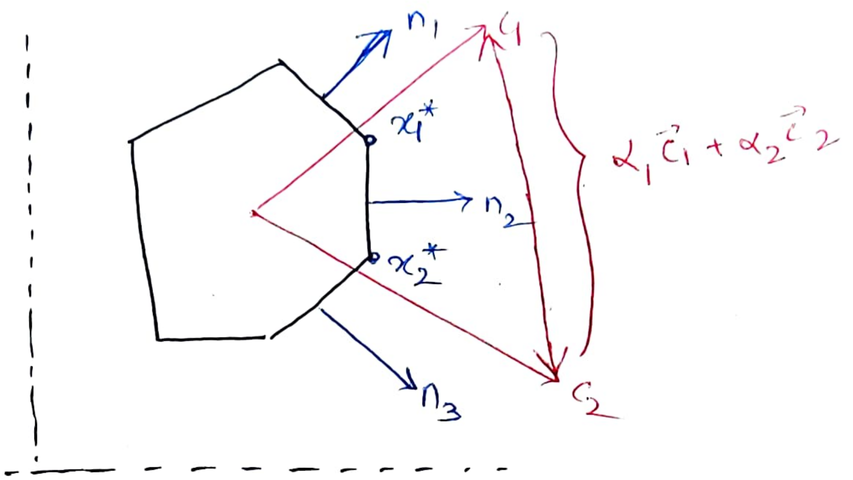
CSCI-675: HW5

$\boxed{\text{w.r.t } c.}$

Q1 b). $OPT(\alpha_1 c_1 + \alpha_2 c_2, b)$

vs $\alpha_1 OPT(c_1, b) + \alpha_2 OPT(c_2 b)$



We know:-

$OPT(c_1, b) = \langle c_1, x_1^* \rangle$

$OPT(c_2, b) = \langle c_2, x_2^* \rangle$

$$\alpha_1 OPT(c_1, b) + \alpha_2 (c_2, b) = \alpha_1 \langle c_1, x_1^* \rangle + \alpha_2 \langle c_2, x_2^* \rangle$$

$OPT(\alpha_1 c_1 + \alpha_2 c_2, b)$ has either x_1^* as the minimum or x_2^* depending on if $\alpha_1 c_1 + \alpha_2 c_2$ lies in the conic combination of (n_1, n_2) or (n_2, n_3).

WLOG, let $\alpha_1 c_1 + \alpha_2 c_2 \in cone(n_1, n_2)$.

Then the optimum occurs at x_1^*.

So $OPT(\alpha_1 c_1 + \alpha_2 c_2) = \langle \alpha_1 c_1 + \alpha_2 c_2, x_1^* \rangle = \alpha_1 \langle c_1, x_1^* \rangle + \alpha_2 \langle c_2, x_1^* \rangle$

$\geq \alpha_1 \langle c_1, x_1^* \rangle + \alpha_2 \langle c_2, x_2^* \rangle$

$[\text{since } \langle c_2, x_2^* \rangle \leq \langle c_2, x_1^* \rangle]$

$= \alpha_1 OPT(c_1, b) + \alpha_2 OPT(c_2, b)$

So $OPT(\alpha_1 c_1 + \alpha_2 c_2, b) \geq \alpha_1 OPT(c_1, b) + \alpha_2 OPT(c_2, b)$

* Hence non-convex.

However, if it was ~~minimisation~~ maximisation, it is convex.

For maximisation, $OPT(\alpha_1 c_1 + \alpha_2 c_2) = \langle \alpha_1 c_1 + \alpha_2 c_2, x_1^* \rangle$

$$= \alpha_1 \langle c_1, x_1^* \rangle + \alpha_2 \langle c_2, x_1^* \rangle$$

$$\leq \alpha_1 \langle c_1, x_1^* \rangle + \alpha_2 \langle c_2, x_2^* \rangle$$

$\boxed{\because OPT(c_2, x_2^*) \geq OPT(c_2, x_1^*)}$

$$= \alpha_1 OPT(c_1, b) + \alpha_2 OPT(c_2, b)$$

So $OPT(\alpha_1 c_1 + \alpha_2 c_2, b) \leq \alpha_1 OPT(c_1, b) + \alpha_2 (c_2, b)$

Hence convex.

