sheet03_notebook

November 10, 2025

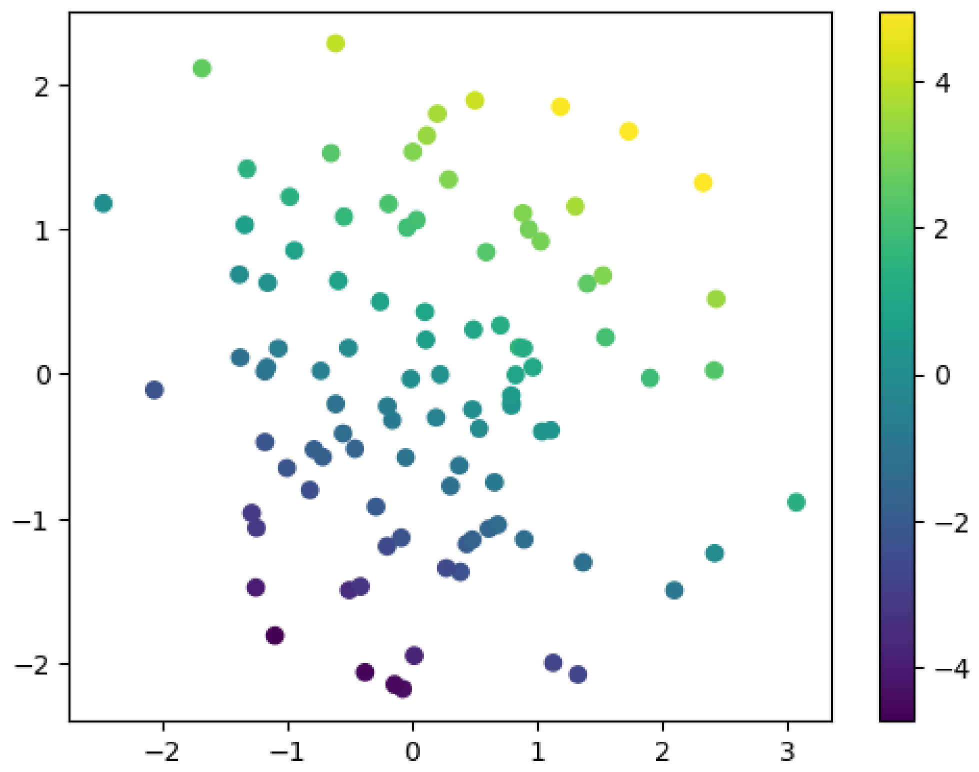
1 Sheet 03

[1]:
```
import numpy as np
from matplotlib import pyplot as plt
from sklearn.linear_model import LinearRegression
```

1.1 (2) Visualize Regularization Contours

[2]:
```
# load the data
data = np.load('data/linreg.npz')
x = data['X']
y = data['Y']
print(f'x.shape: {x.shape}, "y.shape:" {y.shape}')
plt.scatter(*x, c=y);
plt.colorbar()
plt.show()
```

```
x.shape: (2, 100), "y.shape:" (1, 100)
```



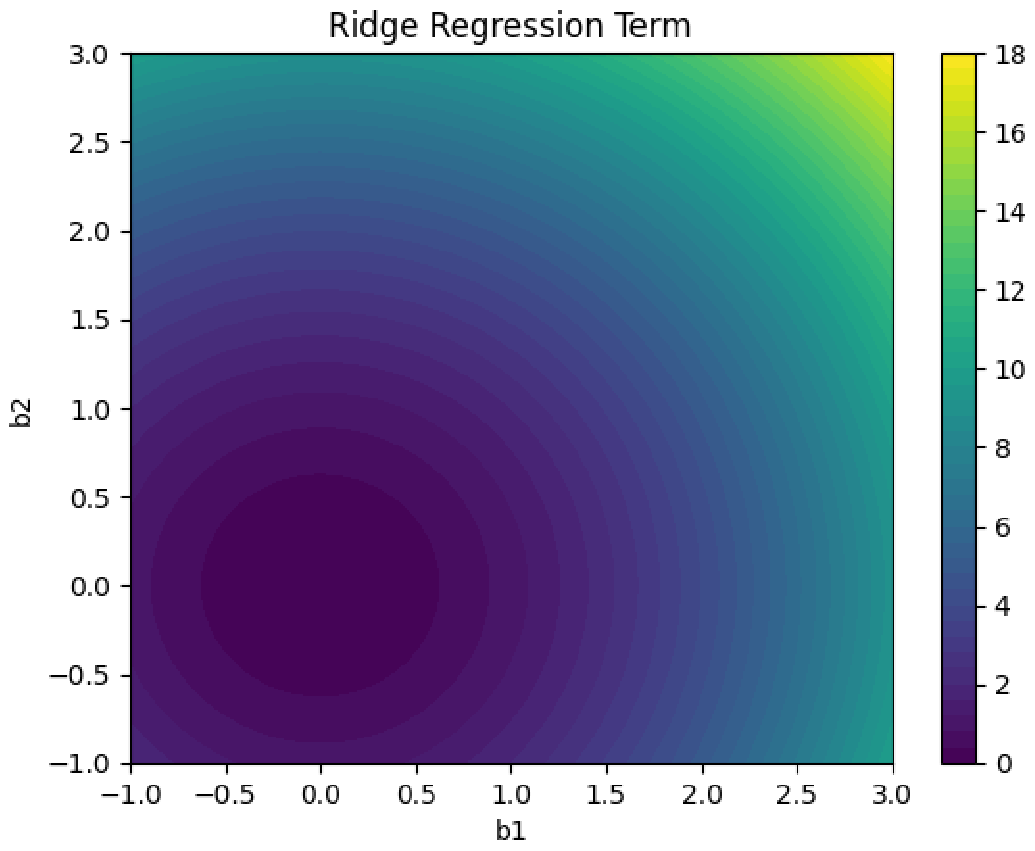
```
[3]: # create a grid of points in the parameter space
     b1, b2 = np.linspace(-1, 3, 101), np.linspace(-1, 3, 101)
     bs = np.stack(np.meshgrid(b1, b2, indexing='ij'), axis=-1)
     bs.shape
```

