Problem 1: Prove the following identity using induction.

$$Pr(\alpha_1, \ldots, \alpha_n \mid \beta) = Pr(\alpha_1 \mid \alpha_2, \ldots, \alpha_n, \beta)$$
$$Pr(\alpha_2 \mid \alpha_3, \ldots, \alpha_n, \beta) \ldots Pr(\alpha_n \mid \beta)$$

<u>Proof by induction</u>

Base case:

$n=1$ $\quad Pr(\alpha_1 \mid \beta) = Pr(\alpha_1 \mid \beta)$ ✓

$n=2$ $\quad Pr(\alpha_1, \alpha_2 \mid \beta) = Pr(\alpha_1 \mid \alpha_2, \beta) \cdot Pr(\alpha_2 \mid \beta)$

$$= \frac{Pr(\alpha_1, \alpha_2, \beta)}{P(\alpha_2, \beta)} \cdot \frac{P(\alpha_2, \beta)}{P(\beta)}$$

$$= \frac{Pr(\alpha_1, \alpha, \beta)}{P(\beta)} = Pr(\alpha_1, \alpha_2 \mid \beta) ✓$$

<u>By induction</u>:

$n = k-1$ case

$$Pr(\alpha_1, \alpha_2, \ldots, \alpha_{k-1} \mid \beta)$$

$$= \frac{Pr(\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \beta)}{Pr(\alpha_2, \ldots, \alpha_{k-1}, \beta)} \cdot \frac{Pr(\alpha_2, \alpha_3, \ldots, \alpha_{k-1}, \beta)}{Pr(\alpha_3, \ldots, \alpha_{k-1}, \beta)} \cdot \ldots\ldots$$

$$\cdot \frac{Pr(\alpha_{k-2}, \alpha_{k-1}, \beta)}{Pr(\alpha_{k-1}, \beta)} \cdot \frac{P(\alpha_{k-1}, \beta)}{P(\beta)}$$

$$= \frac{Pr(\alpha_1, \ldots, \alpha_{k-1}, \beta)}{Pr(\beta)} = Pr(\alpha_1, \alpha_2, \ldots, \alpha_{k-1} \mid \beta)$$

$n = k$ case

$$\Pr(\alpha_1, \ldots, \alpha_k \mid \beta)$$

$$= \frac{\Pr(\alpha_1, \alpha_2, \ldots, \alpha_k, \beta)}{\Pr(\alpha_2, \ldots, \alpha_k, \beta)} \cdot \frac{\Pr(\alpha_2, \alpha_3, \ldots, \alpha_k, \beta)}{\Pr(\alpha_3, \alpha_4, \ldots, \alpha_k, \beta)} \cdot \cdots$$

$$\cdot \frac{\Pr(\alpha_{k-1}, \alpha_k, \beta)}{\Pr(\alpha_k, \beta)} \cdot \frac{\Pr(\alpha_k, \beta)}{\Pr(\beta)}$$

$$= \frac{\Pr(\alpha_1, \alpha_2, \ldots, \alpha_k, \beta)}{\Pr(\beta)} = \Pr(\alpha_1, \ldots, \alpha_k \mid \beta) \quad \blacksquare$$

Problem 2

Given

$P(\text{oil}) = 0.5 \equiv$ probability of oil being present

$P(\text{gas}) = 0.2 \equiv$ probability of natural gas being present

$P(\text{neither}) = 0.3 \equiv$ probability of neither being present

$P(\text{positive} \mid \text{oil}) = 0.9 \equiv$ probability of positive result with oil present

$P(\text{positive} \mid \text{gas}) = 0.3 \equiv$ probability of positive result with natural gas present

$P(\text{positive} \mid \text{neither}) = 0.1 \equiv$ probability of positive result with neither present

