Network Flow Applications

Cutting Off The Water (Min Cut)

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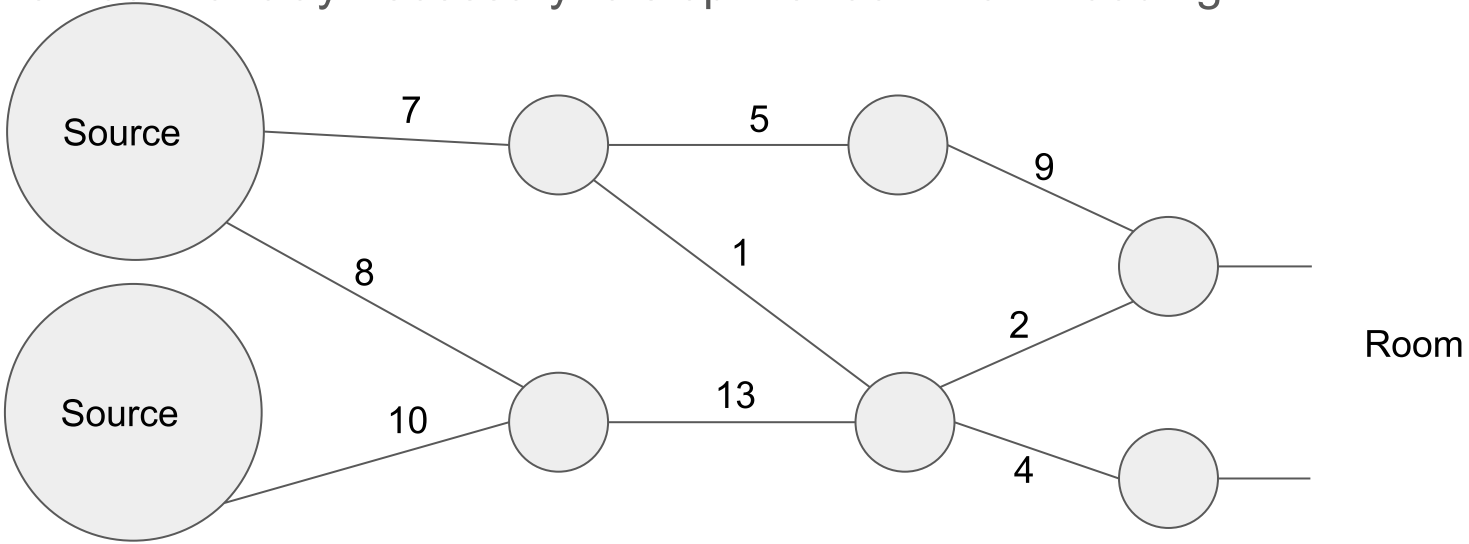

