HOMEWORK - 1

2. A manufacturer wishes to produce an alloy that is, by weight, 30 % metal A and 70 % metal B. Five alloys are available at various prices as indicated below:

Alloy	1	2	3	4	5
%A	10	25	50	75	95
% B	90	75	50	25	5
Price/lb	\$ 5	\$ 4	\$ 3	\$ 2	\$ 1.50

The desired alloy will be produced by combining some of the other alloys. The manufacturer wishes to find the amounts of the various alloys needed and to determine the least expensive combination. Formulate this problem as a linear program.

Let the weights of the alloys 1, 2, 3, 4, 5 be x_1, x_2, x_3, x_4, x_5 while producing the new alloy. Weights are +ve, hence $x_i \geq 0$.

Let $\vec{x} = [x_1 \; x_2 \; x_3 \; x_4 \; x_5]^T$

Let $M = \begin{bmatrix} 0.1 & 0.25 & 0.5 & 0.75 & 0.95 \\ 0.9 & 0.75 & 0.5 & 0.25 & 0.5 \end{bmatrix}$

$\vec{c} = [5 \quad 4 \quad 3 \quad 2 \quad 1.5]^T$

Clearly $\underbrace{[1 \;\; 1]}_{1_2^T} M = \underbrace{[1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]}_{1_5^T}$

Let $\vec{y} = M\vec{x}$, then $(\vec{1}^T\vec{y})^{-1}\vec{y} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$

Constraining $\underbrace{\vec{1}^T\vec{y} = 1}$ gives us $\vec{y} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = M\vec{x}$

$$1_2^T y = 1_2^T M x = (1_2^T M) x = 1_5^T x = 1$$

So the constraints are

$$\begin{cases} 1_5^T x = 1 & \text{1 kg of input } \xrightarrow[\text{to produce}]{\text{Consumed}} \text{1 kg output} \\ Mx = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} & \text{Output} = 0.3\text{kg A} + 0.7\text{kg B} \\ x_i \geq 0 & \end{cases}$$

Cannot use −ve amount of old alloys ≡ cannot subtract/extract old alloys ≡ no old alloy is formed as a by-product during the process of producing the new alloy.

For eg. if $w_3 = -0.1$, then output = 0.1 kg alloy 3 + 0.9 kg new alloy which might cost less than when $w_i \geq 0$ but we want no by-products since the new alloy ≠ 30% A + 70% B iff the by-produced old alloy ≠ 30% A + 70% B (assuming this old alloy is different from the new alloy of the same composition which is a bit against the definition of alloys but we can think of similar problems when alloys are replaced by chemicals which might have same composition by weight but different chemical properties for e.g. isomers

And we need to optimize our cost, hence we minimize $C^T x$.

minimize $c^T x$

s.t. $1_5^T x = 1$ or $\sum x_i = 1$

$Mx = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$ or $\sum M_{\gamma i} x_i = 0.3(\gamma - 1) + 0.7(\gamma - 2)$ where $\gamma \in \{1, 2\}$

$\vec{x} \geq \vec{0}$ or $x_i \geq 0$

* **Note:** If one or more old alloys (or chemicals) have the same composition of 30% A + 70% B, the cost of production may be further optimised by relaxing their weights to not be constrained to +ve values i.e. allowing them to be by-products in the manufacturing process.

8. Convert the following problem to a linear program in standard form:

$$\begin{aligned} &\text{minimize } |x| + |y| + |z| \\ &\text{subject to } x + y \leqslant 1 \\ &\qquad\qquad 2x + z = 3. \end{aligned}$$

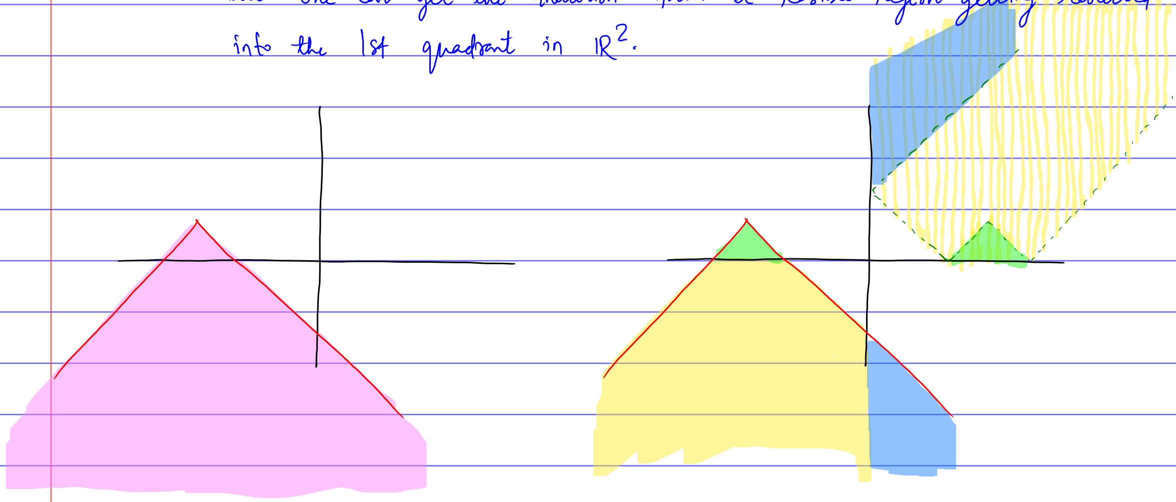
Let us define $f(x,y,z) = |x| + |y| + |z|$

