

# Spatiotemporal Modeling of Extreme Events

**Samuel Morris and Brian Reich**

North Carolina State University

# Motivation

- ▶ Average behavior is important to understand, but it does not paint the whole picture.
  - ▶ e.g. When constructing river levees, engineers need to be able to estimate a 100-year or 1000-year flood levels.
- ▶ In geostatistical analysis, kriging uses spatial correlation to help inform prediction at unknown locations.
- ▶ Want to explore computationally easy methods that are available in higher dimensions

# Standard non-spatial analysis

- ▶ Block maxima:
  - ▶ Uses yearly maxima
  - ▶ Discards many observations
  - ▶ Models are fit using the generalized extreme value distribution
- ▶ Generalized extreme value distribution (GEV):

$$\Pr(Y_j < y) = G_j(y) = \exp \left\{ - \left[ \left( 1 + \xi_j \frac{y - \mu_j}{\sigma_j} \right)_+^{-1/\xi_j} \right] \right\}$$

- ▶ For a spatial analysis, max-stable processes give an appropriate limiting distribution.

# Standard non-spatial analysis

- ▶ Peaks-over-threshold:
  - ▶ Incorporates more data than block maxima
  - ▶ Select a threshold,  $T$ , and fit data above the threshold using the generalized Pareto distribution
  - ▶ Autocorrelation may be an issue between observations (e.g. flood levels don't dissipate overnight)
- ▶ Generalized Pareto distribution (GPD):

$$\Pr(Y_j > y | Y_j > T) = F_j(y) = \left(1 + \xi_j \frac{y - T}{\sigma_j}\right)_+^{-1/\xi_j}$$

# Multivariate analysis

- ▶ Multivariate max-stable and GPD models have nice features, but they are
  - ▶ computationally hard to work with
  - ▶ joint distribution only available in low dimension
- ▶ Pairwise likelihood approach (Huser and Davison, 2014)

# Model objectives

- ▶ Our objective is to build a model that
  - ▶ has a flexible tail
  - ▶ has asymptotic spatial dependence
  - ▶ computation on the order of Gaussian models for large space-time datasets

# Thresholding data

- ▶ We threshold the observed data at a high threshold  $T$ .
- ▶ Thresholded data:

$$Y_t^*(\mathbf{s}) = \begin{cases} Y_t(\mathbf{s}) & Y_t(\mathbf{s}) > T \\ T & Y_t(\mathbf{s}) \leq T \end{cases}$$

- ▶ Allows tails of the distribution to speak for themselves.

- ▶ The  $\chi$  coefficient is a measure of extremal correlation
- ▶ Specifically, we focus on  $\chi(c, \mathbf{h})$  for the upper tail given by

$$\chi(c, \mathbf{h}) = \Pr(Y(\mathbf{s}) > c \mid Y(\mathbf{s} + \mathbf{h}) > c)$$

- ▶ If  $\lim_{c \rightarrow \infty} \chi(c) = 0$ , then high extreme values are not correlated.



# Gaussian spatial model

- ▶ In geostatistical models, observations,  $Y(\mathbf{s})$  are often modeled using a Gaussian spatial model with a likelihood of

$$f(y(\mathbf{s})) = \frac{1}{(2\pi)^{n/2}} |\Sigma|^{-1/2} \exp \left[ -\frac{1}{2} (Y(\mathbf{s}) - \mu(\mathbf{s}))^T \Sigma^{-1} (Y(\mathbf{s}) - \mu(\mathbf{s})) \right]$$

where

- ▶  $\mu(\mathbf{s})$  is a mean vector
- ▶  $\Sigma$  is a covariance matrix
- ▶ Model properties:
  - ▶ Nice computing properties
  - ▶ For a Gaussian spatial model  $\lim_{c \rightarrow \infty} \chi(c) = 0$  regardless of the strength of the correlation in the bulk of the distribution.
  - ▶ Tail is not flexible

# Spatial skew- $t$ distribution

- ▶ Assume observed data  $Y_t(\mathbf{s})$  come from a skew- $t$  (Zhang and El-Shaarawi, 2012)

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \alpha z_t + v_t(\mathbf{s})$$

where

- ▶  $\alpha \in \mathcal{R}$  controls the skewness
- ▶  $z_t \stackrel{iid}{\sim} N_{(0,\infty)}(0, \sigma_t^2)$  is a random effect
- ▶  $v_t(\mathbf{s})$  is a Gaussian process with variance  $\sigma_t^2$  and Matérn correlation
- ▶  $\sigma_t^2 \stackrel{iid}{\sim} \text{IG}(a, b)$

