Math 425a Spring 2023 HW 2

Sampad Mohanty

TOTAL POINTS

30 / 35

QUESTION 1

1 **5 / 5**

✓ **- 0 pts** *Correct*

- 1 pts It's not true if you replace $$f^{-1}$$ by f.

- 4 pts You need to prove the first two arguments first.

QUESTION 2

2 **5 / 5**

✓ **- 0 pts** *Correct*

QUESTION 3

3 **5 / 5**

✓ **- 0 pts** *Correct*

- 1 pts e) is countable.

- 1 pts d) is uncountable.

QUESTION 4

4 **5 / 5**

✓ **- 0 pts** *Correct*

- 1 pts You need to use the Burnside's theorem.

QUESTION 5

5 **5 / 5**

✓ **- 0 pts** *Correct*

- 1 pts (0,1) is uncountable.

QUESTION 6

6 **0 / 5**

- 0 pts Correct

✓ **- 5 pts** *No proof.*

QUESTION 7

7 **5 / 5**

✓ **- 0 pts** *Correct*

1 **5 / 5**

✓ **- 0 pts** *Correct*

- 1 pts It's not true if you replace $$f^{-1}$$ by f.

- 4 pts You need to prove the first two arguments first.

2 **5 / 5**

✓ **- 0 pts** *Correct*

3 **5 / 5**

✓ **- 0 pts** *Correct*

- 1 pts e) is countable.

- 1 pts d) is uncountable.

4 **5 / 5**

✓ **- 0 pts** *Correct*

- 1 pts You need to use the Burnside's theorem.

5 **5 / 5**

✓ **- 0 pts** *Correct*

- 1 pts (0,1) is uncountable.

6 **0 / 5**

- 0 pts Correct

✓ **- 5 pts** *No proof.*

7 **5 / 5**

✓ **- 0 pts** *Correct*

Q1 $f: X \to Y$ is a function.

$\{Y_j\}$ is a collection of subsets of Y.

$$f^{-1}\left(\bigcup_j Y_j\right) = \left\{x \in X \,\middle|\, f(x) \in \bigcup_j Y_j\right\}$$

$$= \left\{x \in X \,\middle|\, \exists j \text{ s.t. } f(x) \in Y_j\right\}$$

$$= \bigcup_j \left\{x \in X \,\middle|\, f(x) \in Y_j\right\}$$

$$= \bigcup_j f^{-1}(Y_j) \quad \blacksquare$$

$$f^{-1}\left(\bigcap_j Y_j\right) = \left\{x \in X \,\middle|\, f(x) \in Y_j \ \forall j\right\}$$

$$= \bigcap_j \left\{x \in X \,\middle|\, f(x) \in Y_j\right\}$$

$$= \bigcap_j f^{-1}(Y_j) \quad \blacksquare$$

Q Are the above properties true if $\{Y_j\}$ is replaced by a collection $\{X_j\}$ of subsets of X & f^{-1} by f. ?

Ans Not for intersection. Let $f: \mathbb{R} \to \mathbb{R}$ & $f(x) = x^2$

If $A = (-1, 0)$ & $B = (0, 1)$

$f(A \cap B) = \phi$ & $f(A) \cap f(B) = (0, 1)$

$$f\left(\bigcup_j X_j\right) = \left\{y \in Y \,\middle|\, \exists j, y = f(x_j)\right\} = \bigcup_j \left\{y \in Y \,\middle|\, y = f(x_j)\right\}$$

$$= \bigcup_j f(X_j) \quad \blacksquare$$

2 a) * Let $f: A \to B$ be an injection (one-one) where $|A| = |B|$.

we want to show f is also surjective (onto)

* For any injective f, $|f(A)| = |A|$ & hence $|f(A)| = |A| = |B|$

However, $f(A) \subseteq B$ by definition of $f(A)$ being the range & B being the codomain.

* $f(A) \subseteq B$ & $|f(A)| = |B|$

$$\Downarrow$$

$$f(A) = B$$

Hence range = Codomain & hence f is surjective (onto)

* Now we want to show that the above is not true when A & B are infinite. Let $A = \mathbb{N} = B$ such that $|A| = |B|$

Now we show by a counter example that the above does not hold. Let $f: A \to B$ be $f(n) = 2n$. Clearly f is one-one but it isn't onto. Ex-2: $f(x) = x/2$, $A = B = [0,1]$

b) Let $f: A \to B$ be a surjection (onto) with $|A| = |B|$.

