Network Flow Algorithms

Traffic

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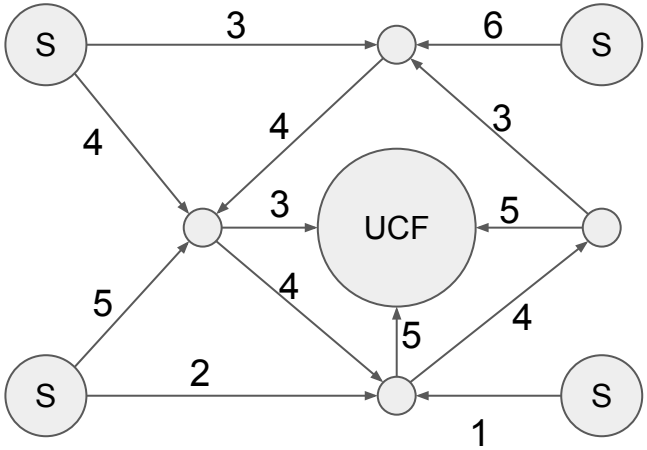

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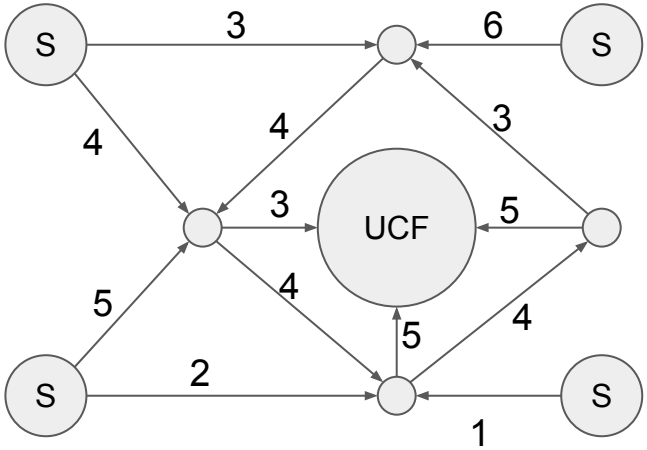

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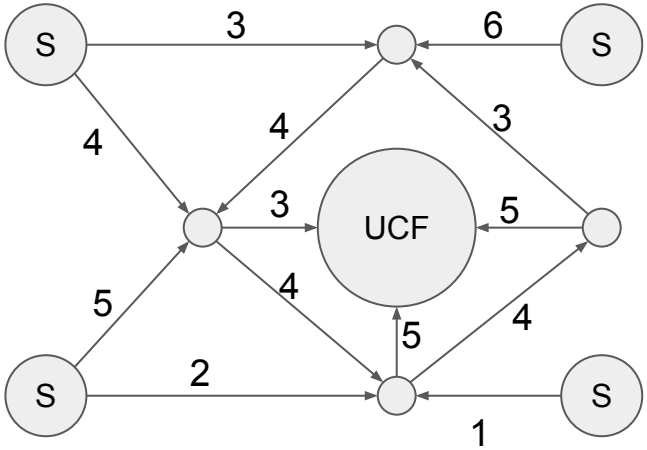

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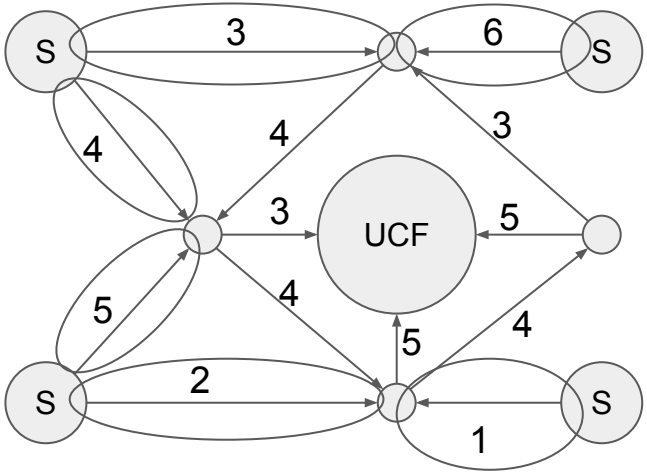
Leaving S = 21



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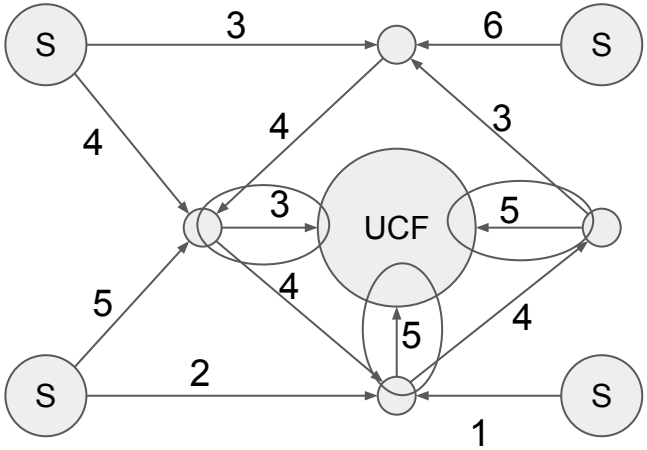
Leaving S = 21
Entering UCF = 13



