CSCI675 : Homework-2

26th February, 2022

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> ========== Textbook: Convex Optimisation, Stephen Boyd and Lieven Vandenberghe ==========

Find the question here - Homework2 Questions

Q1 : Midpoint Convexity and closed set

2.3 *Midpoint convexity.* A set C is *midpoint convex* if whenever two points a, b are in C, the average or midpoint $(a+b)/2$ is in C. Obviously a convex set is midpoint convex. It can be proved that under mild conditions midpoint convexity implies convexity. As a simple case, prove that if C is closed and midpoint convex, then C is convex.

Q2 : Convex Sets

2.12 Which of the following sets are convex?

(a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\}$.

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}$.

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

Axioms

A1) Halfspaces are convex.
A2) Intersection of convex sets is convex.
A3) Union of convex sets is not necessarily convex. A4) Affine transformation of a convex set is convex, i.e convexity is preserved under affine transformations. A5) Set of points closer to point x than to point y is a halfspace, say $H(x, y)$.

part	convex / non-convex	reasoning
a	convex	Slab = intersection of halfspaces $\alpha \leq a^T x$ and $a^T x \leq \beta$. Apply A1 and A2.
b	convex	Rectangle = intersection of axis aligned slabs (i.e a is an element of the standard basis or a multiple of it in the formulation in part a). Apply A1 and A2.
c	convex	Wedge = intersection of halfspaces. Apply A1 and A2.
d	convex	Let S be the described set and x_0 be the point. Let a generic point $y_i \in S$. The halfspace $H(x_0, y_i)$ describe all the points closer to x_0 than the point y_i. Let $S_i = H(x_0, y_i)$. Clearly, $S = \bigcap_i S_i = \bigcup_i H(x_0, y_i)$ since all $y_i \in S$ are closer to x_0 than any point in T. Apply A1 and A2.
e	non-convex	An easy conterexample in $\mathbb{R}^2$ - set A = {(-1,0), (1,0)} and B={(0,0)}. The set Q of all points in A than in B is the union of the halfspaces $x > 0.5$ and $x < -0.5$. In general, Q is the union of the convex sets S_i where S_i is the set described in part (d), i.e S_i is the convex set that is closer to a_i than set B for an arbitrary point $a_i \in A$. Apply axiom A3
f	convex	S_1 is the translation of S_2 by the vector $\vec{x}$. Translation is an affine transformation (f(\vec{p}) = I \vec{p} + \vec{x}). Apply axiom A4.
g	convex	Turns out that this set is the interior of a circle centered at one of the points because $0 \leq \theta \leq 1$.

Q3) Probability Distributions and Convexity

2.15 *Some sets of probability distributions.* Let x be a real-valued random variable with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$, where $a_1 < a_2 < \cdots < a_n$. Of course $p \in \mathbf{R}^n$ lies in the standard probability simplex $P = \{p \mid \mathbf{1}^T p = 1,\ p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

(a) $\alpha \leq \mathbf{E} f(x) \leq \beta$, where $\mathbf{E} f(x)$ is the expected value of $f(x)$, *i.e.*, $\mathbf{E} f(x) = \sum_{i=1}^n p_i f(a_i)$. (The function $f : \mathbf{R} \to \mathbf{R}$ is given.)

(b) $\mathbf{prob}(x > \alpha) \leq \beta$.

(c) $\mathbf{E}\,|x^3| \leq \alpha\,\mathbf{E}\,|x|$.

(d) $\mathbf{E}\,x^2 \leq \alpha$.

(e) $\mathbf{E}\,x^2 \geq \alpha$.

(f) $\mathbf{var}(x) \leq \alpha$, where $\mathbf{var}(x) = \mathbf{E}(x - \mathbf{E}\,x)^2$ is the variance of x.

(g) $\mathbf{var}(x) \geq \alpha$.

(h) $\mathbf{quartile}(x) \geq \alpha$, where $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \leq \beta) \geq 0.25\}$.

(i) $\mathbf{quartile}(x) \leq \alpha$.

part	convex / non-convex	reasoning
a	convex	Let $b_i = f(a_i)$. $\mathbf{E}f(x) = f(a_i)p_i = b_i p_i$. $\alpha \leq \mathbf{E}f(x) \leq \beta \iff \alpha \leq b_i p_i$ and $b_i p_i \leq \beta$. So p_i has to obey two linear inequalities and hence the region defined in p-space is convex.
b	convex	Let $a_k \geq \alpha$ for some $k \in \{1, 2, ...n\}$. $\therefore prob(x > \alpha) \leq \beta = \sum_{i=k}^n p_i \leq \beta$. Above is a linear inequality in p-space when α and β are some given constants.
c	convex	$\mathbf{E}\|x^3\| = \sum_{i=1}^n \|a_i\|^3 p_i$ and $\mathbf{E}\|x\| = \sum_{i=1}^n \|a_i\| p_i$. $\mathbf{E}\|x^3\| \leq \alpha \mathbf{E}\|x\| \iff \mathbf{E}\|x^3\| - \alpha \mathbf{E}\|x\| \leq 0$ or $\|a_i^3\| p_i - \alpha \|a_i\| p_i \leq 0 \iff \|a_i^3\| p_i - \|a_i\| p_i$. $(\|a_i\|^3 - \|a_i\|) p_i \leq 0$. Above is a linear inequality in p-space and hence is convex since a_i and α are constants.
d	convex	$\mathbf{E}x^2 = \sum_{i=1}^n a_i^2 p_i \leq \alpha$. Above is a linear inequality in p-space.
e	convex	$\mathbf{E}x^2 = \sum_{i=1}^n a_i^2 p_i \geq \alpha$. Above is a linear inequality in p-space.

