COMP 211-212

Fall 2014 and Spring 2015

Write code

Write beautiful code

Two notions of beauty

Structure: code communicates an idea

Efficiency: code instructs a computer

Two notions of beauty

Structure: code communicates an idea

logic : COMP 321, COMP 360-1

Efficiency: code instructs a computer

Two notions of beauty

Structure: code communicates an idea

logic : COMP 321, COMP 360-1

Efficiency: code instructs a computer

combinatorics: MATH 228, COMP 312, COMP 301

write code

write code

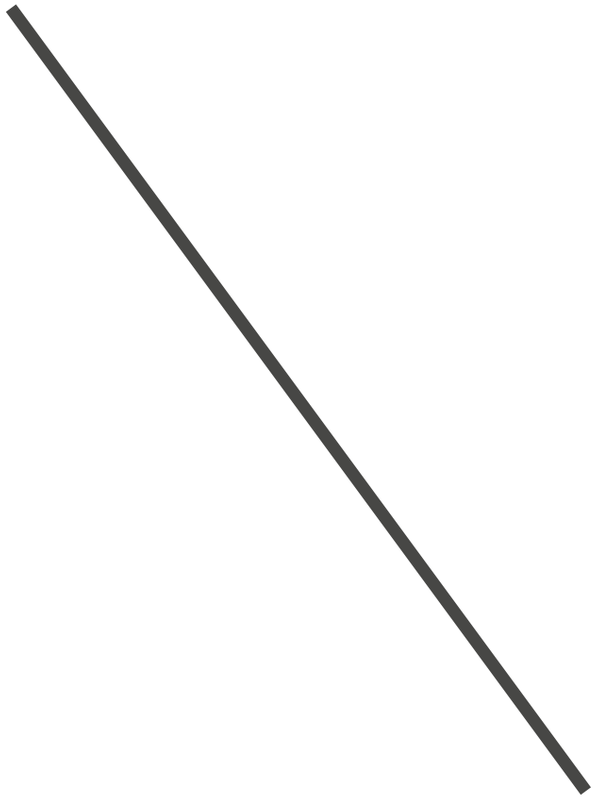
reason about
efficiency

write code

reason about behavior

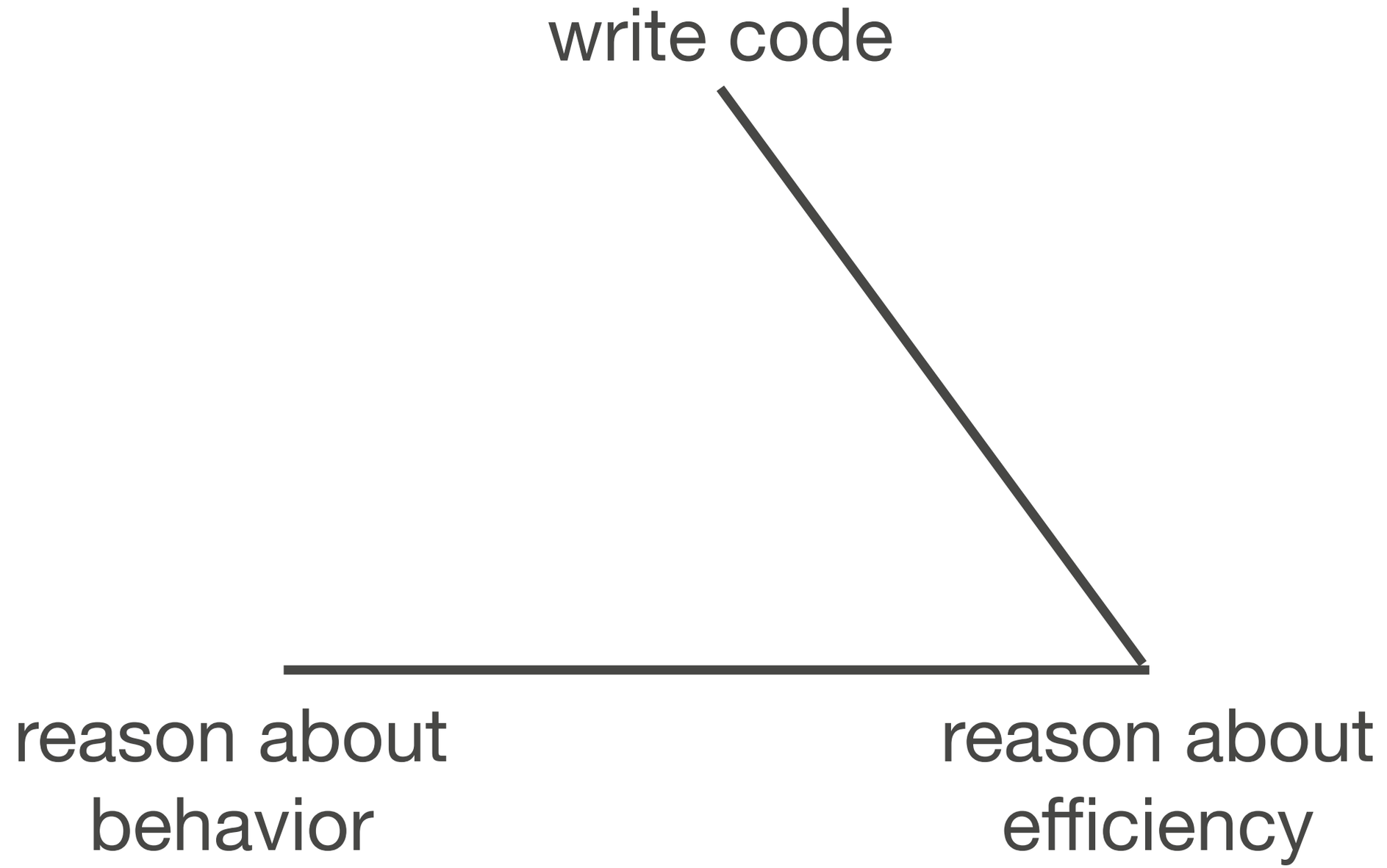
reason about efficiency

write code



reason about behavior

reason about efficiency


write code
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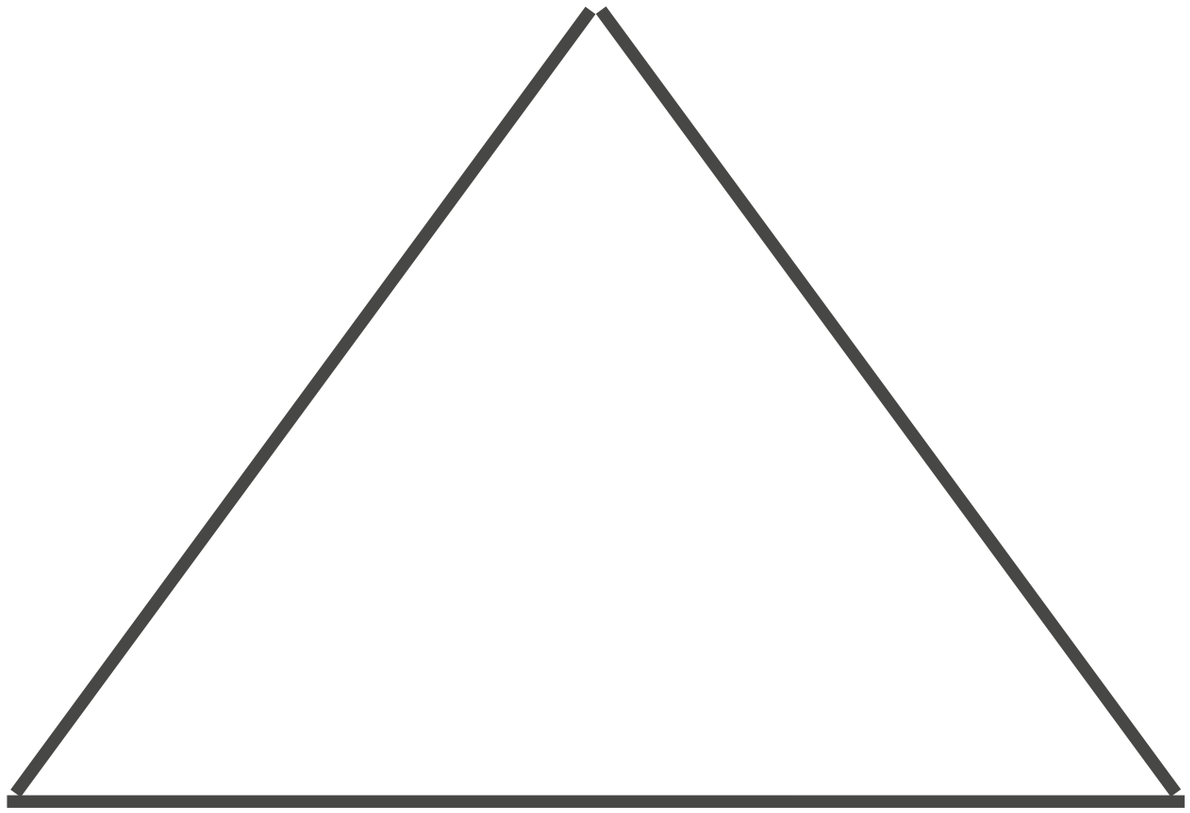

reason about behavior

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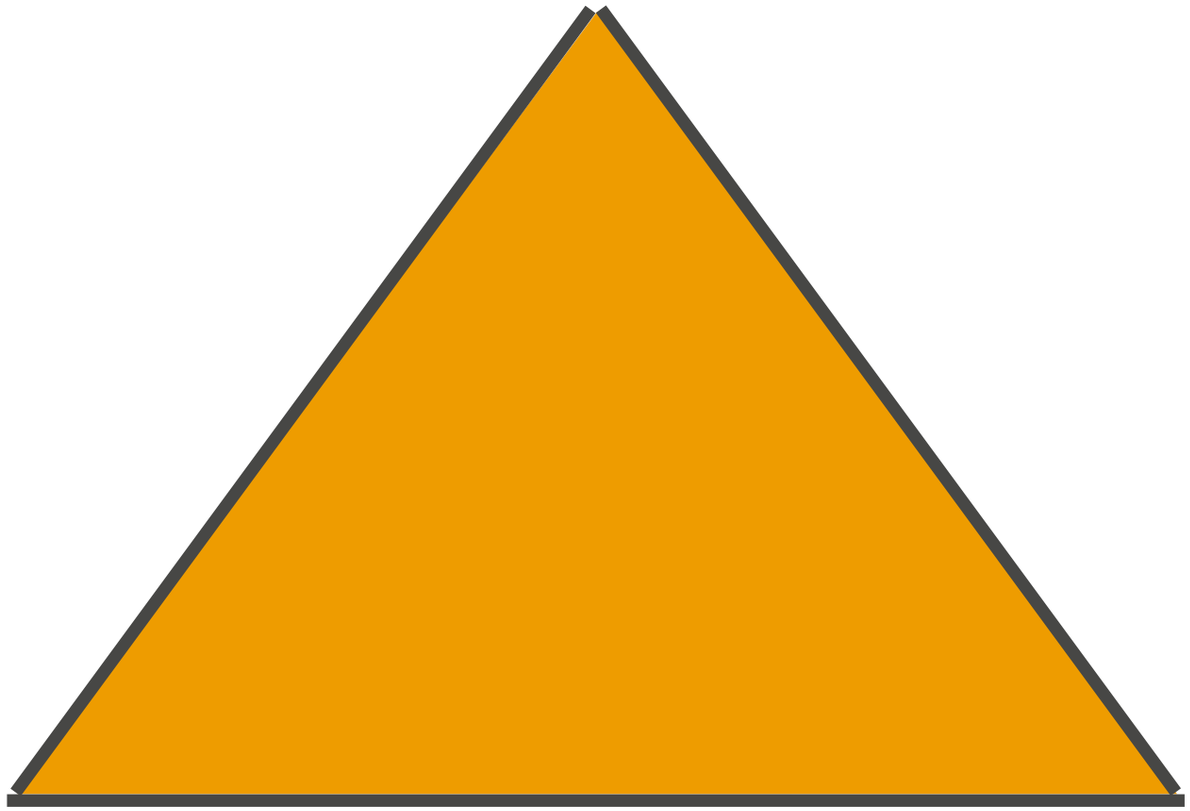

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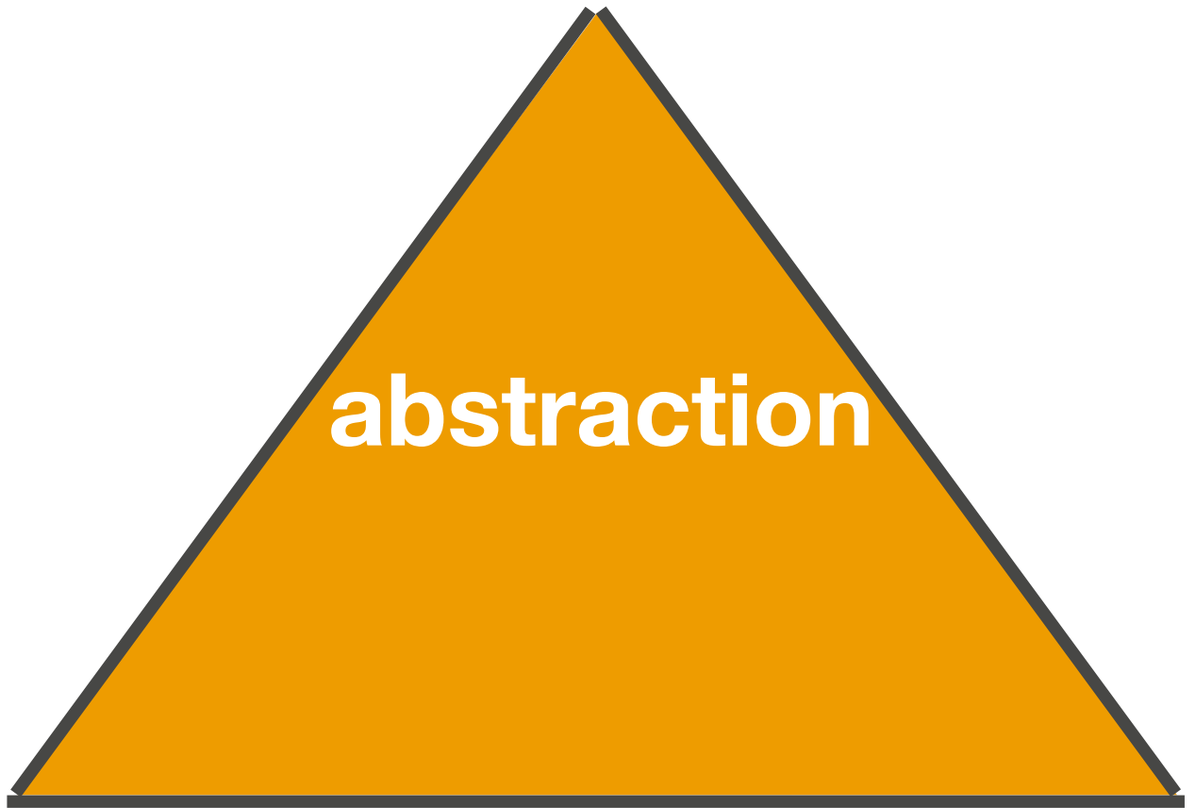

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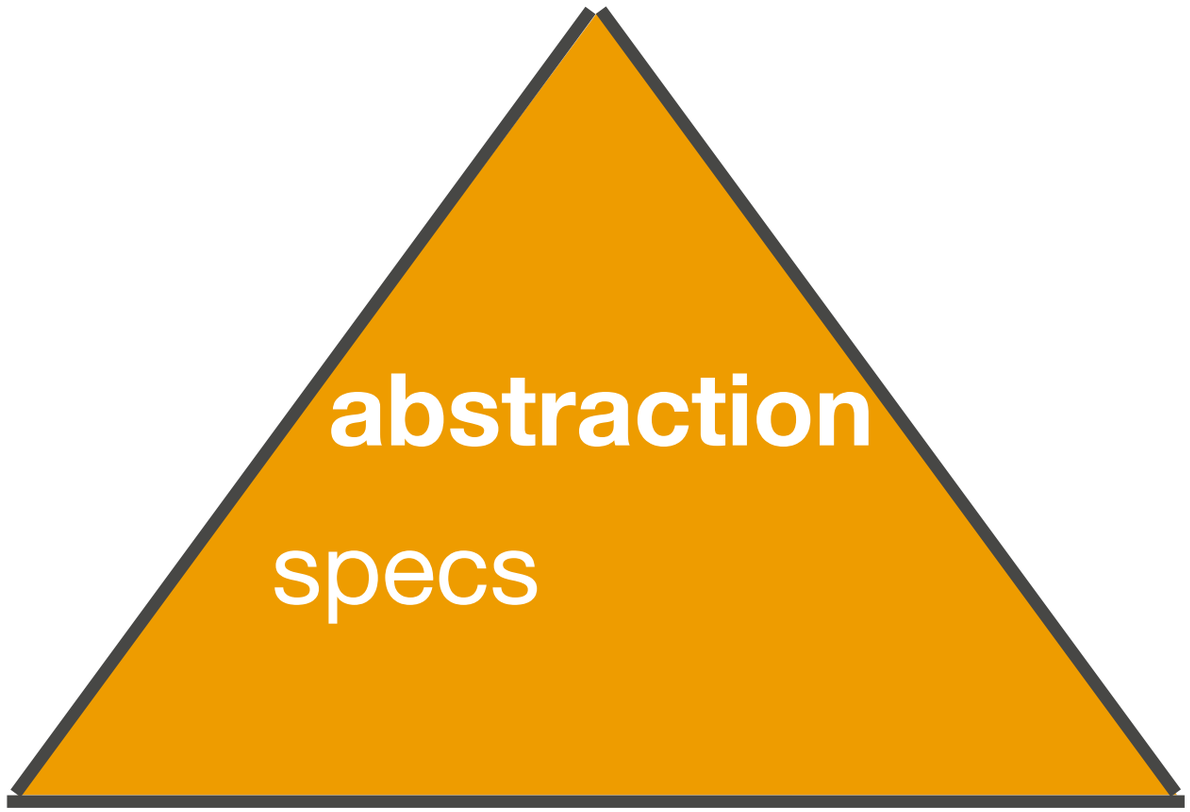

