BIOMETRICS 000, 000–000 DOI: 000
000 0000

Web-based Supplementary Materials for A Space-time Skew-t Model for Threshold Exceedances by Morris, Reich, Thibaud, and Cooley

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This paper has been submitted for consideration for publication in *Biometrics*

Web Appendix A. MCMC details

The MCMC sampling for the model in Section 4 is done using `R` (http://www.r-project.org). Whenever possible, we select conjugate priors (see Web Appendix B); however, for some of the parameters, no conjugate prior distributions exist. For these parameters, we use a random walk Metropolis-Hastings update step. In each Metropolis-Hastings update, we tune the algorithm during the burn-in period to give acceptance rates near 0.40.

Spatial knot locations

For each day, we update the spatial knot locations, $\mathbf{w}_1, \ldots, \mathbf{w}_K$, using a Metropolis-Hastings block update. Because the spatial domain is bounded, we generate candidate knots using the transformed knots $\mathbf{w}_1^*, \ldots, \mathbf{w}_K^*$ (see Section 3.3) and a random walk bivariate Gaussian candidate distribution

$$\mathbf{w}_k^{*(c)} \sim \mathrm{N}(\mathbf{w}_k^{*(r-1)}, s^2 I_2)$$

where $\mathbf{w}_k^{*(r-1)}$ is the location for the transformed knot at MCMC iteration $r-1$, s is a tuning parameter, and I_2 is an identity matrix. Let $\mathbf{Y}_t = [Y(\mathbf{s}_1), \ldots, Y(\mathbf{s}_n)]$ be the vector of observed responses at each site for day t. After candidates have been generated for all K knots, the acceptance ratio is

$$R = \left\{ \frac{l[\mathbf{Y}_t | \mathbf{w}_1^{(c)}, \ldots, \mathbf{w}_K^{(c)}, \ldots]}{l[\mathbf{Y}_t | \mathbf{w}_1^{(r-1)}, \ldots, \mathbf{w}_K^{(r-1)}, \ldots]} \right\} \times \left\{ \frac{\prod_{k=1}^K \phi(\mathbf{w}_k^{(c)})}{\prod_{k=1}^K \phi(\mathbf{w}_k^{(r-1)})} \right\} \times \left\{ \frac{\prod_{k=1}^K p(\mathbf{w}_k^{*(c)})}{\prod_{k=1}^K p(\mathbf{w}_k^{*(r-1)})} \right\}$$

where l is the likelihood given in (17), and $p(\cdot)$ is the prior either taken from the time series (see Section 3.3) or assumed to be uniform over $\mathcal{D}$. The candidate knots are accepted with probability $\min\{R, 1\}$.

Spatial random effects

If there is no temporal dependence amongst the observations, we use a Gibbs update for z_{tk}, and the posterior distribution is given in Web Appendix B. If there is temporal dependence amongst the observations, then we update z_{tk} using a Metropolis-Hastings update. Because this model uses

$|z_{tk}|$, we generate candidate random effects using the z^*_{tk} (see Section 3.3) and a random walk Gaussian candidate distribution

$$z^{*\,(c)}_{tk} \sim \mathrm{N}(z^{*\,(r-1)}_{tk}, s^2)$$

where $z^{*\,(r-1)}_{tk}$ is the value at MCMC iteration $r-1$, and s is a tuning parameter. The acceptance ratio is

$$R = \left\{ \frac{l[\mathbf{Y}_t | z^{(c)}_{tk}, \ldots]}{l[\mathbf{Y}_t | z^{(r-1)}_{tk}]} \right\} \times \left\{ \frac{p[z^{(c)}_{tk}]}{p[z^{(r-1)}_{tk}]} \right\}$$

where $p[\cdot]$ is the prior taken from the time series given in Section 3.3. The candidate is accepted with probability $\min\{R, 1\}$.

