

Dynamic Programming

Introduction

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- Sequence Alignment
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- Matrix Chain Multiplication

Fibonacci

Number sequence

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$$f_1 = 1$$

Fibonacci

Number sequence

$$f_1 = 1; f_2 = 2$$

Fibonacci

Number sequence

$$f_1 = 1; f_2 = 2; f_n = f_{n-1} + f_{n-2} \text{ for } n > 2$$

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Naïve solution,

Fibonacci

Number sequence

$$f_1 = 1; f_2 = 2; f_n = f_{n-1} + f_{n-2} \text{ for } n > 2$$

Naïve solution,

```
Function F (N)
```

```
    If N < 3
```

```
        Return N
```

```
    End If
```

```
    Return F (N-1) + F (N-2)
```

```
End Function
```

Fibonacci

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$$f_1 = 1; f_2 = 2; f_n = f_{n-1} + f_{n-2} \text{ for } n > 2$$

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    If N < 3
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What is the runtime for computing F(N)?

Fibonacci Improvement

Why are we recomputing $F(3)$, $F(4)$, ...

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- After we determine $F(n)$ store the value.
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Initialize answer with INVALID

Function $F(N)$

If $N < 3$

Return N

End If

If answer[N] is not INVALID

Return answer[N]

End If

Return answer[N] = $F(N-1) + F(N-2)$

End Function

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What is the runtime for computing $F(N)$?

What is the memory footprint?

Can we improve the memory usage?

Iterative and Space Saving

Memoization and recursion is a common dynamic programming approach.

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- Figuring out the sub problem solutions prior to needing them.
- Moving in the reverse direction of function returns.

```
answer[1] = 1; answer[2] = 2;  
For i = 3 to N  
    answer[i] = answer[i-1] + answer[i-2]  
End For  
Return answer[N]
```

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The memory footprint is still $O(N)$

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Dynamic programming approaches can be turned into an iterative approach by

- Figuring out the sub problem solutions prior to needing them.
- Moving in the reverse direction of function returns.

```
Previous = 1; Current = 2;  
For i = 3 to N  
    Next = Previous + Current // Compute  
    Previous = Current        // Update answer[N]  
    Current = Next            // Update answer[N-1]  
End For  
Return Current
```

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- Figuring out the sub problem solutions prior to needing them.
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Previous = 1; Current = 2;  
For i = 3 to N  
    Next = Previous + Current // Compute  
    Previous = Current        // Update answer[N]  
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End For  
Return Current
```

The memory footprint is now $O(1)$

0-1 Knapsack

We are going on a trip. We have items we can take. We would like to take everything, but we have a limited carrying capacity. Each item has some weight and value AND can only be take once. We want to know what is the largest value sum we can take.

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Carry capacity 12

Item	Weight	Value
Banana	2	3
Trail Mix	3	6
Toilet Paper	10	15
Sleeping Bag	9	11
Pillow	5	6
Butane Lighter	1	1

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What subproblem would be useful to solving this problem?

0-1 Knapsack Solution

Subproblem:

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What is the best value sum using a knapsack with capacities up to the maximum?

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- What is the best knapsack of cap 0?

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What is the best value sum using a knapsack with capacities up to the maximum?

In other words,

- What is the best knapsack of cap 0?
- What is the best knapsack of cap 1?

0-1 Knapsack Solution

Subproblem:

What is the best value sum using a knapsack with capacities up to the maximum?

In other words,

- What is the best knapsack of cap 0?
- What is the best knapsack of cap 1?
- What is the best knapsack of cap 2?
- What is the best knapsack of cap 3?

0-1 Knapsack Solution

Subproblem:

What is the best value sum using a knapsack with capacities up to the maximum?

In other words,

- What is the best knapsack of cap 0?
- What is the best knapsack of cap 1?
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- ...
- What is the best knapsack of cap $\leq \text{max}$?

0-1 Knapsack Solution

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What is the best value sum using a knapsack with capacities up to the maximum?

In other words,

- What is the best knapsack of cap 0?
- What is the best knapsack of cap 1?
- What is the best knapsack of cap 2?
- What is the best knapsack of cap 3?
- ...
- What is the best knapsack of cap $\leq \text{max}$?

We also need to know what items we have left to select...

0-1 Knapsack DP State

We need to know capacity (either left or used) AND the last item taken.

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The method below is not efficient enough for my tastes...

