

Spatiotemporal Modeling of Extreme Events

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Motivation

- ▶ Average behavior is important to understand, but it does not paint the whole picture.
 - ▶ e.g. When constructing river levees, engineers need to be able to estimate a 100-year or 1000-year flood levels.
- ▶ In geostatistical analysis, kriging uses spatial correlation to help inform prediction at unknown locations.
- ▶ Want to explore computationally easy methods that are available in higher dimensions

Standard non-spatial analysis

- ▶ Block maxima:
 - ▶ Uses yearly maxima
 - ▶ Discards many observations
 - ▶ Models are fit using the generalized extreme value distribution
- ▶ Generalized extreme value distribution (GEV):

$$\Pr(Y_j < y) = G_j(y) = \exp \left\{ - \left[\left(1 + \xi_j \frac{y - \mu_j}{\sigma_j} \right)_+^{-1/\xi_j} \right] \right\}$$

- ▶ For a spatial analysis, max-stable processes give an appropriate limiting distribution.

Standard non-spatial analysis

- ▶ Peaks-over-threshold:
 - ▶ Incorporates more data than block maxima
 - ▶ Select a threshold, T , and fit data above the threshold using the generalized Pareto distribution
 - ▶ Autocorrelation may be an issue between observations (e.g. flood levels don't dissipate overnight)
- ▶ Generalized Pareto distribution (GPD):

$$\Pr(Y_j > y | Y_j > T) = F_j(y) = \left(1 + \xi_j \frac{y - T}{\sigma_j}\right)_+^{-1/\xi_j}$$

Multivariate analysis

- ▶ Multivariate max-stable and GPD models have nice features, but they are
 - ▶ computationally hard to work with
 - ▶ joint distribution only available in low dimension
- ▶ Pairwise likelihood approach (Huser and Davison, 2014)

Model objectives

- ▶ Our objective is to build a model that
 - ▶ has a flexible tail
 - ▶ has asymptotic spatial dependence
 - ▶ computation on the order of Gaussian models for large space-time datasets

Thresholding data

- ▶ We threshold the observed data at a high threshold T .
- ▶ Thresholded data:

$$Y_t^*(\mathbf{s}) = \begin{cases} Y_t(\mathbf{s}) & Y_t(\mathbf{s}) > T \\ T & Y_t(\mathbf{s}) \leq T \end{cases}$$

- ▶ Allows tails of the distribution to speak for themselves.

- ▶ The χ coefficient is a measure of extremal correlation
- ▶ Specifically, we focus on $\chi(c, \mathbf{h})$ for the upper tail given by

$$\chi(c, \mathbf{h}) = \Pr(Y(\mathbf{s}) > c \mid Y(\mathbf{s} + \mathbf{h}) > c)$$

- ▶ If $\lim_{c \rightarrow \infty} \chi(c) = 0$, then high extreme values are not correlated.

Gaussian spatial model

- ▶ In geostatistical models, observations, $Y(\mathbf{s})$ are often modeled using a Gaussian spatial model with a likelihood of

$$f(y(\mathbf{s})) = \frac{1}{(2\pi)^{n/2}} |\Sigma|^{-1/2} \exp \left[-\frac{1}{2} (Y(\mathbf{s}) - \mu(\mathbf{s}))^T \Sigma^{-1} (Y(\mathbf{s}) - \mu(\mathbf{s})) \right]$$

where

- ▶ $\mu(\mathbf{s})$ is a mean vector
- ▶ Σ is a covariance matrix
- ▶ Model properties:
 - ▶ Nice computing properties
 - ▶ For a Gaussian spatial model $\lim_{c \rightarrow \infty} \chi(c) = 0$ regardless of the strength of the correlation in the bulk of the distribution.
 - ▶ Tail is not flexible

Spatial skew- t distribution

- ▶ Assume observed data $Y_t(\mathbf{s})$ come from a skew- t (Zhang and El-Shaarawi, 2012)

