# Spatial methods for extreme value analysis

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# Motivation

- Average behavior is important to understand, but it does not paint the whole picture
  - e.g. When constructing river levees, engineers need to be able to estimate a 100-year or 1000-year flood levels
  - e.g. Probability of exceeding a certain threshold level
- Spatial methods borrow information across space to estimate spatial correlation and make predictions by Kriging at unknown locations
- Want to explore similar methods for extremes

# Standard analysis - Block maxima

- Uses yearly maxima
- Discards many observations
- Models are fit using the generalized extreme value distribution with parameters $\mu, \sigma$, and $\xi$

$$\Pr(Y < y) = \begin{cases} \exp\left\{-\left[1 + \xi\left(\frac{y-\mu}{\sigma}\right)\right]^{-1/\xi}\right\} & \xi \neq 0 \\ \exp\left\{-\exp\left(-\frac{y-\mu}{\sigma}\right)\right\} & \xi = 0 \end{cases}$$

- Standardized distribution is unit Fréchet or GEV(1, 1, 1)

$$\Pr(Z < z) = exp(-z^{-1})$$

# Standard analysis - Peaks over threshold

- Incorporates more data than block maxima
- Select a threshold, $T$, and use the Generalized Pareto distribution (GPD) to model the exceedances
- The generalized Parety distribution has three parameters $\mu, \sigma$, and $\xi$

$$P(Y < y) = \begin{cases} 1 - \left[1 - \xi\left(\frac{y-\mu}{\sigma}\right)\right]^{-1/\xi} & \xi \neq 0 \\ 1 - \exp\left\{\frac{y-\mu}{\sigma}\right\} & \xi = 0 \end{cases}$$

- Temporal dependence may be an issue between observations (e.g. flood levels don't dissipate overnight)

# Introduction to extremes

- For a spatial analysis, max-stable processes give an appropriate limiting distribution (Cooley et al., 2012):
  - Consider a spatial process $x_t(\mathbf{s})$, $t = 1, \ldots, T$.
  - Let $M_T(\mathbf{s}) = \left\{ \bigvee_{t=1}^{T} x_t(\mathbf{s}_1), \ldots, \bigvee_{t=1}^{T} x_t(\mathbf{s}_n) \right\}$
  - If there exists normalizing sequences $a_T(\mathbf{s})$ and $b_T(\mathbf{s})$ such that for all sites, $\mathbf{s}_i, i = 1, \ldots, d$,

$$a_T^{-1}(\mathbf{s}) \left\{ M_T(\mathbf{s}) - b_T(\mathbf{s}) \right\} \xrightarrow{d} Y(\mathbf{s})$$

    which has a non-degenerate distribution, then $Y(\mathbf{s})$ is a max-stable process.

# Multivariate representations

- Multivariate distributions:
  - Assume common standardized max-stable marginal, like unit-Fréchet
  - The multivariate representation for the GEV is

$$\Pr(\mathbf{Z} \leq \mathbf{z}) = G^*(\mathbf{z}) = \exp(-V(\mathbf{z}))$$

$$V(\mathbf{s}) = d \int_{\Delta_d} \bigvee_{i=1}^{d} \frac{w_i}{z_i} H(\mathrm{d}w)$$

  where
    - $\Delta_d = \{\mathbf{w} \in \mathcal{R}_+^d \mid w_1 + \cdots + w_d = 1\}$
    - $H$ is a probability measure on $\Delta_d$
    - $\int_{\Delta_d} w_i H(\mathrm{d}w) = 1/d$ for $i = 1, \ldots, d$.

# Multivariate analysis

- Multivariate max-stable and GPD models have nice features, but they are
  - computationally challenging to work with
  - joint distribution only available in low dimension
- Bayesian hierarchical model (Reich and Shaby, 2012)
- Pairwise likelihood approach (Huser and Davison, 2014)

# Three principal contributions

1. A spatio-temporal model with flexible tails, asymptotic spatial dependence, and computation on the order of Gaussian models for large space-time datasets
2. Predicting exceedances using a spatially dependent generalized extreme-value link function
3. A Bayesian hierarchical model to allow for non-stationary covariance in extreme value models.