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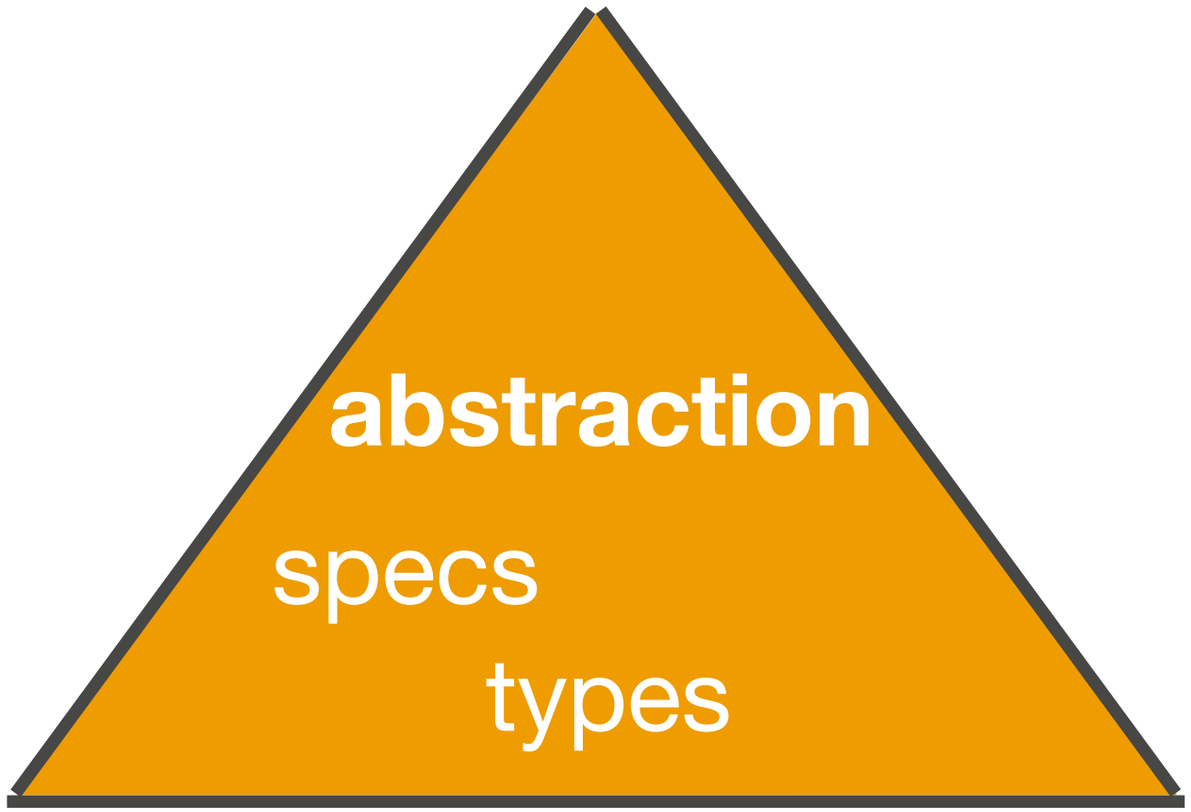

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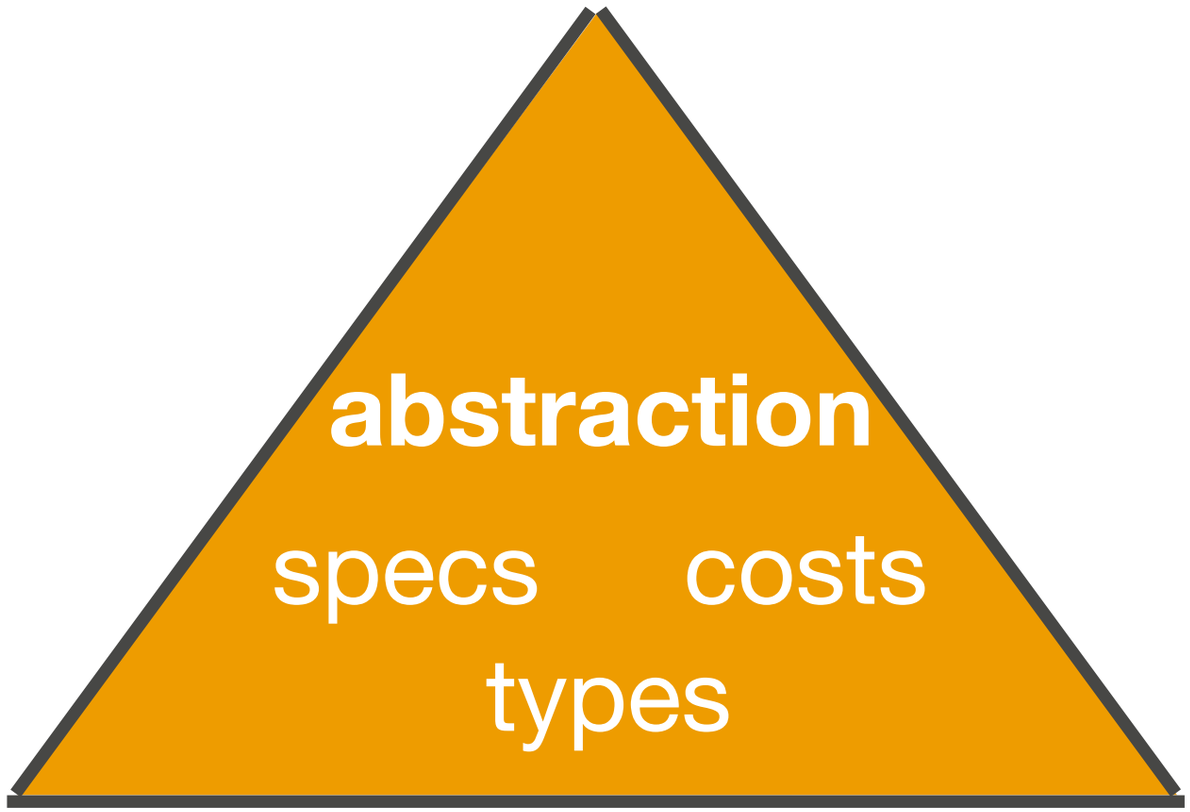


reason about behavior

reason about efficiency

Specifications

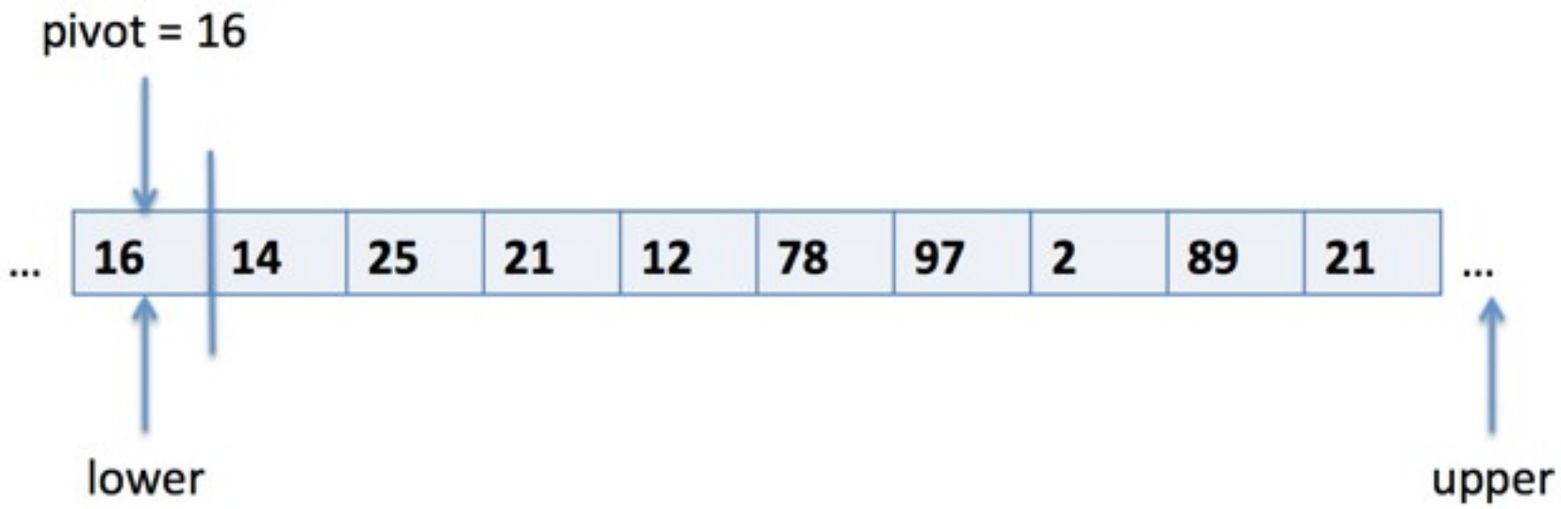
contracts structure code

- pre-conditions, post-conditions
- loop invariants

In-place quicksort

- Motivated by space and time efficiency
- Requires specification and verification to communicate an idea

In-place quicksort



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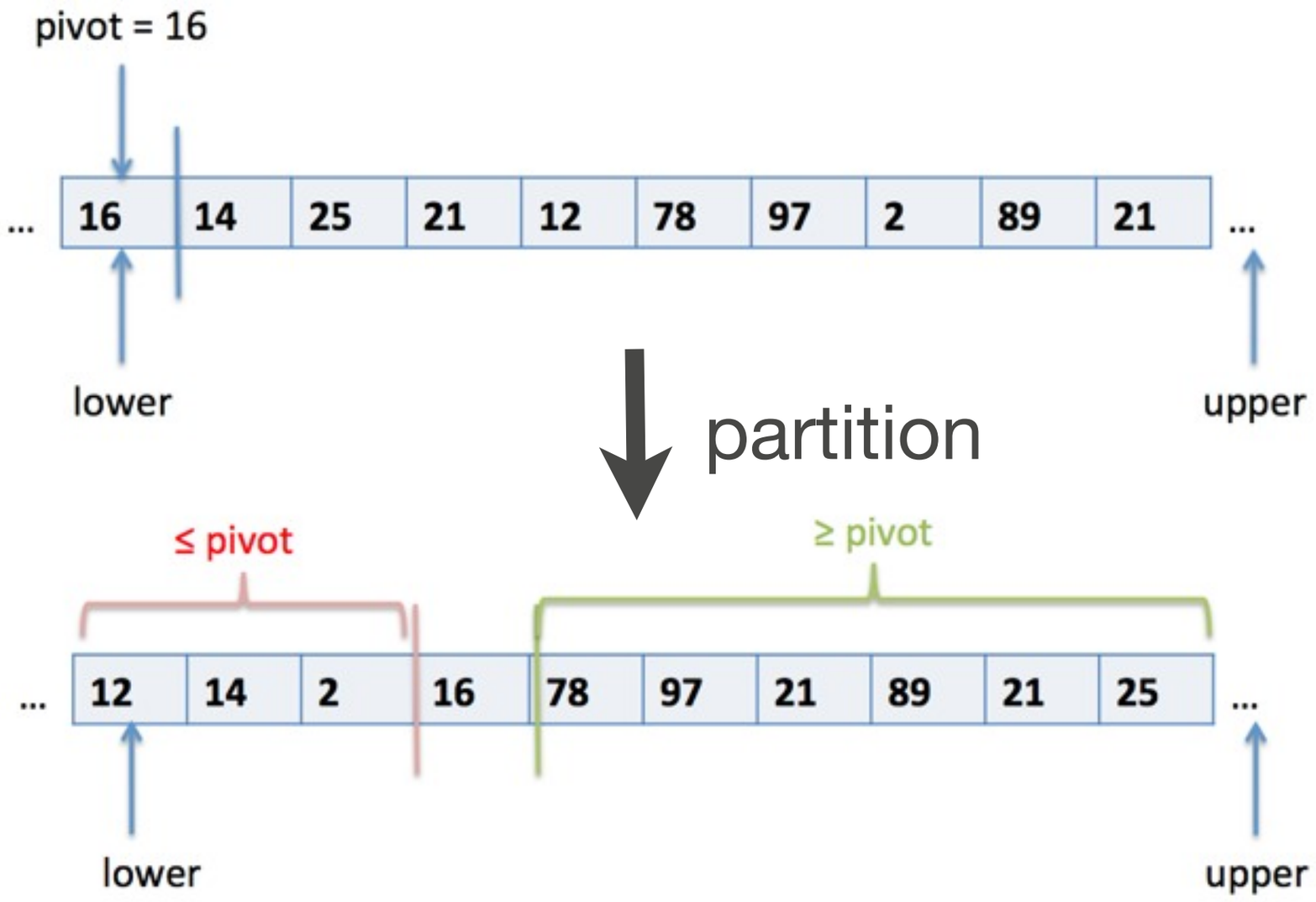

