Heterogeneous Agent Modelling of Asset Price Dynamics and Parameter Estimation

by

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Abstract

Motivated by the ubiquity of a set of peculiar empirical regularities in financial price data, this report presents a general framework for developing mathematical models to replicate and explain them. In particular, we model the dynamics of asset prices as the product of many interactions between agents in a large heterogeneous group of market participants. Following the presentation of several state-of-the-art models, we introduce a simulation-based technique for Bayesian parameter inference. We then present and discuss the numerical results of applying this method to fit a model to the S&P 500 market index value.

Acknowledgements

I wish to express a great deal of gratitude to my supervisors, Dr. Radu Cîmpeanu and Dr. Susana Gomes, for their diligence, stimulating questions and ever-constructive criticism. I am certain that this report would not have been possible without their support, guidance and generosity with their time. It is my sincere hope that readers will derive as much pleasure this research as I have during its production.

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1 Introduction

Investment based on long-term expectation is so difficult as to be scarcely practicable. He who attempts it must surely lead much more laborious days and run greater risks than he who tries to guess better than the crowd how the crowd will behave.
— John Maynard Keynes

Financial markets play an incredibly important role in the day-to-day lives of nearly everyone on Earth. They provide the key allocation mechanism to ensure the smooth running of capitalist economies, allowing companies to raise money to develop new products, governments to fund public spending, and for businesses (in potentially different countries) to buy and sell the raw materials needed for manufacturing, to name a few examples. Improving our understanding of how the prices of assets traded on these markets behave is of the utmost importance, since an accurate description of price action enables one to make better decisions and quantify one's uncertainty about said decisions.

The interdisciplinary field of study dedicated to modelling and understanding the dynamic behaviour of asset prices is termed **economic dynamics**, and appeals to results from stochastic calculus, dynamical systems, synergetics and, of course, economics. This subject is currently witnessing a paradigm shift, as the core modelling assumptions — which have dominated the field for over half a century — are slowly but surely being relaxed to account for the inherently flawed way that humans make financial decisions. In this report, we explore and develop a modelling framework that enables one to investigate the impact of this type of behaviour on asset price dynamics. Generally speaking, an **asset pricing model** is a mathematical description of the dynamical behaviour of the price of a financial asset, such as a stock, bond or option contract. The fluctuations of asset prices are the result of the trading interactions between members of a large, heterogeneous[1] group of market participants. The specific reasons to develop asset price models are threefold. Firstly, real world asset prices behave in a consistently peculiar way, as we shall see, and modelling efforts can help shed light on the mechanisms behind these unusual empirical regularities. Moreover, dynamic pricing models enable one to determine the fair prices for other financial contracts, namely derivatives such as options and swaps. Finally, realistic pricing models have important applications in the testing and optimisation of algorithmic trading strategies. Since there is only one observation (in path space) for each asset price, issues of data scarcity currently plague these techniques, yielding over-fitted strategies that have poor performance in practice. In this report, we propose, analyse and test an **agent-based** model that could potentially replicate and provide explanations for the behaviour of real-world prices. We begin with a brief discussion about the classical rational expectations paradigm and its consequences for asset pricing.

[1]We consider **heterogeneity** in the sense of the agents' beliefs about the future, investment goals, aversion to risk, etc.

1.1 Rationality, Market Efficiency and the Friedman Hypothesis

The classical modelling framework in economics appeals to the assumption of **rational expectations (RE)**, which postulates that all agents have **perfect information** and infinite computational capacity to process said information. In such a homogeneous world, we can model the market by means of a single **representative agent**, the forecasts of whom are unbiased estimators of the true conditional expectations [1]. The RE assumption also implies the **efficient market hypothesis (EMH)**, which states that prices reflect all available information and hence, they follow a random walk (since news is random). Common examples of such processes include geometric Brownian motion, the Ornstein-Uhlenbeck process and other similar Markov processes driven by Brownian motion. The Markov property plays a crucial role in the rationality assumption. It implies that all "information" is contained in the current price of the asset, and the historical prices contains no additional information, namely about the conditional distribution of future prices [1].

RE models are often justified by appealing to some unobservable learning process, the result of which is a homogeneous population of rational agents. One important part of this process is the so-called **Friedman hypothesis** [2], which states that non-rational traders will be *driven out* of the market by rational traders taking positions against them. Specifically, if a pattern were to exist in asset prices, it would be exploited by rational traders, thereby removing it. In reality, agents' subjective probability distributions about the future do *not* necessarily coincide with the true distributions, and agents actively try to *learn* the dynamics as they trade. It has also been shown, in even a simple model, that **boundedly-rational** traders, following simple heuristics, can not only persist but also earn higher expected returns than rational traders [3]. In this situation, the very presence of non-rational agents deters rational traders from taking positions that would otherwise correct mispricings, and it is unclear to the rational traders when prices will revert to their fundamental values, if ever.[2] Building on the theoretical objections to the RE framework, there is a large amount of *empirical* evidence — that is, a set of pervasive statistical regularities — which is not consistent with rational expectations.

1.2 The Stylised Facts

Financial time series, and asset prices in particular, possess a ubiquitous set of statistical properties which do not admit generally-accepted explanations. We briefly detail these so-called **stylised facts**, and refer the interested reader to some excellent surveys [4, 5]. The stylised facts provide compelling evidence for a high degree of heterogeneity in the beliefs and behaviours of market participants at large.

[2] As Keynes famously said, *"markets can stay irrational far longer than you can stay solvent."*

(SF1) High trading volume. The sheer amount of trading that occurs in real markets directly contradicts the homogeneity and rationality assumed by most equilibrium models from mathematical economics [2]. Such assumptions imply the **no-trade theorems**, which state that there are no, or at least very few, situations where an individual has an incentive to trade with another [6, 7].

(SF2) Excess volatility over fundamental values. In many rational equilibrium models, current prices equal the total value of expected future payoffs, discounted to the present time. Therefore, they predict that prices should exhibit a similar amount of volatility as the so-called **fundamental** variables that determine them; yet prices are remarkably more volatile than these fundamentals [8, 9] and large movements often take place with little explanation by macroeconomics [2].

(SF3) Fat tails of the returns. Under the assumption of trader homogeneity, one can appeal to the central limit theorem to predict that the empirical distribution of asset returns is approximately normal. While the distribution is indeed unimodal, the presence of large price changes, at frequencies that are orders of magnitude greater than the normal distribution, causes the tails to exhibit a power-law-type decay. This fact is quantified by the **excess kurtosis** of the distribution, which measures the frequency of extreme values (those far away from the mean) relative to a normal distribution. That said, as the length of the time interval over which asset returns are calculated is increased, the distribution increasingly resembles a normal distribution, which is a phenomenon called **aggregational normality**. Importantly, the *shape* of the distribution is highly dependent on the time scale [4]. Figure 1 illustrates this effect using S&P 500 market index data, the price of which reflects the value of a basket of US stocks.

(SF4) Absence of return autocorrelation. Linear autocorrelations of returns are insignificant over time scales greater than a couple of hours. This particular observation highlights that prices are far more random than one might initially expect.

(SF5) Autocorrelation of absolute returns. In contrast to the previous observation, absolute (and squared) returns *do* exhibit strong autocorrelation, the significance of which decreases as the time horizon increases. The correlogram of absolute returns decays slowly, suggesting a long-range dependence structure. Figure 2 demonstrates these effects for daily and weekly returns, again using data for the S&P 500 market index.

(SF6) Temporal volatility clustering. Volatility estimates exhibit a positive linear autocorrelation; this shows that high-volatility periods have a tendency to cluster in time. Figure 3 presents correlograms for daily and weekly volatility of the S&P 500 market index.

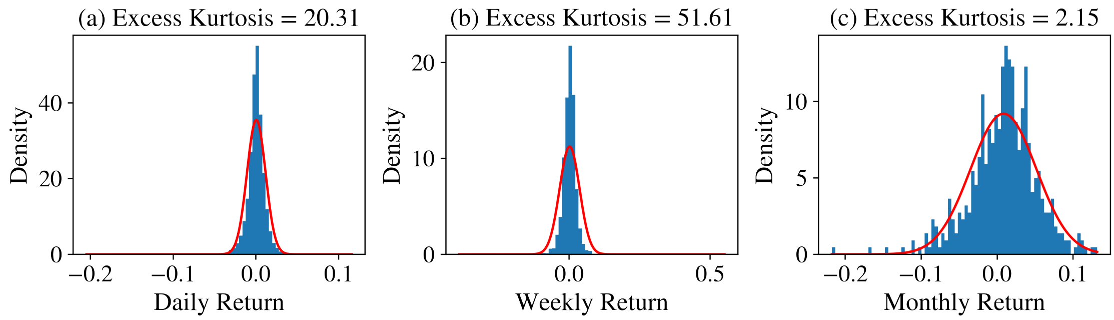

Figure 1: Frequency-density histograms of the returns — that is, percentage changes — of the closing value of the S&P 500 market index taken between 01/01/1980 and 31/12/2021 for (a) daily, (b) weekly and (c) monthly time scales. The Figure overlays the probability density function of a normal random variable with the same mean and variance as the empirical data. As the time horizon grows, the histogram increasingly resembles that of a normal distribution.

(SF7) Persistence of an approximate unit root. Asset prices exhibit a persistent approximate unit root when estimated by an autoregressive process. It is a well-known result that such a process can therefore not be stationary [10]. In an augmented Dicky-Fuller test, in which the null hypothesis is the existence of a unit root for a time series, the S&P 500 daily price data yielded a p-value of $p \approx 1$, from which one can very safely conclude that these data are highly non-stationary, and a unit root persists.

The traditional modelling assumptions of homogeneity and rationality are challenged by the pervasiveness of the stylised facts, in addition to the inability of classical models to replicate them. Over the last 30 years, there has been a growing body of research on the consequences of introducing heterogeneity and bounded rationality on asset price dynamics [2, 5], motivating a new category of asset pricing models, namely heterogeneous agent models (HAMs).

1.3 Heterogeneous Agent Models

The group of mathematical models collectively known as **heterogeneous agent models** seek to explain and replicate the stylised facts by modelling the determination of price with a heterogeneous population of boundedly-rational, interacting agents. In this context, the term **'boundedly-rational'** simply refers to *any* setup in which agents are not assumed to be rational and have perfect information. These agents follow simple heuristics to determine their demand for the asset, which changes endogenously based on the price and the patterns therein. Introducing heterogeneity is one of the most natural modelling directions: after all, if we do not have heterogeneity, then we have no trade [2].

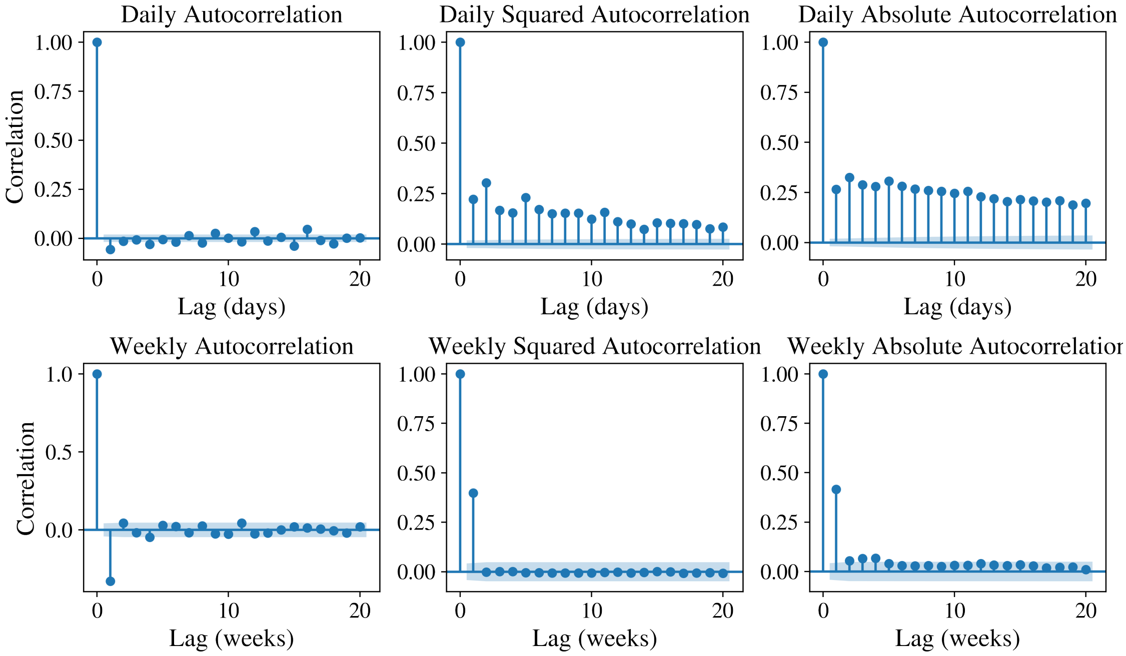

Figure 2: Correlograms for daily and weekly returns of the S&P 500 market index between 01/01/1980 and 31/12/2021. We observe that, while the returns themselves do not exhibit significant autocorrelation, the squared and absolute daily returns *do* show significant autocorrelation that decays slowly with the lag. As the time horizon increased from one day to one week, the autocorrelation values decrease in significance.

