Fenwick Trees

AKA Binary Index Trees

Overview

- The Problem
- The Ugly
- The Bad
- The Good

Motivation

- Competitive programming
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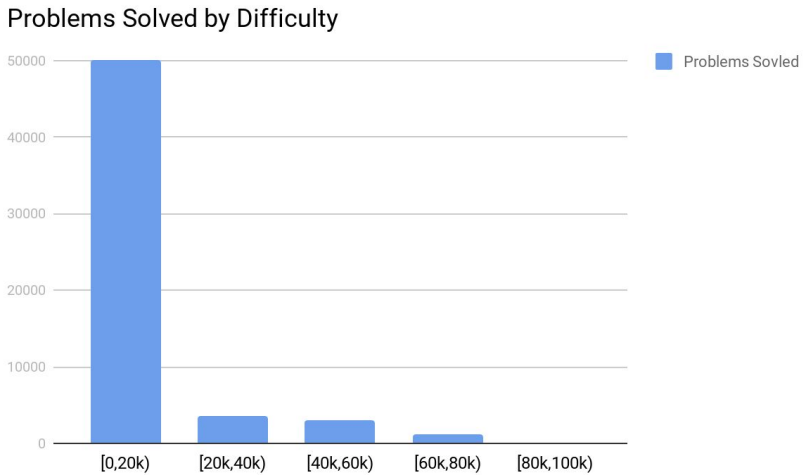

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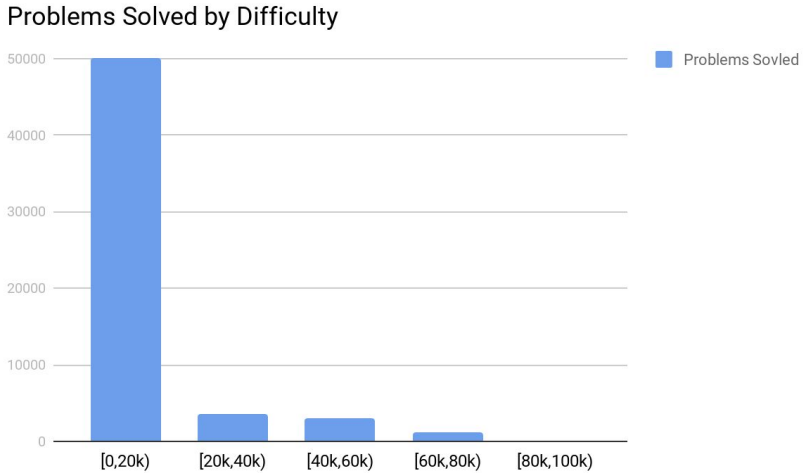
- Each problem has a difficulty of 0 to 100k
- Obviously 0 is hardest



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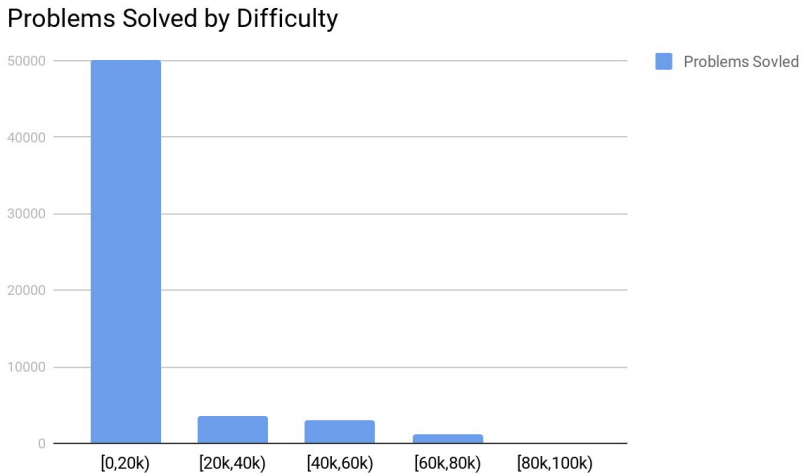

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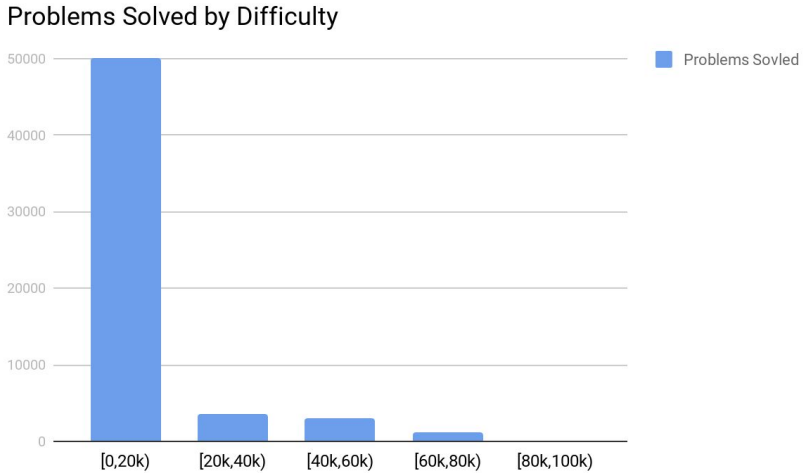

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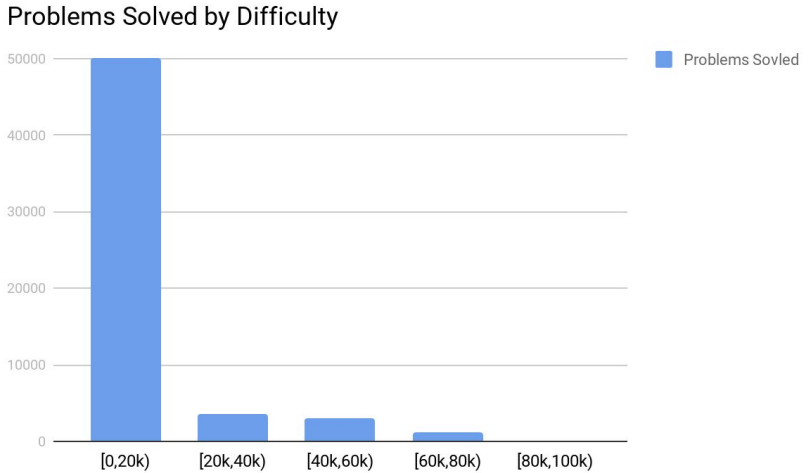

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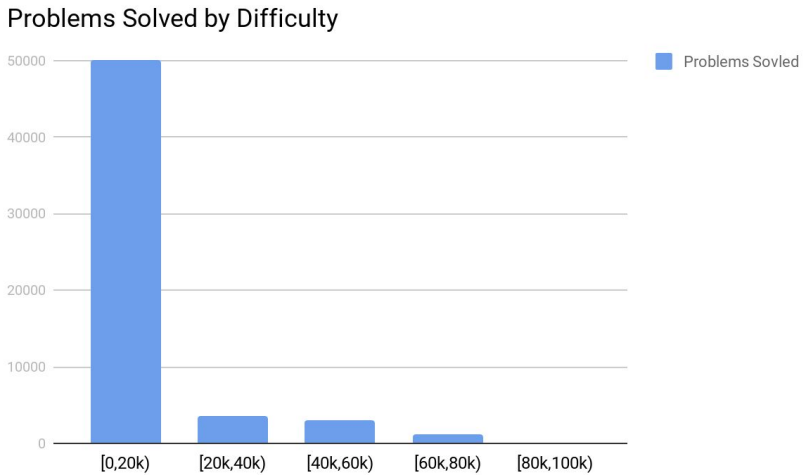