```
Function K(c, i) // cap left and last item taken
    If seen state c and i // Check if we know the solution
        Return memo table at c and i // Return the known solution
    End If
    Initialize memo table at c and i with 0
    For j = i + 1 to Last item // Try all untaken items
        If can take item j with capacity c // Check the capacity is large enough
            Update memo table at c and i with K(c - weight of j, j) + value of j
        End If
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What is the worst runtime?

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The Trick.

State Space AND Transition Time Optimization

The 0-1 knapsack problem can be solved faster using better transition times.

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Take it or don't,

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Take it or don't,

- Take the current item (item moves to next position; update capacity/value)

State Space AND Transition Time Optimization

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The current runtime for the transition is $O(\# \text{ items})$.

The 0-1 knapsack can transition in $O(1)$. How?

Take it or don't,

- Take the current item (item moves to next position; update capacity/value)
- Don't take the current item (item moves to next position)

Updated Algorithm

```
Function K(c, i) // cap left and last item taken
  If i is at the end // Check if we cannot add more items
    Return 0 // Return no extra value
  End If
  If seen state c and i // Check if we know the solution
    Return memo table at c and i // Return the known solution
  End If

  Initialize memo table at c and i with K(c, i + 1) // Don't take the item
  Update memo table at c and i with K(c - weight of i, i) + value of i // Take the item

  Return memo table at c and i // Return the found solution
End Function
```

Iterative 0-1 Knapsack

We loop over each item and...

Update an array that represents the best value for each knapsack capacity.

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We update by looping over the array of different knapsack capacities and...

- Updating the spots using other knapsacks with a lesser capacity.

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Suppose we have the following knapsack answers,

Cap	0	1	2	3	4	5	6
Value	0	1	4	5	5	7	8

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Let's add a weight 2, value 3 to the given list.

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Updating in the forward direction...

Iterative 0-1 Knapsack [Example]

Moving forward updates...


Weight 2 value 3.

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$$0 + 3$$

Iterative 0-1 Knapsack [Example]

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$0 + 3$
no good

Iterative 0-1 Knapsack [Example]

Moving forward updates...

Weight 2 value 3.




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Iterative 0-1 Knapsack [Example]

Moving forward updates...

Weight 2 value 3.




Cap	0	1	2	3	4	5	6
Value	0	1	4	5	5	7	8

$$4 + 3$$

Iterative 0-1 Knapsack [Example]

Moving forward updates...

Weight 2 value 3.




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4 + 3
good!

Iterative 0-1 Knapsack [Example]

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Weight 2 value 3.




Cap	0	1	2	3	4	5	6
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4 + 3
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Iterative 0-1 Knapsack [Example]

Moving forward updates...

Weight 2 value 3.




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$$5 + 3$$

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Moving forward updates...

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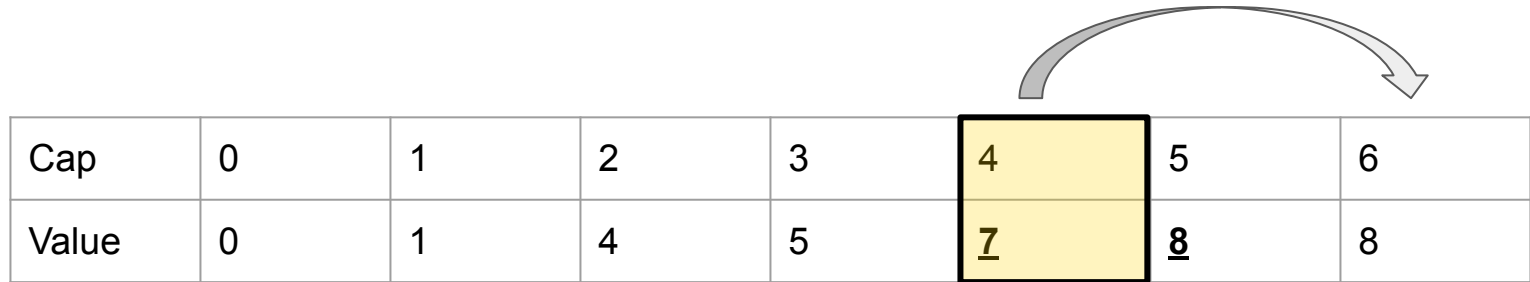
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good!