$f(x_0, y_0, z_0) = k \iff f(\pm x_0, \pm y_0, \pm z_0) = k$

i.e. f is symmetric about all axes and hence in all octants. Hence we can reduce the search space from all of $\mathbb{R}^3$ to just the 1st octant i.e. $x, y, z \geqslant 0$.

Restricted to the 1st octant, $f(x,y,z) = |x| + |y| + |z| = x + y + z$

However, we now have another problem i.e. the constraints are not symmetric about the octants. There should be a way to reflect the feasible convex region onto the 1st octant using algebra and simple linear which can be encoded as linear constraints/inequalities. Then the problem becomes really simple. I will change tracks to find another method for now but one can get the intuition from a feasible region getting reflected into the 1st quadrant in $\mathbb{R}^2$.



Here we change tracks & try another method.

$$\text{minimize } |x| + |y| + |z|$$

$$\equiv \text{minimize } a + b + c$$

$$\text{s.t. } a \geqslant x \ \& \ a \geqslant -x$$
$$b \geqslant y \ \& \ b \geqslant -y$$
$$c \geqslant z \ \& \ c \geqslant -z$$

Clearly this is true since $a \geqslant x$ & $a \geqslant -x$ means $a \geqslant \max\{x, -x\} = |x|$

Also the optimum values of a, b, c will occur at $a = |x|$, $b = |y|$ & $c = |z|$ since if $a > |x|$, a can be reduced unless it is constrained by one of the external inequalities.

Hence we can convert the initial problem into the following -

minimize $a + b + c$

$a \geq x$
$a \geq -x$
$b \geq y$
$b \geq -y$
$c \geq z$
$c \geq -z$

$x + y \leq 1$
$2x + z = 3$

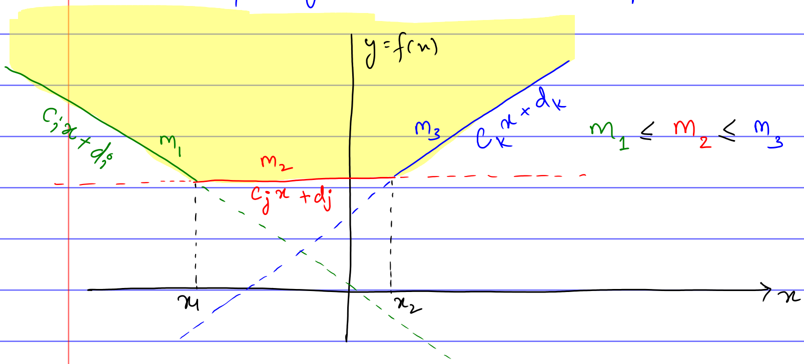
Clearly this can be converted into the standard form by using slack variables and replacing each variable t by $t_1 - t_2$ where $t_1, t_2 \geq 0$.

9. A class of piecewise linear functions can be represented as $f(\mathbf{x}) = \text{Maximum}$ $(\mathbf{c}_1^T\mathbf{x}+d_1,\ \mathbf{c}_2^T\mathbf{x}+d_2,\ \ldots,\ \mathbf{c}_p^T\mathbf{x}+d_p)$. For such a function f, consider the problem

minimize $f(\mathbf{x})$
subject to $\mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0}$.

Show how to convert this problem to a linear programming problem.

Observations : $\max\{c_1^Tx + d_1, c_2^Tx + d_2 + \cdots\}$ will form the sides of a convex polygon that is open upwards. Let us start from $x \to -\infty$. As we move on the x-axis towards the right hand side, at the 1st point $x = x_1$ where we change from one linear function to another because the max of the set of linear function changes, the linear function that is the new max must have a slope higher than that of the previous one.



$*$ in $\mathbb{R}^2$ $c_i^T x = c_i x$

$m_1 \leq m_2 \leq m_3$

Now, minimizing $f(x)$ means finding a point $x = \alpha$ s.t $f(\alpha)$ is minimized. So we are searching for $x \in \mathbb{R}$.

This is equivalent to saying that we are searching for a point $(x, y) \in \mathbb{R}^2$ so that we minimize y constrained to the curve $y = f(x)$.

