STANDARDIZED

1 4 7 9 3 6 5 8 2

2 5 8 7 1 4 6 9 3

3 6 9 8 2 5 4 7 1

8 2 5 4 7 1 3 6 9

9 3 6 5 8 2 1 4 7

7 1 4 6 9 3 2 5 8

6 9 3 2 5 8 7 1 4

4 7 1 3 6 9 8 2 5

5 8 2 1 4 7 9 3 6

dif stand

1 4 7 6 9 3 8 2 5

2 5 8 4 7 1 9 3 6

3 6 9 5 8 2 7 1 4

5 8 2 7 1 4 3 6 9

6 9 3 8 2 5 1 4 7

4 7 1 9 3 6 2 5 8

9 3 6 2 5 8 4 7 1

7 1 4 3 6 9 5 8 2

8 2 5 1 4 7 6 9 3

dif stand2

1 4 7 8 2 5 6 9 3

2 5 8 9 3 6 4 7 1

3 6 9 7 1 4 5 8 2

9 3 6 4 7 1 2 5 8

7 1 4 5 8 2 3 6 9

8 2 5 6 9 3 1 4 7

5 8 2 3 6 9 7 1 4

6 9 3 1 4 7 8 2 5

4 7 1 2 5 8 9 3 6

Dif stand 3

1 4 7 9 3 6 5 8 2

2 5 8 7 1 4 6 9 3

3 6 9 8 2 5 4 7 1

6 9 3 2 5 8 7 1 4

4 7 1 3 6 9 8 2 5

5 8 2 1 4 7 9 3 6

8 2 5 4 7 1 3 6 9

9 3 6 5 8 2 1 4 7

7 1 4 6 9 3 2 5 8

Assume all 24 symmetries hold and the 6 constraints are all satisfied in a given solution.

Claim 1: There exist a nontrivial number sudoku puzzle solutions with all 24 symmetries that arise from an I,j,k,ell system of labelling the locations.

Proof by counterexample:

The claim that there exist no solutions with all 24 symmetries is disproven by the four examples presented above.

Claim 2: In each of these solutions, it must be true that 9 unique 3x3 “blocks” (used interchangeably with tic tac toes) arise.

Part 1: Proof that only cyclic permutations of subrows and subcolumns arise in these solutions.

Proof by contradiction:

WLOG let us view the first subrow in the first block (labelled I\_1,J\_1,k\_1,ell). The three-tuple present is 1,2,3.

There are 6 permutations of 1,2,3. 3 of which are cyclical, and 3 of which are reflective. For each of the cyclical permutations, each number is in a location separate from any other cyclical permutation (ie. 1,2,3 2,3,1 3,1,2). In a reflective permutation, exactly 1 number retains its position relative to the 1,2,3 three-tuple, and the other two swap positions (1,3,2 3,2,1 2,1,3).

WLOG let any reflective permutation above exist as a subcolumn in a block in a solution. For example, let 1,3,2 exist in a block. If the block is in the same row as the first block (the one set in standardizing the solutions), which contains the three-tuple subcolumn 1,2,3, the 1 will be in the top row of both blocks and thus one of the 6 constraints is not satisfied, thus we reach a contradiction proving that no reflective permutation may exist in the same row as the standardized block.

Applying the same logic as above proves that no reflective permutation may exist in the same column as the standardized block (to repeat the logic for clarity, if any of the three numbers is in the same “k” as the one in the standardized block, the 6 constraints are thus not satisfied).

The following implies that transitive permutations must exist in the two blocks in the same row and column as the standardized block. (a subset of the 24 symmetries guarantees that in every block, only permutations of rows/columns in the parent block exist. Since we can eliminate reflective permutations, only the 3 cyclical permutations can exist). Furthermore, since there are 3 cyclical permutations, and 3 blocks in each row and column, there may be exactly 1 block in any row or column that has a given cyclical permutation (by contradiction, if two blocks in the same column/row have the same permutation, not all 6 constraints are satisfied).

