Homework #2
CS675 Spring 2022

Due Monday Feb 28, by midnight

General Instructions The following assignment is meant to be challenging. Feel free to discuss with fellow students, though please write up your solutions independently and acknowledge everyone you discussed the homework with on your writeup. I also expect that you will not attempt to consult outside sources, on the Internet or otherwise, for solutions to any of these homework problems — doing so would be considered cheating.

Several of these problems are drawn from the following texts, each of which is linked on the course website: Luenberger and Ye (4th edition), Korte and Vygen (5th edition), and Boyd and Vendenberghe. Please make sure you are using the correct edition of each of the books by using the links on the course website.

We request that you submit your homework as a pdf file, by email to the TA.

Finally, whenever a question asks you to "show" or "prove" a claim, please provide a formal mathematical proof.

Problem 1. (3 points)
B&V Exercise 2.3.

Problem 2. (5 points)
B&V Exercise 2.12.

Problem 3. (5 points)
B&V Exercise 2.15.

Problem 4. (5 points)
B&V Exercise 2.16.

Problem 5. (3 points)
B&V Exercise 2.21.

Problem 6. (5 points)
B&V Exercise 2.26.

Problem 7. (5 points)
B&V Exercise 2.36. Additionally, describe the dual cone of Euclidean distance matrices as the

conic hull of a set of matrices.
(Note: The latter part of the question will make more sense after we cover geometric duality in class.)

Problem 8. (2 points)
B&V Exercise 3.2.

Problem 9. (3 points)
B&V Exercise 3.12.

Problem 10. (4 points)
B&V Exercise 3.24.

Problem 11. (4 points)
B&V Exercise 3.29.

Problem 12. (4 points)
B&V Exercise 3.30.

Problem 13. (4 points)
B&V Exercise 3.36, parts a and c.

Problem 14. (14 points)
In this problem, we will explore the relationship between polar and Lagrangian duality. Specifically, in the context of linear programming, we will argue that the two are respectively geometric and algebraic formulations of the same idea.

a (2 points). Consider a linear program of the form

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array} \tag{1}$$

Using the rules we saw in class, we can derive the Lagrangian dual of (1) as the following LP

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y = c \\ & y \succeq 0 \end{array} \tag{2}$$

Naturally, scaling each inequality $a_i \cdot x \leq b_i$ of (1) by a constant $\alpha_i > 0$ to get the inequality $\alpha_i a_i \cdot x \leq \alpha_i b_i$ preserves the feasible set and objective function of (1) (and therefore also preserves the optimal solution and objective value). In other words, scaling the inequalities produces a geometrically equivalent optimization problem. Show that the same cannot be said for the Lagrangian dual of (1); specifically, show that scaling the inequalities of (1) changes feasible set and optimal solution of its dual. Conclude that Lagrangian duality is an algebraic transformation, since given two equivalent LPs (same feasible set and objective) represented differently, it yields different dual LPs.

b (2 points). For simplicity, assume $x = \vec{0}$ is a strictly feasible solution of (1) — i.e., the feasible region includes an open ball about the origin.[1] Show that (1) is equivalent, in the sense of having the same feasible set and objective function, to an LP of the following "normalized" form, and has an optimal value $\nu^* \geq 0$.

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \widehat{A}x \preceq \vec{1} \end{array} \tag{3}$$

c (2 points). Using the rules for taking duals, the dual of (3) is the following LP.

$$\begin{array}{ll} \text{minimize} & \vec{1}^T y \\ \text{subject to} & \widehat{A}^T y = c \\ & y \succeq \vec{0} \end{array} \tag{4}$$

Show that (4) and (2) are equivalent up to a simple transformation of the variables, and note that said transformation preserves the optimal value.

d (4 points). Let $P = \left\{ x \in \mathbb{R}^n : \widehat{A}x \preceq \vec{1} \right\}$ denote the feasible set of (3) (and therefore also of (1)), and let P° denote its polar. You will show that one can derive tight bounds on ν^* from the polar P°. Specifically, show that if $\frac{1}{\nu}c \in P^\circ$ for some constant $\nu > 0$, then $\nu^* \leq \nu$. Conversely, show that $\frac{1}{\nu^*}c \in P^\circ$.

e (4 points). Recall from class that if $P = \left\{ x \in \mathbb{R}^n : \widehat{A}x \preceq \vec{1} \right\}$ is a polytope, then its polar P° is equal to the convex hull of the rows of the matrix $\widehat{A}$. More generally, when P is a polyhedron its polar P° is the convex hull of $rows(\widehat{A}) \cup \left\{ \vec{0} \right\}$ (you are invited to verify this for yourself if curious). Explain how LP (4) — and by (c), also LP (2) — can be interpreted as finding the tightest upperbound on ν^* implied by the polar, in the sense of (d). Conclude that Lagrangian duality is an algebraic analog of polar duality, which is a purely geometric relationship between convex sets.

[1] This is without loss of generality in most (though not all) natural applications of LP, since it can be enfored by a combination of projection and a suitable shift of the feasible set.