However, since the trace of $y = f(x)$ has to be the sides of a convex polygon (polytope), we can relax the feasible region to be the shaded convex region without affecting the solution $(\alpha, f(\alpha))$ since we know that the minimum of $y = f(x)$ is guaranteed to occur at the boundary of the upward open shaded convex region. Note that we could not have done this had we to maximize $f(x)$.

So, our optimization can then be formulated as

$$\text{minimize } y$$

$$\text{s.t. } \quad \underbrace{y \geq (c_i x + d) \text{ or } y_i - (c_i x + d) \geq 0}_{\text{Select the region above the line/plane/hyperplane.}}$$

$$Ax = b$$

$$x \geq 0$$

Generalizing this for $z = f(x) = \max\{ c_i^T x + d_i \mid i = 1, 2, \dots k \}$

$$\text{minimize } \quad z$$

$$\text{s.t. } \quad z \geq c_i^T x + d_i \text{ or } z - c_i^T x \geq d_i$$

$$A\vec{x} = \vec{b}$$

$$\vec{x} \geq 0$$

$$\text{minimize } \quad z$$

$$\text{s.t. } \quad \left[\begin{array}{c|c} 1 & -c_1^T \\ 1 & -c_2^T \\ \vdots & \vdots \\ 1 & -c_k^T \end{array}\right] \left[\begin{array}{c} z \\ \hline \vec{x} \end{array}\right] \geq \begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix}$$

$$\left[\begin{array}{c|c} \vec{0}_k & A \end{array}\right] \left[\begin{array}{c} z \\ \hline \vec{x} \end{array}\right] = \vec{b}$$

$$\vec{x} \geq 0$$

This can be converted to the standard form without much effort by setting $z = z_1 - z_2$ where $z_1, z_2 \geq 0$ and introducing slack variables.

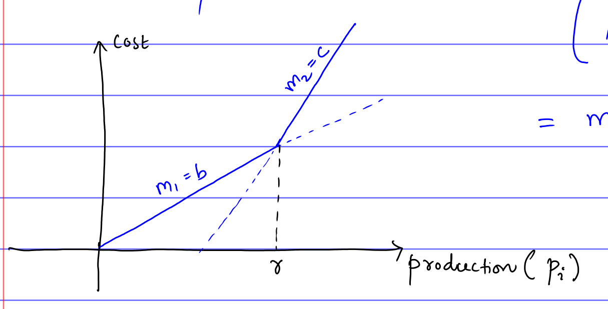
10. A small computer manufacturing company forecasts the demand over the next n months to be d_i, $i = 1, 2, \dots, n$. In any month it can produce r units, using *regular* production, at a cost of b dollars per unit. By using *overtime*, it can produce additional units at c dollars per unit, where $c > b$. The firm can store units from month to month at a cost of s dollars per unit per month. Formulate the problem of determining the production schedule that minimizes cost. (*Hint*: See Exercise 9.)

Let them produce $p_i \geq 0$ units in i^{th} month.

$$\sum_{k=1}^{i} p_i \geq \sum_{k=1}^{i} d_i \quad \forall i \in \{1, 2, \dots, n\}$$ (Running sum of production $\geq$ running sum of demand)

$$\text{Cost of production at month } i = \begin{cases} p_i b & \text{if } p_i \leq r \\ p_i b + (p_i - r)c & \text{if } p_i > r \end{cases}$$

$$= \max\{ p_i b, \; p_i b + (p_i - r)c \}$$



Cost of storing $= \sum_{i=1}^{n} (p_i - d_i)(n-i)s$ {excess at the end of i^{th} month is stored for $(n-i)$ months.}

Total cost $= f(\vec{p}) =$ cost of production + cost of storing

$$= \sum_{i=1}^{n} \max\{p_i b + p_i b + (p_i - r)c\} + \sum_{i=1}^{n} (p_i - d_i)(n-i)s$$

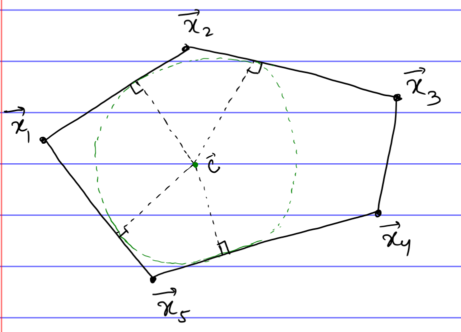
$$f(\vec{p}) = \sum_{i=1}^{n} \max\left\{ p_i b + (p_i - d_i)(n-i)s,\ p_i b + (p_i - r)c + \sum_{i=1}^{n}(p_i - d_i)(n-i)s \right\}$$

So the well formed optimisation problem is as below –

Now clearly this can be converted to standard form using the above question

$$\begin{aligned} &\text{minimize} \quad f(\vec{p}) \\ &\text{s.t.} \quad \sum_{i=1}^{k} p_i \geq \sum_{i=1}^{k} d_i \quad \forall\ k = 1, 2, \ldots n \\ &\quad\quad p_i \geq 0 \end{aligned}$$

4. Show that the problem to compute the largest ball that is a subset of a given polyhedron can be formulated as a linear program.



$$\vec{c} = \sum_{i=1}^{k} \alpha_i \vec{x}_i$$

$$\sum \alpha_i = 1$$

$$\alpha_i \geq 0$$

$\boxed{\max \min \left(\text{distance to the faces}\right)}$

* Let the polyhedron be defined by the halfspaces $A\vec{x} \preceq \vec{b}$ where A is a matrix. $A\vec{x} \preceq \vec{b} \Rightarrow A\vec{x} - \vec{b} \preceq \vec{0}$

* Let the center of a ball of radius R be $\vec{c}$.

