QEC with Rotors and Torsion

Why not put $\mathbb{Z}$ into our chain complexes?

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What is a rotor?

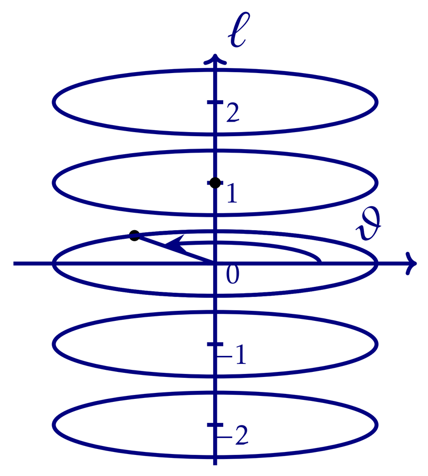
We are used to **qubits** (two basis states), and perhaps **qudits** (d basis states). What happens when we send $d \to \infty$ (heuristically) ?

Definition: rotor [VCT23]

A quantum **rotor** is a system whose states we describe using the **circle** group

$\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, and by its **Pontryagin dual** $\widehat{\mathbb{T}} \cong \mathbb{Z}$,

The angle $\vartheta \in \mathbb{T}$ describes position, and $\ell \in \mathbb{Z}$ angular momentum.



Pontryagin dual: $\widehat{G} := \mathrm{Hom}(G, \mathbb{T})$, the group of continuous characters of a group G, i.e. homomorphisms to the circle. The pairs of groups $(G, \widehat{G})$ are the setting for **Fourier analysis**.

Here, we **focus on one** of these groups: the one **described by** $\mathbb{Z}$. We can do this exactly because a homomorphism $\varphi \in \mathrm{Hom}(\mathbb{T}, \mathbb{T})$ is fully described by how many times, and in which direction, it "goes around" the circle.

Why is this interesting?

For N **qubits**, the state space is $(\mathbb{C}^2)^{\otimes N}$, and we represent it by its **basis** $\mathbb{Z}_2^N$ ($\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$). To study **linear codes**, we use its natural $\mathbb{Z}_2$-**vector space** structure.

Same idea for **rotors**: N rotors should be **described by** $\mathbb{Z}^N$.

But wait!

Above, $\mathbb{Z}_2$ is a **field** and thus $\mathbb{Z}_2^N$ is a **vector space**. This is a very nice and familliar setting, everything works neatly.

However, the ring $\mathbb{Z}$ is **not** a field, and $\mathbb{Z}^N$ is **not** a vector space! Instead, $\mathbb{Z}^N$ is a $\mathbb{Z}$-**module**. This is like a vector space with scalars being integers in $\mathbb{Z}$, but some important properties are very different.

Conclusion: We need **more general** machinery, and unexpected phenomena happen:

✨ TORSION ✨

Summary

Think of $\mathbb{Z}^N$ as a vector space, but $\mathbb{Z}$ is **not a field**, so sometimes **weird stuff** happens.

Okay whatever. Why is this REALLY interesting?

Motivation 1: Building qubit systems from rotors

This is something we **can build** and use, so it's worth studying what it can do. In particular, we can use physical **rotors** to **encode other kinds** of systems (e.g. **qubits**), like in [VCT23].

Motivation 2: Revealing subtleties

The more general machinery reveals **subtle details** that are also present, but **hidden**, in more **conventional** constructions.

Bonus: To study topological qudit codes, we can just lift to the rotor code, and do our calculations there, then translate the result back [Nov24].

Torsion: algebraic version

Definition: torsion

Let M be a $\mathbb{Z}$-module, and let $\underline{v} \in M$. If there exists a **nonzero** (and generally also **non-zero-divisor**) $n \in \mathbb{Z}$, such that

$$n \cdot \underline{v} = \underline{0}$$

then we call $\underline{v}$ a **torsion** element. The set of such $\underline{v}$ is called the **torsion submodule**.

Example

Let $M = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ as $\mathbb{Z}$-module. Then:

$$4 \cdot \bar{1} = \overline{4 \cdot 1} \pmod 2 = \bar{0}.$$

The whole $\mathbb{Z}_2$ is a torsion $\mathbb{Z}$-module!

References

[VCT23] Christophe Vuillot, Alessandro Ciani, and Barbara M. Terhal, *Homological Quantum Rotor Codes: Logical Qubits from Torsion*, September 2023.

[Nov24] Samo Novák, *Homological Quantum Error Correction with Torsion*, Master's thesis, Oxford University, Oxford, May 2024.

[BM07] H. Bombin and M. A. Martin-Delgado, *Homological Error Correction: Classical and Quantum Codes*, Journal of Mathematical Physics **48** (2007), no. 5, 052105.