part	convex / non-convex	reasoning
f	non-convex	Let $\mathbf{E}x = \mu = \sum_{i=1}^n a_i p_i$. $\mathbf{var}(x) = \mathbf{E}[x-\mu]^2 = \mathbf{E}[x^2+\mu^2-2x\mu x]$ $= \mathbf{E}x^2 + \mu^2 - 2\mu\mathbf{E}x = \mathbf{E}x^2 + \mu^2 - 2\mu^2 = \mathbf{E}x^2 - \mu^2$ $= \sum_{i=1}^n a_i^2 p_i - (\sum_{i=1}^n a_i p_i)^2$. Let $c_i = a_i^2$ and $\mathbf{c} = [c_1 ... c_i ... c_n]^T$. Similarly, let $\mathbf{a} = [a_1 ... a_i ... a_n]^T$. Then, $\mathbf{var}(x) = \mathbf{c^T p} - \mathbf{p^T}(\mathbf{aa^T})\mathbf{p}$ $\mathbf{var}(x) \leq \alpha \iff \mathbf{c^T p} - \mathbf{p^T}(\mathbf{aa^T})\mathbf{p} \leq \alpha$ This may not be convex always. For example, if we are in $\mathbb{R}^2$, then the LHS in the above inequality defines a conic section which can be a hyperbola (since there is a -ve sign in the equation). This can cause the failure of convexity if the inequality forces the set to be the two disjoint unbounded sides of the hyperbola.
g	convex	using a similar derivation as in part f, we can get $\mathbf{var}(x) = \mathbf{c^T p} - \mathbf{p^T}(\mathbf{aa^T})\mathbf{p}$ $\mathbf{var}(x) \leq \alpha \implies \mathbf{c^T p} - \mathbf{p^T}(\mathbf{aa^T})\mathbf{p} \geq \alpha$ Unlike the previous case, there is no -ve sign in the LHS and the entires of the vector $\mathbf{c}$ are all positive since $c_i = a_i^2$. Also since $\mathbf{aa^T}$ is always +ve semidefinite for all $\mathbf{a}$, we can guarantee that the corresponding conic section will be a parabola and hence the inequality forces the set to be on the side of parabola that makes the set convex.
h	convex	linear inequality
i	convex	linear inequality

Q4. Operations that preserve convextiy

Operations that preserve convexity

2.16 Show that if S_1 and S_2 are convex sets in $\mathbf{R}^{m+n}$, then so is their partial sum

$$S = \{(x, y_1 + y_2) \mid x \in \mathbf{R}^m,\ y_1,\ y_2 \in \mathbf{R}^n, (x, y_1) \in S_1,\ (x, y_2) \in S_2\}.$$

We will solve this using 4 methods -
1. Using algebraic definition of convex sets
2. Projecting and then taking Minkowski sum of the projections.
4. Using multilinear operator and tensor algebra
2. Using direct sum of convex sets and then projecting

Let $f((x, y_1), (x, y_2)) = (x, y_1 + y_2))$ where $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ is a multilinear function.

Let
$A_1 = (x, a_1), B_1 = (x, b_1)\ s.t\ A_1, B_1 \in S_1$ and
$B_1 = (x, a_2), B_2 = (x, b_2)\ s.t\ A_2, B_2 \in S_2$.

S_1 is convex, $M_1 = \theta A_1 + (1-\theta)B_1 = (x, \theta a_1 + (1-\theta)b_1) \in S_1\ \forall\, \theta \in [0,1]$.

Similarly, $M_2 = \theta A_2 + (1-\theta)B_2 = (x, \theta a_2 + (1-\theta)b_2) \in S_2\ \forall\, \theta \in [0,1]$.

By definition, $F_A = f(A_1, A_2) = (x, a_1 + a_2) \in S_1$ and $F_B = f(B_1, B_2) = (x, b_1 + b_2) \in S_2$.

We need to show that $F = \theta F_A + (1-\theta)F_B \in S\ \forall\, \theta \in [0,1]$.

$$F = \theta F_A + (1-\theta)F_B$$

$$= \theta(x\ ,\ a_1 + a_2) + (1-\theta)(x\ ,\ b_1 + b_2)$$

$$= (\theta x\ ,\ \theta[a_1 + a_2]) + ((1-\theta)x\ ,\ (1-\theta)[b_1 + b_2])$$

$$= (\theta x + (1-\theta)x\ ,\ [\theta a_1 + (1-\theta)b_1] + [\theta a_2 + (1-\theta)b_2])$$

$$= (x\ ,\ [\theta a_1 + (1-\theta)b_1] + [\theta a_2 + (1-\theta)b_2])$$

$$= f\Big((x\ ,\ \theta a_1 + (1-\theta)b_1)\ ,\ (x\ ,\ \theta a_1 + (1-\theta)b_1\Big)$$

$$= f\Big((x\ ,\ \theta a_1 + (1-\theta)b_1)\ ,\ (x\ ,\ \theta a_1 + (1-\theta)b_1\Big)$$

$$= f\Big((x\ ,\ \theta a_1 + (1-\theta)b_1)\ ,\ (x\ ,\ \theta a_1 + (1-\theta)b_1\Big)$$

$$= f\Big(M_1\ ,\ M_2\Big) \in S \quad \blacksquare$$

> Theorem 1: Direct Sum of convex sets is convex.
>
> X, Y convex $\implies X \oplus Y$ is convex.
>
> If $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ are convex sets. Let us embed X and Y in the first m and the last n dimensions of the space $\mathbb{R}^{m+n}$. Let then the direct sum of the two convex sets can be defined as the set $S = \{\langle x, y\rangle |\ x \in X, y \in Y\}$ where $\langle x, y\rangle \in \mathbb{R}^{m+n}$ is the concatenation of the vectors $x \in X$ and $y \in Y$. Then S is convex.
>
> Proof:
> Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Since X and Y are both convex, $x_3 = \alpha x_1 + (1-\alpha)x_2 \in X$ and $y_3 = \alpha y_1 + (1-\alpha)y_2 \in Y$.
Let $s_1 = \langle x_1, y_1 \rangle \in S$ and $s_2 = \langle x_1, y_1 \rangle \in S$.
Clearly $s_3 = \langle x_3, y_3 \rangle \in S$ by definition of direct sum. We need to show that $s = \alpha s_1 + (1-\alpha)s_2 \in S$.

$$
\begin{aligned}
s &= \alpha s_1 + (1-\alpha)s_2 \\
&= \alpha\langle x_1, y_1\rangle + (1-\alpha)\langle x_2, y_2\rangle \\
&= \langle \alpha x_1 + (1-\alpha)x_2 \ , \ \alpha y_1 + (1-\alpha)y_2 \rangle \\
&= \langle x_3, y_3 \rangle = s_3 \in S \quad \blacksquare
\end{aligned}
$$

Theorem 2: Projection operator preserves convexity

Let P be a projector operator such that $P : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$.
Let $X \subset \mathbb{R}^{m+n}$ be convex.
Then the image of X under P is convex, i.e $P(X)$ ius convex.