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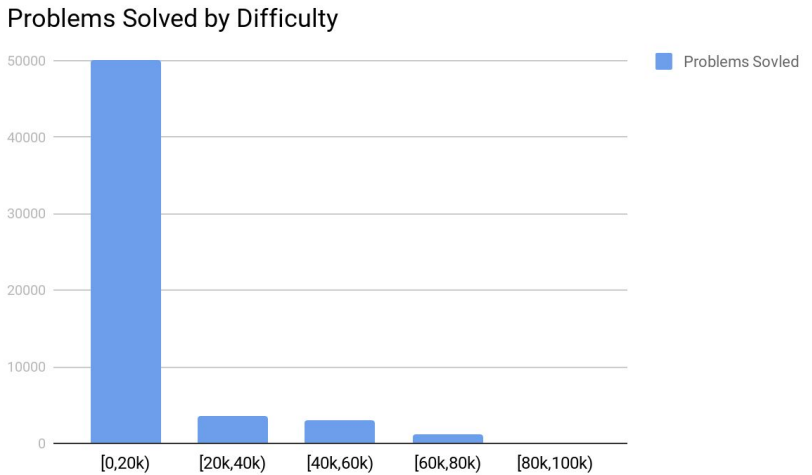

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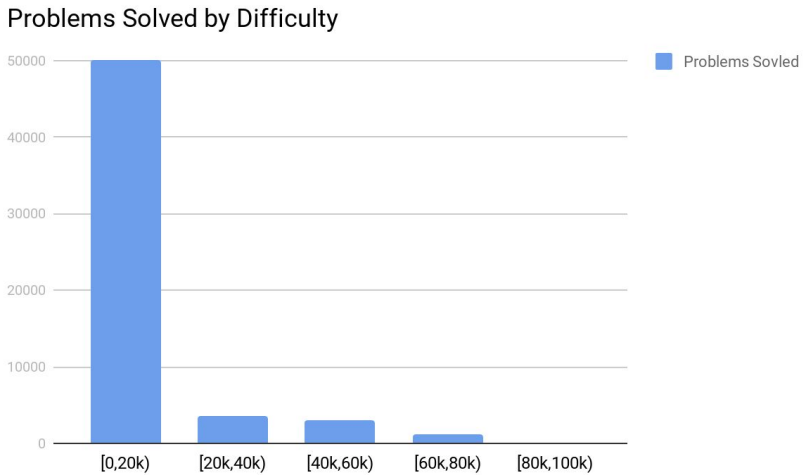

O(NQ) where N is number of problems and Q is number of queries
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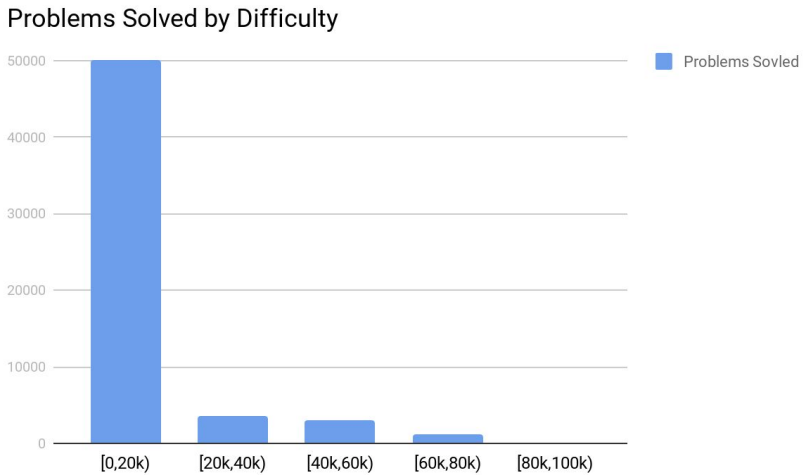

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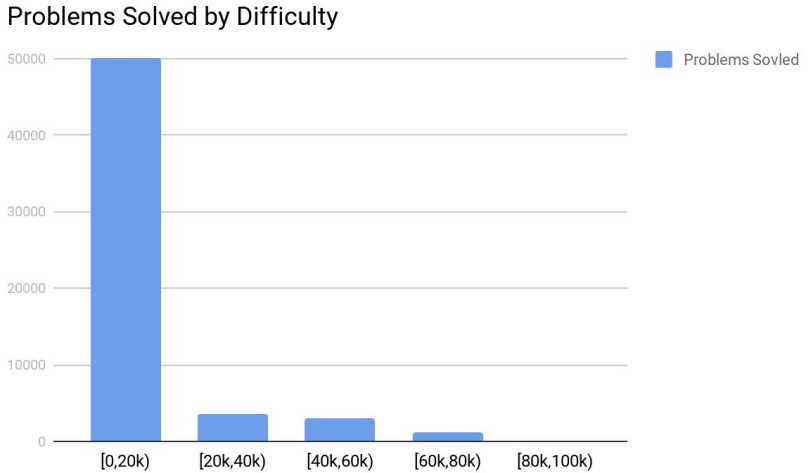

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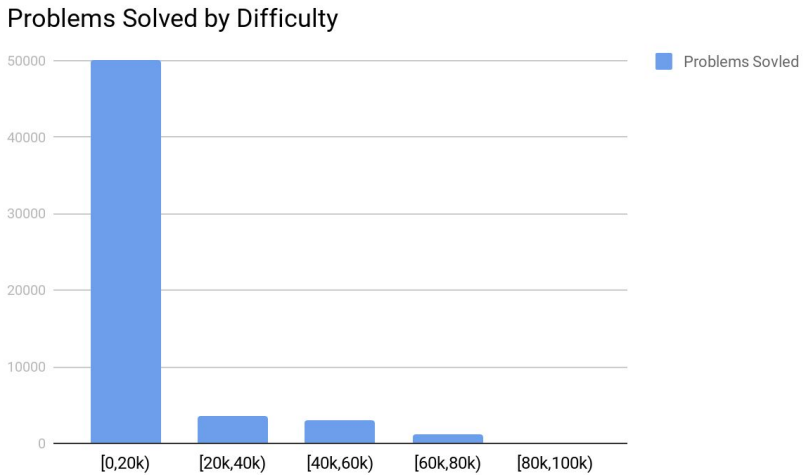

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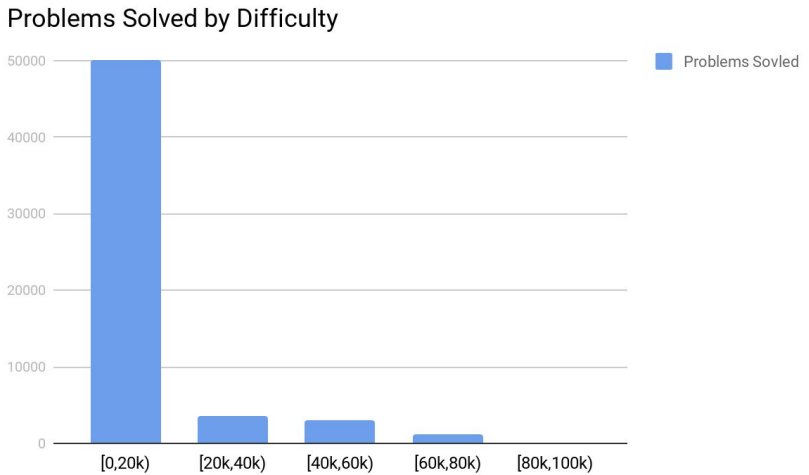