Variance terms

When there is more than one site in a partition, then we update σ^2_{tk} using a Metropolis-Hastings update. First, we generate a candidate for σ^2_{tk} using an IG$(a^*/s, b^*/s)$ candidate distribution in an independence Metropolis-Hastings update where $a^* = (n_{tk} + 1)/2 + a$, $b^* = [\mathbf{Y}^\top_{tk}\Sigma^{-1}_{tk}\mathbf{Y}_{tk} + z^2_{tk}]/2 + b$, n_{tk} is the number of sites in partition k on day t, and $\mathbf{Y}_{tk}$ and Σ^{-1}_{tk} are the observations and precision matrix for partition k on day t. The acceptance ratio is

$$R = \left\{ \frac{l[\mathbf{Y}_t | \sigma^{2\,(c)}_{tk}, \ldots]}{l[\mathbf{Y}_t | \sigma^{2\,(r-1)}_{tk}]} \right\} \times \left\{ \frac{l[z_{tk} | \sigma^{2\,(c)}_{tk}, \ldots]}{l[z_{tk} | \sigma^{2\,(r-1)}_{tk}, \ldots]} \right\} \times \left\{ \frac{p[\sigma^{2\,(c)}_{tk}]}{p[\sigma^{2\,(r-1)}_{tk}]} \right\} \times \left\{ \frac{c[\sigma^{2\,(r-1)}_{tk}]}{c[\sigma^{2\,(c)}_{tk}]} \right\}$$

where $p[\cdot]$ is the prior either taken from the time series given in Section 3.3 or assumed to be IG(a, b), and $c[\cdot]$ is the candidate distribution. The candidate is accepted with probability $\min\{R, 1\}$.

Spatial covariance parameters

We update the three spatial covariance parameters, $\log(\rho)$, $\log(\nu)$, γ, using a Metropolis-Hastings block update step. First, we generate a candidate using a random walk Gaussian candidate distribution

$$\log(\rho)^{(c)} \sim \mathrm{N}(\log(\rho)^{(r-1)}, s^2)$$

where $\log(\rho)^{(r-1)}$ is the value at MCMC iteration $r-1$, and s is a tuning parameter. Candidates are generated for $\log(\nu)$ and γ in a similar fashion. The acceptance ratio is

$$R = \left\{ \frac{\prod_{t=1}^{T} l[Y_t(\mathbf{s})|\rho^{(c)}, \nu^{(c)}, \gamma^{(c)}, \ldots]}{\prod_{t=1}^{T} l[Y_t(\mathbf{s})|\rho^{(r-1)}, \nu^{(r-1)}, \gamma^{(r-1)}, \ldots]} \right\} \times \left\{ \frac{p[\rho^{(c)}]}{p[\rho^{(r-1)]}} \right\} \times \left\{ \frac{p[\nu^{(c)}]}{p[\nu^{(r-1)}]} \right\} \times \left\{ \frac{p[\gamma^{(c)}]}{p[\gamma^{(r-1)}]} \right\}.$$

All three candidates are accepted with probability $\min\{R, 1\}$.

Web Appendix B. Posterior distributions

Conditional posterior of $z_{tk} \mid \ldots$

If knots are independent over days, then the conditional posterior distribution of $|z_{tk}|$ is conjugate. For simplicity, drop the subscript t, let $\tilde{z}_k = |z_k|$, $\tilde{\mathbf{z}}_{k^c}$ be the vector of $[|z(\mathbf{s}_1)|, \ldots, |z(\mathbf{s}_n)|]$ for $\mathbf{s} \notin P_k$, $\mathbf{X} = [\mathbf{X}(\mathbf{s}_1), \ldots, \mathbf{X}(\mathbf{s}_n)]^\top$, let $\mathbf{Y}_k$ and $\mathbf{X}_k$ be the observations and covariate measurements for $\mathbf{s} \in P_k$, and let $\mathbf{Y}_{k^c}$ and $\mathbf{X}_{k^c}$ be the observations and covariate measurements for $\mathbf{s} \notin P_k$ and define

$$\mathbf{R} = \begin{cases} \mathbf{Y}_k - \mathbf{X}_k \boldsymbol{\beta} & \mathbf{s} \in P_k \\ \mathbf{Y}_{k^c} - \mathbf{X}_{k^c}\boldsymbol{\beta} - \lambda \tilde{\mathbf{z}}_{k^c} & \mathbf{s} \notin P_k \end{cases}$$

