Six Vertex Monotonicity

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§1 Introduction

The result to be proven is:

Theorem (Proposition 4.1)

Let $a, b, c > 0$ be weights. Then the following statement is true iff $a, b \leq c$:

Let Λ be a rectangular domain, and let $r, r' \geq 0$ and $m \geq 1$ be integers. Suppose that $\mathbf{f}, \mathbf{g}, \mathbf{f}', \mathbf{g}'$ enter and exit Λ at vertices $u_0, u_{m+1}, u_0', u_{m+1}'$ and $w_0, w_{m+1}, w_0', w_{m+1}'$, respectively. Further let $(\mathbf{u}, \mathbf{w}) and (\mathbf{u}', \mathbf{w}')$ denote two pairs of m-tuples of vertices on Λ that each constitutes an admissible boundary condition on Λ. Assume that $\mathfrak{E}_{\mathbf{f},\mathbf{g}}^{\mathbf{u},\mathbf{w};r}$ and $\mathfrak{E}_{\mathbf{f}',\mathbf{g}'}^{\mathbf{u}',\mathbf{w}';r'}$ are both nonempty.

If $\mathbf{f} \leq \mathbf{f}', \mathbf{g} \leq \mathbf{g}', \mathbf{u} \leq \mathbf{u}', \mathbf{w} \leq \mathbf{w}'$, and $r \leq r'$, then it is possible to couple the standard measures (i.e. the Gibbs energy of the state divided by the partition function) on $\mathfrak{E}_{\mathbf{f},\mathbf{g}}^{\mathbf{u},\mathbf{w};r}$ and $\mathfrak{E}_{\mathbf{f}',\mathbf{g}'}^{\mathbf{u}',\mathbf{w}';r'}$ on a common probability space such that the following holds. If the pair $(\mathcal{P}, \mathcal{P}') \in \mathfrak{E}_{\mathbf{f},\mathbf{g}}^{\mathbf{u},\mathbf{w};r} \times \mathfrak{E}_{\mathbf{f}',\mathbf{g}'}^{\mathbf{u}',\mathbf{w}';r'}$ is chosen with respect to the coupled measure, then $\mathcal{P} \leq \mathcal{P}'$ almost surely.

§2 Necessity

We first show that if a or b is less than c, then the statement does not hold, starting with b. We can scale a, b, c so that $a = 1$ (since the Gibbs measure is homogeneous). Then, consider the following path ensembles:

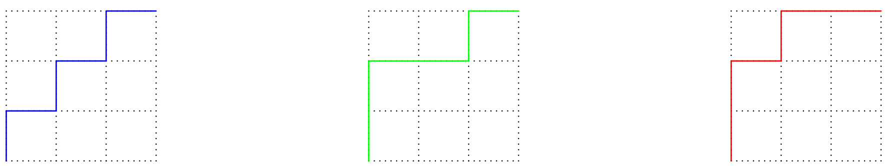


Figure 1: From left to right: $\mathbf{g} = \mathbf{g}'$; $\mathbf{f}'$; $\mathbf{f}$

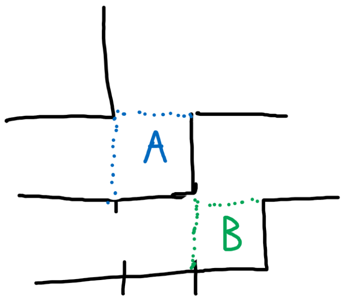
Note that $\mathbf{g}$ and $\mathbf{g}'$ have Gibbs measure c^5, while $\mathbf{f}'$ and $\mathbf{f}$ have measure b^2c^3. Additionally, $\mathbf{g}'$ and $\mathbf{f}'$ are the only paths in between $\mathbf{g}'$ and $\mathbf{f}'$. However, there are four paths between $\mathbf{g}$ and $\mathbf{f}$: those two, $\mathbf{f}'$, and the reflection of $\mathbf{f}'$. So, the partition function for $\mathfrak{E}_{\mathbf{f},\mathbf{g}}^{\mathbf{u},\mathbf{w};r}$ is $3b^2c^3 + c^5$, while the partition function for $\mathfrak{E}_{\mathbf{f}',\mathbf{g}'}^{\mathbf{u}',\mathbf{w}';r'}$ is $b^2c^3 + c^5$. Note that the only two paths in $\mathfrak{E}_{\mathbf{f},\mathbf{g}}^{\mathbf{u},\mathbf{w};r}$ that are greater than or equal to $\mathbf{f}'$ are $\mathbf{f}'$ and $\mathbf{f}$. So, if such a coupling exists, we have that the probability of $\mathbf{f}'$ in $\mathfrak{E}_{\mathbf{f}',\mathbf{g}'}^{\mathbf{u}',\mathbf{w}';r'}$ is at most the sum of the probabilities of $\mathbf{f}'$ and $\mathbf{f}$ in $\mathfrak{E}^{\mathbf{u},\mathbf{w};r}_{\mathbf{f},\mathbf{g}}$, i.e.

