Spatiotemporal Modeling of Extreme Events

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JSM 2014, Boston

Motivation

- Average behavior is important to understand, but it does not paint the whole picture.
  - e.g. When constructing river levees, engineers need to be able to estimate a 100-year or 1000-year flood levels.
- Spatial extreme borrows strength across space to estimate return levels and make predictions at unknown locations..
- In geostatistical analysis, kriging uses spatial correlation for prediction.
- Want to explore similar methods for extremes.

Standard analysis - Block maxima

- Uses yearly maxima
- Discards many observations
- Models are fit using the generalized extreme value distribution
- For a spatial analysis, max-stable processes give an appropriate limiting distribution

Standard analysis - Peaks over threshold

- Incorporates more data than block maxima
- Select a threshold, T, and fit data above the threshold using the generalized Pareto distribution
- Temporal dependence may be an issue between observations (e.g. flood levels don't dissipate overnight)
- Generalized Pareto distribution (GPD) is used for the exceedances.

Multivariate analysis

- Multivariate max-stable and GPD models have nice features, but they are
  - computationally challenging to work with
  - joint distribution only available in low dimension
- Pairwise likelihood approach (Huser and Davison, 2014)

Model objectives

- Our objective is to build a model that
  - has marginal distribution with a flexible tail
  - has asymptotic spatial dependence
  - has computation on the order of Gaussian models for large space-time datasets

Thresholding data

- We threshold the observed data at a high threshold T.
- Thresholded data:

$$Y_t^*(\mathbf{s}) = \begin{cases} Y_t(\mathbf{s}) & Y_t(\mathbf{s}) > T \\ T & Y_t(\mathbf{s}) \leq T \end{cases}$$

- Allows tails of the distribution to speak for themselves.

χ coefficient

- The χ coefficient is a measure of extremal dependence
- Specifically, we focus on $\chi(\mathbf{h})$ for the upper tail given by

$$\chi(\mathbf{h}) = \lim_{c \to \infty} \Pr(Y(\mathbf{s}) > c \mid Y(\mathbf{s} + \mathbf{h}) > c)$$

- If $\chi(\mathbf{h}) = 0$, then observations are asymptotically independent at distance $\mathbf{h}$.
- We expect $\lim_{\mathbf{h} \to \infty} \chi(\mathbf{h}) = 0$.

Gaussian spatial model

- In geostatistics $Y(\mathbf{s})$ are often modeled using a Gaussian process with mean function $\mu(\mathbf{s})$ and covariance function $\rho(\mathbf{h})$.
- Model properties:
  - Nice computing properties (closed-form likelihood)
  - For a Gaussian spatial model $\lim_{c\to\infty} \chi(c) = 0$ regardless of the strength of the correlation in the bulk of the distribution.
  - Tail is not flexible (Gaussian is light tailed)

Spatial skew-t distribution

- Assume observed data $Y_t(\mathbf{s})$ come from a skew-t (Zhang and El-Shaarawi, 2012)

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \alpha z_t + v_t(\mathbf{s})$$

where

  - $\alpha \in \mathcal{R}$ controls the skewness
  - $z_t \overset{iid}{\sim} N_{(0,\infty)}(0, \sigma_t^2)$ is a random effect
  - $v_t(\mathbf{s})$ is a Gaussian process with variance σ_t^2 and Matérn correlation
  - $\sigma_t^2 \overset{iid}{\sim} \mathrm{IG}(a, b)$

Spatial skew-t distribution

- Conditioned on z_t and σ_t^2, $Y_t(\mathbf{s})$ is a Gaussian spatial model
- Can use standard geostatistical methods to fit this model.
- Predictions can be made through Kriging.
- Marginalizing over z_t and σ_t^2 (via MCMC),

$$Y_t(\mathbf{s}) \sim \text{skew-t}(\mu, \Sigma^*, \alpha, \text{df} = 2a)$$

where

  - μ is the location
  - a, b are the IG parameters for σ_t^2
  - $\Sigma^* = \frac{b}{a}\Sigma$ is a scale matrix, and Σ is a Matérn covariance matrix
  - $\alpha \in \mathcal{R}$ controls the skewness

