Minimum Spanning Tree

Graph Terminology

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Graph

- Collection of Nodes and Edges

Edges

Special Edges

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- Loops, multi

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Potential Properties

Edges

Special Edges

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Potential Properties

- Direction

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Special Edges

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Potential Properties

- Direction
- Weight (cost)

Edge/Node Series

Path

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- A series of distinct Nodes

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- Adjacent Node pairs are connected by a **distinct** edge
    - From the first node of the series to the second in the case of directed edges

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Cycle

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Cycle

- Path where the first and last Node are equal

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Types we will work with (for now)

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- Acyclic
  - A graph with no cycles
- Tree
  - Undirected, Connected, Acyclic Graph

Problem

We have country. The country has cities. No citizen can move between cities currently. Two-way Highways between cities will be built. Each highway will have some cost based on the clearing and the material cost required. We can assume that the cost of building one highway will not affect the cost of building others.

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How do we minimize the cost?

Answer: Lazy Builders

Analysis

No Cycles

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Connected Acyclic Undirected Graph (Tree)

To find the **Minimum Spanning Tree**

Two Algorithms

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Prim’s

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- On sparse graphs requires fast incident edge(s) look up AND a Heap

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Kruskal’s

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Kruskal’s

- Requires a Merge Find Set data structure AND a sort

Prim’s

Initialize our answer with 0
Start with a single node in a set of connected nodes
Until all nodes are in the set do the following
    Find the edge that is smallest leaving our set of connected nodes
    Add the edges weight to our answer update the set of connected nodes
Return the answer

Kruskal’s

Initialize the answer to 0
Have each node in their own group
Sort all the edges
Loop through the sorted edges
    If the endpoints of the current edge are in different groups
        Join the groups
        Add the weight to the answer
Return the answer

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Exchange Argument

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Exchange Argument (Proof by Contraction)

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- Start with the “best answer” (not our answer obvi.)
- Show that we can use our answer to make an even better answer
- A better answer than the best answer exists ※ (or ⊥)

Prim’s Exchange

It’s best to assume that all edge costs are distinct.

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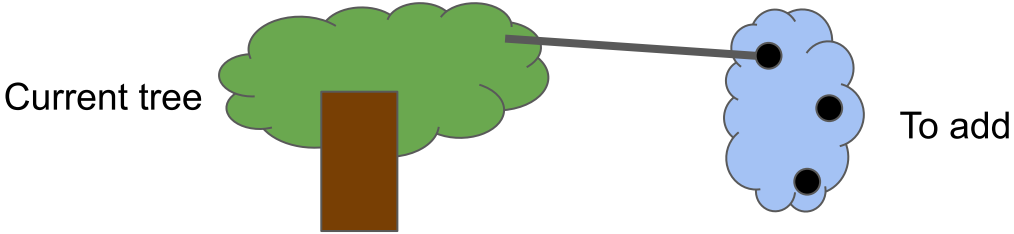

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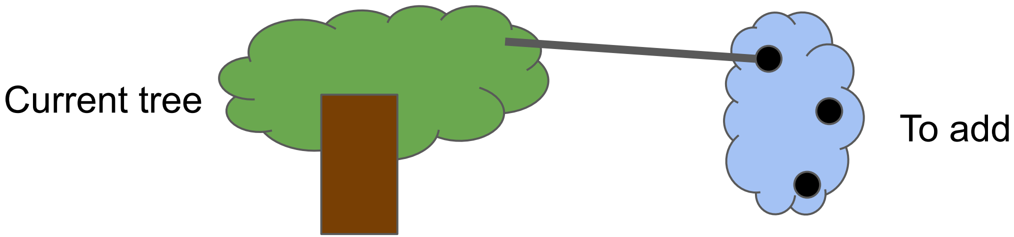
The best answer chose some other path from the tree to that node not in the built up tree.



