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Web-based Supplementary Materials for A Space-time Skew-t Model for Threshold Exceedances by Morris, Reich, Thibaud, and Cooley

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Web Appendix A. MCMC details

The MCMC sampling for the model in Section 4 is done using `R` (http://www.r-project.org). Whenever possible, we select conjugate priors (see Web Appendix B); however, for some of the parameters, no conjugate prior distributions exist. For these parameters, we use a random walk Metropolis-Hastings update step. In each Metropolis-Hastings update, we tune the algorithm during the burn-in period to give acceptance rates near 0.40.

Spatial knot locations

For each day, we update the spatial knot locations, $\mathbf{w}_1, \ldots, \mathbf{w}_K$, using a Metropolis-Hastings block update. Because the spatial domain is bounded, we generate candidate knots using the transformed knots $\mathbf{w}_1^*, \ldots, \mathbf{w}_K^*$ (see Section 3.3) and a random walk bivariate Gaussian candidate distribution

$$\mathbf{w}_k^{*(c)} \sim \mathrm{N}(\mathbf{w}_k^{*(r-1)}, s^2 I_2)$$

where $\mathbf{w}_k^{*(r-1)}$ is the location for the transformed knot at MCMC iteration $r-1$, s is a tuning parameter, and I_2 is an identity matrix. After candidates have been generated for all K knots, the acceptance ratio is

$$R = \left\{ \frac{l[Y_t(\mathbf{s}|\mathbf{w}_1^{(c)}, \ldots, \mathbf{w}_K^{(c)}, \ldots)]}{l[Y_t(\mathbf{s}|\mathbf{w}_1^{(r-1)}, \ldots, \mathbf{w}_K^{(r-1)}, \ldots)]} \right\} \times \left\{ \frac{\prod_{k=1}^K \phi(\mathbf{w}_k^{(c)})}{\prod_{k=1}^K \phi(\mathbf{w}_k^{(r-1)})} \right\} \times \left\{ \frac{\prod_{k=1}^K p(\mathbf{w}_k^{*(c)})}{\prod_{k=1}^K p(\mathbf{w}_k^{*(r-1)})} \right\}$$

where l is the likelihood given in (17), and $p(\cdot)$ is the prior either taken from the time series given in (3.3) or assumed to be uniform over $\mathcal{D}$. The candidate knots are accepted with probability $\min\{R, 1\}$.

Spatial random effects

If there is no temporal dependence amongst the observations, we use a Gibbs update for z_{tk}, and the posterior distribution is given in Web Appendix B. If there is temporal dependence amongst the observations, then we update z_{tk} using a Metropolis-Hastings update. Because this model uses $|z_{tk}|$, we generate candidate random effects using the z_{tk}^* (see Section 3.3) and a random walk

Gaussian candidate distribution

$$z_{tk}^{*\,(c)} \sim \mathrm{N}(z_{tk}^{*\,(r-1)}, s^2)$$

where $z_{tk}^{*\,(r-1)}$ is the value at MCMC iteration $r-1$, and s is a tuning parameter. The acceptance ratio is

$$R = \left\{ \frac{l[Y_t(\mathbf{s})|z_{tk}^{(c)}, \ldots]}{l[Y_t(\mathbf{s})|z_{tk}^{(r-1)}]} \right\} \times \left\{ \frac{p[z_{tk}^{(c)}]}{p[z_{tk}^{(r-1)}]} \right\}$$

where $p[\cdot]$ is the prior taken from the time series given in Section 3.3. The candidate is accepted with probability $\min\{R, 1\}$.

Variance terms

When there is more than one site in a partition, then we update σ_{tk}^2 using a Metropolis-Hastings update. First, we generate a candidate for σ_{tk}^2 using an IG$(a^*/s, b^*/s)$ candidate distribution in an independence Metropolis-Hastings update where $a^* = (n_{tk}+1)/2+a$, $b^* = [Y_{tk}'\Sigma_{tk}^{-1}Y_{tk}+z_{tk}^2]/2+b$, n_{tk} is the number of sites in partition k on day t, and Y_{tk} and Σ_{tk}^{-1} are the observations and precision matrix for partition k on day t. The acceptance ratio is