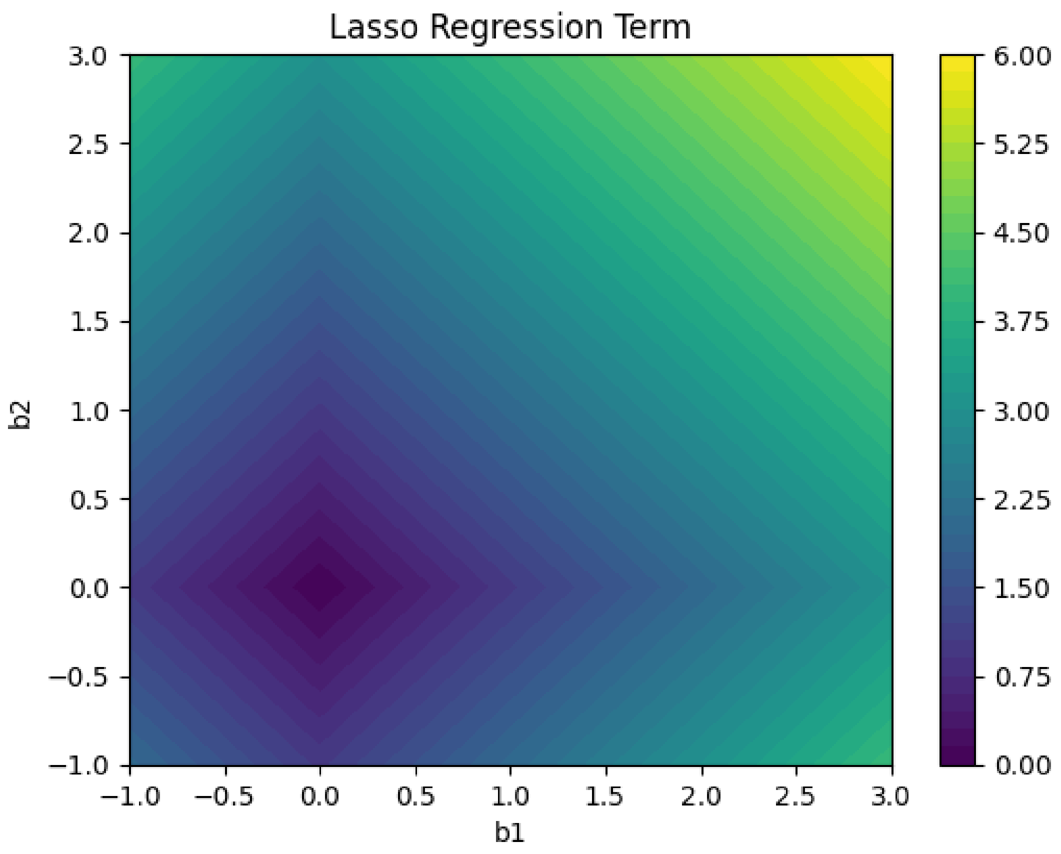
```
[3]: (101, 101, 2)
```

1.1.1 (a)

```
[5]: # Define ridge regression term
     def ridge_term(b, alpha=1.0):
         return alpha * np.sum(b**2, axis=-1)

     # Plot ridge regression contour
     plt.contourf(b1, b2, ridge_term(bs), levels=50)
     plt.colorbar()
     plt.xlabel('b1')
     plt.ylabel('b2')
     plt.title('Ridge Regression Term')
     plt.show()

     # define lasso regression term
```

```python
def lasso_term(b, alpha=1.0):
    return alpha * np.sum(np.abs(b), axis=-1)

# Plot lasso regression contour
plt.contourf(b1, b2, lasso_term(bs), levels=50)
plt.colorbar()
plt.xlabel('b1')
plt.ylabel('b2')
plt.title('Lasso Regression Term')
plt.show()
```