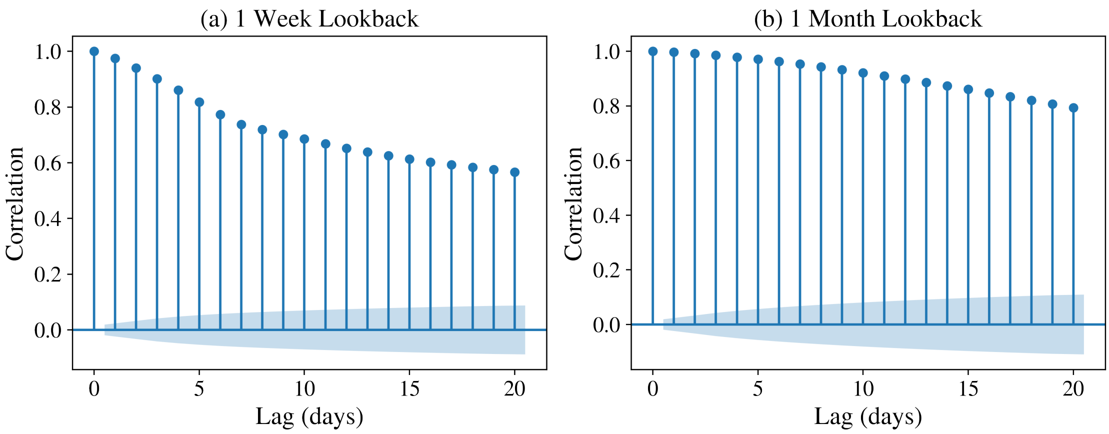

Figure 3: Correlograms of S&P 500 volatility between 01/01/1980 and 31/12/2021, as measured by the Yang-Zhang volatility estimator [11]. We observe the very significant positive linear autocorrelation of volatility over both 1-week and 1-month lookback periods.

In simulated trading markets, the human participants were found to extrapolate from non-existent price 'patterns' which led to price bubbles, *even when* all individuals had complete information [12]. Faced with the limited computational capabilities of the human brain, people prefer to use simple rules-of-thumb to make decisions when faced with uncertainty. Furthermore, surveys of foreign exchange dealers revealed that, in the short-term, over 90% of respondents placed *some* weight on **technical analysis**, whereby forecasts are made based on widely-available price data alone [13]. As the time-horizon was increased, higher weight was placed on **fundamental analysis**, whereby agents form expectations based on "fundamental" variables, examples of which include macroeconomic figures, profit and loss, debt-equity ratios, among *many* others. In HAMs, the heuristics on which agents rely are usually based on a small number of variables and extrapolations from past prices. Agent behaviour can therefore more accurately model the psychological mechanisms that might explain the stylised facts [5].

Simple HAMs have already been shown to be consistent with real-world observations. For example, the discrete-time HAM proposed by Frankel and Froot [14, 15] was designed to investigate the seemingly inexplicable appreciation of the United States dollar against other currencies in the early 1980s; the authors hypothesised the existence of a *"self-confirming speculative bubble"* to explain the observed increases, and the estimated model parameters were consistent with this.

When modelling *any* complex system, it is tempting to introduce as many free parameters and potential behaviours as possible, in the hope that one will be rewarded with realistic results. Indeed, for those readers content to shelve any strong desire for mathematical analysis, a litany of purely computational models, termed **artificial stocks markets (ASMs)**, emerged in the late 1990s. The Santa Fe Institute's ASM is the most well-known, and models the active learning of agents by means of a genetic algorithm, thereby permitting the natural *emergence* of different behaviours. In contrast with ASMs, most of the analytically-tractable HAMs assume *a priori* that agents follow one of a small number of expectation-formation regimes (usually two). Their high degree of complexity makes ASMs less desirable as tools for understanding the links between behavioural heterogeneity and the stylised facts [1]. Some argue that ASMs can generate more realistic dynamics [16]; however, the estimation and validation of agent-based models is an active research area with many open problems, primarily induced by their high dimensionality [17]. Simpler HAMs have been criticised as being too restrictive, due to the common assumption of local homogeneity of each type of agent [5].

The *holy grail* of economic dynamics is *the* HAM which strikes the perfect balance between realism — in terms of replicating the stylised facts — and analytical tractability [16].

1.4 Key Questions and this Report

The goals of this research report are twofold. *Firstly, given our ability to quantify the stylised facts, how can we develop a tractable model that replicates them?* We seek to accurately model the relationship between agent behaviour and price dynamics. Section 2 presents a general structure of a heterogeneous-agent asset-pricing model along with some important examples from the literature, followed by various concrete models. The section concludes by detailing some interesting extensions — namely, additional heterogeneity and a herding mechanism — and discussing how these extensions may facilitate different styles of subsequent analysis.

Secondly, how can one fit HAMs to data? Section 3 details a state-of-the-art method for estimating parameters from data, before presenting our numerical results in doing so for one of the models in Section 2. We discuss the quality of the parameter distributions and investigate their consistency with data.

Section 4 concludes the report with a general discussion of our approach and results. We summarise what we believe to be the key messages of this research, and highlight important directions of future work.

2 Heterogeneous Agent-based Asset Price Modelling

All models are wrong, but some are useful.
— George E. P. Box

The seminal contributions of Zeeman [18], Beja and Goldman [19], Lux [20], Frankel and Froot [14, 15], Day and Huang [21], and Brock and Hommes [22] inspired a large amount of research on heterogeneous agent-based asset pricing models. We begin with a discussion around the general structure of HAMs, with some pertinent examples from the literature. HAMs have been successful in explaining the stylised facts, as the result of complicated interactions between agents with different trading strategies [23]. These models are often formulated mathematically as systems of **coupled differential/difference equations** which describe the dynamic behaviour of the asset price, in addition to other state or latent variables of interest [5]. These equations can also include noise or delay terms, modelling stochastic behaviour and long-range dependencies respectively. Many models in the literature consider discrete-time dynamics; however, we pay particular attention to the continuous-time case, for reasons that will become clear.

Consider a stylised financial market in which agents can invest in two assets at times $t \in \mathcal{I}$, for some partially-ordered time index set $\mathcal{I}$. There is a risk-free asset paying constant interest rate $r > 0$, which can be thought of as a government bond. There is a risky asset with price $(P(t) : t \in \mathcal{I})$ which could be a stock, foreign currency or derivative contract. In the case that $P(t)$ represents the price of an equity share, we assume it to be the cum-dividend price, and that all dividends are instantaneously reinvested. Let $(F(t) : t \in \mathcal{I})$ denote the fundamental asset value that would prevail, $P(t) \equiv F(t)$, in a homogeneous, perfectly rational world, i.e. the total value of expected future payoffs, discounted to the present time [2].

Let $N \in \mathbb{N}$ denote the number of agents in the population. Let $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\in\mathcal{I}}, \mathbb{P})$ denote a complete filtered probability space, where $\Omega \neq \emptyset$ and $\omega \in \Omega$ represents the *state of the world*, according to some abstraction. Of course, this construction shall not be required for deterministic models. For convenience, we take subscript and bracket notation to be equivalent, i.e. $P(t) \equiv P_t$, and interchange between them at will.

2.1 Choice, Expectation and Price Determination

In this stylised market scenario, consider the decision problem faced by an individual agent, denoted by some $j \in \{1, 2, \ldots, N\}$. At each point in time $t \in \mathcal{I}$, the agent must choose how much of the risky asset $X_j(t)$ to demand from the market. This could be an absolute number of units (e.g. shares) of the asset or a proportion of the agent's current wealth, depending on the formulation. *How should the agent choose $X_j(t)$?* There are two approaches in the literature. The first (and by far the most popular) approach involves an *ad hoc* derivation of the functional form of $X_j(t)$ in terms of some model-relevant variables.

For example, an agent may believe the asset price to be reverting to some fixed value $\overline{P}$, in which case, a natural demand function is $X_j(t) \propto \overline{P} - P(t)$. This approach is advantageous not only since one can move rapidly on to specify the system dynamics, but also there is extensive literature on the structure and form of common trading strategies from which one can derive inspiration [24, 25]. An important characteristic of these simple heuristics is that no agent (or type of agent) accounts for the impact of their own activity nor the activity of others on the price dynamics [5].

Alternatively, one can formulate the choice of $X_j(t)$ rigorously as an optimisation problem. To this end, we first assign a complete filtered probability space $\big(\Omega, \mathcal{F}_j, (\mathcal{F}_j(t))_{t\in\mathcal{I}}, \mathbb{P}_j\big)$ to each agent $j \in \{1, 2, \ldots, N\}$, since agents do not necessarily possess the same information. This construction yields natural time-dependent probability measures and expectation operators, given by

$$\begin{aligned} &\mathbb{P}_j(t)[A] := \mathbb{P}_j\big[A \,\big|\, \mathcal{F}_j(t)\big] \quad \text{for all } A \in \mathcal{F}_j,\ t \in \mathcal{I}, \\ \text{and} \quad &\mathbb{E}_j(t)[X] := \int_\Omega X(\omega)\ \mathrm{d}\mathbb{P}_j(t)(\omega) \quad \text{for all } \mathcal{F}_j(t)\text{-measurable } X \in \mathbb{R}^\Omega,\ t \in \mathcal{I}. \end{aligned}$$

For the purposes of this discussion, we shall focus on one agent, and briefly drop any subscripts. We introduce a **utility of wealth** function $u : \mathbb{R} \to \mathbb{R}$ and assume both a fixed demand and conditional normality of the price change or return over a small time interval $[t, t + \Delta t]$, $\Delta t > 0$. Let $(W(t) : t \in \mathcal{I})$ denote the wealth of the agent and let $(\pi(t) : t \in \mathcal{I})$ denote the proportion of said wealth invested in the risky asset. One can approximate the demand function which maximises the expected utility by $X(t) = \pi^*(t)W(t)/P(t)$ where

$$\pi^*(t) = -\frac{u'(W(t))}{u''(W(t))W(t)\Delta t} \times \frac{\mathbb{E}(t)[\rho_{t+\Delta t}] - r\Delta t}{\mathrm{Var}(t)[\rho_{t+\Delta t}]} \quad \text{and} \quad \rho_{t+\Delta t} := \frac{P_{t+\Delta t} - P_t}{P(t)}. \tag{1}$$

Of course, we must take $u \in C^2(\mathbb{R})$. (See Appendix A for details of the derivation.) Equivalently, one can write $X(t)$ in terms of the expected price changes and variances thereof, that is

$$X(t) = -\frac{u'(W(t))}{u''(W(t))\Delta t} \times \frac{\mathbb{E}(t)[P_{t+\Delta t} - (1 + r\Delta t)P(t)]}{\mathrm{Var}(t)[P_{t+\Delta t} - (1 + r\Delta t)P(t)]}. \tag{2}$$

Following that, one can propose estimations for the conditional expected returns (and variances thereof) which agents may possess. It is common in this setup to assume that the utility function, risk aversion and conditional variances are homogeneous over time and the population of agents, such that **the only axis of heterogeneity is that of expectation** [22, 26].

The prior approach has seen greater popularity in the literature, since **expected utility maximisation (EUM)** does not necessarily provide the best foundation for models of this type. Indeed, price dynamics are the result of agent behaviour, which is the result of price dynamics, and so on. Mathematical modelling has to start somewhere. Additionally, the preferences of real-world traders may not be consistent with EUM. We note the von Neumann-Morgenstern utility theorem, which provides a strict set of conditions that an agent's preferences must satisfy in order to guarantee that a utility function exists, and that the agent's behaviour is consistent with the maximisation of this function (in expectation) [27].

Whichever approach is taken, for each agent $j \in \{1, 2, \ldots, N\}$ and time $t \in \mathcal{I}$, one has a demand function $X_j(t)$ (now reintroducing the superscripts). *How does demand determine the asset price?* Again, there are two approaches. In the case that $X_j(t)$ can be written as a function of the price $P(t)$, the general **equilibrium** approach — also known as **market clearing** — yields $P(t)$ by setting total demand equal to some (potentially stochastic) supply, denoted by $(Y(t) : t \in \mathcal{I})$. Models often take $Y(t) \equiv 0$, as done in [21, 22, 28] and as we shall for this entire report. In this case, the demand $X_j(t)$ can also be thought of as *excess* demand over some per-agent supply.

$$\sum_{j=1}^{N} X_j(t) = Y(t). \tag{3}$$

If the maps $P(t) \mapsto X_j(t)$ are nonlinear, one may have to resort to numerical methods to determine $P(t)$. Following [19, 21, 29, 30], the second approach we consider postulates the existence of an additional 'sluggish' **market maker** agent, who sets price changes proportional to total excess demand,

$$\begin{aligned} \mathrm{d}P(t) &= \lambda\left(\sum_{j=1}^{N} X_j(t) - Y(t)\right)\mathrm{d}t \quad \text{in continuous time, or} \\ P_{t+1} - P_t &= \lambda\left(\sum_{j=1}^{N} X_j(t) - Y(t)\right) \quad \text{in discrete time,} \end{aligned}$$

where $\lambda > 0$ represents the **adjustment speed** of the market maker. It is this second approach on which we shall focus for the remainder of this report, due to the mathematical tractability of the models it yields [25]. It is important to note that these equations only specify the dynamics of *price* only, and models will often contain other time-dependent state variables [5]. Unfortunately, there is no microeconomic foundation that would give this price update rule as optimal from the perspective of the market maker; however, linear price adjustment rules *are* found in the literature on price formation under asymmetric information [31]. In recent studies [17, 24, 32], authors prefer to take the first *ad hoc* approach, proceeding to the equations for the system dynamics without paying *too much* attention to how they arise from the aggregation of individual decisions. These models seek to describe how bounded rationality and interaction affects the price dynamics instead of local optimality for each agent.