Let

$$\begin{aligned} \mathbf{R}_1 &= \text{the vector of } \mathbf{R} \text{ for } \mathbf{s} \in P_k \\ \mathbf{R}_2 &= \text{the vector of } \mathbf{R} \text{ for } \mathbf{s} \notin P_k \\ \Omega &= \Sigma^{-1}. \end{aligned}$$

Then

$$\begin{aligned} \pi(z_k | \ldots) &\propto \exp\left\{ -\frac{1}{2} \left[\begin{pmatrix} \mathbf{R}_1 - \lambda \tilde{z}_k \mathbf{1} \\ \mathbf{R}_2 \end{pmatrix}^\top \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \begin{pmatrix} \mathbf{R}_1 - \lambda \tilde{z}_k \mathbf{1} \\ \mathbf{R}_2 \end{pmatrix} + \frac{\tilde{z}_k^2}{\sigma_k^2} \right] \right\} I(z_k > 0) \\ &\propto \exp\left\{ -\frac{1}{2} \left[\Lambda_k \tilde{z}_k^2 - 2\mu_k \tilde{z}_k \right] \right\} \end{aligned}$$

where

$$\mu_k = \lambda(\mathbf{R}_1^\top \Omega_{11} + \mathbf{R}_2^\top \Omega_{21})\mathbf{1}$$
$$\Lambda_k = \lambda^2 \mathbf{1}^\top \Omega_{11}\mathbf{1} + \frac{1}{\sigma_k^2}.$$

Then $\tilde{z}_k | \ldots \sim N_{(0,\infty)}(\Lambda_k^{-1}\mu_k, \Lambda_k^{-1})$

Conditional posterior of $\boldsymbol{\beta}, \lambda \mid \ldots$

For models that do not include a skewness parameter, we update $\boldsymbol{\beta}$ as follows. Let $\boldsymbol{\beta} \sim \mathrm{N}_p(0, \Lambda_0)$ where Λ_0 is a precision matrix. Then

$$\begin{aligned}
\pi(\boldsymbol{\beta} \mid \ldots) &\propto \exp\left\{-\frac{1}{2}\boldsymbol{\beta}^\top \Lambda_0 \boldsymbol{\beta} - \frac{1}{2}\sum_{t=1}^{n_t}[\mathbf{Y}_t - \mathbf{X}_t\boldsymbol{\beta}]^\top \Omega [\mathbf{Y}_t - \mathbf{X}_t\boldsymbol{\beta}]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\beta}^\top \Lambda_\beta \boldsymbol{\beta} - 2\sum_{t=1}^{n_t}(\boldsymbol{\beta}^\top \mathbf{X}_t^\top \Omega \mathbf{Y}_t)\right]\right\} \\
&\propto \mathrm{N}(\Lambda_\beta^{-1}\mu_\beta, \Lambda_\beta^{-1})
\end{aligned}$$

where

$$\mu_\beta = \sum_{t=1}^{n_t} \mathbf{X}_t^\top \Omega \mathbf{Y}_t$$
$$\Lambda_\beta = \Lambda_0 + \sum_{t=1}^{n_t} \mathbf{X}_t^\top \Omega \mathbf{X}_t.$$

For models that do include a skewness parameter, a simple augmentation of the covariate matrix $\mathbf{X}$ and parameter vector $\boldsymbol{\beta}$ allows for a block update of both $\boldsymbol{\beta}$ and λ. Let $\mathbf{X}_t^* = [\mathbf{X}_t, |\mathbf{z}_t|]$ where $\mathbf{z}_t = [z(\mathbf{s}_1), \ldots, z(\mathbf{s}_n)]^\top$ and let $\boldsymbol{\beta}^* = (\beta_1, \ldots, \beta_p, \lambda)^\top$. So to incorporate the $N(0, \sigma_\lambda^2)$ prior on λ, let $\boldsymbol{\beta}^* \sim \mathrm{N}_{p+1}(0, \Lambda_0^*)$ where

$$\Lambda_0^* = \begin{pmatrix} \Lambda_0 & 0 \\ 0 & \sigma_\lambda^{-2} \end{pmatrix}.$$