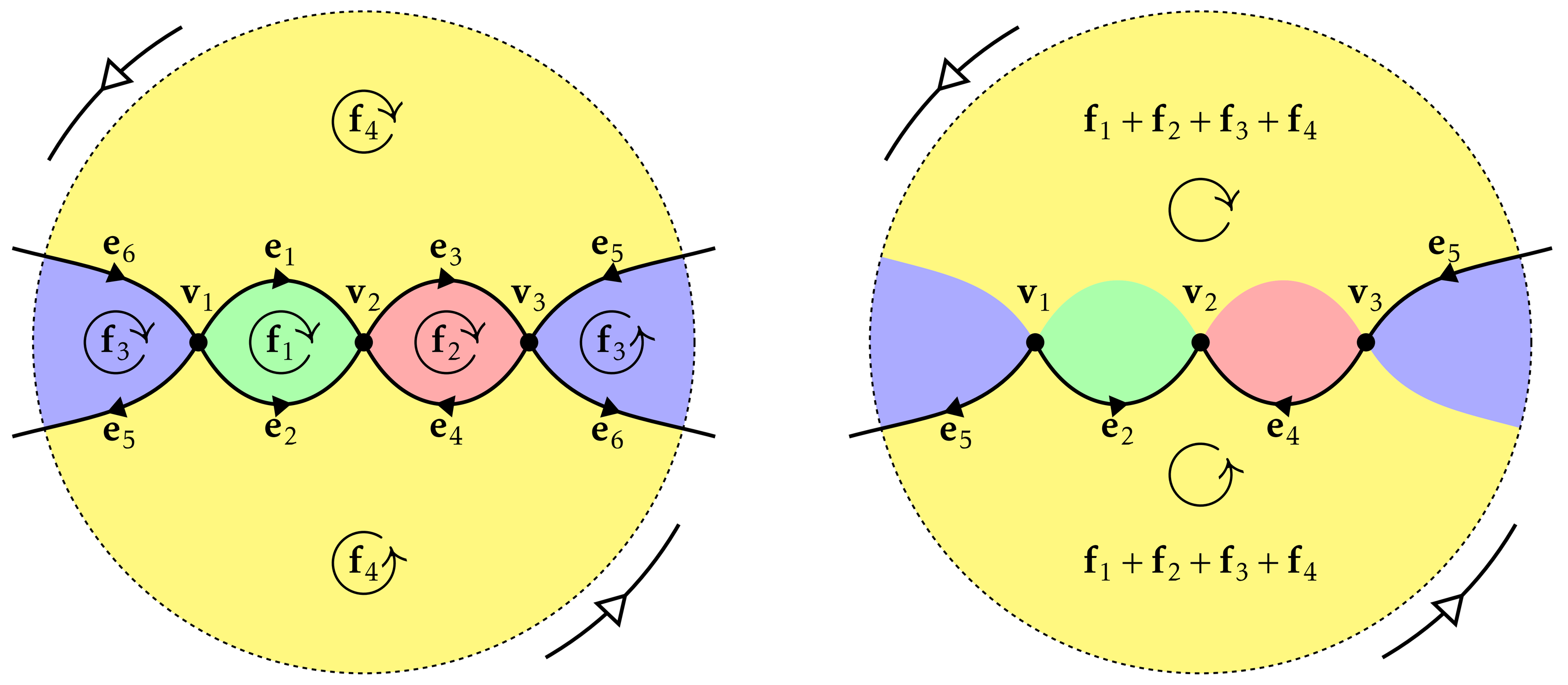
[Gun24] Lane G. Gunderman, *Stabilizer Codes with Exotic Local-dimensions*, Quantum **8** (2024), 1249.

[1] Klein bottle image source: Tttrung, *Klein bottle made with gnuplot 4.0.*, July 2006. https://commons.wikimedia.org/wiki/File:Klein_bottle.svg

Torsion: topological version – Example of $\mathbb{RP}^2$

Torsion describes a kind of **weirdness** of **topological spaces**.

Below: the real **projective plane** $\mathbb{RP}^2$ which is a **disk**, but with **opposite** points of the boundary **glued**. Note that this is **non-orientable**, in particular, if you walk across the boundary, left and right get **reversed**.



Intuition

Torsion, here $\mathbb{Z}_2$, corresponds to a **cycle** (a loop; right image) which must be **traversed multiple times** (2) to **come back** exactly to the **same point** in the **same orientation**.

To build a code, we cellulate the space with **points** $\{\mathbf{v}_1, \ldots, \mathbf{v}_3\}$, **oriented edges** $\{\mathbf{e}_1, \ldots, \mathbf{e}_6\}$, and **oriented faces** $\{\mathbf{f}_1, \ldots, \mathbf{f}_4\}$.