Models of this form clearly have the potential to be very complicated, especially when the number of agents is large or the functional form of $X_j(t)$ varies greatly over the population. Following [19, 21, 22], we divide the population into locally-homogeneous groups, with a view towards a simplified yet sufficiently general version of the above dynamics. Assume that agents possess one of $H \in \mathbb{N}$ demand functions $\{X_h(t)\}_{h=1}^{H}$ and let $n_h(t) \in [0,1]$ denote the proportion of the population with demand function $h \in \{1, 2, \dots, H\}$ at times $t \in \mathcal{I}$. In this case, the models can be specified as,

$$\mathrm{d}P(t) = \lambda\left(\sum_{h=1}^{H} n_h(t)X_h(t) - Y(t)\right)\mathrm{d}t \quad \text{in continuous time, or}$$
$$P_{t+1} - P_t = \lambda\left(\sum_{h=1}^{H} n_h(t)X_h(t) - Y(t)\right) \qquad \text{in discrete time.}$$

We thus have a framework for translating the demand functions of agents into a dynamic model for price action. When expected utility maximisation is used to derive agents' demand functions, one often assumes agents to be homogeneous with respect to their utility function, risk aversion and conditional variances, such that there is a direct link between expectation formation, strategies and price dynamics [2, 17, 22, 30]. The agent population is often split into *two* homogeneous groups following so-called **fundamentalist** and **chartist** strategies respectively [19, 21, 28, 33].

2.2 Fundamentalists versus Chartists

In most studies, examples of which include [14, 21, 22, 33, 34], the authors hypothesise that the behaviour of market participants can be split into two categories: fundamentalist and chartist. This hypothesis has considerable empirical support in the form of survey data of foreign exchange professionals [13], and the agreement between price data and simple models accounting for this type of heterogeneity [15]. **Fundamentalist** agents have access to the fundamental value $F(t)$ — that is, the total expected future payoffs discounted to the present time — and believe that the market price $P(t)$ will revert to this value at some rate. Therefore, they seek to buy the asset (i.e. have positive demand) when the asset price is below the fundamental price $P(t) < F(t)$, and sell (i.e. have negative demand) when $P(t) > F(t)$. Hence, a general form of demand function $X_\mathrm{f}(t)$ for fundamentalists is

$$X_\mathrm{f}(t) = g\big(F(t) - P(t)\big)$$

where $g : \mathbb{R} \to \mathbb{R}$ is an increasing function satisfying $g(0) = 0$. A natural and parsimonious choice for g is the simple linear function $g(x) = \beta_\mathrm{f} x$ as used in [22, 32]. Belonging to the more general class of S-shaped functions[3], another common choice for g is the hyperbolic tangent function $g(x) = \tanh\big(\beta_\mathrm{f} x\big)$ which has been widely used [24, 25, 28, 23]. This is

[3]These functions are defined as satisfying $g(0) = 0$, $g'(x) \geqslant 0$ for all x and $xg''(x) < 0$ for all $x \neq 0$.

motivated by the magnitude of wealth to which banks and institutional investors have access, relative to individuals and retail investors. The former group of agents can afford to pay teams of analysts to accurately estimate the fundamental value and thus, trade in a fundamentalist-like way; for the latter group, budget constraints have a much more profound role in restricting the number of units they can buy and sell in the market [28].

At this point, it is important to highlight the distinction between fundamentalists and rational agents from the RE framework. Fundamentalists simply believe that the market price will revert to the fundamental value of the asset, which they know, making no provision for the existence of other agents. Now, if at time t the market price $P(t)$ has recently increased above its fundamental value, fundamentalists would make the myopic decision to sell the asset, anticipating a decrease in price; however, rational traders with perfect information may not trade in this way. For example, suppose that they are aware of the presence of a large number of agents who will extrapolate the price trend and hence cause a further price increase: in this case, rational traders would actually prefer to *buy* or simply not trade.

Chartists, also known as **technical traders**, do not have access to the fundamental value, and instead believe that changes in demand and supply of the risky asset are indicated by the historical values of its price [5]. They use simple heuristics — namely **moving averages** and other summary statistics, referred to as **signals** — to forecast future price changes. In the example of the **moving average** strategy, also known as the **momentum** strategy, agents seek to buy the asset when the asset price $P(t)$ exceeds a moving average signal $u(t)$ computed over some lookback period of length $\tau > 0$, and sell when $P(t)$ falls below it. This strategy corresponds to buying when the asset is expensive, in anticipation of further capital gains, and selling when the asset is cheap, in anticipation of further losses. Naturally, the moving average can assign more weight to more recent prices. For example, following [28] in continuous time, one could formulate $u(t)$ as an exponentially-weighted moving average,

$$u(t) = \frac{k}{1 - e^{-k\tau}} \int_{t-\tau}^{t} e^{-k(t-s)} P(s)\,\mathrm{d}s, \tag{4}$$

where $k \in (0, \infty)$ represents the decay parameter for the exponential weighting, and the constant multiplying the integral is chosen such that the weights integrate to unity. As $k \to 0$, the weights become uniform, and we have a so-called simple moving average,

$$\lim_{k \to 0} u(t) = \frac{1}{\tau} \int_{t-\tau}^{t} P(s)\,\mathrm{d}s. \tag{5}$$

Similarly, as $k \to \infty$, we have $u(t) \to P(t)$. As above, a natural demand function $X_\mathrm{c}(t)$ for chartists is,

$$X_\mathrm{c}(t) = g\big(P(t) - u(t)\big),$$

where $g : \mathbb{R} \to \mathbb{R}$ is taken from the same family of functions as for fundamentalists. Upon changing the sign of the moving average chartist demand $X_{\mathrm{c}}(t)$, we obtain the demand function for a **mean-reversion** or **contrarian** strategy. In a similar vein to the fundamentalists, these agents believe the market price will revert to some value, but that value is given by a historical average rather than an explicit estimation of the fundamental value. Intuitively, this behaviour corresponds to buying when the asset is relatively cheap and selling when it is relatively expensive, as measured by this historical average. Given the widespread usage of these strategies in the real-world [14, 15], incorporating their impact on asset price dynamics is of utmost importance [23].

Finally, agents may act as **noise traders**, also known as **liquidity traders**, who buy and sell the asset at (virtually) random points in time. Changes in demand due to agents of this type cannot be attributed to any concrete intentions. From a mathematical perspective, it serves no purpose to model the decisions of these agents explicitly, and one often accounts for the additional randomness by means of an additive noise term, obtaining a *stochastic* difference/differential equation for the price.

The seminal modelling work of Beja and Goldman [19], henceforth referred to at the BG model, is exemplary of many other HAMs with fundamentalists and chartists. The model is specified as a two-dimensional system of ODEs with constant proportions of agent types; it provides a foundation from which one can derive intuitive results about price behaviour. Suppose fundamentalists know the (constant) fundamental value F and have demand function $X_{\mathrm{f}}(t) = \alpha(F - P(t))$ for some parameter $\alpha > 0$ representing the rate at which fundamentalists believe $P(t)$ will revert to F. Chartists are taken to be momentum traders, as described above, who extrapolate from previous returns; their demand function is $X_{\mathrm{c}}(t) = \beta\pi(t)$ where $\pi(t)$ is specified to approximate the instantaneous price change. The system dynamics are given by:

$$\begin{cases} \mathrm{d}P(t) = \lambda\Big(\alpha\big(F - P(t)\big) + \beta\pi(t)\Big)\,\mathrm{d}t, \\ \mathrm{d}\pi(t) = \eta\Big(\mathrm{d}P(t) - \pi(t)\,\mathrm{d}t\Big) \quad \text{where } \eta > 0. \end{cases}$$

Here, η represents the speed at which the response variable of chartists $\pi(t)$ changes in response to $P(t)$. A key question that dominates the literature is that of the existence and stability of the so-called **fundamental equilibrium (FE)**, where the market price and fundamental value are equal. This question can be answered in the stylised BG model by first observing that the model is an autonomous first-order linear system,

$$\begin{aligned} \frac{\mathrm{d}P(t)}{\mathrm{d}t} &= \lambda\alpha F - \lambda\alpha P(t) + \lambda\beta\pi(t), \\ \frac{\mathrm{d}\pi(t)}{\mathrm{d}t} &= \eta\lambda\alpha F - \eta\lambda\alpha P(t) + \eta(\lambda\beta - 1)\pi(t). \end{aligned}$$

The only equilibrium point is the fundamental equilibrium $(P^*, \pi^*) = (F, 0)$, when the price is at its fundamental value, and the chartist agents do not expect any subsequent price changes [5]. A sufficient condition for the stability of this FE is

$$\alpha\lambda + \eta(1 - \beta\lambda) > 0 \tag{6}$$

as shown in Appendix B. One can already draw some interesting conclusions from (6). Firstly, a higher sensitivity or strength of fundamentalists α has a stabilising effect on the FE, whereas a higher sensitivity or strength of chartists has a destabilising effect on the FE. Also, the influence of the speed of expectation adjustment of chartists is not clear: if $\beta < 1/\lambda$ the system is always stable; however, if $\beta > 1/\lambda$ then the FE may be unstable, and decreasing η has a stabilising effect. Unfortunately, the model becomes devoid of meaning when the fundamental equilibrium is unstable: as the BG model is linear, one would observe unbounded exponential growth of the price in this case. The original BG model was extended in [2], to permit richer behaviour, as follows:

$$\begin{cases} \mathrm{d}P(t) = \lambda\Big(\alpha\big(F - P(t)\big) + \beta\big(g(t) - \pi(t)\big) + \epsilon(t)\Big)\,\mathrm{d}t, \\ \mathrm{d}\pi(t) = \eta\Big(\mathrm{d}P(t) - \pi(t)\,\mathrm{d}t\Big) \quad \eta > 0, \end{cases}$$

where $(g(t))_{t\geqslant 0}$ represents an *expected* response, and $(\epsilon(t))_{t\geqslant 0}$ represents a perturbation term allowing for the introduction of nonlinearity. While the BG model implicitly fixes the proportions — or *'sensitivities'* in the words of the original authors — of each type of agent, this need not be the case. Several studies have developed HAMs which allow these proportions to change dynamically, by modelling how agents switch their strategies based on their relative performances. We hypothesise that such an endogenous switching mechanism could provide an additional source of randomness, which might explain the excess volatility of market prices over fundamental values **(SF2)** and the heavy tails of returns **(SF3)**.

2.3 Endogenously-determined Market Fractions

In their seminal contribution, Brock and Hommes [22] proposed a discrete-time HAM with fundamentalists and chartists, in which agents can choose between the two strategies based on their relative performances. The proportions of the population following each strategy are additional state variables, which we refer to as the **market fractions** as in [5, 17]. The key feature of this model is that these proportions are determined by an *endogenous* feedback mechanism, whereby agents' selection of strategies impacts the price dynamics and the performance of strategies recursively [28]. Performance can be measured in many ways, such as profit-and-loss, predictive accuracy, utility of wealth, etc.

Subsequent discrete-time developments following [22] have highlighted the fact that, when modelling the dynamics induced by more realistic *chartist* strategies, i.e. using long or heterogeneous look-back periods over which signals are computed, the dimension of the system grows with said look-back period [25, 30]. High-dimensional models are seldom tractable, even *in silico*, so mathematical analysis thereof is generally untenable in discrete-time. The continuous-time formulation permits a more natural investigation into the impact of heterogeneity and other related questions on the price dynamics [28]. For the remainder of this report, we take $\mathcal{I} = \mathbb{R}_{\geqslant 0}$ or $\mathcal{I} = [0, T]$.

2.3.1 Replicator Dynamics

We now introduce a model for the continuous-time evolution of the **market fractions**, as defined above. Our initial presentation holds generally, closely following the discussion in [35, Chap. 7]. Following Section 2.1, we divide the population into $H \in \mathbb{N}$ types and let $n_h(t)$ denote the proportion of the population corresponding to type $h \in \{1, 2, \dots, H\}$, where $\sum_{h=1}^{H} n_h(t) = 1$ and each $n_h : \mathcal{I} \to [0, 1]$ is assumed to be a differentiable function on $\mathcal{I}$. Additionally, let $n(t) = \big(n_1(t), \dots, n_H(t)\big)$ denote the state vector for this system.

Let $f_h(t)$ be the *instantaneous fitness* of type $h \in \{1, 2, \dots, H\}$ at time $t \in \mathcal{I}$, and let $\hat{f}(t)$ be the average (instantaneous) fitness of the population, given by

$$\hat{f}(t) := \sum_{h=1}^{H} n_h(t) f_h(t). \tag{7}$$

We model the rate of change of each $n_h(t)$ as proportional to the difference between the fitness of type h at the average fitness in the population — that is,

$$\mathrm{d}n_h(t) = n_h(t)\Big(f_h(t) - \hat{f}(t)\Big)\,\mathrm{d}t \quad \text{for all } h \in \{1, 2, \dots, H\}. \tag{8}$$

This system is referred to in the literature as the **replicator dynamics**, appealing to the key principle of Darwinian evolution that the individuals (or types in this model) that have higher than average fitness increase in proportion while those with lower than average fitness decrease in proportion. We briefly check that this system behaves as one would expect. Let S_H denote the $(H-1)$-dimensional **simplex**, given by

$$S_H := \left\{ (x_1, x_2, \dots, x_H) \in [0, 1]^H \quad \text{such that} \quad \sum_{h=1}^{H} x_h = 1 \right\}. \tag{9}$$

It is clear that S_H is an invariant set under the replicator dynamics — that is, if $n(0) \in S_H$ then $n(t) \in S_H$ for all $t \in \mathbb{R}$. We consider the dynamics of the sum $\sum_{h=1}^{H} n_h$ as follows,

$$\begin{aligned} \mathrm{d}\left(\sum_{h=1}^{H} n_h(t)\right) &= \sum_{h=1}^{H} \mathrm{d}n_h(t) = \sum_{h=1}^{H} n_h(t)\big(f_h(t) - \hat{f}(t)\big)\,\mathrm{d}t \\ &= \sum_{h=1}^{H} n_h(t) f_h(t)\,\mathrm{d}t - \hat{f}(t) \sum_{h=1}^{H} n_h(t)\,\mathrm{d}t \end{aligned}$$

$$= \left(1 - \sum_{h=1}^{H} n_h(t)\right) \hat{f}(t)\,\mathrm{d}t,$$

from which it follows that $\sum_{h=1}^{H} n_h = 1$ is a stationary point for the dynamics of the sum. Therefore, if $n(0) \in S_H$ then $\sum_{h=1}^{H} n_h(0) = 1$ thus $\sum_{h=1}^{H} n_h(t) = 1$ and $n(t) \in S_H$ for all $t \in \mathbb{R}$.