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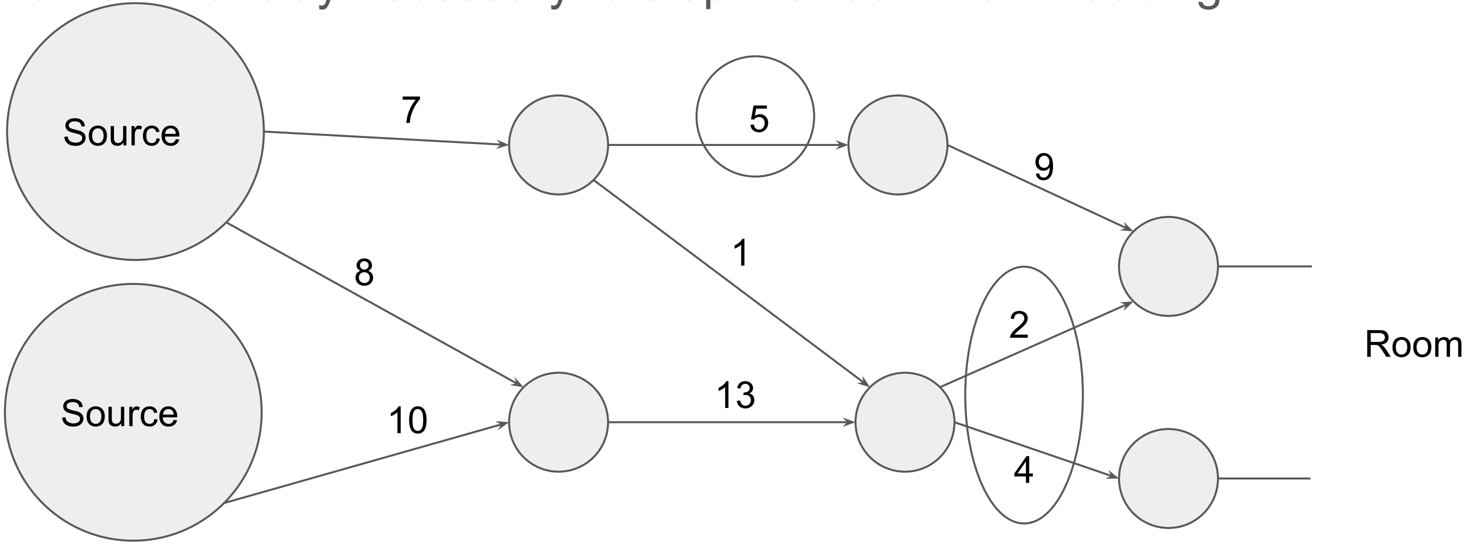

Solution

We can use the flow through the graph to find the minimum sum of values to cut off the source from the sink.

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It’s as easy as that.

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The sum of the capacities of these saturated edges represent the amount of flow that must be moving from s to t.

Correctness (cont.)

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Thus these two values are equal.

Group Problem Project (Bipartite Matching)

A group of students needs to complete some set of tasks to complete their project. Each student want to work on at most one task. Each task will be worked on by at most one task. Some students are incapable of working on certain tasks. Determine the maximum number of tasks the group can complete

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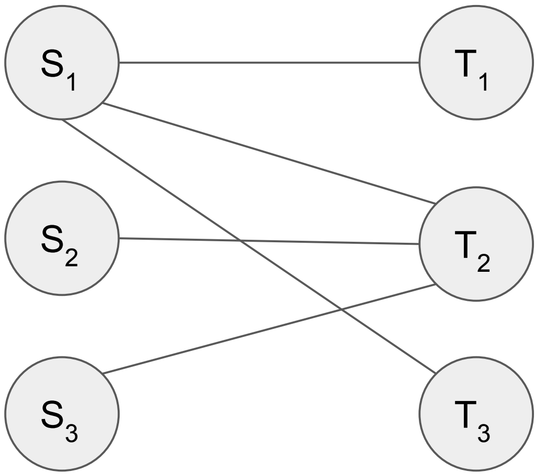

Bipartite Graph

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If you can partition the nodes into 2 groups where there is no edge going between 2 nodes within the same group, then the graph is bipartite.

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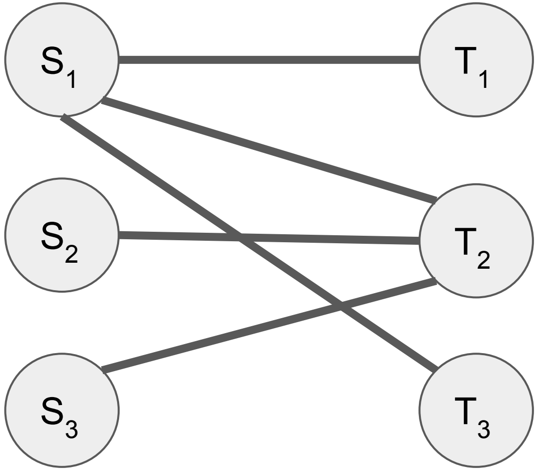

