MODELING COUETTE FLOWS

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ABSTRACT

Couette flows can be modeled with ordinary differential equations derived from conservation principles. A fluid is considered Newtonian if its shear stress is proportional to shear rate, rheopectic if its viscosity increases as time increases, and thixotropic if its viscosity reduces as time increases. In this study, we contrasted the behavior of Newtonian fluids with rheopectic and thixotropic fluids in Couette flows and flows down inclines.

Keywords— Newtonian Fluids, Couette flow, Viscosity, Ordinary Differential Equations, Shear stress, rheopectic Fluids, thixotropic Fluids

1. INTRODUCTION

Coutte flows occur in fluids sandwiched between two surfaces. The bottom surface is stationary, and the top surface moves horizontally with a constant velocity U. Conservation of forces in the y direction of a fluid element can be applied to give

$$\frac{d\tau}{dy} = 0. \tag{1}$$

In many fluids, shear stress is proportional to shear rate (du/dy) plus a yield stress (τ_0) to give

$$\tau = \mu \frac{du}{dy} + \tau_0 \ . \tag{2}$$

with μ represting viscosity. In a Newtonian fluid, μ is constant and τ_0 is zero. Combining (1), (2), and the assumptions for Newtonian fluids, we can solve the ODE analytically for the solution

$$u(y) = \frac{U}{h} y \tag{3}$$

where U is the velocity of the surface, h is the total height of the fluid, and y is the depth of the element. There are multiple types of non-Newtonian fluids which have different relationships of shear stress, shear rate, viscosity, and time. Thixotropic and rheopectic fluids are two non-Newtonian fluids that were examined in this study.

2. METHODOLOGY

2.1.1 Modeling a Thixotropic Fluid

A thixotropic fluid's viscosity reduces as time increases. We will consider a thixotropic fluid with τ_0 = 0 and has viscosity given by the relationship

$$\mu = \frac{1}{(u+1)e^t} \ . \tag{4}$$

Substituting (4) and τ_0 = 0 into (2) gives

$$\tau = \frac{1}{(u+1)e^t} \frac{du}{dy}. \tag{5}$$

This can be written as

$$\tau \, (u+1)e^t = \frac{du}{dy} \ . \tag{6}$$

Rather than attempting to obtain an analytic solution to solve this ordinary differential equation, MATLAB's ode45 solver was used.

2.1.2 Modeling a Rheopectic Fluid

A rheopectic fluid's viscosity increases as time increases. We will consider a rheopectic fluid that has viscosity given by the relationship

$$\mu = \frac{t}{40(t+1)} \tag{7}$$

and a yield stress

$$\tau_0 = \left(15 - \frac{u}{4000}\right) \ . \tag{8}$$

Substituting (7) and (8) into (2) gives

$$\tau = \left(15 - \frac{u}{4000}\right) + \frac{t}{40(t+1)} \frac{du}{dy} \ . \tag{9}$$

This can be rewritten as

$$(\tau - \left(15 - \frac{u}{4000}\right)) * \frac{40(t+1)}{t} \ = \frac{du}{dy}. \tag{10}$$

Ode45 was also used to obtain a solution to this differential equation.

2.2 Flow Down an Incline

If conservation of forces is performed for an element of a fluid flowing down an incline, the following arises

$$\tau = \gamma(h - y)\sin(\theta) \tag{11}$$

where γ is the specific weight of the fluid, θ is the slope of the incline, h is the thickness of the fluid, and y the depth of element. We assume that u(0) is 0. Substituting 11 into (2) gives

$$\gamma(h - y)\sin(\theta) = \mu\frac{du}{dy} + \tau_0. \tag{12}$$

This is the general ODE describing flow down an incline. Ode45 was used to obtain a solution to this differential equation as well.

3. RESULTS

To better understand the behavior of the models, the solutions were plotted. For models that were time dependent, the 'greener' a plot was, the more time had elapsed.

3.1 Couette Flow of a Thixotropic Fluid

To simulate (6), a 0.1 height was used. Figure 1 shows the behavior of the model at U values of 0.5, 1, 1.5 and 2.0 for two seconds.