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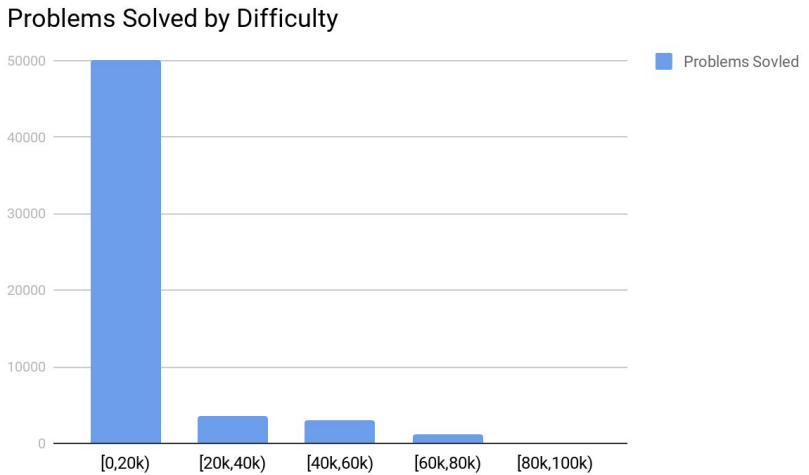

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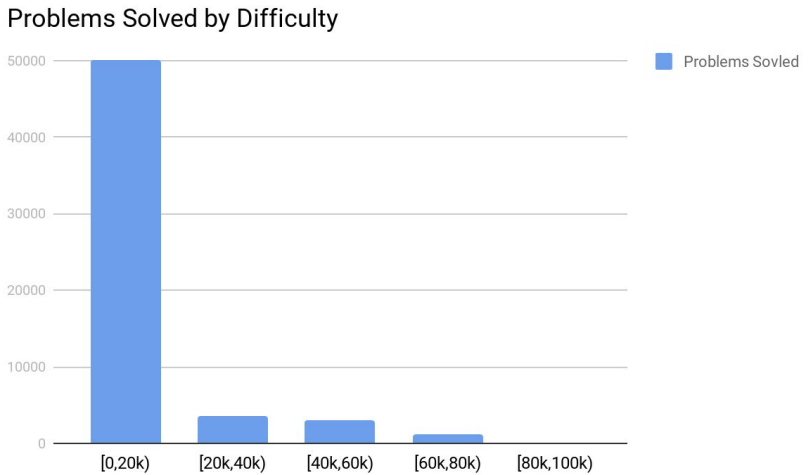

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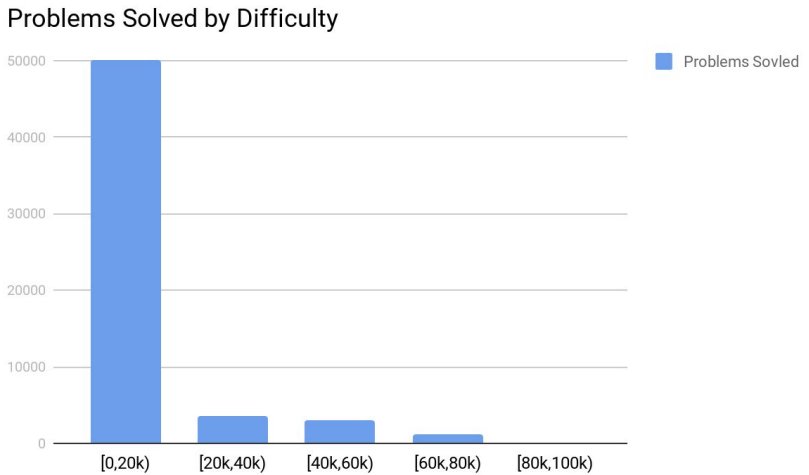

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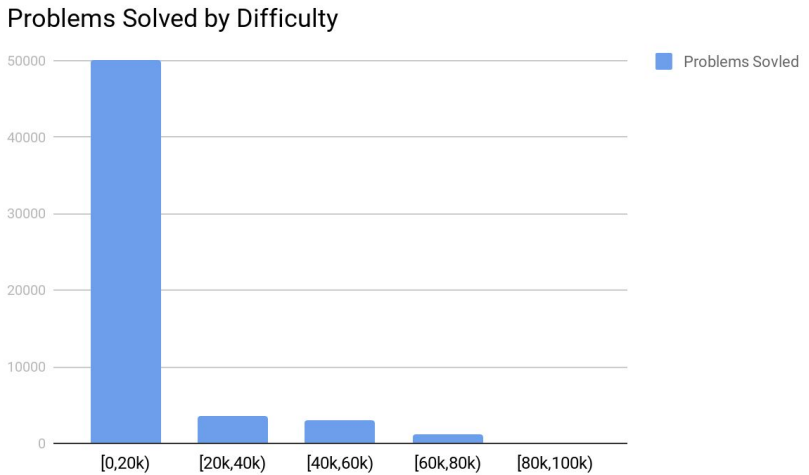
- IF d is around 100k
  - 100k changes is 10^{10}



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- Many solutions seem promising but fall short
  - Modify prefix sum
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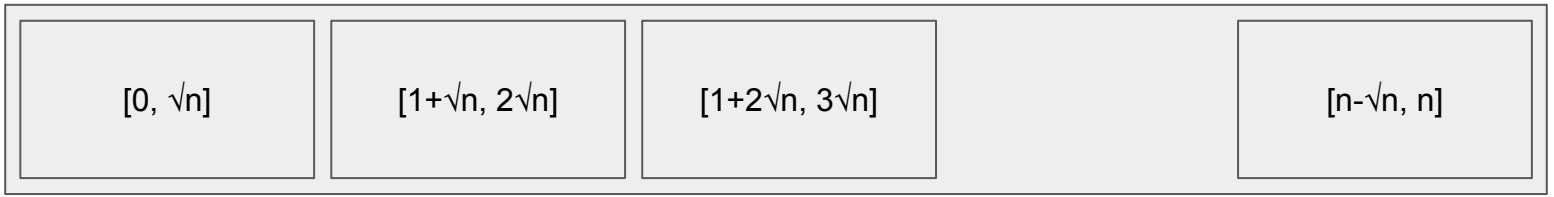
- Don’t do these please
  - Your (and possibly my) computer will cry

The Bad

- A bit complex, but we can break the difficulty into buckets of sqrt N size



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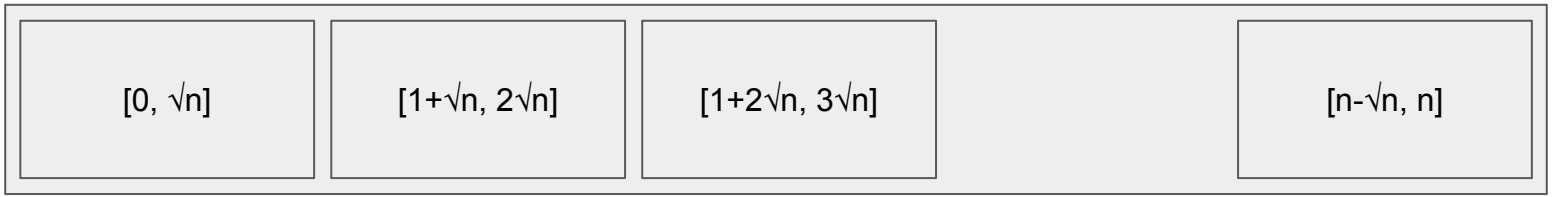

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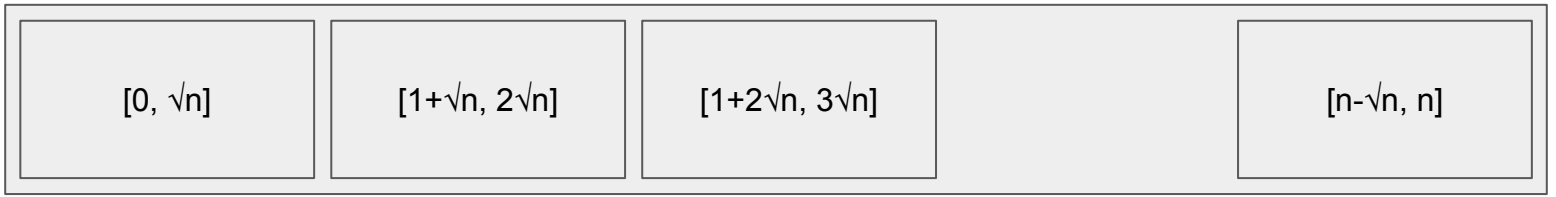
- To sum a range…
  - We add all the buckets completely covered (at most root n)
  - We the values manually for those in the ending buckets