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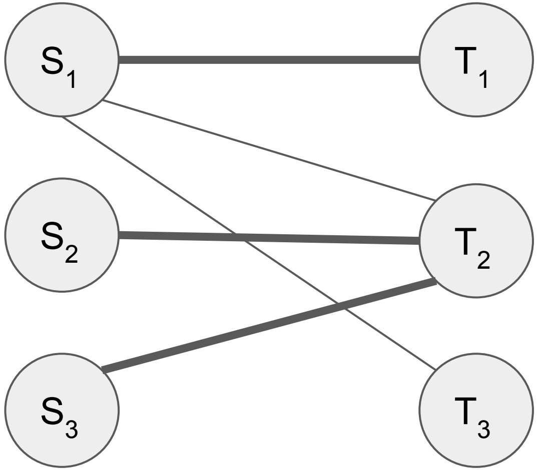

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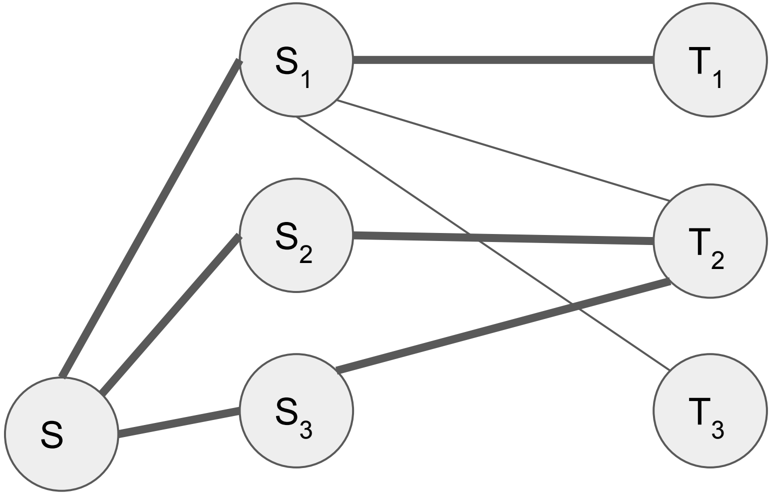

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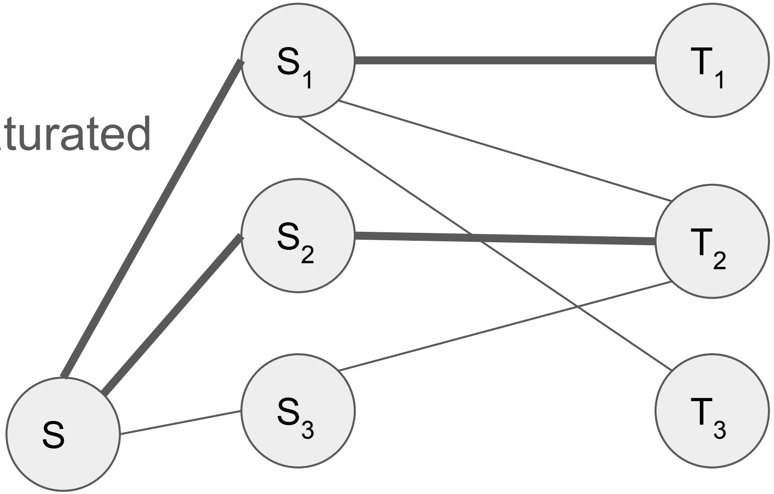