# Spatiotemporal modeling for extreme values

- Model to analysis spatiotemporal extreme values
- Model objectives
  - has marginal distribution with a flexible tail
  - has asymptotic spatial dependence
  - has computation on the order of Gaussian models for large space-time datasets

# Spatial skew-$t$ distribution

- Assume observed data $Y(\mathbf{s})$ come from a skew-$t$ (Zhang and El-Shaarawi, 2012)

$$Y(\mathbf{s}) = \mathbf{X}(\mathbf{s})\boldsymbol{\beta} + \lambda|z| + v(\mathbf{s})$$

where

  - $\lambda \in \mathcal{R}$ controls the skewness
  - $z \sim N(0, \sigma^2)$ is a random effect
  - $v(\mathbf{s})$ is a Gaussian process with variance $\sigma^2$ and Matérn correlation
  - $\sigma^2 \sim \mathrm{IG}(a, b)$

# Spatial skew-$t$ distribution

- Conditioned on $z$ and $\sigma^2$, $Y(\mathbf{s})$ is a Gaussian spatial model
- Can use standard geostatistical methods to fit this model
- Predictions can be made through Kriging

# Spatial skew-$t$ distribution

- Marginalizing over $z$ and $\sigma^2$ (via MCMC),

$$Y(\mathbf{s}) \sim \text{skew-t}(\mathbf{X}(\mathbf{s}), \boldsymbol{\Omega}, \alpha, \text{df} = 2a)$$

where

- $\mathbf{X}(\mathbf{s})\beta$ is the location
- $\boldsymbol{\Omega} = \boldsymbol{\omega}\bar{\boldsymbol{\Omega}}\boldsymbol{\omega}$ is a correlation matrix
- $\boldsymbol{\omega} = \text{diag}\left(\frac{1}{\sqrt{ab}}, \ldots, \frac{1}{\sqrt{ab}}\right)$
- $\bar{\boldsymbol{\Omega}} = (\boldsymbol{\Sigma} + \lambda^2 \mathbf{1}\mathbf{1}^T)$
- $\Sigma$ is a postive definite correlation matrix
- $\alpha = \lambda(1 + \lambda^2 \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1})^{-1/2} \mathbf{1}^T \boldsymbol{\Sigma}^{-1}$ controls the skewness

# Censoring data to focus on tail behavior

- We censor the observed data at a high threshold $T$.
- Censored data:

$$\tilde{Y}_t(\mathbf{s}) = \begin{cases} Y_t(\mathbf{s}) & \delta(\mathbf{s}) = 1 \\ T & \delta(\mathbf{s}) = 0 \end{cases}$$

  where $\delta(\mathbf{s}) = I[Y(\mathbf{s}) > T]$
- Allows tails of the distribution to speak for themselves.

# $\chi$ statistic

- The $\chi$ statistic is a measure of extremal dependence
- Specifically, we focus on $\chi(\mathbf{h})$ for the upper tail given by

$$\chi(h) = \lim_{c \to \infty} \Pr(Y(\mathbf{s}) > c \mid Y(\mathbf{t}) > c)$$

  where $h = ||\mathbf{s} - \mathbf{t}||$
- If $\chi(h) = 0$, then observations are asymptotically independent at distance $\mathbf{h}$.

# Gaussian spatial model

- In geostatistics $Y(\mathbf{s})$ are often modeled using a Gaussian process with mean function $\mu(\mathbf{s})$ and covariance function $\rho(\mathbf{h})$.
- Model properties:
  - Nice computing properties (closed-form likelihood)
  - For a Gaussian spatial model $\lim_{c \to \infty} \chi(\mathbf{h}) = 0$ regardless of the strength of the correlation in the bulk of the distribution
  - Tail is not flexible (Gaussian is light tailed)

# Spatial skew-$t$ distribution

- Model properties
  - Has flexible tail controlled by skewness $\alpha$ and degrees of freedom $2a$
  - For a skew-$t$ distribution $\lim_{c \to \infty} \chi(\mathbf{h}) > 0$ (Padoan, 2011)
  - Computation that is on the order of Gaussian computation
- For this distribution, $\chi(\mathbf{h})$ shows asymptotic dependence that does not approach 0 as $\mathbf{h} \to \infty$
- This occurs because all observations (near and far) share the same $z$ and $\sigma^2$
- We deal with this through a daily random partition (similar to Kim et al., 2005)