$$R = \left\{ \frac{l[Y_t(\mathbf{s})|\sigma_{tk}^{2\;(c)}, \ldots]}{l[Y_t(\mathbf{s})|\sigma_{tk}^{2\;(r-1)}]} \right\} \times \left\{ \frac{l[z_{tk}|\sigma_{tk}^{2\;(c)}, \ldots]}{l[z_{tk}|\sigma_{tk}^{2\;(r-1)}, \ldots]} \right\} \times \left\{ \frac{p[\sigma_{tk}^{2\;(c)}]}{p[\sigma_{tk}^{2\;(r-1)}]} \right\} \times \left\{ \frac{c[\sigma_{tk}^{2\;(r-1)}]}{c[\sigma_{tk}^{2\;(c)}]} \right\}$$

where $p[\cdot]$ is the prior either taken from the time series given in Section 3.3 or assumed to be IG(a, b), and $c[\cdot]$ is the candidate distribution. The candidate is accepted with probability $\min\{R, 1\}$.

Spatial covariance parameters

We update the three spatial covariance parameters, $\log(\rho)$, $\log(\nu)$, γ, using a Metropolis-Hastings block update step. First, we generate a candidate using a random walk Gaussian candidate distribution

$$\log(\rho)^{(c)} \sim \mathrm{N}(\log(\rho)^{(r-1)}, s^2)$$

where $\log(\rho)^{(r-1)}$ is the value at MCMC iteration $r-1$, and s is a tuning parameter. Candidates are generated for $\log(\nu)$ and γ in a similar fashion. The acceptance ratio is

$$R = \left\{ \frac{\prod_{t=1}^{T} l[Y_t(\mathbf{s}) | \rho^{(c)}, \nu^{(c)}, \gamma^{(c)}, \ldots]}{\prod_{t=1}^{T} l[Y_t(\mathbf{s}) | \rho^{(r-1)}, \nu^{(r-1)}, \gamma^{(r-1)}, \ldots]} \right\} \times \left\{ \frac{p[\rho^{(c)}]}{p[\rho^{(r-1)}]} \right\} \times \left\{ \frac{p[\nu^{(c)}]}{p[\nu^{(r-1)}]} \right\} \times \left\{ \frac{p[\gamma^{(c)}]}{p[\gamma^{(r-1)}]} \right\}.$$

All three candidates are accepted with probability $\min\{R, 1\}$.

Web Appendix B. Posterior distributions

Conditional posterior of $z_{tk} \mid \ldots$

If knots are independent over days, then the conditional posterior distribution of $|z_{tk}|$ is conjugate. For simplicity, drop the subscript t, let $\tilde{z}_{tk} = |z_{tk}|$, and define

$$R(\mathbf{s}) = \begin{cases} Y(\mathbf{s}) - X(\mathbf{s})\beta & s \in P_l \\ \\ Y(\mathbf{s}) - X(\mathbf{s})\beta - \lambda \tilde{z}(\mathbf{s}) & s \notin P_l \end{cases}$$

Let

$$\begin{aligned} R_1 &= \text{the vector of } R(\mathbf{s}) \text{ for } s \in P_l \\ R_2 &= \text{the vector of } R(\mathbf{s}) \text{ for } s \notin P_l \\ \Omega &= \Sigma^{-1}. \end{aligned}$$

Then

$$\begin{aligned} \pi(z_l | \ldots) &\propto \exp\left\{ -\frac{1}{2} \left[\begin{pmatrix} R_1 - \lambda \tilde{z}_l \mathbf{1} \\ R_2 \end{pmatrix}' \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \begin{pmatrix} R_1 - \lambda \tilde{z}_l \mathbf{1} \\ R_2 \end{pmatrix} + \frac{\tilde{z}_l^2}{\sigma_l^2} \right] \right\} I(z_l > 0) \\ &\propto \exp\left\{ -\frac{1}{2} \left[\Lambda_l \tilde{z}_l^2 - 2\mu_l \tilde{z}_l \right] \right\} \end{aligned}$$

where

$$\begin{aligned} \mu_l &= \lambda(R_1' \Omega_{11} + R_2' \Omega_{21}) \mathbf{1} \\ \Lambda_l &= \lambda^2 \mathbf{1}' \Omega_{11} \mathbf{1} + \frac{1}{\sigma_l^2}. \end{aligned}$$

Then $\tilde{Z}_l | \ldots \sim N_{(0,\infty)}(\Lambda_l^{-1}\mu_l, \Lambda_l^{-1})$