Then the update for both $\boldsymbol{\beta}$ and λ is done using the conjugate prior given above with $\mathbf{X}_t = \mathbf{X}_t^*$ and $\boldsymbol{\beta} = \boldsymbol{\beta}^*$

Conditional posterior of $\sigma^2 \mid \ldots$

In the case where $L = 1$ and temporal dependence is negligible, then σ^2 has a conjugate posterior distribution. Let $\sigma_t^2 \overset{iid}{\sim} \text{IG}(\alpha_0/2, \beta_0/2)$. For simplicity, drop the subscript t. Then

$$\begin{aligned}
\pi(\sigma^2 \mid \ldots) &\propto (\sigma^2)^{-\alpha_0/2-1/2-n/2-1} \exp\left\{-\frac{\beta_0}{2\sigma^2} - \frac{|z|^2}{2\sigma^2} - \frac{(\mathbf{Y}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{Y}-\boldsymbol{\mu})}{2\sigma^2}\right\} \\
&\propto (\sigma^2)^{-(\alpha_0-1-n)/2-1} \exp\left\{-\frac{1}{2\sigma^2}\left[\beta_0 + |z|^2 + (\mathbf{Y}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{Y}-\boldsymbol{\mu})\right]\right\} \\
&\propto \text{IG}(\alpha^*, \beta^*)
\end{aligned}$$

where

$$\begin{aligned}
\alpha^* &= \frac{\alpha_0 + 1 + n}{2} \\
\beta^* &= \frac{1}{2}\left[\beta_0 + |z|^2 + (\mathbf{Y}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{Y}-\boldsymbol{\mu})\right].
\end{aligned}$$

In the case that $K > 1$, a random walk Metropolis Hastings step will be used to update σ_{kt}^2.

Web Appendix C. Proof that $\lim_{h\to\infty} \pi(h) = 0$

Let c be the midpoint of $\mathbf{s}_1$ and $\mathbf{s}_2$. Define A as the circle centered at c with radius $h/2$ where $h = ||\mathbf{s}_1 - \mathbf{s}_2||$ is the distance between sites $\mathbf{s}_1$ and $\mathbf{s}_2$. Consider a homogeneous spatial Poisson process over A with intensity given by

$$\mu_{PP}(A) = \lambda_{PP}|A| = \lambda_{PP}\pi\left(\frac{h}{2}\right)^2 = \lambda_{PPA}^* h^2.$$

Consider a partition of A into four regions, B_1, B_2, R_1, R_2 as seen in Web Figure 1.

[Figure 1 about here.]

Let N_i be the number of knots in B_i and $L_i = l$ if $\mathbf{s}_i \in P_l$ for $i = 1, 2$. Then

$$P(L_1 \neq L_2) \geqslant P(N_1 > 0, N_2 > 0) \tag{1}$$

since knots in both B_1 and B_2 is sufficient, but not necessary, to ensure that $\mathbf{s}_1$ and $\mathbf{s}_2$ are in different partition sets. By definition of a Poission process, N_1 and N_2 are independent and thus

$P(N_1 > 0, N_2 > 0) = P(N_1 > 0)^2$, and the intensity measure over B_1 is given by

$$\mu_{PP}(B_1) = \lambda_{PP}|B_1| = \lambda_{PP}\frac{h^2}{4}\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= \lambda^*_{PPB1}h^2. \tag{2}$$

So,

$$P(L_1 \neq L_2) >= P(N_1 > 0)^2 = [1 - P(N_1 = 0)]^2 = [1 - \exp\left(-\lambda^*_{PPB1}h^2\right)]^2 \tag{3}$$

which goes to 1 as h goes to infinity.