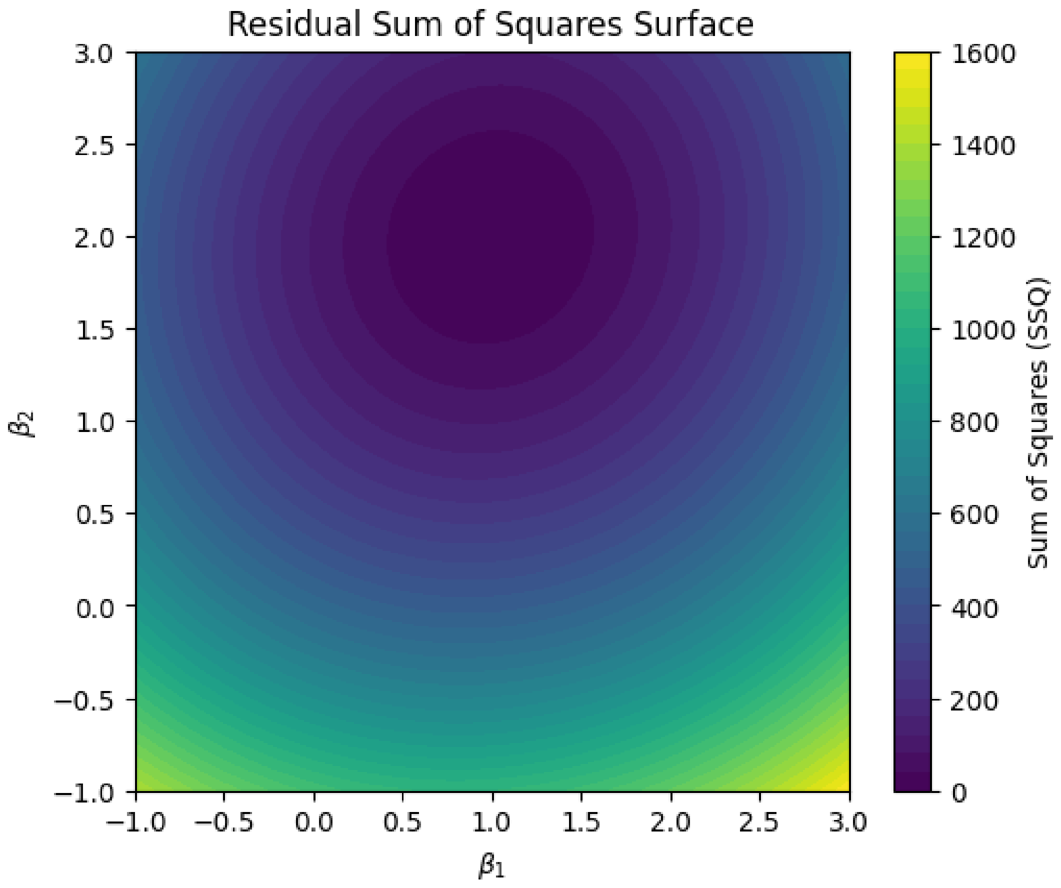


[13]:
```
x.shape
```

[13]:
```
(2, 100)
```

1.1.2 (b)

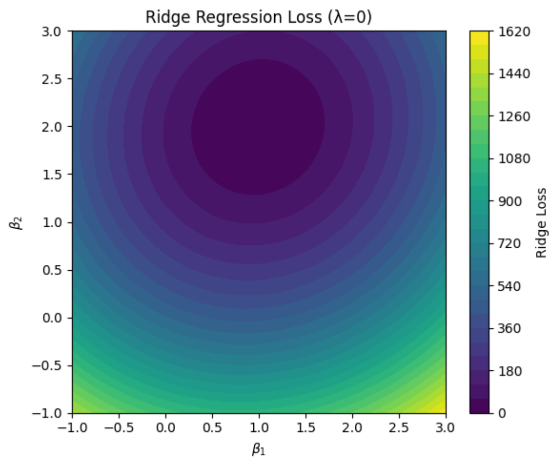
[14]:
```
# Initialise SSQ array
SSQ = np.zeros((len(b1), len(b2)))

# Compute SSQ for each ( ,  ) pair
for i in range(len(b1)):
    for j in range(len(b2)):
        beta = bs[i, j, :]  # ( ,  )
        X_T = x.T  # before (2,100), now (100, 2)
        y_pred = X_T @ beta  # works! (100,2) @ (2,) -> (100,)
        SSQ[i, j] = np.sum((y - y_pred)**2)

# Plot contour
plt.figure(figsize=(6, 5))
plt.contourf(b1, b2, SSQ.T, levels=50, cmap="viridis")
plt.colorbar(label="Sum of Squares (SSQ)")
plt.xlabel(r"$\beta_1$")
plt.ylabel(r"$\beta_2$")
plt.title("Residual Sum of Squares Surface")
plt.show()
```

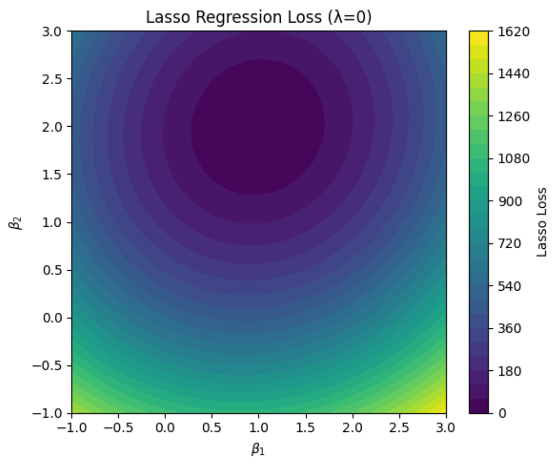


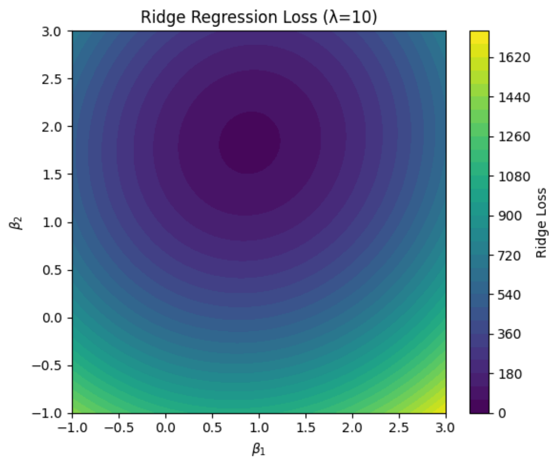
1.1.3 (c)