# Random partition

- Daily random partition allows $z$ and $\sigma^2$ to vary by site

$$Y(\mathbf{s}) = \mathbf{X}(\mathbf{s})\boldsymbol{\beta} + \lambda z(\mathbf{s}) + \sigma(\mathbf{s})v(\mathbf{s})$$

- Consider a set of knots $\mathbf{w}_k \sim$ Uniform that define a random partition $P_1, \ldots, P_K$ such that

$$P_k = \{s : k = \arg\min_\ell ||\mathbf{s} - \mathbf{w}_\ell||\}$$

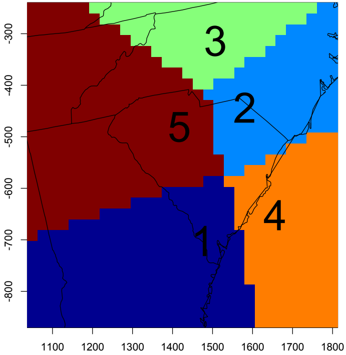
  where $\mathbf{w} = (w_1, w_2)$

- For $\mathbf{s} \in P_k$

$$z(\mathbf{s}) = z_k$$
$$\sigma^2(\mathbf{s}) = \sigma^2_k$$

- Within each partition $Y(\mathbf{s})$ has the same MV skew-t distribution as before

# Example partition



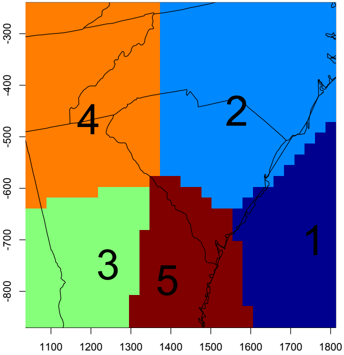




Figure: Two sample partitions (number is at partition center)