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \alpha z_t + v_t(\mathbf{s})$$

where

- ▶ $\alpha \in \mathcal{R}$ controls the skewness
- ▶ $z_t \stackrel{iid}{\sim} N_{(0,\infty)}(0, \sigma_t^2)$ is a random effect
- ▶ $v_t(\mathbf{s})$ is a Gaussian process with variance σ_t^2 and Matérn correlation
- ▶ $\sigma_t^2 \stackrel{iid}{\sim} \text{IG}(a, b)$

Spatial skew- t distribution

- ▶ **Conditioned** on z_t and σ_t^2 , $Y_t(\mathbf{s})$ is a Gaussian spatial model
- ▶ Can use standard geostatistical methods to fit this model.
- ▶ Predictions can be made through kriging.
- ▶ **Marginalizing** over z_t and σ_t^2 (via MCMC),

$$Y_t(\mathbf{s}) \sim \text{skew-}t(\mu, \Sigma^*, \alpha, \text{df} = 2a)$$

where

- ▶ μ is the location
- ▶ a, b are the IG parameters for σ_t^2
- ▶ $\Sigma^* = \frac{b}{a}\Sigma$ is a scale matrix, and Σ is a Matérn covariance matrix
- ▶ $\alpha \in \mathcal{R}$ controls the skewness

Spatial skew- t distribution

- ▶ Model properties
 - ▶ Has flexible tail controlled by α and degrees of freedom
 - ▶ For a skew- t distribution $\lim_{c \rightarrow \infty} \chi(c) > 0$ Padoan, 2011)
 - ▶ Computation that is on the order of Gaussian computation
- ▶ For this distribution, $\chi(c, \mathbf{h})$ shows asymptotic dependence that does not approach 0 as $\mathbf{h} \rightarrow \infty$
- ▶ Deal with this through a daily random partition.

Daily random partition

- ▶ Daily random partition allows z_t and σ_t^2 to vary by site.

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \alpha z_t(\mathbf{s}) + \sigma(\mathbf{s})v_t(\mathbf{s})$$

- ▶ Consider a set of daily knots $\{w_{t1}, \dots, w_{tK}\}$ that define a daily partition P_{t1}, \dots, P_{tK} such that

$$P_{tk} = \{\mathbf{s} : k = \arg \min_{\ell} \|\mathbf{s} - w_{t\ell}\|\}$$

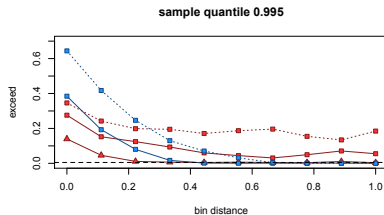
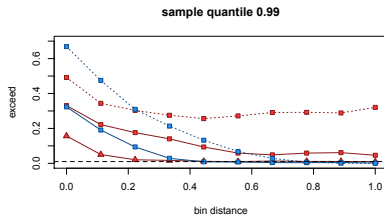
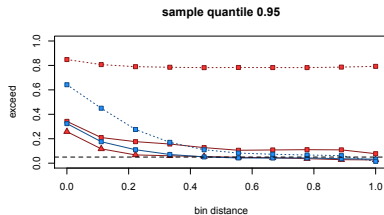
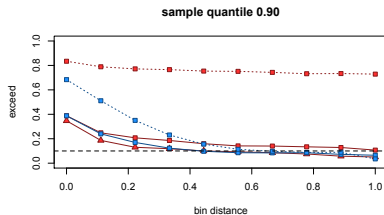
- ▶ For $\mathbf{s} \in P_{tk}$

$$z_t(\mathbf{s}) = z_{tk}$$

$$\sigma_t^2(\mathbf{s}) = \sigma_{tk}^2$$

- ▶ Within each partition $Y_t(\mathbf{s})$ has the same MVT distribution as before.

