Matrix Chain Multiplication (MCM)

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We want to multiply a group of matrices together such that the number of operations is minimized.

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This can be solved using a DP. Let's work on a more fun example.

MCM Problems

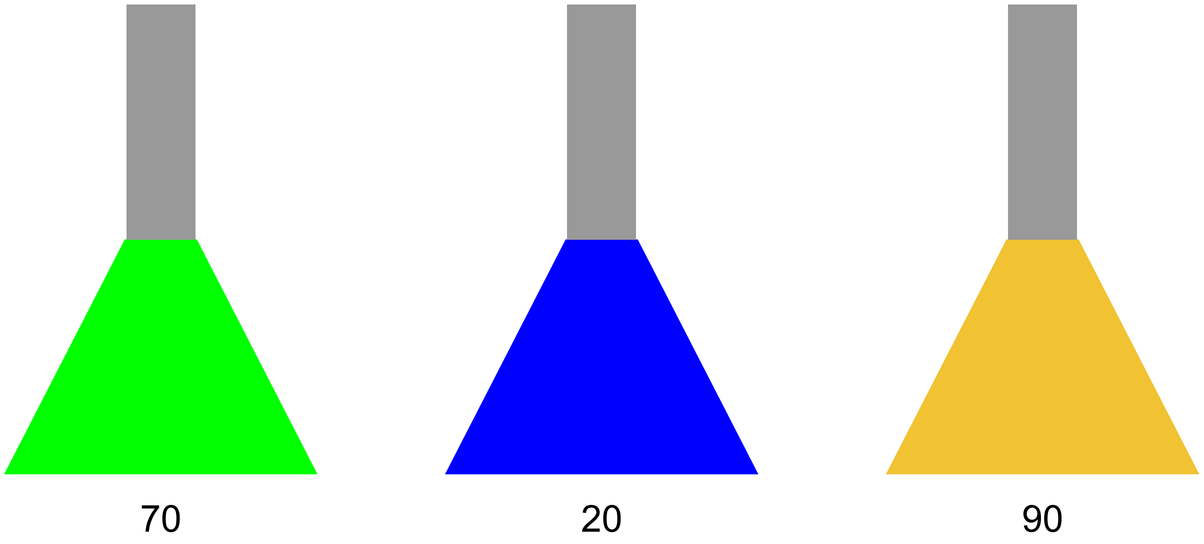
https://www.spoj.com/problems/MIXTURES/

https://www.interviewbit.com/problems/rod-cutting/

Problem (SPOJ)

Harry Potter is going to mix n potions together which are currently line up. The potions have some integer value from 0 to 99. Only adjacent potions can be mixed. Potions that are mixed go back into place in a combined container. Potions of value a and b produce an amount of smoke equal to a*b. The new combined mixture has a value of (a+b)%100. Find the minimum amount of smoke that can be produced after having combined all the potions.

Example



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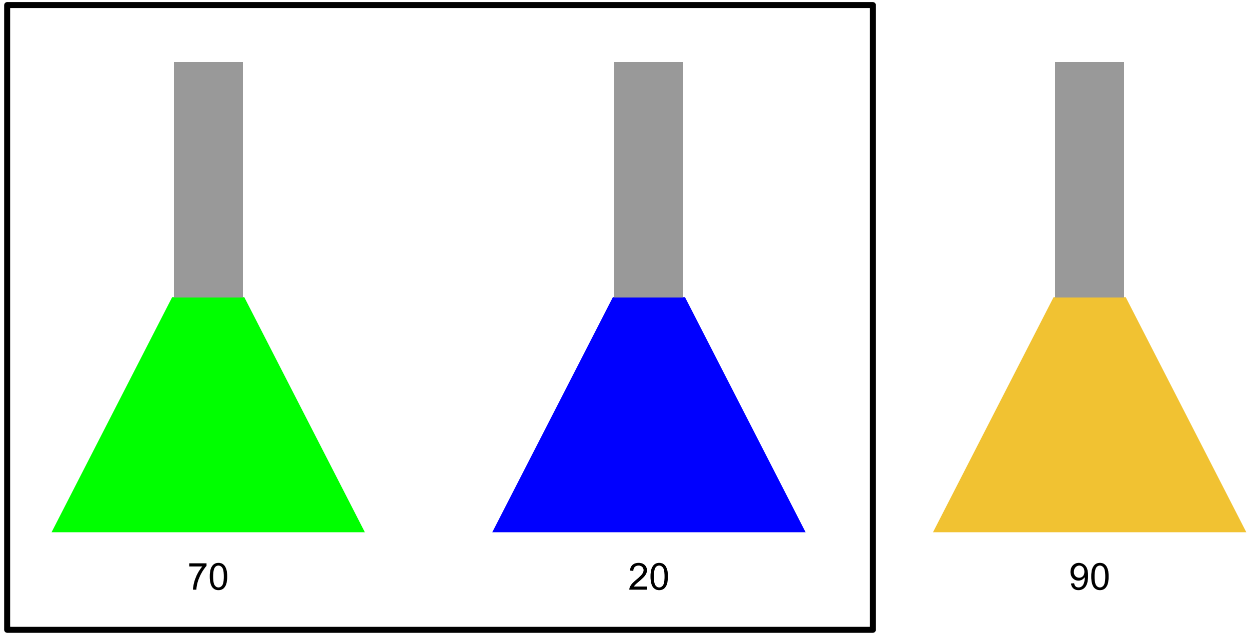

