Q1

$a^T x + b > 0 \quad \forall\, x \in \mathcal{E}_1 \cup \mathcal{E}_2 \ldots \cup \mathcal{E}_k$

$a^T x + b < 0 \quad \forall\, x \in \mathcal{E}_{k+1}, \mathcal{E}_{k+2}, \ldots \mathcal{E}_{k+L}$

$$\mathcal{E}_i = \{ P_i u + q_i \mid \|u\|_2 = 1 \}, \quad i = 1, 2, \ldots, k+L$$

Let $x \in \mathcal{E}_i$, then $\quad a^T x + b > 0 \quad \forall\, i \in \{1, 2, \ldots k\}$

or $a^T (P_i u + q_i) + b > 0 \quad$ s.t $\|u\|_2 = 1$

or $a^T P_i u + a^T q_i + b > 0 \quad$ s.t $\|u\|_2 = 1$

or $a^T P_i u > -a^T q_i - b \quad$ s.t. $\|u_2\| = 1$

Clearly $\underset{\|u\|=1}{a^T P_i u} = (a^T P_i) \cdot \hat{u}$ = projection of $a^T P_i$ on $\hat{u}$ dirn.

$$\underbrace{-\|a^T P_i\|_2}_{\text{occurs at } \hat{u}_{min}} \le (a^T P_i) \cdot \hat{u} \le \underbrace{+\|a^T P_i\|_2}_{\text{occurs at } \hat{u}_{max}} \quad \text{since } -\|a\|_2 \le a \cdot \hat{b} \le \|a\|_2$$

Since $\quad a^T P_i u > -a^T q_i - b \quad \forall \|u\|_2 = 1,$

$a^T P_i u_{min} > -a^T q_i - b$

$\Rightarrow -\|a^T P_i\|_2 > -a^T q_i - b$

$\Rightarrow \|a^T P_i\|_2 < a^T q_i + b. \quad \forall\, i \in \{1, 2, \ldots k\}$

Similarly $\forall\, i \in \{k+1, k+2, \ldots k+L\}$, $a^T x + b < 0$

$a^T x + b < 0 \Rightarrow a^T (P_i u + q_i) + b < 0 \quad \forall \|u\|_2 = 1$

$\Rightarrow a^T P_i u + a^T q_i + b < 0 \quad \forall \|u\|_2 = 1$

$\Rightarrow (a^T P_i) \cdot \hat{u}_{max} < -a^T q_i - b \quad \forall \|u\|_2 = 1$

$\Rightarrow \|a^T P_i\|_2 < -a^T q_i - b \quad \forall\, i \in \{k, k+1, \ldots k+L\}$

So the linear separation problem between the two sets of ellipsoids can be posed as solving for $\vec{a}$ in the following problem.

$$\text{minimize} \quad 1$$

SOCP feasibility
Second Order Cone Programming

$$\text{s.t.} \quad \|a^T P_i\| < a^T q_i + b \quad \forall i \in \{1, 2, \dots K\}$$

$$\|a^T P_i\|_2 < -a^T q_i - b \quad \forall i \in \{K, K+1, \dots K+L\}$$

or alternatively

$$\text{minimize} \quad 1$$

SOCP feasibility

$$\text{s.t.} \quad \|a^T P_i\| < C(i) \cdot (a^T q_i - b) \quad \forall i \in [K+L]$$

$$\text{where } C(i) = \text{class}(i) = \begin{cases} +1 & \text{if } i \leq K \\ -1 & \text{if } i \geq K+1 \end{cases}$$