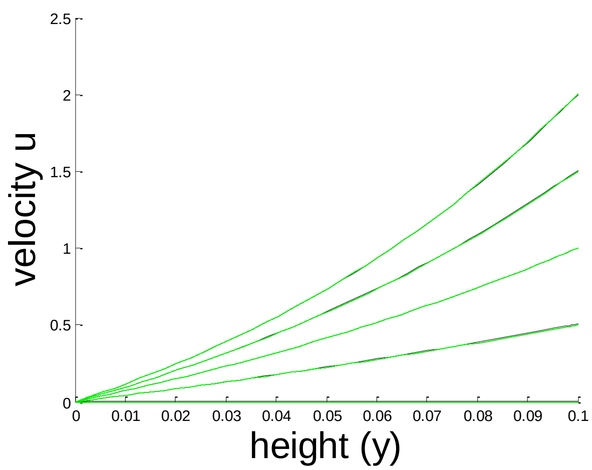

Figure 1: Graph of velocity as time varies for u = 0.5, 1, 1.5, and 2.0.

In Figures 2 through 5, the ratio of shear stress to shear rate is plotted as time varies from 0 to 2 seconds at U values of 0.5, 1, 1.5, and 2.0.

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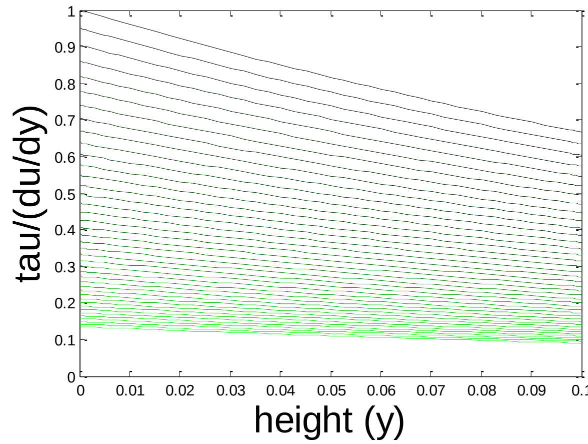

Figure 2: Graph of the ratio of shear stress to shear rate as time varies from 0 to 2 seconds for a thixotropic fluid with U = 0.5.

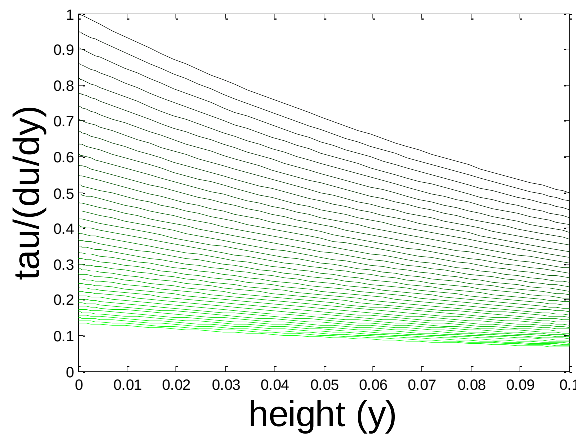

Figure 3: Graph of the ratio of shear stress to shear rate as time varies from 0 to 2 seconds for a thixotropic fluid for U = 1.0.

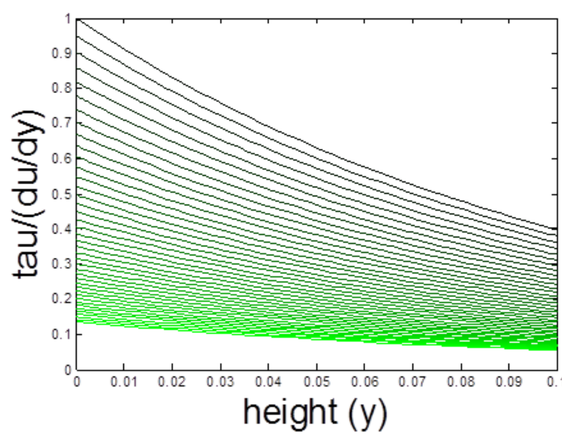

Figure 4: Graph of the ratio of shear stress to shear rate as time varies from 0 to 2 seconds for a thixotropic fluid for U = 1.5.