# $\chi$(h) plot

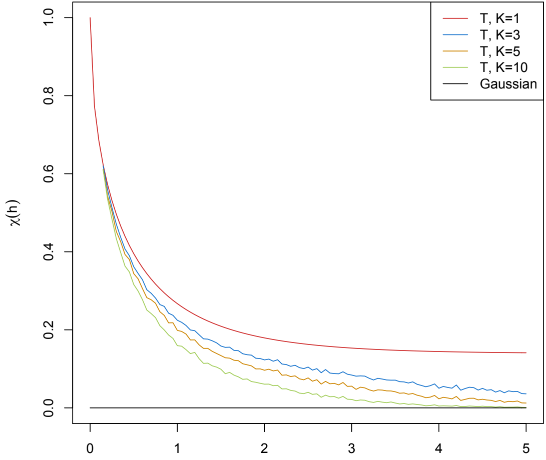



Figure: $\chi$ plot for different data settings

# Sample simulated datasets

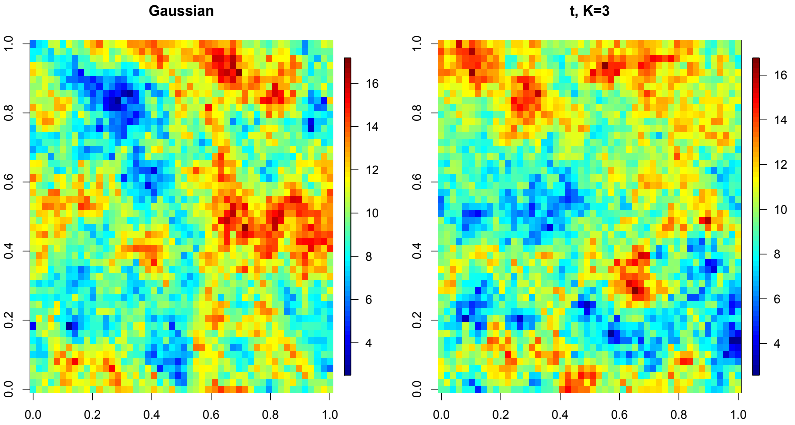



Figure: Gaussian and $t$ with 3 partitions

# Sample simulated datasets

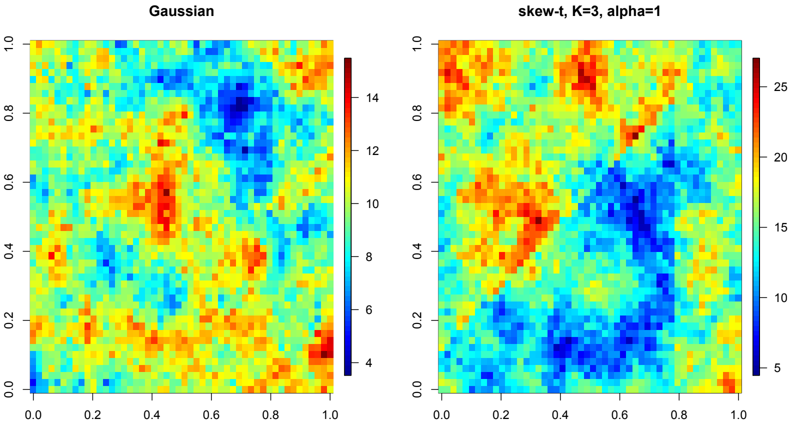



Figure: Gaussian and skew-$t$ with 3 partitions

# Extension to space-time data

- We extend our previous model to be

$$Y_t(\mathbf{s}) = \mathbf{X}_t(\mathbf{s})^T \boldsymbol{\beta} + \lambda\sigma_t(\mathbf{s})|z_t(\mathbf{s})| + \sigma_t(\mathbf{s})v_t(\mathbf{s})$$

where $t = 1, \dots, T$ denotes the day of each observation.

- We incorporate an AR(1) time series on $\mathbf{w}^*_{tk} = (w^*_{tk1}, w^*_{tk2})$, $z_{tk}$, and $\sigma^*_{tk}$ where

$$w^*_{tki} = \Phi^{-1}\left[\frac{w_{tki} - \min(\mathbf{s}_i)}{\text{range}(\mathbf{s}_i)}\right] \quad i = 1, 2$$

$$\sigma^{2*}_t(\mathbf{s}) = \Phi^{-1}\{\text{IG}[\sigma^2_t(\mathbf{s})]\}$$

are transformations to $\mathcal{R}^2$

# MCMC details

- Three main steps:
  1. Impute censored data below $T$
  2. Update parameters with standard random walk Metropolis Hastings or Gibbs sampling
  3. Make spatial predictions
- Priors are selected to be conjugate when possible

# Simulation study

- 6 different data settings:
  - Gaussian vs $t$ vs skew-$t$ marginal distribution
  - $K = 1$ partition vs $K = 5$ partitions
- 5 different models:
  - Gaussian vs skew-$t$ marginal distribution
  - $K = 1$ partition vs $K = 5$ partitions
- Brier score used to determine model that gives best fit

# Brier score

- The Brier score for predicting exceedance of threshold $c$ is

$$[e(c) - P(c)]^2$$

  where

  - $y$ is a test set value
  - $e(c) = I[y > c]$
  - $P(c)$ is the predicted probability of exceeding $c$

- Relative Brier scores:

$$\mathrm{BS}_{\mathrm{rel}} = \frac{\mathrm{BS}_{\mathrm{method}}}{\mathrm{BS}_{\mathrm{Gaussian}}}$$

