Quicksort Analysis

The Algorithm

Input : Array

Output : Sorted Array

The Algorithm

Input : Array

Output : Sorted Array

Base Case (remember; it's recursive)

The Algorithm

Input : Array

Output : Sorted Array

- Base Case
- Find a pivot

The Algorithm

Input : Array

Output : Sorted Array

- Base Case
- Find a pivot
- Split array into less than and greater than pivot segments

The Algorithm

Input : Array

Output : Sorted Array

- Base Case
- Find a pivot
- Split array into less than and greater than pivot segments
- Quicksort segments

The Algorithm

Input : Array

Output : Sorted Array

- Base Case
- Find a pivot
- Split array into less than and greater than pivot segments
- Quicksort segments
- Return Less Than Segment || pivot || Greater Than Segment

Best Case

Best possible pivot selection

Best Case

Best possible pivot selection

- The middle

Best Case

Best possible pivot selection

- The middle

Runtime?

Best Case

Best possible pivot selection

The middle

Runtime? $\Theta(N\log(N))$

Best Case

Best possible pivot selection

The middle

Runtime? $\Theta(N\log(N))$

Why?

Best Case

Best possible pivot selection

The middle

Runtime? Θ(Nlog(N))

Why? Recurrence relations

Best Case

Best possible pivot selection

The middle

Runtime? $\Theta(N\log(N))$

Why? Recurrence relations

$T(N) = 2T(N/2) + O(N)$

Best Case

Best possible pivot selection

The middle

Runtime? $\Theta(N\log(N))$

Why? Recurrence relations

$T(N) = 2T(N/2) + cN$

Best Case

Best possible pivot selection

The middle

Runtime? $\Theta(N\log(N))$

Why? Recurrence relations

$T(N) = 2T(N/2) + cN$

General Form: $T(N) = 2^kT(N/(2^k)) + ckN$

Best Case

Best possible pivot selection

The middle

Runtime? $\Theta(N\log(N))$

Why? Recurrence relations

$T(N) = 2T(N/2) + cN$

General Form: $T(N) = 2^kT(N/(2^k)) + ckN$

Plug in $k = \log(N)$

Best Case

Best possible pivot selection

The middle

Runtime? $\Theta(N\log(N))$

Why? Recurrence relations

$T(N) = 2T(N/2) + cN$

General Form: $T(N) = 2^k T(N/(2^k)) + ckN$

Plug in $k = \log(N) \Rightarrow T(N) = N \cdot 1 + c\log(N)N \in \Theta(N\log(N))$

Average Case

Any possible pivot selection with an equal chance

Average Case

Any possible pivot selection with an equal chance

Index	1	2	3	4	5	...	N-1	N
Prob.								

Average Case

Any possible pivot selection with an equal chance

1/N for each possible pivot

Index	1	2	3	4	5	...	N-1	N
Prob.	1/N	1/N	1/N	1/N	1/N		1/N	1/N

Average Case

Any possible pivot selection with an equal chance

1/N for each possible pivot

Then we sort both sides...

Index	1	2	3	4	5	...	N-1	N
Prob.	1/N	1/N	1/N	1/N	1/N		1/N	1/N

Average Case

Any possible pivot selection with an equal chance

1/N for each possible pivot

Then we sort both sides…

How many elements are on the LHS and RHS when spot 1 is chosen?

Index	1	2	3	4	5	...	N-1	N
Prob.	1/N	1/N	1/N	1/N	1/N		1/N	1/N
|LHS|								
|RHS|								

Average Case

Any possible pivot selection with an equal chance

1/N for each possible pivot

Then we sort both sides…

How many elements are on the LHS and RHS when spot 1 is chosen?

Index	1	2	3	4	5	...	N-1	N
Prob.	1/N	1/N	1/N	1/N	1/N		1/N	1/N
|LHS|	0							
|RHS|	N-1							

Average Case

Any possible pivot selection with an equal chance

1/N for each possible pivot

Then we sort both sides…

How many elements are on the LHS and RHS when spot 1 is chosen? And 2?