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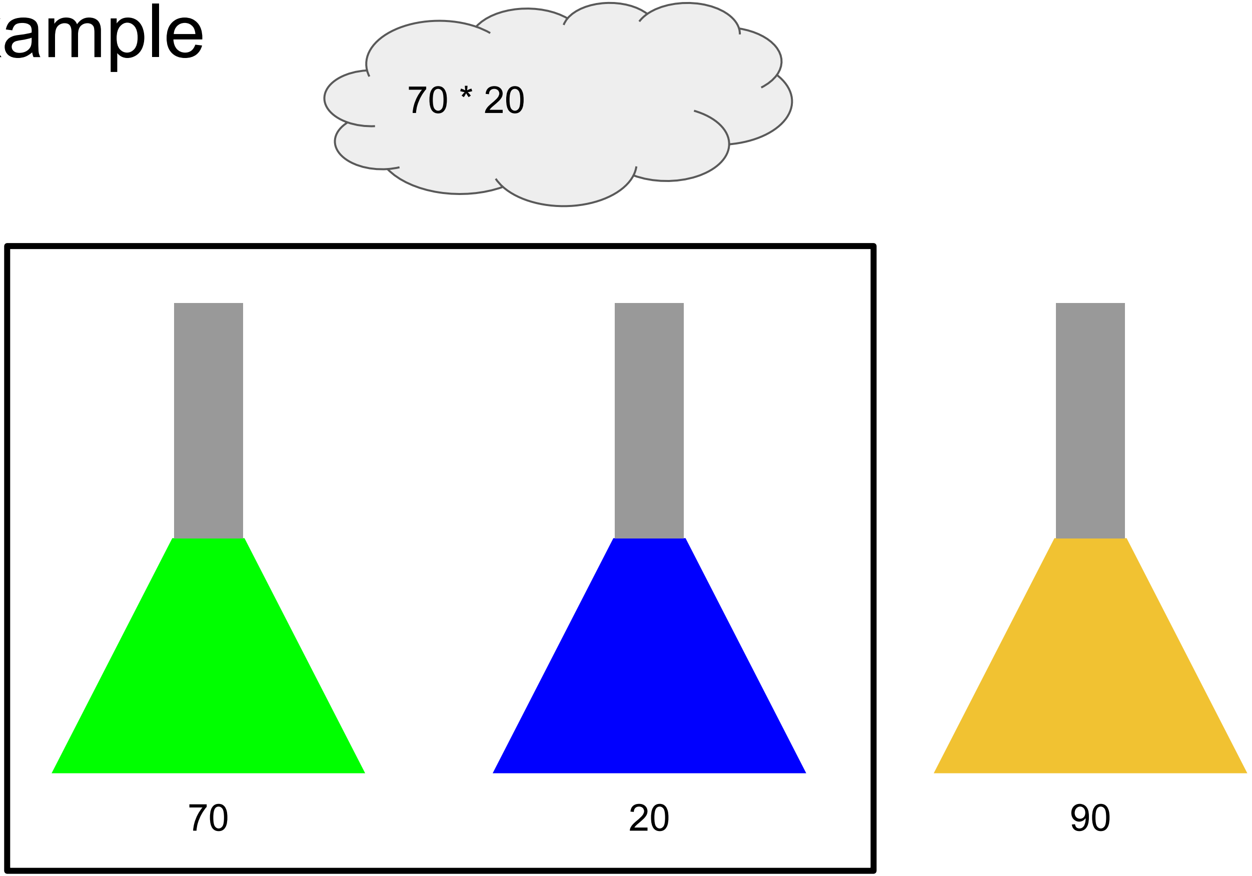

