PROBABILISTIC METHODS IN COMBINATORICS
MIT 18.226 (FALL 2024)
PROBLEM SET

https://sammy-luo.github.io/18-226/

A. Introduction and linearity of expectations

A1. Verify the following asymptotic calculations used in Ramsey number lower bounds:

(a) For each k, the largest n satisfying $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ has $n = \left(\frac{1}{e\sqrt{2}} + o(1)\right) k2^{k/2}$.

(b) For each k, the maximum value of $n - \binom{n}{k}2^{1-\binom{k}{2}}$ as n ranges over positive integers is $\left(\frac{1}{e} + o(1)\right) k2^{k/2}$.

(c) For each k, the largest n satisfying $e\left(\binom{k}{2}\binom{n}{k-2} + 1\right) 2^{1-\binom{k}{2}} < 1$ satisfies $n = \left(\frac{\sqrt{2}}{e} + o(1)\right) k2^{k/2}$.

A2. Prove that, if there is a real $p \in [0, 1]$ such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$$

then the Ramsey number $R(k, t)$ satisfies $R(k, t) > n$. Using this show that

$$R(4, t) \geq c\left(\frac{t}{\log t}\right)^{3/2}$$

for some constant $c > 0$.

ps1 A3. Let G be a graph with n vertices and m edges. Prove that K_n can be written as a union of $O(n^2(\log n)/m)$ isomorphic copies of G (not necessarily edge-disjoint).

A4. Prove that there is an absolute constant $C > 0$ so that for every $n \times n$ matrix with distinct real entries, one can permute its rows so that no column in the permuted matrix contains an increasing subsequence of length at least $C\sqrt{n}$. (A subsequence does not need to be selected from consecutive terms. For example, $(1, 2, 3)$ is an increasing subsequence of $(1, 5, 2, 4, 3)$.)

A5. *Generalization of Sperner's theorem.* Let $\mathcal{F}$ be a collection of subset of $[n]$ that does not contain $k+1$ elements forming a chain: $A_1 \subsetneq \cdots \subsetneq A_{k+1}$. Prove that $\mathcal{F}$ is no larger than taking the union of the k levels of the Boolean lattice closest to the middle layer.

A6. Let G be a graph on $n \geq 10$ vertices. Suppose that adding any new edge to G would create a new clique on 10 vertices. Prove that G has at least $8n - 36$ edges.

Hint in white:

A7. Let $k \geq 4$ and H a k-uniform hypergraph with at most $4^{k-1}/3^k$ edges. Prove that there is a coloring of the vertices of H by four colors so that in every edge all four colors are represented.

ps1 A8. Given a set $\mathcal{F}$ of subsets of $[n]$ and $A \subseteq [n]$, write $\mathcal{F}|_A := \{S \cap A : S \in \mathcal{F}\}$ (its *projection* onto A). Prove that for every n and k, there exists a set $\mathcal{F}$ of subsets of $[n]$ with $|\mathcal{F}| = O(k2^k \log n)$ such that for every k-element subset A of $[n]$, $\mathcal{F}|_A$ contains all 2^k subsets of A.

ps1 A9. Let $A_1, \ldots, A_m$ be r-element sets and $B_1, \ldots, B_m$ be s-element sets. Suppose $A_i \cap B_i = \emptyset$ for each i, and for each $i \neq j$, either $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$. Prove that $m \leq (r+s)^{r+s}/(r^r s^s)$.

ps1⋆ A10. Show that in every non-2-colorable n-uniform hypergraph, one can find at least $\frac{n}{2}\binom{2n-1}{n}$ unordered pairs of edges with each pair intersecting in exactly one vertex.

A11. Let A be a measurable subset of the unit sphere in $\mathbb{R}^3$ (centered at the origin) containing no pair of orthogonal points.

ps1 (a) Prove that A occupies at most $1/3$ of the sphere in terms of surface area.

ps1⋆ (b) Prove an upper bound smaller than $1/3$ (give your best bound).

ps1⋆ A12. Prove that every set of 10 points in the plane can be covered by a union of disjoint unit disks.

A13. Let $\boldsymbol{r} = (r_1, \dots, r_k)$ be a vector of nonzero integers whose sum is nonzero. Prove that there exists a real $c > 0$ (depending on $\boldsymbol{r}$ only) such that the following holds: for every finite set A of nonzero *reals*, there exists a subset $B \subseteq A$ with $|B| \geq c|A|$ such that there do not exist $b_1, \dots, b_k \in B$ with $r_1 b_1 + \cdots + r_k b_k = 0$.

ps1 A14. Prove that every set A of n nonzero integers contains two disjoint subsets B_1 and B_2, such that both B_1 and B_2 are sum-free, and $|B_1| + |B_2| > 2n/3$.

ps1 A15. Let G be an n-vertex graph with pn^2 edges, with $n \geq 10$ and $p \geq 10/n$. Prove that G contains a pair of vertex-disjoint and isomorphic subgraphs (not necessarily induced) each with at least cp^2n^2 edges, where $c > 0$ is a constant.

ps1⋆ A16. Prove that for every positive integer r, there exists an integer K such that the following holds. Let S be a set of rk points evenly spaced on a circle. If we partition $S = S_1 \cup \cdots \cup S_r$ so that $|S_i| = k$ for each i, then, provided $k \geq K$, there exist r congruent triangles where the vertices of the i-th triangle lie in S_i, for each $1 \leq i \leq r$.

ps1⋆ A17. Prove that $[n]^d$ cannot be partitioned into fewer than 2^d sets each of the form $A_1 \times \cdots \times A_d$ where $A_i \subsetneq [n]$.