Homework 1: Real numbers (Due February 3^{rd})

1. a) Prove that the axiom "$0 \leq x$ and $0 \leq y$ implies $0 \leq xy$" is equivalent to "$x \leq y$ and $z \geq 0$ implies $xz \leq yz$.

b) Prove that if the above axiom is replaced by "$0 \leq x$ and $0 \geq y$ implies $0 \leq xy$" in the ordered field axiomatic, it follows that a) $x^2 \leq 0$, b) $1 < 0$. How do the rules of signs change?

2. In the axioms of $\mathbb{R}$ assumed that $0 \neq 1$ to avoid the trivial situation of a one-element field. Prove that one can equivalently impose the condition that there is some element different from 0.

3. Let A be a non-empty subset of an ordered set. Let α be a lower bound and β be an upper bound for A. Prove that $\alpha \leq \beta$.

4. Let A be a non-empty subset or $\mathbb{R}$ which is bounded below. Let $-A := \{-x : x \in A\}$. Prove that

$$\inf A = -\sup(-A).$$

5. a) Given two subsets of real numbers, $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$, define their sum as

$$A + B := \{x + y : (x \in A) \wedge (y \in B)\}$$

Prove that $\sup(A + B) = \sup A + \sup B$.

b) If we define the product of A and B to be

$$AB := \{xy : (x \in A) \wedge (y \in B)\},$$

is it true that $\sup(AB) = \sup A \sup B$? Prove it if true or give a counterexample if false.

6. Use the Archimedean property to show that

$$(x \geq 0) \wedge (\forall n \in \mathbb{N},\ x < \frac{1}{n}) \Longrightarrow x = 0$$

7. Prove that there is always a rational number between two given different real numbers. **Hint.** Use the Archimedean property or Ex. 6.

8. Define the **absolute value** of a real number x as $|x| = \max\{x, -x\}$. Prove the triangle inequality:

$$|x + y| \leq |x| + |y|,$$

for any $x, y \in \mathbb{R}$. When does the equality hold?

9. Use induction on n to prove the generalized triangle inequality

$$|x_1 + x_2 + \ldots x_n| \leq |x_1| + |x_2| + \cdots + |x_n|,$$

for arbitrary real numbers x_1, x_2, $x_3, \ldots x_n$

10. Ex. 9 page 45 from Pugh.