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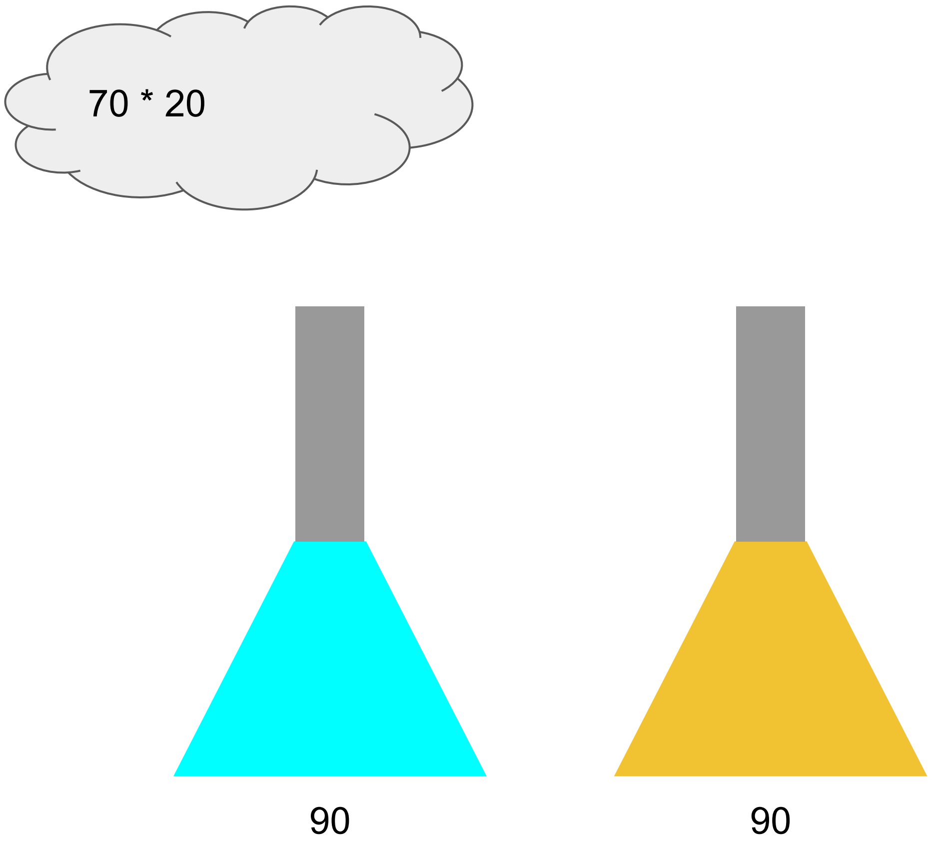

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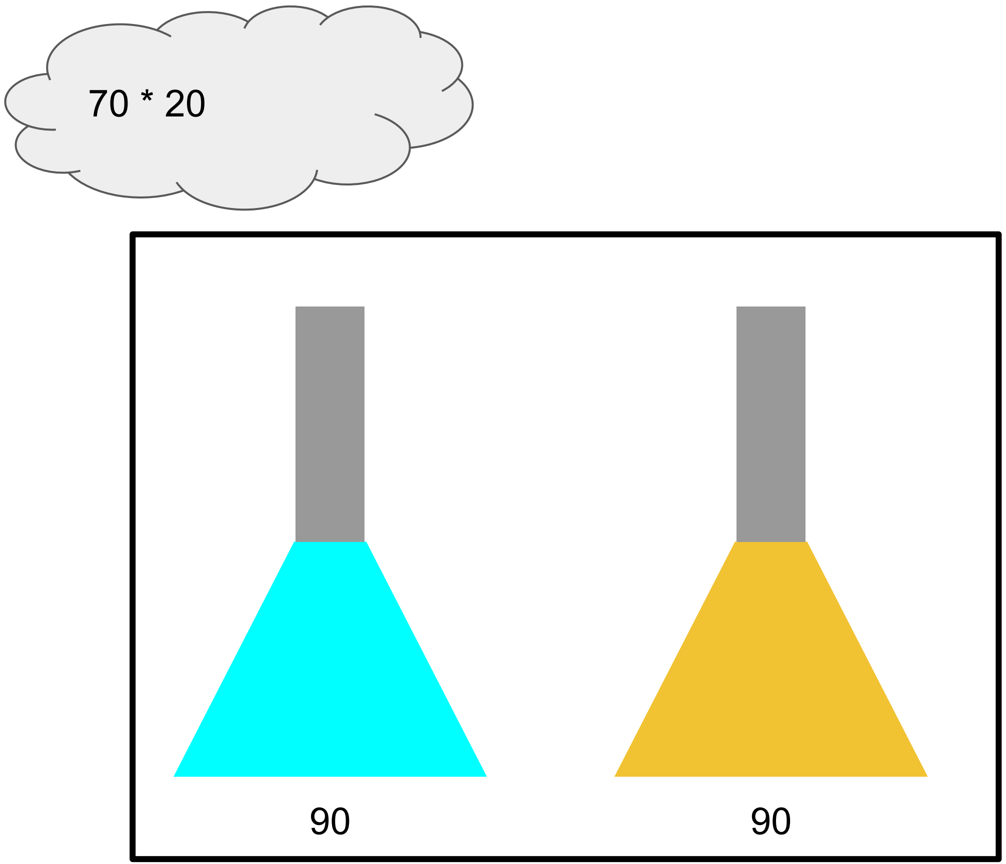

