

A new spatial model for points above a threshold

April 14, 2014

1 Introduction

2 Statistical model

Let $Y_t(\mathbf{s}) \in \mathcal{R}$ be the observed value at location \mathbf{s} on day t . To avoid bias in estimating tail parameters, we model the thresholded data

$$\tilde{Y}_t(\mathbf{s}) = \begin{cases} Y_t(\mathbf{s}) & Y_t(\mathbf{s}) > T \\ T & Y_t(\mathbf{s}) \leq T \end{cases} \quad (1)$$

where T is a pre-specified threshold.

We first specify a model for the complete data, $Y_t(\mathbf{s})$, and then study the induced model for thresholded data, $\tilde{Y}_t(\mathbf{s})$. The full data model is given in Section 2.1 assuming a multivariate normal distribution with a different variance each day. Computationally, the values below the threshold are updated using standard Bayesian missing data methods as described in Section 3.

2.1 Complete data

Consider the spatial process

$$Y_t(\mathbf{s}) = X_t(\mathbf{s})\beta + e_t(\mathbf{s}) \quad (2)$$

$$e_t(\mathbf{s}) = \sigma\delta|u_t(\mathbf{s})| + v_t(\mathbf{s}) \quad (3)$$

where $u_t(\mathbf{s}) = u_{tl}$ if $\mathbf{s} \in P_{tl}$ where P_{t1}, \dots, P_{tL} form a partition, and $u_{tl} \stackrel{iid}{\sim} N(0, 1)$, $\delta \in (-1, 1)$ controls skew, and $v_t(\mathbf{s})$ is a spatial process with mean zero and variance $\sigma^2(1 - \delta^2)$. Then $Y_t(\mathbf{s})$ is skew normal within each partition (Minozzo and Ferracuti, 2012). We model this with a Bayesian hierarchical model as follows. Let w_{t1}, \dots, w_{tL} be partition centers so that P_{tl} includes all spatial locations \mathbf{s} that are within the partition. Then

$$Y_t(\mathbf{s}) \mid \Theta = \mu_t(\mathbf{s}) + v_t(\mathbf{s}) \quad (4)$$

$$\mu_t(\mathbf{s}) = X_t(\mathbf{s})\beta + \sigma\delta|u_{tl}| \quad (5)$$

where $l = \arg \min_j \|\mathbf{s} - w_j\|$ and $\Theta = \{u_{t1}, \dots, u_{tL}, w_{t1}, \dots, w_{tL}, \beta, \rho, \nu, \sigma\}$ are the random effects, knot locations, and parameters for the mean, and spatial covariance.

3 Computation

The MCMC for this model is fairly straightforward. First, we impute values below the threshold. Then, we update Θ using random walk MH or Gibbs sampling when appropriate. Finally, we make spatial predictions. Each requires the joint distribution for the complete data given Θ . As defined in 4, the distribution of $Y_t(\mathbf{s}) \mid \Theta$ is the usual multivariate normal distribution with a Matérn spatial covariance structure.

3.1 Imputation

We can use Gibbs sampling to update $\tilde{Y}_t(\mathbf{s})$ for observations that are below T , the thresholded value. Given Θ , $Y_t(\mathbf{s})$ has truncated normal full conditional with these parameter values. So we sample $Y_t(\mathbf{s}) \sim \text{TN}_{(-\infty, T)}$

3.2 Parameter updates

To update Θ given the current value of the complete data $\mathbf{Y}_1, \dots, \mathbf{Y}_T$, we use a standard Gibbs updates for all parameters except for the knot locations which are done using a Metropolis update. See Appendix A.1 for details regarding Gibbs sampling and $|u_t(\mathbf{s})|$.