* Then, any supporting hyperplanes that contains a facet (See defn in the book) must atleast be at a distance of R or more (atleast one of them will exactly be at R distance)

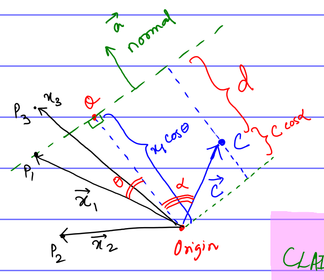
<u>Note</u>: Let $\vec{a}\cdot\vec{x} \leq b \equiv a\vec{x} - \vec{b} \leq 0$ be a halfspace.
Let C be a point contained in the halfspace.

Halfspace $\equiv \vec{a}\cdot\vec{x} \leq b \Rightarrow \frac{\vec{a}\cdot\vec{x}}{a} \leq b/a \Rightarrow \hat{a}\cdot\vec{x} \leq \frac{b}{a}$

Boundary of the halfspace $=$ hyperplane $\equiv \hat{a}\cdot\vec{x} = \frac{b}{a}$

Since C is contained in the halfspace, $\hat{a}\cdot\vec{c} \leq b/a$

$\Rightarrow \hat{a}\cdot\vec{c} - \vec{b}/a \leq 0 \Rightarrow \vec{b} - \hat{a}\cdot\vec{c} \geq 0$



$P_1, P_2 \in$ Halfspace

CLAIM: distance of $\vec{c}$ from the hyperplane $\vec{a}\cdot\vec{x} - b \leq 0$ is given by $d = b/a - \hat{a}\cdot\vec{c} \longrightarrow$ eqn ①

<u>Proof</u>: Any point P_1 on the plane yeilds $\hat{a}\cdot\vec{x}_1 = b/a$

This proof works even if the origin happens to lie outside of the halfspace.

$$\Rightarrow \hat{a}\cdot\vec{x}_1 = b/a$$

$$\Rightarrow \cancel{|\hat{a}|}^{1}\cdot x_1 \cos\theta = b/a$$

$$\Rightarrow \underbrace{x_1 \cos\theta} = b/a$$

$$\Rightarrow OQ = b/a$$

$\Rightarrow$ distance of origin to the hyperplane $= b/a$.

Distance of C from hyperplane $\equiv d \equiv OQ - C\cos\alpha = b/a - C|\hat{a}|\cos\theta$

$$= b/a - \hat{a}\cdot c \quad \blacksquare$$

* The rows of A contain the normals to the facet hyperplanes.
* If we wanted to normalize all the rows of the matrix to unit length, we can carryout the following operations.

Let $C = A^T$ be of shape $m \times n$. So columns of $C \in \mathbb{R}^n$.

Let $\boxed{B = I_{n\times n} * C_{m\times n}}$ where $*$ is the Khatri-Rao product taking the rows/columns of A/C & $I_{n\times n}$ as the blocks. Note: B is $mn \times n$

Since the columns of C are nonzero, so are the columns of B & so are the diagonals of the diagonal matrix B^TB & hence the inverse $(B^TB)^{-1}$ exists with the diagonal entries being the reciprocal of the diagonal entries of B^TB. The diagonal entries of B^TB & $(B^TB)^{-1}$ are +ve and are the lengths of the corresponding row/column vectors of A/C.

Note: B^TB is $n \times n$ as B is $mn \times n$

The polyhedron $A\vec{x} \preceq \vec{b}$ can be written as

$$(B^TB)^{-1}A\vec{x} \preceq (B^TB)^{-1}\vec{b}$$ {Since $(B^TB)^{-1}$ has +ve entries}

Equivalently $(B^TB)^{-1}(A\vec{x} - \vec{b}) \preceq \vec{0}$

family of inequalities are preserved under addition & scaling by +ve numbers.
|||
Probably this is similar to saying convex sets are preserved under conic combination

<u>Note</u>:- $(B^TB)^{-1}$ normalizes the rows of A when multiplied on the left.