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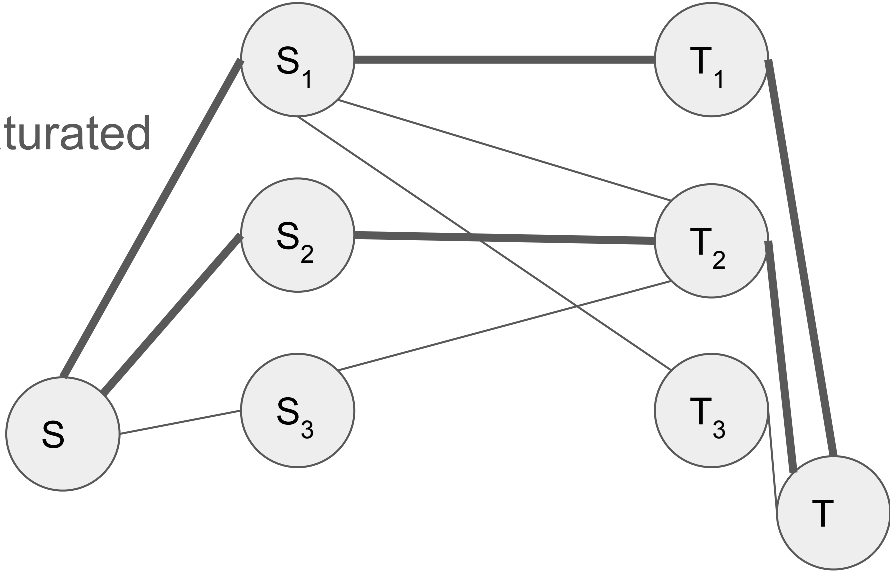
Make sure all the edges are directed.



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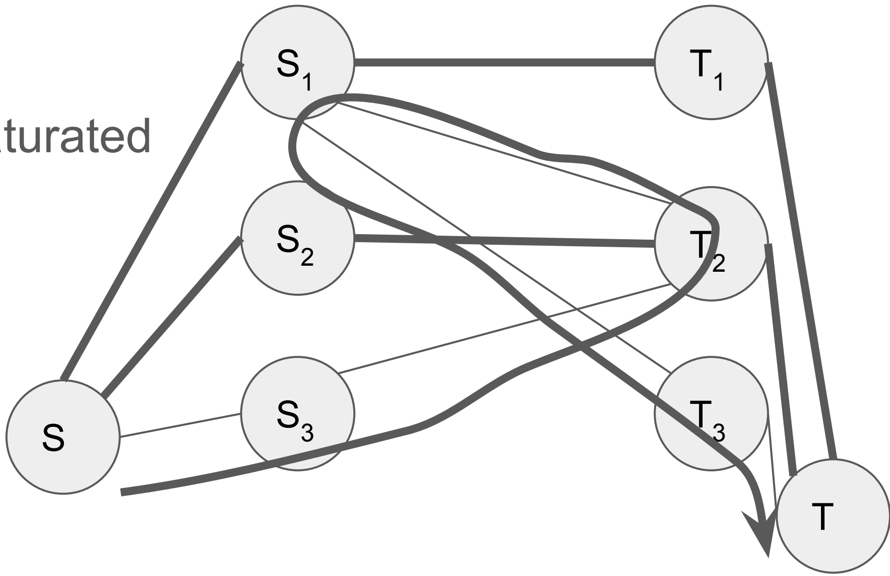
  Don’t push flow backwards



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What if some students could handle working on more than 1 task?

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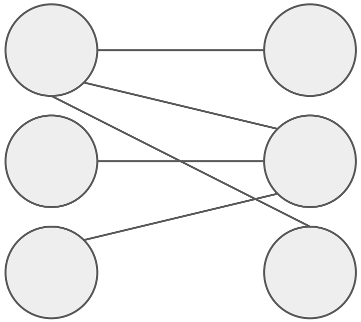

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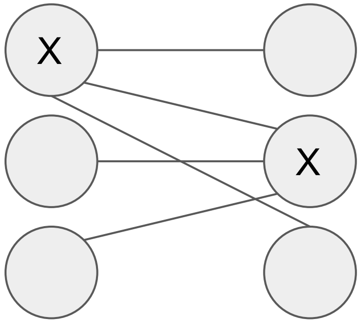

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Each “match” contains 2 nodes.

1 of these nodes should be removed.

Cut the least amount of nodes such that no matches remain.

Non-Trivial Flow Applications

Resource Trading (Circulation with U/L Bounds)

Suppose we have some countries. Each country has some rules on which countries they can trade with (e.g. **A** can send at most 10 resources to **B**). Each country wants to ensure that the amount of resources they export is equal to the resources they import. Additionally, some countries NEED to trade with other countries (e.g. **C** must send at least 3 resources to **D**). Determine if it possible for all the trade requirements to be met.