# Spatial skew- $t$ distribution

- ▶ **Conditioned** on  $z_t$  and  $\sigma_t^2$ ,  $Y_t(\mathbf{s})$  is a Gaussian spatial model
- ▶ Can use standard geostatistical methods to fit this model.
- ▶ Predictions can be made through kriging.
- ▶ **Marginalizing** over  $z_t$  and  $\sigma_t^2$  (via MCMC),

$$Y_t(\mathbf{s}) \sim \text{skew-}t(\mu, \Sigma^*, \alpha, \text{df} = 2a)$$

where

- ▶  $\mu$  is the location
- ▶  $a, b$  are the IG parameters for  $\sigma_t^2$
- ▶  $\Sigma^* = \frac{b}{a}\Sigma$  is a scale matrix, and  $\Sigma$  is a Matérn covariance matrix
- ▶  $\alpha \in \mathcal{R}$  controls the skewness

# Spatial skew- $t$ distribution

- ▶ Model properties
  - ▶ Has flexible tail controlled by  $\alpha$  and degrees of freedom
  - ▶ For a skew- $t$  distribution  $\lim_{c \rightarrow \infty} \chi(c) > 0$  Padoan, 2011)
  - ▶ Computation that is on the order of Gaussian computation
- ▶ For this distribution,  $\chi(c, \mathbf{h})$  shows asymptotic dependence that does not approach 0 as  $\mathbf{h} \rightarrow \infty$
- ▶ Deal with this through a daily random partition.

# Daily random partition

- ▶ Daily random partition allows  $z_t$  and  $\sigma_t^2$  to vary by site.

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \alpha z_t(\mathbf{s}) + \sigma(\mathbf{s})v_t(\mathbf{s})$$

- ▶ Consider a set of daily knots  $\{w_{t1}, \dots, w_{tK}\}$  that define a daily partition  $P_{t1}, \dots, P_{tK}$  such that

$$P_{tk} = \{s : k = \arg \min_{\ell} ||\mathbf{s} - w_{t\ell}||\}$$

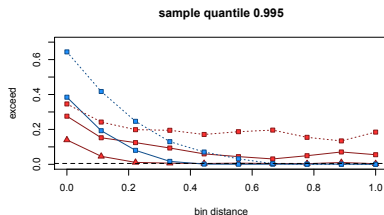
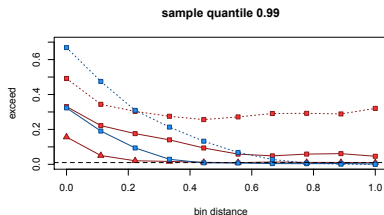
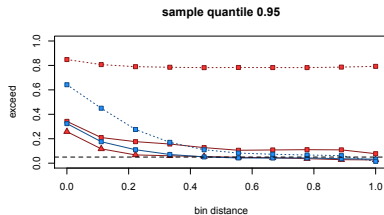
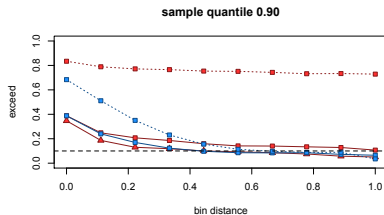
- ▶ For  $\mathbf{s} \in P_{tk}$

$$z_t(\mathbf{s}) = z_{tk}$$

$$\sigma_t^2(\mathbf{s}) = \sigma_{tk}^2$$

- ▶ Within each partition  $Y_t(\mathbf{s})$  has the same MVT distribution as before.

