Strongly Connected Components

Components with Direction

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- Real nodes could only ever reach a real node;
- AND all real nodes were able to reach all other real nodes.

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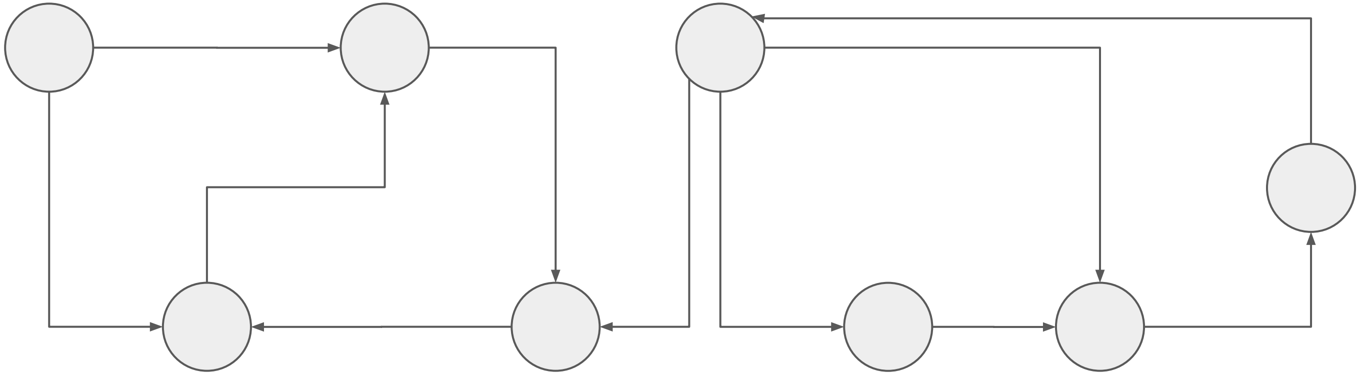

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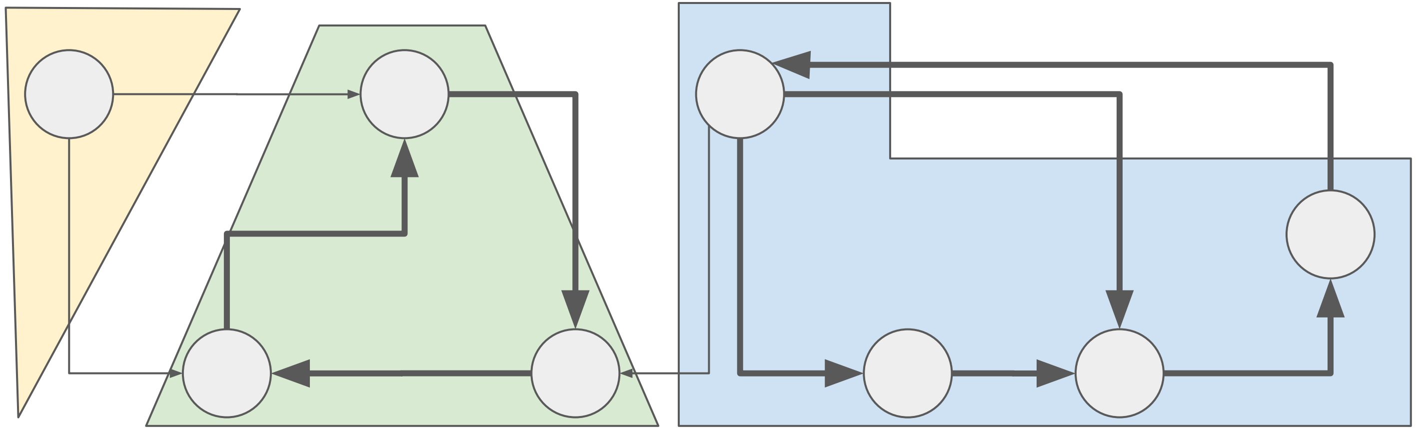

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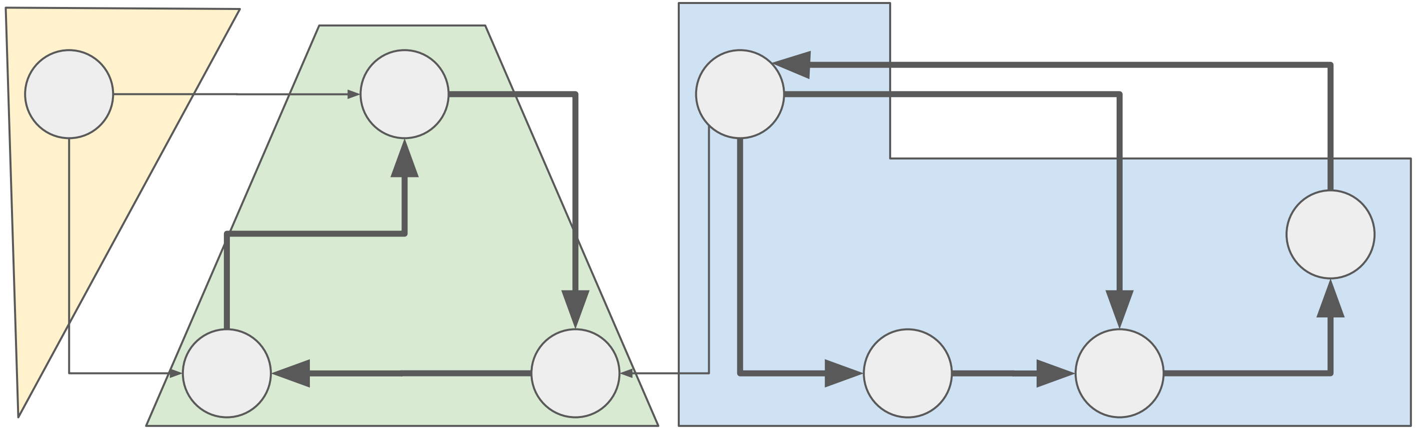