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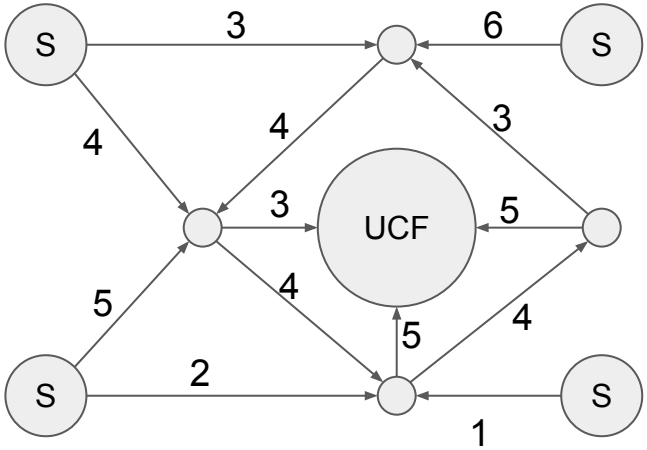

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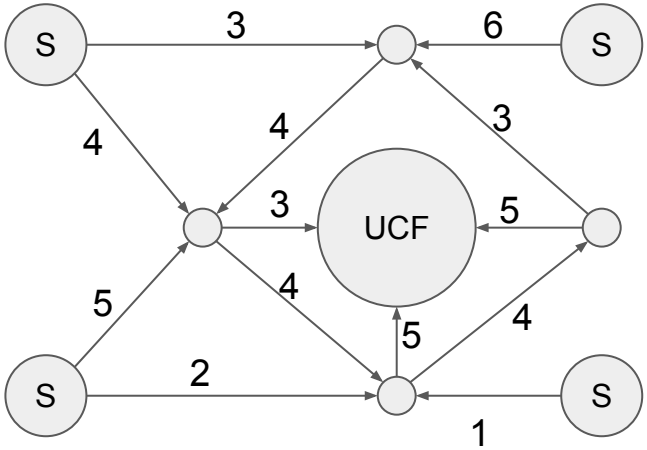

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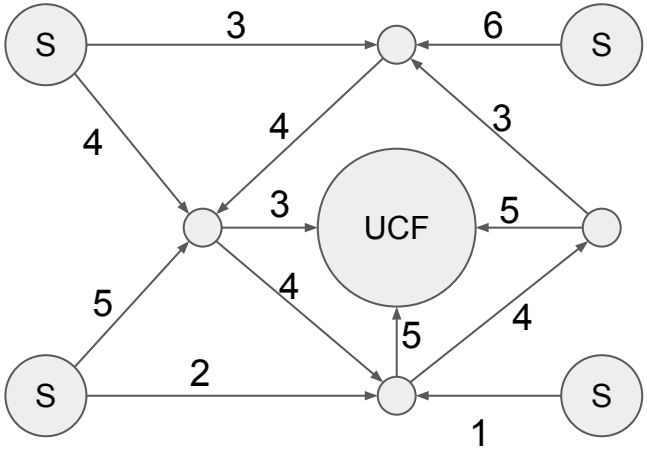
The actual answer is 10.



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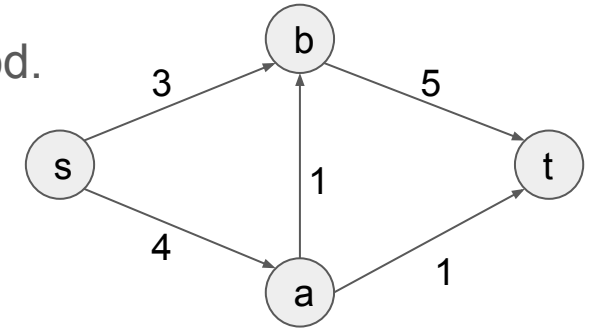

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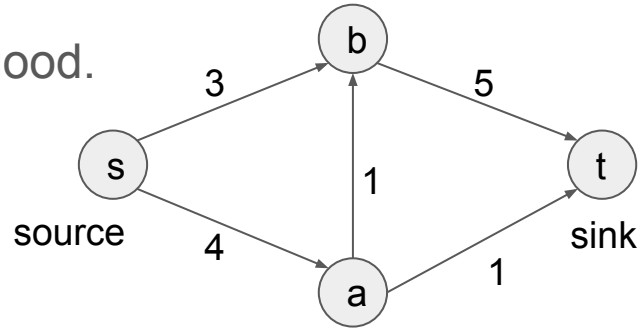

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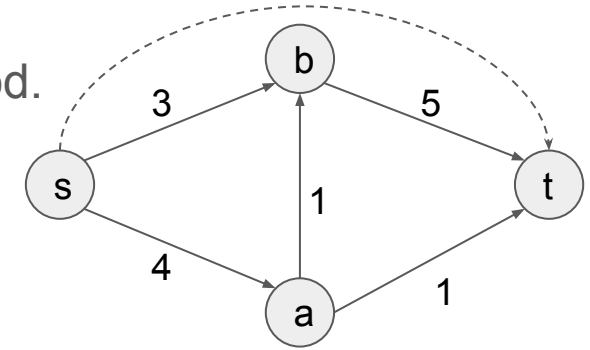

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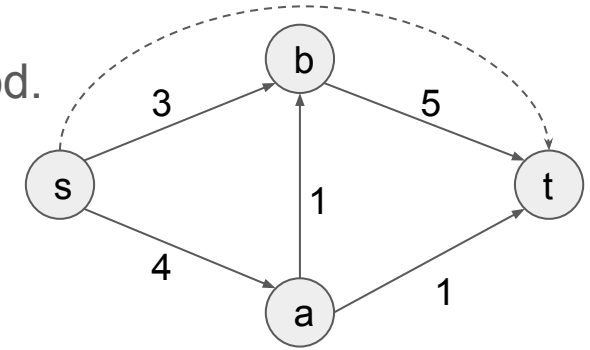