$P(\text{oil} \mid \text{positive}) = ??$

$$\frac{P(\text{positive} \mid \text{oil})P(\text{oil})}{P(\text{positive})} =$$

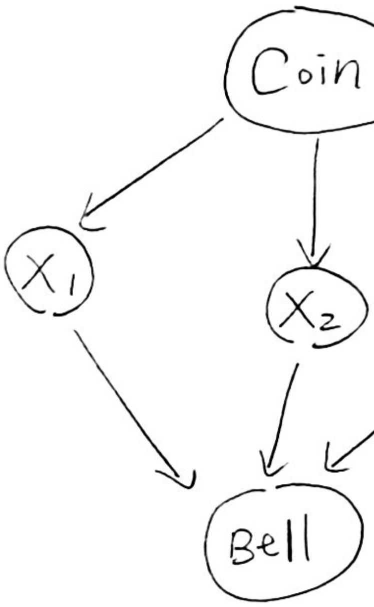
$$= \frac{P(\text{positive} \mid \text{oil})P(\text{oil})}{P(\text{positive} \mid \text{oil})P(\text{oil}) + P(\text{positive} \mid \text{gas})P(\text{gas}) + P(\text{positive} \mid \text{neither})P(\text{neither})}$$

$$= \frac{(0.9)(0.5)}{(0.9)(0.5) + (0.3)(0.2) + (0.1)(0.3)}$$

$$\approx \boxed{0.833333}$$

Problem 3

a) The Bayesian network of the set up



The following values are specified for each variable

$Coin = \{a, b, c\}$

$X_1 = \{H, T\}$; H = head ; T = tail

apply to X_2, X_3

$X_2 = \{H, T\}$

$X_3 = \{H, T\}$

$Bell = \{on, \neg on\}$

ring

off, not ring

b) Define the necessary CPTs

H ≡ head ; T ≡ tail

CPT for the coin

Coin	probability of having X coin (a, b, c)
a	1/3
b	1/3
c	1/3

CPT for outcome X_1

Coin	P(H)	P(T)
a	0.2	0.8
b	0.4	0.6
c	0.8	0.2

CPT for outcome X_2

Coin	P(H)	P(T)
a	0.2	0.8
b	0.4	0.6
c	0.8	0.2

CPT for outcome X_3

Coin	P(H)	P(T)
a	0.2	0.8
b	0.4	0.6
c	0.8	0.2

CPT for the bell

1 ≡ on
0 ≡ off

X_1	X_2	X_3	P(Bell = on)
H	H	H	1
H	H	T	0
H	T	H	0
H	T	T	0
T	H	H	0
T	H	T	0
T	T	H	0
T	T	T	1

Problem 4

(a) The Markovian assumptions according to the given DAG

$I(A, \phi, \{B, E\})$

$I(C, \{A\}, \{B, D, E\})$

$I(B, \phi, \{A, C\})$

$I(D, \{A, B\}, \{C, E\})$

$I(E, \{B\}, \{A, C, D, F, G\})$

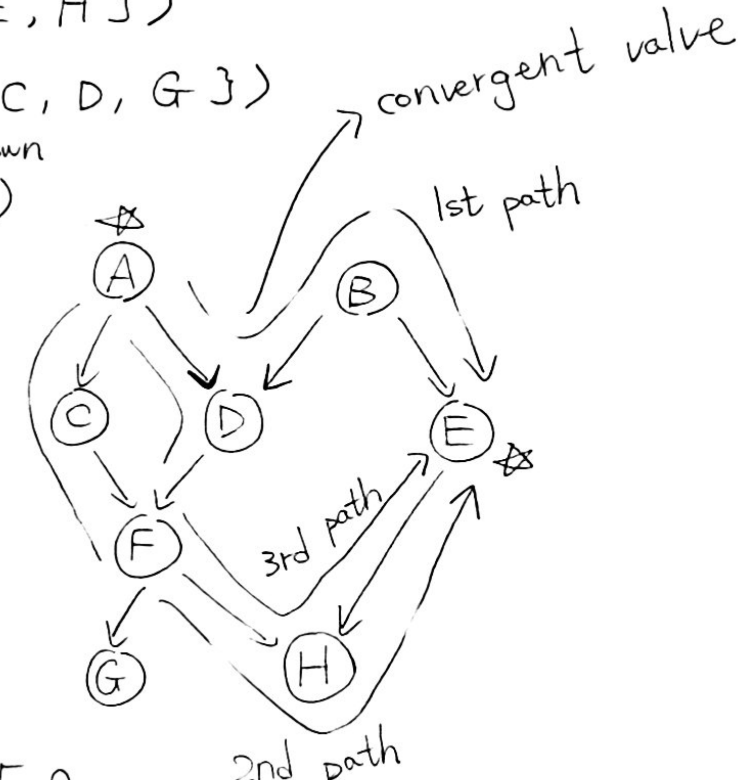
$I(F, \{C, D\}, \{A, B, E\})$