<u>Note: SCCs should only be used on directed graphs</u>

Things to Note

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In the example before yellow could reach green, blue could reach green, but green could not reach other nodes.

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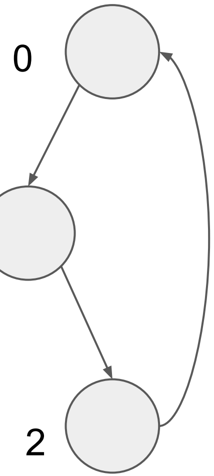

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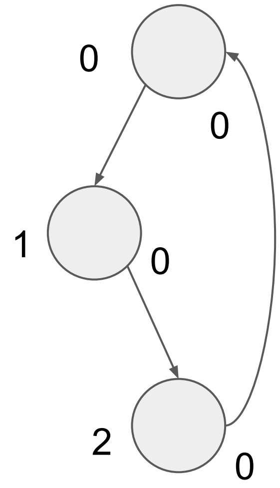

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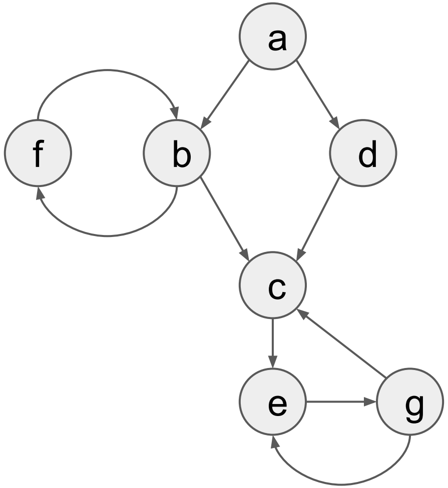
All these nodes have links to nodes that eventually reach the top, and the top can reach all nodes that have greater pre order. Thus the SCC is formed.

Example



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O(E + V)

SCC Application

Checkposts Problems

https://codeforces.com/problemset/problem/427/C

Problem 1: Checkposts

Codeforces Div. 2 C

Given a city composed of junctions and one way roads that connects junctions, place posts to protect the city. At each junction a post can be built. Posts protect the junction they are placed in. Additionally, posts can send police cars to other junctions to protect them as long as the police car can travel back to the post it came from. Cities have a unique cost for building a post.

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Codeforces Div. 2 C

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Determine the minimum cost to protect all junctions **in the fewest number of posts**.

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We want to minimize the cost by finding the minimum cost post in each SCC.

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We want to find out if a solution exists.

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We use arrows to denote implication (b $\rightarrow$ c) ($\neg$c $\rightarrow$ $\neg$b)

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Such a constraint would be impossible to satisfy impossible

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- Turn each variable assignment into a node (both a and ¬a become nodes).
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- If a and ¬a are in the same scc then they can imply each other using transitivity.
- Check each variable and see if they belong to the same SCC.

Problem 2:

https://codeforces.com/problemset/problem/27/D

Problem 2: Ring Road 2

Codeforce Div. 2 D

There is a land with N cities in a circle. Road will be build inside or outside the circle. We don't want roads to cross (outside of a city). Determine if possible and if so print out a way to build the roads (state whether each road is inside or outside the circle.

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If we use 2 SAT we typically want variables that can be set to TRUE or FALSE (two choices). We have this ability. We have roads that can be set to INSIDE or OUTSIDE.

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If we use 2 SAT we typically want variables that can be set to TRUE or FALSE (two choices). We have this ability. We have roads that can be set to INSIDE or OUTSIDE.

If certain roads are inside we could potentially have other roads that must become outside.

Approach 2

We can also greedily assign roads and BFS. :\

Problem 3: 2-SAT 2

TROY Query
https://codeforces.com/gym/100570/problem/D

Problem 3: TROY Query

Codeforces Div. 1 D

We have 2 grids of 10^{18}x10^{18}. Each spot is either -1 or +1. We can flip rows and columns by multiplying every spot in the row by -1 or every spot in the column by -1. We are told 1 spot at a time what the values in each grid are. We want to know for the information given so far if the two grids can be transformed into each other (assuming we choose the unknown bit).

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What can we do to fix?

Note: at some point the Grids will be incapable of reaching each other, and regardless of the extra information it will always be impossible.

Binary Search

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$O(\log(Q)*(V + E))$