Report for Diffusion Process

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1 Introduction

Every particle has some kinetic energy stored in it in form of vibration. When two particles having different kinetic energy and hence difference temperatures come in contact with each other, heat energy is transferred from particle having higher energy to lower energy.

Our problem statement is as follows. We are given a think metal bar. It has constant application of heat and cold at some locations on bar known as hotSites and coldSites. We want to model the diffusion process in this bar and create gif's(animation) for it.

2 The model

For the sake of simplicty, we assume that the metal bar is so thin that the temperature of any point perpendicular to the surface has the same temperatue as that of the surface. We do this assumption so that we can convert our problem from 3D to 2D. For this 2D model, we represent the cross section of the bar as a N * M grid where each cell shows the temperature of that area and also the entire cube situated below it.

With time, the temperature of cell changes depending on it's orignal value and value of it's neighbours. If it depends on 4 neighbours, it is Von Neumann model. If it depends on 8 neighbours it Moore model.

Heat diffusion between any two objects can be modeled by Newton's law of heat transfer. It is as follows.

$$\frac{dQ}{dt} = kA\Delta T = kA(T - T_{environment})$$

where ΔT = difference in temperature of object and surrounsings, k is a heat constant, A is area of contact, T is temperature of cell, $T_{environment}$ is temperature of environment. Also, the change of temperature in Δt of a cell over time depends on the temperature difference of the neighbour's temperature and cell's temperature.

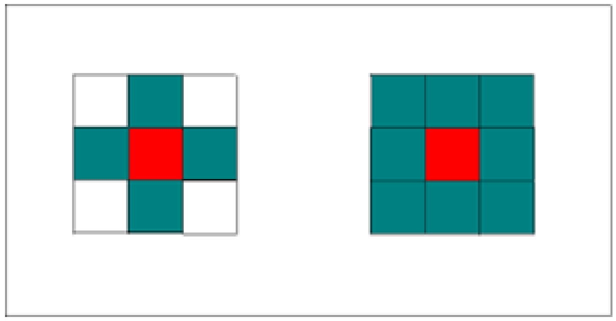

Figure 1: Four neighbour(Von Neumann) model and Eight neighbour(Moore) model

Mathematically writing the equations for the same,
$\Delta Site = r\Sigma(neighbor(i) - Site)$,
where $0 < 8 * r < 1$
$0 < r < 1/8 = 0.125$
Thus at $t + \Delta t$ time temperature is

$$Site(t + \Delta t) = Site(t) + r\Sigma(neighbour(i) - Cell(t))$$

$$Site(t + \Delta t) = (1 - 8 * r)Site(t) + r\Sigma neighbour(i)$$

For a grid of infinite length and width which is isolated from the environment, we can prove that the total temperature of the grid remains constant at each time.
Let define $Site_{ij}$ to be the change in temperature of of cell i due to the neighbour cell j.

$$Change_{ij} = r(Site_i - Site_j)$$

$$Change_{ji} = r(Site_j - Site_i)$$

$$Change_{ij} + Change_{ji} = 0$$

Here we can see that the change of temperature of cell i due to cell j cancels out the change of temperature of cell j due to cell i. So the net change in temperature in whole grid eventually becomes 0. Thus total temperature of grid is remains constant. As energy is proportional to the temperature, total energy is also conserved.

Another model is to apply a weighted sum / filter model for heat diffusion. In that, the temperature of the cell at next time instant depends on it's current temperature and neighbour's current temperature as follow:

$$Site_{t+1} = aSite_t + b_1Neighbour1_t + b_2Neighbour2_t + ... + b_NNeighbourN_t$$

Here it is necessary that $a + b_1 + b_2 + ... + b_N = 1$ for system to conserve total temperature and hence to satisfy law of conservation of energy.

3 Boundary conditions

We define a diffusion function for which parameters are temperatures of 8 neighbours. At boundary cells the absence of neighbourhood cells can cause problems while simulation. Thus to apply diffusion function what we do is extend boundaries by one cell in each of the four directions. We call these cells as ghost cells. We can capture the information of surroundings using these ghost cells. There are three different boundary conditions which can be used to model different type of situations. When to apply which bounday conditions depends on the physical system we are trying to model. These three boundary conditions are as follows:

3.1 Absorbing boundary condition

Here in absorbing boundary condition we apply ambient temperature(25C) on ghost cells which are on all four boundary.