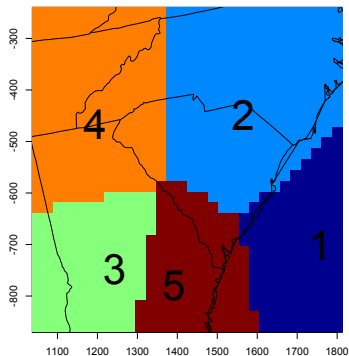
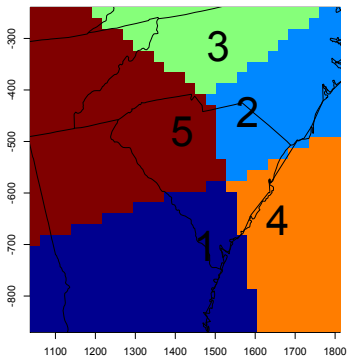
# Simulated $\chi$ plots



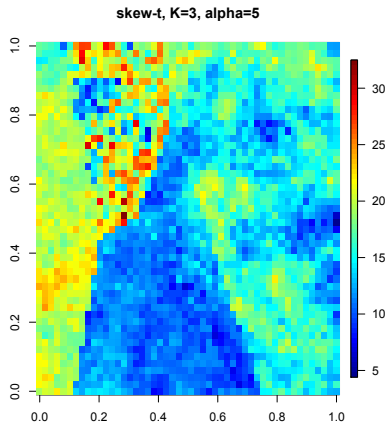
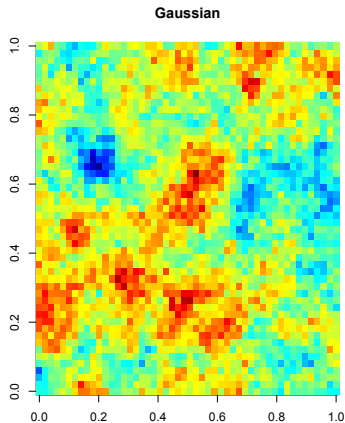
- ▲ Gaussian
- t, K=1
- t, K=5
- skew-t, K=1
- skew-t, K=5

# Example daily partition

Two possible partitions



# Sample simulated datasets



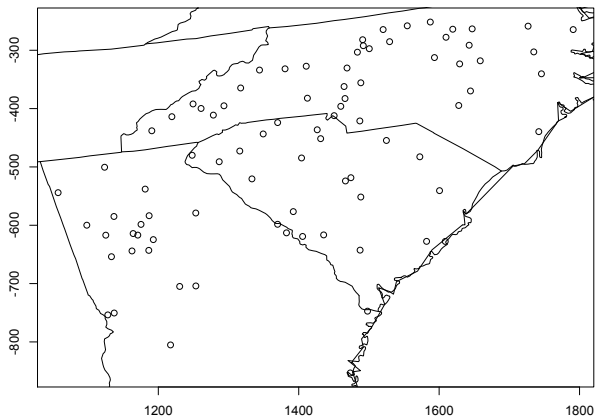


# Spatiotemporal Model

- ▶ We can account for time in one of two ways
  - ▶ The mean: e.g.  $AR(1)$
  - ▶ Three dimensional covariance model (e.g. Huser and Davison, 2014)

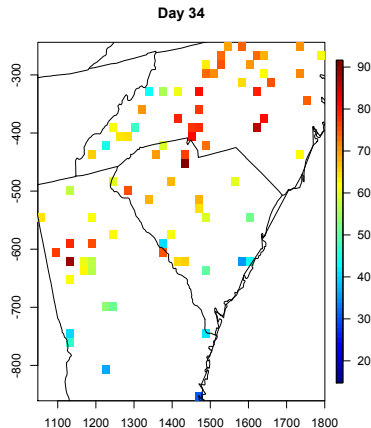
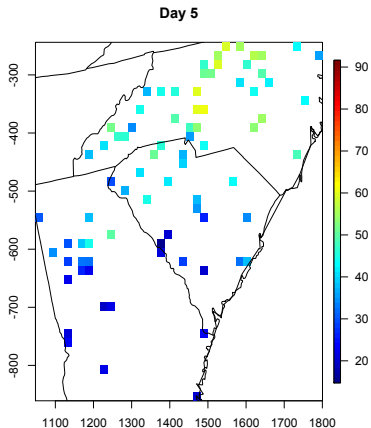
- ▶ Three main steps:
  1. Impute missing observations and data below  $T$
  2. Update parameters with standard random walk Metropolis Hastings or Gibbs sampling
  3. Make spatial predictions
- ▶ Priors are selected to be conjugate when possible.