3.3 Spatial prediction

Given \mathbf{Y}_t the usual Kriging equations give the predictive distribution for $Y_t(\mathbf{s}^*)$ at prediction location (\mathbf{s}^*)

4 Data analysis

5 Conclusions

Acknowledgments

Appendix A.1: Half-Normal results

Half-normal

Let $u = \xi + \sqrt{\eta}|x|$ where $X \sim N(0, 1)$. Then Wiper et al. (2008) show that U follows a half-normal distribution which we shall write as $U \sim \text{HN}(\xi, \theta)$ where $\theta = \frac{1}{\eta}$ is a precision term. The density is given by

$$f_U(u) = \frac{\sqrt{\theta\pi}}{\sqrt{2}} \exp\left(-\frac{(u-\xi)^2\theta}{2}\right), \quad u > \xi. \quad (6)$$

Conditional posterior of $U|Y$

Let $Y|U \sim N(U, \sigma^2)$, let $\tau = 1/\sigma^2$, and let $\pi(U) \propto \exp\left\{-\frac{u^2\theta}{2}\right\}$. Then the conditional posterior of $U|Y$ is

$$\begin{aligned} \pi(U | Y) &\propto \exp\left\{-\frac{u^2\theta}{2}\right\} \exp\left\{-\frac{\tau(y-u)^2}{2}\right\} \\ &\propto \exp\left\{-\frac{1}{2}[u^2\theta + \tau(y^2 - 2yu + u^2)]\right\} \\ &\propto \exp\left\{-\left(\frac{\theta + \tau}{2}\right)\left[u^2 - 2u\frac{\tau y}{(\theta + \tau)}\right]\right\} \\ &\propto \exp\left\{-\frac{1}{2}\left(u - \frac{\tau y}{\theta + \tau}\right)^2 (\sqrt{\theta + \tau})^2\right\} \\ &\propto \text{HN}(\xi^*, \theta^*) \end{aligned} \quad (7)$$

where

$$\begin{aligned} \xi^* &= \frac{\tau y}{\theta + \tau} \\ \theta^* &= \theta + \tau \end{aligned}$$

35 **Conditional posterior of $U_{t1}|\mathbf{Y}_{t1}(\mathbf{s})$**

Consider a multivariate response $Y_t(\mathbf{s})$ as given by 4 using two partitions. Then conditioned on the observations in partition 2,

$$Y_{t1}|Y_{t2} \sim N_{n_1}(\bar{\mu}, \bar{\Sigma}) \quad (8)$$

where $\bar{\mu} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y_{t2} - \mu_2)$, and $\bar{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Then conditional posterior of $U_{t1}|\mathbf{Y}_{t1}$ is

$$\pi(U_{t1}|\mathbf{Y}_{t1}) \propto \exp \left\{ -\frac{1}{2\sigma^2\delta^2}u^2 - \frac{1}{\sigma^2(1-\delta^2)} [\mathbf{Y}_{t1} - \bar{\mu}]^T \bar{\Sigma}^{-1} [\mathbf{Y}_{t1} - \bar{\mu}] \right\} \quad (9)$$

$$\propto \exp \left\{ -\frac{1}{2\sigma^2} \left[\frac{1}{\delta^2}u^2 + \frac{\sigma^2\delta^2}{(1-\delta^2)} u^2 \mathbf{1}^T \bar{\Sigma}^{-1} \mathbf{1} - 2u \mathbf{1}^T \bar{\Sigma}^{-1} [\mathbf{Y}_{t1} - X_{t1}\beta - \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{Y}_{t2} - \mu_2)] \right] \right\} \quad (10)$$

$$\propto \exp \left\{ -\frac{1}{2}(u - \xi^*)^2(\theta^*) \right\} \quad (11)$$

where

$$\xi^* = \frac{\sigma\delta \mathbf{1}^T \bar{\Sigma}^{-1} [\mathbf{Y}_{t1} - X_{t1}\beta - \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{Y}_{t2} - \mu_2)]}{\frac{1}{\delta^2} + \frac{\sigma^2\delta^2 \mathbf{1}^T \bar{\Sigma}^{-1} \mathbf{1}}{(1-\delta^2)}} \quad (12)$$

$$\theta^* = \frac{1}{\sigma^2\delta^2} + \frac{\delta^2 \mathbf{1}^T \bar{\Sigma}^{-1} \mathbf{1}}{(1-\delta^2)} \quad (13)$$

36 **Appendix A.2: MCMC Details**

37 **Priors**

38 **References**

- 39 Minozzo, M. and Ferracuti, L. (2012) On the existence of some skew-normal stationary processes. *Chilean*
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- 41 Wiper, M. P., Girón, F. J. and Pewsey, A. (2008) Objective Bayesian Inference for the Half-Normal and
42 Half- t Distributions. *Communications in Statistics - Theory and Methods*, **37**, 3165–3185.