Q2

$$\text{Maximize} \quad \alpha_4 T \gamma^2$$

$$\text{s.t.} \quad \alpha_1 \frac{T\gamma}{w} + \alpha_2 \gamma + \alpha_3 \gamma w \leq C_{max}.$$

$$T_{min} \leq T \leq T_{max}$$

$$\gamma_{min} \leq \gamma \leq \gamma_{max}$$

$$w_{min} \leq w \leq w_{max}$$

$$w \leq 0.1\gamma$$

which is same as — $\text{Maximize} \quad \alpha_4 T \gamma^2$

$$\text{s.t} \quad \frac{\alpha_1}{C_{max}} T \gamma w^{-1} + \frac{\alpha_2 \gamma}{C_{max}} + \frac{\alpha_3 \gamma w}{C_{max}} \leq 1$$

$$\frac{T}{T_{max}} \leq 1 \quad \& \quad \frac{T_{min}}{T} \leq 1$$

$$\gamma_{min} \gamma^{-1} \leq 1 \quad \& \quad (\gamma_{max})^{-1} \times \gamma \leq 1$$

$$\frac{w_{min}}{w} \leq 1 \quad \& \quad \frac{w}{w_{max}} \leq 1$$

$$\frac{w}{0.1\gamma} \leq 1$$

Q3 $A(x) = A_0 + A_1 x_1 + \cdots + A_n x_n = A_0 + x_1 A_1 + x_2 A_2 + \cdots x_n A_n$,

where $A : \mathbb{R}^m \to S^m$

& $A_i \in S^m$ i.e. A_i has size $m \times m$ & $A_i^T = A_i$

$\lambda_1(x) \geq \lambda_2(x) \geq \lambda_3(x) \ldots \ldots \geq \lambda_n(x)$ are the eigenvalues of $A(x)$.

a) $\lambda_1(x) \leq t \iff A(x) \preceq t.\mathbb{I}$ where $\preceq$ means the usual order on semi definite matrices.

So the minimization of the maximum eigenvalue becomes the SDP (Semi-Definite Program) :-

minimize t

Subject to $A(x) \preceq t\mathbb{I}$

where the decision variables are $x \in \mathbb{R}^n$ & $t \in \mathbb{R}$.

---

b) Minimize the spread of eigenvalues.

i.e. minimize $\lambda_1(x) - \lambda_m(x)$

$$\lambda_m(x) \leq A(x) \preceq \lambda_1(x)$$

$$\Rightarrow Q^T \lambda_m(x) Q = Q^T A(x) Q \preceq Q^T \lambda_1(x) Q$$

where $A(x) = Q D(x) Q^T$ is the SVD/eigenvalue decomposition of the PSD matrix $A(x)$.

$$\Rightarrow Q^T Q \lambda_m(x) \preceq D(x) \preceq Q^T Q \lambda_1(x)$$

$$\Rightarrow \lambda_m(x)\mathbb{I} \preceq D(x) \preceq \lambda_1(x)\mathbb{I}$$

The above problem can be formulated as the SDP

minimize $\lambda_1 - \lambda_m$

Subject to. $\lambda_m I \leq A(x) \leq \lambda_1 I$

where the decision variables are

$x \in \mathbb{R}^n$, $\lambda_m \in \mathbb{R}$ & $\lambda_1 \in \mathbb{R}$

c) minimize λ/γ

s.t. $0 < \gamma I \leq A(x) \leq \lambda I$

Let $y = \frac{x}{\gamma}$ & $t = \lambda/\gamma$ & $s = 1/\gamma$

Then $0 < \gamma I \leq A(x) \leq \lambda I$

$$\Rightarrow 0 < \gamma I \leq A(\gamma y) \leq \lambda I$$

$$\cancel{\Rightarrow 0 < \gamma I \leq \gamma A(y) \leq \lambda I}$$

$$\cancel{\Rightarrow 0 < I \leq A(y) \leq \frac{\lambda}{\gamma} I = tI}$$

$$\cancel{\Rightarrow 0 < I \leq A(y) \leq tI}$$

$$A(x) = A(\gamma y) = A_0 + \gamma y_1 A_1 + \gamma y_2 A_2 \cdots\cdots \gamma y_n A_n$$