Conditional posterior of $\beta \mid \ldots$

Let $\beta \sim \mathrm{N}_p(0, \Lambda_0)$ where Λ_0 is a precision matrix. Then

$$\begin{aligned}
\pi(\beta \mid \ldots) &\propto \exp\left\{-\frac{1}{2}\beta'\Lambda_0\beta - \frac{1}{2}\sum_{t=1}^{n_t}[\mathbf{Y}_t - X_t\beta - \lambda|z_t|]'\Omega[\mathbf{Y}_t - X_t\beta - \lambda|z_t|]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\beta'\Lambda_\beta\beta - 2\sum_{t=1}^{n_t}[\beta' X_t'\Omega(\mathbf{Y}_t - \lambda|z_t|)]\right]\right\} \\
&\propto \mathrm{N}(\Lambda_\beta^{-1}\mu_\beta, \Lambda_\beta^{-1})
\end{aligned}$$

where

$$\begin{aligned}
\mu_\beta &= \sum_{t=1}^{n_t}\left[X_t'\Omega(\mathbf{Y}_t - \lambda|z_t|)\right] \\
\Lambda_\beta &= \Lambda_0 + \sum_{t=1}^{n_t} X_t'\Omega X_t.
\end{aligned}$$

Conditional posterior of $\sigma^2 \mid \ldots$

In the case where $L = 1$ and temporal dependence is negligible, then σ^2 has a conjugate posterior distribution. Let $\sigma_t^2 \overset{iid}{\sim} \mathrm{IG}(\alpha_0, \beta_0)$. For simplicity, drop the subscript t. Then

$$\begin{aligned}
\pi(\sigma^2 \mid \ldots) &\propto (\sigma^2)^{-\alpha_0 - 1/2 - n/2 - 1}\exp\left\{-\frac{\beta_0}{\sigma^2} - \frac{|z|^2}{2\sigma^2} - \frac{(\mathbf{Y} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{Y} - \boldsymbol{\mu})}{2\sigma^2}\right\} \\
&\propto (\sigma^2)^{-\alpha_0 - 1/2 - n/2 - 1}\exp\left\{-\frac{1}{\sigma^2}\left[\beta_0 + \frac{|z|^2}{2} + \frac{1}{2}(\mathbf{Y} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{Y} - \mu)\right]\right\} \\
&\propto \mathrm{IG}(\alpha^*, \beta^*)
\end{aligned}$$

where

$$\begin{aligned}
\alpha^* &= \alpha_0 + \frac{1}{2} + \frac{n}{2} \\
\beta^* &= \beta_0 + \frac{|z|^2}{2} + \frac{1}{2}(\mathbf{Y} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{Y} - \boldsymbol{\mu}).
\end{aligned}$$

In the case that $L > 1$, a random walk Metropolis Hastings step will be used to update σ_{lt}^2.

Conditional posterior of $\lambda \mid \ldots$

TODO: Still needs to be updated for single λ

For convergence purposes we model $\lambda = \lambda_1 \lambda_2$ where

$$\lambda_1 = \begin{cases} +1 & \text{w.p.}0.5 \\ -1 & \text{w.p.}0.5 \end{cases} \tag{1}$$

$$\lambda_2^2 \sim IG(\alpha_\lambda, \beta_\lambda). \tag{2}$$

$$\tag{3}$$

Then

$$\pi(\lambda_2 \mid \ldots) \propto \lambda_2^{2(-\alpha_\lambda - 1)} \exp\left\{ -\frac{\beta_\lambda}{\lambda_2^2} \right\} \prod_{t=1}^{n_t} \prod_{k=1}^{K} \frac{1}{\lambda_2} \exp\left\{ -\frac{z_{tk}^2}{2\lambda_2^2 \sigma_{tk})^2} \right\}$$
$$\propto \lambda_2^{2(-\alpha_\lambda - K * n_t - 1)} \exp\left\{ -\frac{1}{\lambda_2^2} \left[\beta_\lambda + \frac{z^2}{2\sigma_{tk}^2} \right] \right\}$$

Then $\lambda_2 \mid \ldots \sim IG\left(\alpha_\lambda + K * n_t, \beta_\lambda + \frac{z^2}{2\sigma_{tk}^2} \right)$

Web Appendix C. Proof that $\lim_{h \to \infty} \pi(h) = 0$

Let c be the midpoint of $\mathbf{s}_1$ and $\mathbf{s}_2$. Define A as the circle centered at c with radius $h/2$ where $h = ||\mathbf{s}_1 - \mathbf{s}_2||$ is the distance between sites $\mathbf{s}_1$ and $\mathbf{s}_2$. Consider a homogeneous spatial Poisson process over A with intensity given by

$$\mu_{PP}(A) = \lambda_{PP}|A| = \lambda_{PP}\pi \left(\frac{h}{2} \right)^2 = \lambda_{PPA}^* h^2.$$

Consider a partition of A into four regions, B_1, B_2, R_1, R_2 as seen in Web Figure 1.