Index	1	2	3	4	5	...	N-1	N
Prob.	1/N	1/N	1/N	1/N	1/N		1/N	1/N
|LHS|	0							
|RHS|	N-1							

Average Case

Any possible pivot selection with an equal chance

1/N for each possible pivot

Then we sort both sides…

How many elements are on the LHS and RHS when spot 1 is chosen? And 2?

Index	1	2	3	4	5	...	N-1	N
Prob.	1/N	1/N	1/N	1/N	1/N		1/N	1/N
|LHS|	0	1						
|RHS|	N-1	N-2						

Average Case

Any possible pivot selection with an equal chance

1/N for each possible pivot

Then we sort both sides…

How many elements are on the LHS and RHS when spot 1 is chosen? And 2? In general?

Index	1	2	3	4	5	...	N-1	N
Prob.	1/N	1/N	1/N	1/N	1/N		1/N	1/N
|LHS|	0	1						
|RHS|	N-1	N-2						

Average Case

Any possible pivot selection with an equal chance

1/N for each possible pivot

Then we sort both sides…

How many elements are on the LHS and RHS when spot 1 is chosen? And 2? In general?

Index	1	2	3	4	5	...	N-1	N
Prob.	1/N	1/N	1/N	1/N	1/N		1/N	1/N
|LHS|	0	1	2	3	4		N-2	N-1
|RHS|	N-1	N-2	N-3	N-4	N-5		N-(N-1)	N-N