(a) $\boxed{\text{w.r.t. } b}$

$$\min c^T x \quad \text{s.t. } Ax \geq b \quad \equiv \quad \max b^T x \quad \text{s.t. } Ax = c, \; x \geq 0$$

maximization, hence convex

(c)

(a) Concave

(b) Convex

$\boxed{\text{maximize } \langle c, x \rangle}$

$$\max \langle c, x \rangle \quad \text{s.t. } Ax \geq b$$

$\cancel{\min \langle c, x \rangle \quad \text{s.t. } Ax \leq b}$

$\cancel{\max \langle c, x \rangle \quad Ax \leq b}$

$\boxed{Ax \leq b}$

(a) $\min \langle c, x \rangle$ s.t. $Ax \leq b$ || non convex

(b) $\min \langle c, x \rangle$ s.t. $Ax \leq b$ || Convex.

$\boxed{Ax \leq b \quad \max \langle c, x \rangle}$

(a) Convex

(b) non-convex.

Q3 Let $w_{max} = \max\{w_i \mid i \in X\}$

Let us define a new set of weights $\bar{w}_i$ as follows.

$$\bar{w}_i = w_{max} - w_i$$

The ordering $\bar{w}_i$'s is exactly the reverse of w_i's.

Also $\bar{w}_i \geq 0 \quad \forall\, i \in X$.

* Using the greedy algo, we can find the max weight independent set w.r.t the weights $\bar{w}_i$.

* Since any independent set can always be expanded to a base of the matroid & since $\bar{w}_i$s are non-negative, the max weight independent set must be a base as otherwise we can expand the max-weight independent set into a base & the weight can only increase as the weights $\bar{w}_i$ are all non-negative.

Q2 a) Clearly rank$(\phi) = 0$, hence normalized.

Monotonic: Let $A \subseteq B$, then rank$(A) \leq$ rank(B)

Let α be a maximal independent set of A.

$$\text{rank}(A) = \#\alpha$$

Since $\alpha \subseteq A \subseteq B$, $\alpha \subseteq B$.

Since $\text{rank}(B) = \max\left(\{\#S \mid S \subseteq B\}\right)$

$$\text{rank}(B) = \max\left(\{\#S \mid S \subseteq B, S \neq \alpha\} \cup \{\#\alpha\}\right)$$

$$= \max\left[\max\left(\{\#S \mid S \subseteq B, S \neq \alpha\}\right), \max\left(\{\#\alpha\}\right)\right]$$

$$= \max\left[\max\left(\{\#S \mid S \subseteq B, S \neq \alpha\right), \#\alpha\right]$$

$$\geq \#\alpha = \text{rank}(A) \quad \left[\because \max\{a,b\} \geq b\right]$$

$$\Rightarrow \text{rank}(B) \geq \text{rank}(A)$$

Submodular : Let $A \subseteq B$ & $\text{rank}(A) < \text{rank}(B)$

Let α be a maximal independent set of A

Let β be a maximal independent set of B

$* \ \#\alpha < \#\beta$

$$\Delta f(A) = \text{rank}(A \cup \{x\}) - \text{rank}(A) = \begin{cases} 1 & \text{if } x \in \text{Span}(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta f(B) = \text{rank}(B \cup \{x\}) - \text{rank}(B) = \begin{cases} 1 & \text{if } x \in \text{Span}(\beta) \\ 0 & \text{otherwise} \end{cases}$$

$\Delta f(A)$	$\Delta f(B)$	
1	1	✓
1	0	✓
0	1	× ← not submodular if this holds.
0	0	✓

Let $\Delta f(A) = 1$ & $\Delta f(B) = 0$.

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$x \in \text{span}\,\alpha$ $\qquad$ $x \notin \text{span}(\beta)$

We know $\exists\, \bar{\beta}$ s.t $\bar{\beta} \supseteq \alpha$ & $\#\bar{\beta} = \#\beta$ i.e $\bar{\beta}$ is a maximal independent set of B containing α.

* This is because of exchage property of matroids.

* $\bar{\beta}$ can be constructed from α & β.