[Figure 1 about here.]

Let N_i be the number of knots in B_i and $L_i = l$ if $\mathbf{s}_i \in P_l$ for $i = 1, 2$. Then

$$P(L_1 \neq L_2) \geqslant P(N_1 > 0, N_2 > 0) \tag{4}$$

since knots in both B_1 and B_2 is sufficient, but not necessary, to ensure that $\mathbf{s}_1$ and $\mathbf{s}_2$ are in different partition sets. By definition of a Poission process, N_1 and N_2 are independent and thus $P(N_1 > 0, N_2 > 0) = P(N_1 > 0)^2$, and the intensity measure over B_1 is given by

$$\begin{aligned}\mu_{PP}(B_1) &= \lambda_{PP}|B_1| = \lambda_{PP}\frac{h^2}{4}\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \\ &= \lambda^*_{PPB1}h^2. \end{aligned} \tag{5}$$

So,

$$P(L_1 \neq L_2) >= P(N_1 > 0)^2 = [1 - P(N_1 = 0)]^2 = [1 - \exp\left(-\lambda^*_{PPB1}h^2\right)]^2 \tag{6}$$

which goes to 1 as h goes to infinity.

Web Appendix D. Skew-t distribution

Univariate skew-t distribution

We say that Y follows a univariate extended skew-t distribution with location $\xi \in \mathcal{R}$, scale $\omega > 0$, skew parameter $\alpha \in \mathcal{R}$, and degrees of freedom ν if has distribution function

$$f_{\text{EST}}(y) = 2f_T(z;\nu)F_T\left[\alpha z\sqrt{\frac{\nu+1}{\nu+z^2}};\nu+1\right] \tag{7}$$

where $f_T(t;\nu)$ is a univariate Student's t with ν degrees of freedom, $F_T(t;\nu) = P(T < t)$, and $z = (y - \xi)/\omega$.

Multivariate skew-t distribution

If $\mathbf{Z} \sim \text{ST}_d(0, \bar{\boldsymbol{\Omega}}, \boldsymbol{\alpha}, \eta)$ is a d-dimensional skew-t distribution, and $\mathbf{Y} = \xi + \boldsymbol{\omega}\mathbf{Z}$, where $\boldsymbol{\omega} = \text{diag}(\omega_1, \ldots, \omega_d)$, then the density of Y at y is

$$f_y(\mathbf{y}) = det(\boldsymbol{\omega})^{-1}f_z(\mathbf{z}) \tag{8}$$

where

$$f_z(\mathbf{z}) = 2t_d(\mathbf{z}; \bar{\mathbf{\Omega}}, \eta) T \left[\boldsymbol{\alpha}' \mathbf{z} \sqrt{\frac{\eta + d}{\nu + Q(\mathbf{z})}}; \eta + d \right] \tag{9}$$

$$\mathbf{z} = \boldsymbol{\omega}^{-1}(\mathbf{y} - \xi) \tag{10}$$

where $t_d(\mathbf{z}; \bar{\mathbf{\Omega}}, \eta)$ is a d-dimensional Student's t-distribution with scale matrix $\bar{\mathbf{\Omega}}$ and degrees of freedom η, $Q(z) = \mathbf{z}'\bar{\Omega}^{-1}\mathbf{z}$ and $T(\cdot; \eta)$ denotes the univariate Student's t distribution function with η degrees of freedom (Azzalini and Capitanio, 2014).

Extremal dependence

For a bivariate skew-t random variable $\mathbf{Y} = [Y(\mathbf{s}), Y(\mathbf{t})]'$, the $\chi(h)$ statistic (Padoan, 2011) is given by

$$\chi(h) = \bar{F}_{\text{EST}} \left\{ \frac{[x_1^{1/\eta} - \varrho(h)]\sqrt{\eta + 1}}{\sqrt{1 - \varrho(h)^2}}; 0, 1, \alpha_1, \tau_1, \eta + 1 \right\} + \bar{F}_{\text{EST}} \left\{ \frac{[x_2^{1/\eta} - \varrho(h)]\sqrt{\eta + 1}}{\sqrt{1 - \varrho(h)^2}}; 0, 1, \alpha_2, \tau_2, \eta + 1 \right\}, \tag{11}$$

