Topological Sort

Superhero Problem

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- Can an ordering of superheroes satisfy my friend’s opinions?
- Create at least one of the orderings, if it exists.

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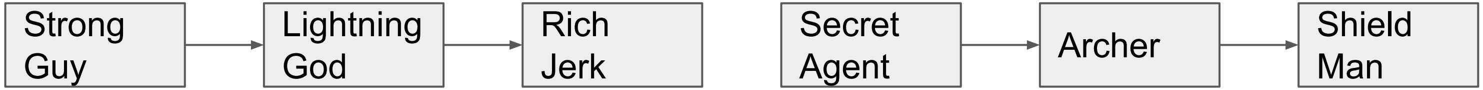

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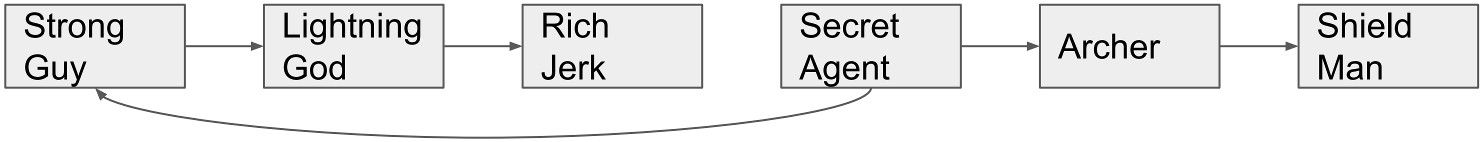

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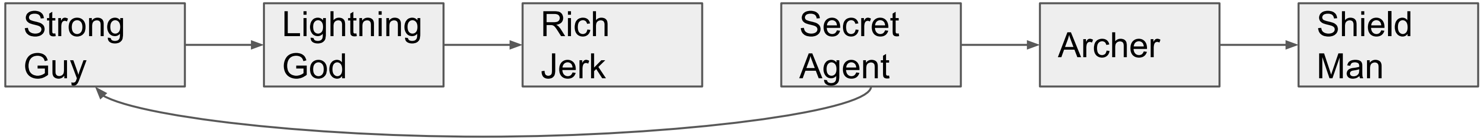

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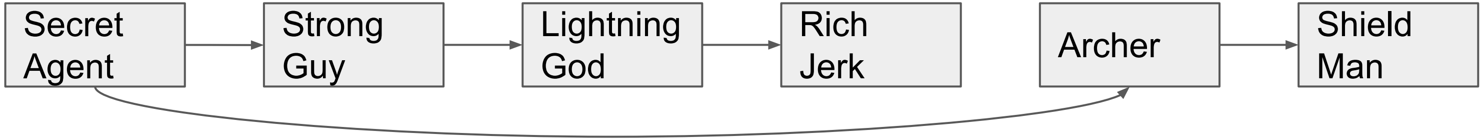

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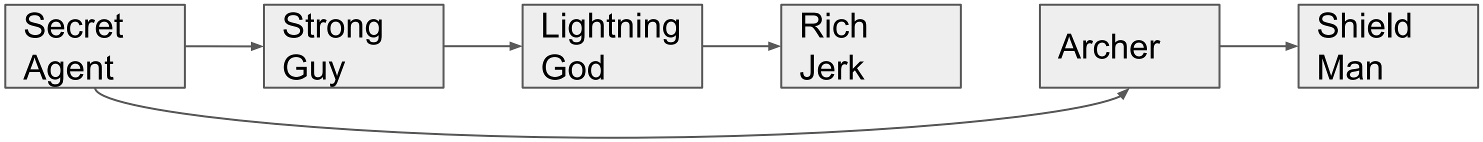