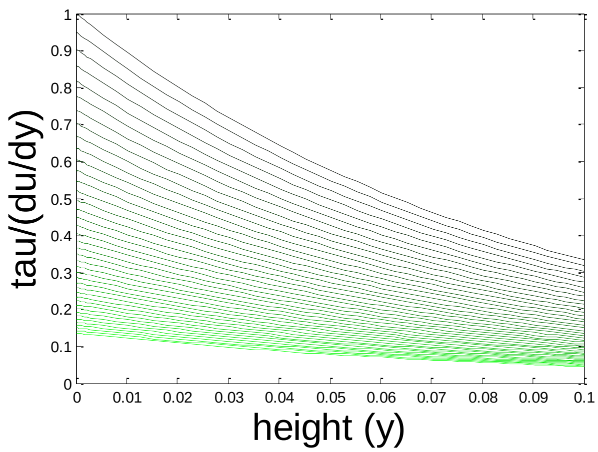

Figure 5: Graph of the ratio of shear stress to shear rate as time varies from 0 to 2 seconds for a thixotropic fluid for U = 2.0.

3.2 Laminar Flow Down an Incline

Figure 6 shows the laminar flow of a Newtonian fluid (SAE 30) moving down an incline due to gravity with $h = 0.05$, $\mu = 0.5$, and $\gamma = 8630$ with $\theta = 0, 7.5, 15, 22.5$ and 30.

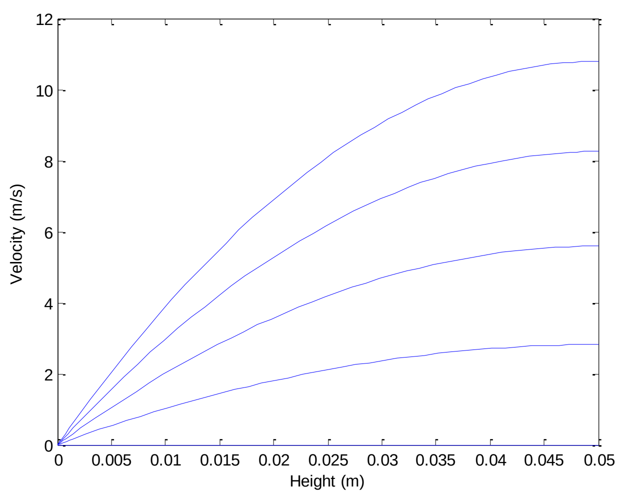

Figure 6: Velocity down an incline for the Newtonian Fluid SAE30 at 0, 7.5, 15, 22.5, and 30 degrees. The behavior of the fluid is independent of time, so there is only one line.

In Figure 7, the velocity for the thixotropic fluid described in (6) down an incline at $\theta = 5, 7.5,$ and 10 with $h = 0.05$ and $\gamma = 9{,}220$ is plotted.

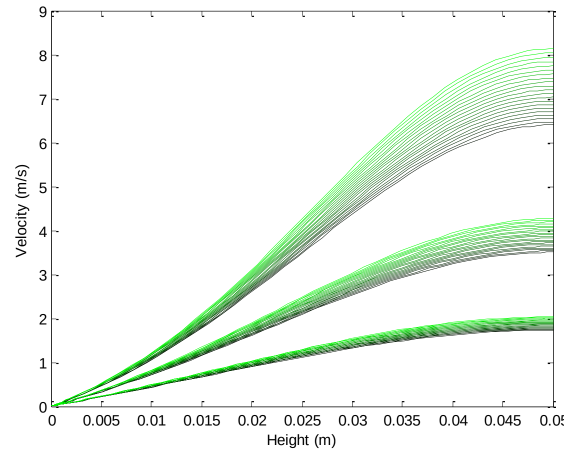

Figure 7: Velocity down an incline of a thixotropic fluid at 5, 7.5, and 10 degrees.