# Simulation study results

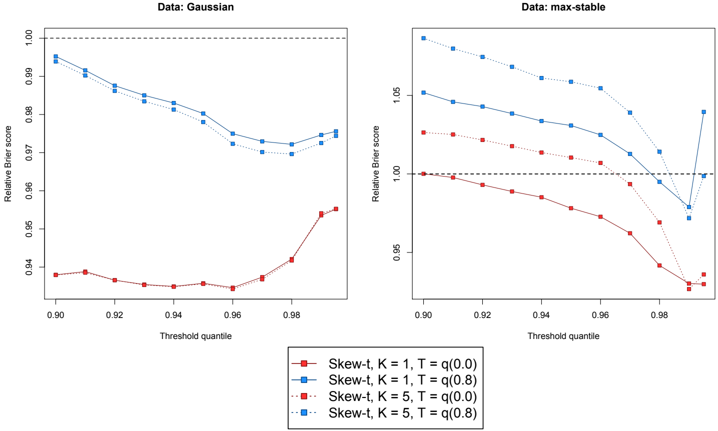



Figure: Relative Brier score results

# Simulation study results

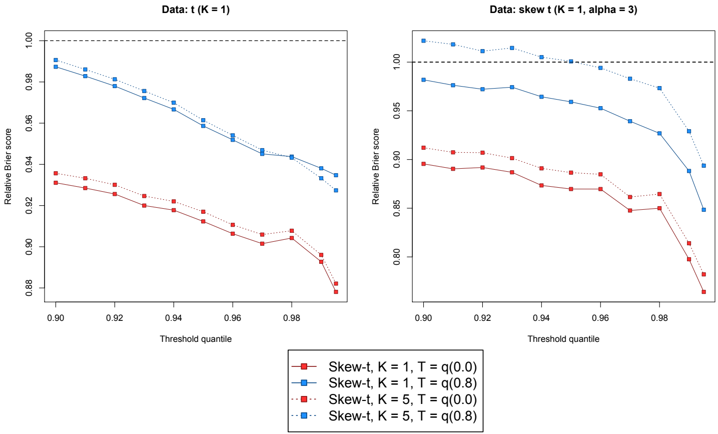



Figure: Relative Brier score results

# Simulation study results

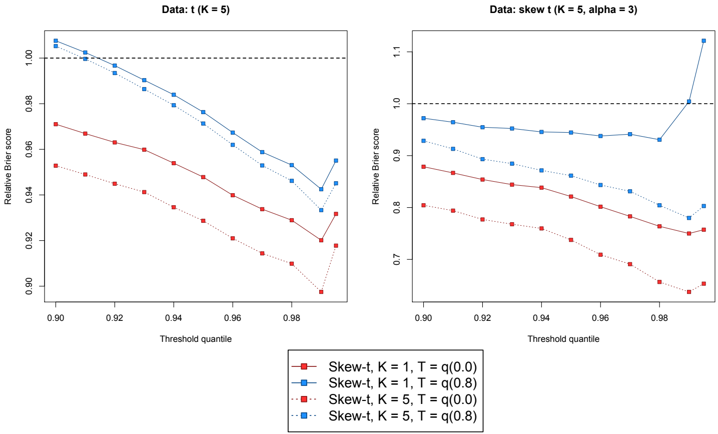



Figure: Relative Brier score results

# Simulation study results

- Key findings:
  - Improvement over Gaussian methods when partitioning
  - Underestimating the number of knots has a detrimental impact
  - In all cases, non-thresholded models perform better than thresholded models

# Data analysis

- Ozone measurements
  - max 8-hour ozone measurements
  - data from 1089 sites
  - July 2005
- We take a stratified sample of $n = 800$ sites:
  - 271 from northeast
  - 96 from northwest
  - 269 from southeast
  - 164 from southwest