Web Appendix D. Skew-t distribution

Univariate skew-t distribution

We say that Y follows a univariate extended skew-t distribution with location $\xi \in \mathcal{R}$, scale $\omega > 0$, skew parameter $\alpha \in \mathcal{R}$, and degrees of freedom ν if has distribution function

$$f_{\text{EST}}(y) = 2f_T(z;\nu)F_T\left[\alpha z\sqrt{\frac{\nu+1}{\nu+z^2}};\nu+1\right] \tag{4}$$

where $f_T(t;\nu)$ is a univariate Student's t with ν degrees of freedom, $F_T(t;\nu) = P(T < t)$, and $z = (y - \xi)/\omega$.

Multivariate skew-t distribution

If $\mathbf{Z} \sim \text{ST}_d(0, \bar{\mathbf{\Omega}}, \boldsymbol{\alpha}, \eta)$ is a d-dimensional skew-t distribution, and $\mathbf{Y} = \xi + \boldsymbol{\omega}\mathbf{Z}$, where $\boldsymbol{\omega} = \text{diag}(\omega_1, \ldots, \omega_d)$, then the density of Y at y is

$$f_y(\mathbf{y}) = det(\boldsymbol{\omega})^{-1}f_z(\mathbf{z}) \tag{5}$$

where

$$f_z(\mathbf{z}) = 2t_d(\mathbf{z};\bar{\mathbf{\Omega}},\eta)T\left[\boldsymbol{\alpha}^\top\mathbf{z}\sqrt{\frac{\eta+d}{\nu+Q(\mathbf{z})}};\eta+d\right] \tag{6}$$

$$\mathbf{z} = \boldsymbol{\omega}^{-1}(\mathbf{y} - \xi) \tag{7}$$

where $t_d(\mathbf{z}; \bar{\mathbf{\Omega}}, \eta)$ is a d-dimensional Student's t-distribution with scale matrix $\bar{\mathbf{\Omega}}$ and degrees of freedom η, $Q(z) = \mathbf{z}^\top \bar{\Omega}^{-1} \mathbf{z}$ and $T(\cdot; \eta)$ denotes the univariate Student's t distribution function with η degrees of freedom (Azzalini and Capitanio, 2014).

Extremal dependence

For a bivariate skew-t random variable $\mathbf{Y} = [Y(\mathbf{s}), Y(\mathbf{t})]^\top$, the $\chi(h)$ statistic (Padoan, 2011) is given by

$$\chi(h) = \bar{F}_{\text{EST}} \left\{ \frac{[x_1^{1/\eta} - \varrho(h)]\sqrt{\eta+1}}{\sqrt{1-\varrho(h)^2}}; 0, 1, \alpha_1, \tau_1, \eta+1 \right\} + \bar{F}_{\text{EST}} \left\{ \frac{[x_2^{1/\eta} - \varrho(h)]\sqrt{\eta+1}}{\sqrt{1-\varrho(h)^2}}; 0, 1, \alpha_2, \tau_2, \eta+1 \right\}, \tag{8}$$

where $\bar{F}_{\text{EST}}$ is the univariate survival extended skew-t function with zero location and unit scale, $\varrho(h) = \text{cor}[y(\mathbf{s}), y(\mathbf{t})]$, $\alpha_j = \alpha_i\sqrt{1-\varrho^2}$, $\tau_j = \sqrt{\eta+1}(\alpha_j + \alpha_i\varrho)$, and $x_j = F_T(\bar{\alpha}_i\sqrt{\eta+1}; 0, 1, \eta)/F_T(\bar{\alpha}_j\sqrt{\eta+1};$ with $j = 1, 2$ and $i = 2, 1$ and where $\bar{\alpha}_j = (\alpha_j + \alpha_i\varrho)/\sqrt{1+\alpha_i^2[1-\varrho(h)^2]}$.

