Math 501
Assignment 2

1. Show that the induced matrix norm corresponding to the $\|\ \ \|_1$ norm on C^n is the maximum absolute column sum.

Let $A = [a_{ij}]$ be a matrix.

By definition, $|A|_1 = \max\limits_{|x|_1 = 1} |Ax|_1$.

Let $J = \underset{j}{\operatorname{argmax}} \sum\limits_i |a_{ij}|$, i.e. J is the column which corresponds to the maximum absolute column sum

Max absolute column sum (MACS) $= \max\limits_j \sum\limits_i |a_{ij}|$

Let us consider the standard unit vector e_J which has all entries as 0 except the Jth entry which is 1.

Clearly $|e_J|_1 = 1$ and hence is a subset of $\{x : |x|_1 = 1\}$

Since the maximum over a subset is always less than or equal to the maximum over the whole set, we have

$$|Ae_J|_1 \leq \max_{|x|_1 = 1} |Ax|_1 = |A|_1$$

$$\Rightarrow \text{MACS} \leq |A|_1 \longrightarrow (1)$$

Now we will try to prove the inequality in the other direction, i.e. MACS $\geq |A|_1$.

$$|Ax|_1 = \sum_i \left|\sum_j a_{ij} x_j\right| \leq \sum_i \sum_j |a_{ij} x_j|$$

$$= \sum_i \sum_j |a_{ij}||x_j|$$

$$= \sum_j \sum_i |a_{ij}||x_j|$$

$$= \sum_j \left(|x_j| \sum_i |a_{ij}|\right)$$

$$\leq \sum_j |x_j| \left(\max_j \sum_i |a_{ij}|\right)$$

$$= \max_j \sum_i |a_{ij}| \cancel{\sum_j |x_j|} \; 1$$

$$= \max_j \sum_i |a_{ij}| \qquad (\text{as } |x|_1 = 1)$$

Hence we have $|Ax|_1 \leq \max_j \sum_i |a_{ij}|$

Taking max over $|x|_1 = 1$, we have

$$\max_{|x|_1=1} |Ax|_1 \leq \max_{|x|_1=1} \max_j \sum_i |a_{ij}| = \max_j \sum_i |a_{ij}|$$

$$\Rightarrow |A|_1 \leq \max_j \sum_i |a_{ij}|$$

$$\Rightarrow |A|_1 \leq \text{MACS} \longrightarrow (2)$$

From (1) & (2), $|A|_1 = \text{MACS}$ ■

2. For $\| \quad \|$ a vector norm on C^n the induced matrix norm is defined by the expression $\|A\| = sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$. Show that (a) $\|A\| = \max_{\|x\|=1} \|Ax\|$ (b) $\|A\| = \max_{\|x\|\leq 1} \|Ax\|$, (c) $\|A\| = \inf\{M: \|Ax\| \leq M\|x\|, x \in C^n\}$

(a) $\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{x\neq 0} \left\| \frac{Ax}{\|x\|} \right\| = \sup_{x\neq 0} \left\| A \frac{x}{\|x\|} \right\|$

$= \sup_{x\neq 0} \|Ay\|$ where $y = \frac{x}{\|x\|}$ or $\|y\| = \left\| \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|} \|x\| = 1$

Hence $\|A\| = \sup_{x\neq 0} \|Ay\| = \sup_{\|y\|=1} \|Ay\|$

Since $\|Ay\|$ is continuous in y & the set $\{y : \|y\| = 1\}$ is closed and bounded, $\|Ay\|$ achieves its maximum on the set. Hence $\|A\| = \sup_{\|y\|=1} \|Ay\| = \max_{\|y\|=1} \|Ay\|$

(b) $\|A\| = \max_{\|x\|=1} \|Ax\|$ as shown in part (a)

Let $y \in \{x : \|x\| \leq 1\} = B$

Define $\hat{y} = \frac{y}{\|y\|}$.