2.3.2 Application to the Market Fractions

Following [28], we now apply the system of replicator dynamics to endogenously determine the market fractions $\{n_h\}_{h=1}^{H}$ based on the cumulative weighted net profits of each strategy. These dynamics are very intuitive in the sense that, if a specific strategy has a higher (resp. lower) than average net profit, then the market fraction corresponding to that strategy will increase (resp. decrease). Let $(\pi_h(t) : t \in \mathcal{I})$ denote the **net profit** per unit time of strategy $h \in \{1, 2, \dots, H\}$. The time-evolution of $\pi_h(t)$ is naturally described by

$$\pi_h(t)\,\mathrm{d}t = X_h(t)\,\mathrm{d}P(t) - C_h\,\mathrm{d}t, \tag{10}$$

where $\{C_h\}_{h=1}^{H}$ are the (constant) costs per unit time for each strategy, which model the (interesting) case where agents must pay to acquire information (e.g. the fundamental value) needed to follow particular strategies. At time $t \in \mathcal{I}$, we define the performance $U_h(t)$ of strategy h to be the cumulative, exponentially-weighted net profit of the strategy over some recent time interval $[t - \tau_h, t]$, that is,

$$U_h(t) := \frac{k_h}{1 - \exp(-k_h \tau_h)} \int_{t-\tau_h}^{t} e^{-k_h(t-s)} \pi_h(s)\,\mathrm{d}s \quad \text{for all } h \in \{1, 2, \dots, H\},$$

where $k_h > 0$ is the rate of decay for the exponential weights, and $\tau_h > 0$ represents the time horizon used to measure the performance. As in (4), the constant multiplying the integral is chosen such that the weights integrate to unity. Using (10), one can write $U_h(t)$ equivalently as

$$U_h(t) = \frac{k_h}{1 - \exp(-k_h \tau_h)} \int_{t-\tau_h}^{t} e^{-k_h(t-s)} X_h(s)\,\mathrm{d}P(s) - C_h \tag{11}$$

Let $\gamma > 0$ be scalar parameter representing the agents' **propensity-to-switch** between strategies. We take the fitness values $\{f_h(t)\}_{h=1}^{H}$ to satisfy $f_h(t)\,\mathrm{d}t = \gamma\,\mathrm{d}U_h(t)$ with average fitness $\hat{f}(t)$ given by

$$\hat{f}(t)\,\mathrm{d}t = \sum_{h=1}^{H} n_h(t) f_h(t)\,\mathrm{d}t = \gamma \sum_{h=1}^{H} n_h(t)\,\mathrm{d}U_h(t) =: \gamma\,\mathrm{d}\hat{U}(t),$$

such that the system of replicator dynamics reads as follows:

$$\mathrm{d}n_h(t) = \gamma n_h(t)\Big(\mathrm{d}U_h(t) - \mathrm{d}\hat{U}(t)\Big) \quad \text{for all } h \in \{1, 2, \dots, H\}. \tag{12}$$

As discussed in [28], in this setup, one can derive the time-dependent solution for $n_h(t)$ as

$$n_h(t) = \frac{\exp\big(\gamma U_h(t)\big)}{\sum_{\ell=1}^{H} \exp\big(\gamma U_\ell(t)\big)} \quad \text{for all } h \in \{1, 2, \dots, H\},\ t \in \mathcal{I}. \tag{13}$$

Taking the market fractions to be *defined by* (13), we have developed a general framework for constructing continuous-time heterogeneous agent-based asset pricing models with and without endogenously-determined market fractions. We turn our attention to specifying some complete models of varying complexity.

2.4 Complete Models

The power of equations lies in the philosophically difficult correspondence between mathematics [...] and an external physical reality.
— Professor Ian Stewart, University of Warwick

We now specify three continuous-time HAMs, all of which are specified by systems of coupled stochastic delay-differential equations, with varying numbers of parameters and degrees of complexity. These models are closest in spirit and structure to some of the most recent modelling work in the field [23, 24, 28, 32]. As above, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a complete, filtered probability space, which satisfies the usual conditions[4]. Let $(B(t))_{t\geqslant 0}$ be a two-dimensional Wiener process which is assumed to exist in the probability space and be adapted to $(\mathcal{F}_t)$, where $B_1(t)$ and $B_2(t)$ denote the two independent scalar Wiener processes which are the components of $B(t)$.

Model (M1). Our first model consists of fundamentalists and chartists with constant market fractions and a constant fundamental value $\overline{F} \geqslant 0$. Let $n \in [0,1]$ (resp. $1-n$) denote the market fraction of fundamentalists (resp. chartists). Following the discussion in Section 2.2 and the recent work in [32], we take both types of agents to have linear demand functions. The demand function of fundamentalists is $X_\mathrm{f}(t) = \beta_\mathrm{f}\big(\overline{F} - P(t)\big)$ where $\beta_\mathrm{f} > 0$ as in [21]. Chartists follow a momentum strategy and thus, have a demand function given by $X_\mathrm{c}(t) = \beta_\mathrm{c}\big(P(t) - u(t)\big)$. In this setup, β_f and β_c denote the sensitivities of fundamentalists and chartists respectively. Following [32], we assume the price signal for chartists, denoted by $(u(t) : t \geqslant 0)$ is given by a simple moving average of the price over a look-back period of length $\tau > 0$ as in (5). The impact of noise traders on the price is modelled by a drift-free Wiener process with constant volatility $\sigma_P \geqslant 0$. The system

[4] The so-called **usual conditions** assert that the filtration is right-continuous, increasing and that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. (See [36, Chap. 10] for details.)

dynamics are therefore given by

$$\begin{cases} \mathrm{d}P_t = \lambda\Big(n\beta_{\mathrm{f}}\big(\overline{F}-P_t\big)+\beta_{\mathrm{c}}(1-n)\big(P_t-u_t\big)\Big)\,\mathrm{d}t+\sigma_P\,\mathrm{d}B_1(t) \quad \forall\, t\geqslant 0, \\ u_t=\frac{1}{\tau}\int_{t-\tau}^{t}P_s\,\mathrm{d}s \quad \forall\, t\geqslant 0, \\ P_t=\overline{P}_t \quad \forall\, t\in[-\tau,0], \end{cases} \tag{14}$$

where $\overline{P}:[-\tau,0]\to\mathbb{R}$ is the initial price data. We have used $B_1(t)$ to drive the SDDE since we only require one-dimensional noise. Furthermore, one can appeal to the fundamental theorem of calculus to rewrite the equation for $u(t)$ in differential form,

$$\begin{cases} \mathrm{d}u_t=\frac{1}{\tau}\Big(P_t-P_{t-\tau}\Big)\,\mathrm{d}t, \\ u_0=\frac{1}{\tau}\int_{-\tau}^{0}\overline{P}_s\,\mathrm{d}s. \end{cases}$$

At first glance, it is not necessarily clear why we have assumed such a simple linear form for both demand functions in (14), especially given our prior discussion about the additional realism that nonlinear functions may endow upon our models. The hyperbolic tangent function in particular sees widespread usage throughout the literature, especially for the demand function of chartists [29, 37, 38]. Firstly, linear demand functions — in combination with the assumption of fixed market fractions — permit a convenient reduction of (14). Defining $\alpha:=\lambda n\beta_{\mathrm{f}}$ and $\beta:=\lambda\beta_{\mathrm{c}}(1-n)$ obtains the following system,

$$\begin{cases} \mathrm{d}P_t=\Big(\alpha\big(\overline{F}-P_t\big)+\beta\big(P_t-u_t\big)\Big)\,\mathrm{d}t+\sigma_P\,\mathrm{d}B_1(t) \quad \forall\, t\geqslant 0, \\ \mathrm{d}u_t=\frac{1}{\tau}\big(P_t-P_{t-\tau}\big)\,\mathrm{d}t \quad \forall\, t\geqslant 0, \\ P_t=\overline{P}_t \quad \forall\, t\in[-\tau,0]. \end{cases} \tag{15}$$

Unfortunately, in doing so we have lost some intuition about the market structure that particular parameter values would suggest. For example, a value of $\alpha=1$ could equally arise from $\lambda=2$, $n=0.5$ and $\beta_{\mathrm{f}}=1$ or $\lambda=5$, $n=0.1$ and $\beta_{\mathrm{f}}=2$. Given the infinite dimensionality of the system (14), induced by initial function $\overline{P}$, one might argue that a reduction from seven to five additional parameters is arbitrary. However, for our purposes, we shall take the initial function $\overline{P}$ to be given by real-world price observations, which are exogenously specified *a priori*, and are thus not actually parameters.[5] Additionally, a first-order Taylor expansion of tanh about zero yields the linear function — that is, $\tanh(x)\approx x$ — and so, one can argue that the linear case is *somewhat* consistent with the nonlinear case, by approximation.

Model (M2). As to the simple moving average in (M1), the assumption that, when making decisions, agents give the same weight to price movements that occurred in the distant past as those that occurred recently is not realistic. Indeed, the so-called **recency bias** is a fundamental pillar of the behavioural economics literature. In our second model,

[5] We therefore have fewer parameters to estimate and no restriction on model behaviour.

we generalise this moving average signal to have exponential weighting with decay parameter $0 < k < \infty$ and look-back period $\tau > 0$, as in (4). With a view towards a model that is valid over both short and long time-horizons, we also challenge the assumption of **(M1)** that the fundamental value is constant. Following [28], we generalise (15) to the case that the fundamental price $F(t)$ is modelled by a geometric Brownian motion with drift $\mu_F \in \mathbb{R}$ and volatility $\sigma_F \geqslant 0$. The system dynamics are given by

$$\begin{cases} \mathrm{d}P_t = \Big(\alpha\big(F_t - P_t\big) + \beta\big(P_t - u_t\big)\Big)\,\mathrm{d}t + \sigma_P\,\mathrm{d}B_1(t) \quad \forall\, t \geqslant 0, \\ \mathrm{d}F_t = \mu_F F_t\,\mathrm{d}t + \sigma_F F_t\,\mathrm{d}B_2(t) \quad \forall\, t \geqslant 0, \\ \mathrm{d}u_t = k\Big(\frac{1}{1-\exp(-k\tau)}\Big(P_t - e^{-k\tau}P_{t-\tau}\Big) - u_t\Big)\,\mathrm{d}t \quad \forall\, t \geqslant 0, \\ P_t = \overline{P}_t \quad \forall\, t \in [-\tau, 0]. \end{cases} \tag{16}$$

We have taken the noise terms in the asset price and fundamental value to be uncorrelated in an attempt to keep the number of model parameters relatively small. An extension to the case of correlated noise is trivial but in our view, the modelling reasons for doing so are not justified by the complexity introduced by the additional parameters.

Model (M3). The third model addresses the issues caused by fixing the market fractions. In our quest towards realism, the assumption that agents adhere to the same strategies and never change is unquestionably objectionable. We extend **(M2)**, adopting the endogenous switching mechanism derived in Section 2.3. Continuing with our two-strategy framework of **(M1)** and **(M2)**, let $k_{\mathrm{f}}, k_{\mathrm{c}} > 0$ and $\tau_{\mathrm{f}}, \tau_{\mathrm{c}} > 0$ denote the decay parameters and look-back period lengths used by fundamentalists and chartists respectively to compute these cumulative weighted profitabilities. Using the notation from Section 2.3, let $U_{\mathrm{f}}(t)$ and $U_{\mathrm{c}}(t)$ denote the performances of fundamentalist and chartist strategies respectively. Let $\gamma > 0$ be the the propensity-to-switch parameter and let $n(t) \in [0, 1]$ denote the market fraction of fundamentalists. This is sufficient since there are only two strategies, the market fractions of which must sum to unity. The system dynamics are therefore given as follows:

$$\begin{cases} \mathrm{d}P_t = \Big(\alpha n_t\big(F_t - P_t\big) + \beta\big(1 - n_t\big)\big(P_t - u_t\big)\Big)\,\mathrm{d}t + \sigma_P\,\mathrm{d}B_1(t)\ \forall t \geqslant 0, \\ \mathrm{d}F_t = \mu_F F_t\,\mathrm{d}t + \sigma_F F_t\,\mathrm{d}B_2(t) \quad \forall\, t \geqslant 0, \\ n_t = \exp\big(\gamma U_{\mathrm{f}}(t)\big)/\Big(\exp\big(\gamma U_{\mathrm{f}}(t)\big) + \exp\big(\gamma U_{\mathrm{c}}(t)\big)\Big) \quad \forall\, t \geqslant 0, \\ \mathrm{d}u_t = k\Big(\frac{1}{1-\exp(-k\tau)}\Big(P_t - e^{-k\tau}P_{t-\tau}\Big) - u_t\Big)\,\mathrm{d}t \quad \forall\, t \geqslant 0, \\ U_{\mathrm{i}}(t) = \frac{k_{\mathrm{i}}}{1-\exp(-k_{\mathrm{i}}\tau_{\mathrm{i}})}\int_{t-\tau_{\mathrm{i}}}^{t} e^{-k_{\mathrm{i}}(t-s)}\pi_{\mathrm{i}}(s)\,\mathrm{d}s \quad \forall\, \mathrm{i} \in \{\mathrm{f}, \mathrm{c}\},\ t \geqslant 0 \\ \pi_{\mathrm{f}}(t)\,\mathrm{d}t = \alpha\big(F_t - P_t\big)\,\mathrm{d}P_t - C_{\mathrm{f}}\,\mathrm{d}t \quad \forall\, t \geqslant 0, \\ \pi_{\mathrm{c}}(t)\,\mathrm{d}t = \beta\big(P_t - u_t\big)\,\mathrm{d}P_t - C_{\mathrm{c}}\,\mathrm{d}t \quad \forall\, t \geqslant 0, \\ P_t = \overline{P}_t \quad \forall\, t \in [-\tau, 0], \end{cases} \tag{17}$$

where the profitabilities π_f and π_c are given by (10), and the initial data $\overline{P}$ and other model parameters are defined in the same way as in **(M1)** and **(M2)**.