Spatial skew-t distribution

- Model properties
  - Has flexible tail controlled by skewness α and degrees of freedom $2a$
  - For a skew-t distribution $\lim_{c\to\infty}\chi(c) > 0$ Padoan, 2011)
  - Computation that is on the order of Gaussian computation
- For this distribution, $\chi(\mathbf{h})$ shows asymptotic dependence that does not approach 0 as $\mathbf{h} \to \infty$
- This occurs because all observations (near and far) share the same z_t and σ_t^2.
- We deal with this through a daily random partition (similar to Huser and Davison).

Daily random partition

- Daily random partition allows z_t and σ_t^2 to vary by site.

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \alpha z_t(\mathbf{s}) + \sigma(\mathbf{s})v_t(\mathbf{s})$$

- Consider a set of daily knots $w_{tk} \sim$ Uniform that define a random daily partition $P_{t1}, \dots, P_{tK}$ such that

$$P_{tk} = \{s : k = \arg\min_\ell ||\mathbf{s} - w_{t\ell}||\}$$

- For $\mathbf{s} \in P_{tk}$

$$z_t(\mathbf{s}) = z_{tk}$$
$$\sigma_t^2(\mathbf{s}) = \sigma_{tk}^2$$

- Within each partition $Y_t(\mathbf{s})$ has the same MV skew-t distribution as before.

Example daily partition

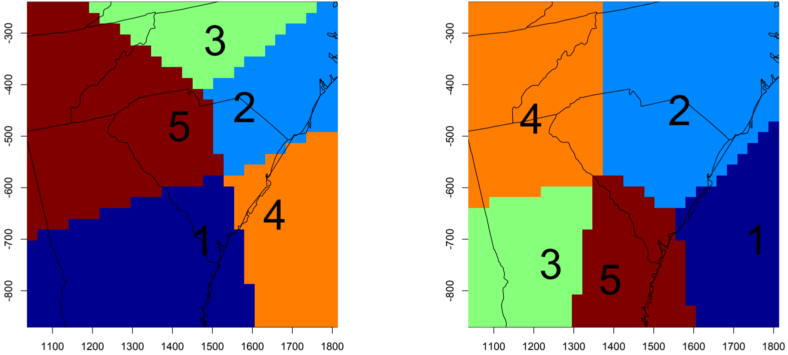
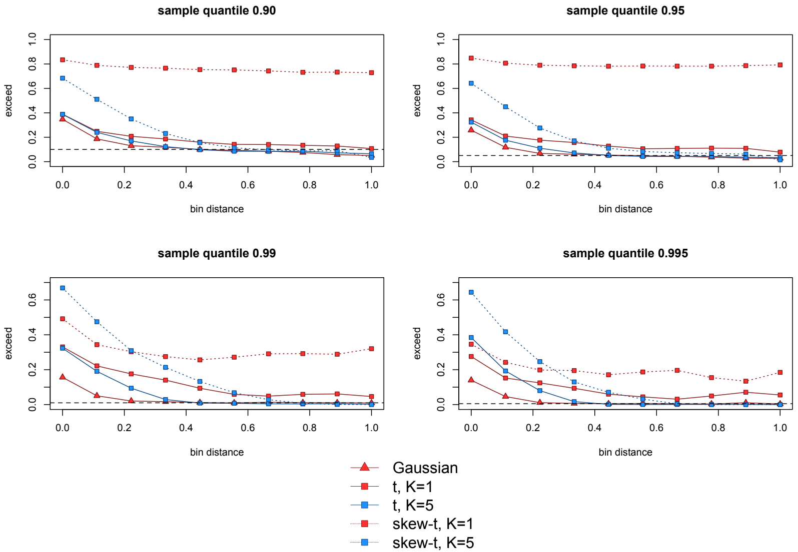