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The sum is 3 + 10 + 2 + **15** + **5** + 0 = 35

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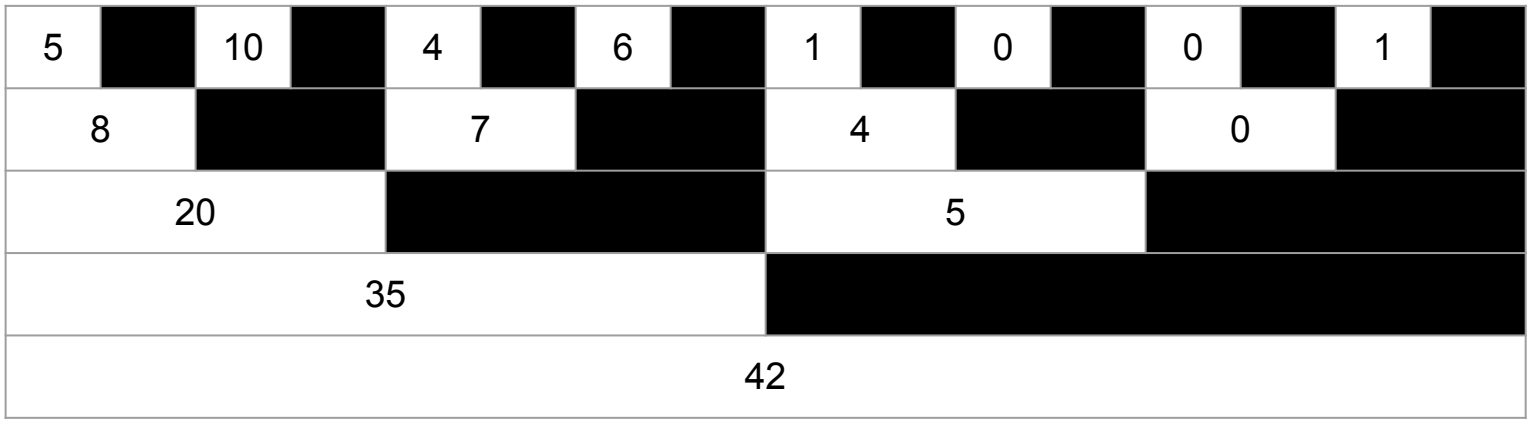

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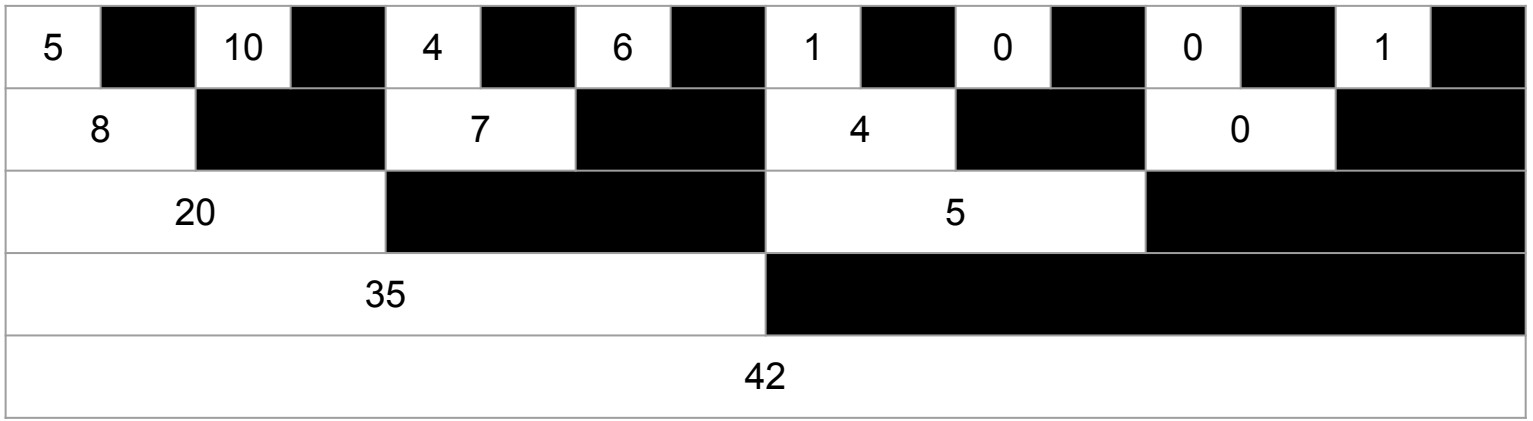

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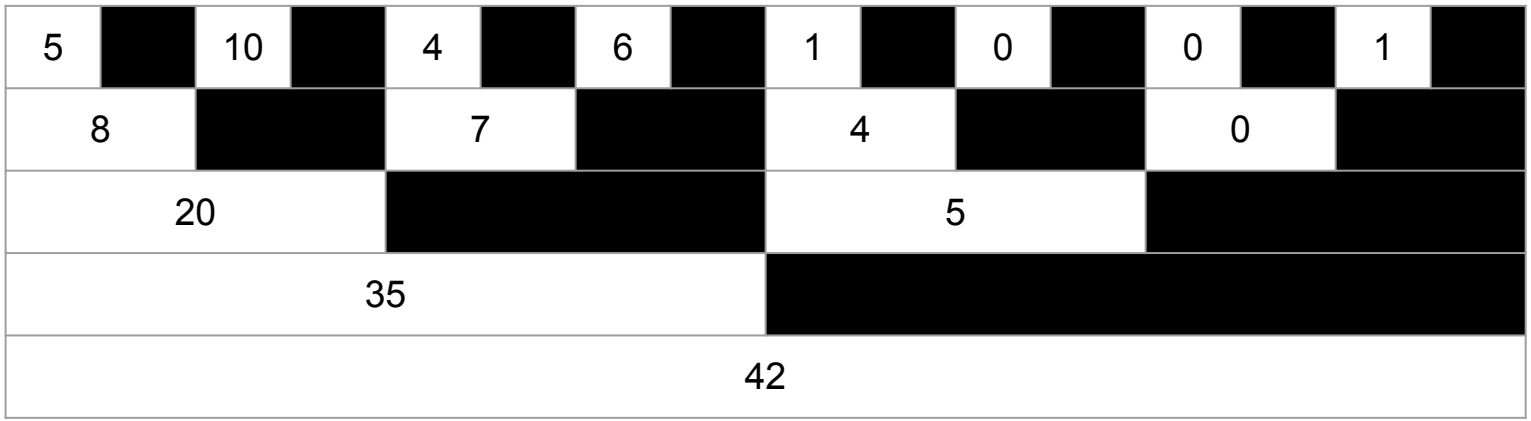