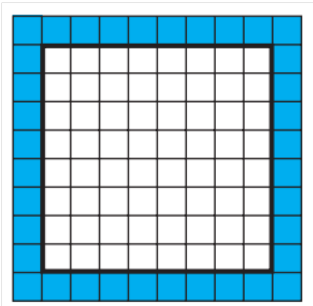

Figure 2: Absorbing boundary condition

Here blue cells denote ghost cells which are at ambient temperature(25C). The white cells represent the inner grid cells which which we are interested in. These type of boundary condition is useful to model systems where an object is kept in environment whose temperature is going to remain constant throughout the process.

3.2 Reflecting Boundary Condition

In this model, we give every extended boundary cell the value of its immediate neighbor which is closest to it. Thus first two rows of extended grid have same temperature. In the case of the spread of temperature, the boundary tends to propagate the current local situation: The air in the room is still, and the air temperature around the bar tends to mimic the temperature of the bar. This type of boundary condition is used to model isolated systems i.e. when the system doesn't interact with the environment.

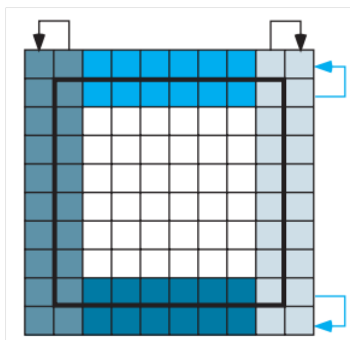

Figure 3: Reflecting boundary condition

3.3 Periodic Boundary Condition

In this case, We create a cylinder from grid by wrapping it around North and South and from it we create a toroid by wrapping around West and East.So on the north side ghost cell will be copy of the south boundary, same thing will apply on the east and west side boundary. So the ghost cells in this case will be the cells of the grid as shown below.Thus in this model first row is the same as last to second row in extended bar. This type of condition is useful to model systems where one boundary depends on another boundary.

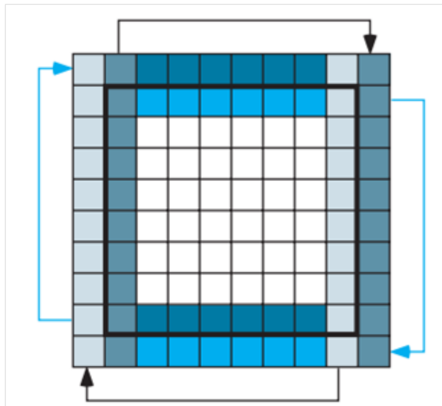

Figure 4: Periodic boundary condition

4 Simulation for above boundary conditions

Now comes the simulation part. For that we used a 20 X 20 grid and gave position of hot and cold source in grid known as hotSites and coldSites. Here is the basic configuration for simulation. We have done several simulation using various boundary condition. We have done simulation for all boundary condition to see the effect of the boundary condition on heat diffusion otherwise the boundary condition for the model is kept to fixed according to the characteristic of the situation. Here this figure is basic configuration for

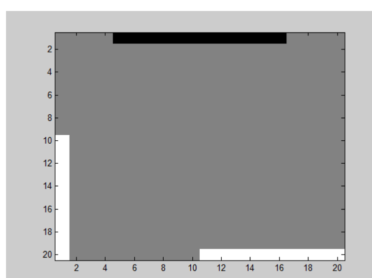

Figure 5: Basic configuration for simulation

simulation here white cells are hot cells and black cells are cold cells while other cells are at ambient temperature.

Here are some instance of the simulation for different boundary condition.

4.1 Results for Absorbing Boundary Condition

For simulating the absorbing boundary condition we remove the hot and cold sources after t=0 time, so hot and cold source will apply only at t=0 time. Here we are using the absorbing boundary condition so this system will stable at ambient temperature after some amount of time.so each cell in bar will at ambient temperature. Below figure is the final state of the simulation.
Here we simulate the model for diffusion constant 0.1 then system will be stable after 128 amount of time and for diffusion constant 0.05 system will be stable after 204 amount of time.