Abnormal Recurrence Relationship

$T(0) = 1$

Abnormal Recurrence Relationship

$T(0) = 1$

$T(N) = 1/N(T(0)+T(N-1)) + 1/N(T(1)+T(N-2)) + \ldots + 1/N(T(N-1)+T(0)) + N$

Abnormal Recurrence Relationship

$T(0) = 1$

$T(N) = 1/N(T(0)+T(N-1)) + 1/N(T(1)+T(N-2)) + \ldots + 1/N(T(N-1)+T(0)) + N$

$NT(N) = T(0)+T(N-1) + T(1)+T(N-2) + \ldots + T(N-1)+T(0) + N^2$

Abnormal Recurrence Relationship

$T(0) = 1$

$T(N) = 1/N(T(0)+T(N-1)) + 1/N(T(1)+T(N-2)) + \dots + 1/N(T(N-1)+T(0)) + N$

$NT(N) = T(0)+T(N-1) + T(1)+T(N-2) + \dots + T(N-1)+T(0) + N^2$

$NT(N) = T(0)+T(0) + T(1)+T(1) + \dots + T(N-1)+T(N-1) + N^2$

Abnormal Recurrence Relationship

$T(0) = 1$

$T(N) = 1/N(T(0)+T(N-1)) + 1/N(T(1)+T(N-2)) + \ldots + 1/N(T(N-1)+T(0)) + N$

$NT(N) = T(0)+T(N-1) + T(1)+T(N-2) + \ldots + T(N-1)+T(0) + N^2$

$NT(N) = T(0)+T(0) + T(1)+T(1) + \ldots + T(N-1)+T(N-1) + N^2$

$NT(N) = 2(T(0) + T(1) +T(2) + \ldots + T(N-1)) + N^2$

Abnormal Recurrence Relationship

$T(0) = 1$

$T(N) = 1/N(T(0)+T(N-1)) + 1/N(T(1)+T(N-2)) + \ldots + 1/N(T(N-1)+T(0)) + N$

$NT(N) = T(0)+T(N-1) + T(1)+T(N-2) + \ldots + T(N-1)+T(0) + N^2$

$NT(N) = T(0)+T(0) + T(1)+T(1) + \ldots + T(N-1)+T(N-1) + N^2$

$NT(N) = 2(T(0) + T(1) +T(2) + \ldots + T(N-1)) + N^2$ Still difficult

Some More Reduction

$NT(N) = 2(T(0) + T(1) + T(2) + \ldots + T(N-1)) + N^2$

Some More Reduction

$NT(N) = 2(T(0) + T(1) + T(2) + \ldots + T(N-1)) + N^2$

$(N-1)T(N-1) = 2(T(0) + T(1) + T(2) + \ldots + T(N-2)) + (N - 1)^2$

Some More Reduction

$NT(N) = 2(T(0) + T(1) + T(2) + \ldots + T(N-1)) + N^2$

$(N-1)T(N-1) = 2(T(0) + T(1) + T(2) + \ldots + T(N-2)) + (N - 1)^2$

Difference of both sides

Some More Reduction

$NT(N) = 2(T(0) + T(1) + T(2) + \ldots + T(N-1)) + N^2$

$(N-1)T(N-1) = 2(T(0) + T(1) + T(2) + \ldots + T(N-2)) + (N-1)^2$

Difference of both sides

$NT(N) - (N-1)T(N-1) =$

Some More Reduction

$NT(N) = 2(T(0) + T(1) + T(2) + \ldots + T(N-1)) + N^2$

$(N-1)T(N-1) = 2(T(0) + T(1) + T(2) + \ldots + T(N-2)) + (N - 1)^2$

Difference of both sides

$$
\begin{aligned}
NT(N) - (N-1)T(N-1) = \\
&2[T(0) + T(1) + T(2) + \ldots + T(N-1)] + N^2 - \\
&2[T(0) + T(1) + T(2) + \ldots + T(N-2)] - (N - 1)^2
\end{aligned}
$$

Some More Reduction

$NT(N) = 2(T(0) + T(1) + T(2) + \ldots + T(N-1)) + N^2$

$(N-1)T(N-1) = 2(T(0) + T(1) + T(2) + \ldots + T(N-2)) + (N - 1)^2$

Difference of both sides

$NT(N) - (N-1)T(N-1) =$

$\quad 2[T(0) + T(1) + T(2) + \ldots + T(N-1)] + N^2 -$

$\quad 2[T(0) + T(1) + T(2) + \ldots + T(N-2)] - (N - 1)^2$

$NT(N) = (N-1)T(N-1) + 2T(N-1) + 2N - 1$

Some More Reduction

$NT(N) = 2(T(0) + T(1) + T(2) + \ldots + T(N-1)) + N^2$

$(N-1)T(N-1) = 2(T(0) + T(1) + T(2) + \ldots + T(N-2)) + (N - 1)^2$

Difference of both sides

$$\begin{aligned} NT(N) - (N-1)T(N-1) = \\ &2[T(0) + T(1) + T(2) + \ldots + T(N-1)] + N^2 - \\ &2[T(0) + T(1) + T(2) + \ldots + T(N-2)] - (N - 1)^2 \end{aligned}$$

$NT(N) = (N-1)T(N-1) + 2T(N-1) + 2N - 1$

$NT(N) = (N+1)T(N-1) + cN$

Some More Reduction

$NT(N) = 2(T(0) + T(1) + T(2) + \ldots + T(N-1)) + N^2$

$(N-1)T(N-1) = 2(T(0) + T(1) + T(2) + \ldots + T(N-2)) + (N - 1)^2$

Difference of both sides

$$\begin{aligned} NT(N) - (N-1)T(N-1) = \\ &2[T(0) + T(1) + T(2) + \ldots + T(N-1)] + N^2 - \\ &2[T(0) + T(1) + T(2) + \ldots + T(N-2)] - (N - 1)^2 \end{aligned}$$

$NT(N) = (N-1)T(N-1) + 2T(N-1) + 2N - 1$

$NT(N) = (N+1)T(N-1) + cN$ for $c \in [1, 2]$

Regular Recurrence Relationship!

$NT(N) = (N+1)T(N-1) + cN$

Regular Recurrence Relationship!

$NT(N) = (N+1)T(N-1) + cN$

$T(N) = (N+1)/N\ T(N-1) + c$

Regular Recurrence Relationship!

$NT(N) = (N+1)T(N-1) + cN$

$T(N) = (N+1)/N \; T(N-1) + c$ (yay!)

Regular Recurrence Relationship!

