Math 425a Spring 2023 HW 6

Sampad Mohanty

TOTAL POINTS

39 / 40

QUESTION 1

1 **5 / 5**

✓ **- 0 pts** *Correct*

QUESTION 2

2 **5 / 5**

✓ **- 0 pts** *Correct*

- 1 pts Miss the last part.

QUESTION 3

3 **5 / 5**

✓ **- 0 pts** *Correct*

QUESTION 4

4 **5 / 5**

✓ **- 0 pts** *Correct*

- 1 pts Why $$g(c+\delta)-g(c)>0$$?

- 1 pts The direction of inequality is not right.

- 1 pts Why you need to require x tends to 0?

- 1 pts Why g(x) is constant?

QUESTION 5

5 **4 / 5**

- 0 pts Correct

- 1 pts For 3) and 4), why?

- 1 pts Your sequence is unbounded, so you cannot use the theorem.

✓ **- 1 pts** *Why [-z,z] is closed interval when z goes to infinity?*

- 1 pts Why you can pick such a and b?

- 1 pts Why the limit of f'(x) equals to 0 implies f'(x)=0?

- 2 pts Why g(c)=0? And why g(x) is decreasing for x<c?

QUESTION 6

6 **5 / 5**

✓ **- 0 pts** *Correct*

QUESTION 7

7 **5 / 5**

✓ **- 0 pts** *Correct*

- 2 pts The answers are incorrect.

QUESTION 8

8 **5 / 5**

✓ **- 0 pts** *Correct*

- 4 pts The answer is not right.

Problem 1 :- $|f(x)-f(y)| \le C|x-y|^{\alpha} \quad \forall x,y$

$$\Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right| \le C|x-y|^{\alpha-1}$$

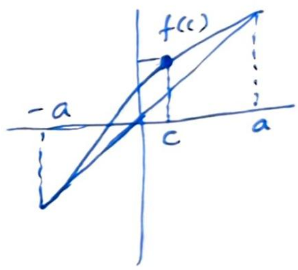
$$\Rightarrow \lim_{x\to y}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{x\to y} C|x-y|^{\alpha-1}$$

$$\Rightarrow |f'(x)| \le \lim_{x\to y} C|x-y|^{\alpha-1} = 0$$

$$\Rightarrow |f'(x)| \le 0 \Rightarrow f'(x) = 0$$

Problem 4 :- We prove this using Lagrange's MVT & contradiction.

Let $\exists\, c \in (-a,a)$ s.t. $f(c) \ne c$.



i) Let $f(c) > c$.

Then $\exists\, d \in (-a,c)$ s.t.

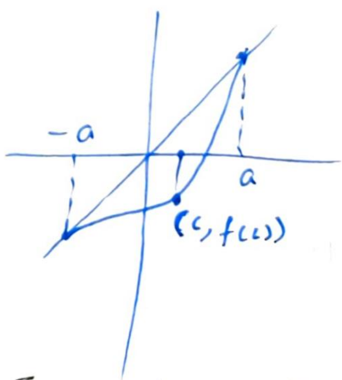
$$f'(d) = \frac{f(c)-f(-a)}{c-(-a)} = \frac{f(c)-(-a)}{c+a} = \frac{f(c)+a}{c+a}$$

$$\Rightarrow f'(d) = \frac{f(c)+a}{c+a} > 1 \text{ as } f(c) > c.$$

But this is a contradiction since $f'(x) \le 1$ in $[a,b]$

ii) Let $f(c) < c$.

Then by LMVT



$$\frac{f(a)-f(c)}{a-c} = f'(e)$$

$$\Rightarrow \frac{a-f(c)}{a-c} = f'(e) \Rightarrow 1 < \frac{a-f(c)}{a-c} = f'(e) \text{ [Contradiction]}$$

$\blacksquare$

1 **5 / 5**

✓ **- 0 pts** *Correct*

Alternatively: Set $g(x) = f(x) - x$ on $[a, -a]$

$$g'(x) = f'(x) - 1 \leq 1 - 1 = 0$$

$\Rightarrow g'(x) \leq 0$ i.e. g is decreasing

However $g(a) = g(-a) = 0$. This is possible only if g is constantly '0'. i.e. $g(x) = 0 \Rightarrow f(x) = x$. ∎

Problem 7:-

$$\lim_{x \to 0} \frac{f(2x) - f(x)}{f(3x) - f(x)} = \lim_{x \to 0} \frac{2f'(2x) - f'(x)}{3f'(3x) - f'(x)} \quad \text{(L'Hopital)}$$

(a) if $f'(0) \neq 0$, we have

$$\lim_{x \to 0} \frac{2f'(2x) - f'(x)}{3f'(3x) - f'(x)} = \frac{2f'(0) - f'(0)}{3f'(0) - f'(0)} = \frac{f'(0)}{2f'(0)} = \frac{1}{2}.$$

(b) if $f'(0) = 0$ but $f''(0) \neq 0$, then

$$\lim_{x \to 0} \frac{2f'(2x) - f'(x)}{3f'(3x) - f'(x)} = \lim_{x \to 0} \frac{4f''(x) - f''(x)}{9f''(3x) - f'(x)} = \frac{4f''(0) - f''(0)}{9f''(0) - f''(0)}$$

$$= \frac{3f''(0)}{8f''(0)} = \frac{3}{8}.$$

Problem: 2

$$\lim_{x \to a} \frac{f(x)}{x - a} = K \text{ (some constant)}$$

$$\lim_{x \to a} \frac{g(x)}{x - a} = 0$$

a) $$\lim_{x \to a} \frac{f(x)g(x)}{x - a} = \lim_{x \to a} \frac{f(x)}{x - a} \lim_{x \to a} g(x) = K \cdot \lim_{x \to a} g(x) = 0$$

Hence $f(x)g(x) = fg(x) = o(x - a)$.