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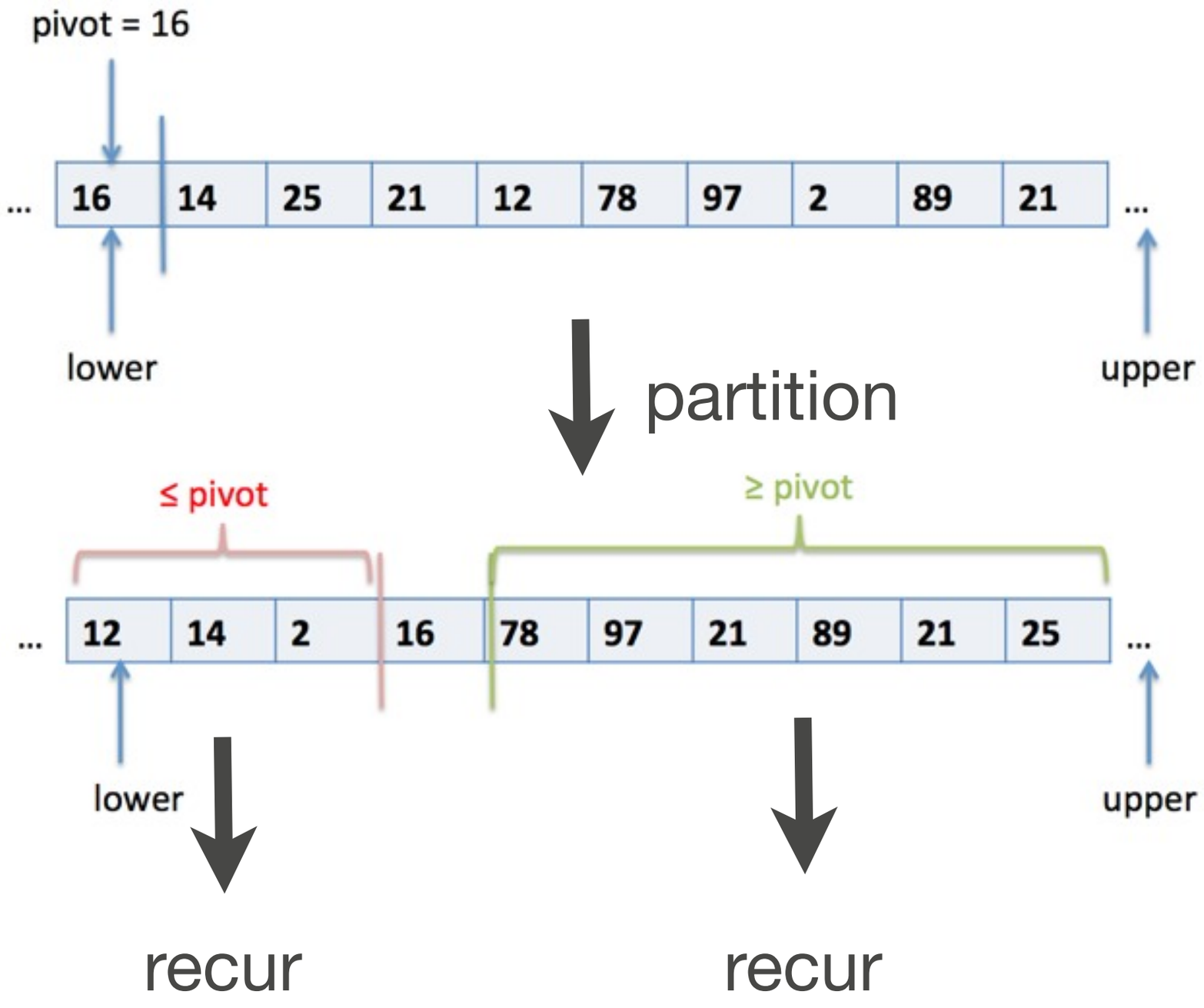

```
int partition(int[] A, int lower, int upper)
{
  int left = lower+1;
  int right = upper;

  while (left < right)
    {
      if (A[left] <= A[lower]) {
        left++;
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O(upper-lower) time

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why does it do that?

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//@requires 0 <= lower < upper <= \length(A);
//@ensures lower <= \result < upper;
//@ensures A[lower, \result)
//          <= A[\result]
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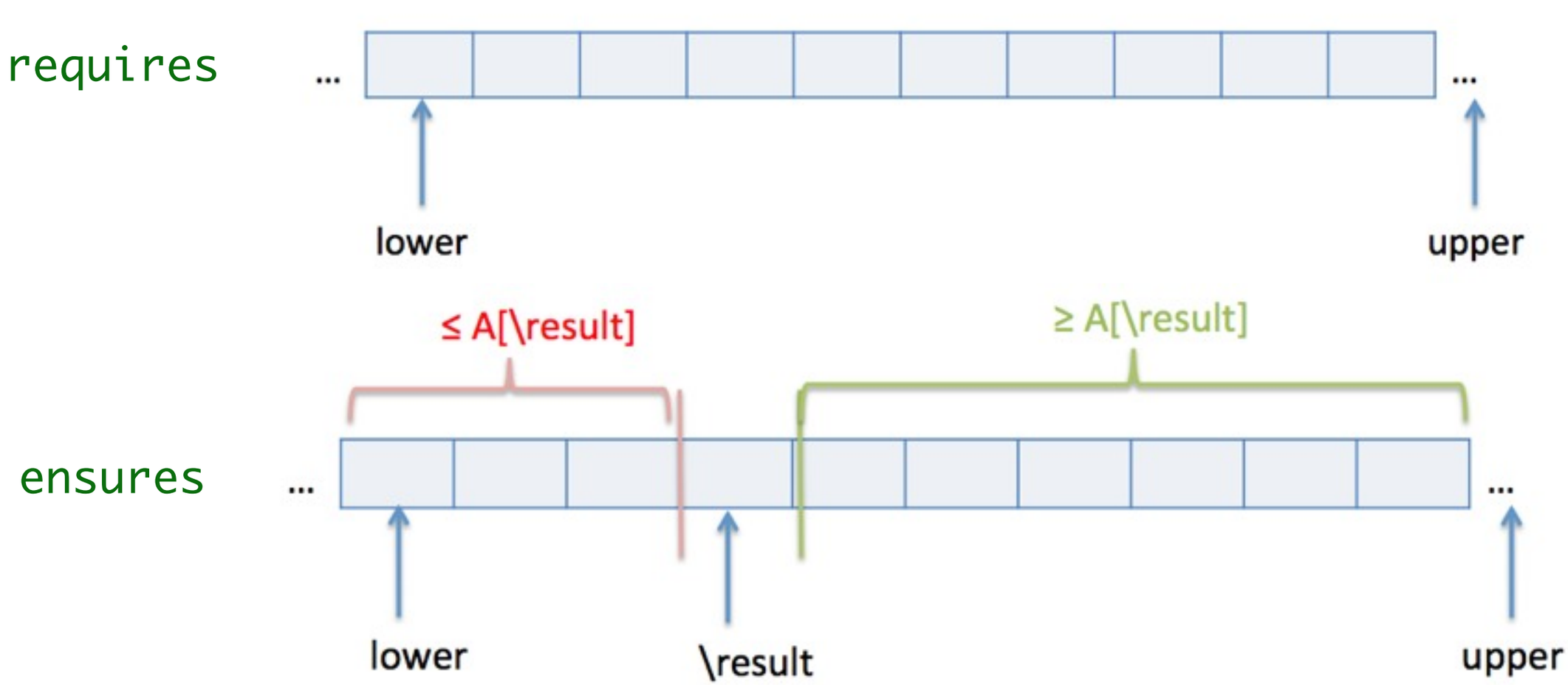

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```

