A new spatial model for points above a threshold

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1 Introduction

2 Statistical model

Let $Y_t(\mathbf{s}) \in \mathcal{R}$ be the observed value at location $\mathbf{s}$ on day t. To avoid bias in estimating tail parameters, we model the thresholded data

$$\tilde{Y}_t(\mathbf{s}) = \begin{cases} Y_t(\mathbf{s}) & Y_t(\mathbf{s}) > T \\ T & Y_t(\mathbf{s}) \leq T \end{cases} \tag{1}$$

where T is a pre-specified threshold.

We first specify a model for the complete data, $Y_t(\mathbf{s})$, and then study the induced model for thresholded data, $\tilde{Y}_t(\mathbf{s})$. The full data model is given in Section 2.1 assuming a multivariate normal distribution with a different variance each day. Computationally, the values below the threshold are updated using standard Bayesian missing data methods as described in Section 3.

2.1 Complete data

Consider the spatial process

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + e_t(\mathbf{s}) \tag{2}$$

$$e_t(\mathbf{s}) = \sigma\delta|u_t(\mathbf{s})| + v_t(\mathbf{s}) \tag{3}$$

where $u_t(\mathbf{s}) = u_{tl}$ if $s \in P_{tl}$ where $P_{t1}, \ldots, P_{tL}$ form a partition, and $u_{tl} \stackrel{iid}{\sim} N(0,1)$, $\delta \in (-1,1)$ controls skew, and $v_t(\mathbf{s})$ is a spatial process with mean zero and variance $\sigma^2(1-\delta^2)$. Then $Y_t(\mathbf{s})$ is skew normal within each partition (Minozzo and Ferracuti, 2012). We model this with a Bayesian hierarchical model as follows. Let $w_{t1}, \ldots, w_{tL}$ be partition centers so that P_{tl} includes all spatial locations $\mathbf{s}$ that are within the partition. Then

$$Y_t(\mathbf{s}) \mid \Theta = \mu_t(\mathbf{s}) + v_t(\mathbf{s}) \tag{4}$$

$$\mu_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \sigma\delta|u_{tl}| \tag{5}$$

where $l = \arg\min_j ||\mathbf{s} - w_j||$ and $\Theta = \{u_{t1}, \ldots, u_{tL}, w_{t1}, \ldots, w_{tL}, \beta, \rho, \nu, \sigma\}$ are the random effects, knot locations, and parameters for the mean, and spatial covariance.

3 Computation

The MCMC for this model is fairly straightforward. First, we impute values below the threshold. Then, we update Θ using random walk MH or Gibbs sampling when appropriate. Finally, we make spatial predictions. Each requires the joint distribution for the complete data given Θ. As defined in 4, the distribution of $Y_t(\mathbf{s}) \mid \Theta$ is the usual multivariate normal distribution with a Matérn spatial covariance structure.

3.1 Imputation

We can use Gibbs sampling to update $\tilde{Y}_t(\mathbf{s})$ for observations that are below T, the thresholded value. Given Θ, $Y_t(\mathbf{s})$ has truncated normal full conditional with these parameter values. So we sample $Y_t(\mathbf{s}) \sim \text{TN}_{(-\infty,T)}$

3.2 Parameter updates

To update Θ given the current value of the complete data $\mathbf{Y}_1, \ldots, \mathbf{Y}_T$, we use a standard Gibbs updates for all parameters except for the knot locations which are done using a Metropolis update. See Appendix A.1 for details regarding Gibbs sampling and $|u_t(\mathbf{s})|$.

3.3 Spatial prediction

Given $\mathbf{Y}_t$ the usual Kriging equations give the predictive distribution for $Y_t(\mathbf{s}^*)$ at prediction location $(\mathbf{s}^*)$

4 Data analysis

5 Conclusions

Acknowledgments

Appendix A.1: Posterior distributions

Half-normal

Let $u = \xi + \sqrt{\eta}|x|$ where $X \sim N(0,1)$. Then Wiper et al. (2008) show that U follows a half-normal distribution which we shall write as $U \sim \text{HN}(\xi, \theta)$ where $\theta = \dfrac{1}{\eta}$ is a precision term. The density is given by

$$f_U(u) = \frac{\sqrt{\theta\pi}}{\sqrt{2}} \exp\left(-\frac{(u-\xi)^2\theta}{2}\right), \quad u > \xi. \tag{6}$$