The distance of a point $\vec{c}$ from all the hyperplanes is given by $\vec{d} = (B^TB)^{-1}(\vec{b} - A\vec{c})$ which follows clearly from eq①.

$$\vec{d} = (B^TB)^{-1}(\vec{b} - A\vec{c}) = \left[(I_{n\times n} * A^T)^T (I_{n\times n} * A^T)\right]^{-1}(\vec{b} - A\vec{c})$$

Using all the information above, we formulate the linear program

maximize R

s.t. $\vec{d} = \left[(\mathbb{I}_{n\times n} * A^T)^T (\mathbb{I}_{n\times n} * A^T) \right]^{-1} (b - A\vec{c}) \preceq R\cdot\vec{1}_n$

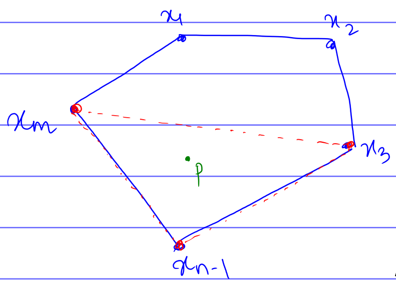
where $\vec{1}_n$ is a n-dimensional column vector of 1s.

$\vec{d} \preceq R\cdot\vec{1}_n$ Captures the constraint that the distance of $\vec{c}$ from each of the facet hyperplanes has to be atmost the radius of the ball.

Problem 6. (3 points)

K&V Chapter 3, Exercise 15. Though the theorem is true regardless, for simplicity you may assume that X is a finite set .
(Hint: Write an LP expressing y as a convex combination of points in X, then use our trick for counting the number of non-zero variables).

15. Prove Carathéodory's theorem:
If $X \subseteq \mathbb{R}^n$ and $y \in \text{conv}(X)$, then there are $x_1, \ldots, x_{n+1} \in X$ such that $y \in \text{conv}(\{x_1, \ldots, x_{n+1}\})$. Convex Hull of X
(Carathéodory [1911])



Let the $X = \{\vec{x}_1, \ldots\ldots \vec{x}_m\} \subseteq \mathbb{R}^n$

For any $\vec{p} \in \text{Conv}(X)$, we have

$$\vec{p} = \sum \alpha_i \vec{x}_i \quad \text{where } \sum \alpha_i = 1 \text{ and } \alpha_i \geq 0$$

Let us consider the L.P below

minimize 1

s.t. $\sum_{i=1}^{m} \alpha_i \vec{x}_i = \vec{p}$ } n tight constraints

$\sum_{i=1}^{m} \alpha_i = 1$ } 1 tight constraint

$\alpha_i \geq 0 \;\; \forall\, i \in \{1, 2, \ldots\ldots m\}$

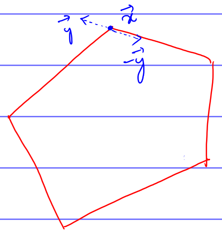
Since we know that if the L.P is feasible then $\exists$ a vertex optimal solution (vertex of the feasible region in α_i space which is in $\mathbb{R}^m$), there must be m tight constraints since a vertex is defined by the intersection of m hyperplanes.

Since $n+1$ tight constraints are already present, out of the m constrains for $\alpha_i \geq 0$, only $m-(n+1)$ are tight for $m-(n+1)$ i's, $\alpha_i = 0$. So for $m-(m-(n+1))$ i's, $\alpha_i > 0 \Rightarrow$ the x_i's corresponding to these $n+1$ i's contain the point $\vec{p}$ in their convex hull since $\sum \alpha_i$ has to be 1. Hence proved.

2 Longer Problems

Problem 7. (5 points)
Let $P \subseteq \mathbb{R}^n$ be a polytope, and let A be a full-rank $n \times n$ matrix. Let $P' = \{Ax : x \in P\}$ be the image of P under A. Show that x is a vertex of P if and only if Ax is a vertex of P'.

$\forall$ vertices $\vec{x} \in P$ $\forall$ $\vec{y} \neq \vec{0}$, $\vec{x}+\vec{y} \in P$ and $\vec{x}-\vec{y} \notin P$ WLOG.



Since $\vec{x} \in P$, $A\vec{x} \in P'$ by definition of P' as image of P.

Assume $A\vec{x}$ not be a vertex

$\Rightarrow \exists\, \vec{z} \neq 0$ s.t. $A\vec{x} + \vec{z} \in P'$ & $A\vec{x} - \vec{z} \in P'$

Let $\vec{z} = A\vec{y}$ or $\vec{y} = A^{-1}\vec{z}$ ($\because A$ is full rank)

$\Rightarrow \exists\, \vec{z} = A\vec{y}$ s.t. $A\vec{x} + A\vec{y} \in P'$ & $A\vec{x} - A\vec{y} \in P'$

$\Rightarrow \exists\, \vec{z} = A\vec{y}$ s.t. $A(\vec{x}+\vec{y}) \in P'$ & $A(\vec{x}-\vec{y}) \in P'$

$\Rightarrow \exists\, \vec{z} = A\vec{y}$ s.t. $\vec{x}+\vec{y} \in P$ & $\vec{x}-\vec{y} \in P$

which contradicts that $\vec{x}$ is a vertex of P
Since $\vec{y} = A^{-1}\vec{z} \neq \vec{0}$ whenever $\vec{z} \neq 0$ & A is fullrank.