$NT(N) = (N+1)T(N-1) + cN$

$T(N) = (N+1)/N\ T(N-1) + c$

$\quad T(N-1) = (N-1+1)/(N-1)\ T(N-1-1) + c$

$\quad T(N-1) = N/(N-1)\ T(N-2) + c$

Regular Recurrence Relationship!

$NT(N) = (N+1)T(N-1) + cN$

$T(N) = (N+1)/N\ \underline{T(N-1)} + c$

$T(N-1) = (N-1+1)/(N-1)\ T(N-1-1) + c$

$\underline{T(N-1)} = \underline{N/(N-1)\ T(N-2) + c}$

$T(N) = (N+1)/N\ (\underline{N/(N-1)\ T(N-2) + c}) + c$

Regular Recurrence Relationship!

$NT(N) = (N+1)T(N-1) + cN$

$T(N) = (N+1)/N \ T(N-1) + c$

$T(N-1) = (N-1+1)/(N-1) \ T(N-1-1) + c$

$T(N-1) = N/(N-1) \ T(N-2) + c$

$T(N) = (N+1)/N \ (N/(N-1) \ T(N-2) + c) + c$

$T(N) = (N+1)/(N-1) \ T(N-2) + (N+1)/N \ (c) + c$

Regular Recurrence Relationship!

$NT(N) = (N+1)T(N-1) + cN$

$T(N) = (N+1)/N\ T(N-1) + c$

$\quad T(N-1) = (N-1+1)/(N-1)\ T(N-1-1) + c$

$\quad T(N-1) = N/(N-1)\ T(N-2) + c$

$T(N) = (N+1)/N\ (N/(N-1)\ T(N-2) + c) + c$

$T(N) = (N+1)/(N-1)\ T(N-2) + (N+1)/N\ (c) + c$

$\quad T(N-2) = (N-2+1)/(N-2)\ T(N-2-1) + c$

$\quad T(N-2) = (N-1)/(N-2)\ T(N-3) + c$

Regular Recurrence Relationship!

$NT(N) = (N+1)T(N-1) + cN$

$T(N) = (N+1)/N\ T(N-1) + c$

$T(N-1) = (N-1+1)/(N-1)\ T(N-1-1) + c$

$T(N-1) = N/(N-1)\ T(N-2) + c$

$T(N) = (N+1)/N\ (N/(N-1)\ T(N-2) + c) + c$

$T(N) = (N+1)/(N-1)\ \underline{T(N-2)} + (N+1)/N\ (c) + c$

$T(N-2) = (N-2+1)/(N-2)\ T(N-2-1) + c$

$\underline{T(N-2)} = \underline{(N-1)/(N-2)\ T(N-3) + c}$

$T(N) = (N+1)/(N-1)\ (\underline{(N-1)/(N-2)\ T(N-3) + c}) + (N+1)/N\ (c) + c$

Regular Recurrence Relationship!

$NT(N) = (N+1)T(N-1) + cN$

$T(N) = (N+1)/N \ T(N-1) + c$

$\quad T(N-1) = (N-1+1)/(N-1) \ T(N-1-1) + c$

$\quad T(N-1) = N/(N-1) \ T(N-2) + c$

$T(N) = (N+1)/N \ (N/(N-1) \ T(N-2) + c) + c$

$T(N) = (N+1)/(N-1) \ T(N-2) + (N+1)/N \ (c) + c$

$\quad T(N-2) = (N-2+1)/(N-2) \ T(N-2-1) + c$

$\quad T(N-2) = (N-1)/(N-2) \ T(N-3) + c$

$T(N) = (N+1)/(N-1) \ ((N-1)/(N-2) \ T(N-3) + c) + (N+1)/N \ (c) + c$

$T(N) = (N+1)/(N-2) \ T(N-3) + (N+1)/(N-1) \ (c) + (N+1)/N \ (c) + c$

General Form

$$T(N) = (N+1)/(N-0)\, T(N-1) + c$$
$$T(N) = (N+1)/(N-1)\, T(N-2) + (N+1)/N\, (c) + c$$
$$T(N) = (N+1)/(N-2)\, T(N-3) + (N+1)/(N-1)\, (c) + (N+1)/N\, (c) + c$$