Proof:
Let $x_1, x_2 \in X$
Since X is convex, $\forall \alpha, \ 0 \leq \alpha \leq 1$ we have $x_3 = \alpha x_1 + (1-\alpha)x_2 \in X$. Let $p_1 = P(x_1) \in \mathbb{R}^m$ and $p_2 = P(x_2) \in \mathbb{R}^m$ be the projections of x_1, x_2 respectively, i.e p_1 and p_2 are the images of x_1 and x_2 under the projection operator and hence by definition of images under operators, $p_1, p_2 \in P(X)$

We need to show that the point $p_3 = \alpha p_1 + (1-\alpha)p_2$ belongs to the image of X under P, i.e $p_3 \in P(X)$.

$$
\begin{aligned}
p_3 &= \alpha p_1 + (1-\alpha)p_2 \\
&= \alpha P(x_1) + (1-\alpha)P(x_2) \\
&= P(\alpha x_1) + P((1-\alpha)x_2) \ \ \{\because P \ is \ linear\} \\
&= P(\alpha x_1 + (1-\alpha)x_2) \ \ \{\because P \ is \ linear\} \\
&= P(x_3) \in P(X)\{\because x_3 \in X\} \quad \blacksquare
\end{aligned}
$$

Theorem 3: Minkowski sum preserves convexity
$A + B = \{a + b \mid a \in A, b \in B\}$

We are not going to prove the following result about Minkowski sums but here is the wiki reference.

$Conv(A + B) = Conv(A) + Conv(B)$
i.e the convex hull of the Minkowski sum is the Minkowski sum of the convex hull of the sets. [Note that this is a special case of the general case of equivariance if we define the Minkowski sum '+' as a binary operator on convex sets.]

If X is a convex set, $\iff Conv(X) = X$.

Using the above two results, we can show that the Minkowski sum of two convex sets A and B is convex, i.e we can show that $S = A + B$ is convex

$$\begin{gathered} Conv(S) = Conv(A+B) \\ = Conv(A) + Conv(B) \\ = A + B \quad \{\because\ A, B \; are \; convex\} \\ = S \\ \\ \implies Conv(S) = S \\ \therefore\ S = A + B \; is \; convex \quad \blacksquare \end{gathered}$$

Now let's prove what we were asked in the question. We need to show that the image under $f((x, y_1), (x, y_2)) = (x, y_1 + y_2)$ is convex. Let $U = P_Y(S_1), V = P_Y(S_2)$ and $W = P_X(S_1) \cap P_X(S_2)$ where P_Y is the projection on to the last n dimensions and P_X is projection onto the first m dimensions. Clearly U and V are convex since S_1, S_2 are convex. Similarly W is convex because it is the intersection of the projection of convex sets S_1 and S_2 and we know that intersections and projections preserce convexity.

It is easy to see that the compsition $Minkowskisum(U, V) \oplus W$ is precisely f. Since all the operations that compose f preserve convexity, f preserves convexity too since convexity is preserved under composition if the operations which themselves preserveconvexity as it is basically a chain of convex images.

$I \; will \; be \; back \; to \; the \; other \; two \; methods \; - \; 3 \; and \; 4 \; when \; I \; get \; time \; later.$

Q5: Set of separating hyperplanes

2.21 *The set of separating hyperplanes.* Suppose that C and D are disjoint subsets of $\mathbf{R}^n$. Consider the set of $(a, b) \in \mathbf{R}^{n+1}$ for which $a^T x \le b$ for all $x \in C$, and $a^T x \ge b$ for all $x \in D$. Show that this set is a convex cone (which is the singleton $\{0\}$ if there is no hyperplane that separates C and D).

Definition of a convex cone: A set S is a cone if $s \in S \iff \alpha s \in S \; \forall \; \alpha > 0$

Clearly when C and D are not separable, $\langle a, b \rangle = \vec{0} \in \mathbb{R}^{n+1}$ and hence is a trivial convex cone.

When there exists a separating hyperplane, let the $\langle a, b \rangle$ be a representation of it. Let us call the set of all such representations H.

$$\begin{gathered} a^T x \le b \; \forall \; x \in C \; and \; a^T y \ge b \; \forall \; y \in D \\ \implies ka^T x \le kb \; and \; ka^T y \ge kb \; \forall \; k \ge 0 \\ \implies (ka)^T x \le kb \; and \; (ka)^T y \ge kb \; \forall \; k \ge 0 \end{gathered}$$

Hence $\langle ka, kb \rangle = k\langle a, b \rangle \; \forall \; k > 0$ is also the representation of the separating hyperplane and hence belongs to the set of the separating hyperplanes. Hence the set of the separating hyperplanes is a cone.

Now we will prove that it is convex as well. Let $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle$ be in the set H. We have to show that $\langle a_3, b_3 \rangle = \theta\langle a_1, b_1 \rangle + (1-\theta)\langle a_2, b_2 \rangle = \langle \theta a_1 + (1-\theta)a_2 \;,\; \theta b_1 + (1-\theta)b_2 \rangle$ is also in the set H when $0 \leq \theta \leq 1$.

$$\begin{gathered}
Let\ x \in C\ and\ y \in D \\[1em]
a_1 x \leq b_1 \; and \; a_1 y \geq b_1 \\
a_2 x \leq b_2 \; and \; a_2 y \geq b_2 \\
\Updownarrow \\
\theta a_1 x \leq \theta b_1 \; and \; \theta a_1 y \geq \theta b_1 \\
(1-\theta)a_2 x \leq (1-\theta)b_2 \; and \; (1-\theta)a_2 y \geq (1-\theta)b_2 \\
\Updownarrow \\
\theta a_1 x + (1-\theta)a_2 x \leq \theta b_1 + (1-\theta)b_2 \; and \; \theta a_1 y + (1-\theta)a_2 y \geq \theta b_1 + (1-\theta)b_2 \\
\Updownarrow \\
a_3 x \leq b_3 \; and \; a_3 y \geq b_3 \\[1em]
\implies \langle a_3, b_3 \rangle \in H \quad \blacksquare
\end{gathered}$$

Hence the set H is convex in addition to being a cone. Hence H, the set of representation of separating hyperplanes is a convex cone. If we restrict the hyperplanes to have a unique representation, then the set might neither be convex or a cone - for example forcing a to be unit length will make the set and arc of the unit circle that is formed when we take the intersection of the convex cone H and the unit circle in $\mathbb{R}^{n+1}$.