Clearly $\hat{y} \in \{x : \|x\| = 1\} = C$

Clearly $C \subset B$

$\forall y \in B, \quad \hat{y} \in C$

Also $\|y\| \le \|\hat{y}\|$ & $y = \|y\|\,\hat{y}$

$$\|Ay\| = \|A\,\|y\|\,\hat{y}\| = \|y\|\,\|A\hat{y}\| \le \|\hat{y}\|\,\|A\hat{y}\| = \|A\hat{y}\|$$

$$\Rightarrow \quad \|Ay\| \le \|A\hat{y}\|$$

$$\Rightarrow \quad \max_{x\in B}\|Ax\| \le \max_{x\in B}\|Ax\| = \max_{x\in C}\|Ax\|$$

$$\Rightarrow \quad \max_{x\in B}\|Ax\| \le \max_{x\in C}\|Ax\| \longrightarrow ①$$

But also since $C \subset B$, we also have

$$\max_{x\in C}\|Ax\| \le \max_{x\in B}\|Ax\| \longrightarrow ②$$

This is because max over a subset (C) is always less than or equal to max over the set (B)

From ① & ②, $\max_{x\in B}\|Ax\| = \max_{x\in C}\|Ax\|$ ■

And to remind, $B = \{x : \|x\| \le 1\}$

$C = \{x : \|x\| = 1\}$

2. Consider R^2 with the norm $\|\cdot\|_p$, as in Example 1, Section 3. Let T be a matrix operator

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

mapping $(R^2, \|\cdot\|_p)$ into $(R^2, \|\cdot\|_p)$.

(a) Compute $\|T\|$ when $b = c$ and $p = 2$.

(b) Compute $\|T\|$ in general.

(a) $\|A\|_2$ is the largest magnitude singular value of A.

For a symmetric matrix, the eigenvalues are the singular values

Since T is symmetric, we can just find the largest eigenvalue by magnitude to find $\|A\|_2$

$$\det(T-\lambda I)=0 \Rightarrow \det\begin{pmatrix} a-\lambda & b \\ b & d-\lambda \end{pmatrix}=0$$

$$\Rightarrow (a-\lambda)(d-\lambda)-b^2=0$$

$$\Rightarrow \lambda^2-(a+d)\lambda+ad-b^2=0$$

$$\Rightarrow \lambda = \frac{(a+d) \pm \sqrt{(a+d)^2-4(ad-b^2)}}{2}$$

$$\Rightarrow \lambda = \frac{(a+d) \pm \sqrt{a^2+d^2+2ad-4ad+b^2}}{2}$$

$$= \frac{(a+d) \pm \sqrt{(a-d)^2+b^2}}{2}$$

The largest singular value is one of the above based on actual values of a & d.

If $a+d<0$, then $|A|_2 = \dfrac{(a+d)-\sqrt{(a-d)^2+b^2}}{2}$

else $|A|_2 = \dfrac{(a+d)+\sqrt{(a-d)^2+b^2}}{2}$

(b) $$\|T\| = \max_{|x|=1} \|Tx\|$$

$$= \max_{x,\lambda} \|Tx\| + \lambda(\|x\|-1)$$

Also to get a bound,

$$Tx = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

$$\|Tx\|_p = \max_{|x|=1} \left(|ax_1+bx_2|^p + |cx_1+dx_2|^p\right)^{1/p}$$

$$\leq \max_{|x|=1} \left[\left(|ax_1| + |bx_2|\right)^p + \left(|cx_1| + |dx_2|\right)^p \right]^{1/p}$$

3. Let M_n denote the space of real $n \times n$ matrices. For $A = (a_{ij}) \in M_n$ define

$$n(A) = \sum_{i,j} |a_{ij}| .$$

(a) Show here that $\|Ax\|_1 \leq n(A)\|x\|_1$, where $x \in R^n$ and R^n have the norm $\|x\|_1 = \sum_{i=1}^n |x_i|$.
(b) Let $A, B \in M_n$. Show that $n(AB) \leq n(A)n(B)$.
(c) Let $\|A\|$ be given by Definition 5.8.1 where $X = Y = R^n$ with the norm $\|x\|_1$. Compare $n(A)$ and $\|A\|$. When are they equal?

a) $\|Ax\|_1 \leq \|A\|_1 \|x\|_1$ by property of matrix norms,

because $\|A\|_1 = \max_y \frac{\|Ay\|_1}{\|y\|_1} \geq \frac{\|Ax\|_1}{\|x\|_1}$

or $\|A\|_1 \|x\|_1 \geq \|Ax\|_1$

Now $\|A_1\|$ = maximum absolute column sum $= \max_j \sum_i |a_{ij}|$

$$n(A) = \sum_{i,j} |a_{ij}| = \sum_j \sum_i |a_{ij}|$$

Let $\alpha_j = \sum_i |a_{ij}| \geq 0$

Then $\|A\|_1 = \max_j \alpha_j$ & $n(A) = \sum_j \alpha_j$

Of course $\max_j \alpha_j \leq \sum_j \alpha_j$ as $\alpha_j \geq 0$

$$\Rightarrow \|A\|_1 \leq n(A)$$

Hence $\|Ax\|_1 \leq \|A\|_1 \|x\|_1 \leq n(A) \|x\|_1$

(b) Let B be written in the block matrix form

as $B = [\, b_1 \mid b_2 \mid b_3 \mid \cdots \mid b_n \,]$

where b_i are the colums of B.