$I(G, \{F\}, \{A, B, C, D, E, H\})$

$I(H, \{E, F\}, \{A, B, C, D, G\})$

$I \equiv$ instantiantions of the DAG

(b) d_separated $(A, \overset{\text{known}}{F}, E)$



1st path

A, D, B, E
D : open
B : open

} valid path

2nd path

A, D, F, H, E
D : open
F : closed

$\underbrace{\hphantom{F : closed}}$ blocked path

3rd path

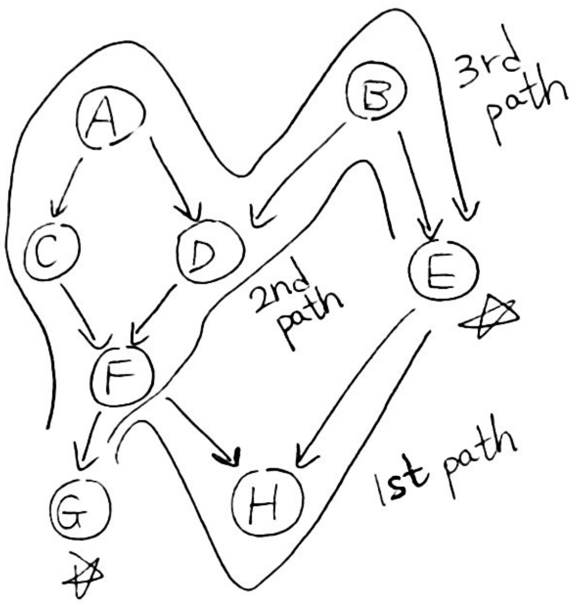
A, C, F, H, E
C : open
F : closed

} blocked path

(b) 1) d-separated (A, F, E), false, because if F is not in Z, there are non-converging path from A to E. In fact, we have a converging path from A to E which is A, D, B, E

known

2) d-separated (G, B, E)



1st path

G, F, H, E
F : open
H : closed
blocked path

2nd path

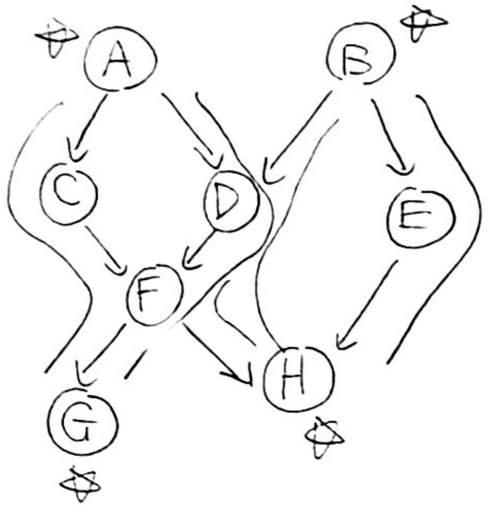
G, F, D, B, E
F : open
D : open
B : closed
blocked path

3rd path

G, F, C, A, D, B, E
F : open
C : open
A : open
D : open
B : closed
blocked path

d-separated (G, B, E), true, because all the possible paths from G to E are blocked given B is in Z.

3) d_separated (A B, CDE, G H)



1st path A, C, F, G

C : closed

2nd path A, D, F, G

D : closed

3rd path A, D, F, H

D : closed

4th path A, D, B, E, H

D : closed

5th path B, D, F, G

D : closed

6th path B, D, F, H

D : closed

7th path B, E, H

E : closed

8th path B, D, A, C, F, G

D : closed

All of the above paths are blocked

d_separated (A B, C D E, G H) is true because all the possible path from A to G
from A to H
from B to G
from B to H
are blocked given {C, D, E} are in Z.