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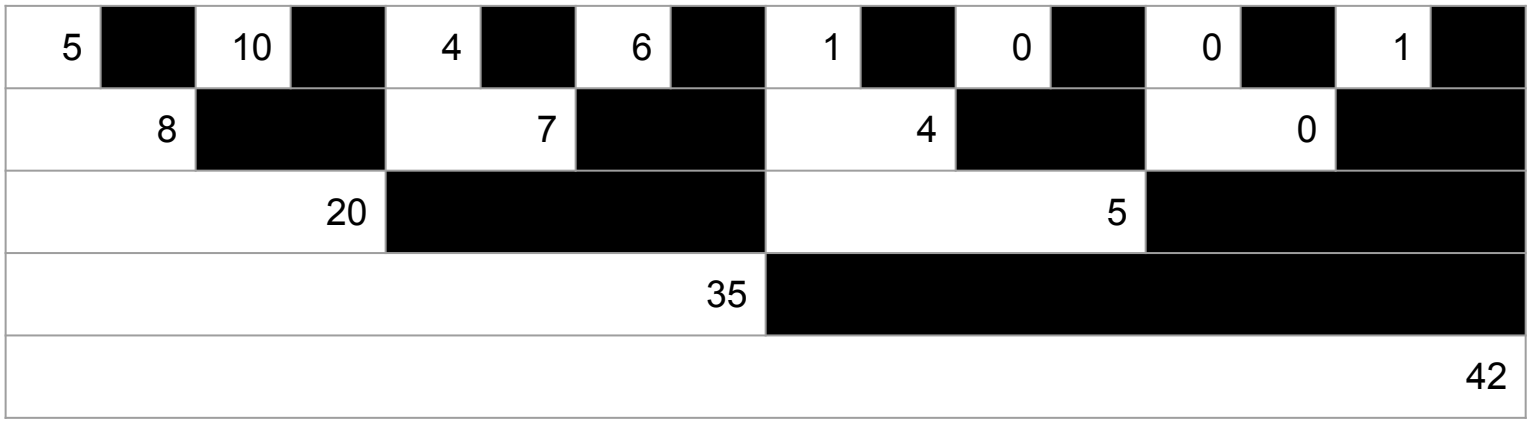

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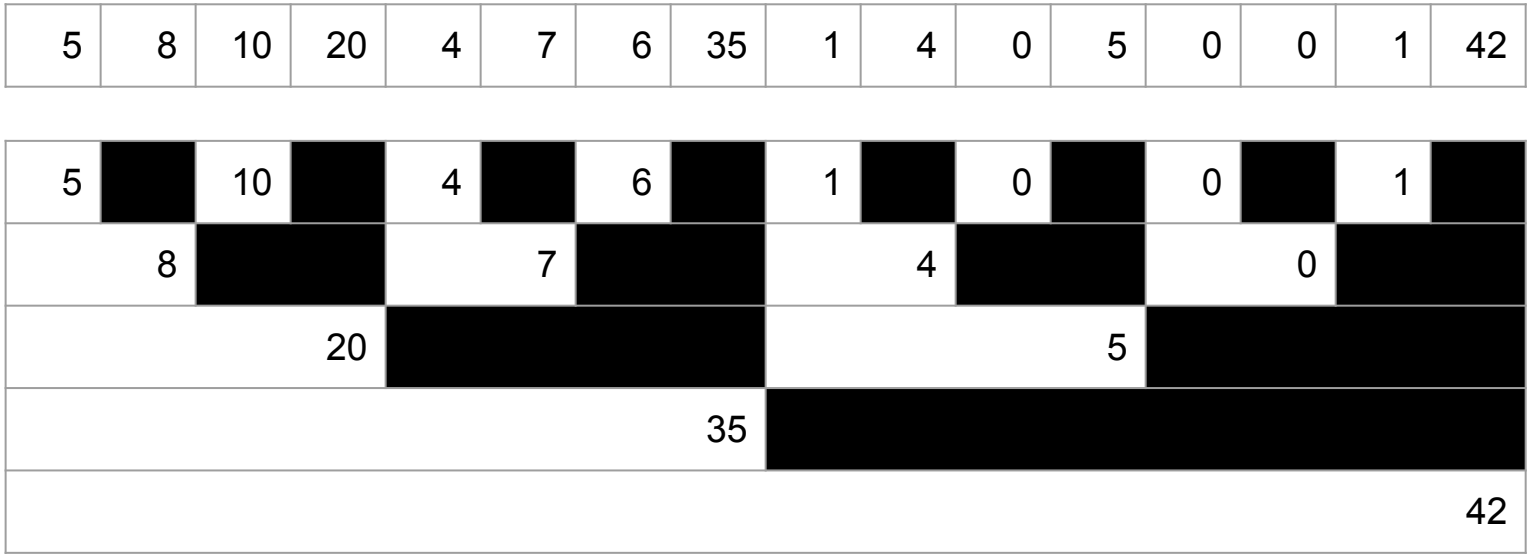

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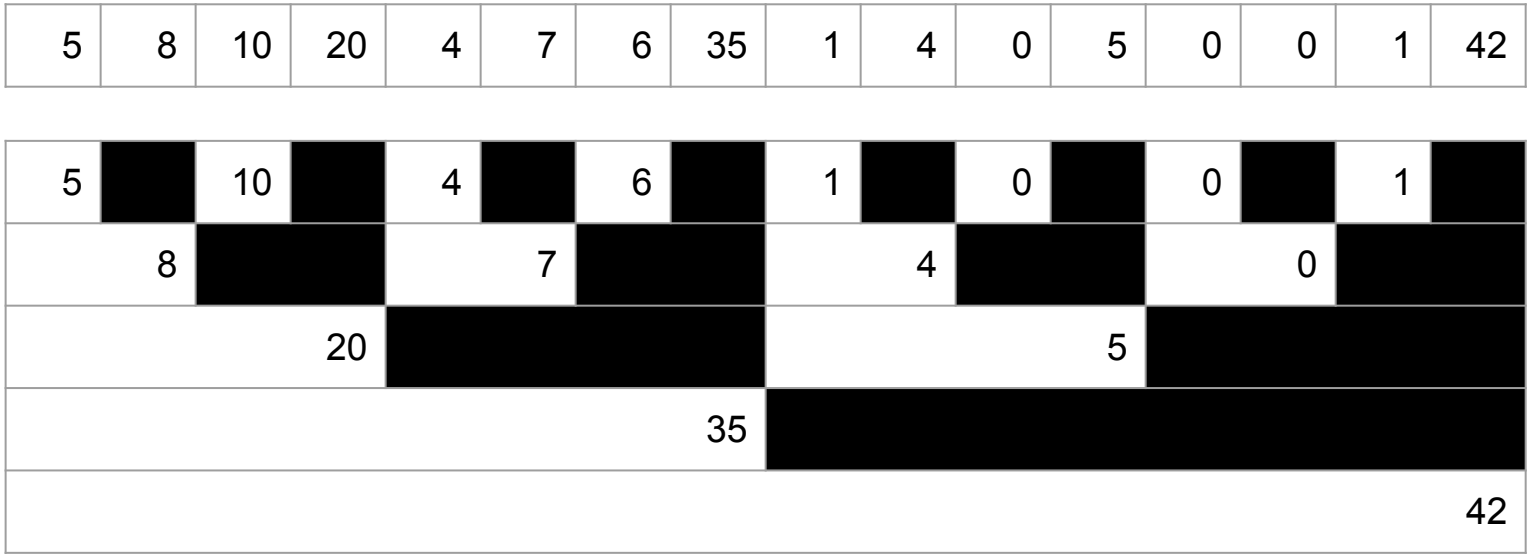