In stark contrast to much of the other literature, a key design feature of the models presented here is that each subsequent model embeds the previous models as special cases [17]. Specifically, taking $\sigma_F = 0$ and $k \to 0$ in **(M2)** obtains **(M1)** and for **(M3)**, taking $\gamma = 0$ and the initial condition $n(0) = n$ recovers **(M2)**. Eagle-eyed readers will notice that, by taking the fundamental value to follow a geometric Brownian motion (GBM) in **(M2)** and **(M3)**, we have *also* embedded the case that the market price itself follows a GBM, namely by taking $\beta = \sigma_P = \gamma = 0$. These relationships are more useful than they may seem *prima facie*. Letting Θ_j denote the parameter space of the jth model for $j \in \{1, 2, 3\}$, we have $\Theta_1 \subset \Theta_2 \subset \Theta_3$, which implies a direct correspondence between expanding the parameter space and increasing the complexity of the model.

2.4.1 Preliminary Simulations

We now present some exploratory simulation results obtained using *ad hoc* parameter choices. Models **(M1)** and **(M2)** have five and seven parameters respectively, excluding the initial function, and so selecting parameters is relatively straightforward. We also have some intuition as to the orders of magnitude of each parameter from the existing literature and a cursory scan of some market data (see Figure 1), which we can use to choose sensible values. Moreover, **(M3)** has 14 parameters, which is twice that of **(M2)**, and we have no intuition about *any* of the seven additional parameters relative to **(M2)**. As noted in various other parts of the literature [22, 25, 32], the sharp increase in the number of parameters when using time-dependent coefficients presents one with unwieldy mathematical objects. In terms of further study, it is incredibly useful to have derived the framework for endogenous switching; however, we desist from any simulation efforts for **(M3)** for the remainder of this report, focussing instead on **(M1)** and **(M2)**.

The purpose of the subsequent discussion is to illustrate how the models vary in behaviour, and how different parameters yield intuitive changes in results, by means of examples. In order to simulate sample paths of the models, we applied a first-order Euler-Maruyama scheme to the systems of SDDEs. This discretisation is justified by the recent work of Zhang and Li [39], who prove a useful theoretical result guaranteeing the mean-square stability of the split-step backward Euler scheme for linear SDDEs. In particular, the coefficients of the diffusion in the *delayed* components of (15), (16) and (17) are not state-dependent, from which it follows that the split-step backward Euler scheme is equivalent to the ordinary Euler-Maruyama scheme [40], and is hence stable, in the mean-square sense. We omit explicit presentation of the discretised systems themselves, in the interest of concision; they are notationally intricate and do not yield any deeper insight into the underlying model structure.

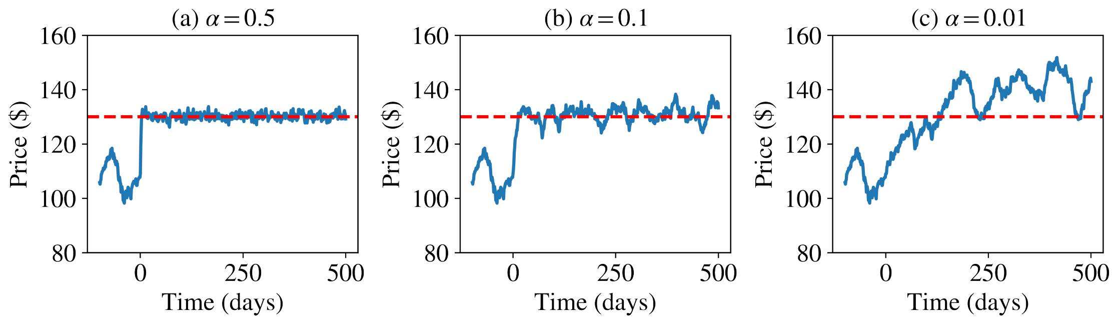

Figure 4: Example sample paths of length $T = 500$ days for model **(M1)**, with the following parameters: $\beta = 0.05$, $\sigma_P = 1$ and $\tau = 25$. The constant fundamental value $\overline{F} = \$130$ is overlayed, with subplots corresponding to: (a) $\alpha = 0.5$, (b) $\alpha = 0.1$ and (c) $\alpha = 0.01$. We observe that, as the strength parameter of fundamentalists α decreases, the chartists strategy behaviour has a greater effect on the system, leading to systematic overpricing (also known as a **price bubble** [21]) that takes some time to be 'corrected', i.e. a return to the fundamental value.

Importantly, for the simulation results presented here — and, in fact, throughout the rest of this report — we scale the time-step Δt such that $\Delta t = 1$ day. We then generate sample paths for models **(M1)** and **(M2)**, using the first 100 end-of-day market closing prices of the S&P500 market index for the initial data $\overline{P}$. Figures 4 and 5 present the resulting exemplary sample paths, for a selection of natural parameter values. We celebrate the ability of the models to replicate some of the most important stylised facts, in particular, the fat tails **(SF3)** and persistent non-stationarity of the returns **(SF7)**, as detailed in Table 1. Furthermore, Figure 6 shows that both models replicate the absence of return autocorrelation, using the parameters from subplot (c) from Figures 4 and 5.

2.5 Extensions

To conclude this section, we briefly step away from the detail a variety of extensions to the models in Section 2.4. These serve not only to demonstrate the power and ubiquity of this modelling framework in general, but also highlight the possible directions of future investigation that may shed even more light on the causal mechanisms underpinning the stylised facts.

2.5.1 Greater Heterogeneity of Strategies

Firstly, the models in Section 2.4 all assume that agents follow one of two simple strategies. While real-world trader behaviour may *generally* be partitioned into fundamental and chartist types, it is unrealistic to assume that *all* fundamentalists and *all* chartists follow exactly the same strategy. Furthermore, we have assumed linear demand functions for both types of agent: as described above, there are natural extensions to nonlinear demand

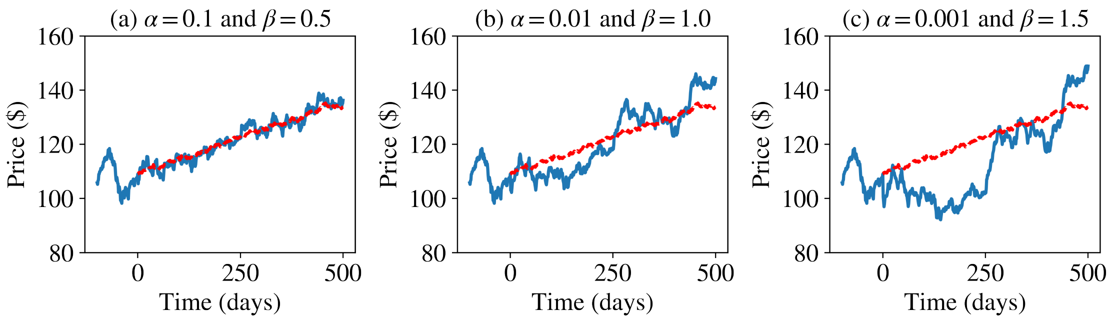

Figure 5: Example sample paths of length $T = 500$ days for model **(M2)**, with the following parameters: $\mu_F = 0.0005$, $\sigma_F = 0.002$, $\sigma_P = 1$, $k = 0.8$ and $\tau = 25$. The (non-constant) fundamental value is overlayed, as in Figure 4, with subplots corresponding to: (a) $(\alpha, \beta) = (0.1, 0.5)$, (b) $(\alpha, \beta) = (0.01, 1.0)$ and (c) $(\alpha, \beta) = (0.001, 1.5)$. As in Figure 4, we also observe the increasing persistence of mispricings as α decreases.

Model	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	**Excess Kurtosis**	***p*-value in ADF Test**
M1	0.5	0.05	0.87	0.0364
M1	0.1	0.05	3.40	0.5730
M1	0.01	0.05	7.96	0.6103
M2	0.1	0.5	1.11	0.9442
M2	0.01	1.0	3.13	0.9549
M2	0.001	1.5	7.23	0.9460

Table 1: All excess kurtosis values are positive, which is encouraging evidence that both models can replicate the fat tails of empirical returns. The excess kurtosis value of daily S&P500 returns was 20.31, but the difference between that and the values presented here is not a major concern; matching the properties of returns shall be the target of our subsequent parameter estimation in Section 3. As described in Section 1.2, the p-value for the augmented Dickey-Fuller test is interpreted as the probability that the time series has a unit root, and is therefore non-stationary. Sample paths of **(M2)** are non-stationary (to a high degree of confidence) for our parameter choices; this is not necessarily the case for **(M1)**.

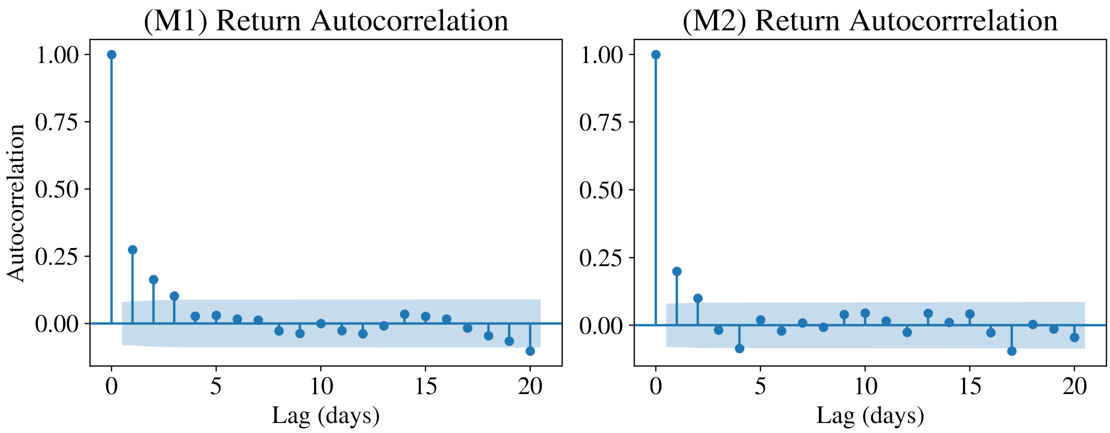

Figure 6: The left correlogram is that of the returns of **(M1)** with parameters $(\alpha, \beta, \sigma_P, \tau, \overline{F}) = (0.01, 0.05, 1, 25, 130)$. The right correlogram is that of the returns of **(M2)** with parameters $(\alpha, \beta, \sigma_P, \mu_F, \sigma_F, k, \tau) = (0.001, 1.5, 1, 0.0005, 0.002, 0.8, 25)$. We note the insignificance of return autocorrelation in both cases.

functions — namely, the hyperbolic tangent function — but such choices yield models which are intractable later down the road of discretisation, simulation and parameter estimation. Having presented the general HAM framework earlier in this section, a natural extension is to include a larger number of strategies, allowing for richer behaviour. One could add mean-reversion strategies as described in Section 2.2. Survey data has shown that real-world traders believe prices to exhibit short-term momentum and long-term mean reversion [13], so strategies of this type have empirical support. For example, a common variant of the contrarian strategy involves buying (resp. selling) when the asset is cheap (resp. expensive) as measured by rolling minima and maxima. An example of a simple demand function $X_{\mathrm{m}}(t)$ that is consistent with this behaviour is as follows,

$$X_{\mathrm{m}}(t) = 1 - 2\left(\frac{P(t) - m(t)}{M(t) - m(t)}\right),$$

where $m(t) := \min\{P(s) : t - \tau \leqslant s \leqslant t\}$ and $M(t) := \max\{P(s) : t - \tau \leqslant s \leqslant t\}$ are the minima and maxima of the price over some look-back period of length $\tau > 0$. Moreover, one could consider a random trading strategy as a strategy in its own right, leading to a time-dependent diffusion coefficient.