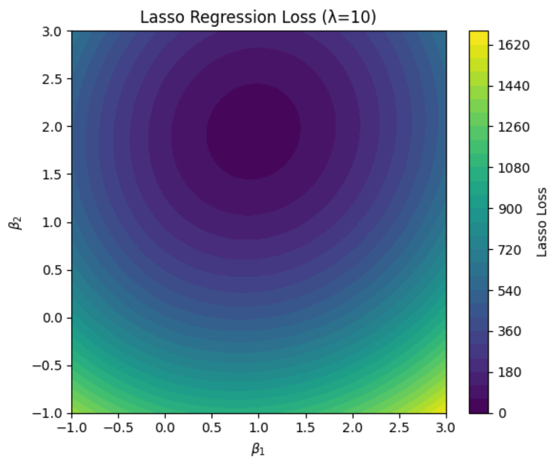
[24]:
```
# TODO: for each lambda, plot both ridge regression and lasso loss functions
lambdas = [0, 10, 50, 100, 200, 300]

for lam in lambdas:
    ridge_loss = SSQ + ridge_term(bs, alpha=lam)
    lasso_loss = SSQ + lasso_term(bs, alpha=lam)

    # Plot ridge regression loss
    plt.figure(figsize=(12, 5))
```

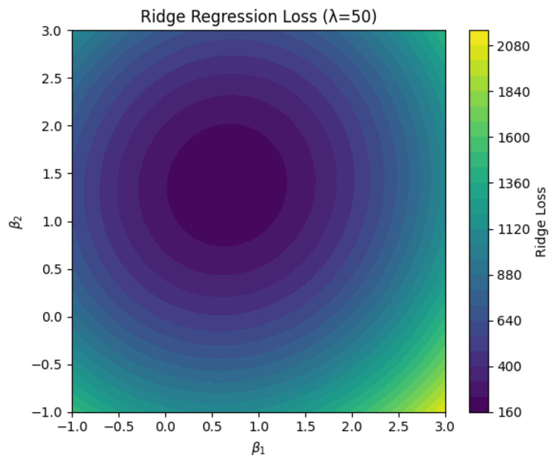
```python
    plt.subplot(1, 2, 1)
    plt.contourf(b1, b2, ridge_loss.T, levels=30, cmap="viridis")
    plt.colorbar(label="Ridge Loss")
    plt.xlabel(r"$\beta_1$")
    plt.ylabel(r"$\beta_2$")
    plt.title(f"Ridge Regression Loss ( ={lam})")

    # Plot lasso regression loss
    plt.subplot(1, 2, 2)
    plt.contourf(b1, b2, lasso_loss.T, levels=30, cmap="viridis")
    plt.colorbar(label="Lasso Loss")
    plt.xlabel(r"$\beta_1$")
    plt.ylabel(r"$\beta_2$")
    plt.title(f"Lasso Regression Loss ( ={lam})")

    plt.tight_layout()
    plt.show()
```