Proof that $\lim_{h\to\infty} \chi(h) > 0$

Consider the bivariate distribution of $\mathbf{Y} = [Y(\mathbf{s}), Y(\mathbf{t})]^\top$, with $\varrho(h)$ given by (2). So, $\lim_{h\to\infty} \varrho(h) = 0$. Then

$$\lim_{h\to\infty} \chi(h) = \bar{F}_{\text{EST}} \left\{ \sqrt{\eta+1}; 0, 1, \alpha_1, \tau_1, \eta+1 \right\} + \bar{F}_{\text{EST}} \left\{ \sqrt{\eta+1}; 0, 1, \alpha_2, \tau_2, \eta+1 \right\}. \tag{9}$$

Because the extended skew-t distribution is not bounded above, for all $\bar{F}_{\text{EST}}(x) = 1 - F_{\text{EST}(x)} > 0$ for all $x < \infty$. Therefore, for a skew-t distribution, $\lim_{h\to\infty} \chi(h) > 0$.

Web Appendix E. Comparisons with other parameterizations

Various forms of multivariate skew-normal and skew-t distributions have been proposed in the literature. In this section, we make a connection between our parameterization in (1) of the main text and another popular version. Azzalini and Capitanio (2014) and Beranger et al. (2016) define a skew-normal process as

$$\tilde{X}(\mathbf{s}) = \tilde{\lambda}|z| + (1 - \tilde{\lambda}^2)^{1/2} v(\mathbf{s}) \tag{10}$$

where $\tilde{\lambda} \in (-1, 1)$, $z \sim N(0, 1)$, and $v(\mathbf{s})$ is a Gaussian process with mean zero, variance one, and spatial correlation function ρ. To extend this to the skew-t distribution, Azzalini and Capitanio (2003) take $\tilde{Y}(\mathbf{s}) = W\tilde{X}(\mathbf{s})$ where $W^{-2} \sim \text{Gamma}(a/2, a/2)$. Returning to the proposed parameterization (with $\boldsymbol{\beta} = 0$), let $W^{-2} = \frac{b}{a}\sigma^{-2} \sim \text{Gamma}(a/2, a/2)$ so that (1) in the manuscript becomes

$$Y(\mathbf{s}) = W\left[\lambda\left(\frac{b}{a}\right)^{1/2}|z| + \left(\frac{b}{a}\right)^{1/2} v(\mathbf{s})\right]. \tag{11}$$

Clearly setting $b = a(1 - \tilde{\lambda}^2) > 0$, and $\lambda = \tilde{\lambda}/(1 - \tilde{\lambda}^2)^{1/2} \in (-\infty, \infty)$ resolves the difference in parameterizations. We note that our parameterization has three parameters (a, b, λ) compared to the two parameters of the alternative parameterization $(a, \tilde{\lambda})$. Since we have assumed that both $v(\mathbf{s})$ and z have unit scale, the additional b parameter in our parameterization is required to control the precision.

Web Appendix F. Temporal dependence

It is very challenging to derive an analytical expression the temporal extremal dependence at a single site **s**. However, using simulated data, we have evidence to suggest that the model does exhibits temporal extremal dependence. To demonstrate that our model maintains temporal extremal dependence, we generate lag-m observations for $m = 1, 3, 5, 10$ from our model setting $\phi_w = \phi_z = \phi_\sigma = \varphi$, for $\varphi = 0, 0.02, 0.04, \ldots, 1$. To estimate the lag-$m$ chi-statistic $\chi(m)$ we first estimate the lag-m F-madogram $\nu_F(m)$ (Cooley et al., 2006) using $\hat{\nu}_F(m) = \frac{1}{2n}\sum_{i=1}^{n}|\hat{F}(y_0) - \hat{F}(y_m)|$ where $\hat{F}(\cdot)$ represents an empirical CDF and y_m is the lag-m observation. The F-madogram is related to the χ statistic as follows

$$\chi = 2 - \frac{1 + 2\nu_F}{1 - 2\nu_F}. \tag{12}$$

Web Figure 2 suggests that the extremal dependence increases as $\phi \to 1$, and that the extremal dependence decreases as m increases.