Q6. Support Functions

2.26 *Support function.* The support function of a set $C \subseteq \mathbf{R}^n$ is defined as

$$S_C(y) = \sup\{y^T x \mid x \in C\}.$$

(We allow $S_C(y)$ to take on the value $+\infty$.) Suppose that C and D are closed convex sets in $\mathbf{R}^n$. Show that $C = D$ if and only if their support functions are equal.

Basically we are looking in the direction of the vector y and trying to see what is the greatest value of the dot product of y with any point/position vector x in the set C.

Why we need the set to be closed is clear from a counterexample. If C is a closed convex set and D is C with a hole cut out in the interior of C, in then the support funcitons are the same but the sets C and D are not.

Why we need convexity for the sets C and D for the claim in the question will become evident when we have proved the statement.

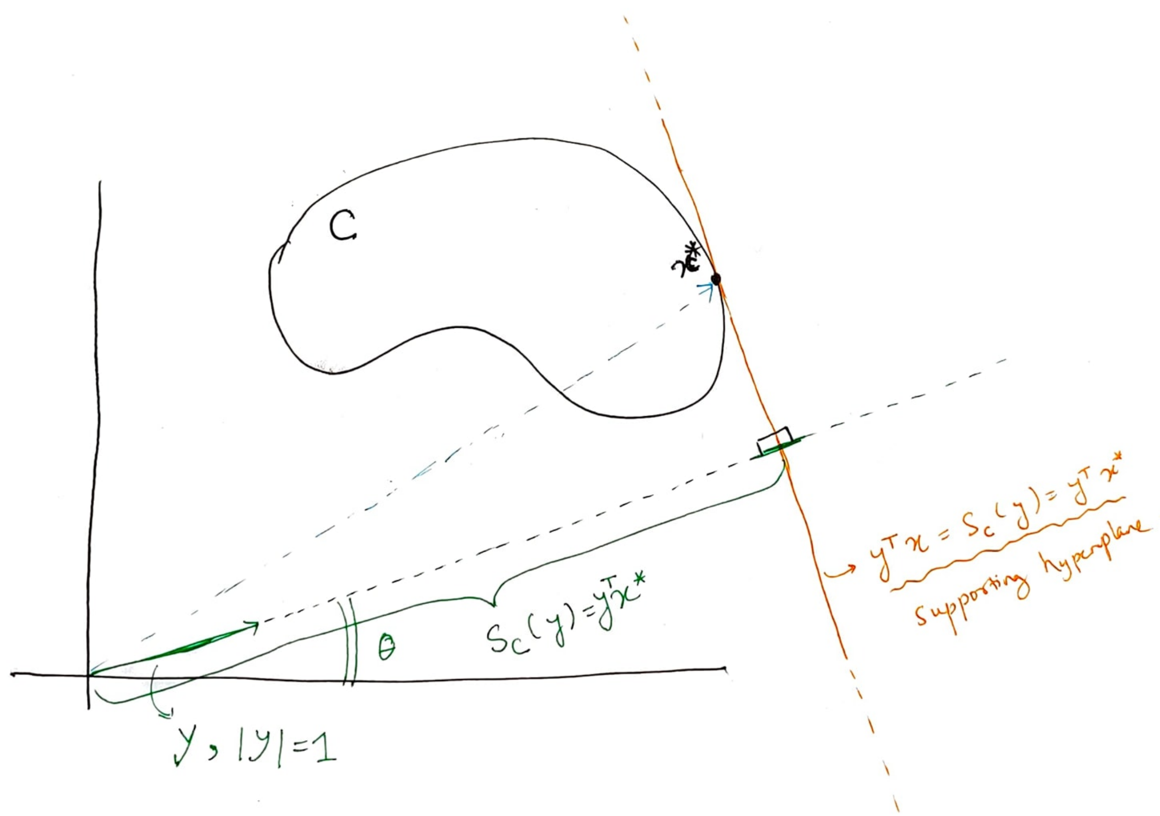
We can easily see that since $S_C(y) = y \cdot x$, $S_C(ky) = ky \cdot x = kS_C \; \forall \; k \in \mathbb{R}$ i.e $S_C ky = kS_C y$. So if C and D are two sets and $S_C(y) = S_D(y)$, then $S_C(ky) = kS_C(y) = kS_D(y) = S_D(ky)$. That means that once we know that the support function for the sets C and D are same given the vector y, it is the same for C and D for all vector in that direction (all linear combinations) of them. So we might aswell just look at the directions and not all the vectors. That is exactly what we will do, we will look at all unit vectors y, i.e $|y| = 1$.

We observe that if the support functions $S_D(y) = S_C(y) \ \forall \ |y| = 1$, then the closed convex sets C and D are the same.

** fact 1: ** Recall that the vector form of a hyperplane that has the unit normal y and is d distance away from the origin along the direction y is given by $y \cdot x = d$ or equivalently $y^T x = d$.

** fact 2: ** $y^T x = y \cdot x = d$ gives the projection of x in the direction of y when $|y| = 1$.

From the above two facts, we can see that $y^T x = S_C(y) \approx max(d)$ is a $hyperplane$ that supports the set C at the point x^* where $S_C(y) = y^T x^*$.



support function

Clearly $S_C(y) = S_C(\theta)$ when $y = 1$ and $S_C(\theta) = sup\{\langle cos\theta, sin\theta\rangle \cdot x \mid x \in C\}$.

Hence $y^T x = S_C(y)\ ,\ |y| = 1$ are the supporting hyperplanes of the set C.