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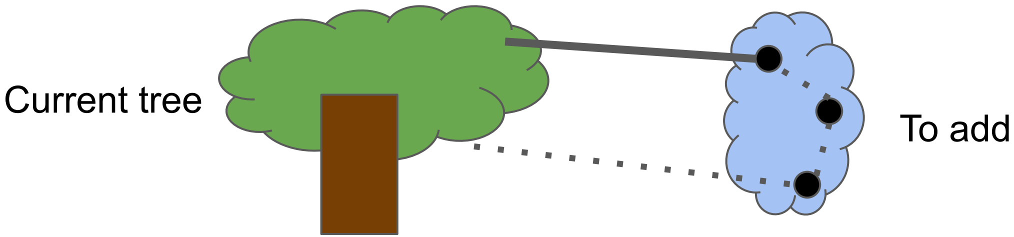

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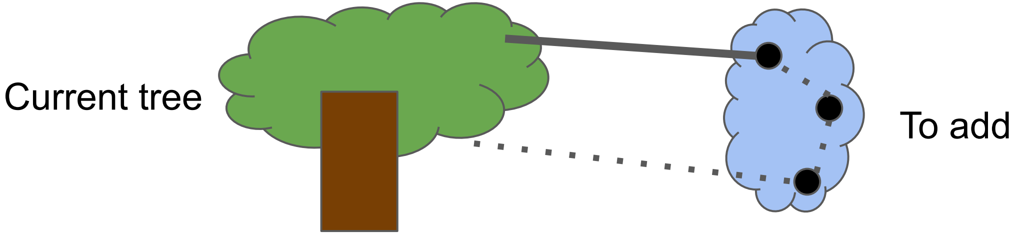

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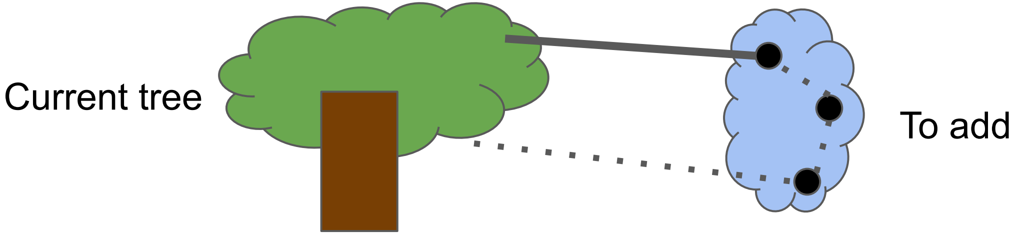
The other edge connecting to the set that formed the cycle will be larger than **our edge**



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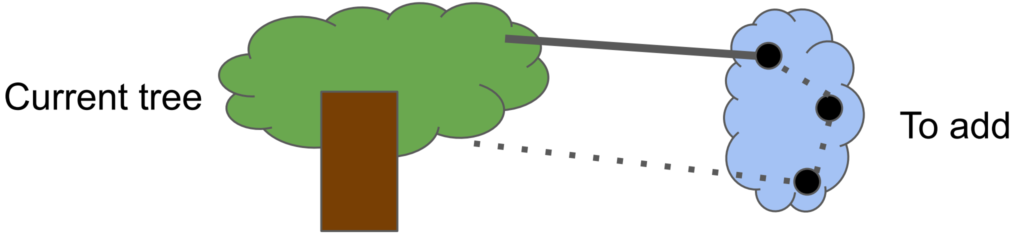

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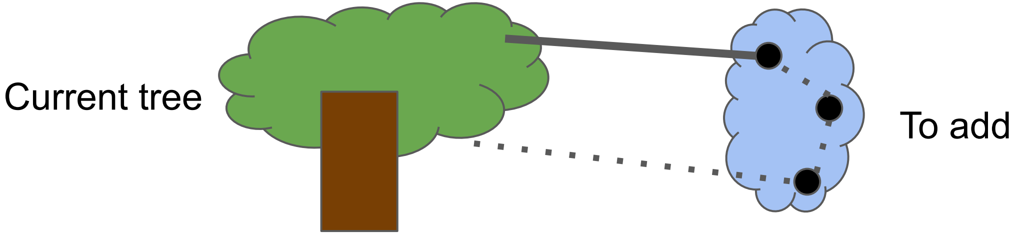