Then $AB = [\, Ab_1 \mid Ab_2 \mid Ab_3 \cdots \mid Ab_n \,]$

$$n(AB) = \sum_i \|Ab_i\|_1 \quad \left(\text{recall that } Ab_i \text{ is a vector } f \;\; \|x\|_1 = \sum_j |x_j| \right)$$

$$\leq \sum_i \|A\|_1 \|b_i\|_1 \quad \left(\because \|Ax\|_1 \leq \|A\|_1 \|x\|_1 \right)$$

$$\leq \sum_i n(A) \|b_i\|_1 \quad \left(\text{we proved this in part (a) i.e. } |A|_1 \leq n(A) \right)$$

$$= n(A) \sum_i \|b_i\|_1$$

$$= n(A)\, n(B) \quad \left(\because n(B) = \sum_i \|b_i\|_1 \right)$$

Hence $n(AB) \leq n(A)\, n(B)$

(c)

Definition 5.8.1

Let X and Y be normed linear spaces and let T be a bounded linear transformation of X into Y. We define the norm of T to be $\|T\| = inf\{M: \|Tx\|_Y \leq M\|x\|_X,\ for\ all\ x\ \in X\}$.

Since the question wants us to take the $\|\cdot\|_1$ norm,

$$\|T\| = \inf \{ M : \|Tx\|_1 \leq M\|x\|_1 \;\; \forall x \in X \}$$

Alternatively $\|T\| = \sup_{x} \dfrac{\|Tx\|_1}{\|x\|_1}$ (think a bit and it's clear)

Hence $\|T\| = \|T\|_1$

Now we know that $\|T\|_1 =$ max abs column sum.

Hence $\underbrace{\|A\| = \|A\|_1}_{\text{Sum of abs. value of one column}} \leq \underbrace{n(A)}_{\text{Sum of abs value of all columns.}}$

When are they equal?

If all the columns of A are zero except one column & the entries of this column are all non-negative, then $\|A\| = \|A\|_1 =$ max abs. column sum

$\quad ||$

Sum of abs value of all columns $= n(A)$.

$$\begin{bmatrix} x_1^T A \overset{\geq 0}{x_1} & x_1^T A x_2 \\ x_2^T A x_1 & x_2^T A \overset{\geq 0}{x_2} \end{bmatrix} = \begin{bmatrix} x_1^T & 0 \\ 0 & x_2^T \end{bmatrix} \begin{bmatrix} A & A \\ A & A \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1^T A & x_1^T A \\ x_2^T A & x_2^T A \end{bmatrix} \left[\begin{array}{c|c} x_1 & 0 \\ \hline 0 & x_2 \end{array}\right]$$

5. Let $x_n \to x$ in a normed linear space $\{X, \| \quad \|\}$. Show that if X is endowed with an equivalent norm, $|||\quad|||$, then it is still true that $x_n \to x$.

$x_n \to x$ if $\forall \epsilon > 0 \ \exists N(\epsilon)$

s.t $n > N \Rightarrow \|x_n - x\| < \epsilon$.

We know that for any equivalent norm $|||x||$, we have

$m\|x\| \leq |||x||| \leq M\|x\|$ for some m, M

Let ϵ be given. Then Consider $\epsilon' = \epsilon/M$.

$\exists N(\epsilon')$ s.t $n > N$, we have $\|x_n - x\| < \epsilon' = \epsilon/M$

$\Rightarrow M\|x_n - x\| < \epsilon$

$\Rightarrow |||x_n - x||| < M\|x_n - x\| < \epsilon$

$\Rightarrow |||x_n - x||| < \epsilon$

Hence there always exists $M(\epsilon) = N(\epsilon/M)$

such that $n > M(\epsilon)$, $|||x_n - x||| < \epsilon$

which proves that $x_n \to x$ under the norm $|||\cdot|||$ as well.

6. Let $f: X \to Y$ where X and Y are both normed linear spaces. Suppose that f is continuous. Show that if X and/or Y are endowed with equivalent norms, then f remains continuous.