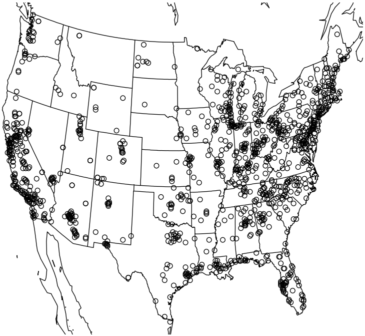


Figure: Ozone monitoring station locations

# Data analysis

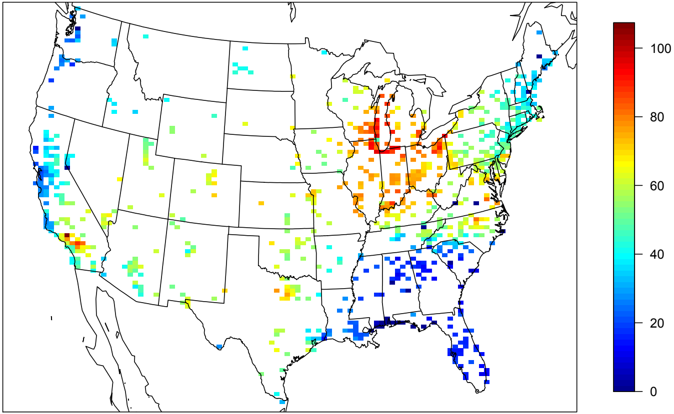



Figure: Max 8-hour ozone measurements on July 10, 2005

# Exploratory data analysis

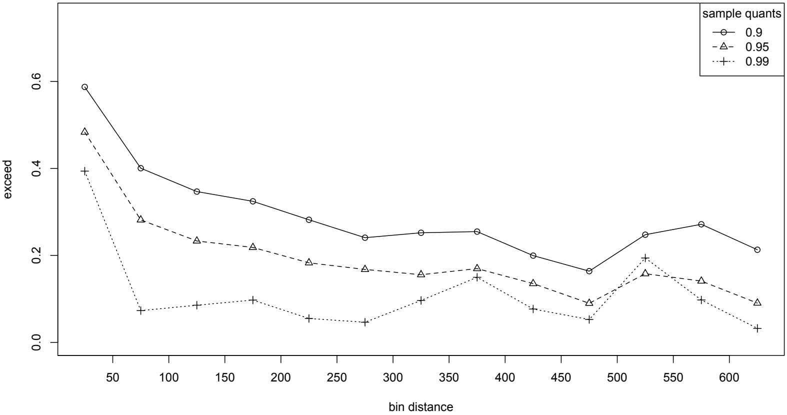



Figure: $\widehat{\chi}$-plot for sample quantiles of ozone observations

# Model comparisons

- 9 different analysis methods incorporating
  - Gaussian vs $t$ vs skew-$t$ marginal distribution
  - $K = 1, 5, 6, 7, 8, 9, 10, 15$ partitions
  - Thresholding at $T = 0, 50, 75,$ and 85 ppb
- All methods use a Matérn or exponential covariance ($\nu = 0.5$)
- Compare quantile and Brier scores using two-fold cross validation (Gneiting and Raftery, 2007)
- Mean function modeled as

$$\beta_0 + \beta_1 \cdot \text{CMAQ}$$

# Two-fold cross-validation results

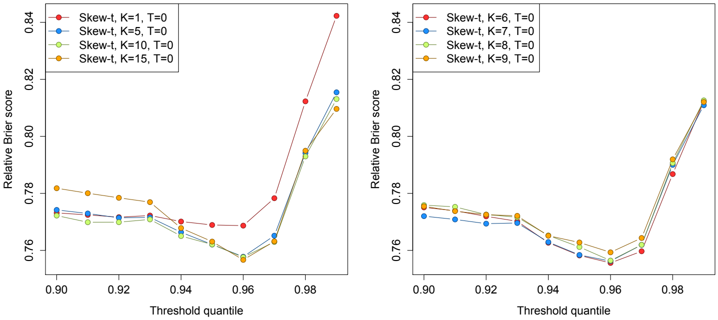



Figure: Relative Brier score results

# Two-fold cross-validation results

- Key findings:
  - Partitioning improves performance across all high thresholds.
  - Models with anywhere from $K = 5$ to $K = 10$ partitions perform similarly
  - In all cases, non-thresholded models perform better than thresholded models

# Discussion and future work

- Improvement of model performance when using partitioned models
- Thresholding makes results worse
  - Possible numerical instability due to truncated normal distribution
- Different ways to incorporate the temporal dependence
  - Three dimensional covariance model for $v_t(\mathbf{s})$ (e.g. Huser and Davison, 2014)
- Different partition structure
  - Distance weighting for each knot vs indicator functions
- Knot selection
  - Possible prior on the probability a knot is in the spatial domain

# Spatial binary regression

- We observe $Y_i = I[Z(\mathbf{s}_i) > T]$, an indicator variable that a continuous latent variable has exceeded a pre-specified threshold $T$.
- We model $\Pr[Y_i = 1]$
- Common examples:
  - Logistic regression

$$\Pr[Y_i = 1] = \frac{\exp(\mathbf{X}(\mathbf{s}_i)\beta)}{1 + \exp(\mathbf{X}(\mathbf{s}_i)\beta)}$$

  - Probit regression

$$\Pr[Y_i = 1] = \Phi[\mathbf{X}(\mathbf{s}_i)\beta]$$

  where $\Phi$ is the standard normal distribution function

# Spatial binary regression

- Logistic and probit regression amount to modeling

$$\Pr[Y_i = 1] = g[\mathbf{X}(\mathbf{s}_i)\beta]$$

where $g[\cdot]$ is a link function transforming from $\mathcal{R}$ to $(0, 1)$.