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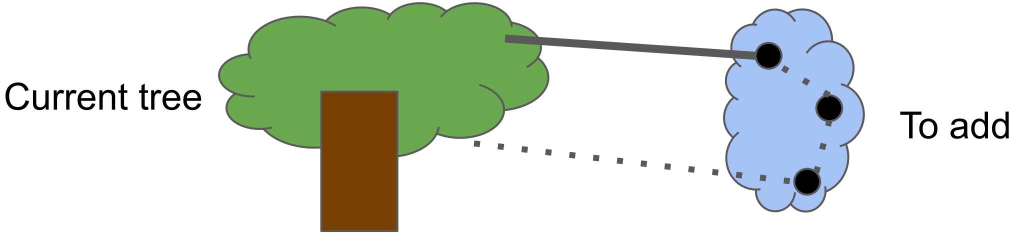
We found a better answer! (Contradiction)



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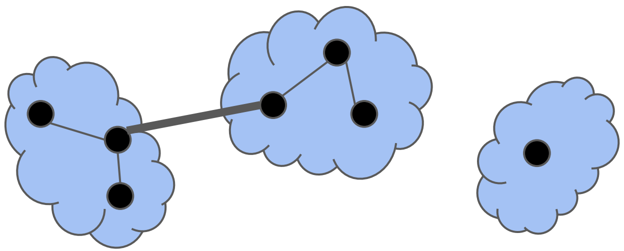

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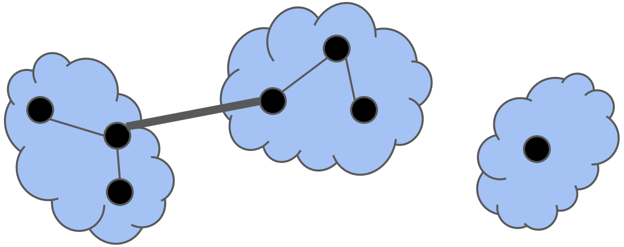

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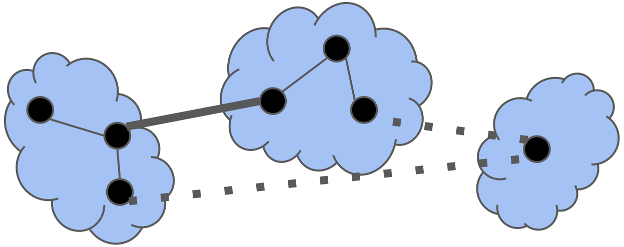

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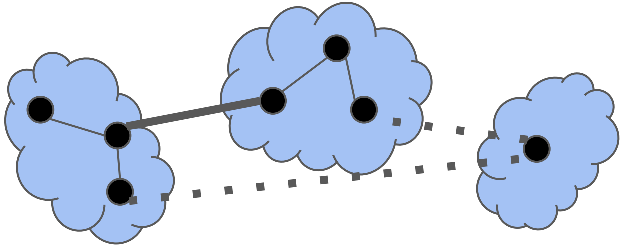

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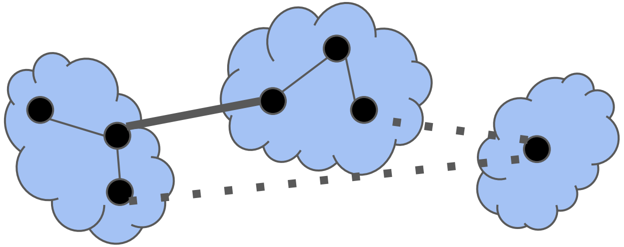
By removing one of the other edges and adding **our edge** we form a better solution.



Similar Problems

Minimum Spanning Arborescence

Steiner Tree** (general version is hard)