Homework - 2

Griffiths Q 1.7 To show :

$$\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle$$

$$\langle p\rangle = m\langle v\rangle = m\frac{d\langle x\rangle}{dt}$$

$$= \cancel{m}\,\frac{-i\hbar}{\cancel{m}} \int \left(\psi^* \frac{\partial \psi}{\partial x}\right) dx$$

$$\Rightarrow \langle p\rangle = -i\hbar \int \psi^* \frac{\partial \psi}{\partial x}\, dx$$

$$\Rightarrow \frac{d\langle p\rangle}{dt} = -i\hbar \int \frac{\partial}{\partial t}\left(\psi^* \frac{\partial \psi}{\partial x}\right) dx$$

$$= -i\hbar \int \left[\frac{\partial \psi^*}{\partial t}\frac{\partial \psi}{dx} + \psi^* \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial x}\right)\right] dx$$

If ψ is smooth enough, we can reverse the order of the partial derivatives.

i.e. $\frac{\partial}{\partial t}\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \psi}{\partial t}$

$$\Rightarrow \frac{d\langle p\rangle}{dt} = -i\hbar \int_{-\infty}^{\infty} \left[\frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial x} + \psi^* \frac{\partial}{\partial x} \frac{\partial \psi}{\partial t} \right] dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \left[\left(-\frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V \psi^* \right) \frac{\partial \psi}{\partial x} \right.$$

$$\left. + \psi^* \frac{\partial}{\partial x} \left(\frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V \psi \right) \right] dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \left\{ -\frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} \frac{\partial \psi}{\partial x} + \cancel{\frac{i}{\hbar} V \psi^* \frac{\partial \psi}{\partial x}} \right.$$

$$+ \frac{i\hbar}{2m} \psi^* \frac{\partial^3 \psi}{\partial x^3} - \frac{i}{\hbar} \psi^* \frac{\partial V}{\partial x} \psi$$

$$\left. - \cancel{\frac{i}{\hbar} V \psi^* \frac{\partial \psi}{\partial x}} \right\} dx$$

$$= -i\hbar \cdot \frac{i\hbar}{2m} \int_{-\infty}^{\infty} \left\{ \psi^* \frac{\partial^3 \psi}{\partial x^3} - \frac{\partial^2 \psi^*}{\partial x^2} \frac{\partial \psi}{\partial x} \right\} dx$$

$$- i\cancel{\hbar} \cdot -\frac{i}{\cancel{\hbar}} \boxed{\int_{-\infty}^{\infty} \psi^* \frac{\partial V}{\partial x} \psi \, dx}$$

$$\left\langle \frac{\partial V}{\partial x} \right\rangle = \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} |\psi|^2 dx$$

$$\Rightarrow \frac{d\langle p\rangle}{dt} = -\frac{i^2\hbar^2}{m}\int_{-\infty}^{\infty}\left(\psi^*\frac{\partial^3\psi}{\partial x^3} - \frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x}\right)dx$$

$$+ i^2\left\langle \frac{\partial V}{\partial x}\right\rangle$$

Let us look at the integral $I = \int_{-\infty}^{\infty}\left(\psi^*\frac{\partial^3\psi}{\partial x^3} - \frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x}\right)dx$

$$I = \int\left[fg''' - f''g'\right] = \int fg''' - \int f''g'$$

$$\Rightarrow I = \left(fg''\Big|_{-\infty}^{\infty} - \int f'g''\right) - \left(f'g'\Big|_{-\infty}^{\infty} - \int f'g''\right)$$

$$= \left[fg'' - f'g'\right]_{-\infty}^{\infty}$$

$$= \underbrace{\left[\psi^*\frac{\partial^2\psi}{\partial x^2} - \frac{\partial\psi^*}{\partial x}\cdot\frac{\partial\psi}{\partial x}\right]_{-\infty}^{\infty}}_{}$$

this expression must go to zero but I am not sure why.

Hence $$\frac{d\langle p\rangle}{dt} = i^2\left\langle \frac{\partial V}{\partial x}\right\rangle = -\left\langle \frac{\partial V}{\partial x}\right\rangle$$

Classically, this is analogous to $F = \frac{dp}{dt} = \frac{-dV}{dx}$

Griffiths Q 1.14 :-

Let $P_{ab}(t)$ be the probability of finding the particle in the range $a < x < b$ at time t. a) Show that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t)$$

where $$J(x,t) = \frac{i\hbar}{2m}\left(\Psi\frac{\partial \Psi^*}{\partial x} - \Psi^*\frac{\partial \Psi}{\partial x}\right)$$