$$A(x) = \gamma (s A_0 + y_1 A_1 + \cdots\cdots + y_n A_n)$$

So,

$$0 < \gamma \mathbb{I} \leq A(x) \leq \lambda \mathbb{I}$$

$$\equiv 0 < \gamma \mathbb{I} \leq \gamma(sA_0 + y_1A_1 + \cdots + y_nA_n) \leq \lambda \mathbb{I}$$

$$\equiv 0 < \mathbb{I} \leq sA_0 + y_1A_1 + \cdots y_nA_n \leq \frac{\lambda}{\gamma}\mathbb{I} = t\mathbb{I}$$

$$\Rightarrow 0 < \mathbb{I} \leq sA_0 + y_1A_1 + \cdots y_nA_n \leq t\mathbb{I}$$

So the SDP is

$$\text{minimize } t$$

$$\text{s.t. } \mathbb{I} \leq sA_0 + y_1A_1 + \cdots y_nA_n \leq t\mathbb{I}$$

$$s \geq 0$$

d) We need to minimize $|\lambda_1(x)| + \cdots + |\lambda_m(x)|$

Hint: $A(x) = A_+ - A_-$ where $A_+ \succeq 0$ & $A_- \succeq 0$

So A_+ has all eigenvalues +ve.

A_- has all eigenvalues +ve.

A_+ contain all the +ve eigenvalues of A } since A is symmetric
A_- contain all the -ve eigenvalues of A. } A has all real eigenvalues & orthonormal eigenvectors.

Also realize that trace(A) = Sum of eigenvalues.

trace(A_+) = sum of +ve eigenvalues of A.

trace(A_-) = sum of -ve eigenvalues of A.

So $\text{tr}(A_+) - \text{tr}(A_-) = |\lambda_1(x)| + \cdots + |\lambda_m(x)|$

So minimize $|\lambda_1| + \cdots |\lambda_m|$ = minimize $tr(A_+^*) - tr(A_-^*)$

s.t $A(x) = A_+^* - A_-^*$ s.t. $A(x) = A_+^* - A_-^*$

$A_+^* \succeq 0$, $A_-^* \succeq 0$ $A_+^* \succeq 0$, $A_-^* \succeq 0$

Let $A(x) = Q \Lambda Q^T$ be the eigenvalue decomp / S.V.D of $A(x)$

Let $\bar{A}_+ = Q^T A_+^* Q$

$\bar{A}_- = Q^T A_+ Q$,

the we can write.

minimize $tr(\tilde{A}_+) + tr(\tilde{A}_-)$

s.t. $\Lambda = \tilde{A}_+^* - \tilde{A}_-$

$\tilde{A}_+^* \succeq 0$, $\tilde{A}_- \succeq 0$

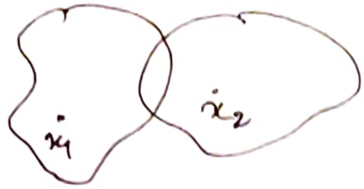
where $\tilde{A}_+$ & $\tilde{A}_-$ are the ~~varia~~ decision variables

<u>Note</u>: $tr A^+ = tr\, Q Q^T A_+^* = tr(Q^T A_+^* Q) = tr(\tilde{A})$

Since trace (ABC) = trace (CAB) = trace (BCA)

as trace is cyclic commutation invariant.

Q4



$$\min \; f_0(x)$$

$$\text{s.t. } \vec{x}_i \in C_i \text{ or } f_i(\vec{x}_i) \leq 0$$

$$\text{and } x = \sum_{i=1}^{q} \theta_i \vec{x}_i \quad \leftarrow \text{not Affine in } x_i \;\&\; \theta_i$$

$$\text{and } 1^T \theta = 1, \; \theta \succeq 0$$

$$\min \; f_0(x)$$

$$\text{s.t. } s_i f_{ij}(z_i/s_i) \leq 0$$

$$1^T s = 1, \; s \succeq 0, \; s_i \in \mathbb{R}$$

$$x = z_1 + \cdots + z_q, \; z_i \in \mathbb{R}^n$$

Since f_{ij} is convex, $s_i f_{ij}(z_i/s_i)$ is convex too as it is a perspective transform.