```
//@loop_invariant lower < left <= right <= upper;
//@loop_invariant     A[lower+1,left)
//                 <= A[lower]
//                 <= A[right,upper);
```

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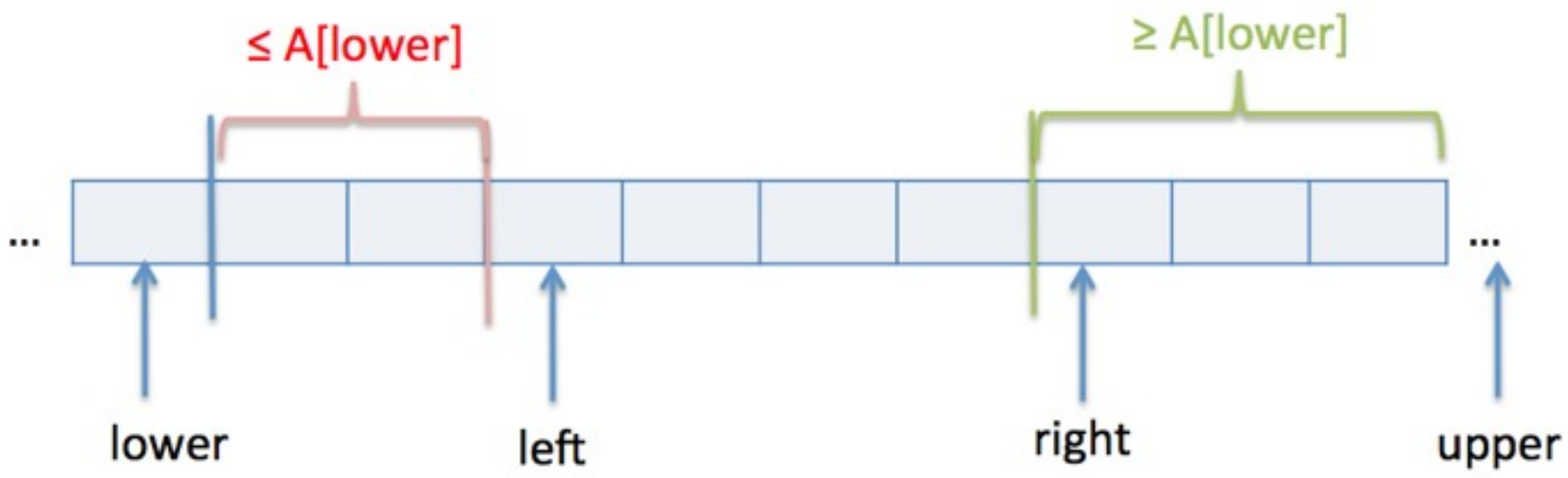

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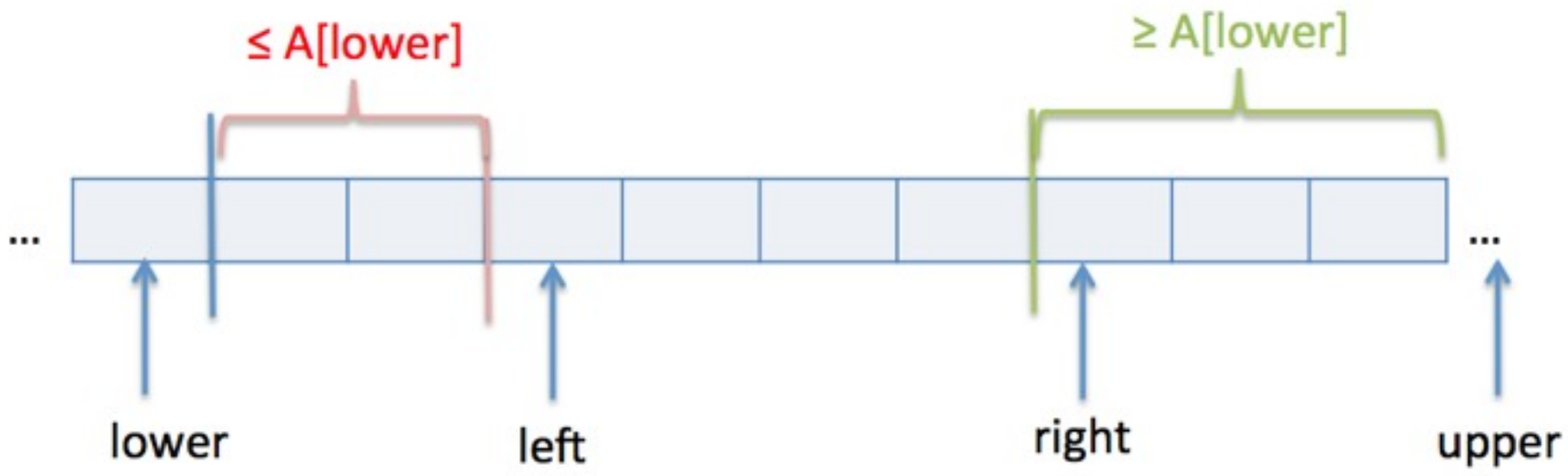
purpose of a test is to establish a fact



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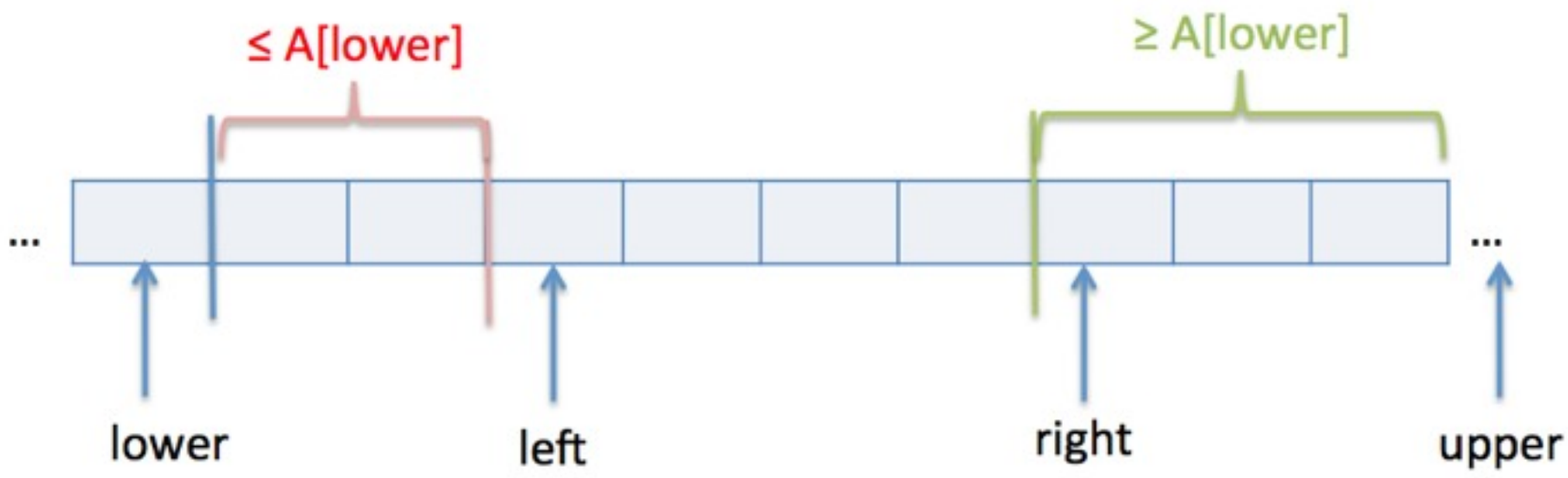
justifies expanding left



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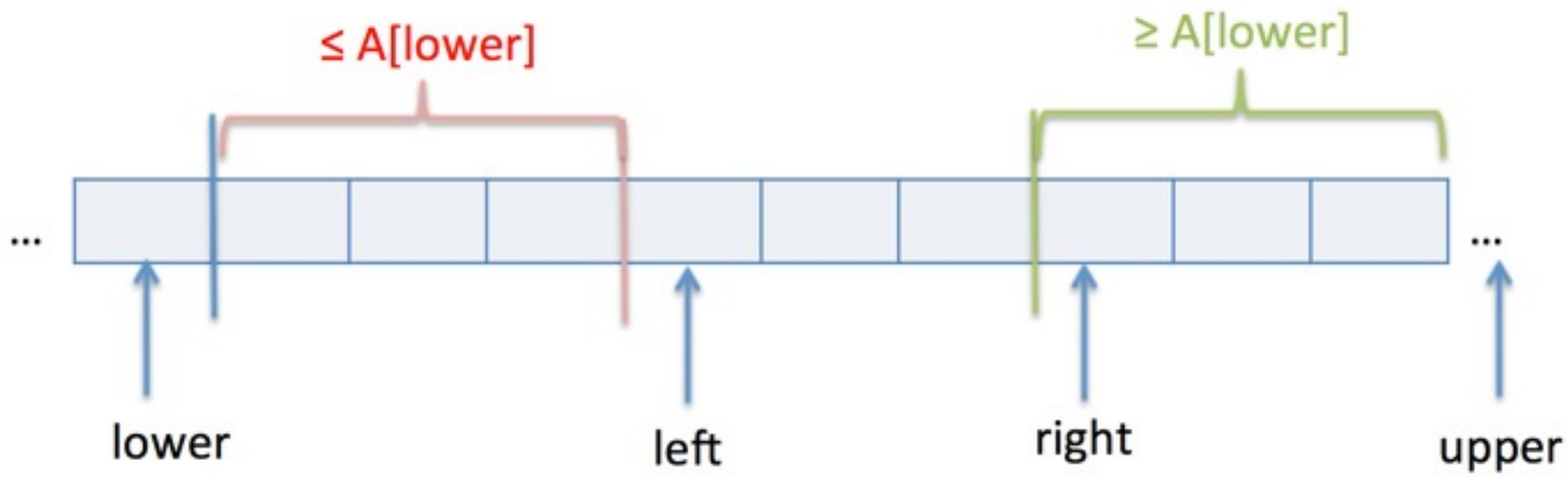

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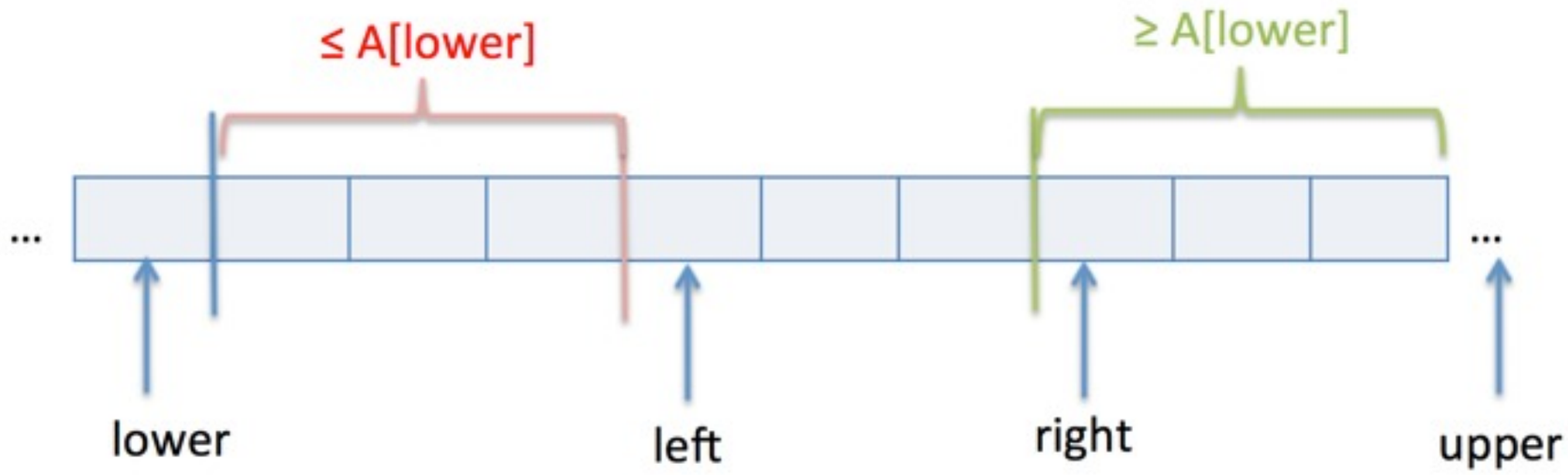
test and swap
justify
expanding right



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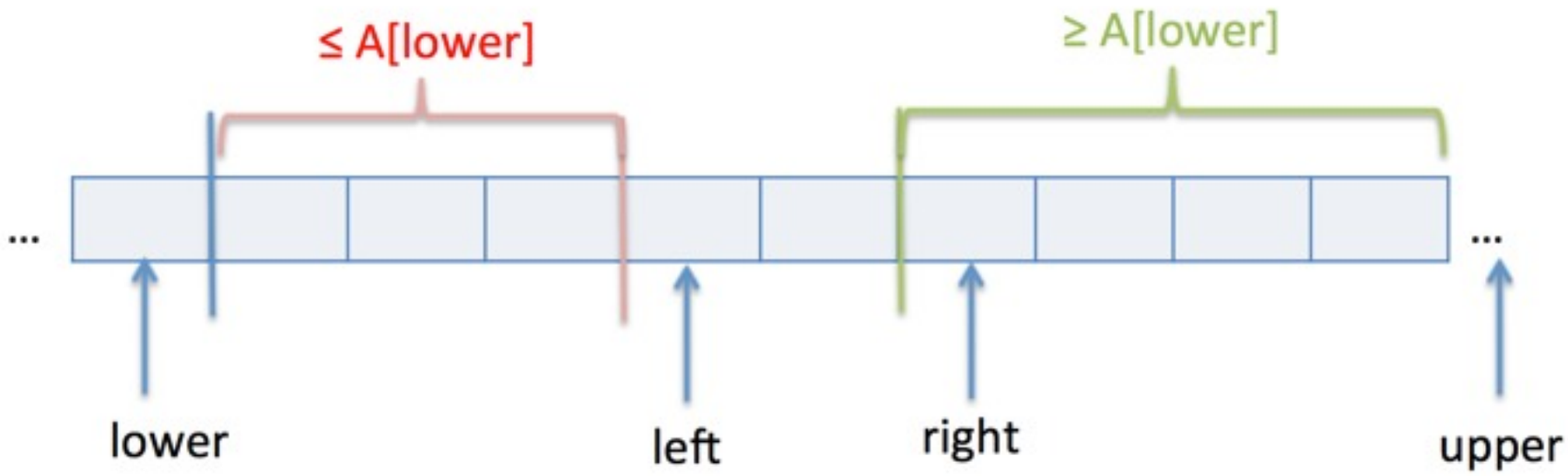

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when loop exits, invariant is true and loop test is false

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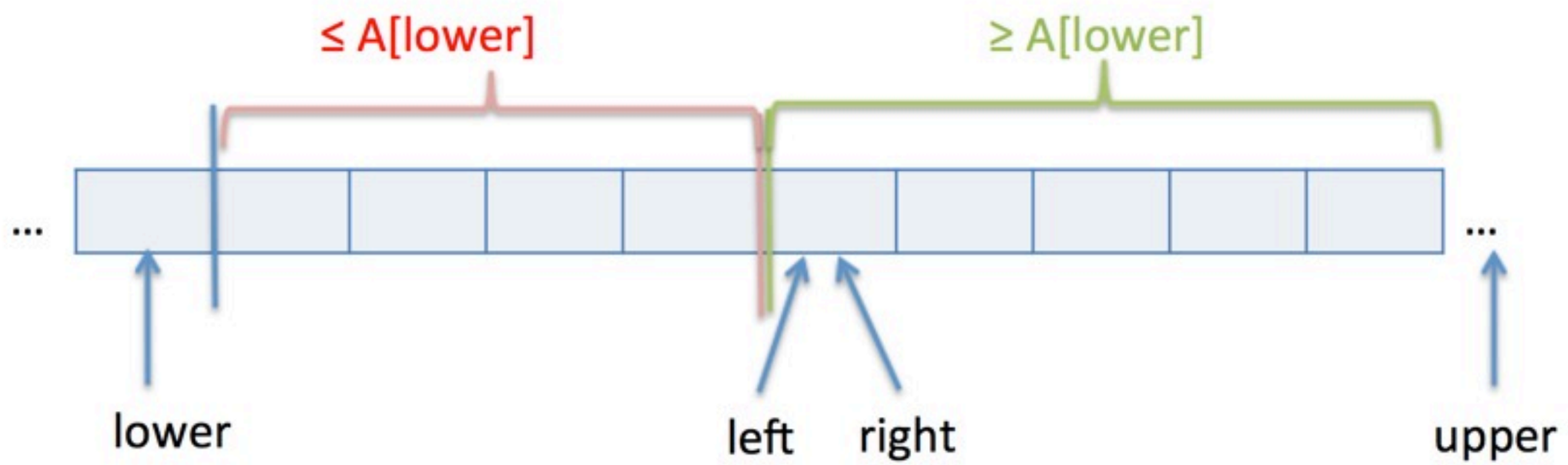

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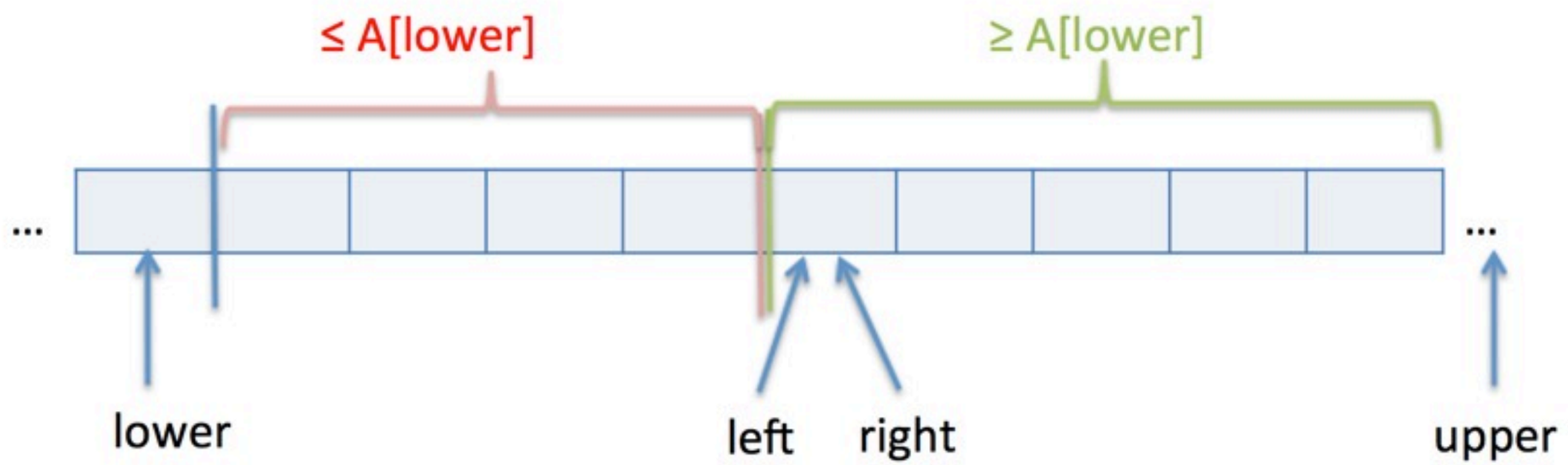
swap makes postcondition true



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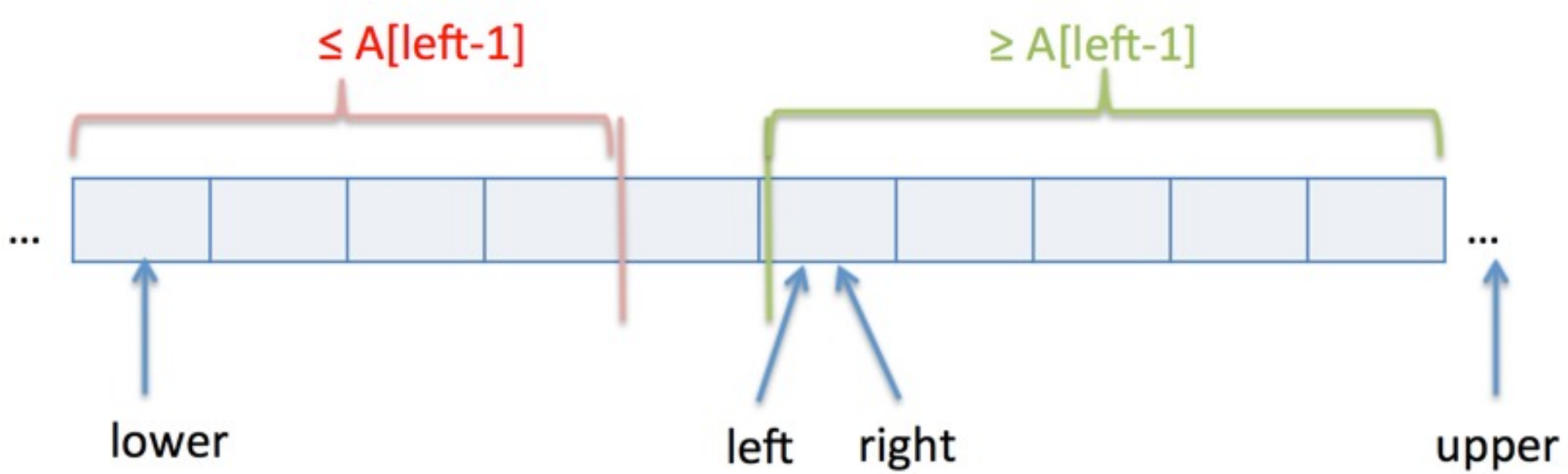
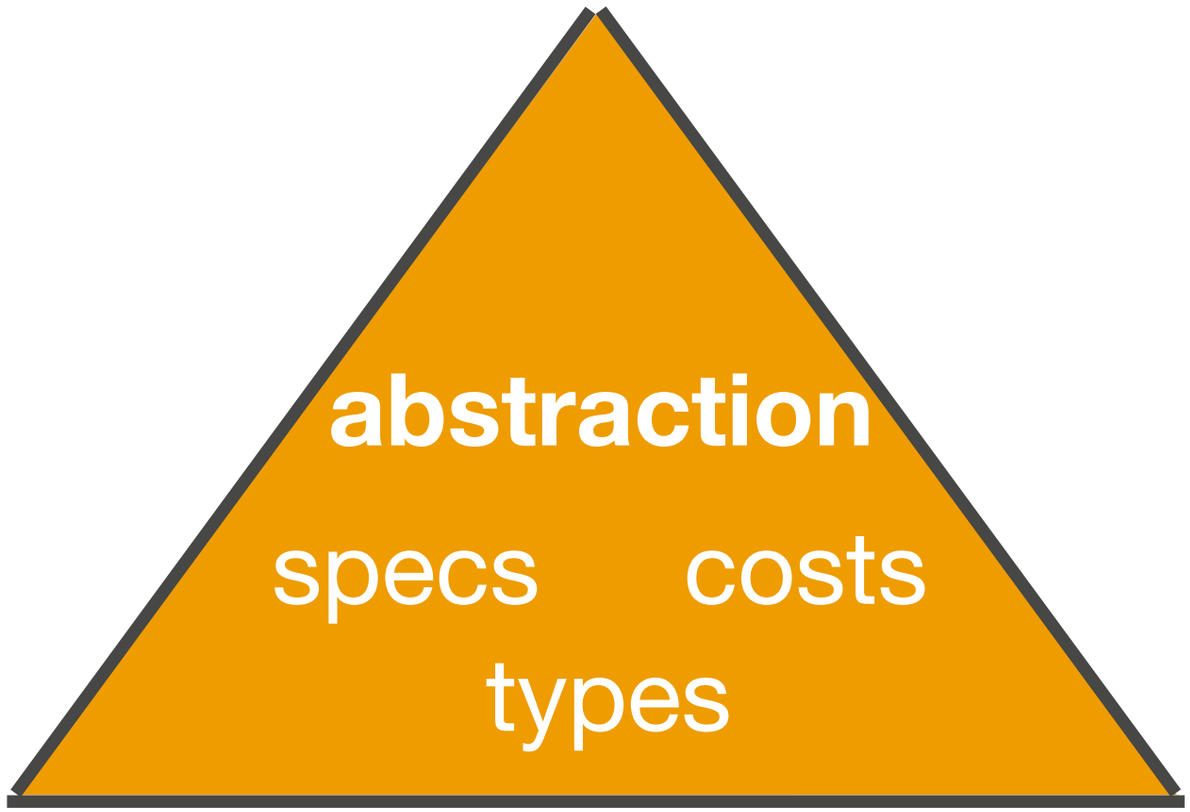


write code



reason about behavior

reason about efficiency

Types

Intro: how do I make something of that type?

Elim: how do I use something of that type?

Types structure code

```
fun f (t : tree) =
  case t of
    Empty => …
  | Leaf x => …
  | Node(l,r) => … f l … f r …
```

Types structure reasoning

Theorem: For all t:tree, …

Proof: structural induction on t.

Case for Empty:

Case for Leaf x:

Case for Node(L,R):
    IH on L:
    IH on R:

Beyond requires/ensures

Theorem: `(map g) o (map f) = map (g o f)`

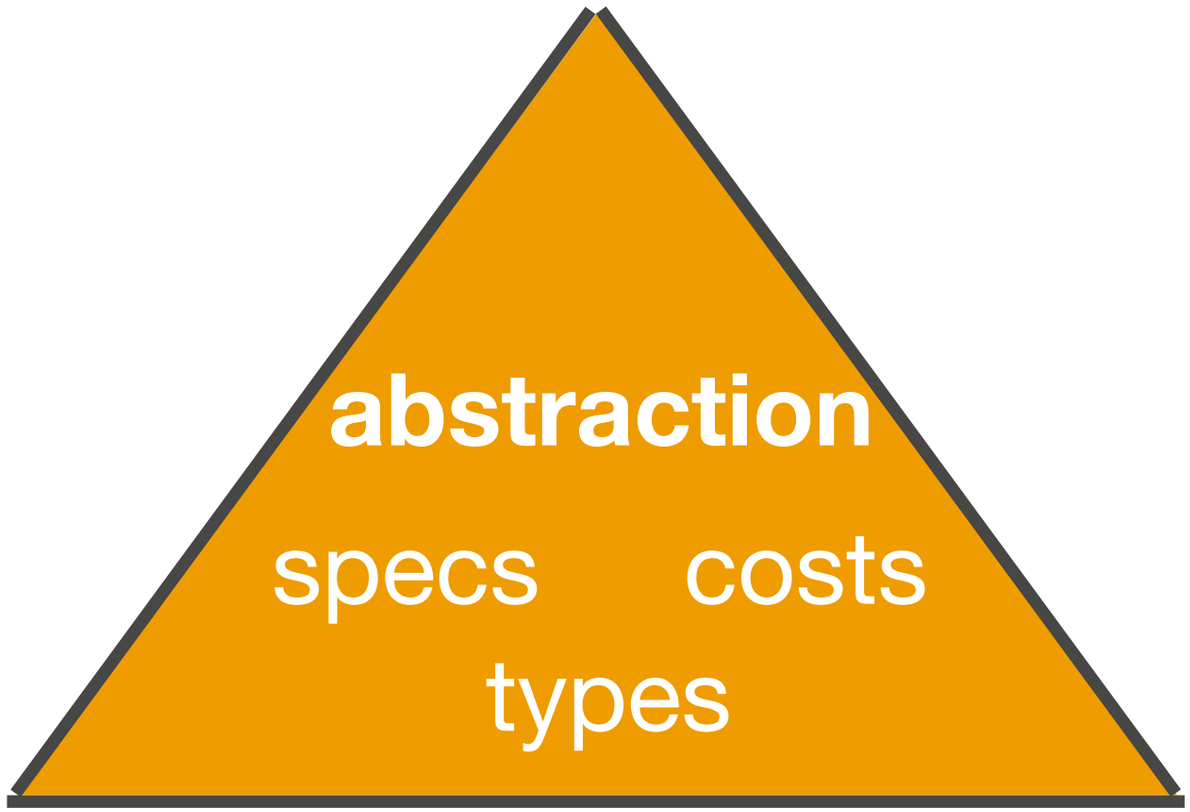
Abstract types

```
signature DICT =
sig
  structure Key : ORDERED
  type 'v dict

  val empty   : 'v dict
  val insert  : 'v dict -> (Key.t * 'v) -> 'v dict

  val lookup  : 'v dict -> Key.t -> 'v option

  val map     : ('a -> 'b) -> 'a dict -> 'b dict

  val merge  : ('v * 'v -> 'v) -> 'v dict * 'v dict -> 'v dict
end
```

Type classes