Problem 4

(c)

$$Pr(a,b,c,d,e,f,g,h)$$
$$= Pr(a) \cdot Pr(c|a) \cdot Pr(b) \cdot Pr(d|a,b) \cdot Pr(e|b) \cdot Pr(f|c,d) \cdot Pr(g|f) \cdot Pr(h|e,f)$$

Problem 4

(d) Compute $Pr(A=1, B=1)$ → case 3 without Z

According to part b from problem 4, A and B are d-separated which means independence.

$$Pr(A=1, B=1) = P(A=1) \cdot P(B=1)$$
$$= 0.2 \cdot 0.7 = \boxed{0.14}$$

Compute $Pr(E=0 | A=0)$ — case 1 without Z

According to part b from problem 4, A and E are d-separated which means independence

$$Pr(E=0 | A=0) = P(E=0)$$
$$= Pr(E=0|B=0) \cdot Pr(B=0) + Pr(E=0|B=1) \cdot Pr(B=1)$$
$$= (0.1)(0.3) + (0.9)(0.7)$$
$$= \boxed{0.66}$$

Problem 5 Joint Probability Distribution.

(a) The models of α = models when α is true

$\alpha : A \Rightarrow B$

$\neg A \vee B$

	A	B	$\neg A$	$\neg A \vee B$	
W_0	T	T	F	T	✓
W_1	T	F	F	F	
W_2	F	T	T	T	✓
W_3	F	F	T	T	✓

$\boxed{W_0, W_2, W_3}$

(b) The probability of $Pr(\alpha)$

$Pr(\alpha) = W_0 + W_2 + W_3 = P(T,T) + P(F,T) + P(F,F)$

$= 0.3 + 0.1 + 0.4 = \boxed{0.8}$

Problem 5 cont.

(c) The conditional probability distribution $Pr(A, B | \alpha)$

$$Pr(T,T|\alpha) = \frac{P(T,T,\alpha)}{\cancel{P(T,\alpha)}} \cdot \frac{\cancel{P(T,\alpha)}}{P(\alpha)}$$

$$= \frac{0.3}{\cancel{0.3}} \times \frac{\cancel{0.3}}{0.8} = \boxed{0.375}$$

$$Pr(T,F|\alpha) = \frac{P(T,F,\alpha)}{P(F,\alpha)} \times \frac{P(F,\alpha)}{P(\alpha)}$$

$$= \frac{0}{0.1+0.4} \times \frac{0.1+0.4}{0.8} = \boxed{0}$$

$$Pr(F,T|\alpha) = \frac{P(F,T,\alpha)}{\cancel{P(T,\alpha)}} \times \frac{\cancel{P(T,\alpha)}}{P(\alpha)}$$

$$= \frac{0.1}{0.8} = \boxed{0.125}$$

$$Pr(F,F|\alpha) = \frac{P(F,F,\alpha)}{\cancel{P(F,\alpha)}} \times \frac{\cancel{P(F,\alpha)}}{P(\alpha)}$$

$$= \frac{0.4}{0.8} = \boxed{0.5}$$

Problem 5

(d) Compute $Pr(A \Rightarrow \neg B \mid \alpha)$

$$A \Rightarrow \neg B$$

$$\neg A \vee \neg B$$

	A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	
W_0	T	T	F	F	F	
W_1	T	F	F	T	T	✓
W_2	F	T	T	F	T	✓
W_3	F	F	T	T	T	✓

$A \Rightarrow \neg B$: W_1, $\underline{W_2, W_3}$ $\qquad$ $\alpha : A \Rightarrow B$: W_0, $\underline{W_2, W_3}$

$$\frac{P(A \Rightarrow \neg B \mid \alpha)}{P(\alpha)} = \frac{0.1 + 0.4}{0.8} = \frac{0.5}{0.8} = \boxed{0.625}$$