Since α, β are indep. & $\#\beta > \#\alpha$, we can always move one element from β to α & still keep the expanded $\bar{\alpha}$ indepdent
At some point we will have to reach $\#\bar{\alpha} = \#\beta$ (otherwise we can add element from β to the intermediate $\bar{\alpha}$ by exchage property).

When $\#\bar{\alpha} = \#\beta$, we declare $\#\bar{\alpha}$ as $\#\bar{\beta}$ & hence $\bar{\beta} = \bar{\alpha} \subseteq \alpha$.

Since, $\Delta f(A) = 0 \Rightarrow \text{rank}(\alpha \cup \{x\}) - \text{rank}(\alpha)$

$\Rightarrow \underline{\text{rank}(\alpha \cup \{x\}) = \text{rank}(\alpha) = \#\alpha}$

Also $\Delta f(B) = 1 \Rightarrow$ $\bar{\beta} \cup \{x\}$ is independent

$\Rightarrow \bar{\alpha} \cup \{x\}$ " "

$\Rightarrow \alpha \cup \{x\} \subseteq \bar{\alpha} \cup \{x\}$ is indep.

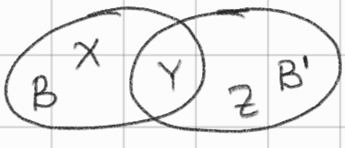
$\Rightarrow \text{rank}(\alpha \cup \{x\}) = \underline{\#\alpha + 1}$

Contradiction.

(b) [4 points]. Prove the *strong exchange property* of matroids: If B and B' are bases of matroid $\mathcal{M}$, then for every $i \in B \setminus B'$ there is some $j \in B' \setminus B$ such that $(B \setminus \{i\}) \cup \{j\}$ is a basis of $\mathcal{M}$. Show that the exchange property is implied by the strong exchange property for every downwards-closed set system. (i.e. we could have equivalently defined matroids using the strong exchange property instead).

B, B' are bases of M.

$\# B = \# B' = \text{rank}(M)$



Let $X = B \setminus B'$

$Y = B \cap B'$

$Z = B' \setminus B$

$\forall\, i \in X$, $B \setminus i$ is independent (subsets of independent sets are indep.)

$\#(B \setminus i) = \text{rank}(M) - 1$

By the exchange property, $\exists\, j \in B' \setminus (B \setminus i) = B' \setminus B$ $\boxed{\text{since } i \notin B' \cap B.}$

$\rightarrow B \setminus i \cup \{j\}$ is independent & hence

$\text{rank}(B \setminus i \cup \{j\}) = \#(B \setminus i \cup \{j\}) = \text{rank}(M)$

Since $Y \subseteq B \setminus i$ ~~4~~ & $j \in B' \setminus (B \setminus i) = B' \setminus B$

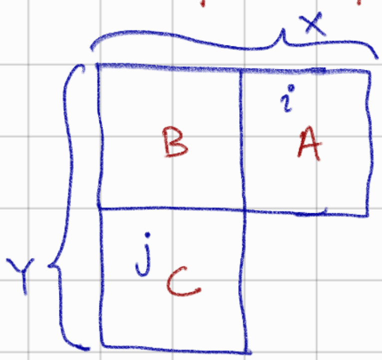
as $j \notin Y \neq B \cap B'$

So $\exists\, j \in B' \setminus B$ s.t $B \setminus i \cup \{j\}$ is indep. & hence $\#(B \setminus i \cup \{j\}) = \text{rank}(M)$ & hence $B \setminus i \cup \{j\}$ is a base.

Proof of : strong exchange property $\Rightarrow$ exchange property for downward-closed set system.

Let X & Y be two bases of M

Let $B = X \cap Y$, $A = X / Y$, $C = Y / X$



By strong exchange property $\exists j \in Y/X$ s.t

$X \setminus i \cup \{j\}$ is a base $\forall i \in X \setminus Y$

or $\exists j \in C$ s.t $X \setminus i \cup \{j\}$ is a base $\forall i \in A$

Q2 c) Define a uniform matroid $M_1 = (X, I_1)$ with all subsets with cardinality less than equal to k being independent.

$M \cap M_1$ is also a matroid as intersection with uniform matroid preserves matroid structure.

Clearly $M_k = M \cap M_1$ & hence M_k is a matroid.

(d) C is a circuit.