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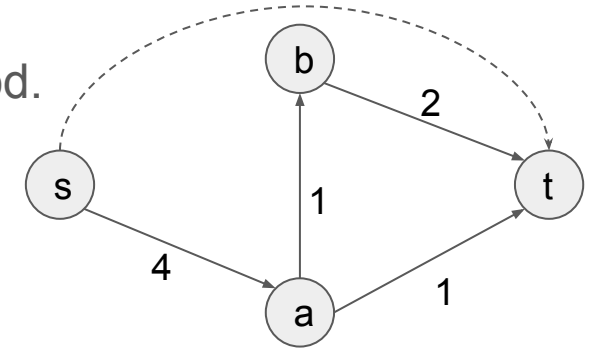

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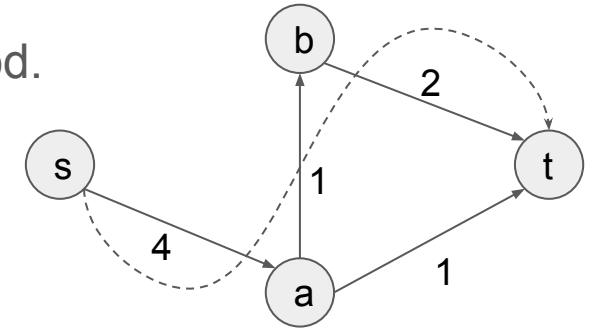

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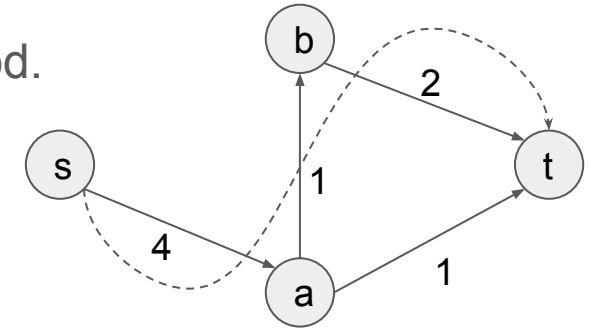

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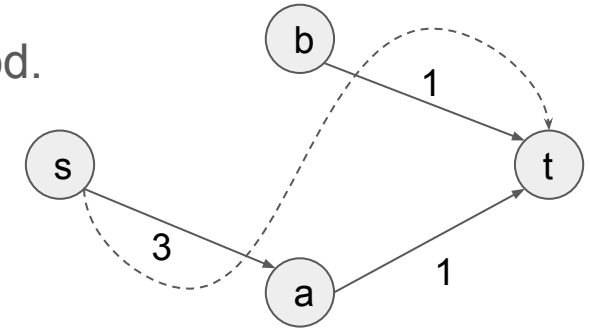

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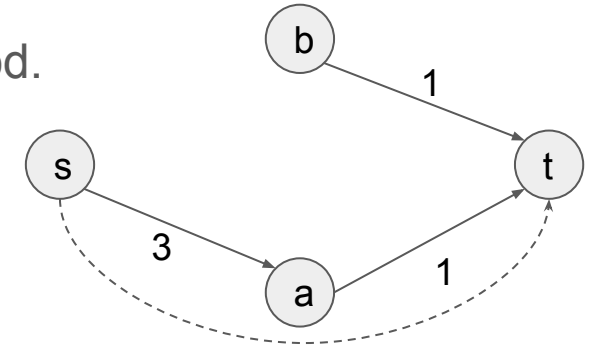

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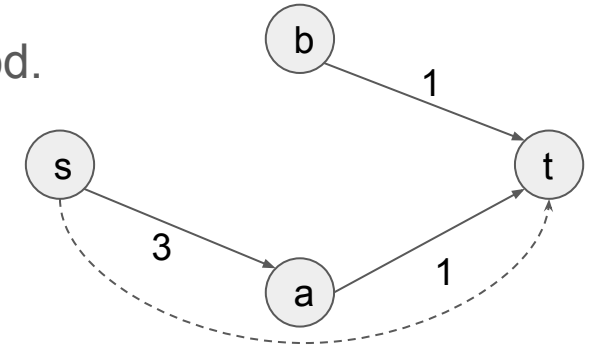

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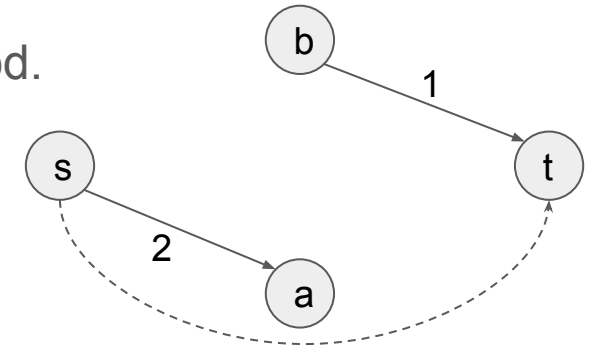

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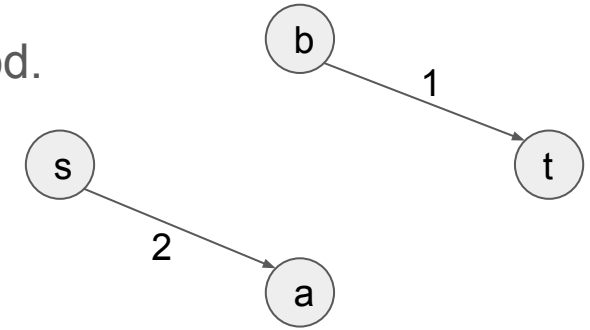
No Remaining Paths



Solution Ideas (cont.)