```
signature ORDERED =
sig
    type t
    val compare : t * t -> order
end
```

write code



reason about behavior

reason about efficiency

Abstract from the machine

Abstract from the machine

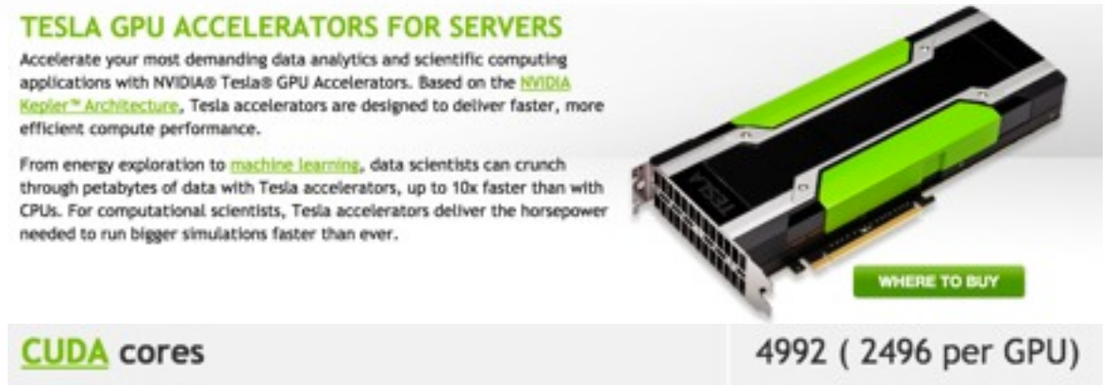
garbage collection :: RAM

Abstract from the machine

garbage collection :: RAM

scheduling :: CPUs

Andrew Chien, 2008



Parallel thinking

required for current and foreseeable platforms

Understand dependencies in an algorithm

Evaluate sequential and parallel efficiency

Parallel thinking

Parallel thinking

Say **what to compute** using
transformations of persistent data structures

Parallel thinking

Say **what to compute** using
transformations of persistent data structures

Analyze **how long it will take** using
language-based cost semantics [Blelloch et al.]

Cost abstraction [Blelloch and Greiner]

Code you write

- what to do
- abstract cost

Compiled code

- scheduled resources
- concrete cost

Cost abstraction [Blelloch and Greiner]

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Bounded implementation

Cost abstraction [Blelloch and Greiner]

Code you write

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Bounded implementation

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time

Bounded implementation

Cost abstraction [Blelloch and Greiner]

Code you write

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time

space

Bounded implementation

Cost abstraction [Blelloch and Greiner]

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space

I/O complexity

Bounded implementation

Cost abstraction [Blelloch and Greiner]

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I/O complexity

parallelism

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Cost abstraction [Blelloch and Greiner]

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processor speed

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Cost abstraction [Blelloch and Greiner]

Code you write

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time

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I/O complexity

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cache policy

Cost abstraction [Blelloch and Greiner]

Code you write

- what to do
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Compiled code

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time

space

I/O complexity

parallelism

Bounded implementation

processor speed

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cache policy

number of processors and scheduling policy

Abstract Costs

Work is usual sequential complexity

Span is length of longest dependency

$$\frac{e_1 \Downarrow_{s_1}^{w_1} v_1 \quad e_2 \Downarrow_{s_2}^{w_2} v_2}{(e_1, e_2) \Downarrow_{\max(s_1,s_2)}^{w_1+w_2} (v_1, v_2)}$$

Abstract Costs

Work is usual sequential complexity

Span is length of longest dependency

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COMP 321

Bounded Implementation

[Brent ’74]

work and *span* → time on *p* processors

$O(\max(work/p, span))$

- Ideally, each processor does work at each step
- Can’t beat the length of the longest dependency

Bounded Implementation

[Brent '74]

work and *span* → time on *p* processors

$O(\max(work/p, span))$

- Ideally, each processor does work at each step
- Can't beat the length of the longest dependency

Parallelizability ratio := *work*/*span*

```
fun mergesort (l : int list) : int list =
    case l of
        [] => []
      | [x] => [x]
      | _ => let val (pile1,pile2) = split l
             in
                 merge (mergesort pile1,
                        mergesort pile2)
             end
```

Work is O(n log n)

[7,1,3,6,8,4,2,5]

Work is O(n log n)

[7,1,3,6,8,4,2,5]

[7,1,3,6] [8,4,2,5]

Work is O(n log n)

[7,1,3,6,8,4,2,5]

[7,1,3,6] [8,4,2,5]

[7,1] [3,6] [8,4] [2,5]

Work is O(n log n)

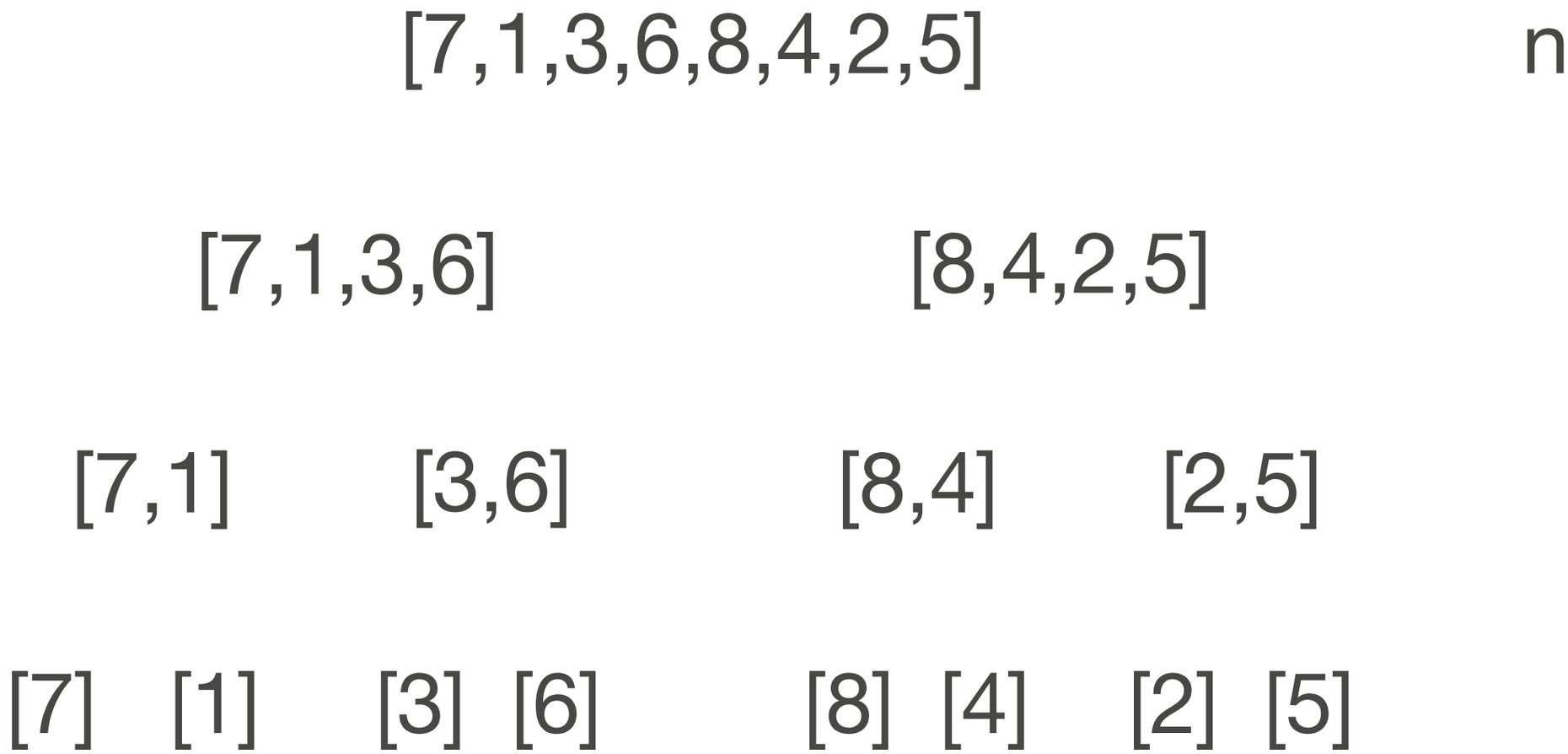
[7,1,3,6,8,4,2,5]

[7,1,3,6] [8,4,2,5]

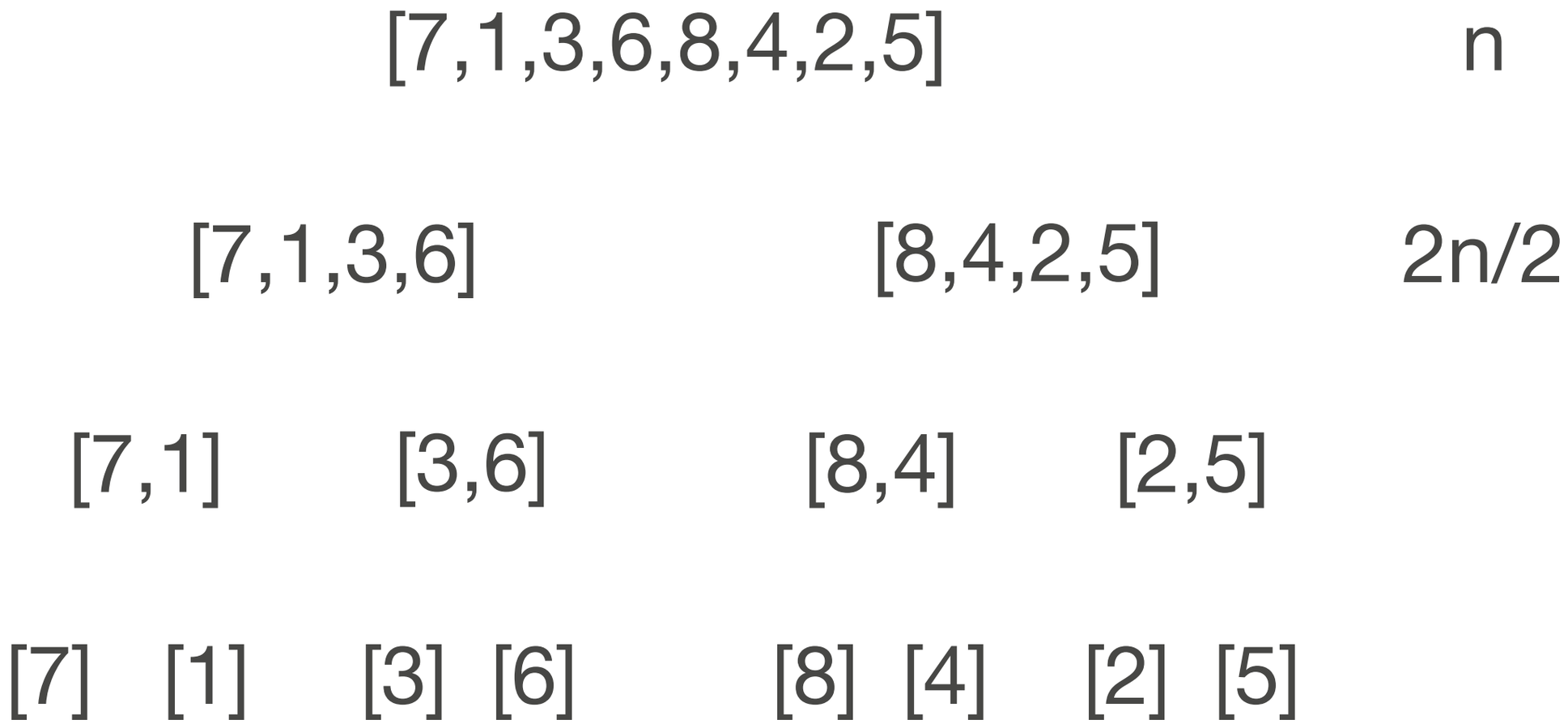
[7,1] [3,6] [8,4] [2,5]

[7] [1] [3] [6] [8] [4] [2] [5]

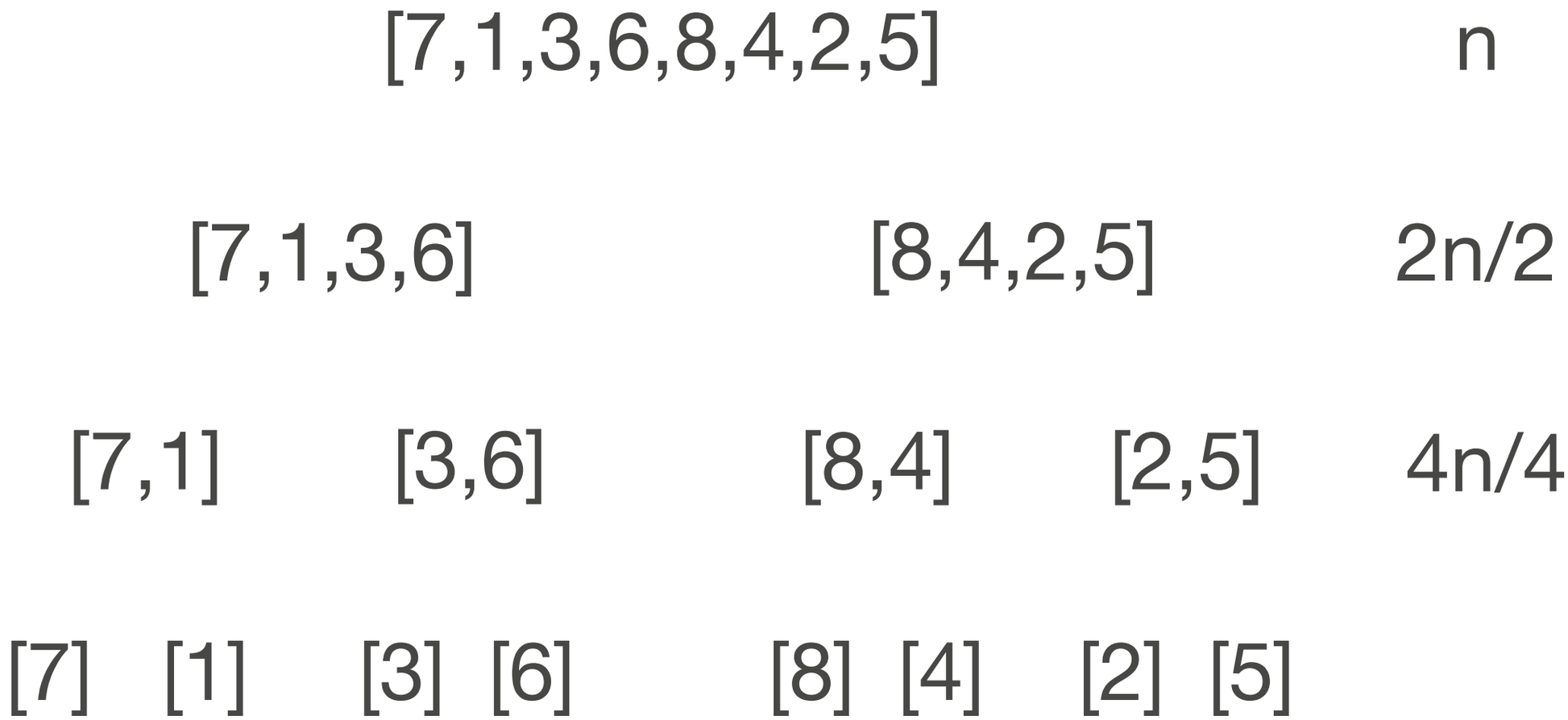
Work is O(n log n)



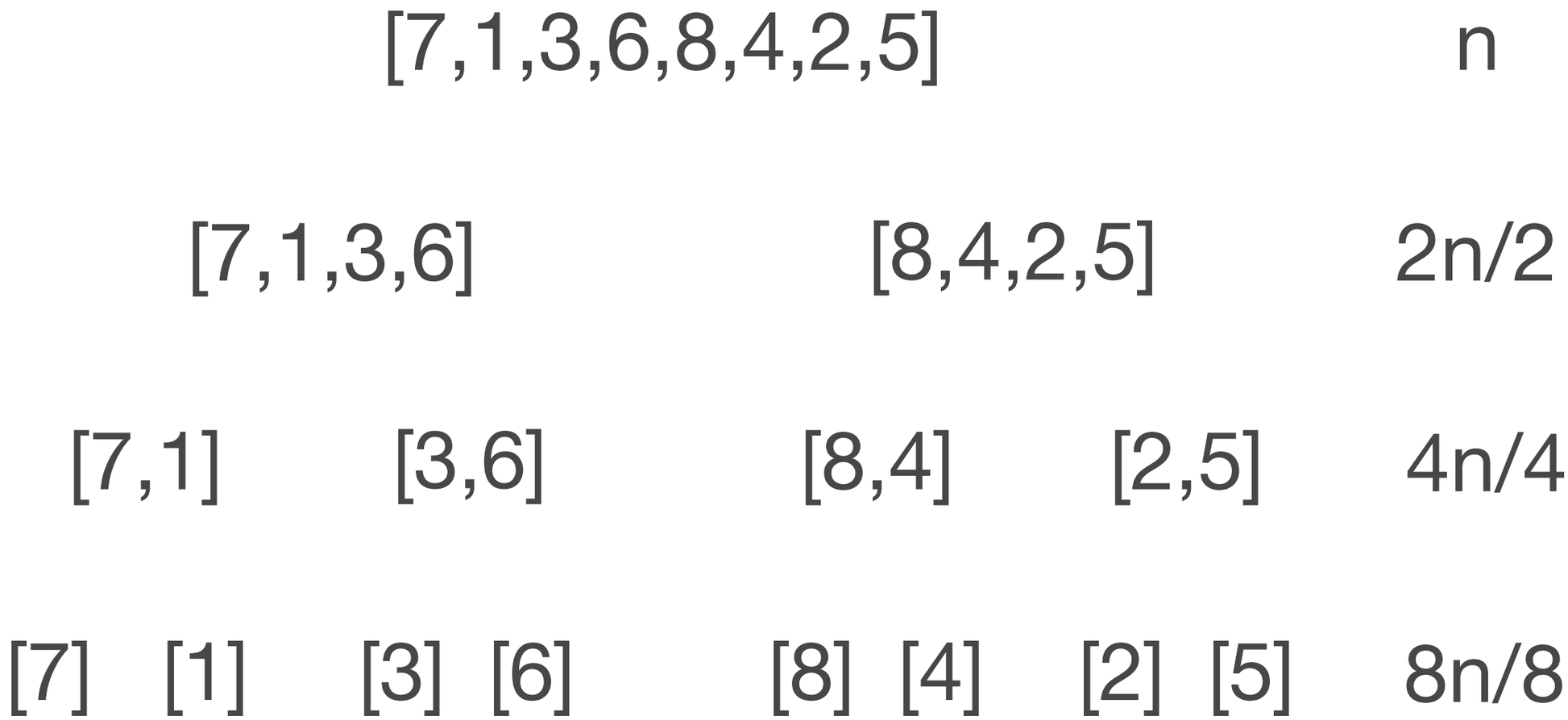
Work is O(n log n)



Work is O(n log n)



Work is O(n log n)



Span