where $\bar{F}_{\text{EST}}$ is the univariate survival extended skew-t function with zero location and unit scale, $\varrho(h) = \text{cor}[y(\mathbf{s}), y(\mathbf{t})]$, $\alpha_j = \alpha_i\sqrt{1 - \varrho^2}$, $\tau_j = \sqrt{\eta + 1}(\alpha_j + \alpha_i\varrho)$, and $x_j = F_T(\bar{\alpha}_i\sqrt{\eta + 1}; 0, 1, \eta)/F_T(\bar{\alpha}_j\sqrt{\eta + 1};$ with $j = 1, 2$ and $i = 2, 1$ and where $\bar{\alpha}_j = (\alpha_j + \alpha_i\varrho)/\sqrt{1 + \alpha_i^2[1 - \varrho(h)^2]}$.

Proof that $\lim_{h\to\infty} \chi(h) > 0$

Consider the bivariate distribution of $\mathbf{Y} = [Y(\mathbf{s}), Y(\mathbf{t})]'$, with $\varrho(h)$ given by (2). So, $\lim_{h\to\infty} \varrho(h) = 0$. Then

$$\lim_{h\to\infty} \chi(h) = \bar{F}_{\text{EST}} \left\{ \sqrt{\eta + 1}; 0, 1, \alpha_1, \tau_1, \eta + 1 \right\} + \bar{F}_{\text{EST}} \left\{ \sqrt{\eta + 1}; 0, 1, \alpha_2, \tau_2, \eta + 1 \right\}. \tag{12}$$

Because the extended skew-t distribution is not bounded above, for all $\bar{F}_{\text{EST}}(x) = 1 - F_{\text{EST}(x)} > 0$ for all $x < \infty$. Therefore, for a skew-t distribution, $\lim_{h\to\infty} \chi(h) > 0$.

Web Appendix E. Comparisons with other parameterizations

Various forms of multivariate skew-normal and skew-t distributions have been proposed in the literature. In this section, we make a connection between our parameterization in (1) of the main text and another popular version. Azzalini and Capitanio (2014) and Beranger et al. (2016) define a skew-normal process as

$$\tilde{X}(\mathbf{s}) = \tilde{\lambda}|z| + (1 - \tilde{\lambda}^2)^{1/2} v(\mathbf{s}) \tag{13}$$

where $\tilde{\lambda} \in (-1, 1)$, $z \sim N(0, 1)$, and $v(\mathbf{s})$ is a Gaussian process with mean zero, variance one, and spatial correlation function ρ. To extend this to the skew-t distribution, Azzalini and Capitanio (2003) take $\tilde{Y}(\mathbf{s}) = W\tilde{X}(\mathbf{s})$ where $W^{-2} \sim$ Gamma$(a/2, a/2)$. Returning to the proposed parameterization (with $\boldsymbol{\beta} = 0$), let $W^{-2} = \frac{2b}{a}\sigma^{-2} \sim$ Gamma$(a/2, a/2)$ so that (1) becomes

$$Y(\mathbf{s}) = W\left[\lambda\left(\frac{a}{2b}\right)^{1/2}|z| + \left(\frac{a}{2b}\right)^{1/2} v(\mathbf{s})\right]. \tag{14}$$

Clearly setting $b = \frac{a}{(1-\tilde{\lambda}^2)} > 0$, and $\lambda = \tilde{\lambda}/(1 - \tilde{\lambda}^2)^{1/2} \in (-\infty, \infty)$ resolves the difference in parameterizations. We note that our parameterization has three parameters (a, b, λ) compared to the two parameters of the alternative parameterization $(a, \tilde{\lambda})$. Since we have assumed that both $v(\mathbf{s})$ and z have unit scale, the additional b parameter in our parameterization is required to control the precision.

Web Appendix F. Simulation study pairwise difference results

The following tables show the methods that have significantly different Brier scores when using a Wilcoxon-Nemenyi-McDonald-Thompson test. In each column, different letters signify that the methods have significantly different Brier scores. For example, there is significant evidence to suggest that method 1 and method 4 have different Brier scores at $q(0.90)$, whereas there is not significant evidence to suggest that method 1 and method 2 have different Brier scores at $q(0.90)$.

In each table group A represents the group with the lowest Brier scores. Groups are significant with a familywise error rate of $\alpha = 0.05$.

[Table 1 about here.]

[Table 2 about here.]

[Table 3 about here.]

[Table 4 about here.]

[Table 5 about here.]