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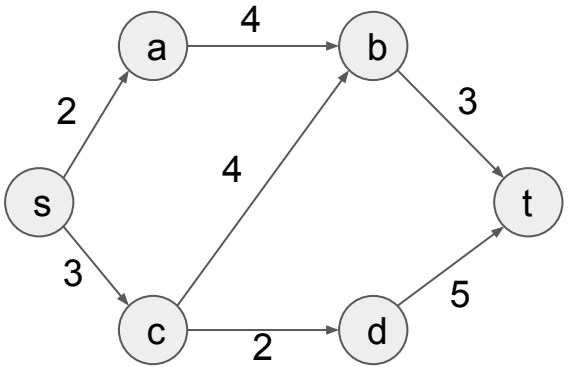

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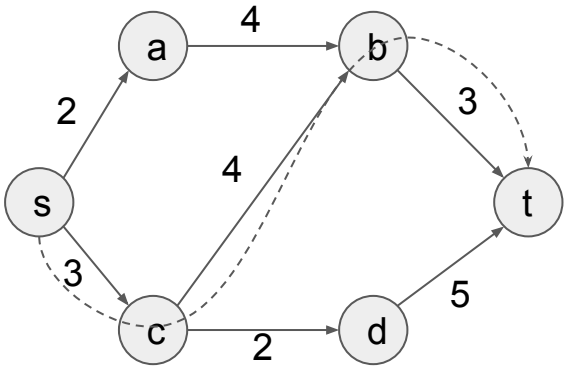

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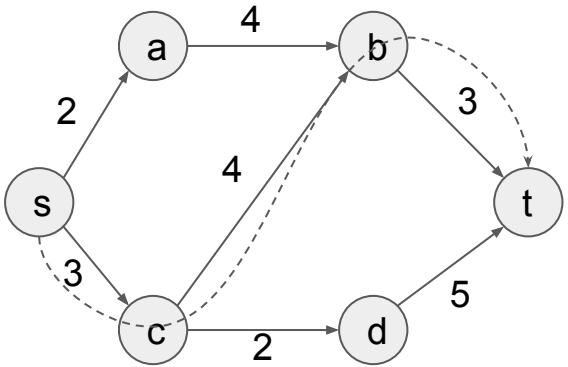

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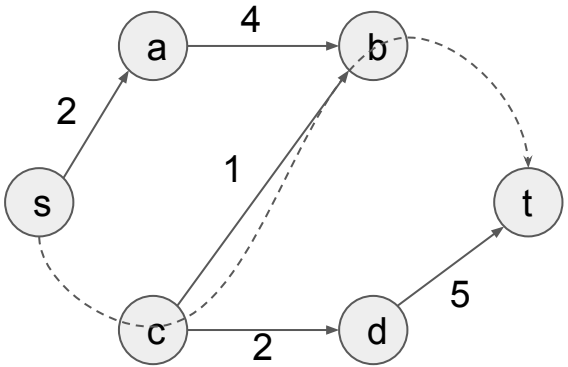

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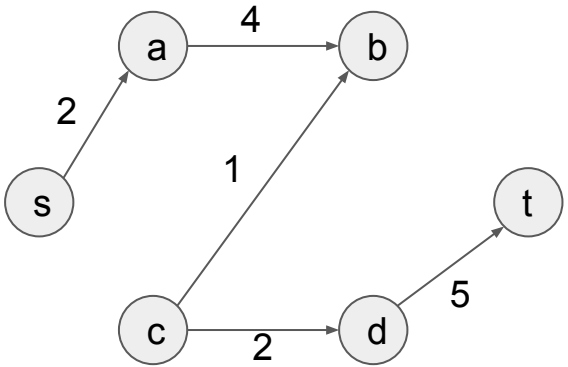

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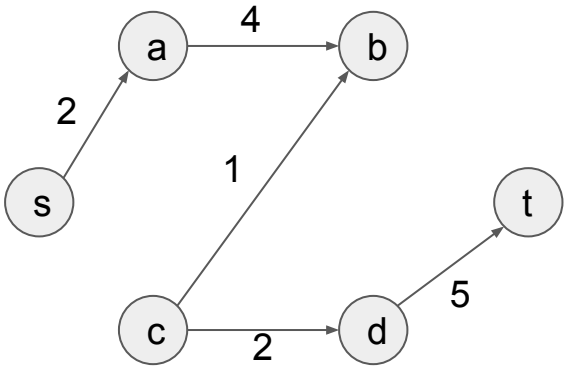

Send 3 cars
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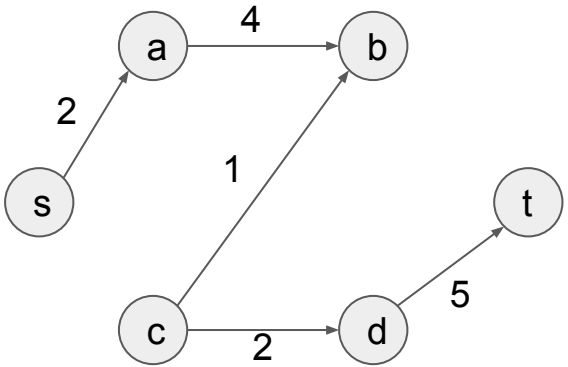

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No path.
But what was the answer?

