ is a univariate Student's t with ν degrees of freedom, $F_T(t; \nu) = P(T < t)$, and $z = (y - \xi)/\omega$.

Multivariate skew- t distribution

If $\mathbf{Z} \sim \text{ST}_d(0, \bar{\boldsymbol{\Omega}}, \boldsymbol{\alpha}, \eta)$ is a d -dimensional skew- t distribution, and $\mathbf{Y} = \xi + \boldsymbol{\omega}\mathbf{Z}$, where $\boldsymbol{\omega} = \text{diag}(\omega_1, \dots, \omega_d)$, then the density of Y at y is

$$f_y(\mathbf{y}) = \det(\boldsymbol{\omega})^{-1} f_z(\mathbf{z}) \quad (8)$$

where

$$f_z(\mathbf{z}) = 2t_d(\mathbf{z}; \bar{\boldsymbol{\Omega}}, \eta) T \left[\boldsymbol{\alpha}^T \mathbf{z} \sqrt{\frac{\eta + d}{\nu + Q(\mathbf{z})}}; \eta + d \right] \quad (9)$$

$$\mathbf{z} = \boldsymbol{\omega}^{-1}(\mathbf{y} - \xi) \quad (10)$$

where $t_d(\mathbf{z}; \bar{\boldsymbol{\Omega}}, \eta)$ is a d -dimensional Student's t -distribution with scale matrix $\bar{\boldsymbol{\Omega}}$ and degrees of freedom η , $Q(z) = \mathbf{z}^T \bar{\boldsymbol{\Omega}}^{-1} \mathbf{z}$ and $T(\cdot; \eta)$ denotes the univariate Student's t distribution function with η degrees of freedom (Azzalini and Capitanio, 2014).

Extremal dependence

For a bivariate skew- t random variable $\mathbf{Y} = [Y(\mathbf{s}), Y(\mathbf{t})]^T$, the $\chi(h)$ statistic (Padoan, 2011) is given by

$$\chi(h) = \bar{F}_{\text{EST}} \left\{ \frac{[x_1^{1/\eta} - \varrho(h)]\sqrt{\eta+1}}{\sqrt{1 - \varrho(h)^2}}; 0, 1, \alpha_1, \tau_1, \eta + 1 \right\} + \bar{F}_{\text{EST}} \left\{ \frac{[x_2^{1/\eta} - \varrho(h)]\sqrt{\eta+1}}{\sqrt{1 - \varrho(h)^2}}; 0, 1, \alpha_2, \tau_2, \eta + 1 \right\}, \quad (11)$$

where \bar{F}_{EST} is the univariate survival extended skew- t function with zero location and unit scale,

$\varrho(h) = \text{cor}[y(\mathbf{s}), y(\mathbf{t})]$, $\alpha_j = \alpha_i \sqrt{1 - \varrho^2}$, $\tau_j = \sqrt{\eta+1}(\alpha_j + \alpha_i \varrho)$, and $x_j = F_T(\bar{\alpha}_i \sqrt{\eta+1}; 0, 1, \eta) / F_T(\bar{\alpha}_j \sqrt{\eta+1}; 0, 1, \eta)$;

with $j = 1, 2$ and $i = 2, 1$ and where $\bar{\alpha}_j = (\alpha_j + \alpha_i \varrho) / \sqrt{1 + \alpha_i^2 [1 - \varrho(h)^2]}$.

Proof that $\lim_{h \rightarrow \infty} \chi(h) > 0$

Consider the bivariate distribution of $\mathbf{Y} = [Y(\mathbf{s}), Y(\mathbf{t})]^T$, with $\varrho(h)$ given by (??). So, $\lim_{h \rightarrow \infty} \varrho(h) = 0$. Then

$$\lim_{h \rightarrow \infty} \chi(h) = \bar{F}_{\text{EST}} \left\{ \sqrt{\eta + 1}; 0, 1, \alpha_1, \tau_1, \eta + 1 \right\} + \bar{F}_{\text{EST}} \left\{ \sqrt{\eta + 1}; 0, 1, \alpha_2, \tau_2, \eta + 1 \right\}. \quad (12)$$

Because the extended skew- t distribution is not bounded above, for all $\bar{F}_{\text{EST}}(x) = 1 - F_{\text{EST}(x)} > 0$ for all $x < \infty$. Therefore, for a skew- t distribution, $\lim_{h \rightarrow \infty} \chi(h) > 0$.

Web Appendix E. Simulation study pairwise difference results

The following tables show the methods that have significantly different Brier scores when using a Wilcoxon-Nemenyi-McDonald-Thompson test. In each column, different letters signify that the methods have significantly different Brier scores. For example, there is significant evidence to suggest that method 1 and method 4 have different Brier scores at $q(0.90)$, whereas there is not significant evidence to suggest that method 1 and method 2 have different Brier scores at $q(0.90)$. In each table group A represents the group with the lowest Brier scores. Groups are significant with a familywise error rate of $\alpha = 0.05$.