Now lets begin with $A\vec{x}$ being a vertex of P'

$\Rightarrow \forall\, \vec{z} = A\vec{y} \neq 0$, $A\vec{x} + \vec{z} \in P'$ & $A\vec{x} - \vec{z} \notin P'$ WLOG.

$\Rightarrow A\vec{x} + A\vec{y} \in P'$ & $A\vec{x} - A\vec{y} \notin P'$

$\Rightarrow A(\vec{x}+\vec{y}) \in P'$ & $A(\vec{x}-\vec{y}) \notin P'$

$\Rightarrow \vec{x}+\vec{y} \in P$ & $\vec{x}-\vec{y} \notin P$ $\forall$ $\vec{y} = A^{-1}\vec{z} \neq \vec{0}$

$\Rightarrow \vec{x}$ is a vertex of P.

Problem 9. (8 points)
Given a matrix $A \in \mathbb{R}^{m \times n}$, show that exactly one of the following systems has a solution

- $Ax \succ 0$, $x \in \mathbb{R}^n$ (note: Ax is entry-wise *strictly* greater than zero)
- $A^{\intercal}y = 0$, $y \succeq 0$, and $y \in \mathbb{R}^m$ is non-zero

Proof

2.1 Farkas Lemma

Theorem 3 (Farkas Lemma). *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following sets must be empty:*

(i) $\{x \mid Ax = b, x \geq 0\}$

(ii) $\{y \mid A^T y \leq 0, b^T y > 0\}$

We know by Farkas Lemma that only one of the sets is non-singular $\rightarrow$

i) $\{x \mid Ax = b, x \geq 0\} \equiv \{y \mid -C^T y = b, y \geq 0\}$

$\equiv \{y \mid C^T y = -b, y \geq 0\}$

$= \{y \mid C^T y = -b, y \geq 0\} \longrightarrow$ ①

ii) $\{y \mid A^T y \le 0,\ b^T y > 0\} \equiv \{x \mid (-\vec{c}^T)^T x \le 0,\ b^T x > 0\}$

$$\equiv \{x \mid -Cx \le 0,\ b^T x > 0\}$$

$$\equiv \{x \mid Cx \geq 0,\ b^T x > 0\}$$

$$\equiv \{x \mid Cx \geq 0,\ -b^T x < 0\} \longrightarrow (2)$$

from (1) & (2),

$$\{x \mid Cx \geq 0,\ -b^T x < 0\} \equiv \{x \mid Cx \geq 0,\ b^T x \geq 0\}$$

$$\{y \mid C^T y = -b,\ y \geq 0\} \equiv \{y \mid C^T y = -b,\ y \geq 0\}$$

Now set $\vec{b} = \vec{0}$ which gives $\rightarrow$

$$\{x \mid Cx \geq 0,\ x \in \mathbb{R}^n\} \leftarrow \{x \mid Cx \geq 0,\ 0^T x \geq 0\}$$

$$\{y \mid C^T y = 0,\ y \geq 0\} \leftarrow \{y \mid C^T y = -0,\ y \geq 0\}$$

This is exactly what we needed to prove. Just rename C to A.

Problem 8. (10 points)

L&Y Chapter 4, Exercise 9. In part (b), additionally show that Y is guaranteed payoff at least $-B$ no matter what $\mathbf{x}$ is chosen by X.

Clarification: Note that each a_{ij} may be positive or negative, where a negative entries means that player Y wins amount $-a_{ij}$ from X. The average payoff to player Y is the negation of the average payoff to player X, namely $-\mathbf{x}^T\mathbf{A}\mathbf{y}$.

9. *Game theory* is in part related to linear programming theory. Consider the game in which player X may select any one of m moves, and player Y may select any one of n moves. If X selects i and Y selects j, then X wins an amount a_{ij} from Y. The game is repeated many times. Player X develops a *mixed* strategy where the various moves are played according to probabilities represented by the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, where $x_i \geqslant 0$, $i = 1, 2, \ldots, m$ and $\sum_{i=1}^m x_i = 1$. Likewise Y develops a mixed strategy $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, where $y_i \geqslant 0$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n y_i = 1$. The average payoff to X is then $\mathbf{P}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\mathbf{A}\mathbf{y}$.

(a) Suppose X selects $\mathbf{x}$ as the solution to the linear program

$$\begin{aligned} &\text{maximize } A \rightarrow \text{let's call this } \alpha \\ &\text{subject to } \sum_{i=1}^m x_i = 1 \\ &\qquad \sum_{i=1}^m x_i a_{ij} \geqslant A,\ j = 1, 2, \ldots, n \\ &\qquad x_i \geqslant 0, \qquad i = 1, 2, \ldots, m. \end{aligned}$$

Show that X is guaranteed a payoff of at least A no matter what $\mathbf{y}$ is chosen by Y.