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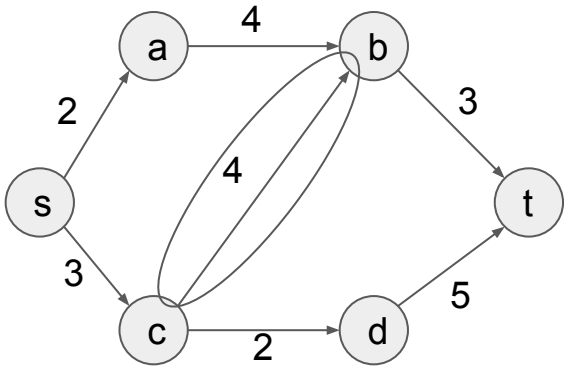

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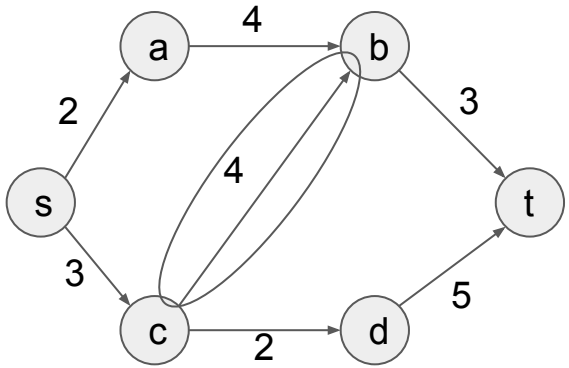

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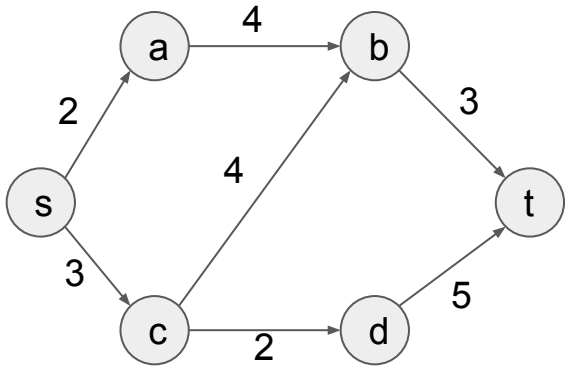

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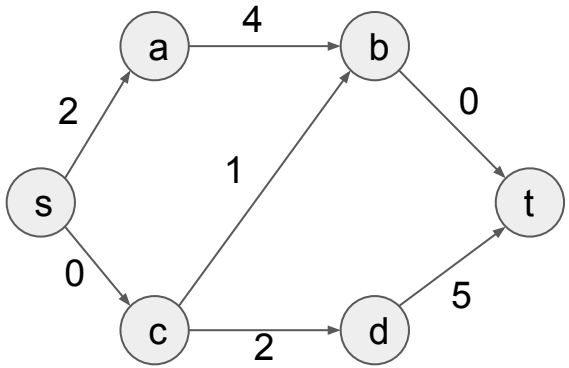

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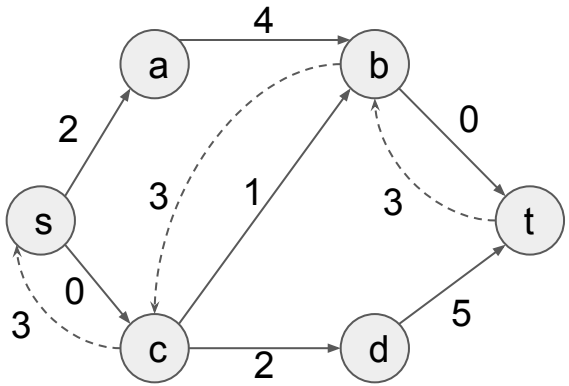

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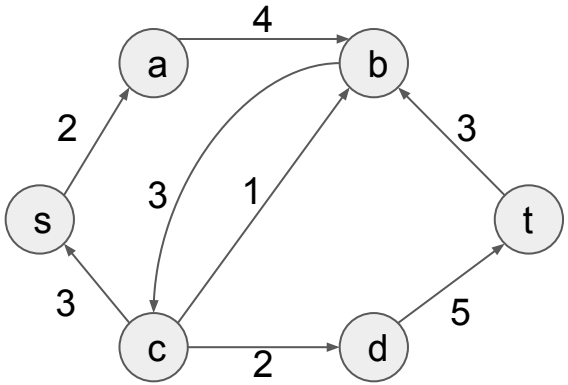

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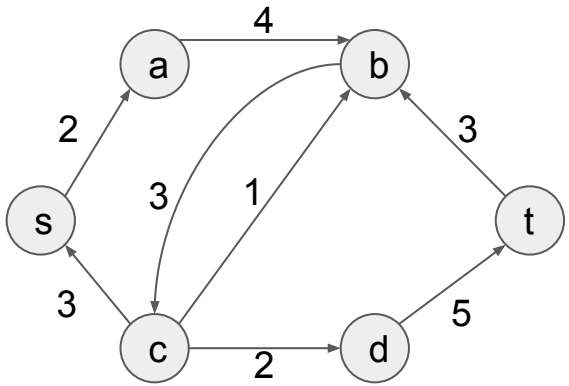

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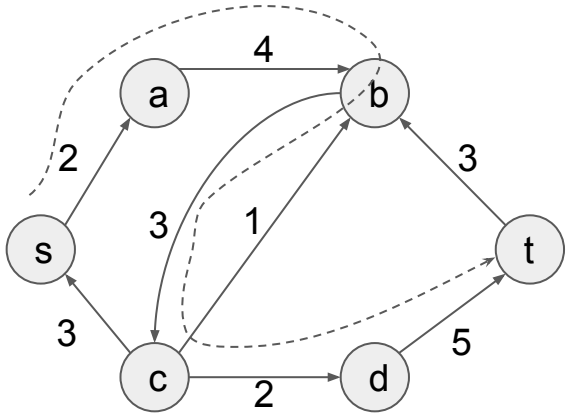

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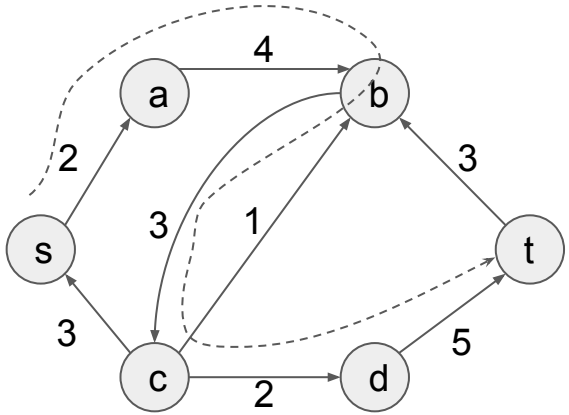