The two problems are equivalent because they are feasible & infeasible simultaneously with change of variables $z_i = \theta_i x_i$ & $s_i = \theta_i$.

clearly since $f_{ij}(x_i) \leq 0 \Rightarrow s_i f_{ij}(x_i) \leq 0$ as $s_i \geq 0$

$$\Rightarrow s_i f_{ij}(z_i/s_i) \leq 0.$$

Also if $s_i f_{ij}(z_i/s_i) \leq 0$ & $s_i \geq 0$ & $\sum_i s_i = 1$

then $f_{ij}(z_i/s_i) \leq 0$

$$\Rightarrow f_{ij}(x_i) \leq 0.$$

Also $x = z_1 + \cdots z_q = \sum s_i x_i$ where $s_i \geq 0$ & $\sum s_i = 1$

Hence $\vec{x}$ is a convex combination of $\vec{x}_i$ & hence lies in the convex hull

Q5

minimize $c^T x$

s.t. $f(x) \le 0$

Refer material on Duality from MIT OCW 15-084j "Duality Theory of Constrained Optimization". Robert M. Freund 2004

Step 1: Create the Lagrangian.

$$L(x, \lambda) = c^T x + \lambda f(x)$$

Step 2: Create the dual function.

$$L^*(\lambda) = \underset{x}{\text{minimum}}\ L(x, \lambda) = \min_x \{c^T x + \lambda f(x)\}$$

$$= \lambda \min_x \left(\frac{c}{\lambda}\right) x + \lambda f(x)$$

$$= -\lambda f^*\left(-\frac{c}{\lambda}\right)$$

Dual → $\min.\ -\lambda f_i^*(-c/\lambda)$

s.t $\lambda \geq 0$.

Q6

minimize $-\sum_{i=1}^{m} \log(b_i - a_i^T x)$

where $x \in \{x \mid a_i^T x < b_i,\ i=1, \dots m\}$

Let $y_i = b_i - a_i^T x$

So

$$\min -\sum_{i=1}^{m} y_i$$

s.t $y_i = b_i - a_i^T x$ or $\vec{y} = \vec{b} - A\vec{x}$.

Lagrangian is $L(x, y, \lambda) = -\sum_{i=1}^{m} \log y_i + \lambda^T (y - b + Ax)$

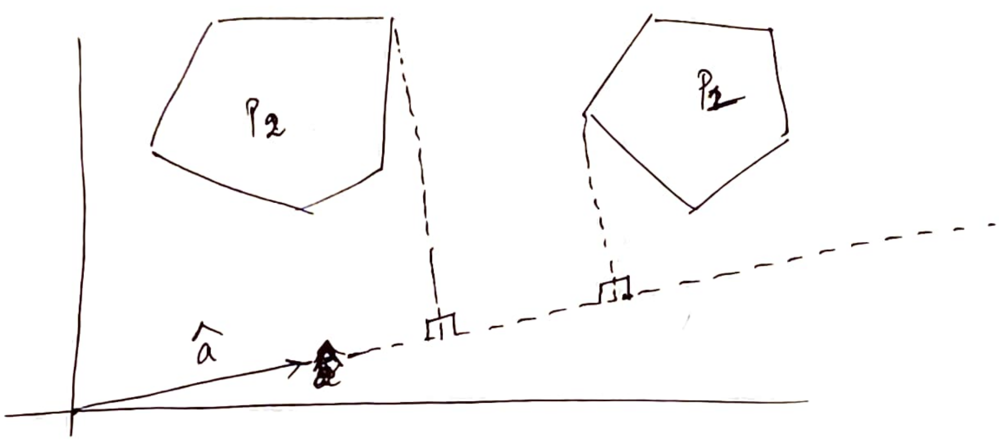
Dual function → $g(\lambda) = \min_{x,y} \left(-\sum_{i=1}^{m} \log y_i + \lambda^T (y - b + Ax)\right)$

Rigid problem

$$\max$$

Q7 $P_1 = \{x \mid Ax \le b\}$, $P_2 = \{x \mid Cx \le d\}$

$a^T x > y$ for $x \in P_1$ & $a^T x < y$ for $x \in P_2$.