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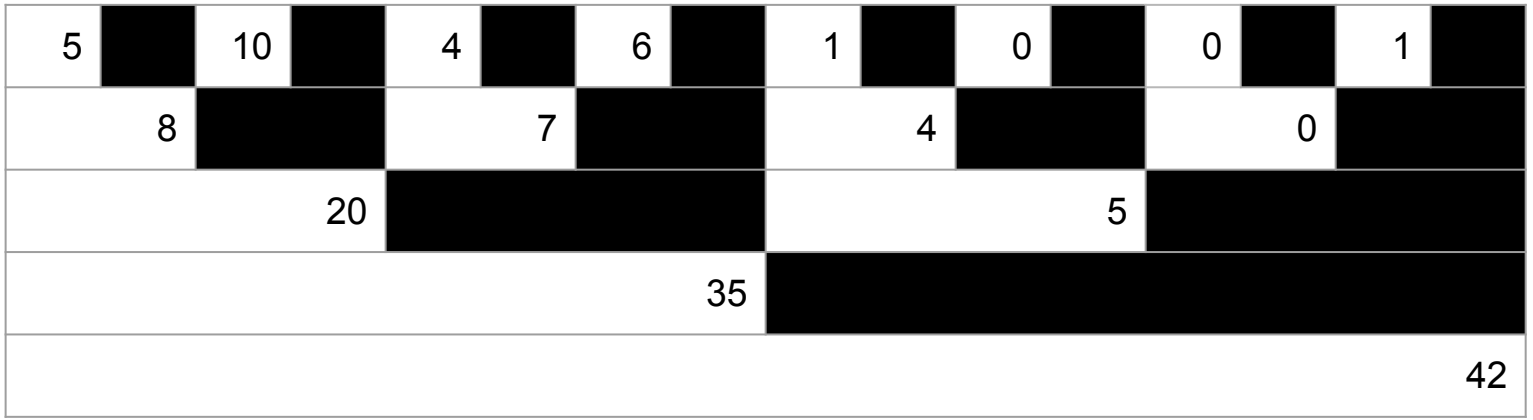

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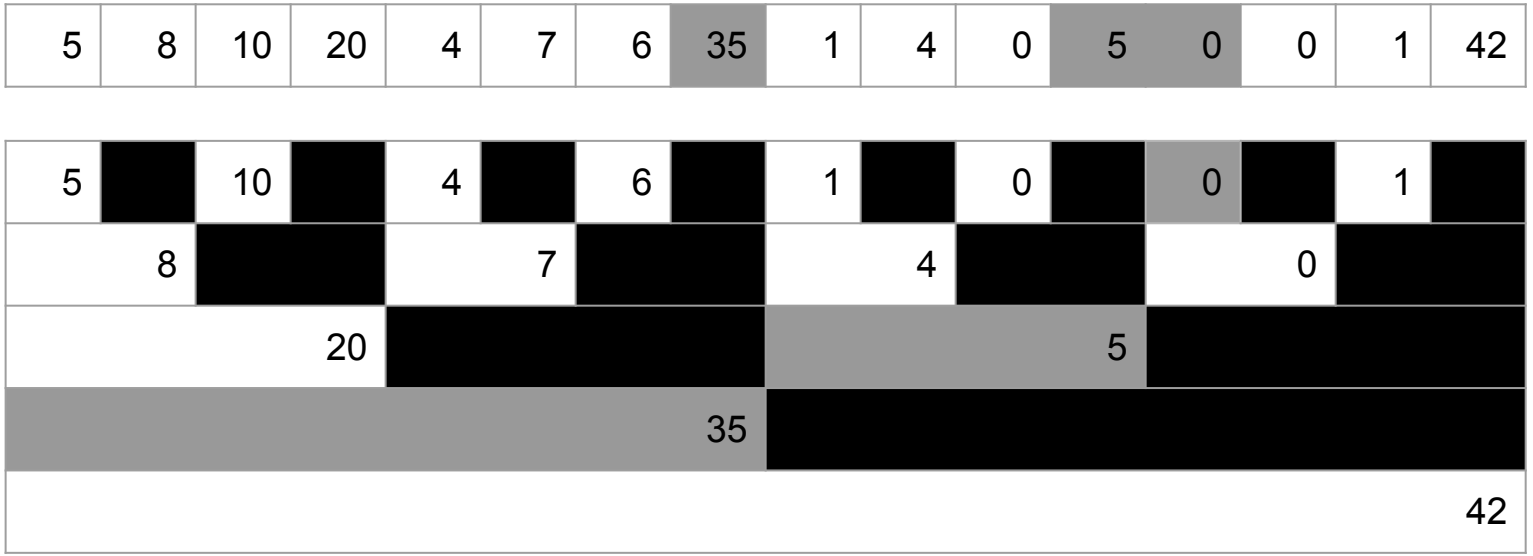

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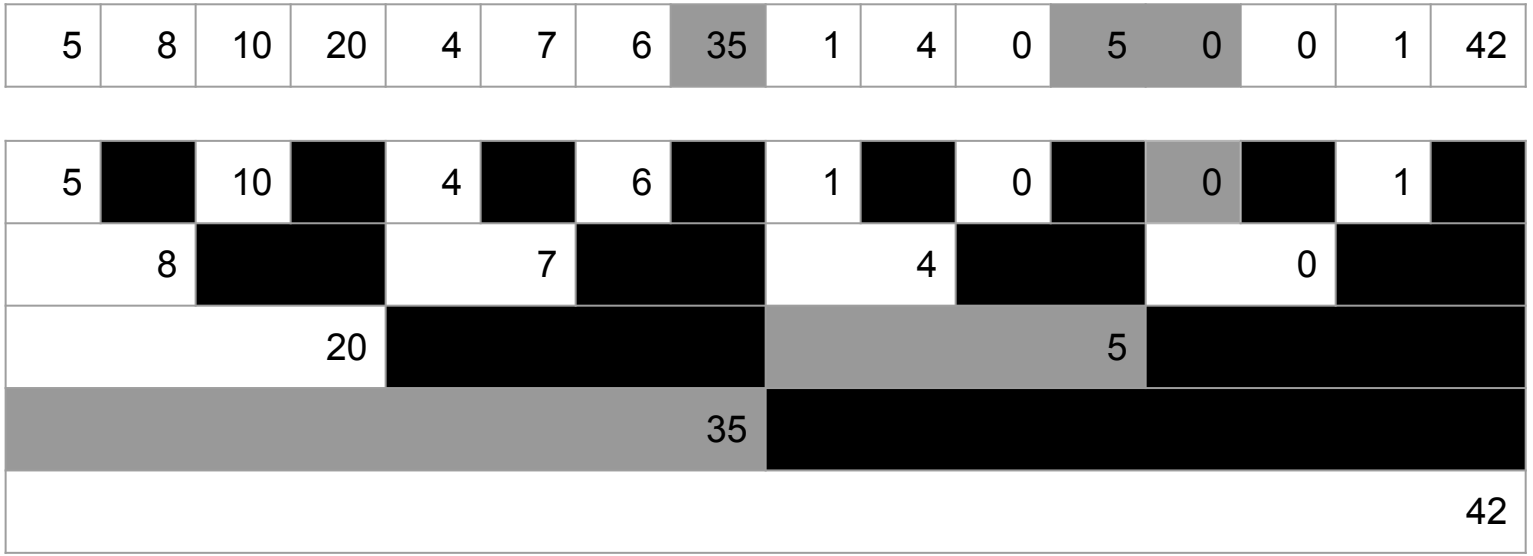

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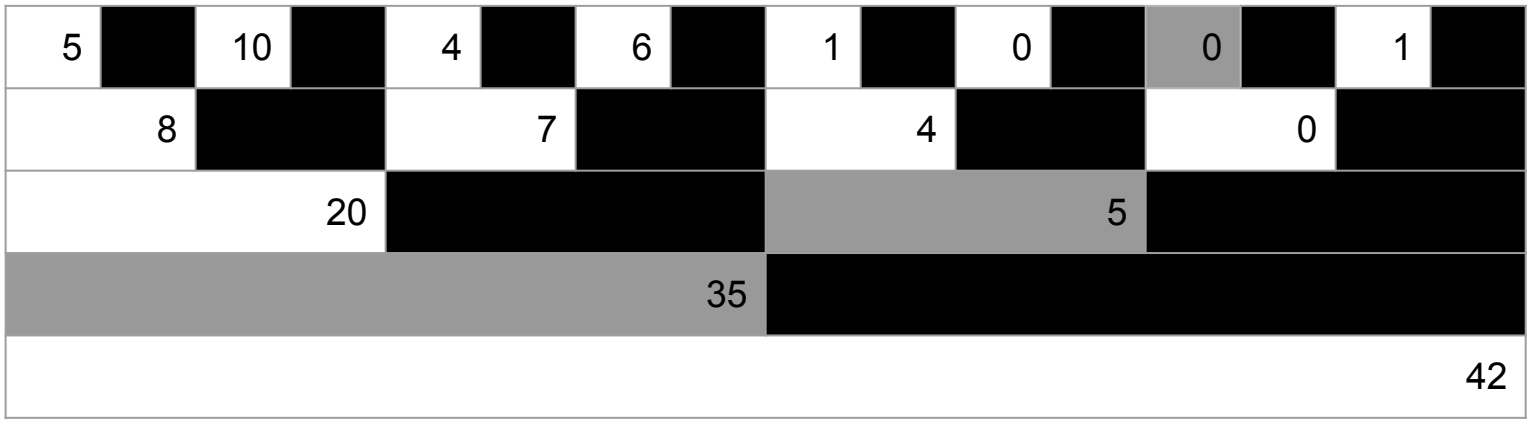

