A Note on Recovering Matrices in Linear Families from Generic Matrix-Vector Products *

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In this note we consider a recovery problem for an unknown matrix A lying in a known linear subspace $\mathcal{A} \subset \mathbb{R}^{m\times n}$ of real $m\times n$ matrices. Examples of such subspaces or "linear families" of matrices include Toeplitz, Hankel, circulant, and tridiagonal matrices. With fixed nonnegative integers q_R and q_L, the goal of the recovery problem is to choose a matrix $X_R \in \mathbb{R}^{n\times q_R}$ and a matrix $X_L \in \mathbb{R}^{m\times q_L}$ so that $A \in \mathcal{A}$ is uniquely determined when $Y_R = AX_R$ and $Y_L = A^T X_L$ are known. The columns of Y_R and Y_L are formed by matrix-vector products of A and A^T with the columns of X_R and X_L. To be precise, (X_R, X_L) solves the recovery problem and we write $(X_R, X_L) \in R(\mathcal{A}, q_R, q_L)$ if and only if the linear map $L_{(X_R,X_L)} : \mathcal{A} \to \mathbb{R}^{m\times q_R} \times \mathbb{R}^{n\times q_L}$ defined by

$$L_{(X_R,X_L)} : A \mapsto (AX_R,\ A^T X_L) \tag{1}$$

is injective. For certain linear families $\mathcal{A}$ including those named above, Halikias and Townsend [2] provide cleverly constructed $(X_R, X_L) \in R(\mathcal{A}, q_R, q_L)$ where $q_R + q_L$ is as small as possible. We strengthen these results by proving the following:

Proposition 1. *If $R(\mathcal{A}, q_R, q_L)$ is nonempty, then its complement in $\mathbb{R}^{n\times q_R} \times \mathbb{R}^{m\times q_L}$ has zero Lebesgue measure.*

Proof. Choosing a basis $\{A_k\}_{k=1}^p$ for $\mathcal{A}$, we have $(X_R, X_L) \in R(\mathcal{A}, q_R, q_L)$ if and only if $\hat{L}_{(X_R,X_L)} : \mathbb{R}^p \to \mathbb{R}^{m\times q_R} \times \mathbb{R}^{n\times q_L}$ defined by

$$\hat{L}_{(X_R,X_L)} : x \mapsto L_{(X_R,X_L)}\left(\sum_{k=1}^p x_k A_k\right) = \left(\sum_{k=1}^p x_k A_k X_R,\ x_k A_k^T X_L\right) \tag{2}$$

is injective. This map is injective if and only if its $p \times p$ Gram matrix

$$G_{(X_R,X_L)} = \hat{L}^*_{(X_R,X_L)} \hat{L}_{(X_R,X_L)} = \left[\operatorname{Tr}\left(X_R^T A_i^T A_j X_R\right) + \operatorname{Tr}\left(X_L^T A_i A_j^T X_L\right)\right]_{1\le i,j\le p} \tag{3}$$

is invertible. The map $\phi : (X_R, X_L) \mapsto \det\left[G_{(X_R,X_L)}\right]$ is a polynomial function on the Euclidean space $\mathbb{R}^{n\times q_R} \times \mathbb{R}^{m\times q_L}$. The key property of this map is that it satisfies $\phi(X_R, X_L) \neq 0$ if and only if $(X_R, X_L) \in R(\mathcal{A}, q_R, q_L)$. Moreover, ϕ is not a constant because $\phi(0, 0) = 0$ and $R(\mathcal{A}, q_R, q_L)$ is assumed to be nonempty. Since ϕ is a non-constant polynomial, the level set $\phi^{-1}(0) = (\mathbb{R}^{n\times q_R} \times \mathbb{R}^{m\times q_L}) \setminus R(\mathcal{A}, q_R, q_L)$ has zero Lebesgue measure in $\mathbb{R}^{n\times q_R} \times \mathbb{R}^{m\times q_L}$ thanks to the main result in [1]. ■

As corollaries to this proposition and the results of Halikias and Townsend [2], an $n \times n$ Toeplitz matrix A is uniquely determined by AX for almost every $X \in \mathbb{R}^{n\times 2}$ with respect to Lebesgue measure. The same holds for an $n \times n$ Hankel matrix. An $n \times n$ circulant matrix A is uniquely determined by Ax for almost every $x \in \mathbb{R}^n$. An $n \times n$ tridiagonal matrix A is uniquely determined by AX for almost every $X \in \mathbb{R}^{n\times 3}$.

References

[1] R. Caron and T. Traynor, *The zero set of a polynomial*, WSMR Report 05-02, (2005).

[2] D. Halikias and A. Townsend, *Matrix recovery from matrix-vector products*, arXiv preprint arXiv:2212.09841, (2022).

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