(b) Show that the dual of the problem above is

$$\begin{aligned} &\text{minimize } B \\ &\text{subject to } \sum_{j=1}^n y_j = 1 \\ &\qquad \sum_{j=1}^n a_{ij} y_j \leqslant B,\ i = 1, 2, \ldots, m \\ &\qquad y_j \geqslant 0, \qquad j = 1, 2, \ldots, n. \end{aligned}$$

(c) Prove that $\max A = \min B$. (The common value is called the *value* of the game.)

(d) Consider the "matching" game. Each player selects heads or tails. If the choices match, X wins \$1 from Y; if they do not match, Y wins \$1 from X. Find the value of this game and the optimal mixed strategies.

(e) Repeat Part (d) for the game where each player selects either 1, 2, or 3. The player with the highest number wins \$1 unless that number is exactly 1 higher than the other player's number, in which case he loses \$3. When the numbers are equal there is no payoff.

a) Let's call the average payoff α

Since $\vec{x}$ is soln to the LP,

$$\left(\sum_{i=1}^m x_i a_{ij}\right) \geq \alpha \ \forall j$$

$$\Rightarrow y_j \left(\sum_{i=1}^m x_i a_{ij}\right) \geq y_j \alpha \ \forall j \ \&\ y_j \geq 0$$

$$\Rightarrow \sum_{j=1}^n y_j \left(\sum_{i=1}^m x_i a_{ij}\right) \geq \sum_{j=1}^n y_j \alpha = \alpha \sum_{j=1}^n y_j$$

$$\Rightarrow \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij} y_j \geq \alpha \sum_{j=1}^n y_j$$

$$\Rightarrow x^T A y \geq \alpha \sum_{j=1}^n y_j$$

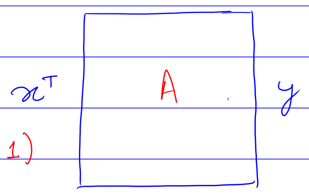
Iff $y^T \vec{1} = \sum_{j=1}^n y_j = 1$, then $x^T A y \geq \alpha$ no matter what $\vec{y}$ is as long as the entries sum to 1.

b)

	a_{ij}	

The dual of the game is from the perspective of the 2nd player who wants to minimize the loss that he has to incur.

So the second player has the following strategy →



$$\text{minimize } \beta$$

s.t. $\sum_i y_i = 1$ (probabilities sum to 1)

$\sum_{j=1}^{n} a_{ij} y_j \le \beta \quad \forall i$ (for each row of A that the 1st player can select, i.e. for any pure strategy of player 1, make sure that my strategy is such as to not allow player 1 to get more than β payoff)

$y_i \geq 0$

To show that X can only get atmost β payoff no matter what strategy (pure or mixed) $\vec{x}$ it chooses →

$$\sum_{j=1}^{n} a_{ij} y_j \le \beta \quad \forall i \in \{1, 2, \dots m\}$$

$$\Rightarrow \quad x_i \sum_{j=1}^{n} a_{ij} y_j \le x_i \beta \quad \forall i \in \{1, 2, \dots m\} \;\&\; x_i \geq 0$$

$$\Rightarrow \sum_{i=1}^{m} x_i \sum_{j=1}^{n} a_{ij} y_j \le \sum_{i=1}^{m} x_i \beta = \beta \cancel{\sum_{i=1}^{m} x_i}^{\,1}$$

$$\Rightarrow \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j}_{} \le \beta \cdot 1$$

$$\Rightarrow x^T A y \le \beta$$

Since this is a zerosum game, the payoff for X = −ve of payoff for Y.

Since payoff for $X \le \beta$

$(-\text{payoff for } Y) \le \beta$

$\Rightarrow$ payoff for $Y \geq -\beta$

i.e. the payoff for Y is atleast $-\beta$ no matter what strategy $\vec{x}$ is chosen by X. Hence proved.

c) The expected payoff that X can get independent of what strategy Y chooses is atleast α (part ⓐ) at at max β no matter what strategy X chooses (part ⓑ). Hence $\alpha \le \beta$ (weak duality)

* Using strong duality, $\alpha = \beta$.

d)

X \ Y	H	T
H	1\$	−1\$
T	−1\$	1\$

$x^T A y$

X wants to maximize his gain

So the L.P is maximize α

s.t. $x_i A_{ij} \geq \alpha \;\forall j$

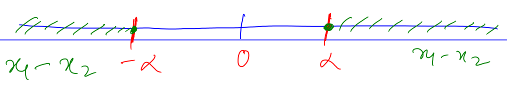
$\sum x_i = 1$

$x_i \geq 0$

$x_1 - x_2 \geq \alpha$

$-x_1 + x_2 \geq \alpha \equiv x_1 - x_2 \le -\alpha$

α has to be zero if the two inequalities can ever be satisfied simultaneously.