[Figure 2 about here.]

Web Appendix G. Simulation study pairwise difference results

The following tables show the methods that have significantly different Brier scores when using a Wilcoxon-Nemenyi-McDonald-Thompson test. In each column, different letters signify that the methods have significantly different Brier scores. For example, there is significant evidence to suggest that method 1 and method 4 have different Brier scores at $q(0.90)$, whereas there is not significant evidence to suggest that method 1 and method 2 have different Brier scores at $q(0.90)$. In each table group A represents the group with the lowest Brier scores. Groups are significant with a familywise error rate of $\alpha = 0.05$.

[Table 1 about here.]

[Table 2 about here.]

[Table 3 about here.]

[Table 4 about here.]

[Table 5 about here.]

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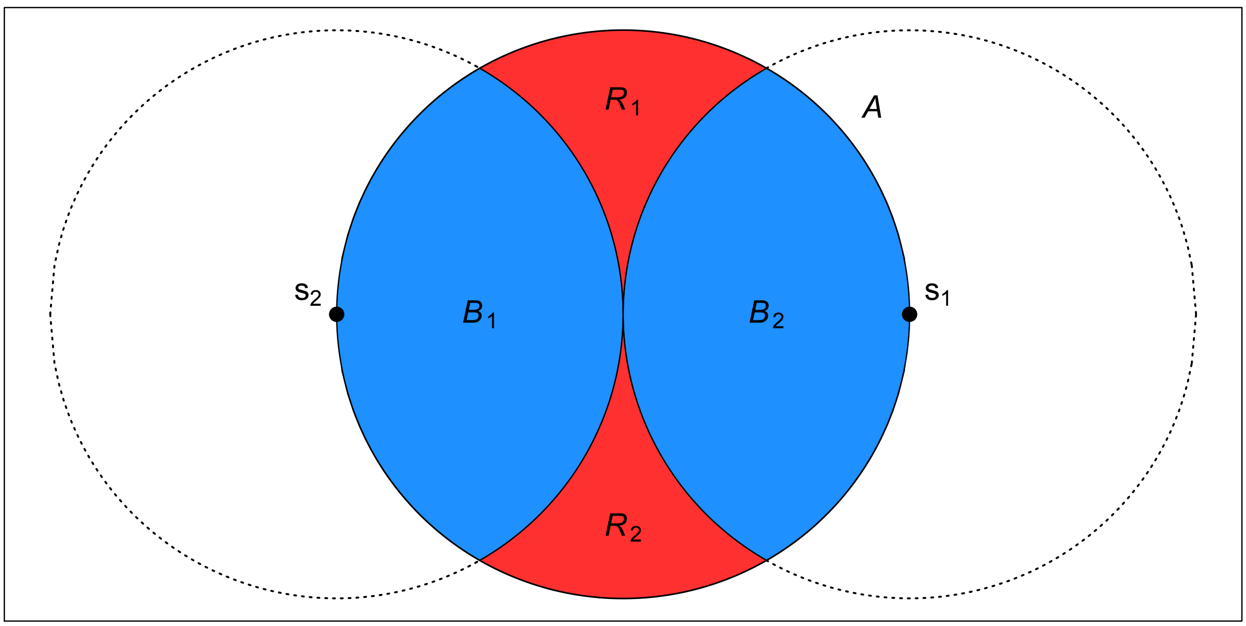
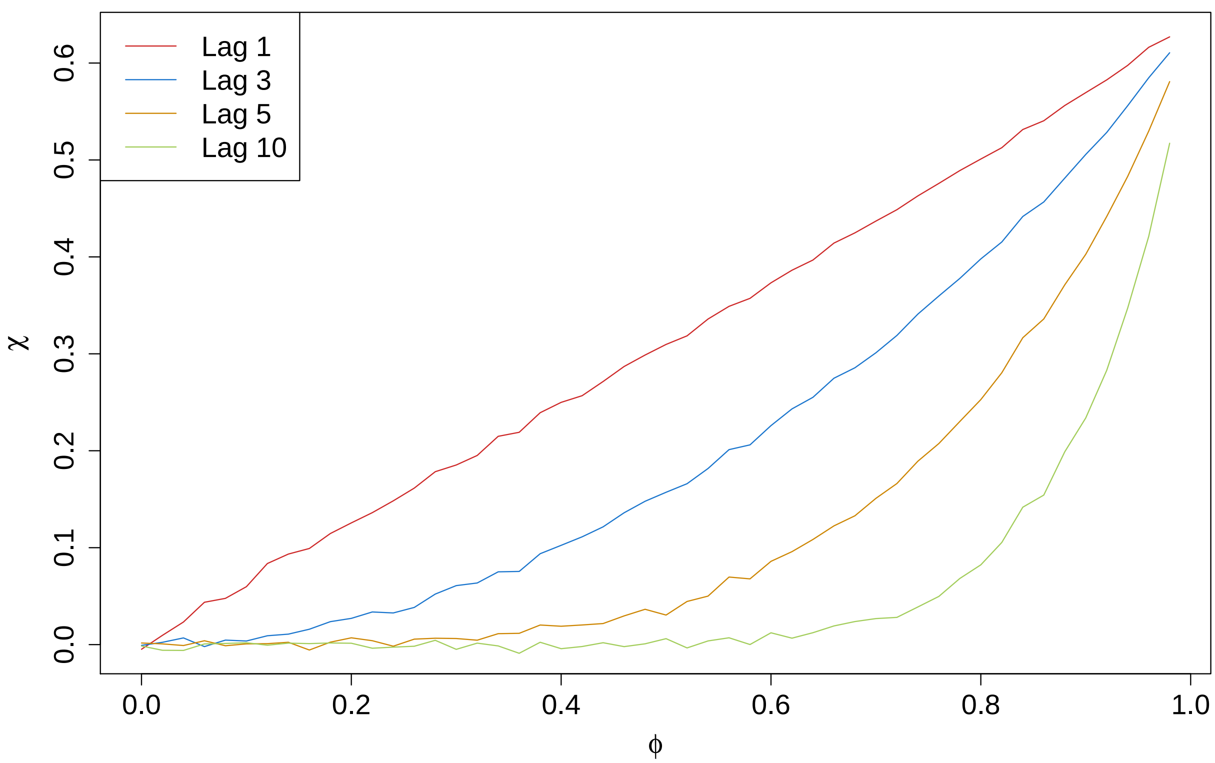