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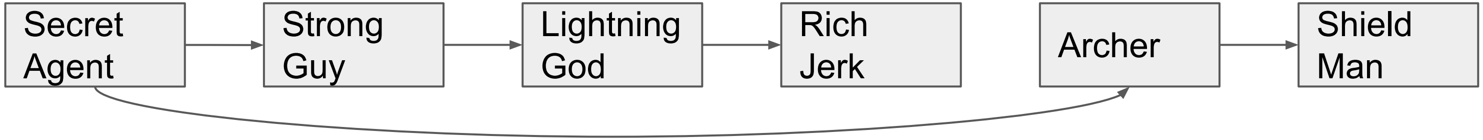

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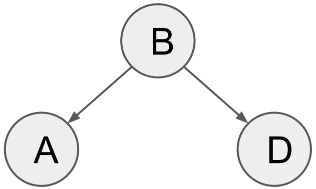

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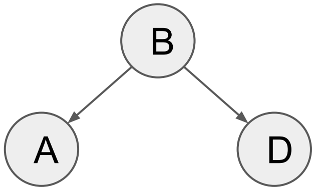

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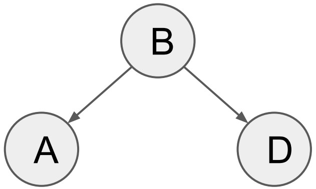

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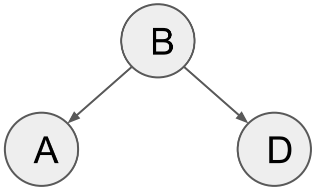

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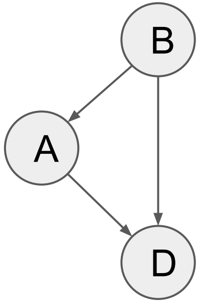

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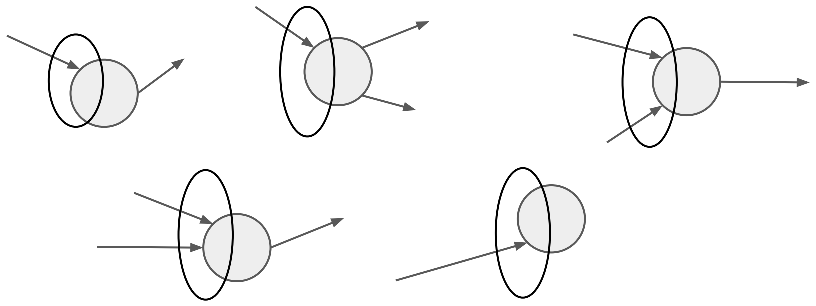

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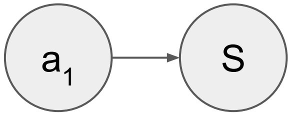

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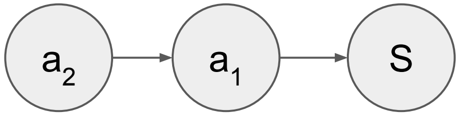

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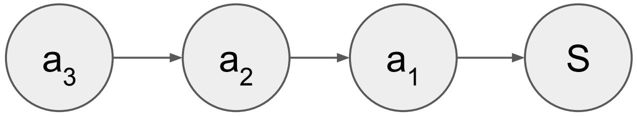

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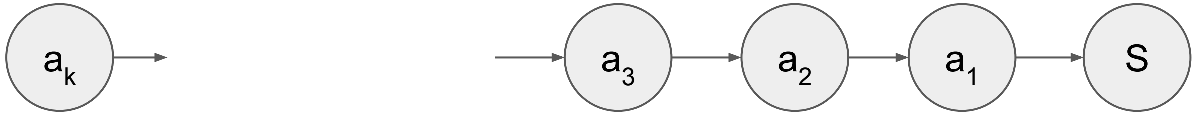

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Induction Hypothesis. All DAGs with ***n*** nodes have a valid topological Order.

DAG Implies Topo. Ordering (Proof) (cont.)

Induction Step. Let G be a DAG with $\boldsymbol{n}$ + 1 nodes. The DAG (by the lemma) has a node, $\boldsymbol{x}$, with no incoming edges.