Figure: Two sample partitions (number is at partition center)

Simulated χ plots



Sample simulated datasets

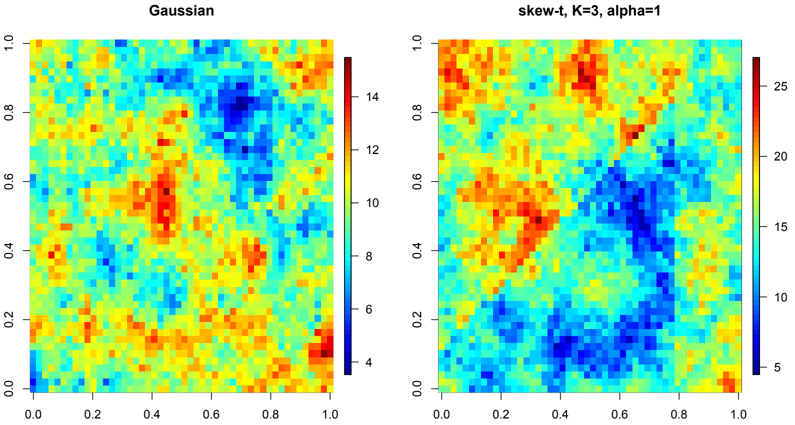


Figure: Gaussian and skew-t with 3 partitions

Spatiotemporal Model

- We can account for time in one of two ways
  - The mean: e.g. AR(1)
  - Three dimensional covariance model (e.g. Huser and Davison, 2014)

MCMC details

- Three main steps:
  1. Impute missing observations and censored data below T
  2. Update parameters with standard random walk Metropolis Hastings or Gibbs sampling
  3. Make spatial predictions
- Priors are selected to be conjugate when possible.

Data analysis

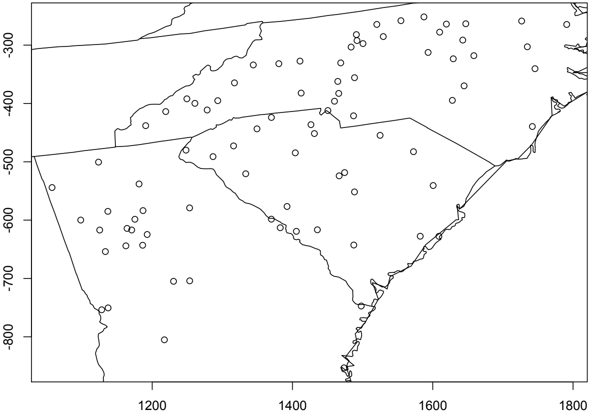


Figure: Ozone monitoring station locations

Data analysis

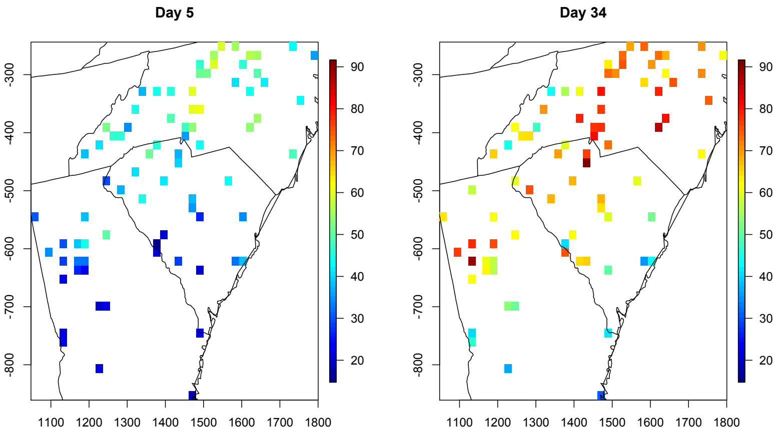


Figure: Max 8-hour ozone measurements at 85 sites in NC, SC, and GA for days 5 and 34.

