Homework 2: Sets, functions and cardinality (Due Tuesday, Feb. 21st)

1. Let $f : X \to Y$ be a function and let $\{Y_j\}$ be a collection of subsets of Y. Prove that the inverse image behaves well under unions and intersections *i.e.*

$$f^{-1}(\bigcup_j Y_j) = \bigcup_j f^{-1}(Y_j); \qquad f^{-1}(\bigcap_j Y_j) = \bigcap_j f^{-1}(Y_j).$$

Are the above properties true if $\{Y_j\}$ is replaced by a collection $\{X_j\}$ of subsets of X and f^{-1} is replaced by f? **Hint.** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Let $A = (-1, 0)$ and $B = (0, 1)$. What are $f(A \cap B)$ and $f(A) \cap f(B)$?

2. *a*) Prove that if A and B are finite sets with $|A| = |B|$, then any injection $f : A \to B$ is also a surjection. Show this is not necessarily true if A and B are not finite.

b) Prove that if A and B are finite sets with $|A| = |B|$, then any surjection $f : A \to B$ is also an injection. Show this is not necessarily true if A and B are not finite.

3. Determine whether the following sets are finite, countably infinite or uncountable, justifying in each case.

a) $\{1/n : n \in \mathbb{Z} \setminus \{0\}\}$;
b) The collection of all finite subsets of $\mathbb{N}$;
c) The set of all functions $f : \mathbb{N} \to \mathbb{N}$;
d) The set of all non-decreasing functions $f : \mathbb{N} \to \mathbb{N}$;
e) The set of all non-increasing functions $f : \mathbb{N} \to \mathbb{N}$;

4. Use the Cantor-Bernstein-Schröeder theorem to show that the real intervals $(0, 1)$ and $[0, 1)$ have the same cardinality. **Hint.** It is easy to find linear injections from each of the intervals into the other.

5. In class, we went over the diagonal argument used by Cantor to show that $\mathbb{R}$ (equivalently, (0, 1)) is uncountable. Give an alternative proof using the following facts:

a) Any real number $x \in (0,\ 1)$ can be written in binary representation, $0.a_1\, a_2\, a_3\, \ldots$ where a_i is either 0 or 1.

b) Binary expressions like the above can be put in a bijective correspondence with subsets of natural numbers (explain how namely).

c) Cantor's theorem (proved in class) stating that $\mathrm{Card}(X) < \mathrm{Card}(\mathcal{P}(X))$

6. Prove that if B is a countable set then $\mathrm{Card}\,(\mathbb{R} \cup B) = \mathrm{Card}\,(\mathbb{R})$.
Hint : For simplicity assume first that $\mathbb{R} \cap B = \emptyset$ and that B is countably infinite, and construct a bijection $f : \mathbb{R} \cup B \to \mathbb{R}$ such that $f(x) = x$ for every $x \in \mathbb{R} \setminus \mathbb{N}$ and that $f(\mathbb{N} \cup B) = \mathbb{N}$.

7. Prove that the set of all circles in $\mathbb{R}^2$ with center $P = (x, y)$ and radius r, such that $r \geq 0$ is a positive rational number and such that $x, y \in \mathbb{Z}$, is countable.