```
fun mergesort (l : int list) : int list =
    case l of
        [] => []
      | [x] => [x]
      | _ => let val (pile1,pile2) = split l
             in
                 merge (mergesort pile1,
                        mergesort pile2)
             end
```

Span

```
fun mergesort (l : int list) : int list =
    case l of
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      | [x] => [x]
      | _ => let val (pile1,pile2) = split l
             in
                 merge (mergesort pile1,
                        mergesort pile2)
             end
```

two recursive calls can be evaluated in parallel!

Span is

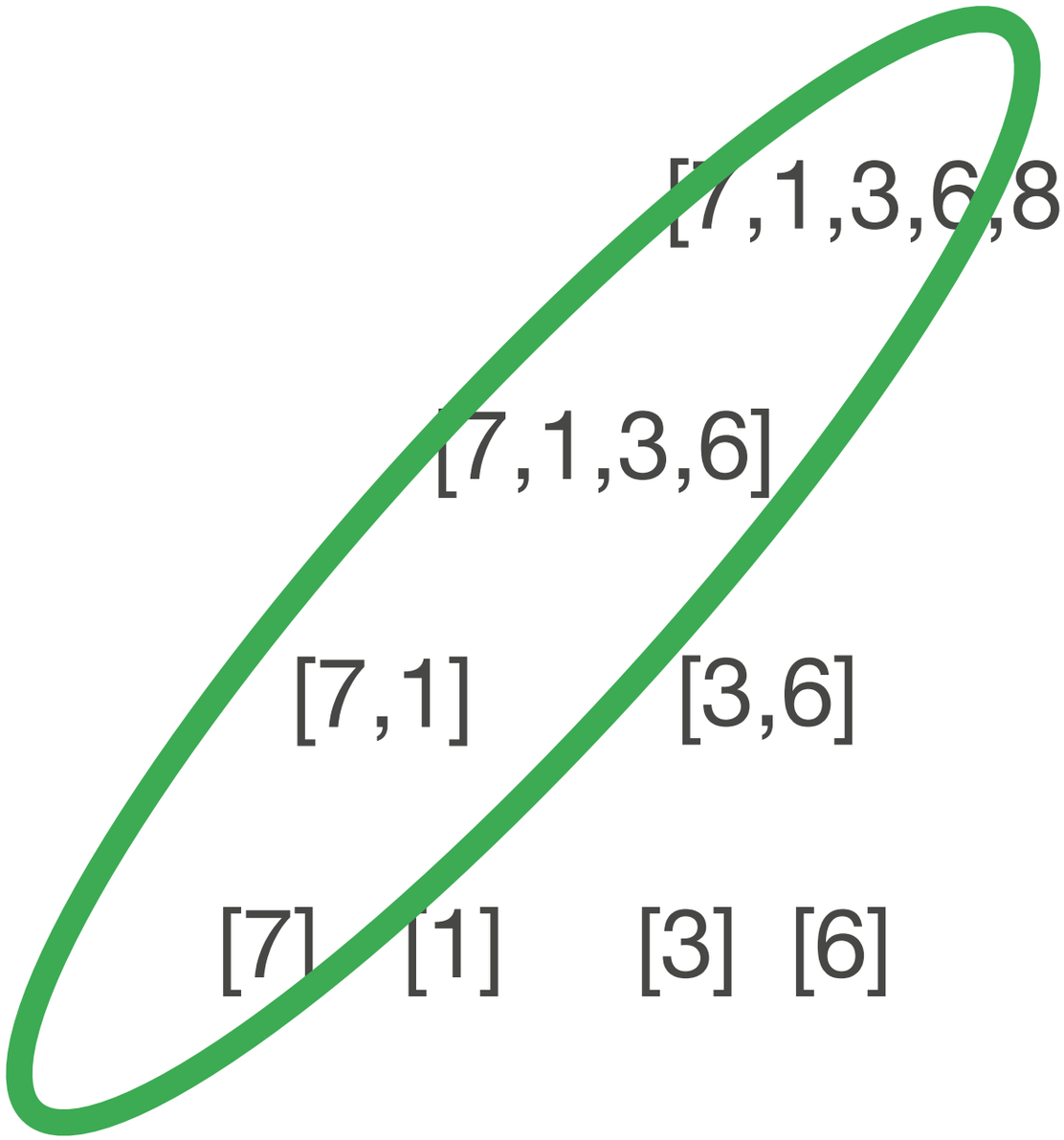
[7,1,3,6,8,4,2,5]

[7,1,3,6] [8,4,2,5]

[7,1] [3,6] [8,4] [2,5]

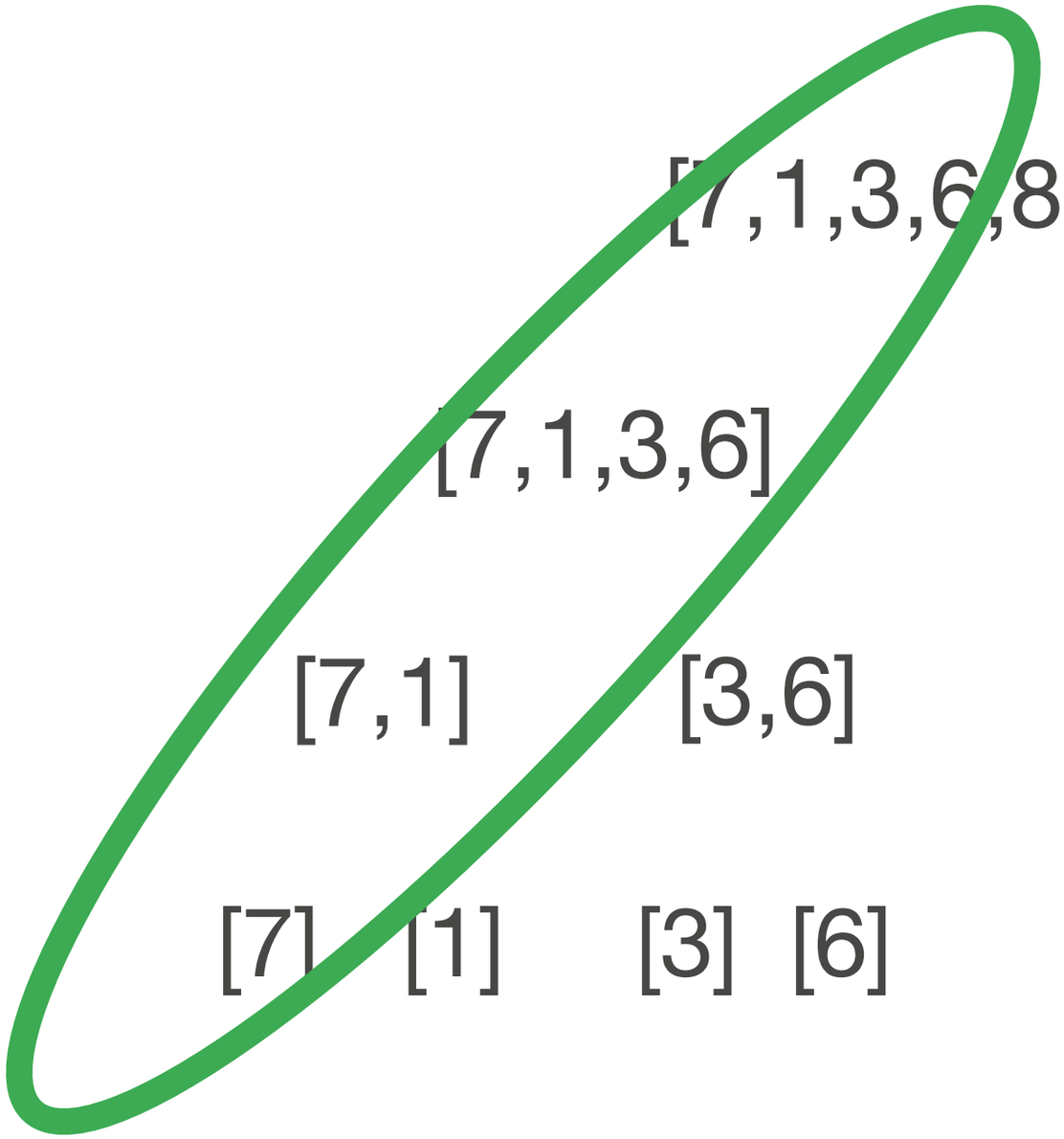
[7] [1] [3] [6] [8] [4] [2] [5]

Span is



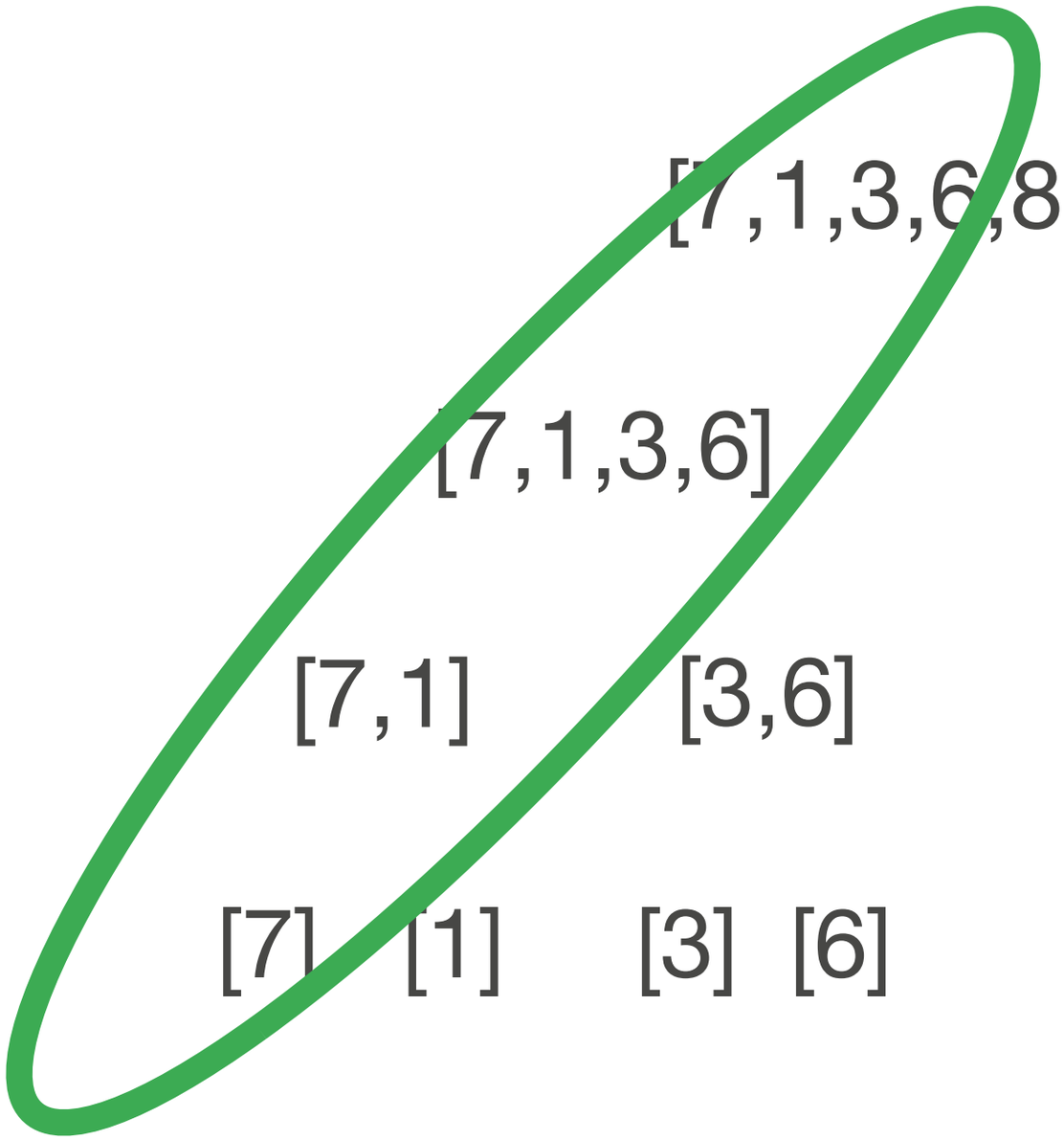
Span is

splitting and merging lists are sequential operations



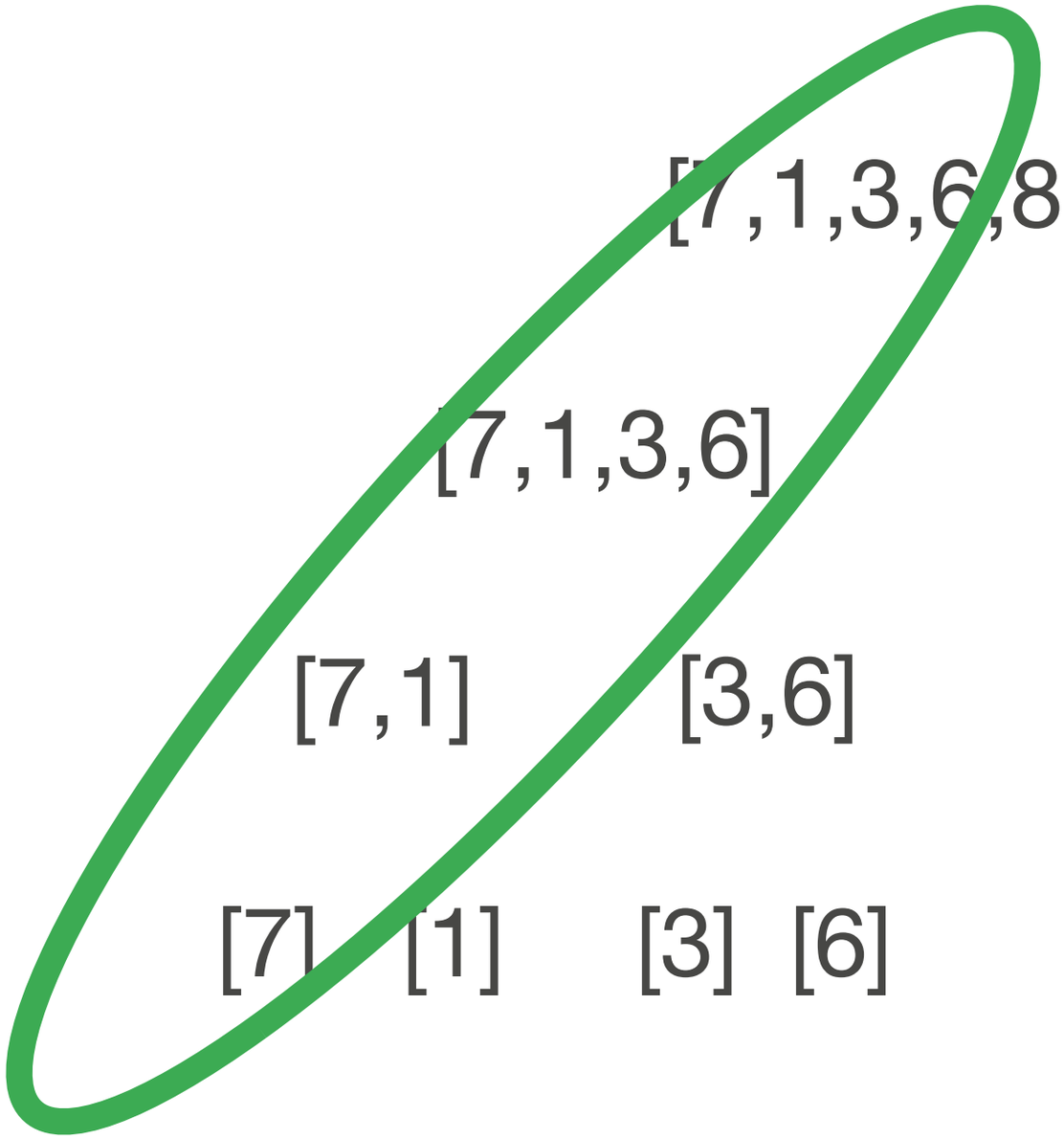
Span is

splitting and merging lists are sequential operations