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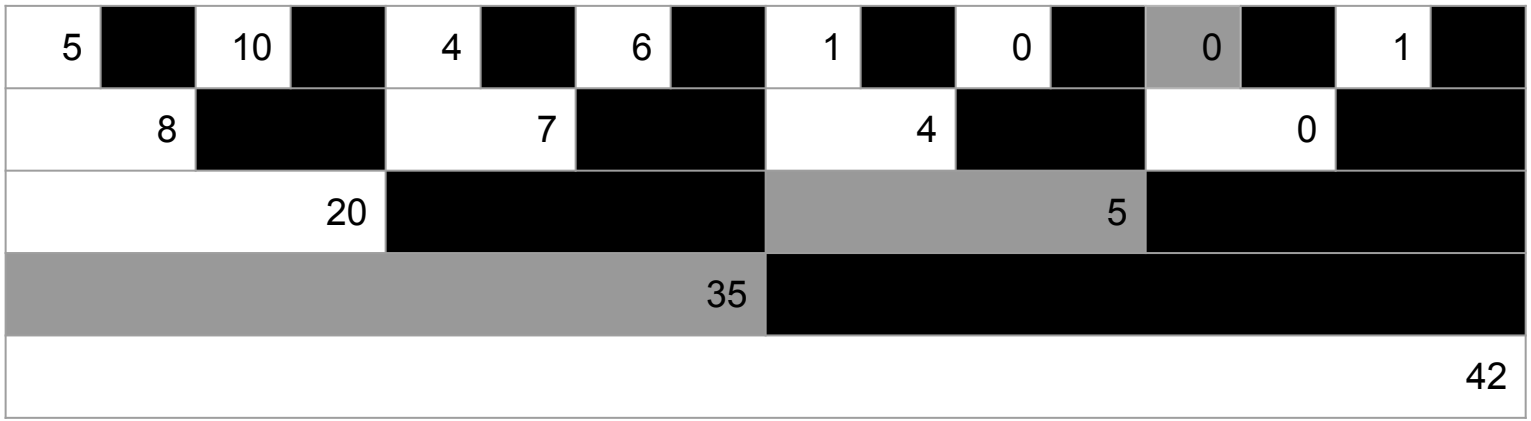
  - 12 => 13 = $(1101)_2$
  - We have the 2^0 bit on, the 2^2 bit on, and the 2^3 bit on...
  - This tells me we have a 2^0 bucket, a 2^2 bucket, and a 2^3 bucket.
  - The buckets are 13 $(1101)_2$, 12 $(1100)_2$, and 8 $(1000)_2$



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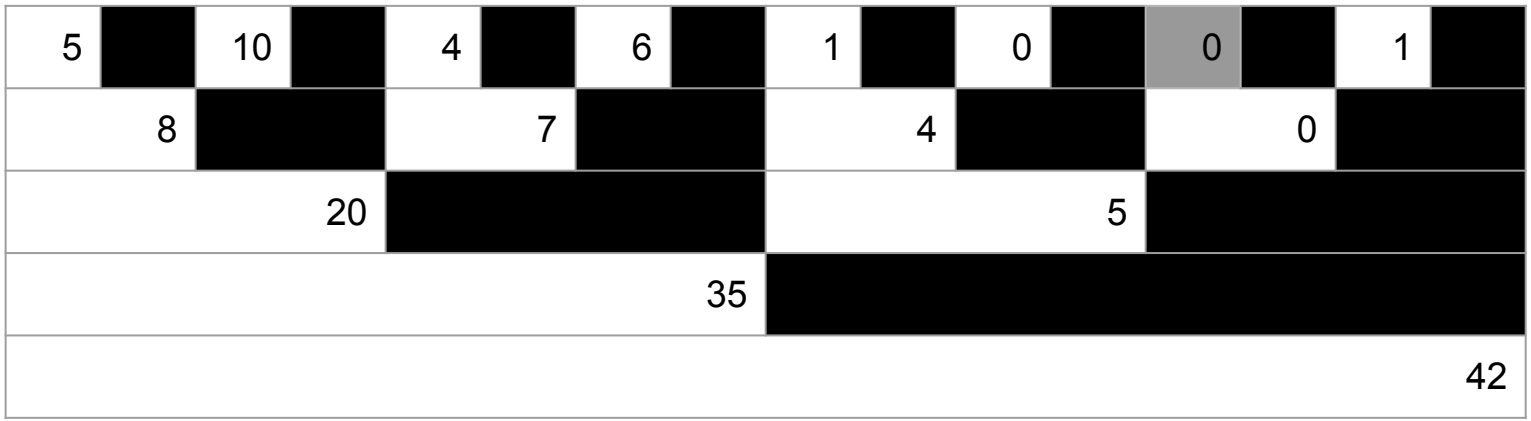
```
int sum (int x) {
   int ans = bucket[++x]; // 1-index and initial bucket
   while (x != 0) ans += bucket[x^=(x&-x)];
   return ans;
}
```