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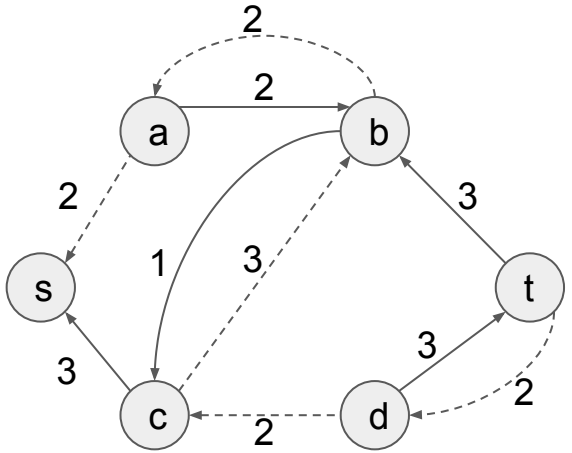

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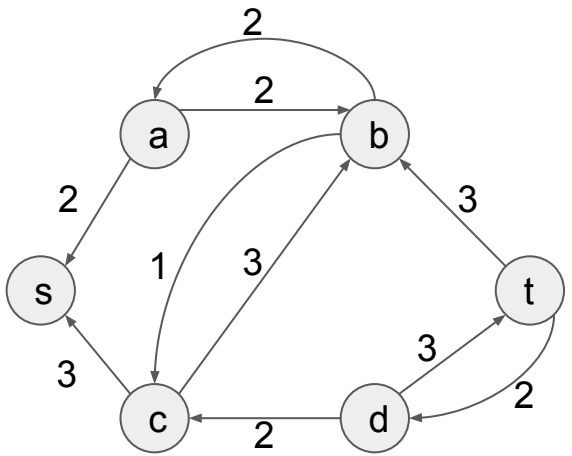

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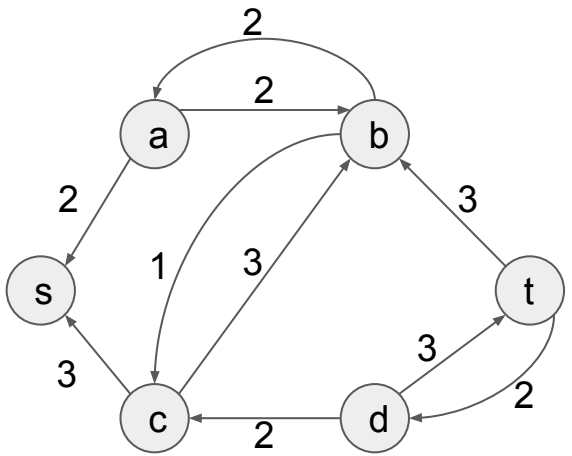

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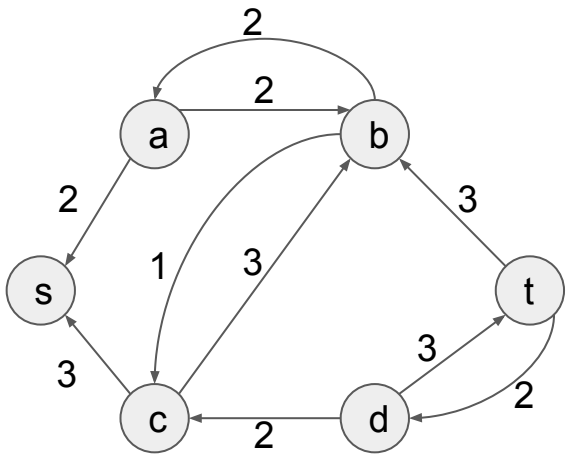

Send 2 cars
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No path from s to t!

Correctness

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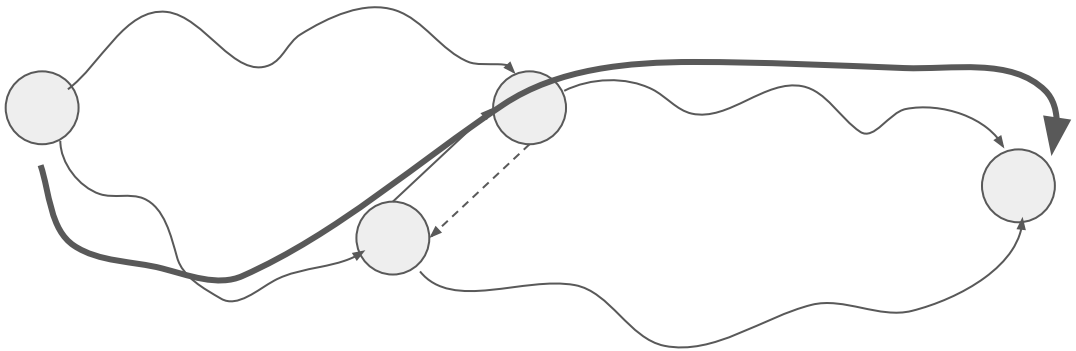