Furthermore, one could model the price dynamics arising due to agents with demand functions that have the same form, but different parameters. One could easily extend *any* of the models in Section 2.4 to include multiple look-back periods or decay parameters for the momentum strategy followed by chartists. Consider a model in which there are no fundamentalists but two types of chartists, both of which follow a momentum strategy as in **(M1)** but differ as to their look-back periods, which are given by $\tau_1 > 0$ and $\tau_2 > 0$

respectively. An exemplary model *might* bear resemblance to

$$\mathrm{d}P_t = \lambda\Big(n \tanh\big(\beta_\mathrm{c}(P_t - u_1(t))\big) + (1-n)\tanh\big(\beta_\mathrm{c}(P_t - u_2(t))\big)\Big)\,\mathrm{d}t + \sigma_P\,\mathrm{d}B_1(t),$$

$$\text{where} \quad u_i(t) = \frac{k}{1-\exp(-k\tau_i)}\int_{t-\tau_i}^{t} e^{-k(t-s)}P(s)\,\mathrm{d}s \quad \text{for } i \in \{1,2\},$$

where $\beta_\mathrm{c} > 0$ denote the sensitivity of chartists and n denotes the proportion thereof which follow the momentum strategy which uses $u_1(t)$. Additionally, we have hyperbolic tangent demand functions as a simple example.

2.5.2 Herding

Secondly, **herding** mechanisms have received considerable attention in the HAM literature [20, 34, 41, 42]. Rather intuitively, they model an agents increased likelihood to follow a particular strategy when a sufficiently large proportion of the population also follows that strategy. For example, it has been shown that in the presence of agents that exhibit herding behaviour, the decision to 'follow the crowd' can be rational [43]. In a similar vein to [42], our extension introduces an additional term to (12) which increases the rate at which agents switch to strategies that are more popular, i.e. those which have a larger market fraction, as follows:

$$\mathrm{d}n_h(t) = \beta n_h(t)\Big(\phi n_h(t) + \mathrm{d}U_h(t) - \mathrm{d}\hat{U}(t)\Big)\,\mathrm{d}t \quad \text{for all } h \in \{1,2,\ldots,H\}, \tag{18}$$

where $\phi \geqslant 0$ is an additional parameter quantifying the strength of the herding effect.

3 Parameter Estimation

In this section, we explore the problem of using real-world price data to calibrate and validate the heterogeneous agent models of study, in a Bayesian manner. Following the presentation in [44, Chap. 12], we begin by describing the framework of **approximate Bayesian computation (ABC)**, also known as likelihood-free Markov chain Monte Carlo. Subsequently, we apply an algorithm — namely, ABC rejection — to estimate the parameters for model **(M2)** described in Section 2.4. We conclude with a discussion of the nature of the approximate parameter distributions and the quality of the fitted model.

3.1 Approximate Bayesian Computation

Throughout this section, let $(\mathcal{Y}, \mathcal{A}, \mathbb{P})$ be a probability space, where $\mathcal{Y} \subseteq \mathbb{R}^d$ is a sample space, $\mathcal{A}$ is a σ-algebra and $\mathbb{P}$ is a probability measure. Let $\{M(\theta) : \theta \in \Theta\}$ be a parameterised set of models, i.e. random variables, which one can sample, where $\Theta \subseteq \mathbb{R}^p$ is a parameter space. We assume all of the random variables X used to be absolutely continuous with respect to Lebesgue measure, and let $\pi(X)$ denote their corresponding densities. Let $D \in \mathcal{Y}$ be a fixed data set. We seek to determine the parameters $\theta^* \in \Theta$ such that samples of $M(\theta^*)$ are most consistent with the data D, in some sense.

Define the **likelihood function** to be $\ell(\theta) := \pi(D \,|\, \theta)$, that is, the density associated with probability of observing the data D under the model $M(\theta)$. In the conventional frequentist sense, also known as maximum likelihood, we choose θ^* to maximise $\ell(\theta)$; the 'best' model is that which assigns the highest probability to the observed data. In the Bayesian sense, we consider a *distribution* over parameter space and apply Bayes' theorem, to obtain

$$\pi(\theta \,|\, D) = \frac{\pi(D \,|\, \theta)\pi(\theta)}{\pi(D)} = \frac{\ell(\theta)\pi(\theta)}{\pi(D)}, \tag{19}$$

where $\pi(\theta)$ is referred to as the **prior distribution** and $\pi(\theta \,|\, D)$ is referred to as the **posterior distribution** over Θ. The denominator $\pi(D)$ is simply a normalisation term that shall therefore play a minor part in our estimation; we write $\pi(\theta \,|\, D) \propto \pi(D \,|\, \theta)\pi(\theta)$ for simplicity. Common to both frequentist and Bayesian approaches to inference is the need for explicit computation of the likelihood function $\ell(\theta)$. Due to the inherently complex formulation of the models in Section 2.4 as systems of stochastic delay-differential equations, we have no access to *any* sort of likelihood function. Therefore, we resort to the family of techniques referred to as **approximate Bayesian computation (ABC)**, which circumvent this necessity by using simulations to approximate the likelihood function.[6]

We begin by augmenting a potential parameter sample $\theta \in \Theta$ with a data set $D' \in \mathcal{Y}$ which is sampled from the corresponding model, $D' \sim M(\theta)$. Applying Bayes' theorem

[6] There are similar methods for frequentist inference, e.g. the simulated method of moments (SMM), however we assume the Bayesian vantage point for its simplicity and ease of implementation.

to $\pi(D', \theta \mid D)$ obtains $\pi(D', \theta \mid D) \propto \pi(D \mid D', \theta)\pi(D' \mid \theta)\pi(\theta)$. An approximation for the posterior distribution $\tilde{\pi}(\theta \mid D)$ is obtained by marginalisation over all possible $D' \in \mathcal{Y}$,

$$\tilde{\pi}(\theta \mid D) \propto \pi(\theta) \int_{\mathcal{Y}} \pi(D \mid D', \theta)\pi(D' \mid \theta)\, \mathrm{d}(D').$$

By definition, $\pi(\theta)$ is specified *a priori*, and $\pi(D' \mid \theta)$ is approximated by sampling the model $M(\theta)$, as described. Therefore, we have a choice for $\pi(D \mid D', \theta)$. If it is equality between the approximate and actual posterior distributions that we desire, then the *only* possible choice is, of course, $\pi(D \mid D', \theta) = \delta_D\big(D'\big)$, where $\delta_D(D')$ denotes the point mass centered at $D' = D$. In reality, the probability of sampling data sets D' which are *exactly* the same as the observed data D is very small, and decreases as the dimensionality of the data increases. Following [45], we approximate $\delta_D(D')$ by means of a uniform density, centered at $D' = D$, given by

$$\pi(D \mid D', \theta) \propto \begin{cases} 1 & \text{if } D' \text{ is similar to } D, \\ 0 & \text{otherwise.} \end{cases}$$

This particular choice underpins one of the most basic ABC techniques — namely, **ABC rejection sampling** — which performs estimation in the following way. Given a prior distribution $\pi(\theta)$, we first sample a parameter $\theta' \sim \pi(\theta)$, which is used to sample a data set $D' \sim M(\theta')$. If the generated data set D' is sufficiently similar to the observed data D, then θ' is accepted as a sample from the posterior distribution $\pi(\theta \mid D)$; otherwise θ' is rejected. This process is repeated until a adequately large number of samples are obtained.

Measuring the similarity between D' and D is not trivial and there are many approaches in the literature [44]. We define a **statistical distance** $\rho : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_{\geqslant 0}$ to be any function that measures the distance between two random objects or samples thereof.[7] We also introduce a **tolerance** $\epsilon \geqslant 0$ and take

$$\pi(D \mid D', \theta) \propto \begin{cases} 1 & \rho(D', D) \leqslant \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

In a simple world, one might choose ρ to be the Euclidean metric on $\mathcal{Y} \times \mathcal{Y}$ — that is, $\rho(D', D) = \|D' - D\|_2$ — however, real-world data is often high-dimensional and the probability of sampling data sets which are similar to the observed data is therefore quite small. In order to lift this curse of dimensionality, we can replace direct comparisons between D and D' with comparisons of their salient features in a lower-dimensional space. Let $T : \mathcal{Y} \to \mathcal{Z}$ be a map which captures some important properties of the data, e.g. summary statistics, where we have $\dim(\mathcal{Z}) \leqslant \dim(\mathcal{Y})$. Another natural choice is $\rho(D', D) = \|T(D') - T(D)\|$. In order to *guarantee* theoretically that there is no addi-

[7] Naturally, implicit in this definition is the requirement that $\rho(x, x) = 0$ for all $x \in \mathcal{Y}$.

tional error introduced by performing such an approximation, one must require that $T(D)$ be a **sufficient statistic** [46], meaning that no additional information about θ can be gleamed from the data D than is contained in $T(D)$ [47]. This subtle consideration is given relatively little attention in contemporary applications of ABC, which has had great success using (potentially) insufficient statistics [45]. The complete algorithm is presented in Algorithm 1.

Algorithm 1 ABC rejection sampling

Input: Prior distribution $\pi(\theta)$, tolerance $\epsilon > 0$, number of samples $n \geqslant 1$, data $D \in \mathcal{Y}$ and a statistical distance $\rho : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_{\geqslant 0}$
Output: n samples from the approximate posterior distribution

```
i ← 0
while i < n do
    done ← false
    while done = false do
        Sample a parameter vector θ′ from the prior distribution, θ′ ← π(θ)
        Sample a dataset D′ ∈ 𝒴 from the model M(θ′)
        if ρ(D′, D) ⩽ ϵ then
            Accept θ′ as a sample from π̃(θ | D)
            done ← true
        end if
    end while
    i ← i + 1
end while
```

Before applying the ABC rejection algorithm to estimate the model parameters, we must be precise about how we compare simulated and observed price data. In particular, we now detail two choices for ρ in the above setup to characterise different types of agreement between the data sets.

3.2 Statistical Distances

We seek to define $\rho(D', D)$ in such a way that the approximate posterior distributions from ABC rejection are consistent with the observed price data. Firstly, let $D = \{P_t\}_{t=0}^{T}$ be a data set of real-world prices, for example, the price of the S&P 500 market index over some time period. Similarly, let $\theta' \sim \pi(\theta)$ be a parameter sampled from the prior distribution and let $D' = \{P'_t\}_{t=0}^{T}$ denote a set of simulated price data from the model $M(\theta')$. In our case, each model from Section 2.4 is a parameterised system of SDDEs; we consider a sampled data set D' from one of these models to be a discretised sample of the price variable only, considering all other variables to be unobserved. This is well justified since we generally only have readily-available price data to compare with simulations.

For each data set and time $t \in \{1, 2, \ldots, T\}$, define the **return** to be $R_t := P_t / P_{t-1} - 1$. Similarly, we define the **returns distribution** for each data set (D or D') to be the empirical distribution obtained from the set of returns $\{R_t\}_{t=1}^{T}$. Specifically, for the data set D (resp. D'), we assume the returns to be independent and identically-distributed samples of some random variable R (resp. R') with support given by those samples.

With a view towards a simple distance function, comparing sample moments of the returns distribution is indeed a good place to start. However, in light of recent criticism of this very approach [48], we instead present two statistical distances based on the first Wasserstein metric.

(D1) First Wasserstein distance between returns distributions.

The first statistical distance we consider directly compares the returns distributions of the simulated and observed price data, by computing the first Wasserstein distance between them. This is a norm on the space of all such probability distributions. In a similar vein to the definition of the L^p norms on function spaces, the Wasserstein distance can be defined for any $p \geqslant 1$; however, we take $p = 1$ for simplicity, giving the following definition [49],

$$\rho_W(D', D) := \inf \Big\{ \mathbb{E}\big[|x - y|\big] : (x, y) \sim \Gamma \quad \text{where} \quad \Gamma \in \Pi\big(R', R\big) \Big\}, \tag{20}$$

where $\Pi\big(R', R\big)$ is the set of joint distributions with marginal distributions equivalent to the distributions of R' and R. This metric is also known as the **Earth mover's distance** and is implemented and solved as a linear program; we refer the reader to the wider literature for rigorous details of the solution method [50, 51].

(D2) First Wasserstein distance between backtest returns distributions.

To introduce the second statistical distance of consideration, we note that returns — either observed or simulated — are those obtained by following a so-called **buy-and-hold** strategy. For this strategy, agents hold a fixed number of units of the risky asset, thus the returns of the portfolio are equivalent to the price returns of the asset itself. In practice however, we are often interested in analysing and testing the behaviour of a particular trading strategy, whereby the number of units held can change over time [52]. Hence, it is perhaps more important to accept parameters which give consistency between the returns distributions corresponding to the particular strategy (or strategies) of interest. This degree of similarity between these two distributions can be measured by the first Wasserstein distance, as in **(D1)**. Let $S(D')$ and $S(D)$ denote the **backtest values**, defined as the resulting portfolio value obtained by using a particular strategy of interest to trade the risky asset with price given by D' and D respectively. Then, the Wasserstein distance between the backtest returns is given by $\rho_S(D', D) := \rho_W(S(D'), S(D))$. To exemplify the approach in a straightforward way, we use a simple moving average strategy, as described in Section 2.2, with an arbitrary 25-day lookback period.