So $\#C = \text{rank}(M) + 1$

Let $x \in C$ be the element with lowest weight

* So there are rank (M) elements that have weight higher than x.

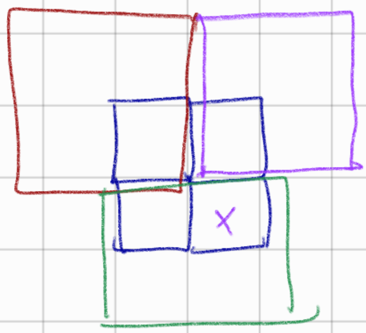
* For any maximum weight basis, we have the set X to choose the basis from & $C \subseteq X$.

* Clearly the maximum weight basis will never have to choose x since instead it can alway choose something from $C \setminus x$ to replace x & get a higher weight.

* An analogue is that if we have a team from USC consisting of 5 members and we have to send a team of 4 that consists of the strongest players from California, then when we have the option to choose members from

teams of USC, UCLA, UCSD, UCI, etc.,
it is guaranteed that the person ranked 5^{th}
in the USC team will never make it to the
team representing California.

(e)



* Let's say the maximum weight element $e \in C$ is not part of some maximum weight basis.

* Let B_c be the maximum weight basis in the set $\{B \mid e \in B\}$.

$$B_c = \underset{\substack{B \\ \text{s.t } e \in B}}{\arg\max} \sum_{i \in B} w_i$$

B_c has to be the max weight basis for all bases of the matroid.

If it is not, then there exist another basis that contains $f \in C$ & has weight higher than B_C.

4 a) $P(M_1) \cap P(M_2)$ is convex hull of indep sets.

$C = \sum x_i = rank(M_1) = rank(M_2)$ { if $rank(M_1) \neq rank(M_2)$, we can immediately return no feasible sol^n }

$P(M_1) \cap P(M_2) \cap C$ is the convex hull of the indep. bases of the matroids M_1 & M_2.

Proof
$$P(M_1) \equiv \sum_{i \in S} x_i \leq rank(S), \quad S \subseteq M_1$$
$$x_i \geq 0$$

$$P(M_2) \equiv \sum_{i \in S} x_i \leq rank(S), \quad S \subseteq M_2$$
$$x_i \geq 0$$

Notice that setting $S = X$ gives us $\rightarrow$

for $M_1 \rightarrow \sum_{i \in X} x_i \leq rank(X) = rank(M_1) \rightarrow (1)$

for $M_2 \rightarrow \sum_{i \in X} x_i \leq rank(X) = rank(M_2) \rightarrow (2)$

Clearly $\sum x_i = rank(X) = rank(M_1) = rank(M_2)$

makes (1) & (2) tight.

Intersecting C makes constraints tight means we have a face of the polytope $P(M_1) \cap P(M_2)$ which is also convex.

Since C contains all the bases but not indep. sets that are not bases, the face is convex hull of the common independent bases since $P(M_1) \cap P(M_2)$ restricts to common indep. sets on C.

So $P(M_1) \cap P(M_2) \cap C$ is convex hull of a modular function

Convex bases & optimizing over it will always get us atleast one vertex sol^n which will be a common base.

Now we just optimize $w^T x$ where $\vec{w}$ is the weight vector using an LP which is polytime.

We have independence oracles for M_1 & M_2 & for C we just implement one ourselves easily by checking if $\sum x_i = \text{rank}(x)$

* If $\text{rank}(M_1) = \text{rank}(M_2)$ but no common bases, then the above algo will return not feasible.

(b) Consider the graph with the self loops.

$$\left.\begin{array}{l} M_1 = (X, I_1) \\ M_2 = (X, I_2) \end{array}\right\} X = \text{edges}$$

Can be shown to be matroids easily

I_1 = Set of edges such that indegree on all vertices = 1 except at s

I_2 = Set of edges such that outdegree on all vertices = 1 except at t.

Let $W_\infty = \sum_i |w_i| = \|\vec{w}\|_1$

* Set the weights of self loops to w_∞.

$M_1 \cap M_2$ can give us →
- → disjoint loops
- → disjoint paths from some node to another node.

{ Now minimizing $w^T x$ over M_1 & M_2 will yeild the shortest path from s to t if one exists.