2 **5 / 5**

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- 1 pts Miss the last part.

3 **5 / 5**

✓ **- 0 pts** *Correct*

Problem 1 :- $|f(x)-f(y)| \le C|x-y|^{\alpha} \quad \forall x,y$

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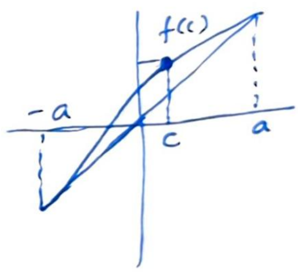

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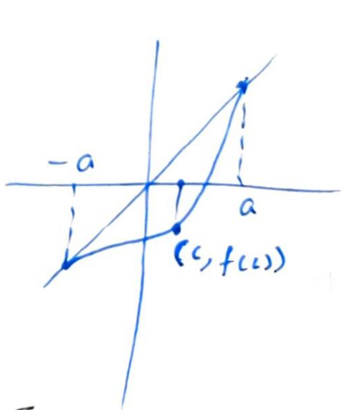

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Then by LMVT

$$\frac{f(a)-f(c)}{a-c} = f'(e)$$

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4 **5 / 5**

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- **- 1 pts** Why you need to require x tends to 0?
- **- 1 pts** Why g(x) is constant?

5 **4 / 5**

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- **- 1 pts** Your sequence is unbounded, so you cannot use the theorem.
- ✓ **- 1 pts** *Why [-z,z] is closed interval when z goes to infinity?*
- **- 1 pts** Why you can pick such a and b?
- **- 1 pts** Why the limit of f'(x) equals to 0 implies f'(x)=0?
- **- 2 pts** Why g(c)=0? And why g(x) is decreasing for x<c?

b) $\lim_{x \to a} \frac{[f(x)]^2}{x-a} = \lim_{x \to a} \frac{f(x)}{x-a} \lim_{x \to a} f(x) = 0 \cdot 0 = 0$

Hence $[f(x)]^2 = O(x-a)$

c) $(f-g)x = f(x) - g(x)$

$\lim_{x \to a} \frac{f(x) - g(x)}{x-a} = \underset{x \to a}{Lt} \frac{f(x)}{x-a} - \underset{x \to a}{Lt} \frac{g(x)}{x-a} = K - 0 = K$

Hence $(f-g)x = O(x-a)$

d) $g/f\,(x) = \frac{g(x)}{f(x)}$.

$\lim_{x \to a} \frac{\frac{g(x)}{f(x)}}{x-a} = \lim_{x \to a} \frac{g(x)}{(x-a)f(x)} = \lim_{x \to a} \frac{\frac{g(x)}{x-a}}{f(x)}$

If $g(x) \propto \bar{O}((x-a)^2)$, then we get $1/K$

Problems 6: Since f is continuous & differentiable everywhere except 0, we can use LMVT.

$\lim_{x \to 0} f'(x) = L \in \mathbb{R}$.

Left hand limit $= \lim_{x \to 0^-} f'(x) = \lim_{h \to 0^+} \frac{f(x) - f(x-h)}{h} = \lim_{h \to 0^+} f'(\alpha)$ by LMVT $\alpha \in [-h, 0]$

$\quad\quad \| $

$\lim_{\alpha \to 0^-} f'(\alpha) = L$

Similarly Right Hand Limit $= \lim_{x \to 0^+} f'(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} f'(\beta)$

$\quad\quad \|$

$L = \lim_{\beta \to 0^+} f'(\beta)$

$\boxed{\text{Hence } f'(0) \text{ exists}}$

6 **5 / 5**

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Alternatively: Set $g(x) = f(x) - x$. on $[a, -a]$

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a) $$\lim_{x \to a} \frac{f(x)g(x)}{x - a} = \lim_{x \to a} \frac{f(x)}{x - a} \lim_{x \to a} g(x) = K \cdot \lim_{x \to a} g(x) = 0$$

Hence $f(x)g(x) = fg(x) = o(x - a)$.

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PROBLEM-5 $\lim_{s\to-\infty} f(s) = 0$ & $\lim_{t\to\infty} f(t) = 0$.

$$\lim_{z\to\infty} f(z) - f(-z) = 0 - 0 = 0$$

Since f is differentiable by LMVT $\exists\, c \in [-z, z]$ s.t. $f'(c) = 0$

So $\exists\, c \in \mathbb{R}$ s.t. $f'(c) = 0$.

PROBLEM-3 :

$$\lim_{x\to+\infty} [f(x+h) - f(x)] = \lim_{x\to+\infty} f'(\alpha) h$$

for some $\alpha \in (x, x+h)$

by Lagrange's MVT.

$$\|$$

$$0 = \lim_{\alpha\to+\infty} f'(\alpha) h$$

Hence proved.

PROBLEM-8 :

$$x \sin\frac{1}{x} = 2\varepsilon_c \sin\frac{1}{\varepsilon_c} - \cos\frac{1}{\varepsilon_c}$$

$$\Rightarrow \cos\frac{1}{\varepsilon_c} = x\sin\frac{1}{x} - 2\varepsilon_c \sin\frac{1}{\varepsilon_c}$$

When applying $\lim_{x\to0}$, we don't know how fast $\varepsilon_c \to 0$ as $x \to 0$.