Web Figure 1. Illustration of the partition of A.



Web Figure 2. Simulated lag-m χ for varying levels of φ.

Web Table 1

Setting 1 – Gaussian marginal, $K = 1$ knot

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	A	A	A	A
Method 2	A	A	A	A
Method 3	B	B	C	A
Method 4	A	A	A B	A
Method 5	B	B	B C	A
Method 6	C	C	D	B

Web Table 2

Setting 2 – Skew-t marginal, $K = 1$ knot

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	B	B	B	B
Method 2	A	A	A	A
Method 3	A B	A B	A B	A B
Method 4	A B	A B	A B	A B
Method 5	C	C	C	C
Method 6	D	D	D	C

Web Table 3

Setting 3 – Skew-t marginal, $K = 5$ knots

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	C	C	B	B
Method 2	C	C	B	B
Method 3	B	B	A	A
Method 4	A	A	A	A
Method 5	A	A	A	A
Method 6	D	D	C	C

Web Table 4

Setting 4 – Max-stable

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	A B	B	B	C
Method 2	B	B	B	B C
Method 3	C D	C	B	B
Method 4	D	D	C	C
Method 5	C	B C	B	B C
Method 6	A	A	A	A

Web Table 5

Setting 5 – Brown Resnick

	$q(0.90)$	$q(0.95)$	$q(0.98)$	$q(0.99)$
Method 1	D	C	C	C
Method 2	D	C	C	C
Method 3	A B	A	A B	B
Method 4	C	B	B	B
Method 5	A	A	A	A B
Method 6	B C	A	A	A