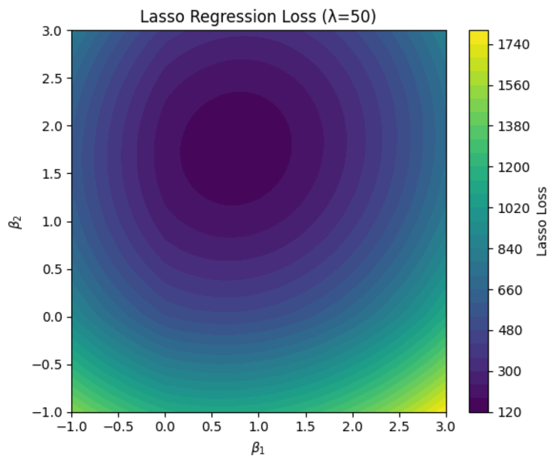


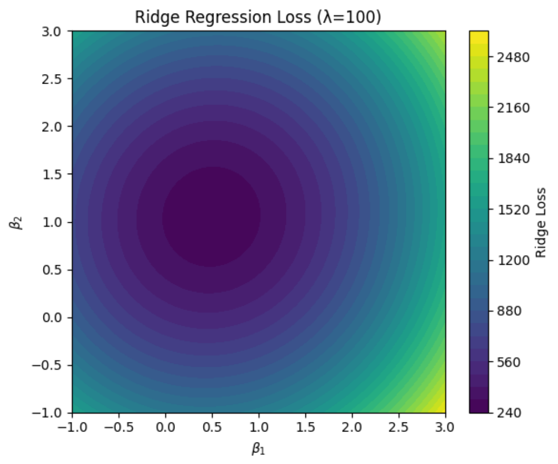


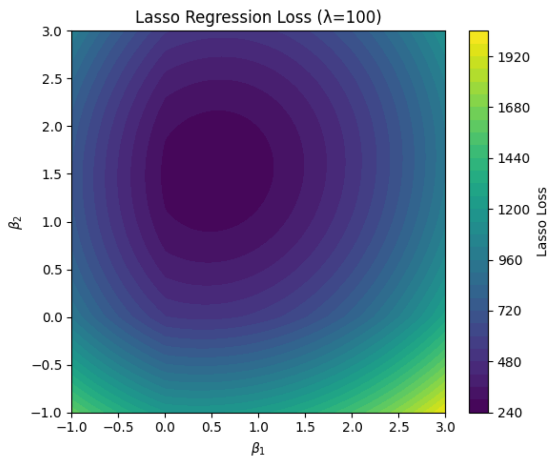


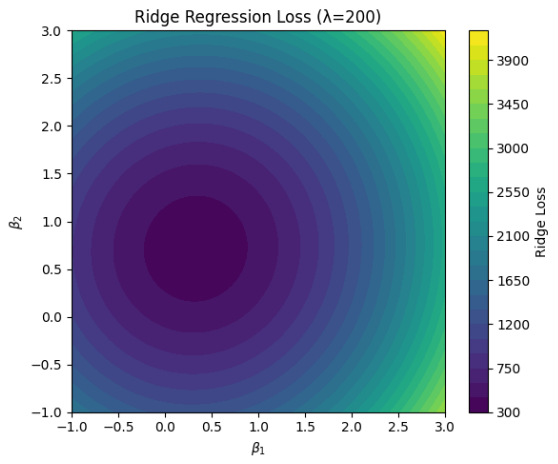


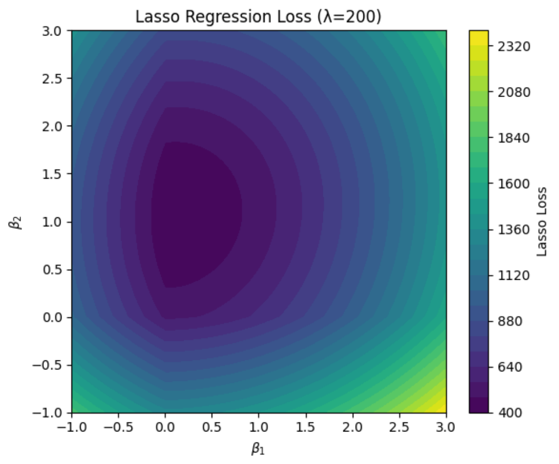

Ridge Regression Loss (λ=50)
β_1
β_2
Ridge Loss
160
400
640
880
1120
1360
1600
1840
2080

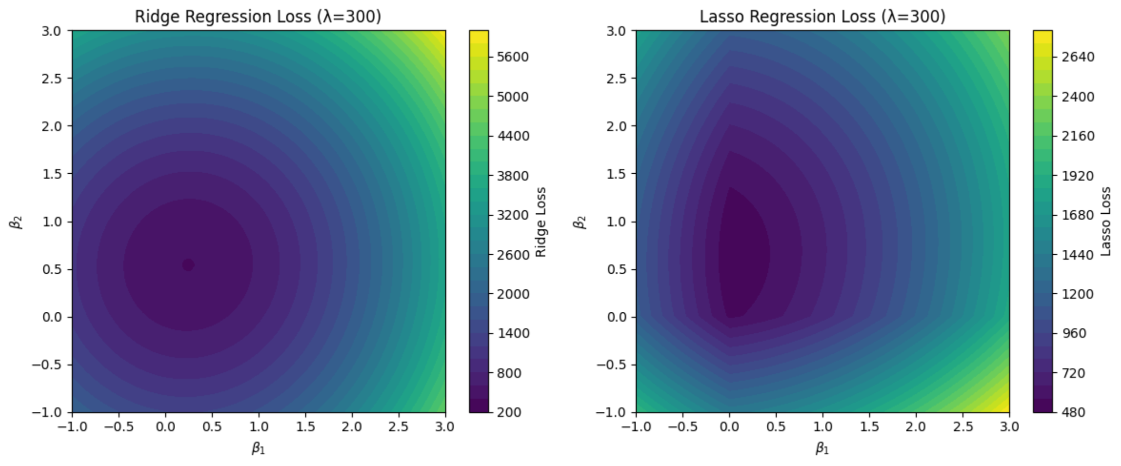

Lasso Regression Loss (λ=50)
β_1
β_2
Lasso Loss
120
300
480
660
840
1020
1200
1380
1560
1740


Ridge Regression Loss (λ=100)
β_1
β_2
Ridge Loss
240
560
880
1200
1520
1840
2160
2480


Lasso Regression Loss (λ=100)
β_1
β_2
Lasso Loss
240
480
720
960
1200
1440
1680
1920


Ridge Regression Loss (λ=200)
β_1
β_2
Ridge Loss
300
750
1200
1650
2100
2550
3000
3450
3900


Lasso Regression Loss (λ=200)
β_1
β_2
Lasso Loss
400
640
880
1120
1360
1600
1840
2080
2320