3.3 Parameter Estimation Results

In this section, we detail the results of using Algorithm 1 to fit Model **(M2)** to the daily price data for the S&P 500 market index between 1980 and 2021. To this end, one must first specify a prior distribution over parameter space. The model **(M2)** has seven parameters, the prior distributions of which we take to be independent and be specified as follows:

$$\begin{cases} \alpha \sim \mathrm{Unif}(0.0, 2.0), \\ \beta \sim \mathrm{Unif}(0.0, 2.0), \\ \sigma_P \sim \mathrm{Unif}(0.0, 2.0), \\ \mu_F \sim \mathrm{Unif}(0.0, 1.0), \\ \sigma_F \sim \mathrm{Unif}(0.0, 1.0), \\ k \sim \mathrm{Unif}(0.0, 2.0), \\ \tau \sim \mathrm{Unif}\{1, 2, \ldots, 99\}. \end{cases}$$

Since we have no information *a priori* about the possible parameter distributions, we take them all to have a uniform distribution over some appropriate interval. We represent parameter vectors as $\theta = \left(\alpha, \beta, \sigma_P, \mu_F, \sigma_F, k, \tau\right)$ and the parameter space $\Theta \subseteq \mathbb{R}^7$ in which they exist is given by $\Theta = (0,2)^3 \times (0,1)^2 \times (0,2) \times \{1, 2, \ldots, 99\}$.

To approximate the posterior distributions given this prior, we execute the ABC rejection algorithm twice, once for each statistical distance function in Section 3.2. Figures 7 and 8 show the marginal histograms of the accepted parameters in the approximate posterior distribution for **(D1)** and **(D2)** respectively. It is perhaps important to note that we applied a slightly perturbed version of Algorithm 1 to derive these results: for each statistical distance, the prior distribution was sampled $n = 2500$ times, a sample path of **(M2)** was simulated under these parameters, and the corresponding distance between the sample path and the S&P500 was computed. Whereas Algorithm 1 would take the tolerance $\epsilon \geqslant 0$ to be fixed *a priori*, our implementation saves all samples and their corresponding statistical distances such that ϵ was set *ex post* to obtain a particular target **acceptance rate**.

It is clear from the results that **(D2)** had a much more profound impact on the prior distributions than **(D1)**. We hypothesise that this is due to the fact that the observed price data, the S&P 500 in this case, behaves very similarly to **(M2)** for a large number of $\theta \in \Theta$. Applying the simple moving average strategy for **(D2)** introduces an additional constraint that restricts the parameter space more significantly.

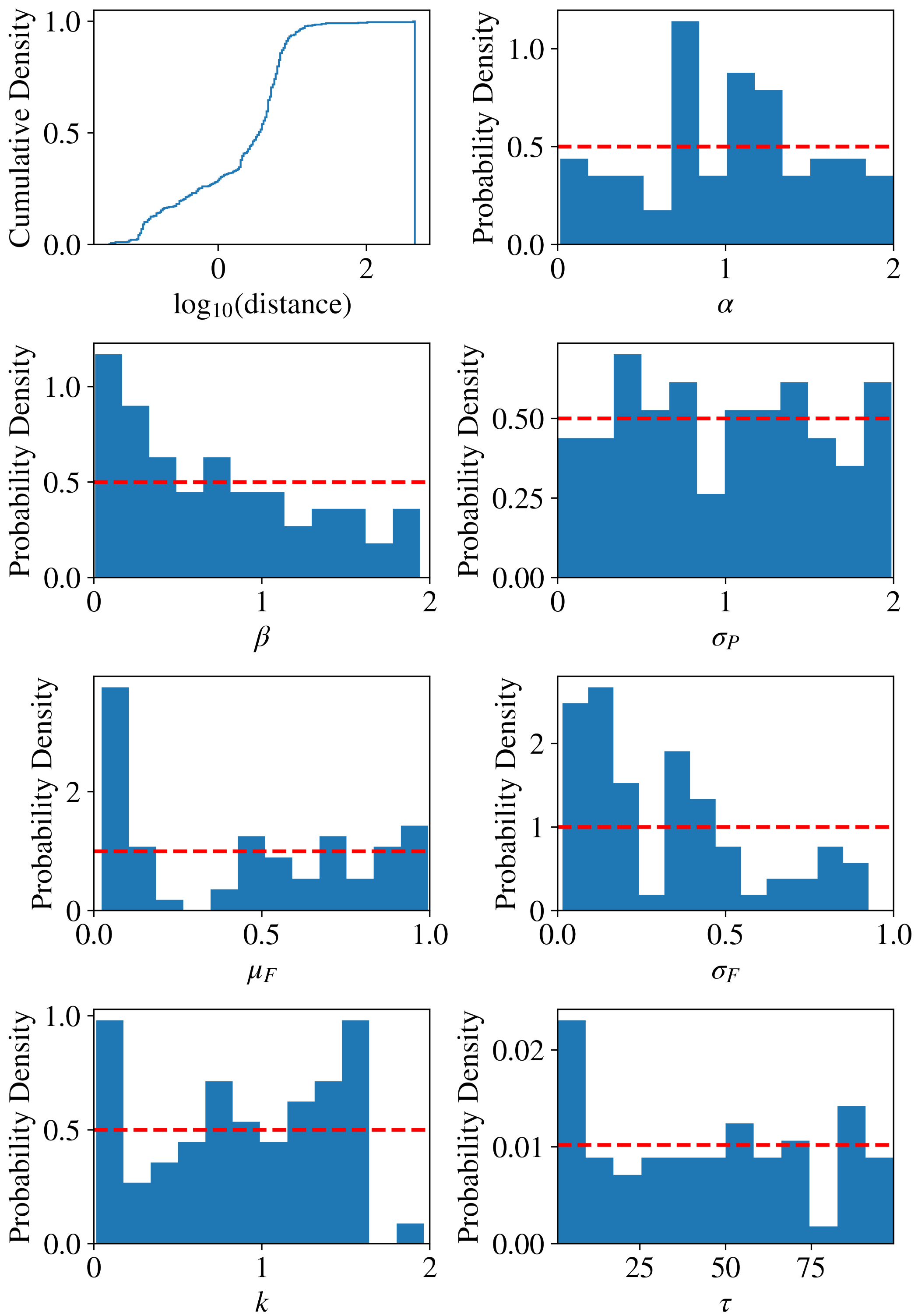

Figure 7: ABC results for **(M2)** with uniform priors, and Wasserstein distance **(D1)** between returns distributions as rejection criterion. The Figure shows marginal histograms for each parameter of $(\alpha, \beta, \sigma_P, \mu_F, \sigma_F, k, \tau) \in \Theta$ with $\epsilon = 0.3$ giving an acceptance rate $\approx 3\%$.

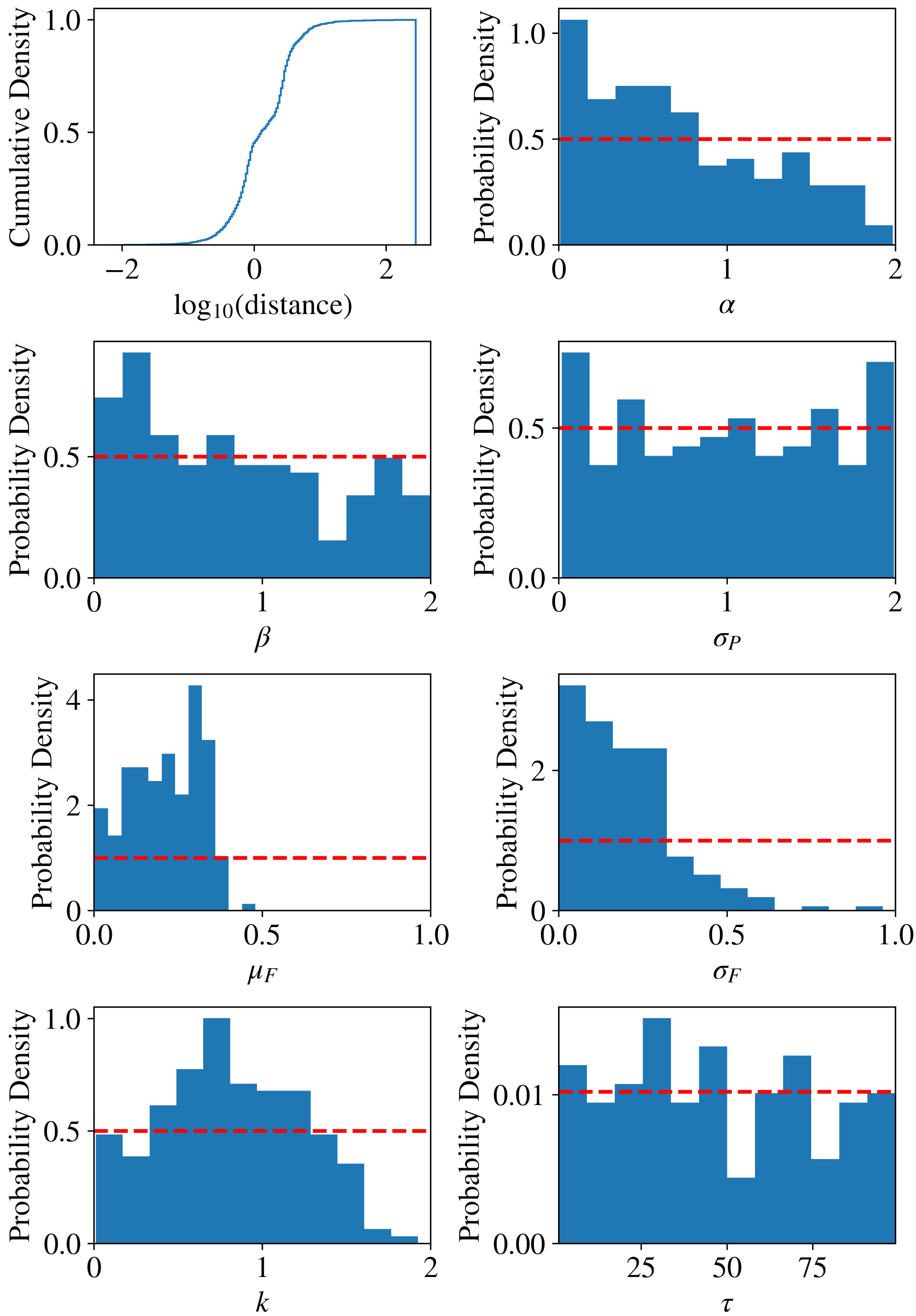

Figure 8: ABC results for **(M2)** with uniform priors, and Wasserstein distance **(D2)** between returns distributions of momentum strategy as rejection criterion. The Figure shows marginal histograms for each parameter of $(\alpha, \beta, \sigma_P, \mu_F, \sigma_F, k, \tau) \in \Theta$ with $\epsilon = 0.35$ giving an acceptance rate $\approx 8\%$.

4 Conclusion

The human mind abhors the emptiness created by its inability to understand.
— P. De Grauwe and M. Grimaldi [53]

In this report, we presented a state-of-the-art mathematical framework for building realistic heterogeneous agent-based models of asset pricing. These models are general, and infinitely extensible, providing the researcher with a rapid and convenient means to translate agent behaviour into a dynamical description of price action. Additionally, we detailed, implemented and applied a modern technique of approximate Bayesian computation, in order to perform parameter estimation for a reasonably complex model **(M2)**. We have shown that these models agree with, and can fit to, observed price data; therefore, we are confident that they provide a formidable alternative framework for modelling asset prices. This is especially the case in situations where disagreements between the real-world and the predictions of simpler models, e.g. geometric Brownian motion, manifest themselves as the stylised facts, which are a set of omnipresent features of financial time series that embarrass these conventional models and their underlying assumptions. Mainstream finance dismisses these stylised facts as 'anomalies' [5]; however, we have demonstrated how they can be modelled as the result of the aggregation of individual trading decisions in an agent-based setup. Finally, in terms of implementation, this research project has yielded a solid codebase which will lend impetus to future research efforts.

4.1 Directions for Future Research

We now conclude this report by detailing and discussing five key axes that demand future work and attention.

(1) Firstly, from a theoretical perspective, it is crucial that further efforts are made to reconcile the market microstructure literature, with the linear price update rule presented in our framework. The fact that update rule has been shown to be compatible with data gives rise to questions of boundedly-rational market makers, in *addition to* boundedly-rational traders. This will dramatically help to increase the validity of the models, the *ad hoc* nature of which has been criticised by some authors [5]. In a similar direction, further work to consolidate the functional forms used for agents' demands, with some model of agents' beliefs could also give credibility to the existing modelling work which dispensed with any such worries.

(2) Furthermore, one interesting avenue of further research would be to leverage recent developments in optimal control theory for systems of SDDEs [54, 55] to answer questions about optimal trading or portfolio allocation in an continuous-time HAM with fundamentalists and chartists. To an extent, this has already been done in [32]; however, the model the authors presented is closest to **(M1)** — the simplest of the models presented in Section 2.4 — and there is room for further novel developments.

(3) From a numerical perspective, we also argue that additional research needs to be conducted as to the discretisation and stability properties of SDDEs, since Euler-Maruyama schemes are not available in general [39]. As discussed in Section 2.2, introducing nonlinearity is a natural axis of extension of the models presented in this report; a lack of a concrete theoretical guarantee or verification process to ensure that results obtained using these equation are indeed valid, i.e. stable, is a current barrier to the further research using nonlinear SDDEs.

(4) In terms of parameter estimation schemes, we elected to use ABC rejection in this report, with a view towards parsimony and concision; however, this algorithm is among one of *the* slowest ABC algorithms. Further research should be done to consider more efficient sampling algorithms, such that more complicated models can be estimated. Additionally, one should extend the empirical fitting to prices of assets other than the S&P500.

(5) Finally, a key question that needs to be addressed is that of the *correct* level of model complexity required to replicate the stylised facts to a reasonable degree of accuracy, whilst also being simple enough to be analytically-tractable. The astute reader might recall that this was our definition of the "*holy grail* of economic dynamics" in Section 1. The results obtained during the course of this research project have provided a faint breeze of intuition towards what will hopefully become a concrete answer to that question in the future. In particular, recent developments in Bayesian model selection algorithms [56] provide an interesting route of extension.