$$\frac{b^2c^3}{b^2c^3 + c^5} \leq \frac{2b^2c^3}{3b^2c^3 + c^5}$$

Simplifying, we get

$$3b^2 + c^2 \leq 2b^2 + 2c^2 \implies b \leq c$$

Next, we show that $c \geq a = 1$.



The diagram above shows a path ensemble and two quadrilateral faces. We can "flip" either quadrilateral face (or both) to get a new path ensemble. We define $\mathbf{g} = \mathbf{g}'$ to be the original ensemble, $\mathbf{f}'$ to be the ensemble when we flip just quadrilateral face A, and $\mathbf{f}$ to be the ensemble when we flip both quadrilateral faces. Let the Gibbs measure of the initial ensemble be G. Then the Gibbs measure of $\mathbf{f}'$ is $G \cdot \frac{a^2}{c^2}$. The Gibbs measure when we flip just B, and when we flip both quadrilateral faces, is the same. By the same logic as above, we get the inequality

$$\frac{G \cdot \frac{a^2}{c^2}}{G \cdot \frac{a^2}{c^2} + G} \leq \frac{2G \cdot \frac{a^2}{c^2}}{3G \cdot \frac{a^2}{c^2} + G}$$

Simplifying,

$$\frac{a^2}{c^2} \leq 1 \implies a \leq c$$

§3 Sufficiency

Now, assume that $a, b \leq c$, and scale so that $a = 1$ once again. Define an upwards switching and downwards switching the same as in the original paper (i.e. flip a right arrow then up arrow into a up arrow then right arrow). We define the following Markov Chain on $\mathfrak{E}^{\mathbf{u},\mathbf{w};r}_{\mathbf{f},\mathbf{g}}$:

Definition 3.1. For each quadrilateral face F between $\mathbf{f}$ and $\mathbf{g}$, assign two exponential clocks of unit rate. Call the first the upwards clock, and the second the downwards clock. When the upwards clock rings on a face F where an upwards switching operation is possible, sample a uniform random variable between 0 and 1, inclusive. If the value sampled is less than the ratio of the Gibbs measures of the new and current path ensembles, then perform the upwards switching operation. Otherwise, do nothing. The downwards clocks are used analogously. These are called the Glauber dynamics on $\mathfrak{E}^{\mathbf{u},\mathbf{w};r}_{\mathbf{f},\mathbf{g}}$.

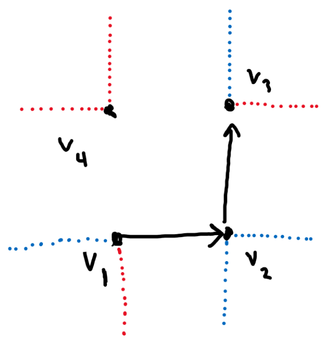
Note that this is identical to running the Metropolis-Hastings algorithm on $\mathfrak{E}^{\mathbf{u},\mathbf{w};r}_{\mathbf{f},\mathbf{g}}$, and so this Markov chain converges to the uniform stationary distribution on $\mathfrak{E}^{\mathbf{u},\mathbf{w};r}_{\mathbf{f},\mathbf{g}}$.