Basically, if the two regions P_1 & P_2 are linearly separable, then there exists a direction $\hat{a}$ such that the value

$$\max_{x \in P_1} \left|\text{proj}_{\hat{a}}\, x\right| > y \equiv \max_{x \in P_1} a^T x \quad \text{s.t. } \|a\|_2 = 1$$

and

$$\min_{x \in P_2} \left|\text{proj}_{\hat{a}}\, x\right| < y \equiv \min_{x \in P_2} a^T x \quad \text{s.t. } \|a\|_2 = 1$$

So

$$\max_{x \in P_2} a^T x < y < \min_{x \in P_1} a^T x \quad \text{s.t. } \|a\| = 1$$

Let $x \in P_1$ & $y \in P_2$,

then $\max\limits_{y \in P_2} a^T y < \gamma < \min\limits_{x \in P_1} a^T x$ for some $\|a\|_2 = 1$

If P_1 & P_2 are linearly separable, then there exist x, y & γ such that the following has a soln :-

OPT-1

$$\begin{cases} \max. \left(\min\limits_{x \in P_1} a^T x - \max\limits_{y \in P_2} a^T y \right) \\ \text{s.t. } Ax \leq b \text{ or } x \in P_1 \\ Cy \leq d \text{ or } y \in P_2 \\ \|a\|_2 = 1 \end{cases}$$

Let $e_1 = \min\limits_{x \in P_1} a^T x$ & $e_2 = \max\limits_{y \in P_2} a^T y$

$$e_1 = \min \; a^T x \quad \text{s.t. } Ax \leq b \qquad \Bigg| \qquad e_2 = \max \; a^T x \quad \text{s.t. } Cx \leq d$$

$$e_1 = \max \; b^T z_1 \quad \text{s.t. } A^T z_1 = a, \; z_1 \geq 0 \qquad \Bigg| \qquad e_2 = \min \; d^T z_2 \equiv \max -d^T z_2 \quad \text{s.t. } C^T z_2 = a, \; z_2 \geq 0$$

So OPT-1 is reformulated as →

$$\max \; e_2 - e_1 \quad \text{s.t. } Ax \leq b, \;\; \text{s.t. } Cy \leq d, \;\; \|a\|_2 = 1$$

$$\min \left\{ \max b^T z_1 - \max(-d^T z_2) \right\}$$

$$\equiv \max \left\{ \max (b^T z_1 + d^T z_1) \right\}$$

$$\text{s.t. } A^T z_1 = a$$

$$C^T z_2 = a$$

$$z_1 \geq 0, \; z_2 \geq 0$$

$$\|a\|_2 = 1$$

OR

$$- \max \{ \max (b^T z_1 + d^T z_2) \}$$
s.t. $A^T z_1 = a$
$c^T z_2 = a$
$z_1 \geq 0,\ z_2 \geq 0,\ \|a\|_2 = 1$

$\equiv$

$$\max -b^T z_1 - d^T z_2$$
s.t. $A^T z_1 = a$
$c^T z_2 = a$
$z_1 \geq 0,\ z_2 \geq 0$
$\|a\|_2 \leq 1$

Since we are maximising, relaxing $\|a\|_2 = 1$ to $\|a\|_2 \leq 1$ is alright.

Q8 Equality Constrained Least Squares,

$$\text{minimize } \|Ax - b\|_2^2$$
s.t. $Gx = h$

where $A \in \mathbb{R}^{m \times n}$ with rank $A = n$ & $G \in \mathbb{R}^{p \times n}$ with rank $G = p$.