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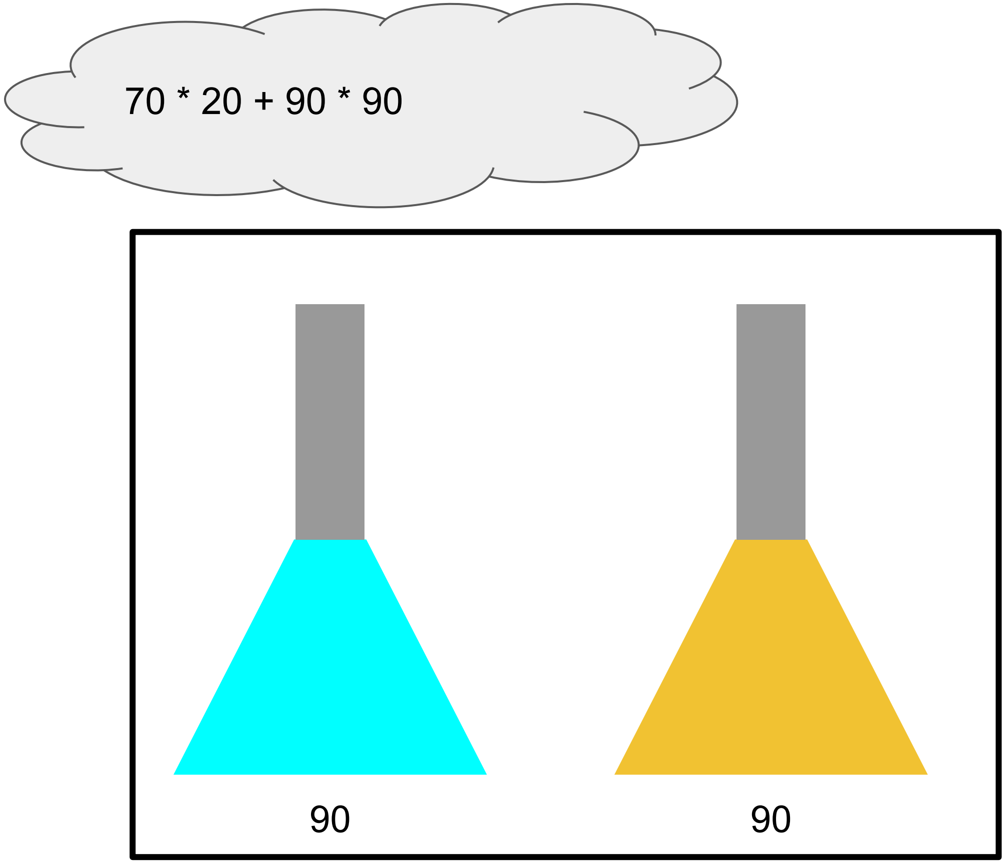