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A Single-period Inter-temporal Choice

Although most of the modelling work does not provide rigorous derivations from microeconomics to justify its choices of expectations, this does not mean we cannot attempt to develop them. We briefly show that this is possible and provide some insight into the modelling conclusions. Following Section 2, consider the behaviour of an individual agent with wealth $(W_t : t \in \mathcal{I})$ who invests a proportion $(\pi_t : t \in \mathcal{I})$ of said wealth in the risky asset. Suppose that $\mathcal{I} = [0, T]$ with $T > 0$. We present the continuous-time formulation for no other reason than it is easier to discretise a continuous-time equation than to take the continuous limit of a difference equation. The goal of this section is not to solve the optimal allocation problem exactly; indeed, this is seldom possible in closed form. Instead, we sketch an informal derivation of an approximate optimal allocation π_t^* to which we appeal in subsequent modelling efforts. We are trying to gain intuition about agents' strategies, the structure of which plays an important role in the price dynamics.

Much of the literature *begins* by assuming a description of the price process, often in the form of an SDE. We assume that agents believe their own wealth process to be an Itô process and shall use results from this discussion to derive the price dynamics. Suppose each agent believes their wealth process to satisfy,

$$\mathrm{d}W_t = \mu(W_t)\,\mathrm{d}t + \sigma(W_t)\,\mathrm{d}B_t, \tag{21}$$

where $(B_t)_{t \geqslant 0}$ denotes the standard one-dimensional Wiener process which is assumed to exist in the probability space and be adapted to the filtration $(\mathcal{F}_t)_{t \geqslant 0}$.

The models in Section 2 take the form of stochastic delay-differential equations (SDDEs); these processes are not generally Markovian, and thus neither is the resulting wealth process, as we have implicitly assumed here. One might expect this inconsistency to invalidate our subsequent derivation. However, one can show that the dynamics of an SDDE system can be well-approximated by a Markov process when the delay is *small*[1] relative to the time horizon T [57]. For our purposes, this assumption holds true, and thus we consider the above process to be sufficiently accurate.

To quantify the preferences of the agent, let $u \in C^2(\mathbb{R})$ be an invertible, strictly concave **utility function**. Following [33], suppose the agent invests a proportion π_t of their wealth W_t in the risky asset, at time $t \in \mathcal{I}$. The agent seeks to choose π_t to maximise her expected utility of wealth at some time point $t + \Delta t$ in the future, given by $\mathbb{E}_t[u(W_{t+\Delta t})]$. Recalling that S_t denotes the asset price, let $\rho_{t+\Delta t}$ denote the **simple return** over the time period $[t, t + \Delta t]$. Suppose that the agent **assumes $\boldsymbol{\rho_{t+\Delta t}}$ is conditionally Gaussian**, when

[1] This reduces to performing a Taylor expansion to yield a stochastic Langevin system. The approach can also be used to approximate the time-dependent density by the Fokker-Planck equation.

conditioned on $\mathcal{F}_t$,

$$\rho_{t+\Delta t}\big|_{\mathcal{F}_t} := \frac{S_{t+\Delta t} - S_t}{S_t} \quad \text{and} \quad \rho_{t+\Delta t}\big|_{\mathcal{F}_t} \sim \mathcal{N}\Big(\mathbb{E}_t[\rho_{t+\Delta t}], \mathrm{Var}_t[\rho_{t+\Delta t}]\Big).$$

This crucial assumption will prove key to approximating the expected utility and the optimal choice of π_t. Equivalently, let $(\xi_t : t \in \mathcal{I})$ denote a set of independent, identically-distributed $\mathcal{N}(0, \Delta t)$ random variables then we can write $\rho_{t+\Delta t}$ as

$$\rho_{t+\Delta t}\big|_{\mathcal{F}_t} = \mathbb{E}_t[\rho_{t+\Delta t}] + \sqrt{\mathrm{Var}_t[\rho_{t+\Delta t}]}\xi_t, \quad \text{where} \quad \xi_t \sim \mathcal{N}(0, \Delta t) \quad \text{for all } t.$$

We now approximate the wealth dynamics under these assumptions. The first-order Euler-Maruyama discretisation (see [40, Sec. 5.2]) of (21) is given by

$$W_{t+\Delta t} - W_t \approx \mu(W_t)\Delta t + \sigma(W_t)\xi_t.$$

Assuming a constant allocation on $[t, t+\Delta t]$, that is, $\pi_s = \pi_t$ for all $s \in [t, t+\Delta t]$, then the change in wealth is simply given as the sum of interest on non-invested wealth and capital gains on the invested wealth, as follows:

$$\begin{aligned} W_{t+\Delta t} - W_t &\approx r(1-\pi_t)W_t\Delta t + \rho_{t+\Delta t}\pi_t W_t, \\ &= r(1-\pi_t)W_t\Delta t + \Big(\mathbb{E}_t[\rho_{t+\Delta t}] + \sqrt{\mathrm{Var}_t[\rho_{t+\Delta t}]}\xi_t\Big)\pi_t W_t, \\ &= \Big(r(1-\pi_t)\Delta t + \pi_t\mathbb{E}_t[\rho_{t+\Delta t}]\Big)W_t + \pi_t W_t\sqrt{\mathrm{Var}_t[\rho_{t+\Delta t}]}\xi_t. \end{aligned}$$

Therefore, at time t, the coefficients of the wealth process are approximated by,

$$\begin{cases} \mu(W_t) \approx r(1-\pi_t)W_t\Delta t + \pi_t\mathbb{E}_t[\rho_{t+\Delta t}]W_t, \\ \sigma(W_t) \approx \pi_t W_t\sqrt{\mathrm{Var}_t[\rho_{t+\Delta t}]}. \end{cases} \tag{22}$$

Towards optimising the *utility of* wealth, let $X_t := u(W_t)$. We have assumed u to be invertible (which is reasonable given von Neumann-Morgenstern utility theory [27]) so let v denote the inverse function such that $W_t = v(X_t)$. Since $u \in C^2(\mathbb{R})$, we can apply Itô's lemma to obtain an SDE for $(X_t : t \in \mathcal{I})$.

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}(u(W_t)) = u'(W_t)\,\mathrm{d}W_t + \frac{1}{2}u''(W_t)\,\mathrm{d}\langle X_t\rangle \\ \langle X_t\rangle &= \int_0^t \sigma(W_s)^2\,\mathrm{d}s \implies \mathrm{d}\langle X_t\rangle = \sigma(W_t)^2\,\mathrm{d}t \\ \mathrm{d}X_t &= u'(W_t)\Big(\mu(W_t)\,\mathrm{d}t + \sigma(W_t)\,\mathrm{d}B_t\Big) + \frac{1}{2}u''(W_t)\sigma(W_t)^2\,\mathrm{d}t \\ &= \Big(u'(W_t)\mu(W_t) + \frac{1}{2}u''(W_t)\sigma(W_t)^2\Big)\,\mathrm{d}t + u'(W_t)\sigma(W_t)\,\mathrm{d}B_t \end{aligned}$$

$$= \underbrace{\left(u'(v(X_t))\mu(v(X_t)) + \frac{1}{2}u''(v(X_t))\sigma(v(X_t))^2\right)}_{\tilde{\mu}(X_t)} \mathrm{d}t + \underbrace{u'(v(X_t))\sigma(v(X_t))}_{\tilde{\sigma}(X_t)} \mathrm{d}B_t$$

$$\mathrm{d}X_t = \tilde{\mu}(X_t)\,\mathrm{d}t + \tilde{\sigma}(X_t)\,\mathrm{d}B_t$$

Therefore, we have shown that (X_t) is also an Itô process, and the same first-order discretisation as above yields:

$$\begin{aligned}
X_{t+\Delta t} &\approx X_t + \tilde{\mu}(X_t)\Delta t + \tilde{\sigma}(X_t)\xi_t \\
\mathbb{E}_t[X_{t+\Delta t}] &\approx X_t + \tilde{\mu}(X_t)\Delta t + \tilde{\sigma}(X_t)\mathbb{E}_t[\xi_t] \quad \text{by measurability of } X_t \\
&= X_t + \tilde{\mu}(X_t)\Delta t \quad \text{since } \mathbb{E}_t[\xi_t] = 0 \\
\mathbb{E}_t[u(W_{t+\Delta t})] &\approx u(W_t) + \left(u'(W_t)\mu(W_t) + \frac{1}{2}u''(W_t)\sigma(W_t)^2\right)\Delta t
\end{aligned}$$

Using the above approximations (22), we estimate the expected utility of wealth, at time $t + \Delta t$, as a function of π_t:

$$\mathbb{E}_t[u(W_{t+\Delta t})] \approx u(W_t) + u'(W_t)\Big(r(1-\pi_t)\Delta t + \pi_t\mathbb{E}_t[\rho_{t+\Delta t}]\Big)W_t + \frac{\Delta t}{2}u''(W_t)\pi_t^2 W_t^2 \mathrm{Var}_t[\rho_{t+\Delta t}].$$

The first-order optimality condition for the right-hand side, taken with respect to π_t is given by

$$\left.\frac{\mathrm{d}\mathbb{E}_t[u(W_{t+\Delta t})]}{\mathrm{d}\pi_t}\right|_{\pi_t=\pi_t^*} = 0 \implies u'(W_t)\Big(\mathbb{E}_t[\rho_{t+\Delta t}] - r\Delta t\Big) + \pi_t^* u''(W_t)\mathrm{Var}_t[\rho_{t+\Delta t}]W_t\Delta t = 0,$$

which gives

$$\pi_t^* = -\frac{u'(W_t)}{u''(W_t)W_t\Delta t} \times \frac{\mathbb{E}_t[\rho_{t+\Delta t}] - r\Delta t}{\mathrm{Var}[\rho_{t+\Delta t}]}.$$

From the assumption of strict concavity — that is, $u'' < 0$ — we confirm that π_t^* is indeed a maximum point, by the second-derivative test:

$$\frac{\mathrm{d}^2\mathbb{E}_t[u(W_{t+\Delta t})]}{\mathrm{d}\pi_t^2} = u''(W_t)\mathrm{Var}_t[\rho_{t+\Delta t}]W_t\Delta t < 0.$$

Without loss of generality, we can scale time such that $\Delta t = 1$. To conclude this section, we consider how this approximate optimal allocation behaves among three popular choices for the utility function u.

Constant absolute risk aversion (CARA). The first example is the exponential utility function, with coefficient of risk aversion $\gamma > 0$, given by $u(w) = -\exp(-\gamma w)$, which implies

$$\pi_t^* = \frac{\mathbb{E}_t[\rho_{t+1}] - r}{\gamma W_t \mathrm{Var}[\rho_{t+1}]}.$$

Constant relative risk aversion (CRRA). This utility function is given by $u(w) = w^\gamma/\gamma$ for a coefficient of risk aversion $\gamma \in (0, 1)$, which yields

$$\pi_t^* = \frac{\mathbb{E}_t[\rho_{t+1}] - r}{(1-\gamma)\mathrm{Var}[\rho_{t+1}]}.$$

Furthermore, in the case $\gamma \to 0^+$, L'Hôpital's rule yields a logarithmic utility function, $u(w) = \log(w)$, in which case we obtain

$$\pi_t^* = \frac{\mathbb{E}_t[\rho_{t+1}] - r}{\mathrm{Var}_t[\rho_{t+1}]}.$$

The agents allocation problem was parameterised in terms of the proportion of wealth that the agent seeks to invest in the risky asset; we have an approximately-optimal proportion π_t^* in terms of the expected excess return of the asset over the risk-free rate and the variance of said return, with a constant that depends on the wealth and utility function. We could equally have chosen to formulate the optimisation problem in terms of the absolute number of units of the risky asset (e.g. shares) the agent demands. It is also interesting to note that this result corresponds directly to the framework for systematic trading presented in [52, Chap. 7], whereby portfolio allocations are determined proportional to some estimation of expected excess return and inversely proportional to the volatility of said returns.

B Stability of the Beja-Goldman Model

We first rewrite the BG model as a linear system,

$$\begin{aligned}
\frac{\mathrm{d}P(t)}{\mathrm{d}t} &= \lambda\alpha F - \lambda\alpha P(t) + \lambda\beta\pi(t), \\
\frac{\mathrm{d}\pi(t)}{\mathrm{d}t} &= \eta\lambda\alpha F - \eta\lambda\alpha P(t) + \eta(\lambda\beta - 1)\pi(t).
\end{aligned}$$

The Jacobian matrix is given by $\mathcal{J}$ and the eigenvalues are given by the roots of the characteristic polynomial of $\mathcal{J}$.

$$\mathcal{J} = \begin{bmatrix} -\lambda\alpha & \lambda\beta \\ -\lambda\alpha\eta & (\lambda\beta - 1)\eta \end{bmatrix}$$

The eigenvalues $x \in \mathbb{R}$ are the roots of the the characteristic polynomial,

$$x^2 + (\lambda\alpha + \eta(1 - \lambda\beta))x + \lambda\alpha\eta = 0.$$

Since we have asumed $\lambda > 0$, $\alpha > 0$ and $\eta > 0$, the eigenvalues have the same sign; hence, a sufficient condition for stability of the fundamental equilibrium is $\lambda\alpha + \eta(1 - \lambda\beta) > 0$, since the negative sum of the roots — which we need to be negative to obtain stability — is given as the coefficient of x in this characteristic polynomial.