Buffon’s Needle

Samuel A. Martinez Koss

2025-04-20

Scenario

The Buffon’s Needle problem poses the following question: If a needle is dropped above a hardwood floor (supposing the wood strips are of equal width), what are the chances that the needle lies across a line between two strips? This is often reframed in terms of dropping sticks over an array of parallel and equidistant lines.

The solution to this scenario, from a naive perspective, is unexpected. It elucidates some key insights involving rotational symmetry and geometric probability, and can even be taken advantage of to approximate pi.

Some references to this problem:

- https://en.wikipedia.org/wiki/Buffon%27s_needle_problem
- https://www.exploratorium.edu/exhibits/pi-toss
- https://youtu.be/sJVivjuMfWA?si=kNZymNkyLpTj2qE7

Math

From a frequentist approach, the probability we are concerned with can be viewed as the ratio $\frac{\text{number of sticks crossing}}{\text{number of sticks dropped}}$. We define the following variables:

$$\begin{aligned}
l &: \text{length of sticks} \\
s &: \text{number of sticks} \\
c &: \text{number of sticks crossing} \\
d &: \text{distance between lines}
\end{aligned}$$

Next, we interrogate what it means for a stick to be dropped randomly above our lines. In order to ascertain whether a dropped stick is crossing a line, we need positional information as well as rotational information.

Consider that the center of a randomly dropped stick might fall arbitrarily close to a line. If the stick is not rotated to an angle substantially extreme, then the perpendicular length of the stick to the line will not be sufficient to cross over the line. Therefore the state of a stick's crossing is contingent on these two parameters:

$$\begin{aligned}
x &: \text{distance of stick center from nearest line} \\
\theta &: \text{rotation of stick along its center}
\end{aligned}$$

Now we examine what condition exists for a stick to be crossing a line. Note however that if $l > d$, it is possible for a single stick to cross multiple lines. For the purposes of these calculations, we will consider only the case where $l \leq d$. We then see that $x \in \left[0, \frac{d}{2}\right]$. We can similarly put bounds on θ by considering its symmetries and realizing that the behavior of its rotation on crossing in one quadrant extends to all other quadrants, so we use $\theta \in \left[0, \frac{\pi}{2}\right)$. Finally, see that when $x \leq \frac{l\sin(\theta)}{2}$ a stick can be considered to be crossing a line, as the length of stick perpendicular to the array of lines would be greater than the distance of the center of the stick to the nearest line.

Because the distributions of x and of θ are uniform and independent, it thus follows that the probability of a stick crossing a line is:

$$\begin{aligned}
\mathbb{P} &= \int_0^{\pi/2} \int_0^{l\sin(\theta)/2} \frac{4}{d\pi} dx d\theta \\
&= \int_0^{\pi/2} \left[\frac{4}{d\pi} x\right]_0^{l\sin(\theta)/2} d\theta \\
&= \int_0^{\pi/2} \frac{4l\sin(\theta)}{2d\pi} d\theta \\
&= \int_0^{\pi/2} \frac{2l\sin(\theta)}{d\pi} d\theta \\
&= \left[\frac{-2l\cos(\theta)}{d\pi}\right]_0^{\pi/2} \\
&= 0 - \frac{-2l}{d\pi} \\
&= \frac{2l}{d\pi}.
\end{aligned}$$

Thus if we choose $l = 1$ and $d = 1$, we find:

$$\mathbb{P} = \frac{2l}{d\pi} = \frac{2 \cdot 1}{1 \cdot \pi} = \frac{2}{\pi} \approx 0.63662.$$

Simulations

This probability can be estimated using simulations as $\mathbb{P} \approx c/s$.