[Table 1 about here.]

[Table 2 about here.]

[Table 3 about here.]

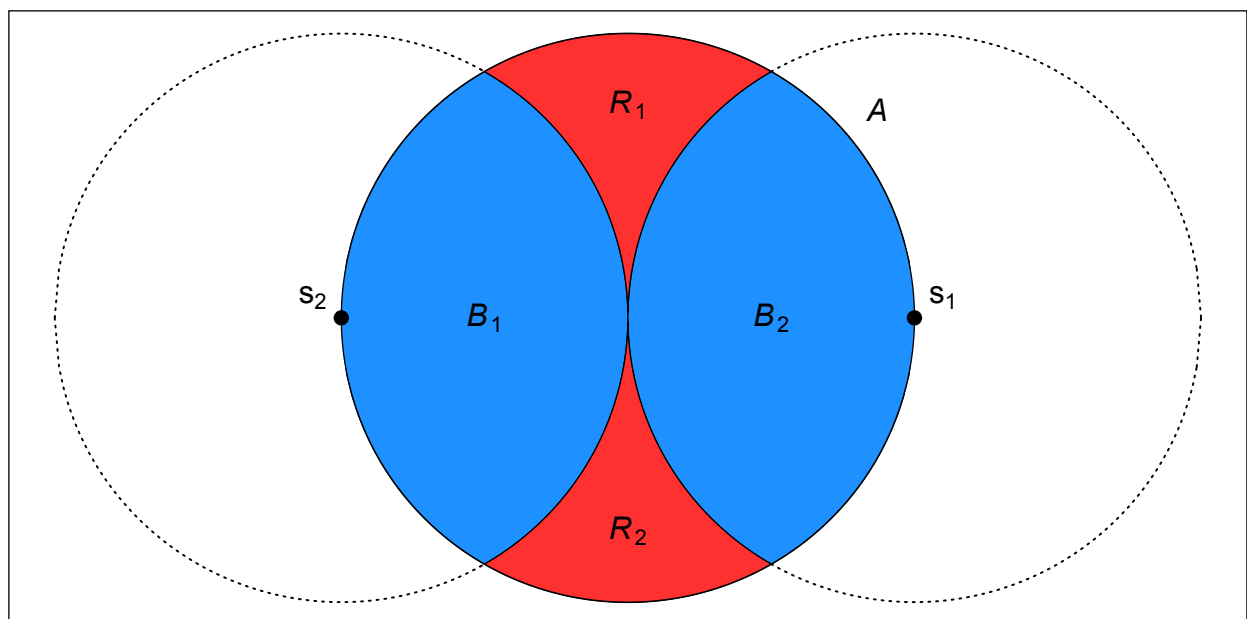
[Table 4 about here.]

[Table 5 about here.]

References

Azzalini, A. and Capitanio, A. (2014). *The Skew-Normal and Related Families*. Institute of Mathematical Statistics Monographs. Cambridge University Press.

Padoan, S. A. (2011). Multivariate extreme models based on underlying skew- and skew-normal distributions. *Journal of Multivariate Analysis* **102**, 977–991.



Web Figure 1. Illustration of the partition of A .

Web Table 1*Setting 1 – Gaussian marginal, $K = 1$ knot*

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	A	A	A	A B
Method 2	A	A	A	A
Method 3	B	B	C	B
Method 4	A	A	A B	A B
Method 5	B	B	B C	A B
Method 6	C	C	D	C

Web Table 2									
Setting 2 – Skew- <i>t</i> marginal, <i>K</i> = 1 knot									
		<i>q</i> (0.90)		<i>q</i> (0.95)		<i>q</i> (0.98)		<i>q</i> (0.99)	
Method 1		C		B		B C		B	
Method 2	A			A		A		A	
Method 3		B	C	A	B	A	B	A	B
Method 4	A	B		B		B		A	
Method 5		D		C		C		B	
Method 6		E		D		D		C	

Web Table 3*Setting 3 – Skew-t marginal, $K = 5$ knots*

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	B	C	B	B
Method 2	B	C	B	B
Method 3	A	B	B	B
Method 4	A	A	A	A
Method 5	A	A	A	A
Method 6	C	D	C	C

Web Table 4
Setting 4 – Max-stable

	$q(0.90)$		$q(0.95)$		$q(0.98)$	$q(0.99)$	
Method 1	A	B	B		B	C	
Method 2	B		B	C	B	B	C
Method 3	C D		C		B	B	
Method 4	D		D		C	C	
Method 5	C		C		B	B	C
Method 6	A	A		A		A	

Web Table 5*Setting 5 – Transformation below $T = q(0.80)$*

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	C	B	C	C
Method 2	B	B	B	A B
Method 3	A	A	A	A
Method 4	B C	B	B	B C
Method 5	B	B	B C	C
Method 6	D	C	D	D