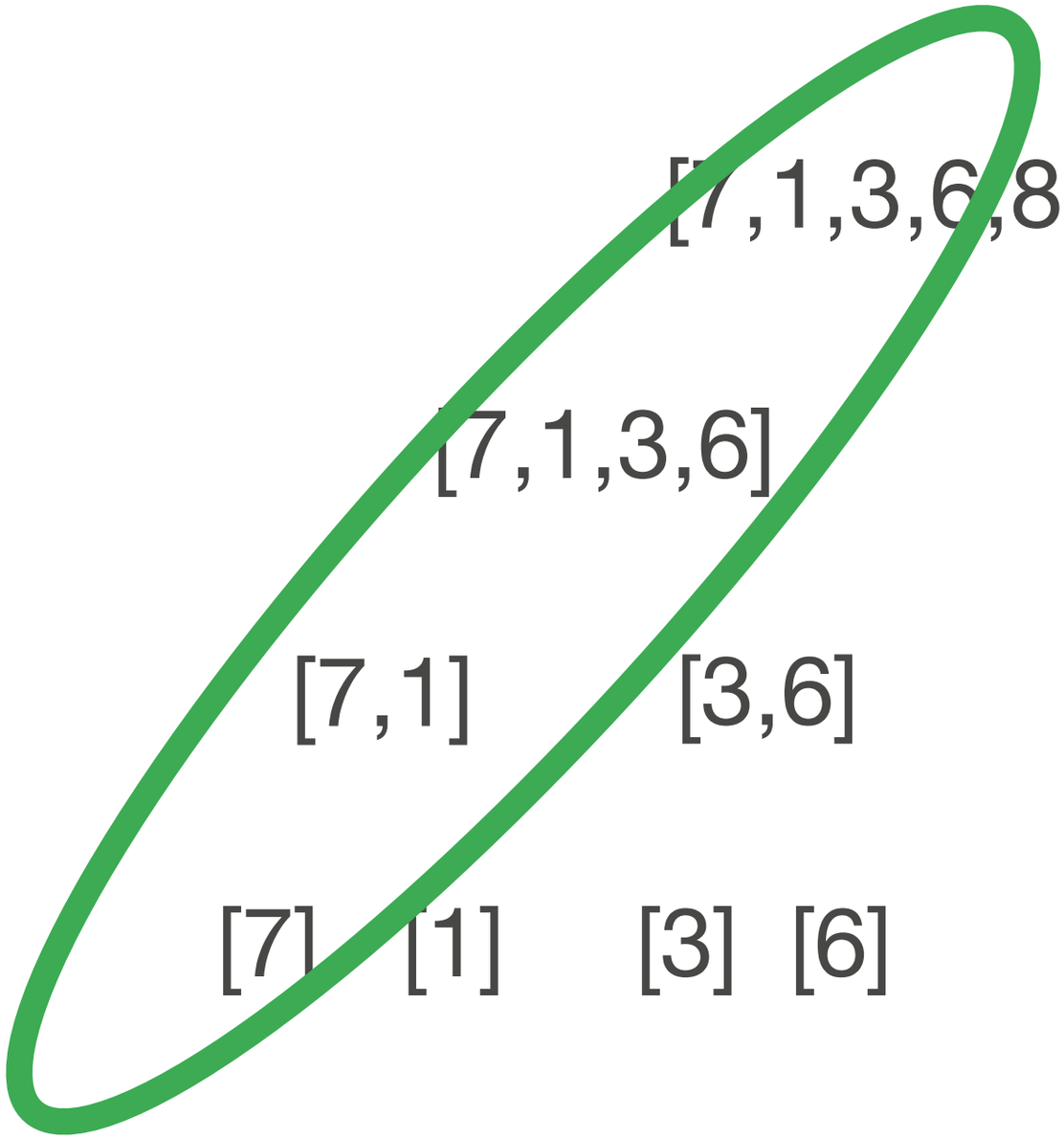
Span is

splitting and merging lists are sequential operations



Span is

splitting and merging lists are sequential operations



[7,1,3,6,8,4,2,5] n

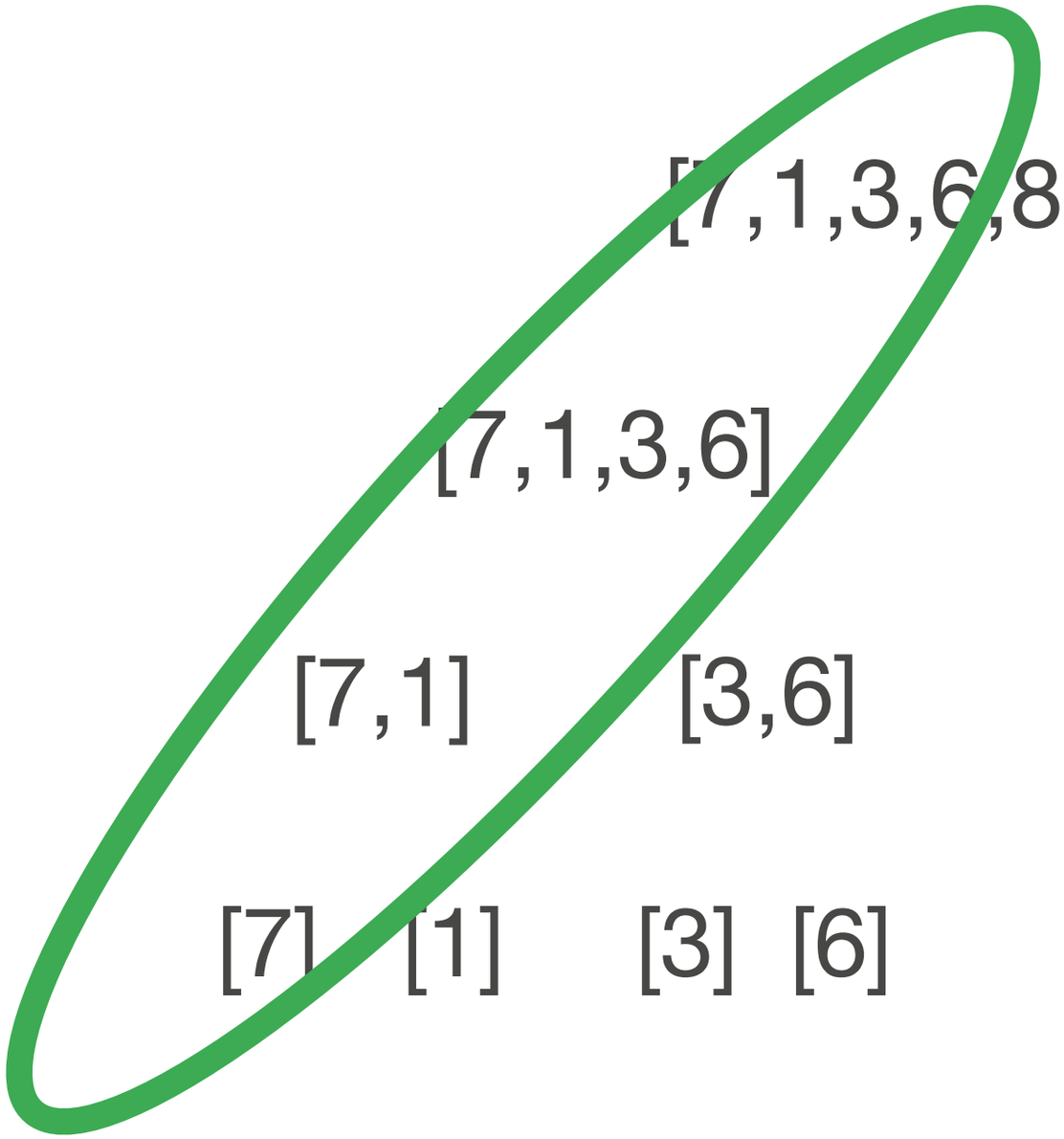
[7,1,3,6] [8,4,2,5] n/2

[7,1] [3,6] [8,4] [2,5] n/4

[7] [1] [3] [6] [8] [4] [2] [5]

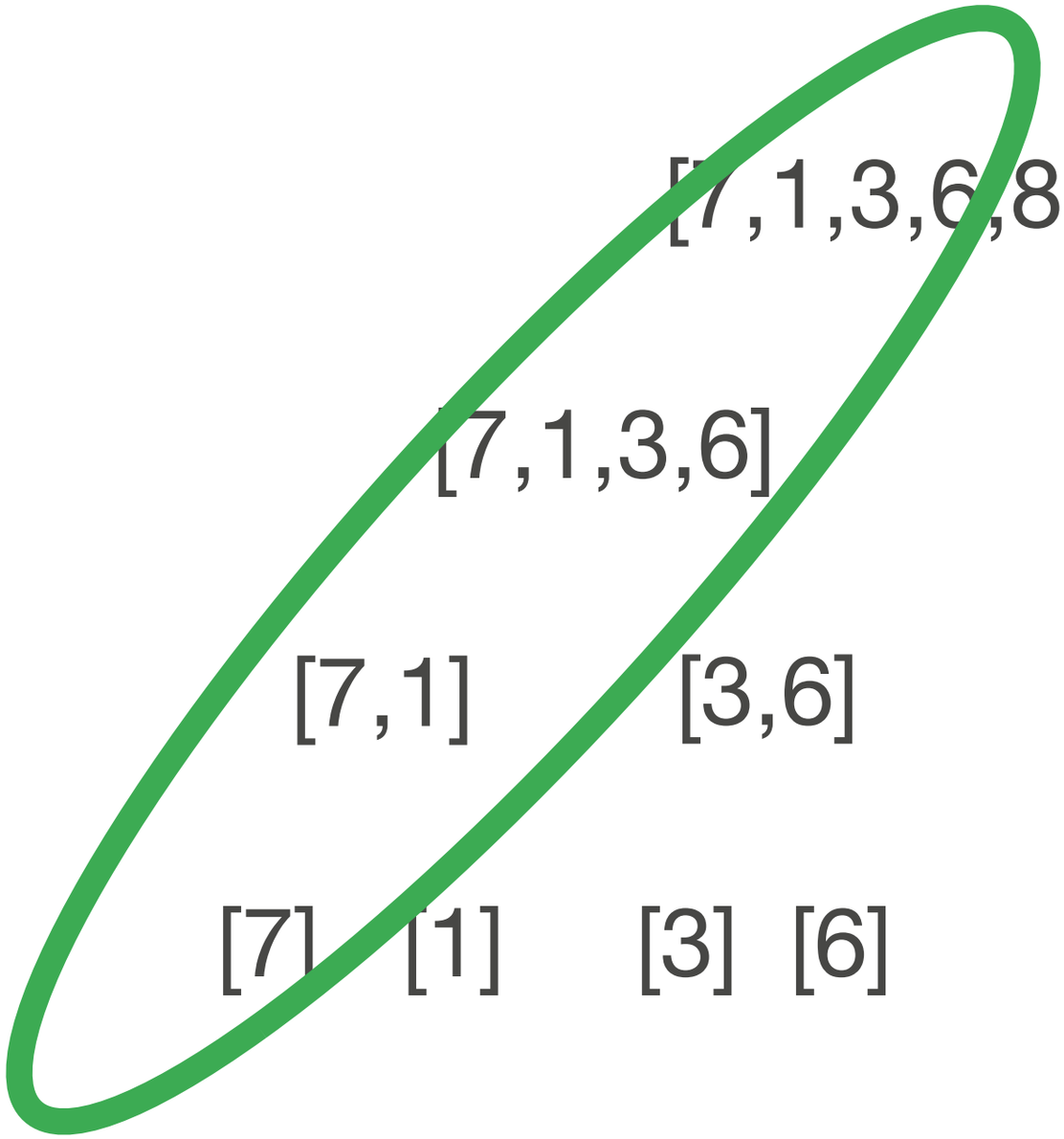
Span is

splitting and merging lists are sequential operations



Span is O(n)

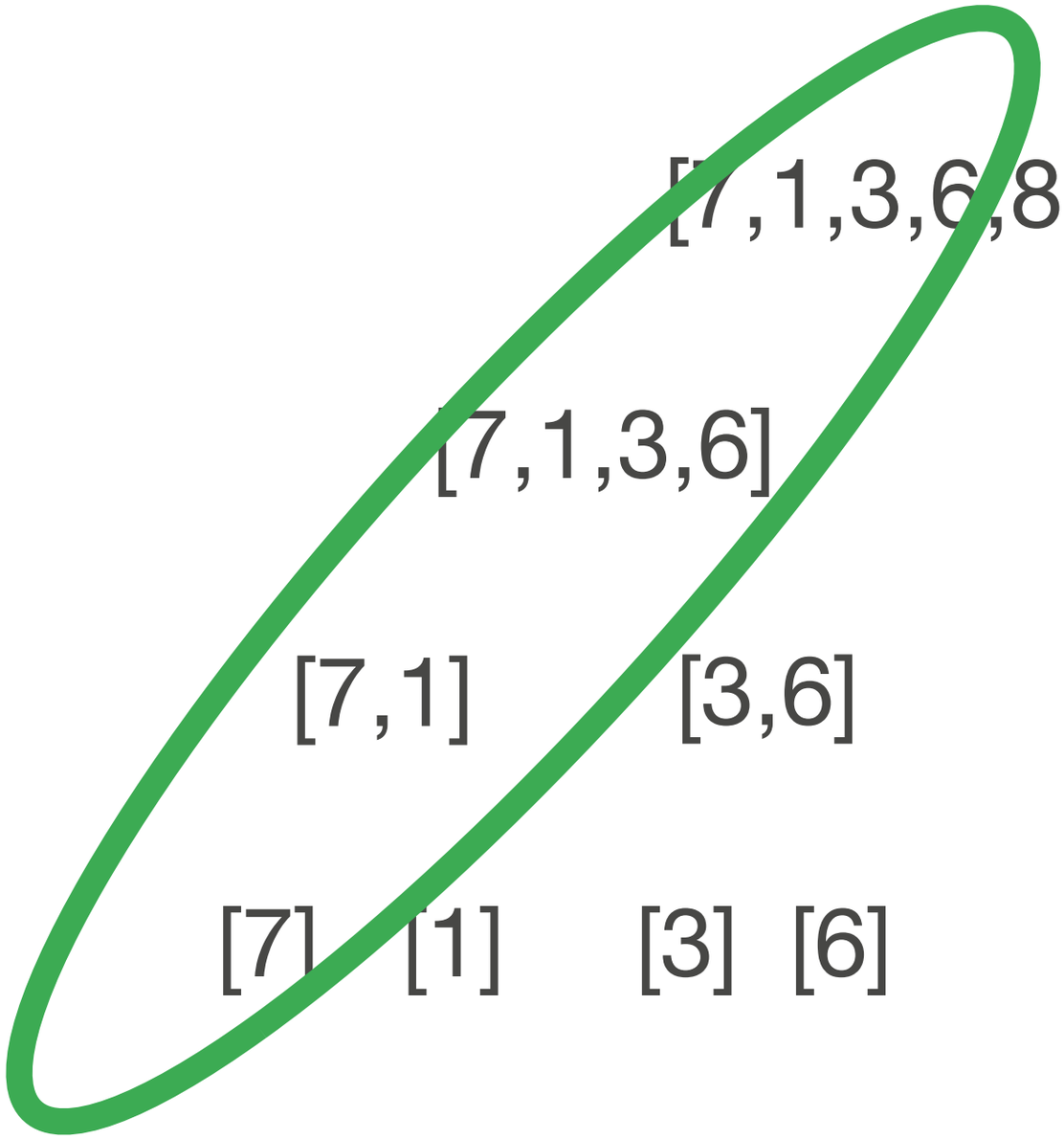
splitting and merging lists are sequential operations



[7,1,3,6,8,4,2,5]					n
[7,1,3,6]			[8,4,2,5]		n/2
[7,1]	[3,6]		[8,4]	[2,5]	n/4
[7] [1]	[3] [6]		[8] [4]	[2] [5]	n/8

Span is O(n)

splitting and merging lists are sequential operations



[7,1,3,6,8,4,2,5]				n
[7,1,3,6]		[8,4,2,5]		n/2
[7,1]	[3,6]	[8,4]	[2,5]	n/4
[7] [1]	[3] [6]	[8] [4]	[2] [5]	n/8

Parallelizability: log n

Data structures matter

Data structures matter

- Lists are bad for parallelism

Data structures matter

- Lists are bad for parallelism
- Trees are better