$f: X \to Y$ is continuous if for all convergent sequences $x_n \to x$, $y_n = f(x_n)$ converges to $y = f(x)$.

Let a & b be norms in X & Y.
Then we have the two metric spaces (X, d_a) & (Y, d_b) where $d_a(x_1, x_2) = \|x_1 - x_2\|_a$ & $d_b(x_1, x_2) = \|x_1 - x_2\|_b$

To be precise, $f: (X, d_a) \to (Y, d_b)$ is given to be continuous

☆ Hence for all convergent sequences $x_n \to x$, we have

$f(x_n) \to f(x)$.

Let $\|\cdot\|_A$ & $\|\cdot\|_B$ be equivalent norms in X & Y.
i.e. $\|\cdot\|_A$ & $\|\cdot\|_a$ are equivalent
& $\|\cdot\|_B$ & $\|\cdot\|_b$ are equivalent

Let any sequence x_n in (X, d_a) be convergent i.e $x_n \to x$
From Q5, x_n is convergent to x in (X, d_A) too.

Let $y_i = f(x_i)$. We know that y_n converges to $y = f(x)$ in (Y, d_b).
Again by Q5, y_n also converges to $y = f(x)$ in (Y, d_B)

Hence, for any $x_n \to x$ in (X, d_A), we have $f(x_n) \to f(x)$ in (Y, d_B).

8. Let $\|\cdot\|_F$ denote the Frobenius norm on $C^{m\times n}$ (i.e. $\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{i,j}|^2\right)^{\frac{1}{2}}$). Show that $\|\cdot\|_F$ is a norm and that $\|AB\|_F \le \|A\|_F\|B\|_F$

Hence $f: (X, d_A) \to (Y, d_B)$ is continuous too ■

Properties we need to prove

i) $\|A\|_F \geq 0$ & $\|A\|_F = 0$ iff $A = 0$

ii) $\|\alpha A\|_F = |\alpha| \, \|A\|_F^2$

iii) $\|A + B\|_F \leq \|A\|_F + \|B\|_F$

iv) $\|Av\|_F \leq \|A\|_F \|v\|_F$

v) $\|AB\|_F \leq \|A\| \, \|B\|_F$

i) $$\|A\|_F = \left(\sum_j \sum_i |a_{ij}|^2 \right)^{1/2} \geq 0$$

$$A = 0 \Rightarrow \|A\|_F^2 = \sum_j \sum_i |a_{ij}|^2 = \sum_j \sum_i 0^2 = 0 \Rightarrow \|A\|_F = 0$$

$$\|A\|_F = 0 \Rightarrow \|A\|_F^2 = 0 \Rightarrow \sum_{i,j} |a_{ij}|^2 = 0 \Rightarrow |a_{ij}| = 0 \Rightarrow a_{ij} = 0$$
$$\Rightarrow A = 0$$

(ii) $$\|\alpha A\|_F^2 = \sum_{ij} |\alpha a_{ij}|^2 = \alpha^2 \sum_{ij} |a_{ij}|^2$$
$$= \alpha^2 \|A\|_F^2$$
$$\Rightarrow \|\alpha A\|_F = \alpha \|A\|_F$$

iii) Notice that $\|A\|_F^2 = \langle A, A \rangle$ where $\langle \cdot , \cdot \rangle$ is an inner product on matrices.

$$\|A\|_F^2 = \langle A, A \rangle = \text{trace}(A^* A)$$

$$\|A+B\|_F^2 = \langle A+B, A+B \rangle = \langle A, A+B \rangle + \langle B, A+B \rangle$$
(by linearity of inner products)

$$= \langle A, A\rangle + \langle A, B\rangle + \langle B, A\rangle + \langle B, B\rangle$$

$$= \|A\|_F^2 + \langle A, B\rangle + \langle B, A\rangle + \|B\|_F^2$$

$$\leq \|A\|_F^2 + \underbrace{\|A\|_F \|B\|_F + \|B\|_F \|A\|_F}_{\text{by Cauchy Schwartz on } \langle \cdot , \cdot \rangle} + \|B\|_F^2$$

$$= \|A\|_F^2 + 2\|A\|_F \|B\|_F + \|B\|_F^2$$

$$= (\|A\|_F + \|B\|_F)^2$$

$$\Rightarrow \|A+B\|_F \leq \|A\|_F + \|B\|_F$$

iv) Notice that for a vector v, $\|v\|_F = \|v\|_2$

So $\|Av\|_F = \|Av\|_2 \leq \|A\|_2 \|v\|_2$

If we can show $\|A\|_2 \leq \|A\|_F$, we are good.