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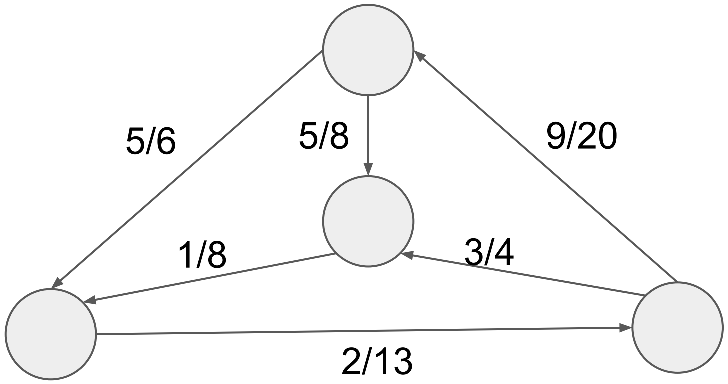
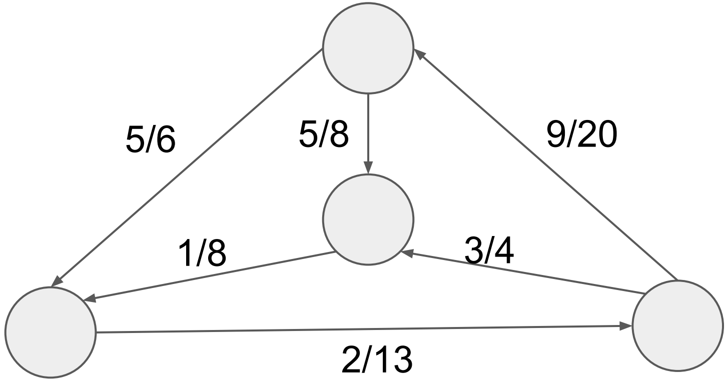


Solution

We must utilize some of the edges with a specified minimum.



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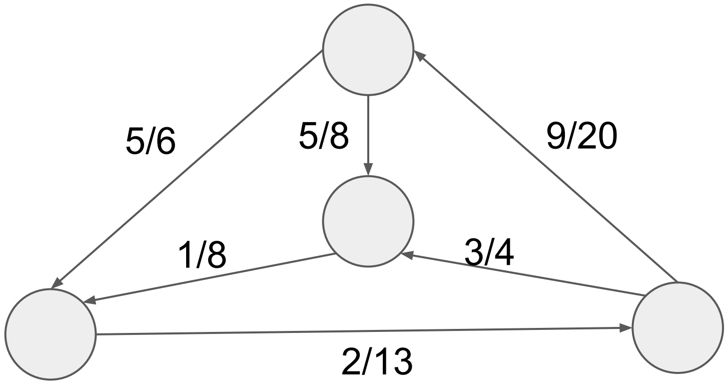
We must utilize some of the edges with a specified minimum.

To ensure we use an edge we can connect corresponding nodes to source and sinks.



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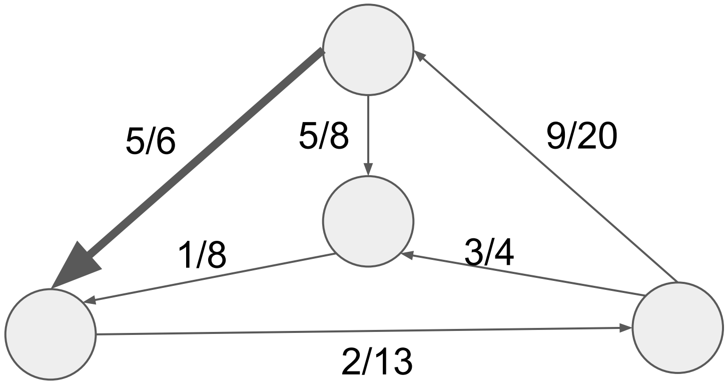