In Figure 8, the velocity for a rheopectic fluid described in (10) down an incline at $\theta = 5,\ 5.5,$ and 6 with h = 0.05 and and $\gamma = 7850$ is plotted.

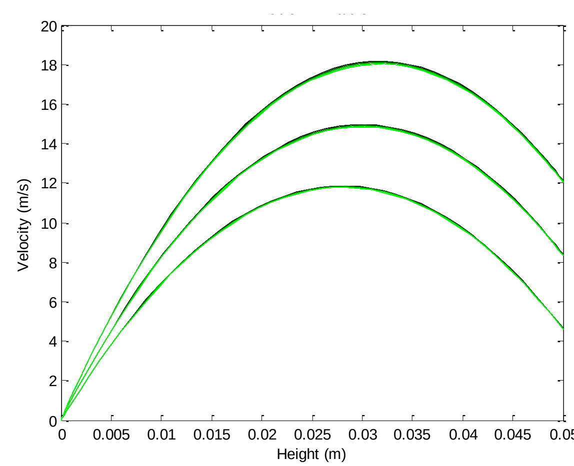

Figure 8: Velocity down an incline of a rheopectic fluid at 5, 5.5, and 6 degrees.

In Figure 9, the average velocity of this fluid is plotted at different inclination angles.

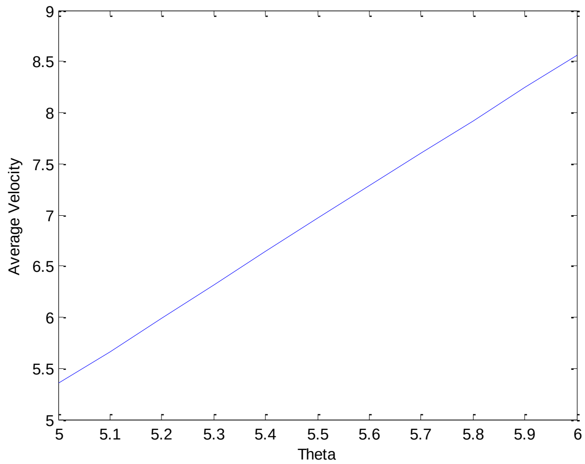

Figure 9: Average velocity of rheopectic fluid down an incline at different angles.

4. DISCUSION

In Figure 1, it appears as if the relationship between velocity and height is mostly linear. This implies that the thixotropic fluid is behaving near Newtonian. It does appear, however, that the greater the velocity, the less linear the relationship, and, therefore, the less Newtonian the thixotropic fluid behaves. As further evidence, consider Figures 2 through 5. For a Newtonian fluid, the rate of shear stress to shear rate should approach a constant value. The greater the velocity, the less constant the ratio of shear stress to shear rate was especially at earlier times (shown in black). As time continues, the ratio of shear stress to shear rate becomes more constant. This implies that over time, the thixotropic fluid behaves more Newtonian.

Interestingly, Newtonian fluids down an incline do not behave linearly as seen in Figure 6. Instead, the fluid rapidly increases in velocity near the bottom of the surface before slowing down to reach a maximum speed determined by the angle of the incline. It appears as if the angle of inclination is linearly related with the maximum speed.

The thixotropic fluid down the incline behaves similar to the Newtonian fluid. The shape of the green curves in Figure 7 (the thixotropic fluid) closely matches the shape of the curve in Figure 6 (the Newtonian Fluid). On the other hand, the rheopectic fluid shown in Figure 8 behaves completely different from a Newtonian fluid. Surprisingly, the maximum velocity of the rheopectic fluid does not occur at the top surface as all of the other fluids examined have. Instead, the maximum velocity occurs in the middle fluid. Despite this interesting behavior, Figure 9 demonstrates that velocity is, as expected, linearly related with angle of inclination. Another property worth noting of this rheopectic fluid is that it is almost completely time independent, as the green and black lines closely overlay each other as seen in Figure 8. This contrasts with the thixotropic fluid which behavior changes significantly as time progresses as seen in Figure 7.