Since $S_C(y) = S_D(y) \ \forall \ y$, hence $S_C(y) = S_D(y)$ for $|y| = 1$.

$$S_C(y) = S_D(y)\ \forall\ |y| = 1$$

$$\implies y^T x = S_C(y) \equiv y^T x = S_D(y)$$

$$supporting\ hyperplanes\ of\ C \equiv supporting\ hyperplanes\ of\ D.$$

If the set of supporting hyperplanes of two convex sets C and D are the same, then the sets are the same since the convex sets are also defined by the set of the separating hyperplanes instead of the set of points. ■

Q7. Euclidean Distance Matrices

2.36 *Euclidean distance matrices.* Let $x_1, \ldots, x_n \in \mathbf{R}^k$. The matrix $D \in \mathbf{S}^n$ defined by $D_{ij} = \|x_i - x_j\|_2^2$ is called a *Euclidean distance matrix.* It satisfies some obvious properties such as $D_{ij} = D_{ji}$, $D_{ii} = 0$, $D_{ij} \geq 0$, and (from the triangle inequality) $D_{ik}^{1/2} \leq D_{ij}^{1/2} + D_{jk}^{1/2}$. We now pose the question: When is a matrix $D \in \mathbf{S}^n$ a Euclidean distance matrix (for some points in $\mathbf{R}^k$, for some k)? A famous result answers this question: $D \in \mathbf{S}^n$ is a Euclidean distance matrix if and only if $D_{ii} = 0$ and $x^T Dx \leq 0$ for all x with $\mathbf{1}^T x = 0$. (See §8.3.3.)

Show that the set of Euclidean distance matrices is a convex cone.

The distance matrix satisfies $D_{ii} = 0$ and $x^T Dx \leq 0$ for all $1^T x = 0$.

We need to show that the space of distance matrices is a convex cone.

It is not very difficult to see that the space of such matrices form a cone. Let $C = kD$ where $k \geq 0, k \in \mathbb{R}$.

$$C_{ii} = k * D_{ii} = k * 0 = 0 \quad \blacksquare$$

For $1^T x = 0$

$$\begin{aligned} x^T Dx &\leq 0 \\ \implies kx^T Dx &\leq 0 \quad \forall k \geq 0 \\ \implies x^T kDx &\leq 0 \\ \implies x^T Cx &\leq 0 \quad \blacksquare \end{aligned}$$

Now let us show that the space of such matrices is infact convex.

Let D and E be two euclidean distance matrices. Let $F = \alpha D + (1-\alpha)E$ for $0 \leq \alpha \leq 1$

$$F_{ii} = \alpha D_{ii} + (1-\alpha)E_{ii} = \alpha * 0 + (1-\alpha) * 0 = 0 \quad \blacksquare$$

for $1^T x = 0$

$$x^T Dx \leq 0 \; and \; x^T Ex \leq 0$$

$$\alpha x^T Dx \leq 0 \; and \; (1-\alpha)x^T Ex \leq 0$$

$$x^T \alpha Dx \leq 0 \; and \; x^T(1-\alpha)Ex^T \leq 0$$

$$\implies x^T[\alpha D + (1-\alpha)E]x \leq 0$$

$$\implies x^T Fx \leq 0 \quad \blacksquare$$

So the space of euclidean distance matrices is infact a convex cone.

Problem 7. (5 points)
B&V Exercise 2.36. Additionally, describe the dual cone of Euclidean distance matrices as the

conic hull of a set of matrices.
(Note: The latter part of the question will make more sense after we cover geometric duality in class.)

We need to find the dual cone of the cone of the euclidean distance matrices.

The polar of a set Z is defined as the set Y = $y \mid y^T z \leq 1 \; \forall z \in Z$.

For a cone Z, the polar is equilvalently defined as the set Y = $y \mid y^T z \leq 0 \; \forall z \in Z$. The dual is just the negative of the polar. Hence the dual of the cone Z is given by Y = $y \mid y^T z \geq 0 \; \forall z \in Z$.

> **Useful remarks**
>
> 1. $w^T Mw = (ww^T) \cdot M$ **where $\cdot$ represents the dot product in the space of matrices.**

We already know that for all euclidean distance matrices D and $1^T x = 0$

$$x^T Dx \leq 0 \quad \forall \; D$$

$$\implies (xx^T)\cdot D \le 0 \quad \forall D$$

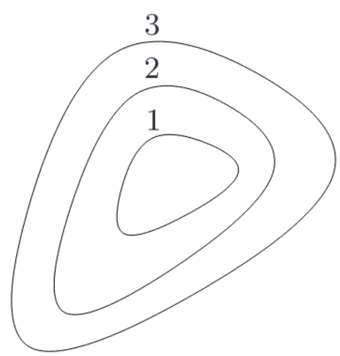
$$\implies (-xx^T)\cdot D \ge 0 \quad \forall D$$

$$\implies -xx^T \; is\ in\ the\ dual\ cone\ of\ the\ set\ of\ euclidean\ distance\ matrices$$

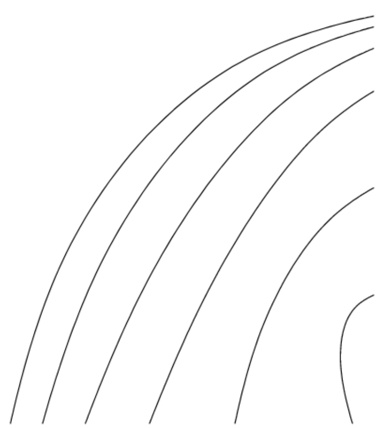
We know that the dual of a cone is a cone and the dual cone is the cone defined by the matrices $-xx^T$ that can be constructed from x under the constraint $1^Tx = 0$.

Q8. Level sets of convex, concave, quasiconvex, and quasiconcave functions

3.2 *Level sets of convex, concave, quasiconvex, and quasiconcave functions.* Some level sets of a function f are shown below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc.