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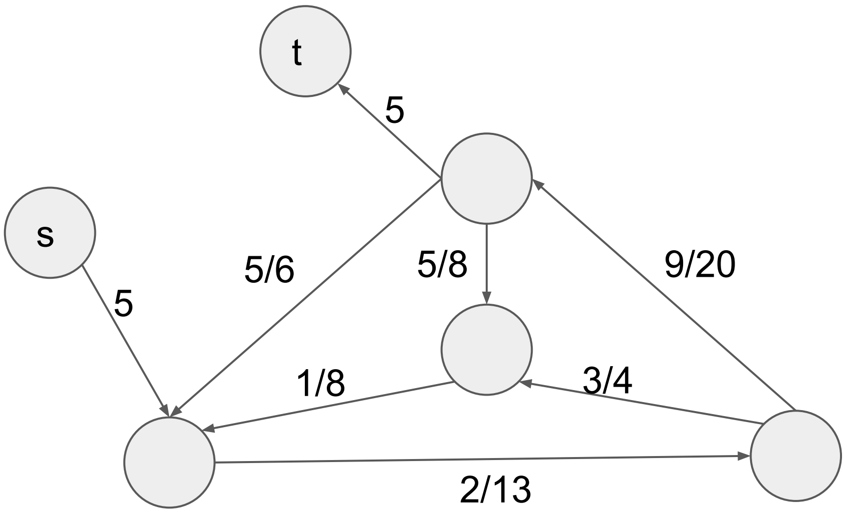

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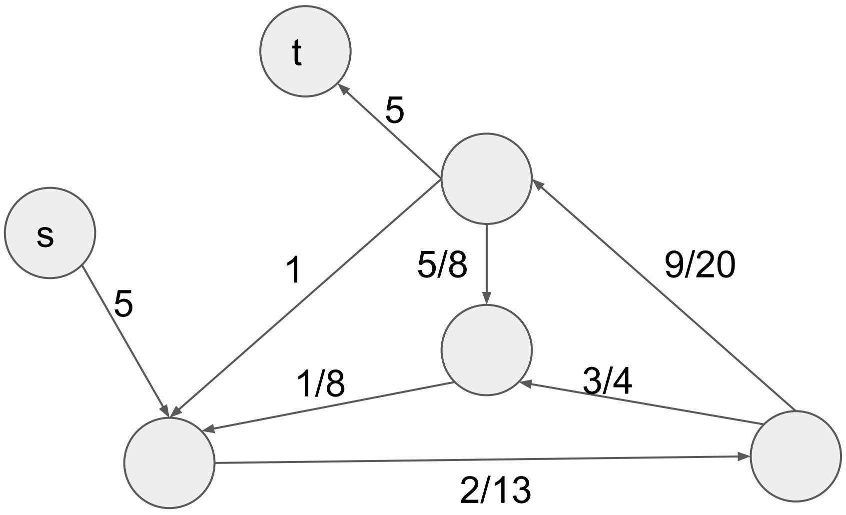

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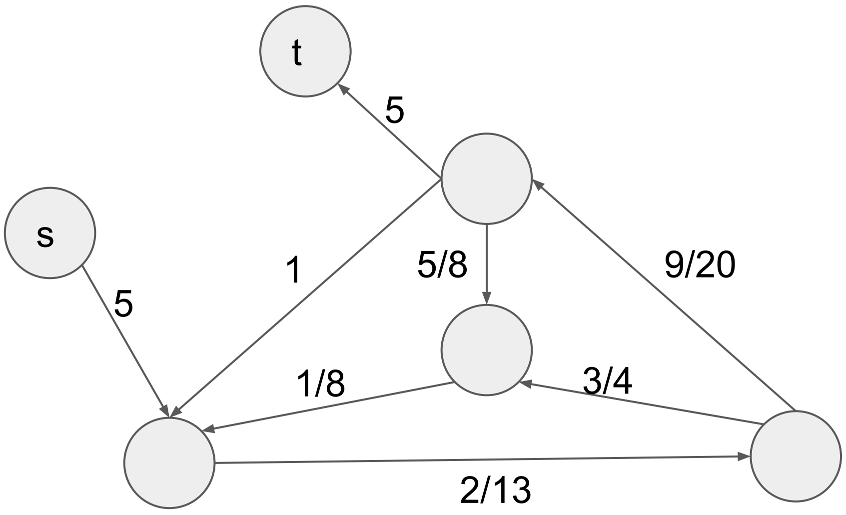