The Lagrangian is given by

$$L(x, \lambda) = \|Ax - b\|_2^2 + \lambda^T (Gx - h)$$
$$= (Ax - b)^T (Ax - b) + \lambda^T (Gx - h)$$
$$= x^T A^T A x - b^T A x - x^T A^T b + b^T b + \lambda^T (Gx - h)$$
$$= x^T A^T A x + (\lambda^T G - 2b^T A) x - \lambda^T h + b^T b$$

We can think of this as $ax^2 + bx + c$ & the minimum occurs at $2ax^* + b = 0$ or $x^* = \frac{-b}{2a}$.

So the minimum occurs at

$$x^* = -\frac{1}{2}(A^TA)^{-1} \times (\lambda^T G - 2b^TA)$$

So dual function is obtained as $g(\lambda) = L(x^*, \lambda)$

Again comparing to $ax^2 + bx + c$, the minimum occurs at

$x^* = \frac{-b}{2a}$ & the minimum value is $a\frac{b^2}{4a^2} - \frac{b^2}{2a} + c$

$$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

$$= -\frac{b^2}{4a} + c$$

So $g(\lambda) = L(x^*, \lambda) = -\frac{1}{4}(\lambda^T G - 2b^TA)^T (A^TA)^{-1} (\lambda^T G - 2b^TA)$
$- \lambda^T h + b^Tb$

For minimum, $\frac{\partial}{\partial \lambda} L(x^*, \lambda) = \frac{\partial}{\partial \lambda} g(\lambda) = 0.$

$\Rightarrow \cancel{\frac{\partial}{\partial x}\left[x^TA^TAx + (G^T\lambda - 2A^Tb)x - \lambda^Th\right] = 0}$

$\Rightarrow \cancel{2x^TA^TA + (G^T\lambda - 2A^Tb) = 0}$

$$g(\lambda) = -\frac{1}{4}\left[G^T\lambda (A^TA)^{-1} \lambda^T G - 4\lambda^T G (A^TA)^{-1} b^TA\right.$$
$$\left. + 4 b^T A^T (A^TA)^{-1} A^T b\right]$$
$$- \lambda^T h + b^Tb$$

$\Rightarrow g'(\lambda) = -\frac{1}{4}\left[2\lambda^T G (A^TA)^{-1} G^T - 4 A^Tb (A^TA)^{-1} G^T\right]$
$- h^T = 0$

$\Rightarrow \quad 2\lambda^T G (A^TA)^{-1} G^T = 4 A^Tb (A^TA)^{-1} G^T + h^T$

$\Rightarrow \quad \lambda^T G (A^TA)^{-1} G^T = 2 A^T b (A^TA)^{-1} G^T + \frac{h^T}{2}$

$$\Rightarrow G (A^TA)^{-1} G^T \lambda = 2 G (A^TA)^{-1} b^T A + \frac{h}{2}$$

$$\Rightarrow \lambda^* = (G (A^TA)^{-1} G^T)^{-1} \left[2 G (A^TA)^{-1} b^T A + \frac{h}{2} \right]$$

Q9 We recall the problem in Q4.

Let epigraph of $f_i(x) = e_i = \{ (\vec{x}, t) \mid t \geq f_i(\vec{x}) \}$

Clearly epigraph of $g = \mathrm{Conv}(\bigcup_i e_i)$

or $e_g = \mathrm{Conv}(\bigcup_i e_i)$

Now, we can minimize s

s.t. $(x_i, t_i) \in e_i$

$\langle x, s \rangle = \sum \theta_i \langle x_i, t_i \rangle$

$0 \leq \vec{\theta}$

$1^T \vec{\theta} = 1$

i.e. minimize $\overbrace{[0, 0, \ldots 0}^{n}\ 1] \begin{bmatrix} \vec{x} \\ \hline s \end{bmatrix}$