1.2 (3) CT Reconstruction

First, set up the design matrix. (Run this once to save it to the disk.)

```
[25]: # create design matrix
      # don't change any of this, just run it once to create and save the design␣
       ↪matrix
      import os

      n_parallel_rays = 70
      n_ray_angles = 30
      res = (99, 117)
      print("Number of pixels in the 2d image:", np.prod(res))
      print("Total number of rays:", n_parallel_rays * n_ray_angles)

      def rot_mat(angle):
          c, s = np.cos(angle), np.sin(angle)
          return np.stack([np.stack([c, s], axis=-1), np.stack([-s, c], axis=-1)],␣
       ↪axis=-1)

      kernel = lambda x: np.exp(-x**2/sigma**2/2)

      if not os.path.exists('data/design_matrix.npy'):
          xs = np.arange(0, res[1]+1) - res[1]/2 # np.linspace(-1, 1, res[1] + 1)
          ys = np.arange(0, res[0]+1) - res[0]/2 # np.linspace(-1, 1, res[0] + 1)

          # rays are defined by origin and direction
          ray_offset_range = [-res[1]/1.5, res[1]/1.5]
          n_rays = n_parallel_rays * n_ray_angles
```

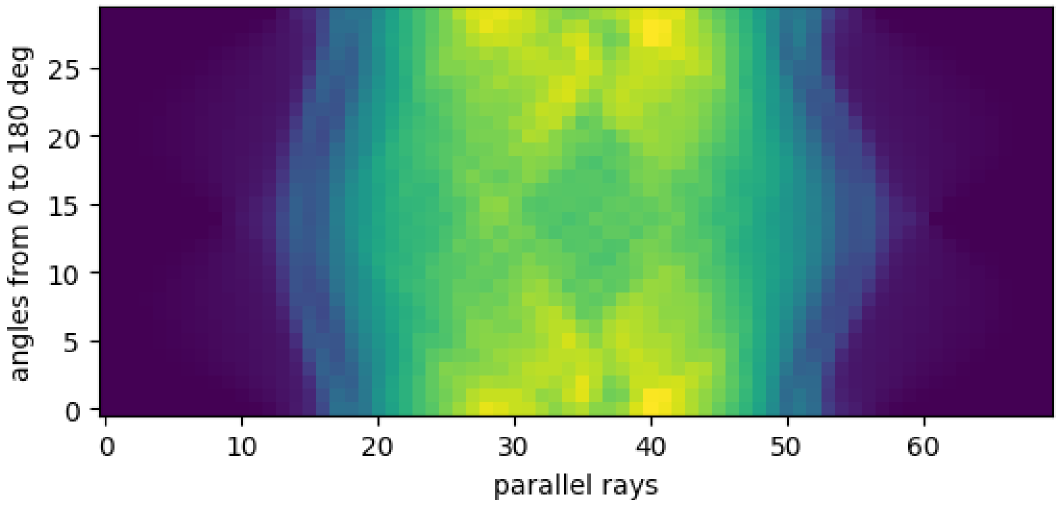
```
    ray_angles = np.linspace(0, np.pi, n_ray_angles, endpoint=False) + np.pi/
↪n_ray_angles

    # offsets for ray_angle = 0, i.e. parallel to x-axis
    ray_0_offsets = np.stack([np.zeros(n_parallel_rays), np.
↪linspace(*ray_offset_range, n_parallel_rays)], axis=-1)
    ray_0_directions = np.stack([np.ones(n_parallel_rays), np.
↪zeros(n_parallel_rays)], axis=-1)



    ray_rot_mats = rot_mat(ray_angles)

    ray_offsets = np.einsum('oi,aij->aoj', ray_0_offsets, ray_rot_mats).
↪reshape(-1, 2)
    ray_directions = np.einsum('oi,aij->aoj', ray_0_directions, ray_rot_mats).
↪reshape(-1, 2)

    sigma = 1
    xsc = (xs[1:] + xs[:-1]) / 2
    ysc = (ys[1:] + ys[:-1]) / 2
    b = np.stack(np.meshgrid(xsc, ysc), axis=-1).reshape(-1, 2)
    a = ray_offsets
    v = ray_directions
    v = v / np.linalg.norm(v, axis=-1, keepdims=True)
    p = ((b[None] - a[:, None]) * v[:, None]).sum(-1, keepdims=True) * v[:,␣
↪None] + a[:, None]
    d = np.linalg.norm(b - p, axis=-1)
    d = kernel(d)
    design_matrix = d.T

    np.save('data/design_matrix.npy', design_matrix)
    print(f'created and saved design matrix of shape {design_matrix.shape} at␣
↪data/design_matrix.npy')
```

```
Number of pixels in the 2d image: 11583
Total number of rays: 2100
created and saved design matrix of shape (11583, 2100) at data/design_matrix.npy
```

[26]:
```
sino = np.load('data/sino.npy')

print(f'sino shape: {sino.shape}')

# visualize sinogram as image
n_parallel_rays = 70
n_angles = 30
```

```
plt.imshow(sino.reshape(n_angles, n_parallel_rays), origin='lower')
# plt.colorbar()
plt.xlabel('parallel rays')
plt.ylabel('angles from 0 to 180 deg')
plt.show();
```