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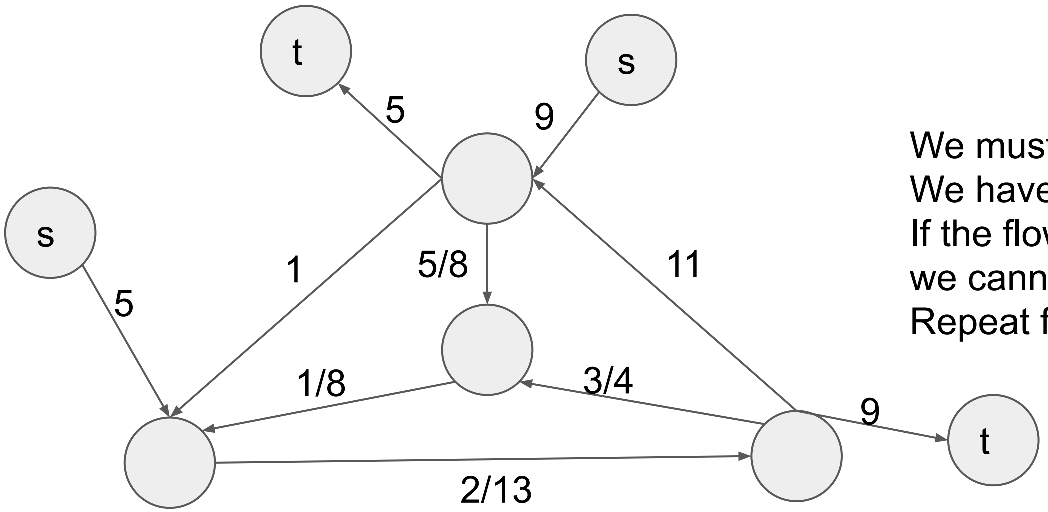

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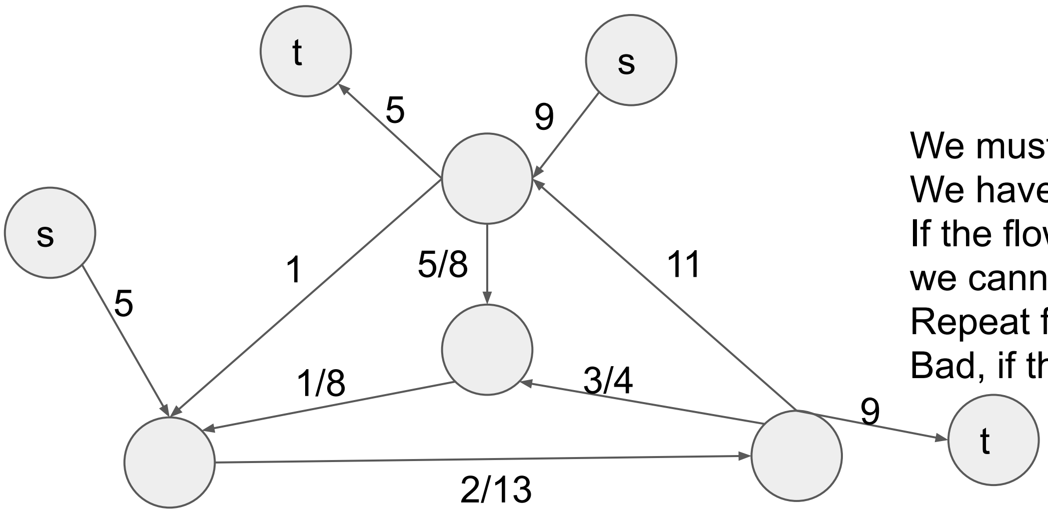

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Profit Maximization (Project Selection)

We have some projects we can work on. Each project will reward us with some money. Each project has some required machines to work on. Each machine has some 1 time cost. Some projects will share machines. We want to find the most money we can make with optimum purchasing.