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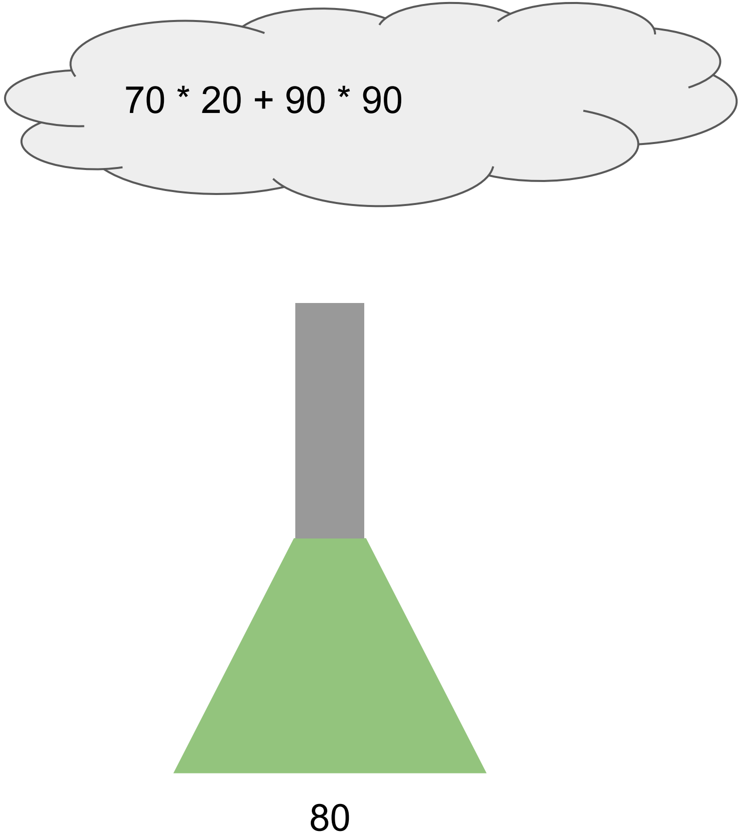

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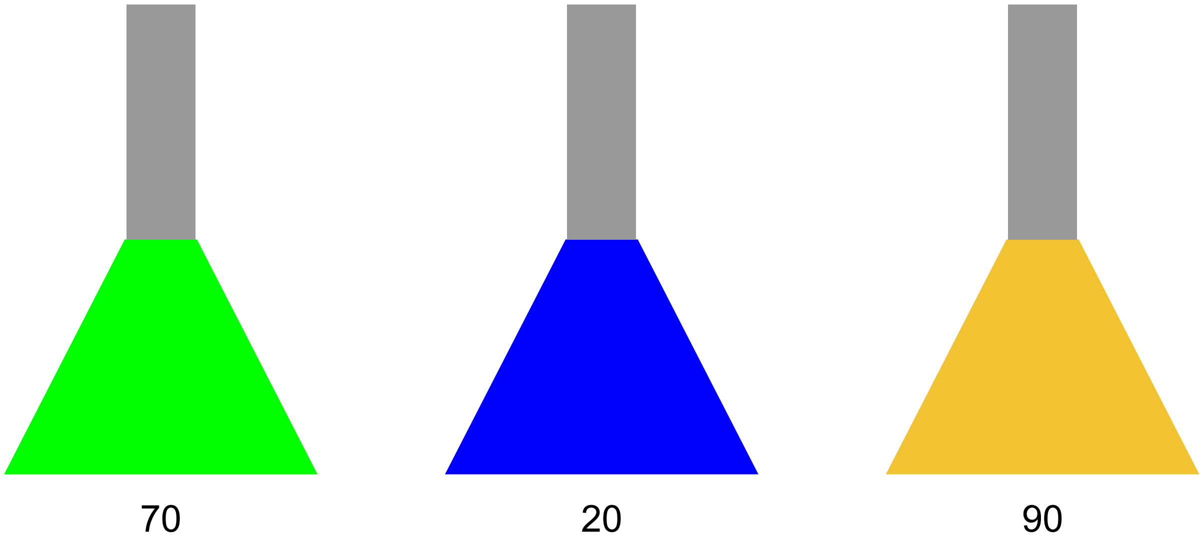