Exploratory data analysis

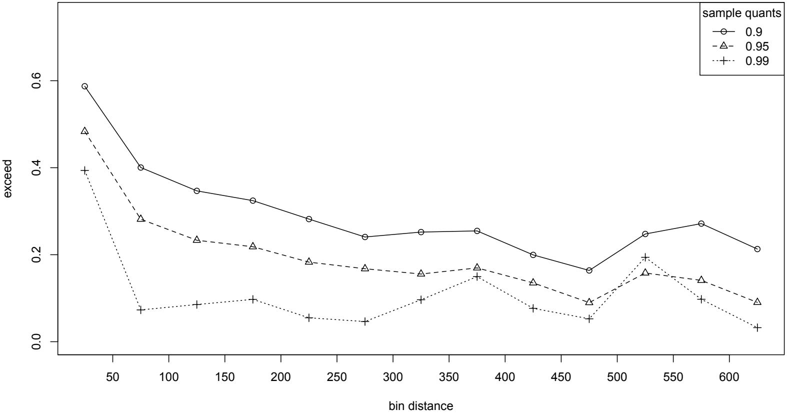


Figure: χ-plot for residuals of ozone sample quantiles

Model comparisons

- 9 different analysis methods incorporating
  - Gaussian vs t vs skew-t marginal distribution
  - $K = 1$ partition vs $K = 3$ partitions
  - No thresholding vs thresholding at $T = 0.90$ sample quantile
- All methods use a Matérn or exponential covariance ($\nu = 0.5$)
- Compare quantile and Brier scores using 5-fold cross validation (Gneiting and Raftery, 2007)
- Mean function modeled using a first-order spatial trend

Quantile score for cross-validation

- The quantile score for the τth quantile is

$$2\{I[y < \widehat{q}(\tau)] - \tau\}(\widehat{q} - y)$$

where:

  - y is a test set value
  - $\widehat{q}(\tau)$ is the estimated τth quantile

Brier score

- The Brier score for predicting exceedance of threshold c is

$$[e(c) - P(c)]^2$$

  where
  - y is a test set value
  - $e(c) = I[y > c]$
  - $P(c)$ is the predicted probability of exceeding c

Five-fold cross-validation results

Marginal	K	T	Quantile 0.950	0.980	0.990	0.995	0.999
Gaussian	1	0	39.820	17.539	9.167	4.720	1.057
t	1	0	**31.008**	**13.898**	7.229	**3.405**	0.879
t	3	0	31.213	13.920	**7.218**	3.498	0.918
t	1	0.9	32.221	14.519	7.549	3.604	0.896
t	3	0.9	38.842	16.781	8.434	4.180	1.020
skew-t	1	0	31.845	14.542	7.533	3.645	**0.844**
skew-t	1	0.9	32.132	14.296	7.484	3.497	0.890
skew-t	3	0	33.653	15.453	8.119	4.338	1.188
skew-t	3	0.9	32.157	14.727	7.794	3.825	0.917

Table: Selected Brier score results from five-fold cross-validation ($\times$1000)

- Quantile score results are similar.

Simulation study

- 6 different data settings:
  - Gaussian vs t vs skew-t marginal distribution
  - $K = 1$ partition vs $K = 5$ partitions
- Results are similar to the results from the data analysis
- Biggest gains come from thresholding.
- Using skew models give additional gain, but small relative to gain for thresholding.

Future work

- Comparison with extreme value analysis methods
- Reporting of compliance with EPA ozone standards
- Including time in the model via standard spatiotemporal Gaussian models.

Questions

- Questions?
- Thank you for your attention.
- Acknowledgment: This work was funded by EPA STAR award R835228.

References

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