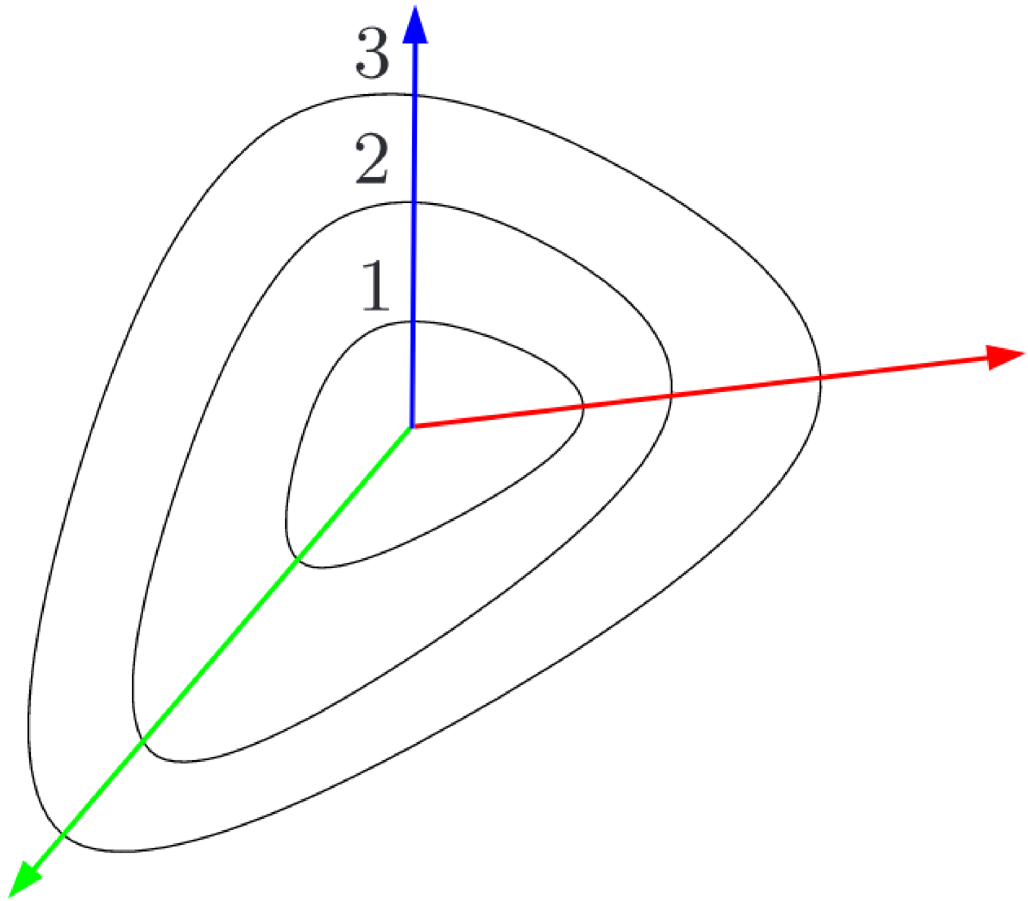


Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.



1 2 3 4 5 6

part 1



Since the level sets show that as we move away from the center, the function increase, the only posibilities are that the function is either convex or quasi convex.

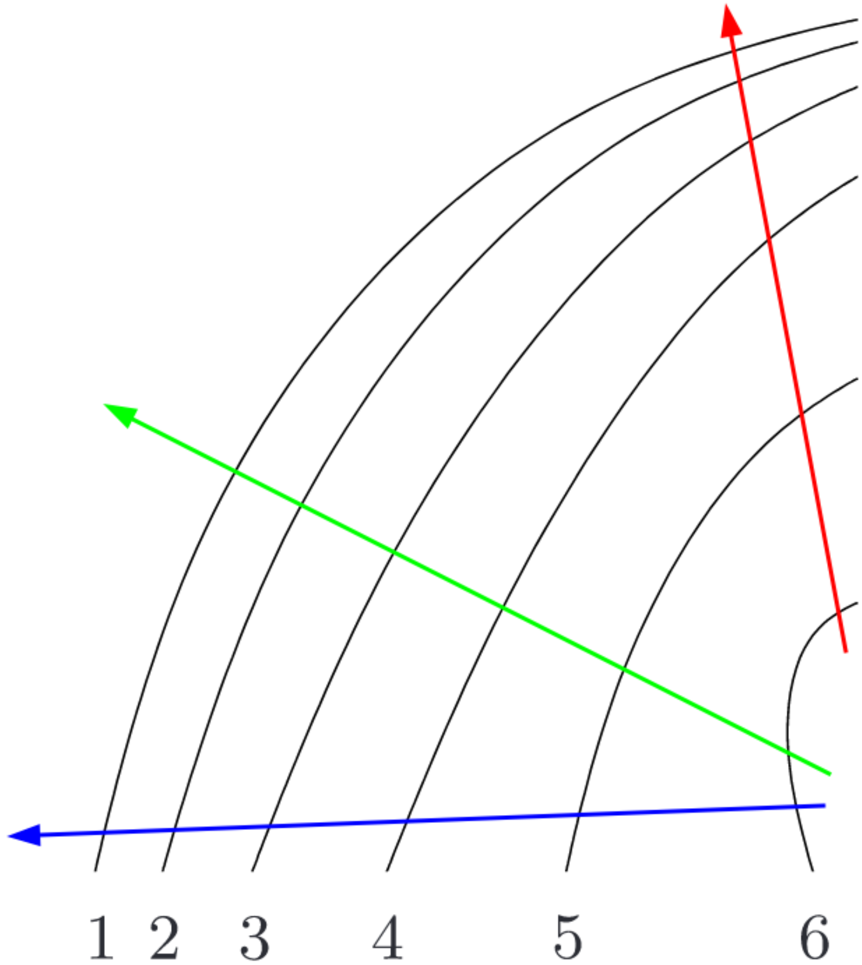
Any higher dimensional convex funciton is also convex when we take the restriction of the function on a one dimensional line in its domain. For our case, the convex function has domain in $\mathbb{R}^2$. So a line in the domain like the red, green and the blue lines must yield a convex function on the corresponding restrictions.

On the one dimensional restrition, the slope of a convex function can only increase as we move on the line (we can assume any direction on the line to be positive). Now, since the slope has to increase, the density of the level curves should increase as w move along any line.

Along the green line, the distance between the level curves does decrease (hence density increases). Along the blue line, the density remain constant which indicates that along that direction the function increase linearly Along the red line, the density decreases. This shows that the function cannot be a convex funciton.

Since the sublevel sets are covnex, the function is quasiconvex.

part 2



As we can see from the level curves, the finction keeps decraseing as we move away from the center and hence the only possibilities are that the function is concave or quasi convex. However, all directions we take in the domain to for a one dimensional line to create a restriction for the function, we see that the density of the level sets at increases and hence the funciton is convex.

Since quasiconcave functions are supersets of concave functions, we conclude that the funciton is quasiconvex as well. We can also say that because the superlevel set of te function as re convex regions/sets in the domain.

Q9.