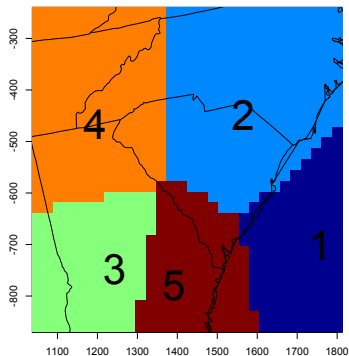
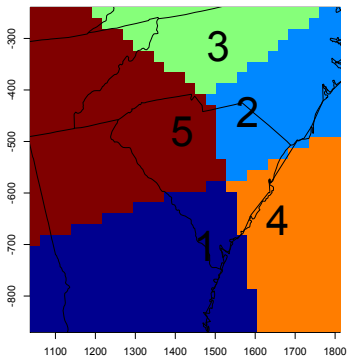
Simulated χ plots



- ▲ Gaussian
- t, K=1
- t, K=5
- skew-t, K=1
- skew-t, K=5

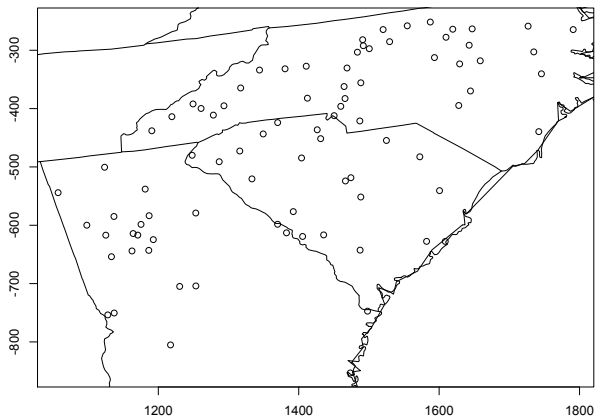
Example daily partition

Two sample partitions



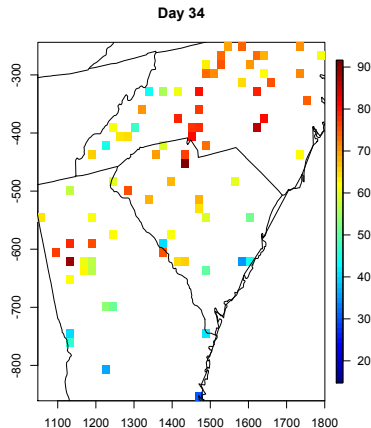
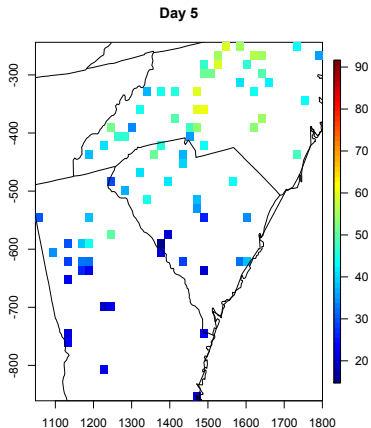
- ▶ Three main steps:
 1. Impute missing observations and data below T
 2. Update parameters with standard random walk Metropolis Hastings or Gibbs sampling
 3. Make spatial predictions
- ▶ Priors are selected to be conjugate when possible.

Ozone monitoring station locations



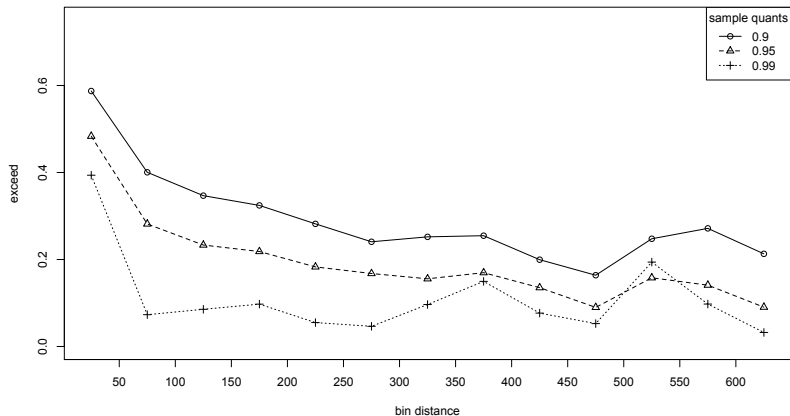
Data analysis

Max 8-hour ozone measurements at 85 sites in NC, SC, and GA
for days 5 and 34.