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Induction Step. Let G be a DAG with $n + 1$ nodes. The DAG (by the lemma) has a node, x, with no incoming edges.

The graph G without node x is a DAG with n nodes, which has a topological order by the Induction hypothesis.

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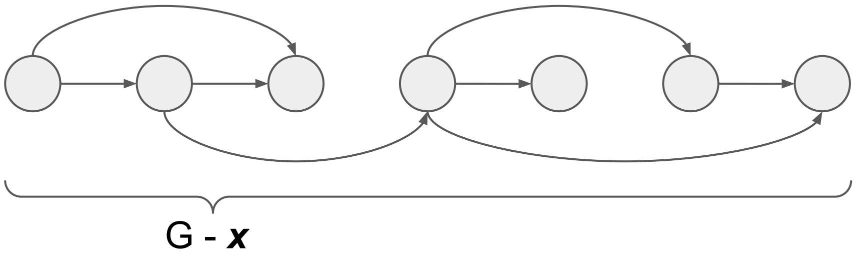

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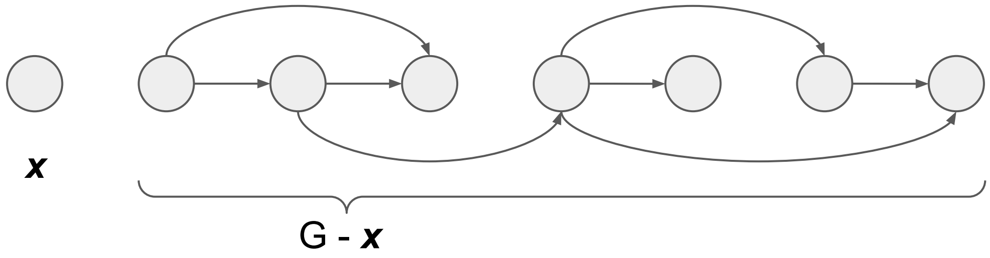

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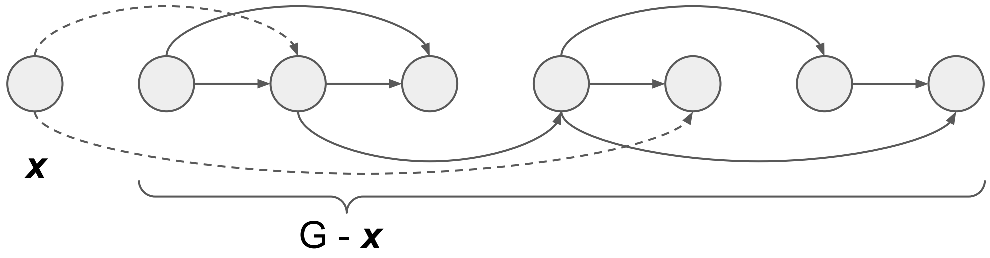

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By induction a topological ordering will exist for DAGs with any natural number of nodes.

High-Level Kahn’s Algorithm

```
Add to a queue all nodes that have no incoming edges
While the queue has an element
   Let x be the first node of the queue
   Remove x from the queue
   Output x
   Remove x from the graph
   For every node, y, that NOW has no incoming edges
      Add y to the queue
   End For
End While
```

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- Loop over the edges (x, y) that are leaving the node x.
  - Decrement the in degree for the y node
  - If the in degree for the y node is 0 add it to the queue

DFS Approach

```
Array on stack is initialized to false
Array visited is initialized to false
DFS from ALL Nodes that have no incoming edges
Function DFS on Node n
   If visited n Then
      Return
   End If
   If n is on stack Then
      BADNESS
   End If
   Let on stack AND visited of n be TRUE
   For all Nodes x that can be reached by an edge leaving n
      DFS on Node x
   Add n to the front of the output list
   Let on stack be FALSE
End Function
```