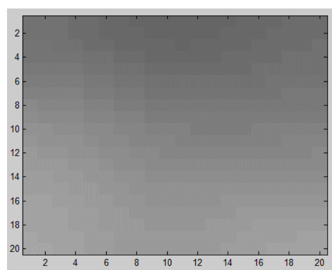

Figure 6: Equilibrium state for Absorbing Boundary Condition

4.2 Results for Reflecting Boundary Condition

For the Reflecting Boundary, if we take the value of a diffusion constant to be 0.10 unit, the system gets the stability after 221 units of time. And for the diffusion constant 0.05 unit, the system stabilized at 321 unit of time.The stabilized grid at equilibrium looks as follows:
Below figure is the intermediate instance if the simulation for the model of the reflecting boundary condition. Here all cell will not be stable at same temperature.

4.3 Results for Periodic Boundary Condition

For the Periodic Boundary, if we take the value of rate of diffusion constant r to be 0.10 unit, the system gets the stability after 195 units of time. And

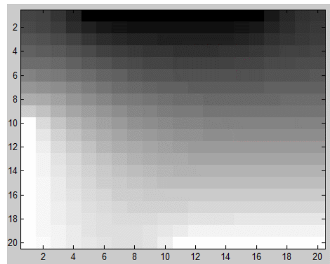

Figure 7: Equilibrium state for Reflecting Boundary Condition

for the diffusion constant r = 0.05 unit, the system stabilized after 285 units of time. The stabilized grid at equilibrium looks as follows:.

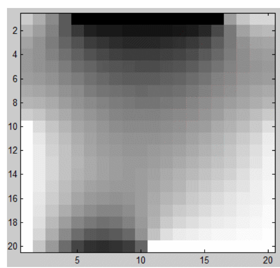

Figure 8: Equilibrium state for Periodic Boundary Condition

4.4 Filter Model

Filter model is another way to model the diffusion process. Here, instead of taking the difference of a sites' temperature with its neighbours we take a weighted sum of the temperature of the given site and its neighbours to determine the temperature at next time instant. The formula for the same is described as follows:

$$site_{t+1} = w * site_t + \Sigma w_i * neighbour_i$$

such that $w + \Sigma w_i = 1$

The weights can be represented as 3X3 matrix. The weights for the case of a filter that is symmetric in north, south, east, west and north-east, north-west, south-east, south west respectively is as shown:

6.25%	12.5%	6.25%
12.5%	25%	12.5%
6.5%	12.5%	6.25%

A bar of 0 degree unit is kept in 25 degree unit temperature environment with absorbing boundary condition and following filter. Here the diffusion from east to west is quicker than west to east and so we get such an animation.

10%	13.5%	6%
10%	25%	6%
10%	13.5%	6%

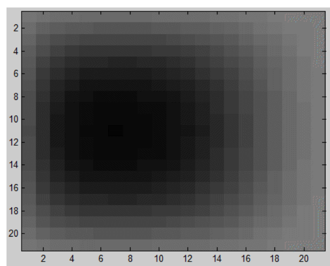

Figure 9: Asymmetric filter Simulation

Here in the figure, we can see that the heat transfer from right neighbours to left neighbour is faster than to the left to right as the coefficient of right neighbour is higher than left neighbours.

5 Real Life Applications

The model of diffusion of heat can be used in various problems. One of such application is to model the diffusion of pollutant in water. Here pollutant are added in left side of the pond such that the pollutant concentration remains constant on the left side of the pond. And we will assume reflecting boundary condition on the other three sides.

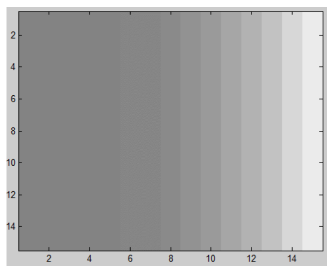

Figure 10: Pollution

We model a bar at 100 degree Celsius with ambient temperature of 25 degree Celsius on surrounding. We generate plots of temperature of cells at corner and middle of the bar versus time.

Here we can see that the temperature of corner cell decreases exponentially. But the temperature of cell in center decreases slowly at first and then start decreasing exponentially. This is so because it takes more time for surroundings to effect center than corner.

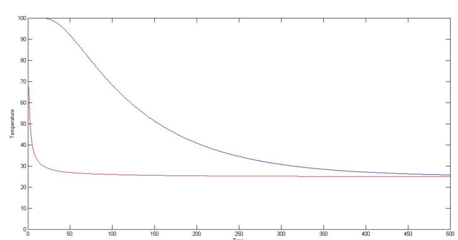

Figure 11: Corner vs Center