We want to show that f is also an injection.

Clearly, since f is onto, $f(A) = B$ & hence $|f(A)| = |B|$

& since $|B| = |A|$, we have $|f(A)| = |A|$. Since $|f(A)| \leq |A|$ for any mapping, $|f(A)| = |A|$ is only true if f is one-one.

Hence f is injective.

Now we want to show that the above doesn't hold when A & B are infinite. Ex: $A = B = [-1, 1]$ & $f: A \to B$ with $f(x) = 1 - 2x^2$

3) a) $\{ 1/n : n \in \mathbb{Z} \setminus \{0\} \}$ ← countable infinite because $\mathbb{Z}$ is countable & infinite.

b) The collection of all finite subsets of $\mathbb{N}$.

↑ countably infinite because set of finite subsets of a countable set is countable.

c) The set of all functions $f : \mathbb{N} \to \mathbb{N}$
Let $A = \{0, 1\} \in \mathbb{N}$.
Now, Let $g : \mathbb{N} \to \{0,1\}$ be a function from natural numbers to $\{0,1\}$. We can have a bijection from $g \to P(\mathbb{N})$ where $P(x)$ is powerset of x. Just like in 5 b), this bijection if $h : g \to \mathbb{P}(\mathbb{N})$ with $h(g) = \{ n \mid \text{if } g(n) = 1 \}$. Hence the set of such f is uncountable. If anything, $|f| > |g| = \infty$.

d) The set of all non-decreasing functions $f : \mathbb{N} \to \mathbb{N}$.
The set such f can be mapped to a non-decreasing sequence in $\mathbb{N}$.
The sequences can be then mapped to a set containing elements appearing in the sequence. These sets are infinite subsets of $\mathbb{N}$ and hence uncountably infinite because the collection of such subsets is powerset of $\mathbb{N}$.

e) The set of all non-increasing functions $f : \mathbb{N} \to \mathbb{N}$.
There exist a bijection from the functions f to non-increasing sequences in $\mathbb{N}$. Since such sequences inevitably ~~flatten out~~ converge at some index i either to 0 or some finite value c, we take the pair (i, c) where $i \in \mathbb{N}$ & $c \in \mathbb{N} \cup \{0\} = \mathbb{Z}_+$
$(i, c) \in \mathbb{N} \times \mathbb{Z}_+$ is infinite but countable. So we have to account for the initial part of the sequences before index i. Since i is finite, we are looking at countable number of such initial parts. Hence the total is countably infinite.

④ Cantor - Bernstein - Schröeder theorem

If $f: A \to B$ & $g: B \to A$ are both injections, then there exists a bijection between $A \leftrightarrow B$.

We want to show that $A = (0,1)$ & $B = [0,1]$ have the same cardinality using the above theorem.

Let $f(x) = x$ & $g(x) = \frac{1-x}{2}$

$f: A \to B$ $g: B \to A$

Clearly both f & g are injections.

Hence there exists a bijection between A & B

Hence $|A| = |B|$.

5. a) Any $x \in (0,1)$ can be written as $0.a_1 a_2 a_3 \ldots$

b) We can put the binary representations in bijective correspondence with subsets of natural numbers by defining the bijection as

$$f: x \to \{ i \mid a_i = 1 \text{ in binary representation of } x \}$$

c) Cantor's theorem says that $\text{Card}(X) < \text{Card}(P(X))$ where $P(X)$ is the powerset of X. Since there is a bijection from $x \in (0,1)$ to $P(N)$, hence $|(0,1)| = |P(N)| > |N| = \infty$

where $|\cdot| = \text{Card}(\cdot)$

Hence $\text{Card}((0,1)) = |(0,1)| = \infty$

6) B is countable.

To prove $Card(\mathbb{R} \cup \mathbb{B}) = Card(\mathbb{R})$

or $|\mathbb{R} \cup \mathbb{B}| = |\mathbb{R}|$

Let $\mathbb{R} \cap B = \phi$ & B be countably infinite.

Let $f : \mathbb{R} \cup B \to \mathbb{R}$ be such that $f(x) = x \quad \forall\, x \in \mathbb{R} \setminus \mathbb{N}$

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The set of these circles with center (x, y) & radius r can be mapped into the 3-tuple (x, y, r) where $x, y \in \mathbb{Z}$ & $r \in \mathbb{Q}$.

Since $\mathbb{Z}$ and $\mathbb{Q}$ are each countably infinite, $(x, y, r) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Q}$ form a set that is countably infinite too.