Soln :- $$P_{ab}(t) = \int_a^b |\Psi(x,t)|^2 dx$$

$$= \int_a^b \Psi^*\Psi \, dx$$

$$\Rightarrow \frac{dP_{ab}}{dt} = \frac{d}{dt}\int_a^b \psi^* \psi \, dx$$

$$= \int_a^b \frac{\partial}{\partial t}(\psi^* \psi)\, dx$$

$$= \int_a^b \left(\frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}\right) dx$$

But we know from Schrödinger eq^n

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V \psi$$

Also $\displaystyle \frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V \psi^*$

$$\therefore \frac{dP_{ab}}{dt} = \int_a^b \left[\left(-\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V \psi^*\right)\psi + \psi^*\left(\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V \psi\right)\right] dx$$

$$\Rightarrow \frac{dP_{ab}}{dt} = \int_a^b \left[\frac{i\hbar}{2m}\left(\psi^* \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial x^2}\psi\right) + \cancel{\frac{i}{\hbar} V \psi^* \psi} - \cancel{\frac{i}{\hbar}\psi^* V \psi}\right] dx$$

$$= \frac{i\hbar}{2m}\int_a^b \left(\boxed{\psi^*}\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial x^2}\psi\right) dx$$

Let us look at the integral with $f = \psi^*$ & $g = \psi$

$$\Rightarrow \frac{dP_{ab}}{dt} = \frac{i\hbar}{2m}\int_a^b \left(\boxed{f} g'' - f'' g\right) dx$$

$$= \frac{i\hbar}{2m}\left\{ fg'\Big|_a^b - \cancel{\int_a^b f'g'} - f'g\Big|_a^b + \cancel{\int_a^b f'g'} \right\}$$

$$\frac{i\hbar}{2m}\left(fg' - f'g\right)\Big|_a^b$$

$$\frac{i\hbar}{2m}\left(f'g - fg'\right)\Big|_b^a = \frac{i\hbar}{2m}\left(\frac{\partial \psi^*}{\partial x}\psi - \psi^* \frac{\partial \psi}{\partial x}\right)\Big|_b^a$$

$$= J\Big|_b^a = J(a,t) - J(b,t)$$

What are the units of $J(x,t)$?

Comment: J is called the probability current, because it tells you the rate at which probability is flowing past the point x. If $P_{ab}(t)$ is increasing, then more probability is flowing into the region at one end than flows out at the other.

<u>Sol^n</u>: Since P_{ab} is unit less, $\frac{dP_{ab}}{dt}$ has the units of s^{-1}.

Since $\frac{dP_{ab}}{dt} = J(a,t) - J(b,t)$,

J must have units of s^{-1} aswell.

<u>Griffiths Q 1.14 (b)</u>

Find the probability current for the wave function in Problem 1.9

<u>Sol^n</u> In 1.9, $\Psi(x,t) = Ae^{-a\left[\frac{mx^2}{\hbar} + it\right]}$

$$J(x,t) = \frac{i\hbar}{2m}\left[\frac{\partial \psi^*}{\partial x}\psi - \psi^*\frac{\partial \psi}{\partial x}\right]$$

$$\psi(x,t) = A e^{-a\left[\frac{mx^2}{\hbar} + it\right]}$$

$$\Rightarrow \frac{\partial \psi}{\partial x} = A e^{-a\left[\frac{mx^2}{\hbar} + it\right]} \cdot \left(-a\left[\frac{2m}{\hbar}x\right]\right)$$

$$\psi^* = A e^{-a\left[\frac{mx^2}{\hbar} - it\right]}$$

$$\Rightarrow \frac{\partial \psi^*}{\partial x} = A e^{-a\left[\frac{mx^2}{\hbar} - it\right]} \cdot \left(-a\left[\frac{2mx}{\hbar}\right]\right)$$

Hence $\frac{\partial \psi^*}{\partial x}\psi = A e^{-a\left[\frac{mx^2}{\hbar} - it\right]}\left(-a\frac{2mx}{\hbar}\right)$
$A e^{-a\left[\frac{mx^2}{\hbar} + it\right]}$

$$= -A^2 a e^{-a\left[\frac{2mx^2}{\hbar}\right]}\left(\frac{2mx}{\hbar}\right)$$

Similarly, $\psi^*\frac{\partial \psi}{\partial x} = A e^{-a\left[\frac{mx^2}{\hbar} - it\right]}$
$A e^{-a\left[\frac{mx^2}{\hbar} + it\right]}\left(-a \cdot \frac{2mx}{\hbar}\right)$

$$= -A^2 a e^{-a\left[\frac{2mx^2}{\hbar}\right]}\left(\frac{2mx}{\hbar}\right)$$

$\therefore J(x,t) = 0$