Since there are no -ve weight cycles, we will never get a -ve optimum.

If it does not, the the optimum value $w^T x^*$ is greater that equal to w_∞.

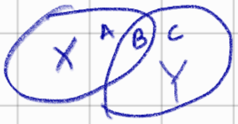
Q6

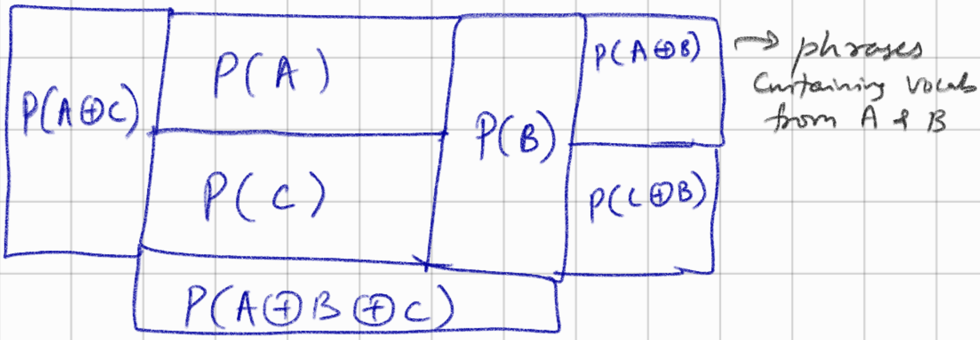
maximizing $\left(\sum_{p \in P(V)} w(p)\right) - |V|$.

Sum of Supermodular functions is supermodular.
Also Supermodular + modular = Supermodular.

$|V|$ is modular.

* So we just need to prove $f(V) = \sum_{p \in P(V)} w(p)$ is Supermodular.





$$P(X) = P(A) \cup P(B) \cup P(A \oplus B)$$
$$P(Y) = P(C) \cup P(B) \cup P(C \oplus B)$$

$$P(X \cup Y) = P(A) \cup P(B) \cup P(C) \cup P(A \oplus C) \cup P(A \oplus B) \cup P(C \oplus B) \cup P(A \oplus B \oplus C)$$

$$P(X \cap Y) = P(B)$$

$$f(X \cup Y) + f(X \cap Y)$$

$$= \sum_{i \in X \cup Y} w_i + \sum_{i \in X \cap Y} w_i$$

$$= \sum_{i \in P(A)} w_i + \sum_{i \in P(B)} w_i + \sum_{i \in P(C)} w_i + \sum_{i \in P(A \oplus C)} w_i + \sum_{i \in P(A \oplus B)} w_i + \sum_{i \in P(B \oplus C)} w_i + \sum_{i \in P(A \oplus B \oplus C)} w_i + \sum_{i \in P(B)}$$

$$= \sum_{i \in P(A) \cup P(B) \cup P(A \oplus B)} w_i + \sum_{i \in P(B) \cup P(C) \cup P(B \oplus C)} w_i + \sum_{i \in P(A \oplus C) \cup P(A \oplus B \oplus C)} w_i$$

$$f(X \cup Y) + f(X \cap Y) = \sum_{i \in P(X)} w_i + \sum_{i \in P(Y)} w_i + \sum_{\substack{i \in P(A \oplus C) \\ \cup P(A \oplus B \oplus C)}} w_i$$

$$= f(X) + f(Y) + \text{Something non-negative}$$

$$\Rightarrow f(X \cup Y) + f(X \cap Y) \geq f(X) + f(Y)$$

$\Rightarrow f$ is Supermodular.

Now we have a minimization over a supermodular function which is easily solved in poly time.

Q7 (b) $\min f(S)$ is N.P hard.
s.t. $|S| \geq k$

* We can easily show $g(V) = f(V/S)$ is also submodular.

$$f(A \cup \{u\}) - f(A) \geq f(B \cup \{u\}) - f(B)$$

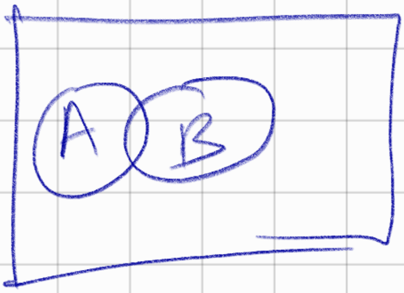
$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$$

$g(A \cup B) + g(A \cap B)$

$= f(V/A\cup B) + f(V/A \cap B)$

$= f(V/A \cap V/B) + f(V/A \cup V/B)$

$\leq f(V/A) + f(V/B) = g(A) + g(B)$



$*$ So $g(A\cup B) + g(A \cap B) \leq g(A) + g(B)$

Hence $g(S) = f(V/S)$ is submodular.