Let us prove some useful things first.

Let Q be unitary, i.e. $Q^* Q = I$

Then $\|Qv\|_2 = \|v\|_2$

Proof $\|Qv\|_2^2 = (Qv)^* Qv$

$$= v^* Q^* Q v = v^* I v = v^* v$$

$$= \|v\|_2^2$$

Also $\|QM\|_F = \|A\|_F^2$

Let $M = [\, m_1 \mid m_2 \mid m_3 \mid \cdots m_k \,]$

where m_i are columns of M.

We know that $\|M\|_F^2 = \sum_i \|m_i\|_2^2$

$$QM = [\, Qm_1 \mid Qm_2 \mid \cdots Qm_k \,]$$

So $\|QM\|_F^2 = \sum_i \|Qm_i\|_2^2 = \sum_i \|m_i\|_2^2$

$$= \|M\|_F^2$$

$$\Rightarrow \|QM\|_F = \|M\|_F$$

We can also show that $\|MQ\|_F = \|M\|_F$

Now let $A = U\Sigma V^T$

Then $\|A\|_F^2 = \|U\Sigma V^T\|_F = \|\Sigma\|_F^2$

$$= \sum_i \sigma_i^2$$

$\|A\|_2^2 = \sigma_1^2$

Clearly $\sum_i \sigma_i^2 \geq \sigma_1^2$

$$\Rightarrow \|A\|_F^2 \geq \|A\|_2^2$$

$$\Rightarrow \|A\|_F \geq \|A\|_2$$

Hence $\|Av\|_F = \|Av\|_2 \leq \|A\|_2 \|v\|_2$

$$\leq \|A\|_F \|v\|_2$$

$$= \|A\|_F \|v\|_F$$

$$\Rightarrow \|Av\|_F \leq \|A\|_F \|v\|_F$$

v) $\|AB\|_F^2 = \left\| \left[Ab_1 ; Ab_2 ; \cdots Ab_m \right] \right\|_F^2$

$$= \sum_i \|Ab_i\|_2^2 \leq \sum_i \|A\|_F^2 \|b_i\|_F^2$$

$$= \sum_i \|A\|_F^2 \|b_i\|_2^2$$

$$= \|A\|_F^2 \sum_i \|b_i\|_2^2$$

$$= \|A\|_F^2 \|B\|_F^2$$

$$\Rightarrow \|AB\|_F^2 \leq \|A\|_F^2 \|B\|_F^2$$

$\Rightarrow \|AB\|_F \le \|A\| \, \|B\|$

9. Show that norm equivalence is an equivalence relation.

Equivalence relation $\rightarrow$

i) reflexive $a \sim a$

ii) Symmetric $a \sim b \Leftrightarrow b \sim a$

iii) transitive $a \sim b, \; b \sim c$
$\Rightarrow a \sim c$

Let $\|\cdot\|_a$ & $\|\cdot\|_b$ be two equivalent norms. Let us call $\|\cdot\|_a \sim \|\cdot\|_b$ if $\exists\; m, M \geq 0$ s.t.

$$m \|\cdot\|_a \le \|\cdot\|_b \le M \|\cdot\|_a$$

Symmetry :- $m \|\cdot\|_a \le \|\cdot\|_b$

$$\Rightarrow \|\cdot\|_a \le \frac{1}{m} \|\cdot\|_b$$

$$\|\cdot\|_b \le M \|\cdot\|_a \Rightarrow \frac{1}{M} \|\cdot\|_b \le \|\cdot\|_a$$

So $\frac{1}{M}\|\cdot\|_b \le \|\cdot\|_a \le \frac{1}{m}\|\cdot\|_b$

$\Rightarrow \quad \|\cdot\|_b \sim \|\cdot\|_a$ as $\frac{1}{M}, \frac{1}{m} \geq 0.$

<u>Reflexivity</u> : $\|\cdot\|_a \sim \|\cdot\|_a$

Clearly $\frac{1}{2}\cdot\|\cdot\|_a \le \|\cdot\|_a \le 2\|\cdot\|_a$

So $\|\cdot\|_a \sim \|\cdot\|_a$

<u>Transitivity</u> : Let $\|\cdot\|_a \sim \|\cdot\|_b$

& $\|\cdot\|_b \sim \|\cdot\|_c$

$\|\cdot\|_a \sim \|\cdot\|_b \Rightarrow m\|\cdot\|_a \le \|\cdot\|_b \le M\|\cdot\|_a$