## Ozone monitoring station locations



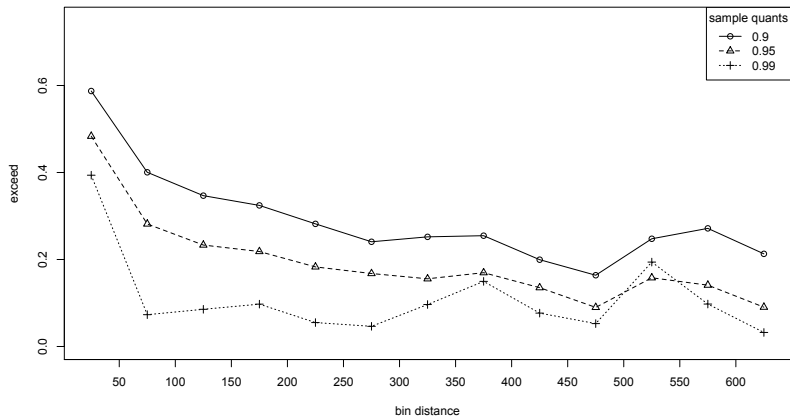
# Data analysis

Max 8-hour ozone measurements at 85 sites in NC, SC, and GA  
for days 5 and 34.



# Exploratory data analysis

$\chi$ -plot for residuals selected ozone sample quantiles



# Model comparisons

- ▶ 9 different analysis methods incorporating
  - ▶ Gaussian vs  $t$  vs skew- $t$  marginal distribution
  - ▶  $K = 1$  partition vs  $K = 5$  partitions
  - ▶ No thresholding vs thresholded
    - ▶ Thresholded data at  $T = 0.90$  sample quantile
- ▶ All methods use a Matérn or exponential covariance ( $\nu = 0.5$ )
- ▶ Compare quantile and Brier scores using 5-fold cross validation (Gneiting and Raftery, 2007)
- ▶ Mean function modeled using a first-order spatial trend

# Quantile score

- ▶ The quantile score for the  $\tau$ th quantile is

$$2\{I[y < \hat{q}(\tau)] - \tau\}(\hat{q} - y)$$

where:

- ▶  $y$  is a test set value
- ▶  $\hat{q}(\tau)$  is the estimated  $\tau$ th quantile

- ▶ Brier score for predicting exceedance of threshold  $c$

$$[e(c) - P(c)]^2$$

where

- ▶  $y$  is a test set value
- ▶  $e(c) = I[y > c]$
- ▶  $P(c)$  is the predicted probability of exceeding  $c$



# Five-fold cross-validation results

Marginal	$K$	$T$	Quantile				
			0.900	0.950	0.990	0.995	0.999
Gaussian	1	0	16.38	15.76	14.52	14.08	13.22
$t$	1	0	16.15	15.51	14.00	13.43	12.32
$t$	5	0	13.61	12.66	10.96	10.40	9.34
skew $t$	1	0	9.24	7.27	4.13	3.27	1.96
skew $t$	5	0	15.81	14.46	11.57	10.57	8.60
$t$	1	0.9	5.52	3.58	1.77	1.47	1.10
$t$	5	0.9	5.98	4.27	2.41	2.03	1.49
skew $t$	1	0.9	<b>4.91</b>	<b>3.16</b>	<b>1.45</b>	<b>1.16</b>	<b>0.82</b>
skew $t$	3	0.9	5.58	3.78	1.93	1.58	1.11

- Brier score results are similar.

# Simulation study

- ▶ 6 different data settings:
  - ▶ Gaussian vs  $t$  vs skew- $t$  marginal distribution
  - ▶  $K = 1$  partition vs  $K = 5$  partitions
- ▶ Results are similar to the results from the data analysis
- ▶ Biggest gains come from thresholding.
- ▶ Using skew models give additional gain, but small relative to gain for thresholding.

# Future work

- ▶ Comparison with extreme value analysis methods
- ▶ Reporting of compliance with EPA ozone standards
- ▶ Spatial mapping of high quantiles

# Questions

- ▶ Any questions?
- ▶ Thank you for your attention.

# References

- ▶ Demarta, S. and McNeil, A. J. (2007) The  $t$  copula and related copulas. *International Statistical Review*, **73**, 111–129.
- ▶ Huser, R. and Davison, A. C. (2014) Space-time modelling of extreme events. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **76**, 439–461.
- ▶ Padoan, S. A. (2011) Multivariate extreme models based on underlying skew- $t$  and skew-normal distributions. *Journal of Multivariate Analysis*, **102**, 977–991.
- ▶ Zhang, H. and El-Shaarawi, A. (2010) On spatial skew-Gaussian processes and applications. *Environmetrics*, **21**, 33–47.