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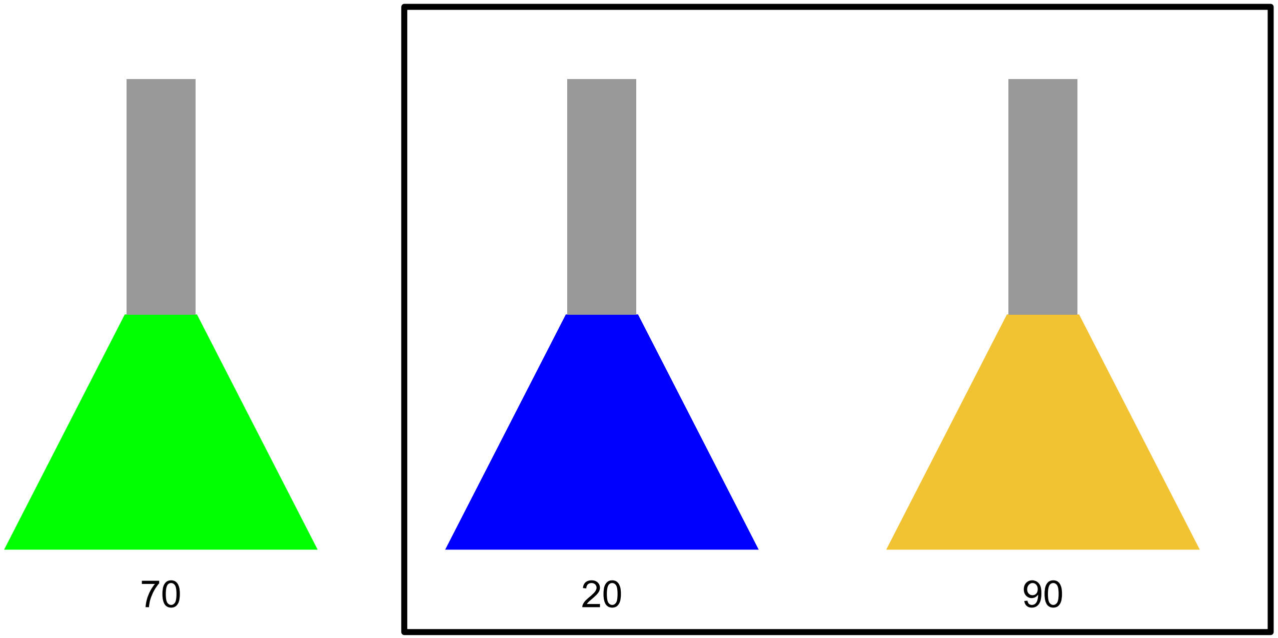

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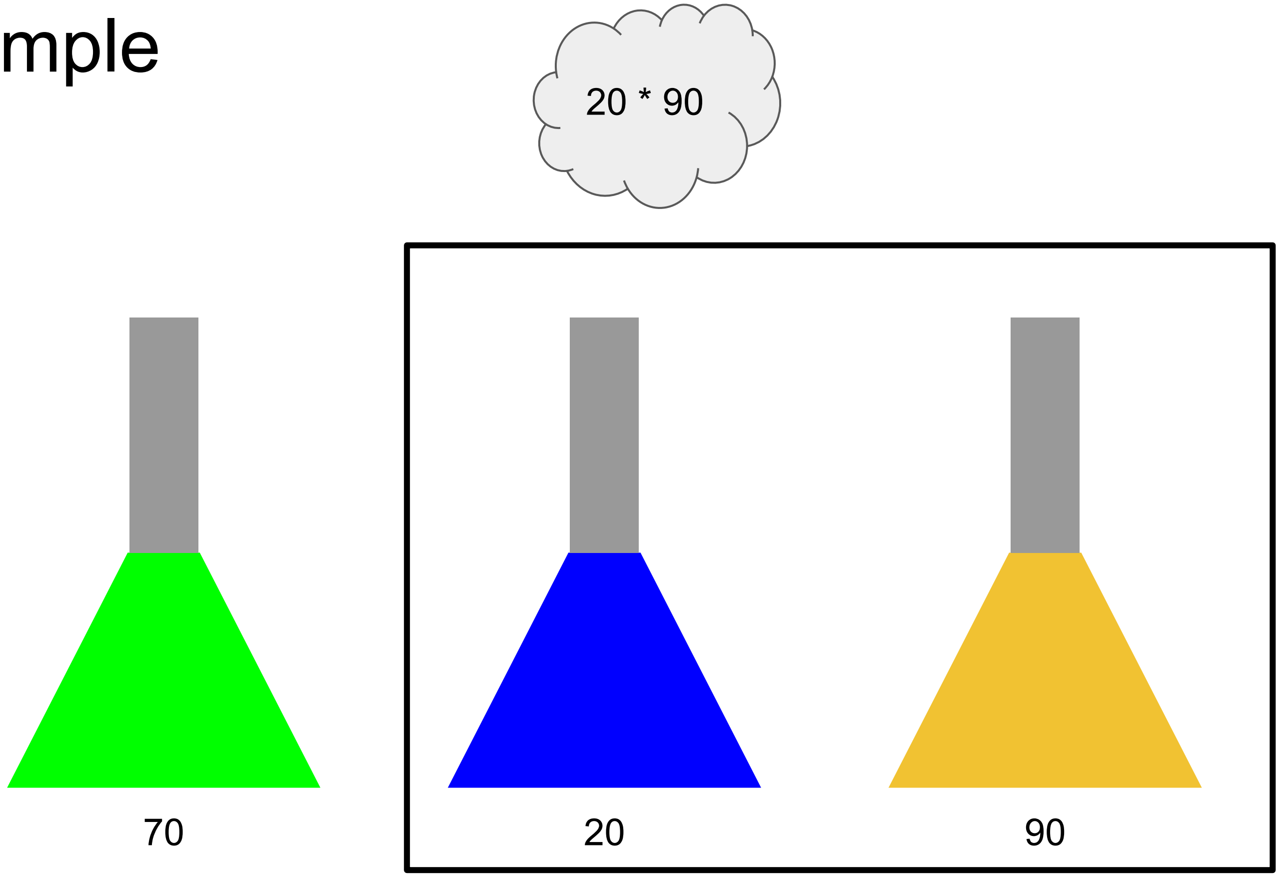