So $\min f(V/S)$ s.t. $|V/S| \geq k$ $\equiv$ $\min f(V/S)$ s.t. $|V| - |S| \geq k$

$\equiv$

$\min f(V/S)$

s.t $\underbrace{|V| - k}$ $\geq |S|$

or $k' \geq |S|$

$\equiv$

minimizing submodular function $\Leftarrow$ $\min g(S)$

Subject to Cardinality upper bound on S s.t $|S| \leq k'$

is N.P hard.

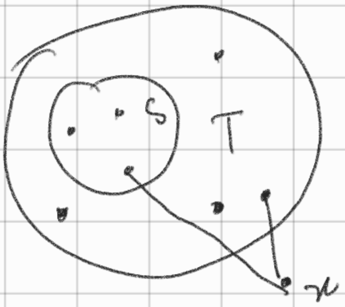
(a) We know from (b) that submodular minimization under cardinality upper & lower bounds are equivalent.

So we are going to look at cardinality upper bound.

We can easily show that $f(S)$ = no. of edges that have endpoints in $S \subseteq V$ is Supermodular.

Proof

$$f(T \cup \{x\}) - f(T) = \#\{e : x \to y \mid y \in T\}$$



$$f(S \cup \{x\}) - f(S) = \#\{e : x \to y \mid y \in S\}$$

$$S \subseteq T \Rightarrow \{e : x \to y \mid y \in S\} \subseteq \{e : x \to y \mid y \in T\}$$

$$\therefore \# \{e : x \to y \mid y \in S\} \leq \{e : x \to y \mid y \in T\}$$

$$\Rightarrow f(S \cup \{x\}) - f(S) \leq f(T \cup \{x\}) - f(T)$$

$\therefore f(S)$ is Supermodular.

Maximizing Supermodular $\equiv$ minimizing Submodular.

So $\max f(S)$ s.t. $|S| \leq k$ can solve the decision problem of if there exist a k-clique in the graph if we check that the maximum value $< {}^kC_2$ since that is the maximum possible.

if max value $= {}^kC_2$, then there exists a k-clique.

Since k - Clique is NP complete, the Submodular minimisation must be NP-Hard.

(c) The arguement here is very similar to part (b).

* Assume that we can have a polynomial approximation to submodular minimization (supermodular maximisation) under upper/lower cardinality bound.

* If above is true then the function $f(S) = [\text{edges that are insident on } S \subseteq V] + [|S|]$ (Supermodular function) that can be approximately maximized in poly time under the constraint $|S| \leq k$ & the sol^n x^* will give us the densest subgraph of size k.

* Note we add $|S|$ to the no. of edges incident on in $S \subseteq V$ since if there is an isolated node, the the algorithm might yeild a sol^n that is densest with $|S| < k$ & not $|S| = k$ (densest k-subgraph needs $|S| = k$).

$$f(S) = \underbrace{E(S)}_{\#\text{ edges incident exclusively on } S \subseteq V} + |S|$$ will always get us $|S| = k$

* So we can approx. solve the densest k-subgraph problem in polytime. But this cannot be true if the exponential time hypothesis (ETH) is true.

* So Submodular (Supermodular) minimization (maximization) under upper/lower cardinality constraints cannot be approximately solved in poly time if ETH is to be believed.

Q5 (a) $P(M) = \sum_{i \in S} x_i \leq \text{rank}(S)$ (a)

$x_i \geq 0$ (b)

$$\{S \subseteq \mathcal{X} : x(S) = rank_{\mathcal{M}}(S)\}$$ ← is a constraint that makes one of (a) tight for a fixed S.

So basically we get faces of the polytope

Intersections of faces are still faces (of lower dimensions)

Unions of faces are also faces, so they form a lattice.