- Wang and Dey (2010): Generalized extreme value link function

$$g[\mathbf{x}(\mathbf{s}_i)\beta] = 1 - \exp\left[-(1 + \xi\mathbf{X}_i\beta)^{-1/\xi}\right]$$

# Spatial binary regression

- Proposed method will
  - use the GEV link function
  - use the hierarchical likelihood from Reich and Shaby (2012)
- Model parameters fit using MCMC

# Spatial binary regression

- We fit parameters $\xi$ and $\beta$ in order to transform the data to GEV(1, 1, 1) marginal distributions.
- Using the link function

$$p_i = 1 - \exp\left[-(1 + \xi \mathbf{X}(\mathbf{s}_i)\boldsymbol{\beta})^{-1/\xi}\right]$$

  we set the latent variable $z_i = -\frac{1}{\log(1-p_i)}$
- Then we evaluate the joint likelihood using $z_i$.

# Likelihood function

- We use a multivariate generalized extreme value distribution with asymetric Laplace deendence function given by

$$G(\mathbf{z}) = \Pr[Z_1 < z_z, \ldots, Z_n < z_n] = \exp\left\{ -\sum_{l=1}^{L} \left[ \sum_{i=1}^{n} \left( \frac{w_l(\mathbf{s}_i)}{z_i} \right)^{1/\alpha} \right]^{\alpha} \right\}$$

where

  - $w_l$ is a weighting function subject to the constraint that $\sum_{l=1}^{L} w_l = 1$.
  - $\alpha$ controls spatial dependence
    - $\alpha = 0$ is strong dependence
    - $\alpha = 1$ is joint independence

# Weighting function

- We use the Gaussian weights proposed by Reich and Shaby (2012) given by

$$w_l(\mathbf{s}_i) = \frac{\exp\left[-0.5\left(\frac{||\mathbf{s}-\mathbf{v}_l||}{\rho}\right)^2\right]}{\sum_{l=1}^{L}\exp\left[-0.5\left(\frac{||\mathbf{s}-\mathbf{v}_l||}{\rho}\right)^2\right]}$$

where

  - $\mathbf{v}_l$ are spatial knots
  - $\rho$ is a bandwidth term for the kernel function

# Joint likelihood

- Let $K_t = \sum_{i=1}^{n} Y_{it}$ be the number of exceedances that occur on day $t$.
- Rearrange the sites so
  - $Y_1, \ldots, Y_K$ are the observations where $Y(\mathbf{s}_i) = 1$
  - $Y_{K+1}, \ldots, Y_n$ are the observations where $Y(\mathbf{s}_i) = 0$
- Then for $K = 0, 1, 2$

$$\Pr(Y_1 = y_1, \ldots, Y_n = y_n) = \begin{cases} G(\mathbf{z}) & K = 0 \\ G(\mathbf{z}_{(1)}) - G(\mathbf{z}) & K = 1 \\ G(\mathbf{z}_{(12)}) - G(\mathbf{z}_{(1)}) - G(\mathbf{z}_{(2)}) + G(\mathbf{z}) & K = 2 \end{cases}$$

  where $G(\mathbf{z}_{(1)} = \Pr(Z_2 < z_2, \ldots, Z_n < z_n)$.
- $K > 2$ can be derived similarly

# Non-stationary covariance for extreme values

- Knot-specific bandwidth

# Joint likelihood

- For small $K$, we can evaluate the likelihood directly.
- For large $K$, we use the hierarchical model of Reich and Shaby (2012).

# Thesis outline

- Chapter 1: Extreme value theory August 2015
- Chapter 2: Spatiotemporal model for extreme value analysis based on the skew-$t$ distribution February 2015
- Chapter 3: Spatial binary regression May / June 2015
- Chapter 4: Non-stationary covariance through knot-specific bandwidth August 2015

# Questions

- Questions?
- Thank you for your attention.
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# References


- Demarta, S. and McNeil, A. J. (2007) The $t$ copula and related copulas. *International Statistical Review*, **73**, 111–129.
- Huser, R. and Davison, A. C. (2014) Space-time modelling of extreme events. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **76**, 439–461.
- Padoan, S. A. (2011) Multivariate extreme models based on underlying skew-$t$ and skew-normal distributions. *Journal of Multivariate Analysis*, **102**, 977–991.
- Zhang, H. and El-Shaarawi, A. (2010) On spatial skew-Gaussian processes and applications. *Environmetrics*, **21**, 33–47.