General Form

$T(N) = (N+1)/(N-0)\ T(N-1) + c$

$T(N) = (N+1)/(N-1)\ T(N-2) + (N+1)/N\ (c) + c$

$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + c$

$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + (N+1)/(N+1)c$

General Form

$T(N) = (N+1)/(N-0)\ T(N-1) + c$

$T(N) = (N+1)/(N-1)\ T(N-2) + (N+1)/N\ (c) + c$

$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + c$

$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + (N+1)/(N+1)c$

$T(N) = (N+1)/(N-(k-1))\ T(N-k) + c(N+1)$ sum i from 1 to k of $1/(N+2-i)$

General Form

$T(N) = (N+1)/(N-0)\ T(N-1) + c$

$T(N) = (N+1)/(N-1)\ T(N-2) + (N+1)/N\ (c) + c$

$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + c$

$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + (N+1)/(N+1)c$

<u>$T(N) = (N+1)/(N-(k-1))\ T(N-k) + c(N+1)$ sum i from 1 to k of $1/(N+2-i)$</u>

Let $k = N$

General Form

$$T(N) = (N+1)/(N-0)\ T(N-1) + c$$

$$T(N) = (N+1)/(N-1)\ T(N-2) + (N+1)/N\ (c) + c$$

$$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + c$$

$$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + (N+1)/(N+1)c$$

T(N) = (N+1)/(N-(k-1)) T(N-k) + c(N+1) sum i from 1 to k of 1/(N+2-i)

Let k = N

T(N) = (N+1)/(N-(N-1)) T(N-N) + c(N+1) sum from 1 to N of 1/(N+2-i)

General Form

$$T(N) = (N+1)/(N-0)\ T(N-1) + c$$

$$T(N) = (N+1)/(N-1)\ T(N-2) + (N+1)/N\ (c) + c$$

$$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + c$$

$$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + (N+1)/(N+1)c$$

<u>T(N) = (N+1)/(N-(k-1)) T(N-k) + c(N+1) sum i from 1 to k of 1/(N+2-i)</u>

Let $k = N$

$$T(N) = (N+1)/(N-(N-1))\ T(N-N) + c(N+1) \text{ sum from 1 to N of } 1/(N+2-i)$$

$$T(N) = (N+1)/(1)\ T(0) + c(N+1)\ (1/2 + 1/3 + 1/4 + \ldots + 1/(N+1))$$

General Form

$T(N) = (N+1)/(N-0)\ T(N-1) + c$

$T(N) = (N+1)/(N-1)\ T(N-2) + (N+1)/N\ (c) + c$

$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + c$

$T(N) = (N+1)/(N-2)\ T(N-3) + (N+1)/(N-1)\ (c) + (N+1)/N\ (c) + (N+1)/(N+1)c$

$\underline{T(N) = (N+1)/(N-(k-1))\ T(N-k) + c(N+1) \text{ sum i from 1 to k of } 1/(N+2-i)}$

Let $k = N$

$T(N) = (N+1)/(N-(N-1))\ T(N-N) + c(N+1) \text{ sum from 1 to N of } 1/(N+2-i)$

$T(N) = (N+1)/(1)\ T(0) + c(N+1)\ (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + 1/(N+1))$

$T(N) = N+1 + c(N+1)\ (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + 1/(N+1))$

Harmonic Series!

$T(N) = N+1 + c(N+1)\ (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + 1/(N+1))$

Harmonic Series!

$T(N) = N+1 + c(N+1)\ (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + 1/(N+1))$

Is in $\Theta(N + N\ (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + 1/(N+1)))$

Harmonic Series!

$T(N) = N+1 + c(N+1)\ (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + 1/(N+1))$

Is in $\Theta(N + N\ (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + 1/(N+1)))$

Which is $\Theta(N + N \log(N)) = \Theta(N \log(N))$