3.12 Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex, $g : \mathbf{R}^n \to \mathbf{R}$ is concave, $\mathbf{dom}\, f = \mathbf{dom}\, g = \mathbf{R}^n$, and for all x, $g(x) \leq f(x)$. Show that there exists an affine function h such that for all x, $g(x) \leq h(x) \leq f(x)$. In other words, if a concave function g is an underestimator of a convex function f, then we can fit an affine function between f and g.

Q10.

3.24 *Some functions on the probability simplex.* Let x be a real-valued random variable which takes values in $\{a_1, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$, with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$. For each of the following functions of p (on the probability simplex $\{p \in \mathbf{R}^n_+ \mid \mathbf{1}^T p = 1\}$), determine if the function is convex, concave, quasiconvex, or quasiconcave.

(a) $\mathbf{E}\, x$.

(b) $\mathbf{prob}(x \geq \alpha)$.

(c) $\mathbf{prob}(\alpha \leq x \leq \beta)$.

(d) $\sum_{i=1}^n p_i \log p_i$, the negative entropy of the distribution.

(e) $\mathbf{var}\, x = \mathbf{E}(x - \mathbf{E}\, x)^2$.

(f) $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \leq \beta) \geq 0.25\}$.

(g) The cardinality of the smallest set $\mathcal{A} \subseteq \{a_1, \ldots, a_n\}$ with probability $\geq 90\%$. (By cardinality we mean the number of elements in $\mathcal{A}$.)

(h) The minimum width interval that contains 90% of the probability, *i.e.*,

$$\inf\{\beta - \alpha \mid \mathbf{prob}(\alpha \leq x \leq \beta) \geq 0.9\}.$$

part	convex / non-convex	reasoning
a	convex, concave, quasiconvex, quasiconcave	expectation is linear in both a_i and p_i
b	convex, concave, quasiconvex, quasiconcave	linear in p_i
c	convex, concave, quasiconvex, quasiconcave	linear in p_i
d	convex, quasiconvex	xlogx is a convex function (second derivative is always positive). Sum of convex functions is convex.
e	convex, quasiconvex	quadratic function with PSD hessian
f	not convex not concave	discontinous
g	not convex not concave	discontinous
h	not convex not concave	from counterexamples

Q11 : Representation of piecewise-linear convex functions

3.29 *Representation of piecewise-linear convex functions.* A convex function $f : \mathbf{R}^n \to \mathbf{R}$, with $\mathbf{dom}\, f = \mathbf{R}^n$, is called *piecewise-linear* if there exists a partition of $\mathbf{R}^n$ as

$$\mathbf{R}^n = X_1 \cup X_2 \cup \cdots \cup X_L,$$

where $\mathbf{int}\, X_i \neq \emptyset$ and $\mathbf{int}\, X_i \cap \mathbf{int}\, X_j = \emptyset$ for $i \neq j$, and a family of affine functions $a_1^T x + b_1, \ldots, a_L^T x + b_L$ such that $f(x) = a_i^T x + b_i$ for $x \in X_i$.
Show that such a function has the form $f(x) = \max\{a_1^T x + b_1, \ldots, a_L^T x + b_L\}$.

Q12: Convex hull or envelope of a function

3.30 *Convex hull or envelope of a function.* The *convex hull* or *convex envelope* of a function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$g(x) = \inf\{t \mid (x,t) \in \mathbf{conv}\,\mathbf{epi}\,f\}.$$

Geometrically, the epigraph of g is the convex hull of the epigraph of f.

Show that g is the largest convex underestimator of f. In other words, show that if h is convex and satisfies $h(x) \le f(x)$ for all x, then $h(x) \le g(x)$ for all x.

Q13: Conjugate functions

Conjugate functions

3.36 Derive the conjugates of the following functions.

$\boxed{\text{(a)}}$ *Max function.* $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbf{R}^n$.

(b) *Sum of largest elements.* $f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbf{R}^n$.

$\boxed{\text{(c)}}$ *Piecewise-linear function on* $\mathbf{R}$. $f(x) = \max_{i=1,\ldots,m}(a_i x + b_i)$ on $\mathbf{R}$. You can assume that the a_i are sorted in increasing order, *i.e.*, $a_1 \leq \cdots \leq a_m$, and that none of the functions $a_i x + b_i$ is redundant, *i.e.*, for each k there is at least one x with $f(x) = a_k x + b_k$.

(d) *Power function.* $f(x) = x^p$ on $\mathbf{R}_{++}$, where $p > 1$. Repeat for $p < 0$.

(e) *Negative geometric mean.* $f(x) = -(\prod x_i)^{1/n}$ on $\mathbf{R}^n_{++}$.

(f) *Negative generalized logarithm for second-order cone.* $f(x, t) = -\log(t^2 - x^T x)$ on $\{(x, t) \in \mathbf{R}^n \times \mathbf{R} \mid \|x\|_2 < t\}$.

Conjugate of $f(x)$ is defined as $f^*(y) = \sup\{y^T x - f(x) \mid x \in dom(f)\}$

a) $f^*(y) = \sup\{y^T x - max(x_i) \mid x \in \{x_1, x_2, ..., x_n\}\}$

Q14: Relation between polar and Lagrangian duality.

Problem 14. (14 points)
In this problem, we will explore the relationship between polar and Lagrangian duality. Specifically, in the context of linear programming, we will argue that the two are respectively geometric and algebraic formulations of the same idea.

a (2 points). Consider a linear program of the form

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array} \tag{1}$$

Using the rules we saw in class, we can derive the Lagrangian dual of (1) as the following LP

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y = c \\ & y \succeq 0 \end{array} \tag{2}$$

Naturally, scaling each inequality $a_i \cdot x \leq b_i$ of (1) by a constant $\alpha_i > 0$ to get the inequality $\alpha_i a_i \cdot x \leq \alpha_i b_i$ preserves the feasible set and objective function of (1) (and therefore also preserves the optimal solution and objective value). In other words, scaling the inequalities produces a geometrically equivalent optimization problem. Show that the same cannot be said for the Lagrangian dual of (1); specifically, show that scaling the inequalities of (1) changes feasible set and optimal solution of its dual. Conclude that Lagrangian duality is an algebraic transformation, since given two equivalent LPs (same feasible set and objective) represented differently, it yields different dual LPs.

b (2 points). For simplicity, assume $x = \vec{0}$ is a strictly feasible solution of (1) — i.e., the feasible region includes an open ball about the origin.[1] Show that (1) is equivalent, in the sense of having the same feasible set and objective function, to an LP of the following "normalized" form, and has an optimal value $\nu^* \geq 0$.