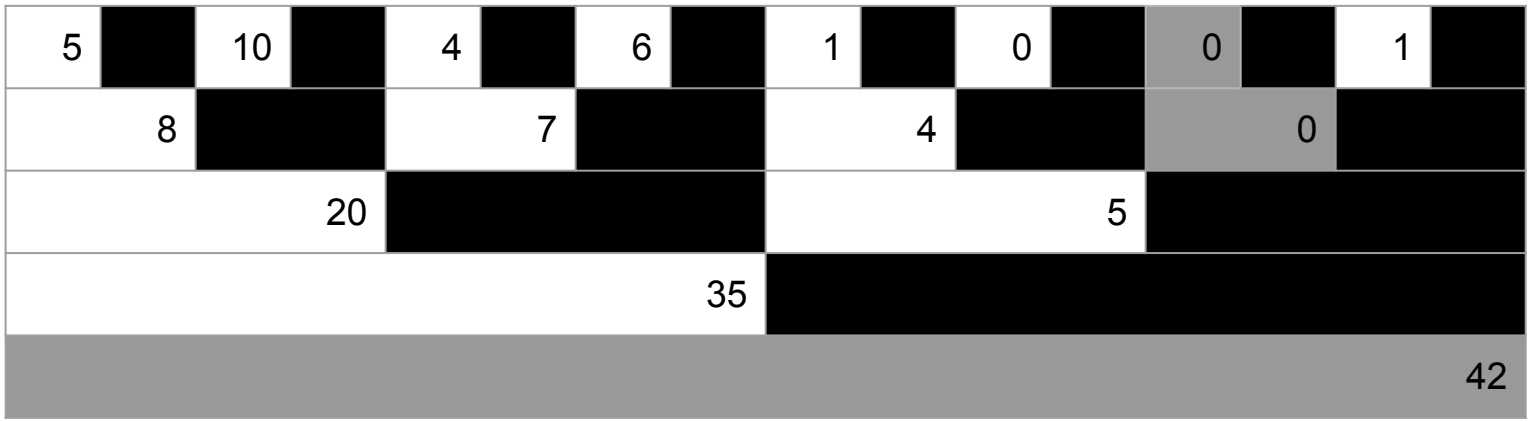
The Good Update

- Example 12



The Good Update

- Example 12
- Update the overlapping buckets



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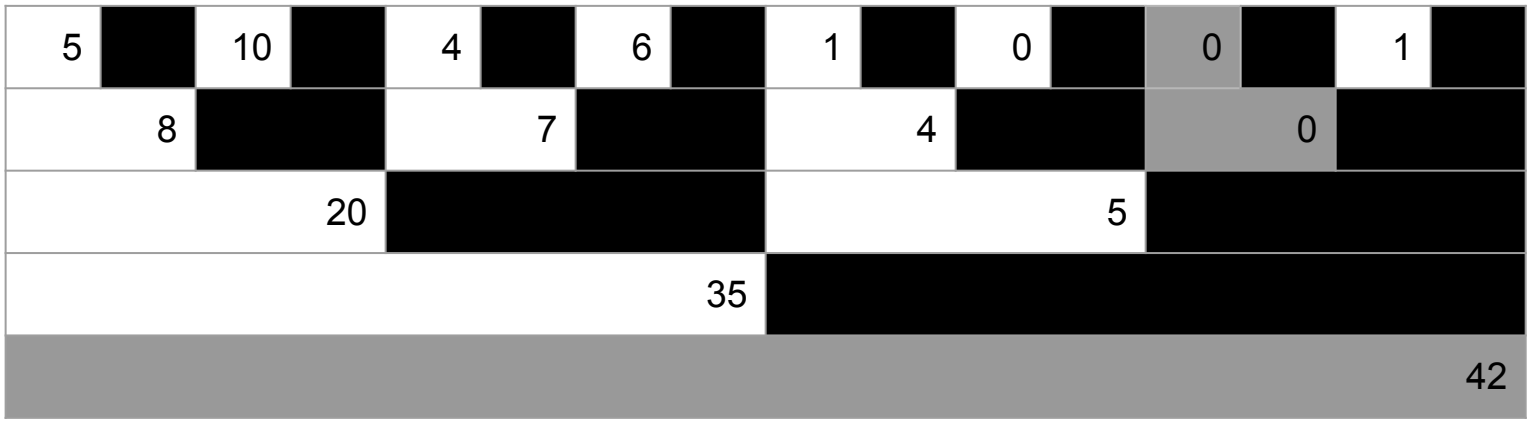
- Example 12
- Update the overlapping buckets
- Again 1 index
  - 13 $(1101)_2$, 14$(1110)_2$, and 16 $(10000)_2$



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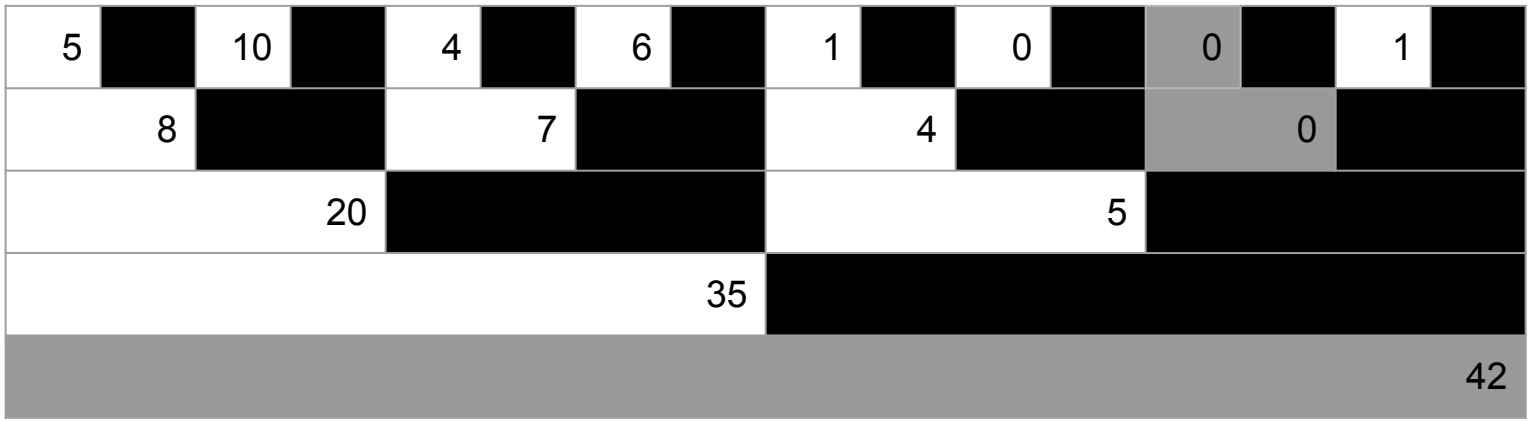
- Example 12
- Update the overlapping buckets
- Again 1 index
  - 13 $(1101)_2$, 14$(1110)_2$, and 16 $(10000)_2$
- Not as obvious, but we are ADDING the lowest 1 bit.



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```
void add(int x, int val) {
   x++; // 1-index
   while (x<max) {
      bucket[x] += val;
      x+=(x&-x)
   }
}
```

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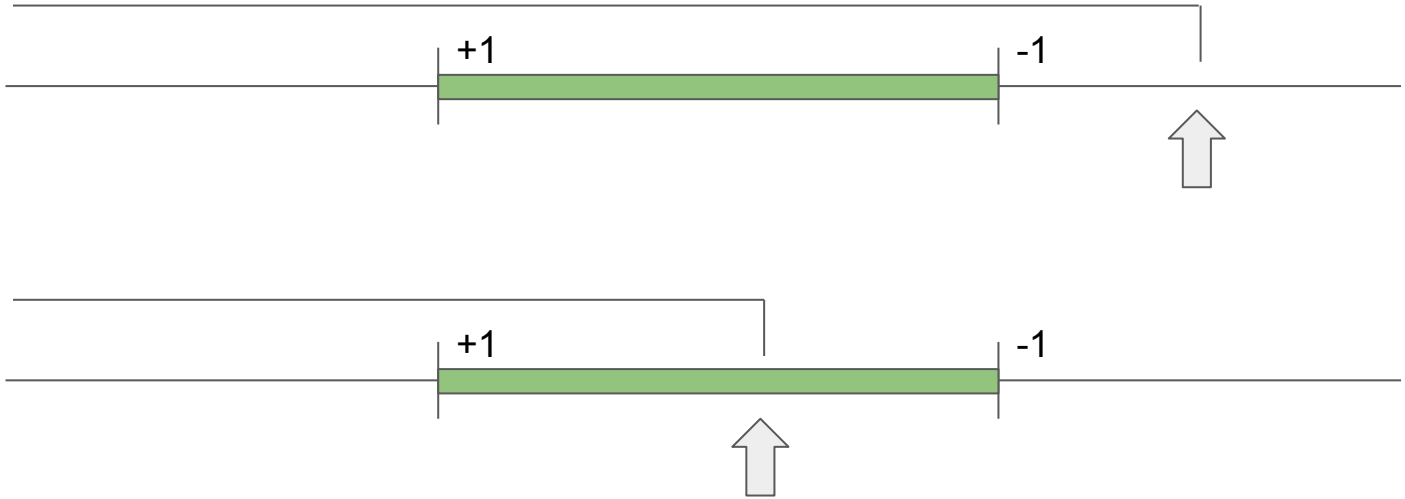
- I can add 1 at the start of an interval and subtract 1 at the end of an interval
- This means
  - if the query point is before the interval the interval is not counted
  - If the query point is within the interval only the 1 is added to the query
  - If the query point is after the interval (outside) the 1 and -1 cancel out

VS Point Query Range Update



Types

- Point Query Range Update
- Point Update Range Query
- XOR stuff