Because we are no longer dealing with integral calculations we can generate $\theta \in [0, 2\pi)$, but will continue to generate $x \in \left[0, \frac{d}{2}\right]$.

```
crossing <- function(length, angle, dist_center_from_line){
  if (angle %% pi > pi / 2){
    angle <- pi - (angle %% pi)
  }
  perpendicular_half_length <- abs(sin(angle)) * length / 2

  return(perpendicular_half_length >=  dist_center_from_line)
}
```

```
buffon_crossings <- function(length, distance_between_lines, num_sims){
  angles <- runif(num_sims, 0, 2 * pi)
  dist_cent_to_line <- runif(num_sims, 0, distance_between_lines / 2)

  crossings <- 0
  for (i in 1: num_sims){
    crossings <- crossings + crossing(length, angles[i], dist_cent_to_line[i])
  }

  return(crossings)
}
```

```
s <- 10000
l <- 1
d <- 1
c <- buffon_crossings(l, d, s)

print(paste("Estimated P:", c / s))
```

```
## [1] "Estimated P: 0.634"
```

Pi

See that

$$\mathbb{P} = \frac{2l}{d\pi}, \mathbb{P} \approx \frac{c}{s} \to \frac{c}{s} \approx \frac{2l}{d\pi} \implies \pi \approx 2\frac{ls}{cd}, l \leq d.$$

Pi can then also be estimated using a simulation approach!

```r
s <- 10000
l <- 1
d <- 1
c <- buffon_crossings(l, d, s)

print(paste("pi:", 2 * l * s / c / d))
```

```
## [1] "pi: 3.09358081979892"
```

Targeting Pi Fractions

Let's here arbitrarily choose $l = 5$ and $d = 8$. Then

$$\pi \approx \frac{5}{8} \cdot \frac{s}{c}.$$

We can choose to "target" a pi fraction accurate to 10 decimal places as follows:

```
From Sloan's:
A002485: 0, 1, 3, 22, 333, 355, 103993, 104348, 208341, 312689, ...
A002486: 1, 0, 1,  7, 106, 113,  33102,  33215,  66317,  99532, ...
A114526: -, -, 0,  2,   4,   6,      9,      9,      9,     10, ...
```

$$\pi \approx \frac{312689}{99532} = 2 \cdot \frac{5}{8} \cdot \frac{s}{c}$$
$$\implies c = \frac{124415 \cdot s}{312689}.$$

Now s should be chosen as a multiple of 312689 so that $\frac{124415 \cdot s}{312689}$ is an integer and c can satisfy the fraction.

See that this holds:

```r
sprintf("%.10f", 2 * 5 * 312689 / 8 / 124415)
```

```
## [1] "3.1415926536"
```

Finally, pi may be estimated, and simulations may be done repeatedly to artificially approach a highly accurate approximation to pi. Do note however that high accuracy in this case, particularly with this pi fraction, occurs with extraordinarily low probability.

```r
s <- 312689 * 1
l <- 5
d <- 8
c <- buffon_crossings(l, d, s)

print(paste("pi:", 2 * l * s / c / d))
```

```
## [1] "pi: 3.14414506813392"
```

Lazzarini famously attempted to generate a pi estimate like this, targeting the fraction 355/113, but his methodology has been questioned and his counts were reportedly doctored.

We show a simple and successful version of this approach on the following page.

Targeting 22/7

We explicitly target the pi approximation of 22/7 by repeatedly running a small simulation and summing the counts until a ratio of c to s is met.

Choose $l = 1$ and $d = 1$.

$$\pi \approx \frac{22}{7} = 2 \cdot \frac{1}{1} \cdot \frac{s}{c}$$
$$\implies c = \frac{7 \cdot s}{11}.$$

```
stop <- 0
reps <- 0

s <- 11
l <- 1
d <- 1
c <- 0

while (!stop){
  reps <- reps + 1
  c <- c + buffon_crossings(l, d, s)

  if (c / s / reps == 7 / 11){
    stop <- 1
  }
}
```

```
## [1] "pi (22/7):                         3.14285714285714"

## [1] "number of repetitions:             3"

## [1] "total number of sticks dropped:    33"

## [1] "total number of sticks crossing:   21"

## [1] "stop condition satisfied by:       21/33 == 7/11"
```