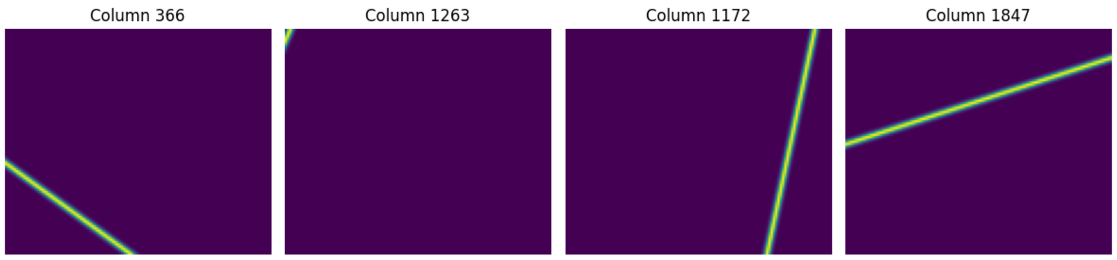
```
sino shape: (1, 2100)
```



1.2.1 (a)

[50]:
```
design_matrix = np.load('data/design_matrix.npy')

# TODO: visualize four random columns as images, using an image shape of (99,␣
↪117)

fig, axs = plt.subplots(1, 4, figsize=(12, 4))
for i in range(4):
    # select random column index
    col_idx = np.random.randint(design_matrix.shape[1])
    print(design_matrix[:, col_idx].shape)

    # reshape column to image shape
    img = design_matrix[:, col_idx].reshape(99, 117)

    # plot the image
    axs[i].imshow(img, origin='lower')
    axs[i].set_title(f'Column {col_idx}')
    axs[i].axis('off')
plt.tight_layout()
plt.show()
```

```
(11583,)
(11583,)
(11583,)
(11583,)
```

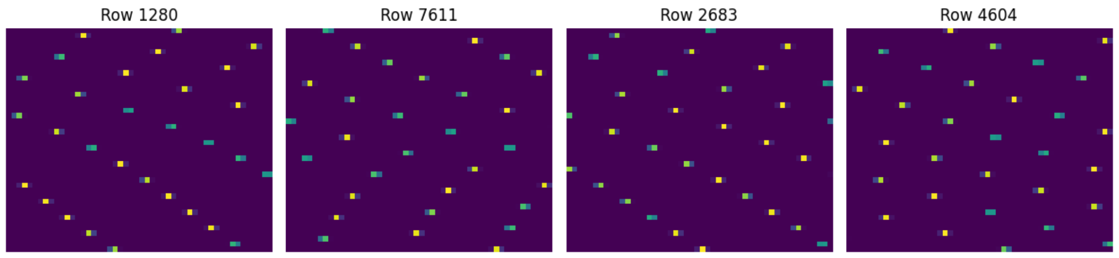


The columns give the different detector readings, meaning one specific column corresponds to a single pixel of the image and how it contributes to all detector readings.

1.2.2 (b)

[40]:
```
# TODO: visualize four random rows as images
fig , axs = plt.subplots(1, 4, figsize=(12, 4))
for i in range(4):
    # Select random row index
    row_idx = np.random.randint(design_matrix.shape[0])
    print(design_matrix[row_idx, :].shape)

    # Reshape row to image shape (2100 = 42*50)
    img = design_matrix[row_idx, :].reshape(42, 50)

    # Plot the image
    axs[i].imshow(img, origin='lower')
    axs[i].set_title(f'Row {row_idx}')
    axs[i].axis('off')
plt.tight_layout()
plt.show()
```

```
(2100,)
(2100,)
(2100,)
(2100,)
```



The rows give the different 2D slices. One individual row corresponds to a single detector reading.

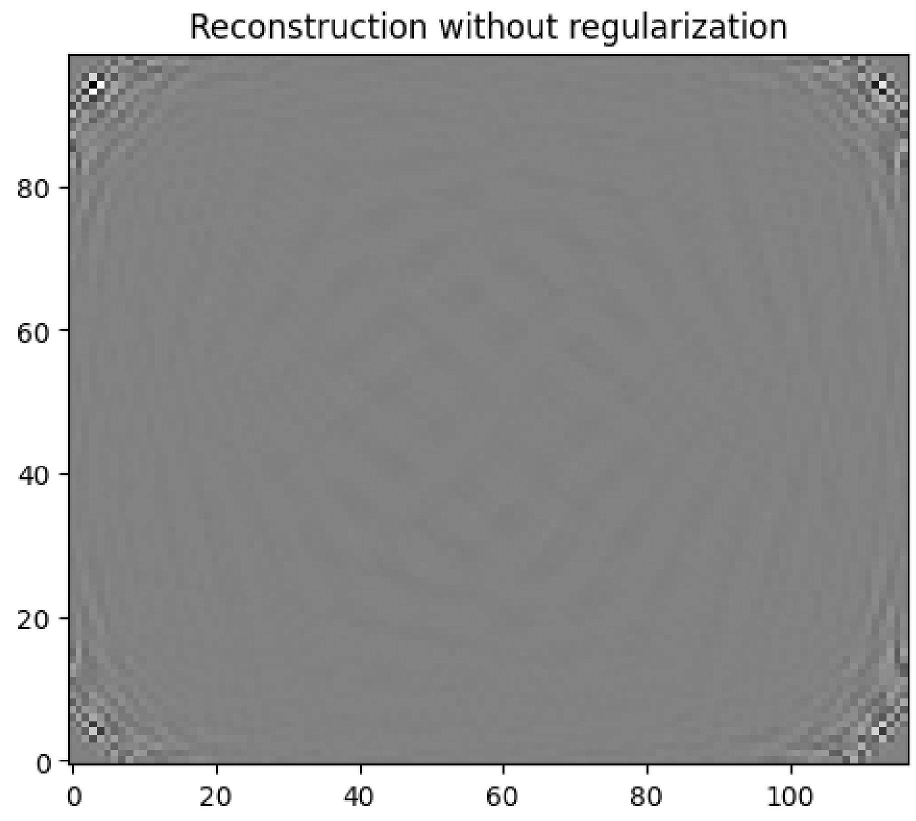
1.2.3 (c)