Exploratory data analysis

χ -plot for residuals selected ozone sample quantiles



Model comparisons

- ▶ 9 different analysis methods incorporating
 - ▶ Gaussian vs t vs skew- t marginal distribution
 - ▶ $K = 1$ partition vs $K = 5$ partitions
 - ▶ No thresholding vs thresholded
 - ▶ Thresholded data at $T = 0.90$ sample quantile
- ▶ All methods use a Matérn or exponential covariance ($\nu = 0.5$)
- ▶ Compare quantile and Brier scores using 5-fold cross validation (Gneiting and Raftery, 2007)
- ▶ Mean function modeled using a first-order spatial trend

- ▶ The quantile score for the τ th quantile is

$$2\{I[y < \hat{q}(\tau)] - \tau\}(\hat{q} - y)$$

where:

- ▶ y is a test set value
- ▶ $\hat{q}(\tau)$ is the estimated τ th quantile

- ▶ Brier score for predicting exceedance of threshold c

$$[e(c) - P(c)]^2$$

where

- ▶ y is a test set value
- ▶ $e(c) = I[y > c]$
- ▶ $P(c)$ is the predicted probability of exceeding c

Five-fold cross-validation results

Marginal	K	T	Quantile				
			0.900	0.950	0.990	0.995	0.999
Gaussian	1	0	16.38	15.76	14.52	14.08	13.22
t	1	0	16.15	15.51	14.00	13.43	12.32
t	5	0	13.61	12.66	10.96	10.40	9.34
skew t	1	0	9.24	7.27	4.13	3.27	1.96
skew t	5	0	15.81	14.46	11.57	10.57	8.60
t	1	0.9	5.52	3.58	1.77	1.47	1.10
t	5	0.9	5.98	4.27	2.41	2.03	1.49
skew t	1	0.9	4.91	3.16	1.45	1.16	0.82
skew t	3	0.9	5.58	3.78	1.93	1.58	1.11

- Brier score results are similar.

Simulation study

- ▶ 6 different data settings:
 - ▶ Gaussian vs t vs skew- t marginal distribution
 - ▶ $K = 1$ partition vs $K = 5$ partitions
- ▶ Results are similar to the results from the data analysis
- ▶ Biggest gains come from thresholding.
- ▶ Using skew models give additional gain, but small relative to gain for thresholding.

Future work

- ▶ Comparison with extreme value analysis methods
- ▶ Reporting of compliance with EPA ozone standards
- ▶ Spatial mapping of high quantiles
- ▶ Including time in the model
 - ▶ AR(1): $Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \phi Y_{t-1}(\mathbf{s}) + \alpha z_t(\mathbf{s}) + v_t(\mathbf{s})$

Questions

- ▶ Any questions?
- ▶ Thank you for your attention.

References

- ▶ Demarta, S. and McNeil, A. J. (2007) The t copula and related copulas. *International Statistical Review*, **73**, 111–129.
- ▶ Huser, R. and Davison, A. C. (2014) Space-time modelling of extreme events. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **76**, 439–461.
- ▶ Padoan, S. A. (2011) Multivariate extreme models based on underlying skew- t and skew-normal distributions. *Journal of Multivariate Analysis*, **102**, 977–991.
- ▶ Zhang, H. and El-Shaarawi, A. (2010) On spatial skew-Gaussian processes and applications. *Environmetrics*, **21**, 33–47.