We can talk about **linear combinations** of these objects, e.g. the path $\mathbf{e}_1 - \mathbf{e}_2$ that goes along $\mathbf{e}_1$ (following its orientation) and then along $\mathbf{e}_2$ (opposite its orientation $\implies$ minus sign).

Torsion more precisely: Topology meets algebra

The **cycle** $\underline{x} = -\mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_5$ is **not** a **boundary** of a face. But its **multiple** $2\underline{x}$ is the boundary of $\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \mathbf{f}_4$. This will give us $\mathbb{Z}_2$ **torsion**.

A way to construct a code is to **cellulate** a topological **space** and build a **chain complex**:

Definition: chain complex CSS code

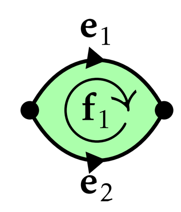
A **chain complex** representing a **rotor CSS code** is:

$$C_2 = \mathbb{Z}^{r_X} \xrightarrow{\partial_2 = H_X^\top} C_1 = \mathbb{Z}^N \xrightarrow{\partial_1 = H_Z} C_0 = \mathbb{Z}^{r_Z}$$

where $\partial_1 \partial_2 = 0$, or equivalently $\operatorname{im} \partial_2 \subseteq \ker \partial_1$. The spaces are:

- $C_2 = \langle \mathbf{f}_1, \ldots, \mathbf{f}_{r_X} \rangle_\mathbb{Z}$ spanned by **faces**, corresponding to X-**stabilizers**,
- $C_1 = \langle \mathbf{e}_1, \ldots, \mathbf{e}_N \rangle_\mathbb{Z}$ spanned by **edges**, corresponding to **physical rotors**,
- $C_0 = \langle \mathbf{v}_1, \ldots, \mathbf{v}_{r_Z} \rangle_\mathbb{Z}$ spanned by **vertices**, corresponding to Z-**syndromes**.

The **maps** ∂_n describe the (oriented) **incidence** of n-dim cells on their $(n-1)$-dim **boundaries**, e.g. $\partial_2(\mathbf{f}_1) = +\mathbf{e}_1 - \mathbf{e}_2$.

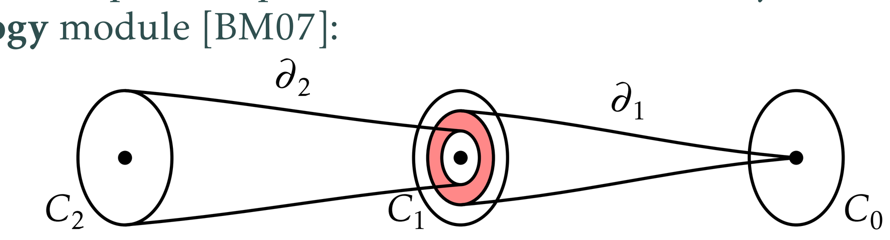


These boundary maps give **parity check matrices** H_X and H_Z.

In a **chain complex** representing a CSS code the **logical operators** are made of those physical operators **not detected as errors** ($\in \ker \partial_1$), that are **not stabilizers** ($\notin \operatorname{im} \partial_2$).

Furthermore, logicals are **equivalent** up to composition with stabilizers. They correspond to the elements of the **first homology** module [BM07]:

$$H_1 := \ker \partial_1 / \operatorname{im} \partial_2$$



Key idea

Torsion comes from a **cycle** of edges that is **not** a **boundary**, but its **multiple is**. This is given by the **image** $\operatorname{im} \partial_2$.

What does torsion mean?

In the $\mathbb{RP}^2$ example, the **homology** is $H_1(\mathbb{RP}^2) = \mathbb{Z}_2$, even though it is constructed from rotors ($\mathbb{Z}$). This is **torsion**, and it means our **rotor** system **encodes a qubit** ($\mathbb{Z}_2$).

Key takeaway

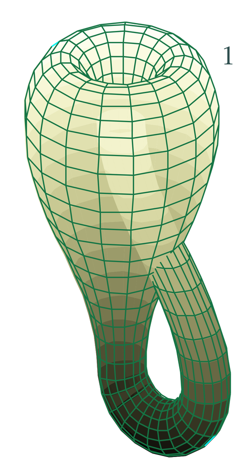
In the **general** setting of **rotors**, we can use one kind of system to **encode another kind**.

Example: Klein bottle

A **Klein bottle** $\mathbb{K}^2$ has homology

$$H_1(\mathbb{K}^2) = \mathbb{Z} \oplus \mathbb{Z}_2.$$

A rotor code defined on a cellulation of $\mathbb{K}^2$ encodes a **rotor** and a **qubit**.



Key observation

We can obtain **mixed-dimension** systems [Nov24, Gun24].