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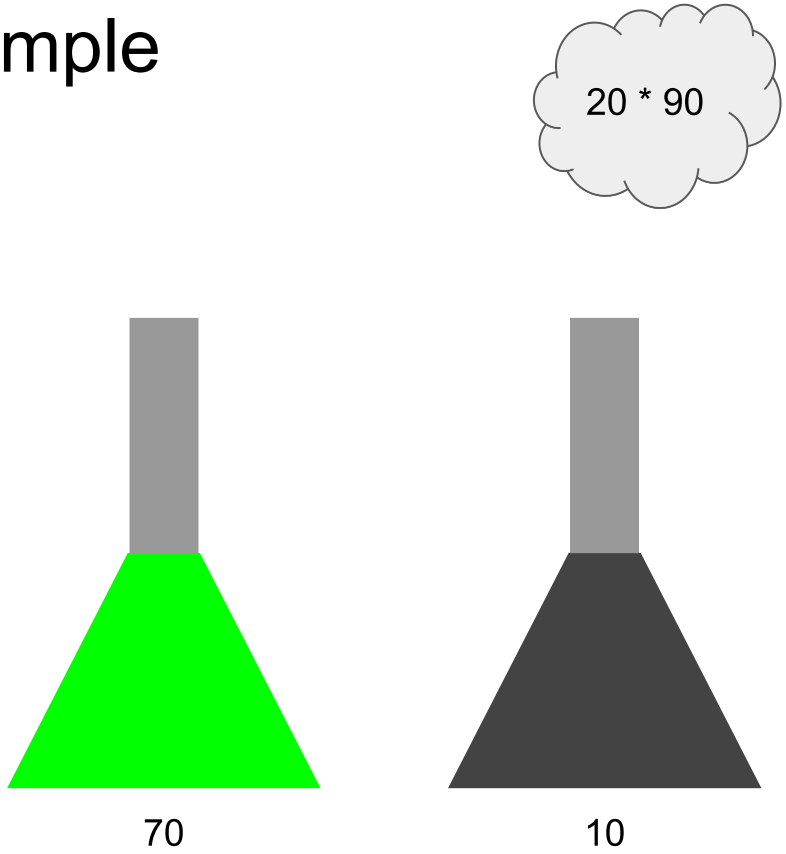