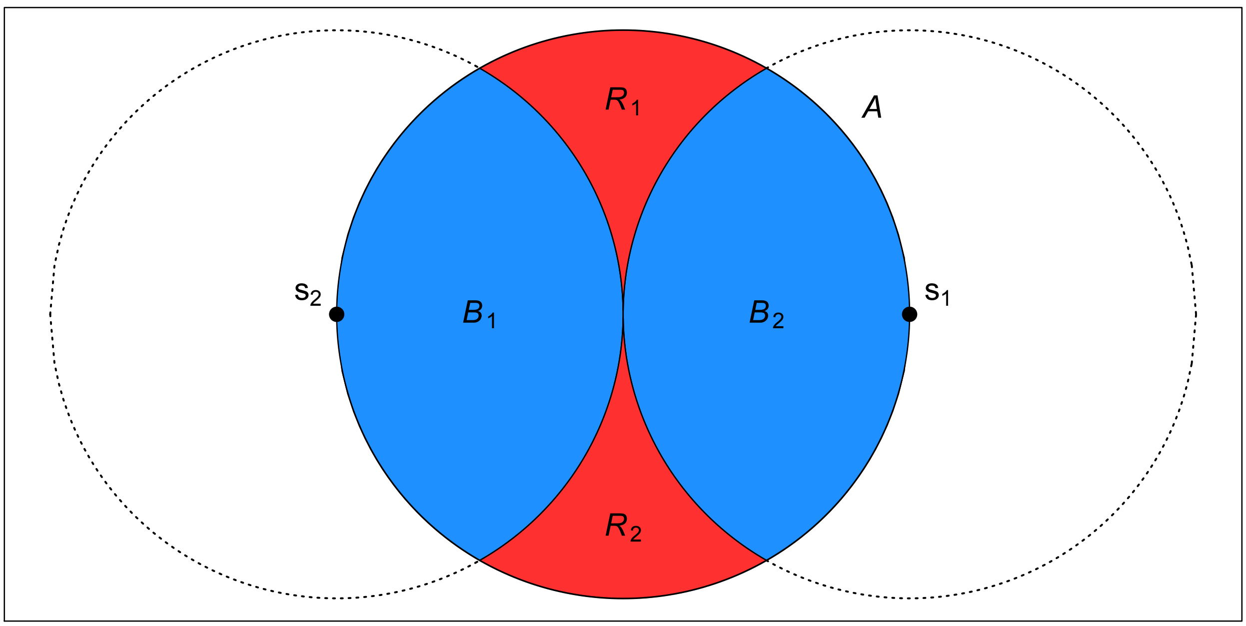
References

Azzalini, A. and Capitanio, A. (2003). Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **65,** 367–389.

Azzalini, A. and Capitanio, A. (2014). *The Skew-Normal and Related Families*. Institute of Mathematical Statistics Monographs. Cambridge University Press.

Beranger, B., Padoan, S. A., and Sisson, S. A. (2016). Models for extremal dependence derived from skew-symmetric families. *ArXiv e-prints* .

Padoan, S. A. (2011). Multivariate extreme models based on underlying skew- and skew-normal distributions. *Journal of Multivariate Analysis* **102,** 977–991.



Web Figure 1. Illustration of the partition of A.

Web Table 1

Setting 1 – Gaussian marginal, $K = 1$ knot

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	A	A	A	A B
Method 2	A	A	A	A
Method 3	B	B	C	B
Method 4	A	A	A B	A B
Method 5	B	B	B C	A B
Method 6	C	C	D	C

Web Table 2

Setting 2 – Skew-t marginal, $K = 1$ knot

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	C	B	B C	B
Method 2	A	A	A	A
Method 3	B C	A B	A B	A B
Method 4	A B	B	B	A
Method 5	D	C	C	B
Method 6	E	D	D	C

Web Table 3

Setting 3 – Skew-t marginal, $K = 5$ knots

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	B	C	B	B
Method 2	B	C	B	B
Method 3	A	B	B	B
Method 4	A	A	A	A
Method 5	A	A	A	A
Method 6	C	D	C	C

Web Table 4

Setting 4 – Max-stable

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	A B	B	B	C
Method 2	B	B C	B	B C
Method 3	C D	C	B	B
Method 4	D	D	C	C
Method 5	C	C	B	B C
Method 6	A	A	A	A

Web Table 5

Setting 5 – Transformation below $T = q(0.80)$

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	C	B	C	C
Method 2	B	B	B	A B
Method 3	A	A	A	A
Method 4	B C	B	B	B C
Method 5	B	B	B C	C
Method 6	D	C	D	D