$\|\cdot\|_b \sim \|\cdot\|_c \Rightarrow n\|\cdot\|_b \le \|\cdot\|_c \le N\|\cdot\|_b$

$m\|\cdot\|_a \le \|\cdot\|_b \Rightarrow mn\|\cdot\|_a \le n\|\cdot\|_b \le \|\cdot\|_c$

$\longrightarrow$ (1)

$\|\cdot\|_b \le M\|\cdot\|_a \Rightarrow N\|\cdot\|_b \le MN\|\cdot\|_a$

$\Rightarrow \|\cdot\|_c \le N\|\cdot\|_b \le MN\|\cdot\|_a$

$\longrightarrow$ (2)

from ① & ②

$$mn\|\cdot\|_a \leq \|\cdot\|_c \leq MN\|\cdot\|_a$$

& Since $mn, MN \geq 0$, we have

$$\|\cdot\|_a \sim \|\cdot\|_c$$

Hence norm equivalence is an equivalence relation.

7. Let A be a square matrix and let λ be a complex number for which $|\lambda| > \|A\|$ where $\| \quad \|$ denotes some induced matrix norm. Show that the matrix $\lambda I - A$ is nonsingular and that

$$(\lambda I - A)^{-1} = \frac{1}{\lambda}\sum_{k=0}^{\infty} \frac{1}{\lambda^k} A^k.$$

$(\lambda I - A)$ is non-singular $\Rightarrow$ all eigenvalues of $(\lambda I - A)$ are non-zero.

If α is an eigenvalue of A, then $\lambda - \alpha$ is an eigenvalue of $\lambda I - A$.

If $A\vec{v} = \alpha\vec{v}$

then $(\lambda I - A)\vec{v} = \lambda I\vec{v} - A\vec{v} = \lambda\vec{v} - \alpha\vec{v} = (\lambda - \alpha)\vec{v}$

Hence $(\lambda - \alpha)$ is an eigenvalue of $\lambda I - A$.

If α_i are the eigenvalues of A, then $\lambda - \alpha_i$ are the eigenvalues of $\lambda I - A$.

If $\lambda > \alpha_i$, then $\lambda - \alpha_i > 0$ & hence we can show that $\lambda I - A$ has all eigenvalues greater than zero, & hence is invertible.

We are given that $\lambda \geq \|A\|$ for some induced matrix norm. We will show that any norm $\|A\|$ upper bounds the eigenvalues of A.

Let $\alpha, \vec{v}$ be an eigenvalue, eigenvector pair of A.
Then,

$$A\vec{v} = \alpha \vec{v}$$
$$\Rightarrow \|Av\| = \|\alpha v\| = \alpha \|v\|$$
$$\Rightarrow \|A\| \|v\| \geq \|Av\| = \alpha \|v\|$$

where $\|v\|$ is the vector norm on V that induces the matrix norm $\|A\|$.

So $\|A\| \|v\| \geq \|Av\| = \alpha \|v\|$

$$\Rightarrow \|A\| \geq \alpha$$

Hence, we proved that $\|A\|$ is greater than any eigenvalue 'α' of A.

Now, we are given that $\lambda \geq \|A\|$. So $\lambda \geq \|A\| \geq \alpha$.
$\Rightarrow \lambda \geq \alpha$
$\Rightarrow \lambda - \alpha \geq 0 \Rightarrow$ eigenvalues of $I\lambda - A$ are all +ve.

Hence $\lambda I - A$ is invertible.

Now, we want to show

$$(\lambda I - A)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} A^k$$

Now, $\left(\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} A^k\right)(\lambda I - A)$

$$= \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} A^k \cdot \lambda I - \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} A^k \cdot A$$

$$= \sum_{k=0}^{\infty} \frac{1}{\lambda^k} A^k - \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} A^{k+1}$$

$$= \left(I + \cancel{\frac{1}{\lambda} A} + \cancel{\frac{1}{\lambda^2} A^2} + \cdots\right) - \left(\cancel{\frac{1}{\lambda} A} + \cancel{\frac{1}{\lambda^2} A^2} + \cdots\right)$$

$$= I$$

$$\therefore \quad (\lambda I - A)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} A^k$$