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \widehat{A}x \preceq \vec{1} \end{array} \tag{3}$$

c (2 points). Using the rules for taking duals, the dual of (3) is the following LP.

$$\begin{array}{ll} \text{minimize} & \vec{1}^T y \\ \text{subject to} & \widehat{A}^T y = c \\ & y \succeq \vec{0} \end{array} \tag{4}$$

Show that (4) and (2) are equivalent up to a simple transformation of the variables, and note that said transformation preserves the optimal value.

d (4 points). Let $P = \left\{x \in \mathbb{R}^n : \widehat{A}x \preceq \vec{1}\right\}$ denote the feasible set of (3) (and therefore also of (1)), and let P° denote its polar. You will show that one can derive tight bounds on ν^* from the polar P°. Specifically, show that if $\frac{1}{\nu}c \in P^\circ$ for some constant $\nu > 0$, then $\nu^* \leq \nu$. Conversely, show that $\frac{1}{\nu^*}c \in P^\circ$.

e (4 points). Recall from class that if $P = \left\{x \in \mathbb{R}^n : \widehat{A}x \preceq \vec{1}\right\}$ is a polytope, then its polar P° is equal to the convex hull of the rows of the matrix $\widehat{A}$. More generally, when P is a polyhedron its polar P° is the convex hull of $rows(\widehat{A}) \cup \left\{\vec{0}\right\}$ (you are invited to verify this for yourself if curious). Explain how LP (4) — and by (c), also LP (2) — can be interpreted as finding the tightest upperbound on ν^* implied by the polar, in the sense of (d). Conclude that Lagrangian duality is an algebraic analog of polar duality, which is a purely geometric relationship between convex sets.

part a

Scaling the inequalities of [1], we get

$$\text{maximize } c^T x$$

$$\text{subject to } \alpha A x \preccurlyeq \alpha b$$

The dual of it is

$$\text{minimize } \alpha b^T y$$
$$\text{subject to } \alpha A^T = c \equiv A^T = \frac{x}{\alpha}$$
$$y \succcurlyeq 0$$

which is not the same as the duality we have originally.

part b

$\vec{0}$ is a strictly feasible solution and hence $A\vec{0} \prec \vec{b}$ which implies $\vec{b} \succ \vec{0}$, i.e $\vec{b}$ has each element strictly greater than 0.

The objectives of (3) and (1) are the same.

So all we need to show is that the feasible set of (1) and (3) are the same.

Let the feasible set of (1) be F_1 and that of (3) be F_3.

$$F_1 \equiv A\vec{x} \preceq \vec{b}$$

$$\implies A_{[i,:]}\vec{x} \leq b_i \; \forall \; i = \{1, 2, 3, ...\}$$

$$\implies \frac{1}{b_i} A_{[i,:]}\vec{x} \leq 1 \; \forall \; i = \{1, 2, 3, ...\}$$

$$\implies \hat{A}_{[i,:]}\vec{x} \leq 1 \; where \; \hat{A}_{[i,:]} = A_{[i,:]}$$

$$\implies \hat{A}\vec{x} \preceq 1 \equiv F_3 \quad \blacksquare$$

part c

Define $\vec{z} = \vec{b} \odot \vec{y}$ or $z_i = b_i y_i$ where $\odot$ is the Hadamard Product of two matrices/vectors.

We just need to show that the objectives and the constraints/feasible regions do not change by the above change of variables.

Clearly $b^T y = \sum b_i y_i = 1^T z$

The objectives are the same

$$\text{minimize } b^T y \equiv \text{minimize } \sum_i b_i y_i = \sum_i z_i = 1^T z$$

Clearly $A^T y = c \implies \hat{A}^T z = c$ where $\hat{A}_{[i,:]} = \frac{1}{b} A_{[i,:]}$

So the feasible region (or the constraints) do not change either.

Finally renaming the variable z to y proves the statement in the question.

part d

Definition of polar of a set P is given by $Y = \{y \mid y^T x \leq 1 \; \forall \; x \in P\}$.

Let $g = \frac{1}{\nu}c$ be in the polar P° for some $\nu > 0$

Then, by definition of the polar,

$$g^T x \leq 1 \; \forall \; x \in P$$

$$\implies (\frac{1}{\nu}c)^T x \leq 1 \; \forall \; x \in P$$

$$\implies \frac{1}{\nu}c^T x \leq 1 \; \forall \; x \in P$$

$$\implies c^T x \leq 1 * \nu \; \forall \; x \in P \quad \{\because \nu > 0\}$$

$$\implies c^T x \leq \nu \; \forall \; x \in P$$

Let x^* be the optimimum solution and the optimum objective value be ν^*, i.e $c^T x^* = \nu^*$.

Since $c^T x \leq 0$ for any $x \in P$ and since $x^* \in P$, definitely $c^T x^* \leq \nu$ which implies $v^* \leq \nu$.

Hence ν is an upper bound on ν^* whenever $\frac{1}{\nu}c$ belongs to the polar P°.

Now we will show that $h = \frac{1}{\nu^*}c$ is in the polar P°.

We know that x^* maximizes the objective $c^T x$, i.e $c^T x^* = \nu^*$ is the maximum value that can be attained for the objective.

So $\nu^* = c^T x^* \geq c^T x$ for any arbitrary $x \in P$ and hence $1 \geq \frac{c^T x}{\nu^*}$ since we assume $\nu^* > 0$ as otherwise $\frac{c}{\nu^*}$ is not well defined in part d.

For any arbitrary $x \in P$,

$$h^T x = (\frac{1}{\nu^*}c)^T x = (\frac{1}{\nu^*})c^T x = \frac{c^T x}{\nu^*} \leq 1$$

$$\implies h^T x \leq 1 \; \forall \; x \in P \text{ and hence } h \text{ is in the polar} \quad \blacksquare$$

part e

> Things we know -
>
> 1. Polar P° is the convex combination of the rows of $\hat{A}$
> 2. $\hat{A}^T z$ is the convex compination of the rows of $\hat{A}^T$