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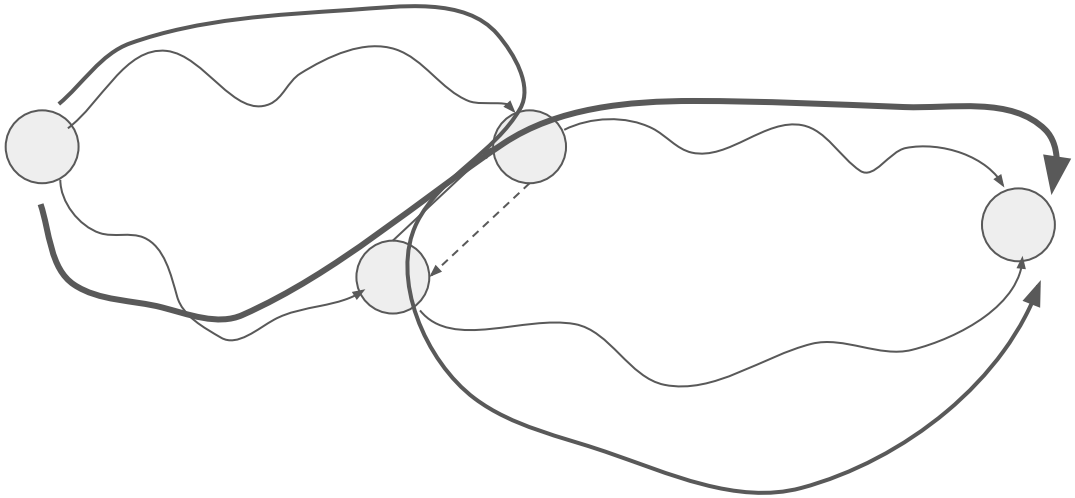

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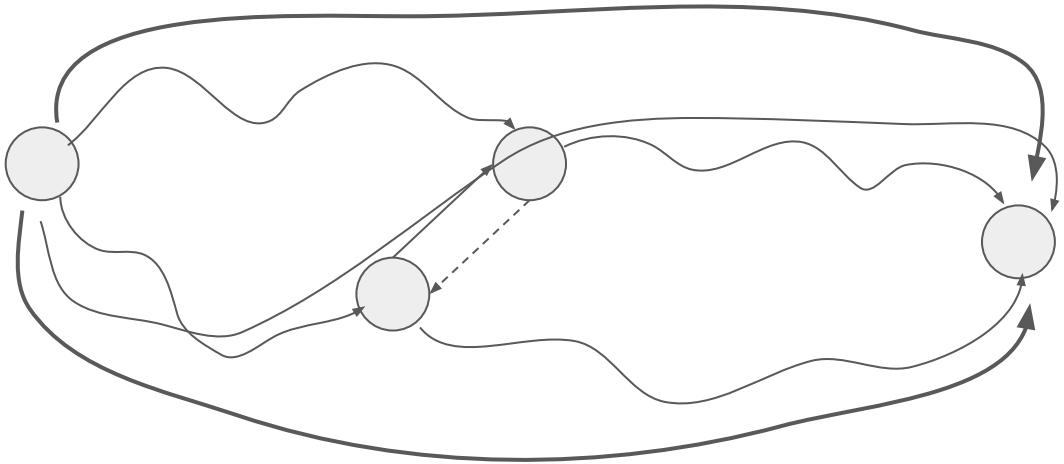

Ford-Fulkerson Algorithm (FFA)

```
Let the answer be 0
While a path exists
    Find the path using a DFS
    Increment answer by the minimum capacity along the path
    Adjust capacities along the path
    Add residual edges along the path
```

FFA Analysis

Runtime?

FFA Analysis

Runtime?

(Runtime of DFS) TIMES (Number of paths)

FFA Analysis

Runtime?

(Runtime of DFS) TIMES (The Answer in worst case)

FFA Analysis

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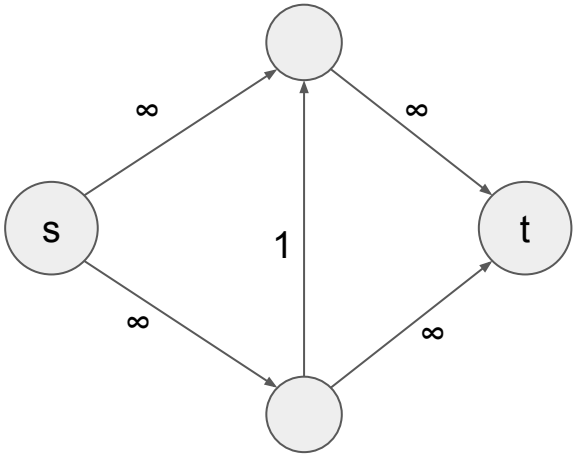

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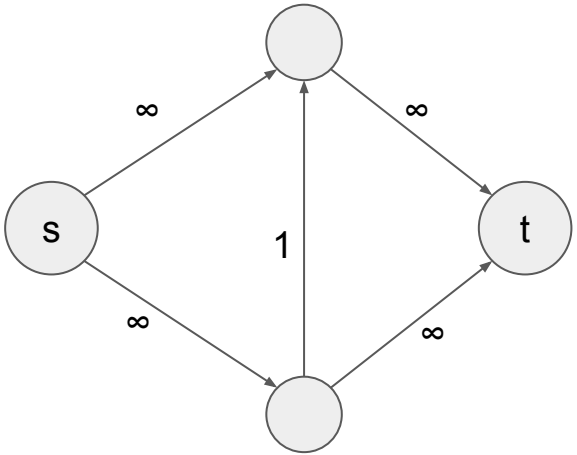

Not really ∞, but a very large value.

