Birla Institute of Technology & Science, Pilani
Mid Semester Test (Closed Book), Second Semester 2016 – 2017
Mathematics–II (MATH F112)

Date: 10th March, 2017 | **Max. Time: 90 Minutes** | **Max. Marks: 105**

Note. Answer all questions. Start answering each question on a fresh page.

1. (a) Consider the system of equations $AX = B$, where $A \in M_{m\times n}$, $X \in \mathbb{R}^n$ and $B(\neq 0_m) \in \mathbb{R}^m$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two solutions of this system and $\lambda, \mu \in \mathbb{R}$. What condition(s) should be imposed on λ and μ so that $\lambda\mathbf{u} + \mu\mathbf{v}$ is also a solution of $AX = B$? Justify. [6]

   (b) Determine the condition(s) on a, b and c so that the following system has a solution(s):
$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 6y - 11z &= b \\ x - 2y + 7z &= c. \end{aligned}$$
   Does this system has a unique solution? Justify. [8]

   (c) Let $V = \mathbb{R}^2$ with operations $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $c \odot v = c(Av)$, where $A = \begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix}$. Prove or disprove that $(V, \oplus, \odot)$ is a vector space. [7]

2. Let V be a subspace of $\mathbb{R}^3$ spanned by the set $S = \{(2,1,-1),(0,-1,2)\}$ and W be a subspace of $\mathbb{R}^3$ spanned by the set $T = \{(x,y,z) \mid x+y=2,\ y+z=1,\ x,y,z \in \mathbb{R}\}$. Then

   (a) Find the vectors $v \in V$ and $w \in W$ such that $v + w = (2,4,5)$. [16]

   (b) Find a vector $u \in T$ such that $S \cup \{u\}$ spans $\mathbb{R}^3$. [5]

3. (a) Consider the subspaces $W_1 = \{(a,b,c,d) : b - 2c + d = 0\}$ and $W_2 = \{(a,b,c,d) : a = d, b = 2c\}$ of $\mathbb{R}^4$. Find a basis and the dimension of $W_1 \cap W_2$. [8]

   (b) Consider the matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 0 \end{pmatrix}$.

      i. Find all the eigenvalues of A.

      ii. Find the eigenspace corresponding to the smallest eigenvalue of A. [3+10]

4. (a) Let $B = \{(1,0,1),(1,1,0),(0,1,1)\}$, $C = \{(1,1,1),(0,1,1),(1,0,1)\}$ be two ordered bases for the vector space $\mathbb{R}^3$. Find the transition matrix from the basis B to the basis C. [10]

   (b) Let $L : \mathbb{R}^3 \to P_2$ be a linear transformation defined by $L(x_1,x_2,x_3) = x_1 + (x_1 - x_2)x + (x_2 - x_3)x^2$. Let $B = \{(1,1,1),(1,2,3),(1,0,0)\}$ and $C = \{x+1, x^2+x+1, x^2-x\}$ be the ordered bases for $\mathbb{R}^3$ and P_2 respectively. Then find the matrix of the above linear transformation L with respect to B and C. [11]

5. (a) Let V be a vector space with ordered basis $B = \{v_1, v_2, \ldots, v_n\}$. Let $L : V \to V$ be a linear operator defined by
$$L(v_1) = v_3,\ L(v_2) = v_4,\ \ldots,\ L(v_{n-2}) = v_n,\ L(v_{n-1}) = v_1,\ L(v_n) = v_2.$$
   Find Range(L) and Ker(L) ? [7]

   (b) Does there exist a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\ker(L) = \text{range}(L)$? Justify [3]

   (c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by
$$T(x_1,x_2,x_3) = (x_1 + x_3, -2x_1 + 3x_2, 2x_2 + x_3).$$
      i. Find T^{-1}, if exists. If it does not exist, justify your answer.

      ii. Is T an isomorphism ? Justify. [9+2]

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