So $x_1 - x_2 \geq 0$
$x_1 - x_2 \leq -0$ $\Big\} \Rightarrow x_1 - x_2 = 0$
$\Rightarrow x_1 = x_2$

Also since $x_1 + x_2 = 1 \Rightarrow 2x_1 = 2x_2 = 1$
$\Rightarrow x_1 = x_2 = 1/2$

So the best strategy for X is $\vec{x} = [\frac{1}{2} \ \frac{1}{2}]^T$.
By symmetry arguement Y has the best strategy.

e)

X \ Y	1	2	3
1	0	+3\$	-1\$
2	-3\$	0	+3\$
3	1\$	-3\$	0

} winning prices of X

$x^T A y$

L.P formulation to maximize the reward for X no matter what strategy Y chooses:

Maximize α

s.t $0 - 3x_2 + x_3 \geq \alpha \Rightarrow -3x_2 + x_3 \geq \alpha$
$3x_1 + 0x_2 - 3x_3 \geq \alpha \Rightarrow 3x_1 - 3x_3 \geq \alpha$
$x_1 - 3x_2 + 0 \geq \alpha \Rightarrow x_1 - 3x_2 \geq \alpha$
$x_1 + x_2 + x_3 = 1$
$x_1, x_2, x_3 \geq 0$

$-x_1 - 4x_2 \geq \alpha - 1 \quad = -3x_2 + (1 - x_1 - x_2) \geq \alpha$
$6x_1 + 3x_2 \geq \alpha + 3 \quad = 3x_1 - 3(1 - x_1 - x_2) \geq \alpha$
$x_1 - 3x_2 \geq \alpha \quad = x_1 - 3x_2 \geq \alpha$

$7x_1 \geq \alpha + 4$
$x_1 \geq \frac{\alpha}{7} + 4$

$-24x_2 + 3x_2 \geq 6\alpha - 6 + \alpha + 3$
$-21x_2 \geq 7\alpha - 3$
$x_2 \leq \frac{7\alpha}{-21} + \frac{3}{21} = -\frac{\alpha}{3} + \frac{1}{7}$

$x_1 + x_2 \leq 1$
$\Rightarrow \frac{\alpha}{7} + 4 - \frac{\alpha}{3} + \frac{1}{7} \leq 1$
$\Rightarrow \alpha\left(\frac{1}{7} - \frac{1}{3}\right) \leq -3 - \frac{1}{7} = \frac{-22}{7}$
$\Rightarrow \alpha\left(\frac{-4}{21}\right) \leq -\frac{22}{7}$
$\Rightarrow \alpha \geq -\frac{22}{7} \Big/ \frac{-4}{21} = \frac{-22}{-4} \times \frac{21}{7} = \frac{11}{2} \times 3 = \frac{33}{2}$

$\Rightarrow \alpha \geq \frac{33}{2}$

Problem 10. (8 points)
Recall the optimal production problem we introduced in class. We will now consider a generalization of that problem to a setting with multiple colluding firms. As in the optimal production problem, there are n products and m resources, where producing a unit of the j'th product consumes A_{ij} units of the i'th resource, and each unit of the j'th product can be sold at a profit of c_j. There are K firms, the k'th of whom is endowed with b_i^k units of the i'th resource — we use $b^k \in \mathbb{R}_+^m$ to denote the endowment of the k'th firm. We allow a subset $S \subseteq \{1, \ldots, K\}$ of the firms to *collude*, in which case the firms pool their resources in order to maximize their collective profit, effectively solving the following optimization problem.

$$
\begin{aligned}
OPT(S) = \quad & \text{maximize} \quad && c^\intercal x \\
& \text{subject to} \quad && Ax \preceq \textstyle\sum_{k \in S} b^k \\
& && x \succeq 0
\end{aligned}
$$

We say a coalition of firms is *stable* if the firms can share profit in such a way so that no subset of the coalition can gain by breaking with the group and forming a coalition of their own. Formally, coalition $S \subseteq \{1, \ldots, K\}$ is stable if there are profit shares $p \in \mathbb{R}_+^S$ such that

1. $\sum_{k \in S} p_k = OPT(S)$. i.e. the total profit distributed equals the aggregate profit of the coalition.

2. $\sum_{k \in T} p_k \geq OPT(T)$ for all $T \subseteq S$. i.e. no subset of the coalition can collectively increase their profit by breaking off from S.

Show that the grand coalition $\{1, \ldots, K\}$ is stable.

Let us look at the dual of the above L.P

$$\text{minimize} \quad y^T (B \vec{1})$$

$$A^T y \geqslant C$$

$$y \geqslant 0$$

Due to strong duality $OPT(\{1, 2, \ldots K\})$

= optimum for y when all the inputs are pooled.