s.t. $\langle \vec{x}_i, t_i \rangle \in e_i$ or $t_i \geq f_i(\vec{x}_i)$

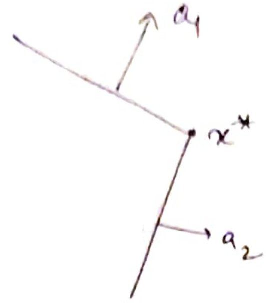
$\langle \vec{x}, s \rangle = \theta_1 \langle \vec{x}_1, t_1 \rangle + \cdots + \cdots + \theta_m \langle \vec{x}_m, t_m \rangle$

$\theta_i \leq 0$

$\sum_{i=1}^{m} \theta_i = 1$

from Q4 we know that this can be posed as a convex optimisation problem using the perspective transform.

Q10 a)



Let $x^* \in V$ be a vertex.

Clearly any $\vec{c} \in \{ \vec{n} = \alpha_i \vec{a}_i \mid \alpha_i \geq 0 \}$ will make x^* the optimum.

* Here $\vec{a}_i$ are the normals to the hyperplanes that correspond to the constraints when we are at vertex x^*.

Not only that, but it is a convex cone

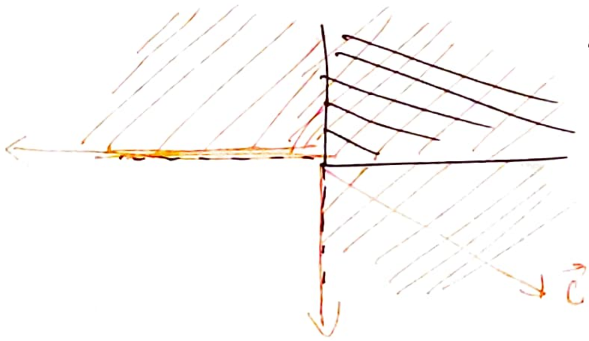
* This is always true because the normals $\vec{a}_i$ are always facing away from the feasible region & to have a vertex, we always have an acute angle (this is a general acute angle in n-dimension).

b) Take $\vec{c} = \alpha_i \vec{a}_i$ where $\vec{a}_i$ are the normals of the hyperplanes corresponding to the tight constraints where

$\sum \alpha_i = 1$ & $\alpha_i \geq 0$.

In particular, we can pick $\alpha_i = \frac{1}{n}$ i.e average of the normal vectors $\vec{a}_i$ corresponding to the tight constraint hyperplanes

c) Counter example



Let the 1st quadrant be the feasible region. Clearly $\boxed{\text{any } \vec{c}}$ in the cone formed by the quadrants II, III & IV will give optimal value $= \infty$.

$\boxed{\text{This is a cone but not a convex cone.}}$

* In almost all cases, the C_∞ is a non-convex cone except when the feasible region is isomorphic to a halfspace.

for example, if the feasible region is Quadrant I & II, which corresponds to $y \geq 0$, then C_∞ is ~~quadrant~~ 1st & 2nd quadrant (i.e. a halfspace) & $C_\infty \cup \{0\}$ is a convex cone.

* In all other cases, C_∞ is a non-convex cone.

d) Clearly, we can subtract C_∞ from the full domain of $\mathbb{R}^n$ to get C_V. Since we already showed that almost all C_∞ are non-convex cones, $\mathbb{R}^n - C_\infty$ is a convex cone for almost all cases.

The only case that could be a problem is when C_∞ is a halfspace but that is automatically resolved as the complement of the halfspace ($\mathbb{R}^n$ - halfspace) is the complementary halfspace which is also convex.

* So C_V is always convex.