[62]:
```
# check shapes and solve (adapt names A and sino to your variables)
A = design_matrix.T             # expected shape (m, n)
b = sino.T                      # expected shape (m,)  or (m, 1) or (m, k)

print("A.shape (m,n) =", A.shape)
print("b.shape (m,) or (m,k) =", b.shape)

# ensure b is a 1D or 2D array with matching first dimension
b = b.ravel() if b.ndim == 2 and b.shape[1] == 1 else b
assert A.shape[0] == b.shape[0], "number of rows in A must equal length of b␣
  ↪(m)"
```

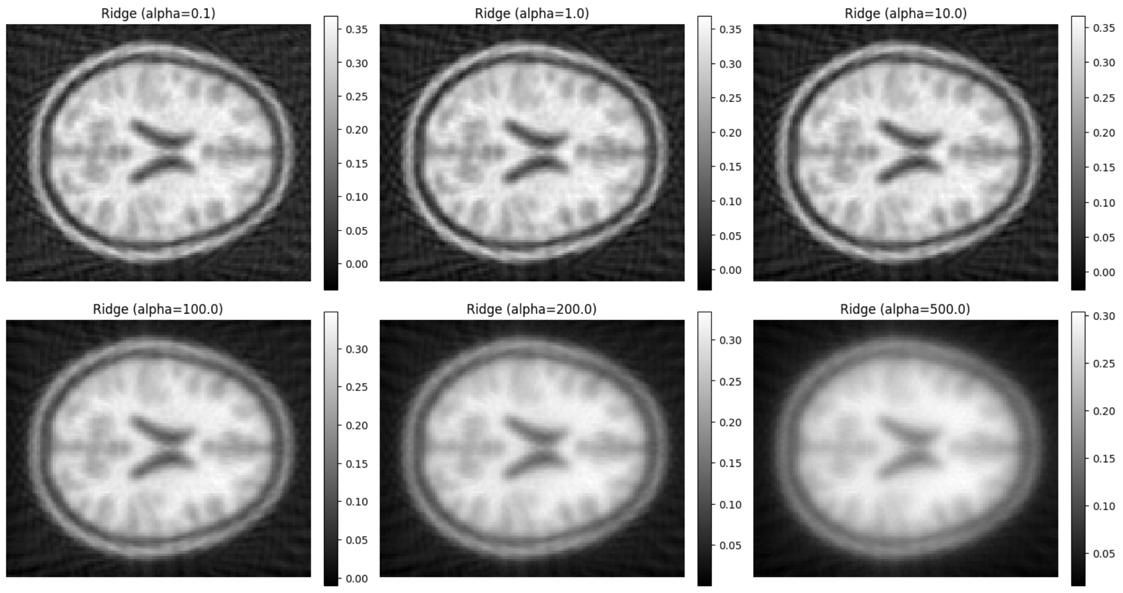
```
A.shape (m,n) = (2100, 11583)
b.shape (m,) or (m,k) = (2100, 1)
```

[63]:
```
# TODO: solve the reconstruction with linear regression and visualize the result
I_hat = np.linalg.lstsq(A, b, rcond=None)[0]
image_recon = I_hat.reshape(99, 117)
plt.imshow(image_recon, cmap='gray', origin='lower')
plt.title('Reconstruction without regularization')
plt.show()
```

```
(2, 100) (1, 100) (11583,)
```



[69]:
```
# TODO: Same with ridge regression for different alphas
from sklearn.linear_model import Ridge

# define range of alphas to test
alphas = [0.1, 1.0, 10.0, 100.0, 200.0, 500.0]
fig, axs = plt.subplots(2, 3, figsize=(15, 8))

# loop over alphas and plot reconstructions
for ax, alpha in zip(axs.flat, alphas):
    # fit ridge regression model
    ridge = Ridge(alpha=alpha, fit_intercept=False)
    ridge.fit(A, b)
    I_ridge = ridge.coef_.ravel()

    # reshape and plot the reconstructed image
    im = ax.imshow(I_ridge.reshape(99, 117), cmap='gray', origin='lower')
    ax.set_title(f'Ridge (alpha={alpha})')
    ax.axis('off')
    fig.colorbar(im, ax=ax, fraction=0.046, pad=0.04)

plt.tight_layout()
plt.show()
```



With ridge regularisation a brain cut is visible. For larger regularisation strengths it becomes more blurry.