Conditional posterior of $U|Y$

Let $Y_i|U \sim \text{N}(U, \sigma^2)$, $i = 1, \ldots, n$, let $\tau = 1/\sigma^2$, and let $\pi(U) \propto \exp\left\{-\frac{u^2\theta}{2}\right\}$. Then the conditional posterior of $U|Y$ is

$$\begin{aligned}
\pi(U \mid Y) &\propto \exp\left\{-\frac{u^2\theta}{2}\right\} \exp\left\{-\sum_{i=1}^{n} \frac{\tau(y_i - u)^2}{2}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[u^2\theta + \sum_{i=1}^{n} \tau(y_i^2 - 2y_i u + u^2)\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(u - \frac{\tau\sum_{i=1}^{n} y_i}{\theta + n\tau}\right)^2 (\theta + n\tau)\right\} \\
&\propto \text{HN}(\xi^*, \theta^*)
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\xi^* &= \frac{\tau\sum_{i=1}^{n} y_i}{\theta + n\tau} \\
\theta^* &= \theta + n\tau
\end{aligned}$$

Conditional posterior of $U_{tl}|\mathbf{Y}_{tl}(\mathbf{s})$

Consider a multivariate response $Y_t(\mathbf{s})$ as given by 4 using two partitions. Then conditioned on the observations in partition 2,

$$Y_{t1}|Y_{t2} \sim \mathrm{N}_{n_1}(\overline{\mu}, \overline{\Sigma}) \tag{8}$$

where $\overline{\mu} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y_{t2} - \mu_2)$, and $\overline{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Then conditional posterior of $U_{t1}|\mathbf{Y}_{t1}$ is

$$\pi(U_{t1}|\mathbf{Y}_{t1}) \propto \exp\left\{-\frac{1}{2\sigma^2\delta^2}u^2 - \frac{1}{\sigma^2(1-\delta^2)}\left[\mathbf{Y}_{t1} - \overline{\mu}\right]^T \overline{\Sigma}^{-1}\left[\mathbf{Y}_{t1} - \overline{\mu}\right]\right\} \tag{9}$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{1}{\delta^2}u^2 + \frac{\sigma^2\delta^2}{(1-\delta^2)}u^2\mathbf{1}^T\overline{\Sigma}^{-1}\mathbf{1} - 2u\mathbf{1}^T\overline{\Sigma}^{-1}[\mathbf{Y}_{t1} - X_{t1}\beta - \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{Y}_{t2} - \mu_2)]\right]\right\} \tag{10}$$

$$\propto \exp\{-\frac{1}{2}(u - \xi^*)^2(\theta^*)\} \tag{11}$$

where

$$\xi^* = \frac{\sigma\delta\mathbf{1}^T\overline{\Sigma}^{-1}\left[\mathbf{Y}_{t1} - X_{t1}\beta - \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{Y}_t 2 - \mu_2)\right]}{\frac{1}{\delta^2} + \frac{\sigma^2\delta^2\mathbf{1}^T\overline{\Sigma}^{-1}\mathbf{1}}{(1-\delta^2)}} \tag{12}$$

$$\theta^* = \frac{1}{\sigma^2\delta^2} + \frac{\delta^2\mathbf{1}^T\overline{\Sigma}^{-1}\mathbf{1}}{(1-\delta^2)} \tag{13}$$

Conditional posterior of $\beta| \ldots$

Let $\beta \sim \mathrm{N}_p(0, \Lambda_0)$ where Λ_0 is a precision matrix. Then

$$\begin{aligned}\pi(\beta|\text{rest}) &\propto \exp\left\{-\frac{1}{2}\beta^T\Lambda_0\beta - \frac{1}{2}[\mathbf{Y}_t(\mathbf{s}) - X_t(\mathbf{s})\beta - \sigma\delta|u_t|]^T\Sigma^{-1}[\mathbf{Y}_t(\mathbf{s}) - X_t(\mathbf{s})\beta - \sigma\delta|u_t|]\right\} \\ &\propto \exp\left\{-\frac{1}{2}\beta^T\Lambda_n\beta - 2[\beta^T X_t(\mathbf{s})\Sigma^{-1}(\mathbf{Y}_t(\mathbf{s}) + \sigma\delta|u_t|)]\right\} \\ &\propto \mathrm{N}_p(\mu_p, \Lambda_p)\end{aligned} \tag{14}$$

where

$$\begin{aligned}\mu_p &= \Lambda_p^{-1}X_t(\mathbf{s})^T\Sigma^{-1}(\mathbf{Y}_t(\mathbf{s}) + \sigma\delta|u_t|) \\ \Lambda_p &= (\Lambda_0 + X_t(\mathbf{s})^T\Sigma^{-1}X_t(\mathbf{s}))\end{aligned}$$

and Λ_p is a precision matrix.

Appendix A.2: MCMC Details

Priors

References

Minozzo, M. and Ferracuti, L. (2012) On the existence of some skew-normal stationary processes. *Chilean Journal of Statistics (ChJS)*, **3**, 157–170.

Wiper, M. P., Girón, F. J. and Pewsey, A. (2008) Objective Bayesian Inference for the Half-Normal and Half- t Distributions. *Communications in Statistics - Theory and Methods*, **37**, 3165–3185.