Part 2: Proof that there are 9 unique blocks

Proof by combinatorics:

Since we know that each block in the row or column with the first block contains a cyclical permutation of 1,2,3 in a subcolumn, the blocks in the same row or column as these blocks must have the same property. The contradiction arises in the exact same way as with the first standardized block.

We now are certain that in every block, a cyclical permutation of the three-tuple 1,2,3 present in the first block exists in a subcolumn. This must also be the case for the other 2 subcolumns. Thus, each block must contain 3 subcolumns with cyclical permutations of the subcolumns in the first block.

Claim 3: Since there are 3 cyclical permutations for each subcolumn/subrow, and 3 subcolumns/subrows in a block, there exists 3\*3 possible blocks.

In order to place the cyclical permutation of 1,2,3 into a block, label the subcolumns ell1,ell2,ell3. Pick one (3 ways). Now pick a permutation (3 ways). Once that is picked (for example 3,1,2 in the ell2 subcolumn) the subrow permutations within the block are forced and thus the block is completely determined.

Since the subrow permutations also must be cyclical, fixing one number (1,2, or 3 each belong to a separate subrow permutation) implies that the other 2 are uniquely determined. Since 1,2,3 are places 9 ways, and for each placement there is only 1 solution, there are exactly 9 unique blocks that exist.

There are 8 different 24 symmetry solutions, 4 of which are unique and the other for are simply the transpose of the 4 unique ones.

Proof:

Due to one of the 6 constraints, it must be true that the numbers 1 through 9 appear in the top left corner (WLOG) of a tic tac toe, thus we may create a bijection between each tic tac toe and the number in its top left corner. For example, the tic tac toe highlighted in yellow above will be mapped to the number 1, gold will be mapped to 2, gray to 3 and so on.

Note that 1 may not be in the same row or column as 2, because at the bottom of 2, we see the 1,4,7 permutation exactly as is which makes the solution invalid (a 1 in the first row of two tic tac toes in the same row or column is against the constraints). This same logic may be applied to all tic tac toes where:

1. 1,4,7 is present in that exact permutation (2,3)
2. A permutation of 1,4,7 is present in the first row of the tic tac toe (4,7)

Therefor, in the 4 boxes not in a row or column with 1, (4,2,3,7) must be present.

Furthermore, 2 and 3 may not be next to each other, and neither can 4 or 7, for the exact same reasons listed above.

This leaves 8 possibilities for solutions:

1 . . | 1 . . |1 . . |1 . . |

. 4 2 |. 7 2|. 2 7|. 3 4|

. 3 7|. 3 4|. 4 3|. 7 2|

And their transpositions.

To prove that the dots are filled in uniquely, let us enumerate the possibilities and show what needs to replace a dot.

If we use:

1 . .

. 4 2

. 3 7

And look at the dot above the 4 and to the right of the 1, there is exactly one remaining block that may be inserted. That is block 8.

To recall, the remaining blocks are 5,6,8,9.

5 does not work in the same row or column as 4 because it contains the same permutations in each row, thus the first column of 4 contains the same numbers as the first column of 5 so they may not be stacked. They may not be placed next to each other for the same reason (also labeled b) earlier in this proof).

This same reasoning proves that 6 may not be adjacent to 4.

9 may not be in the same row or column as 3 thus the only remaining block that may be on top of 4 and to the right of 1 is 8. Note that the same reasoning proves that 3,4,8 must be in a row or column together, and we already have shown that 3 and 4 must be in a row or column together thus we are justified in representing each of the 8 possibilities as:

1 8 . | 1 . . |1 . . |1 . . |

. 4 2 |. 7 2 |. 2 7 |8 3 4|

. 3 7| 8 3 4|8 4 3|. 7 2|

And their transpositions.

Continuing with the same reasoning guarantees that each of the 8 are unique:

1 8 6 |1 5 9|1 9 5|1 5 9|

9 4 2|6 7 2|6 2 7|8 3 4|

5 3 7|8 3 4|8 4 3|6 7 2|

And their transpositions.

When represented this way, all possible solutions form a magic square with all rows and columns adding up to 15.

Each of these 8 solutions have been tested and are correct.