Lemma 3.2

Let $t < 0$, $P \in \mathfrak{C}^{\mathbf{u},\mathbf{w};r}_{\mathbf{f},\mathbf{g}}$, $P' \in \mathfrak{C}^{\mathbf{u}',\mathbf{w}';r'}_{\mathbf{f}',\mathbf{g}'}$, with $P \leq P'$. Run the Glauber dynamics on both P and P' for time t. Then the image of P is less than or equal to the image of P', almost surely.

Proof. It suffices to show the claim for a single clock ring. Furthermore, we will only show the upwards switching case, as the downwards case is analogous. Let the clock that rings be at quadrilateral face F. Let v_1, v_2, v_3, v_4 be the vertices of F, with v_1 being the southwest corner and continuing counterclockwise. Also let S, S' denote the images of P, P' after the upwards switching operation.

Since $P \leq P'$, the only way that P' can "pass" P is if they both have paths $\mathbf{p}'_k \in P', \mathbf{p}_k \in P$ that contain the arrows (v_1, v_2) and (v_2, v_3), but not the other two arrows of the face. Furthermore, at that face, P' needs to successfully switch, while P does not switch. Due to the nature of the uniform random variable, this can only happen if the ratio of the Gibbs measures of S' and P' is strictly greater than that ratio for S and P. Now, consider the below diagram:



An upwards switching of F only changes the vertex types of the four vertices of F. We therefore consider each vertex separately. For $i = 1, 2, 3, 4$, define the **switching ratios** s_i, s'_i to be the ratio of the vertex type (a, b, c) of v_i before and after switching P and P', respectively. For example, if v_1 switches from a b to a c, then $s_1 = \frac{c}{b}$. It then suffices to prove that $s_1 s_2 s_3 s_4 \geq s'_1 s'_2 s'_3 s'_4$.

At v_1, the path can enter the vertex from either the red dotted line or the blue dotted line. If both P and P' take the same entrance path, then $s_1 = s'_1$. Otherwise, since $P \leq P'$, P must take the blue path, while P' takes the red path. Then $s_1 = \frac{c}{b}$, while $s'_1 = \frac{b}{c}$. Thus, since $c \geq b$, in all cases, $s_1 \geq s'_1$. The exact same logic applies to the exit arrow from v_3.

At v_2, any path can either contain both the blue lines or not contain either of them. If P and P' are identical at v_2, then $s_2 = s'_2$. Otherwise, since $P \leq P'$, the only way that they can differ is if P contains the blue lines while P' does not. Then, $s_2 = \frac{c}{a}$ and $s'_2 = \frac{a}{c}$. Since $c \geq a$, in all cases, $s_2 \geq s'_2$. The same logic applies to the red lines at v_4.

Thus, since $s_i \geq s'_i$ for all i, and all of them are positive, $s_1 s_2 s_3 s_4 \geq s'_1 s'_2 s'_3 s'_4$ as desired. Therefore, P' can never pass P on an upward switching, so monotonicity is preserved. $\square$

Now, we can finish the proof. Given two probability spaces $\mathfrak{C}^{\mathbf{u},\mathbf{w};r}_{\mathbf{f},\mathbf{g}}$, $\mathfrak{C}^{\mathbf{u}',\mathbf{w}';r'}_{\mathbf{f}',\mathbf{g}'}$ with $\mathbf{f} \leq \mathbf{f}', \mathbf{g} \leq \mathbf{g}', \mathbf{u} \leq \mathbf{u}', \mathbf{w} \leq \mathbf{w}'$, and $r \leq r'$, pick paths $D \in \mathfrak{C}^{\mathbf{u},\mathbf{w};r}_{\mathbf{f},\mathbf{g}}$, $D' \in \mathfrak{C}^{\mathbf{u}',\mathbf{w}';r'}_{\mathbf{f}',\mathbf{g}'}$, such that $D \leq D'$ (it is easy to verify this can be done), and run the Glauber dynamics on both spaces simultaneously. Since the coupled space is finite, this process will reach a limit point. The distribution of this limit point is thus a coupling of the two spaces obeying monotonicity, proving the result.