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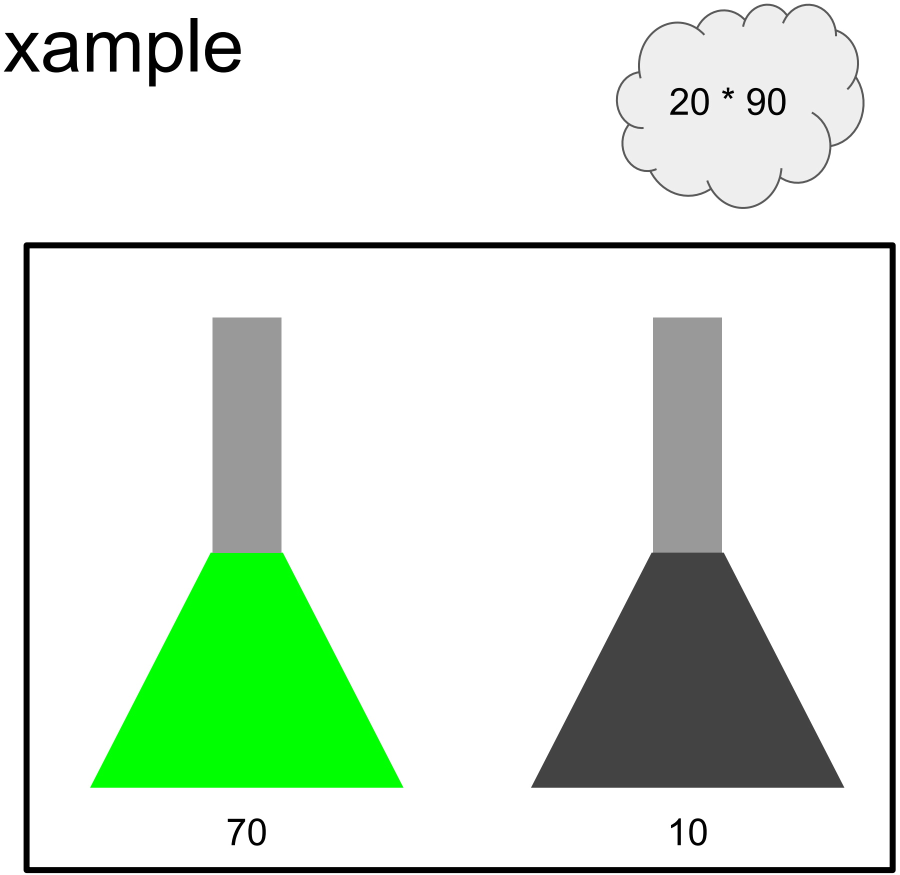

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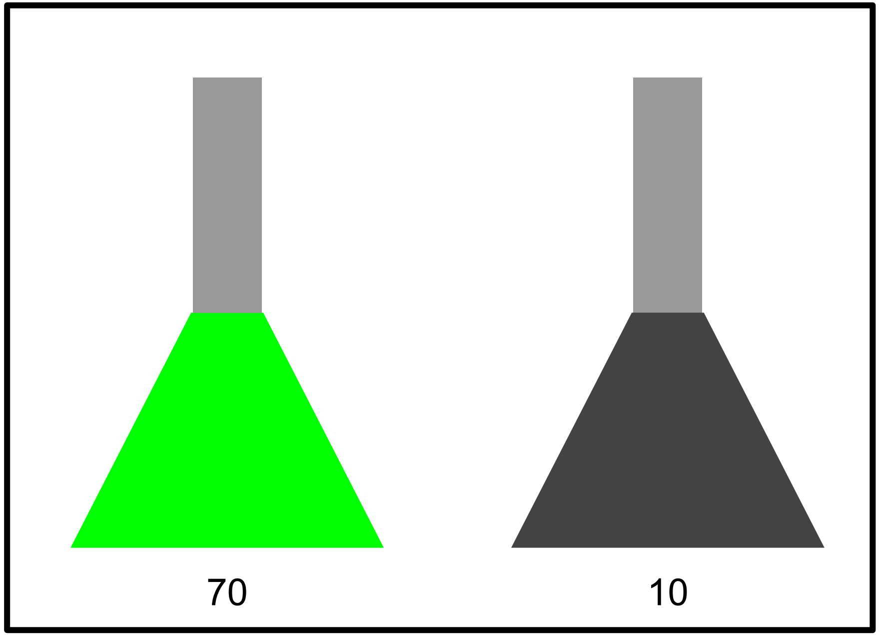

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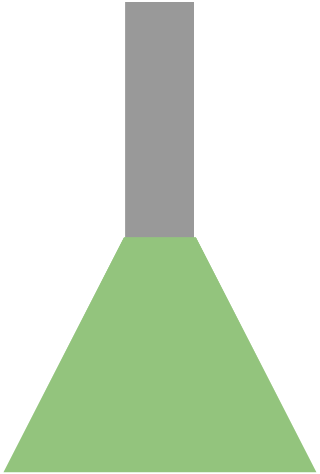

80

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Harry has 100 potions...

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$A_1 + A_2 + A_3 + \ldots + A_k$ AND $A_{k+1} + A_{k+2} + \ldots + A_N$

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Just a tad over 10^8

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B/C It can be done for determine number of operations when multiply matrices

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The question is what is the least effort to find ABCD...X?

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Do that harry potter DP!