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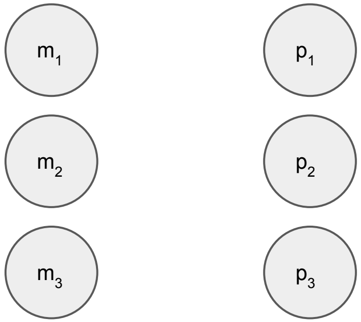

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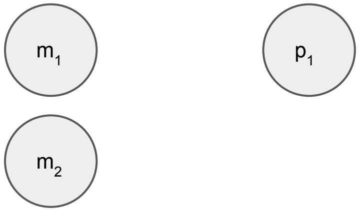
Suppose we have 1 project and 2 machines



Profit Max Flow Graph

Taxi Problem (Minimum Hiring Job Coverage)

Suppose we have a list of jobs a taxi company must fulfill. The taxi will start a some location at a specified time and will reach a destination at a specified time. The problem is that these jobs might need different taxis due to the time constraints. We know the time it takes to travel between the end point of one taxi job and the start point of another taxi job. What is the least number of taxis needed?

Taxi Problem Flow Graph

Image Segmentation

We have some pixels that have some probability of belonging to a foreground, and some probability of belonging to the background. The foreground is usually a clump of pixels. Pixels that are close to each other have a probability of belonging to the same group.

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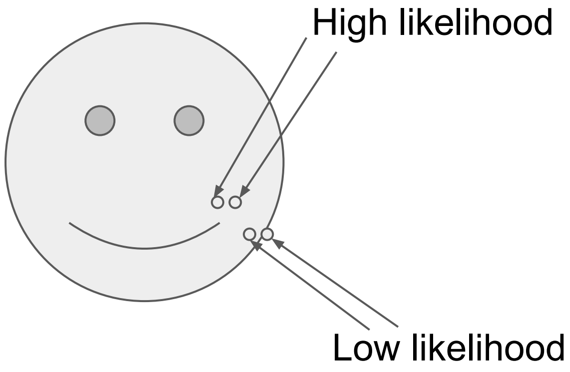

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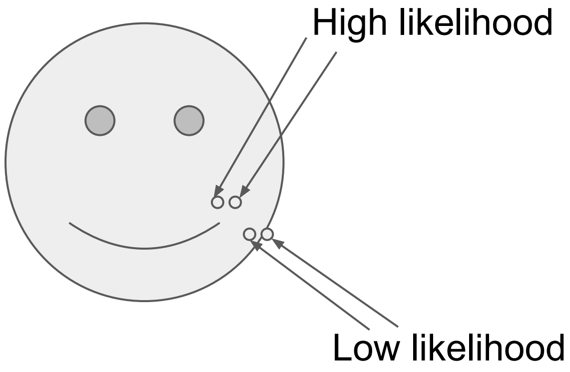

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Using the concepts of logs, the problem can become maximize a sum of values.

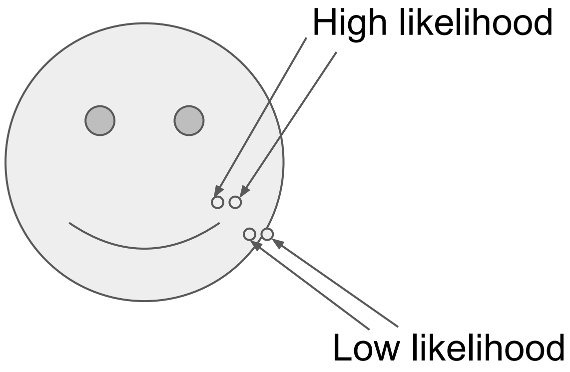

Image Segmentation Flow Graph

Baseball “Elimination”

Give a game schedule and the results of a few games. Determine which teams have the potential to finishing a season with the most number of wins.

Baseball Elimination Flow Graph