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Moving forward updates...

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Moving forward updates...

Weight 2 value 3.



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Value	0	1	4	5	<u>7</u>	<u>8</u>	8

7 + 3
crap!

Iterative 0-1 Knapsack [Example]

Moving forward updates...

Weight 2 value 3.

We actually want to move in the reverse to prevent using an item twice!

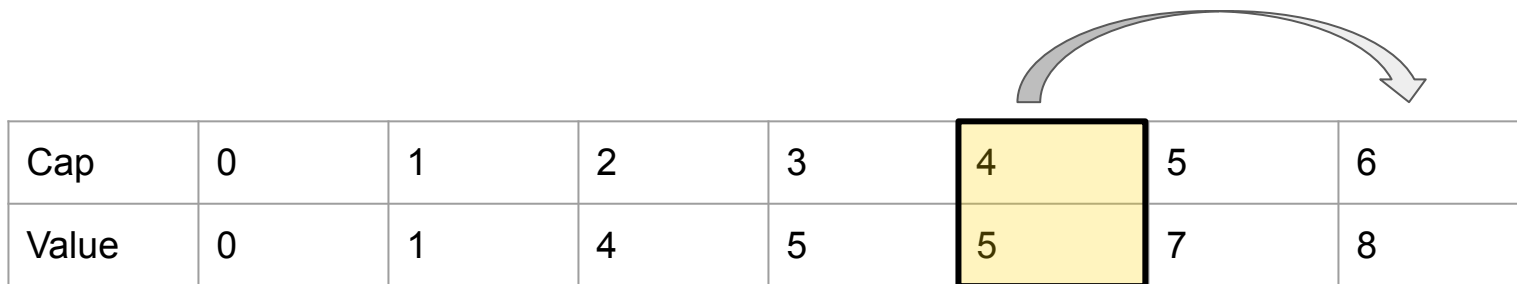
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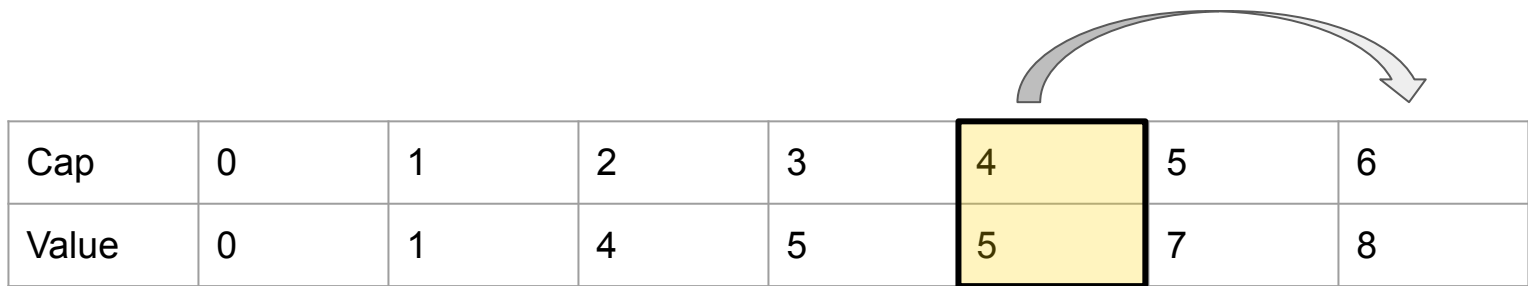
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5 + 3
better-ish!

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Moving forward updates...

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Iterative 0-1 Knapsack [Example]

Moving forward updates...

Weight 2 value 3.

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2 updates later...



Cap	0	1	2	3	4	5	6
Value	0	1	4	5	<u>7</u>	<u>8</u>	8

0 + 3
done!

Iterative 0-1 Knapsack [Example]


Moving forward updates...

Weight 2 value 3.

We actually want to move in the reverse to prevent using an item twice!

Runtime?

2 updates later...



Cap	0	1	2	3	4	5	6
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0 + 3
done!

Other Knapsack Problems

0- ∞ knapsack

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Knapsack with items selected any number of times.

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Do the forward updating iterative method!

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Knapsack with items selected any up to k times.

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Knapsack with items selected any up to k times.

Do the reverse updating iterative method k times for each item!