<u>Algebraic proof</u> : Let $\vec{c}_1 \in C_V$ & $\vec{c}_2 \in C_V$.

$$\text{Optimum } c_1^T x < \infty \quad \Rightarrow \quad \text{optimum } \alpha c_1^T x < \infty$$
$$\text{s.t } Ax \leq b \qquad\qquad \text{if } \alpha \geq 0$$

$$\Rightarrow \alpha \vec{c}_1 \in C_V$$

$\boxed{\text{So } C_V \text{ is a cone}}$

Also if optimum $c_1^T x < \infty$

& optimum $c_2^T x < \infty$

$\Rightarrow \alpha_1 c_1^T x + \alpha_2 c_2^T x < \infty$ iff $\alpha_1, \alpha_2 \geq 0$

$\Rightarrow (\alpha_1 c_1^T + \alpha_2 c_2^T) x < \infty$

$\Rightarrow \alpha_1 c_1 + \alpha_2 c_2 \in C_V$

So $\forall c_1, c_2 \in C_V$, $\alpha_1 c_1 + \alpha_2 c_2 \in C_V$ $\forall \alpha_1, \alpha_2 \geq 0$

Hence C_V is a convex cone, if we set $\alpha_2 = 1 - \alpha_1$ & $\alpha_1 \leq 1$

Is it a convex cone? YES

Q 12

Let $A \equiv G$ has perfect matching

$B \equiv |N(S)| \geq |S|$ $\forall S \subseteq L$

$A \Rightarrow B$ is obvious & easy since the matching gives atleas $|N(S)| = |S|$ which are the partners on the rightside.

We want to show $B \Rightarrow A$.

We will show this by contradiction. → we will show that if min-vertex cover is less than 'n' then B is violated.

Every node v in L has atleast one edge emerging/incident on it since v has $N(\{v\}) \geq |\{v\}| = 1$

~~On any subset $S \subseteq L$, Let $H = \{S\} \cup \{N(S)\}$.~~

~~We can show by exchange arguement that since $|S| \leq |N(S)|$~~
~~and that each vertex in S has atleast one edge incident on it,~~
~~the~~

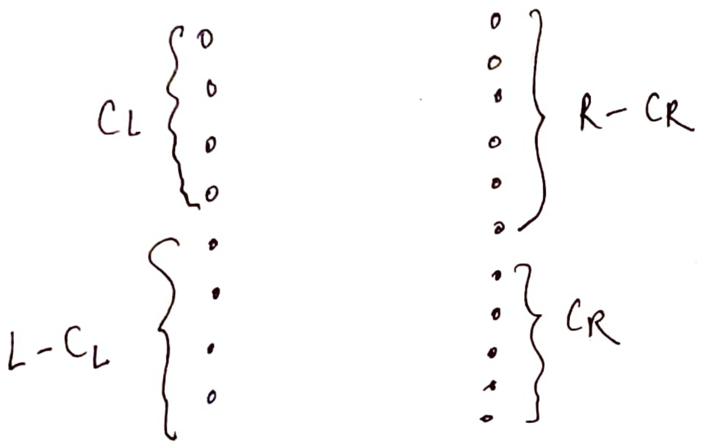
Let min vertex cover $< n$.

Let there be n_L vertices in L & n_R vertices in right that are part of the vertex cover. Let the set of vertices that are the part of the minimum cover be $C_L \cup C_R$

where $C_L \subseteq L$ & $C_R \subseteq R$

L ↓ left nodes

R ↓ right nodes.



Clearly, $|C_L| = n_L$

$|C_R| = n_R$

& $n_L + n_R < n = |L| = |R|$

~~If $n_L < n_R$, then C_R covers the edges incident on $L - C_L$.~~

~~Since $n_L + n_R < n$~~

~~& $n_R < n_L$~~

⇒ $n_R < n - n_L$

⇒ $|C_R| < |L - C_L|$

~~⇒ $n_L < n - n_L$~~ However, there cannot be any edge between $L - C_L$ & $R - C_R$ as that edge will not be covered by $C_L \cup C_R$.

So $N(L - C_L) = C_R$ but $|C_R| < |L - C_L|$. Let $S = L - C_L$, $|N(S)| < |S|$

Contradiction to statement B.