FFA Analysis

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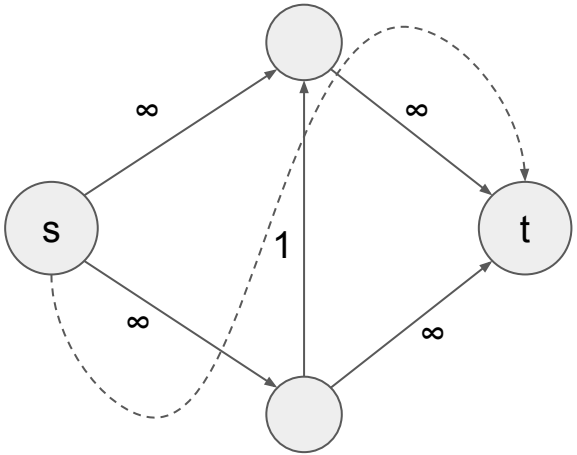

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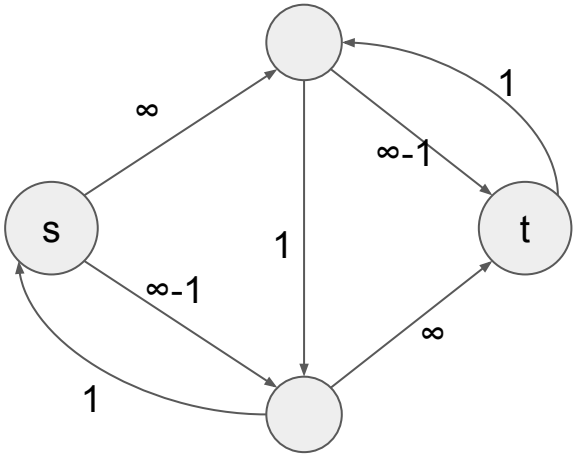

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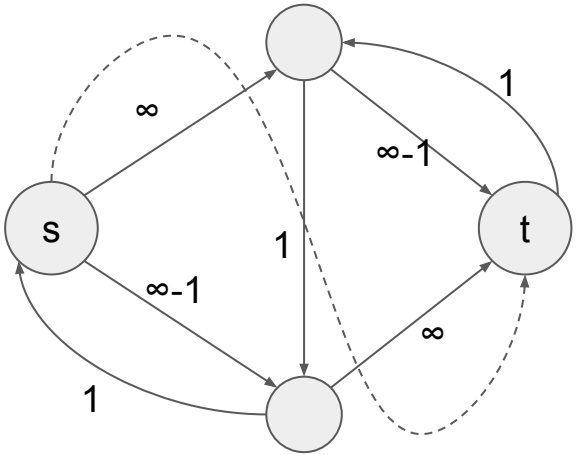

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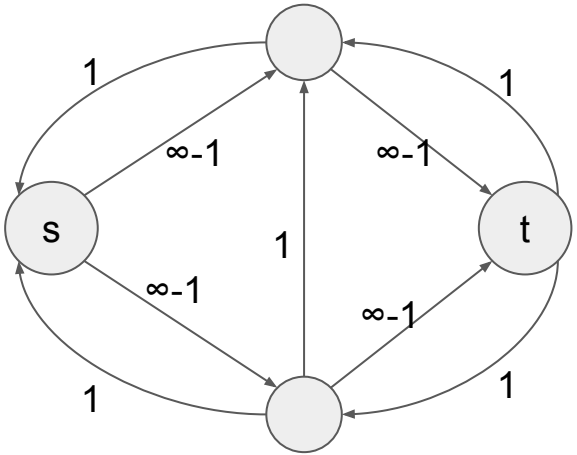

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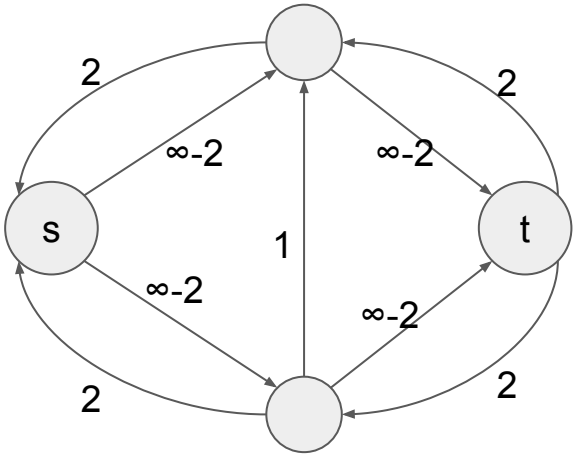

After 2 more DFS's

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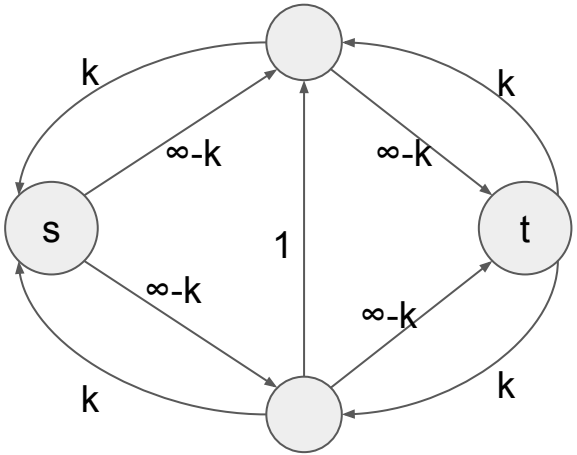

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FFA can be thought of as $O((|E|+|V|)(F))$, where F is the maximum flow.

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To fix FFA we change our DFS to a BFS.

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Using the BFS makes the algorithm something called Edmonds-Karp Algorithm.

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$O((|V|+|E|)(|E|)(|V|))$ or $O(|E|^2|V|)$

Dinitz (Dinic's) Algorithm

Instead of only pushing 1 path of flow…

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The blocking flow provably has a worst case of $O(|E||V|^2)$

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An optimized version has $O(|V|^2|E|^{0.5})$

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https://ioinformatics.org/journal/v12_2018_25_41.pdf