```
fun mergesort (t : tree) : tree =
  case t of
       Empty => Empty
  | Node (l , x , r) =>
                  (insert(x,
                                merge (mergesort l ,
                                       mergesort r))
```

Span O((log n)^3) if balanced

```
fun mergesort (t : tree) : tree =
  case t of
       Empty => Empty
  | Node (l , x , r) =>
                  (insert(x,
                                merge (mergesort l ,
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```

splitting is given

Span O((log n)^3) if balanced

traversing entire tree multiplies span by log(n)

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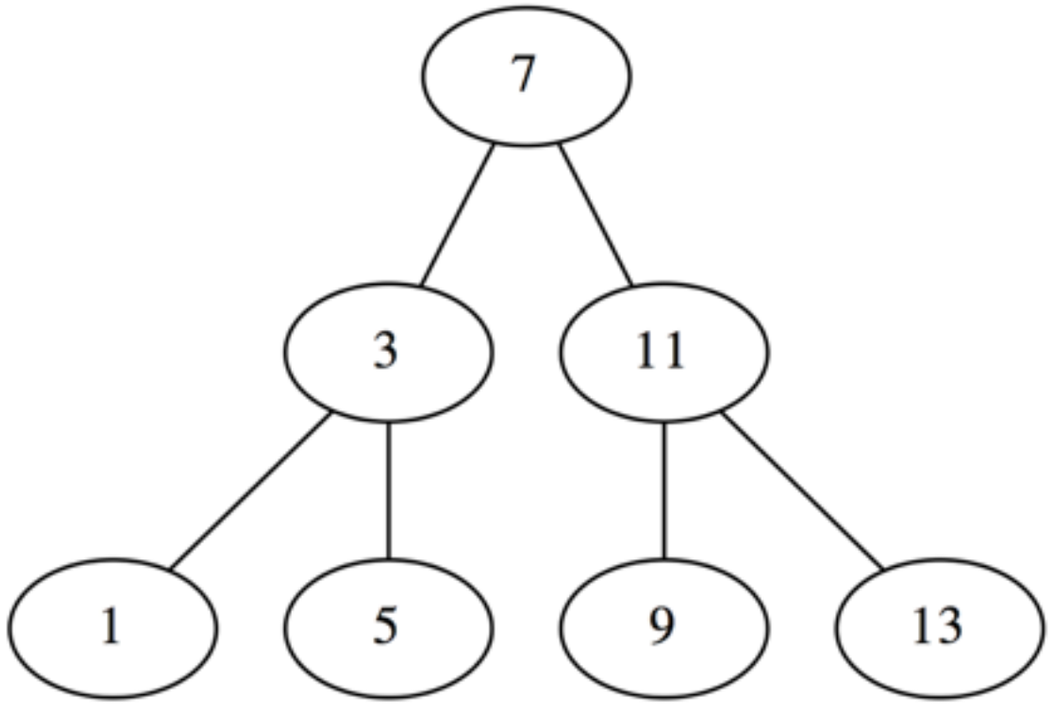
```

splitting is given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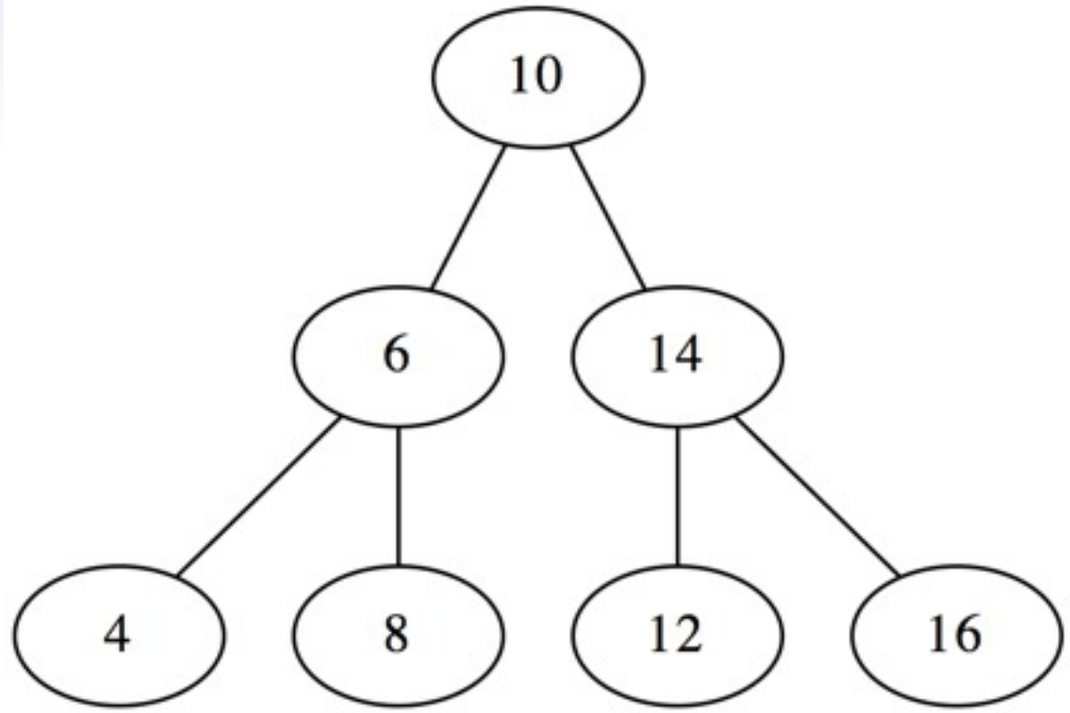
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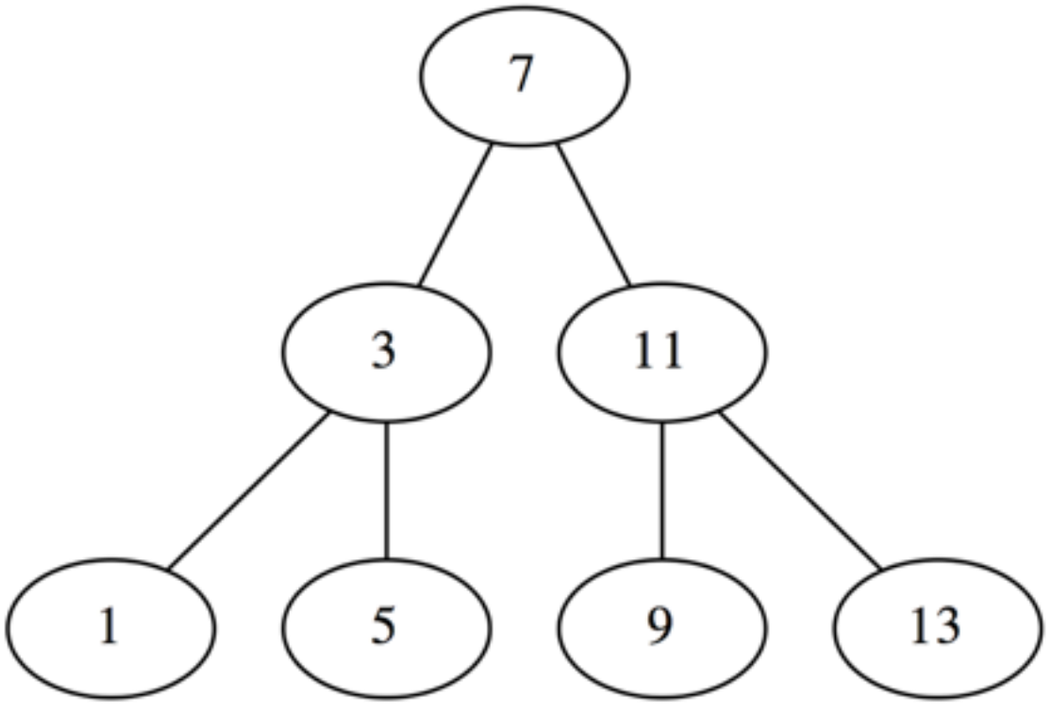
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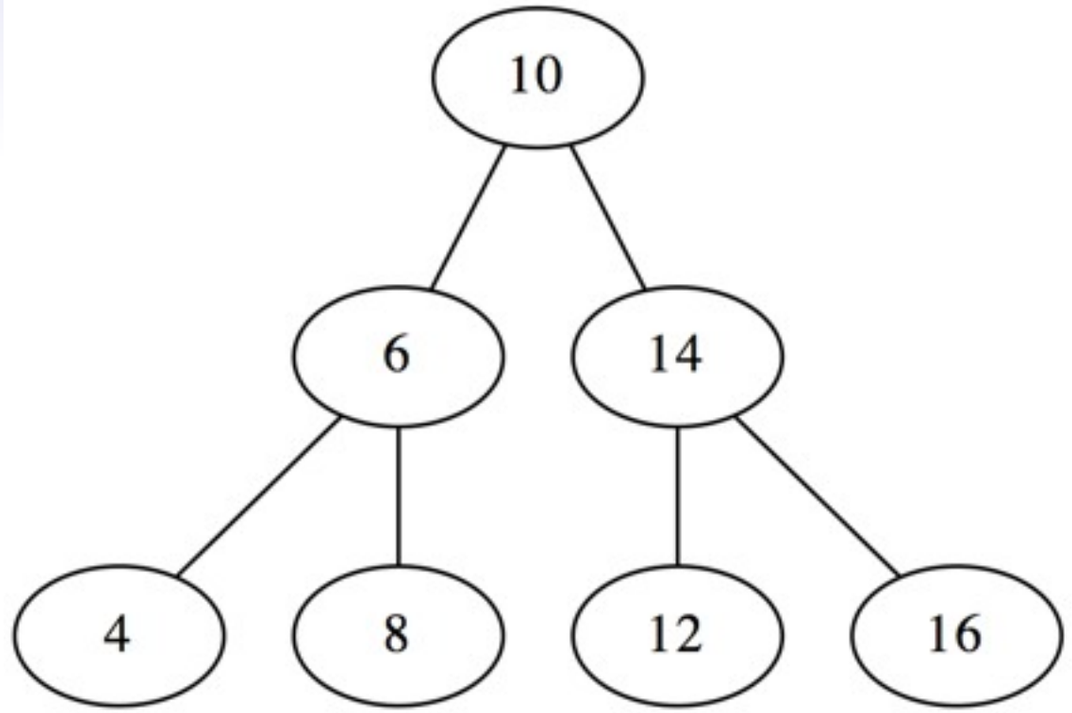
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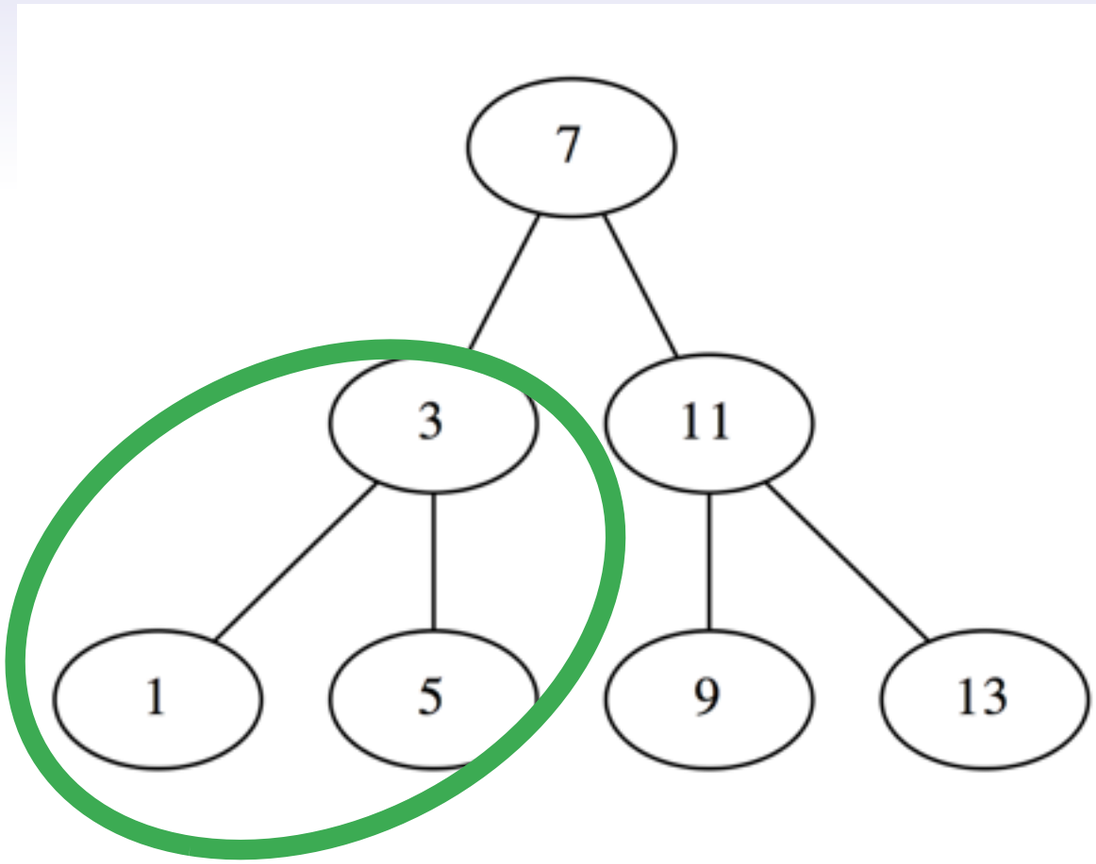
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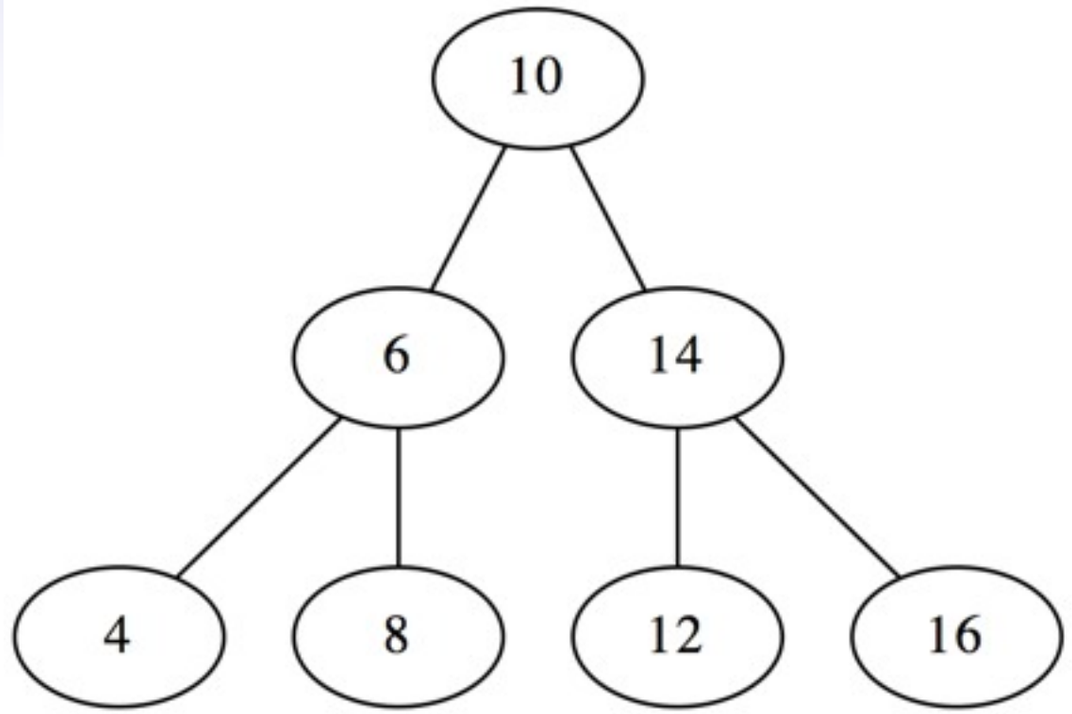
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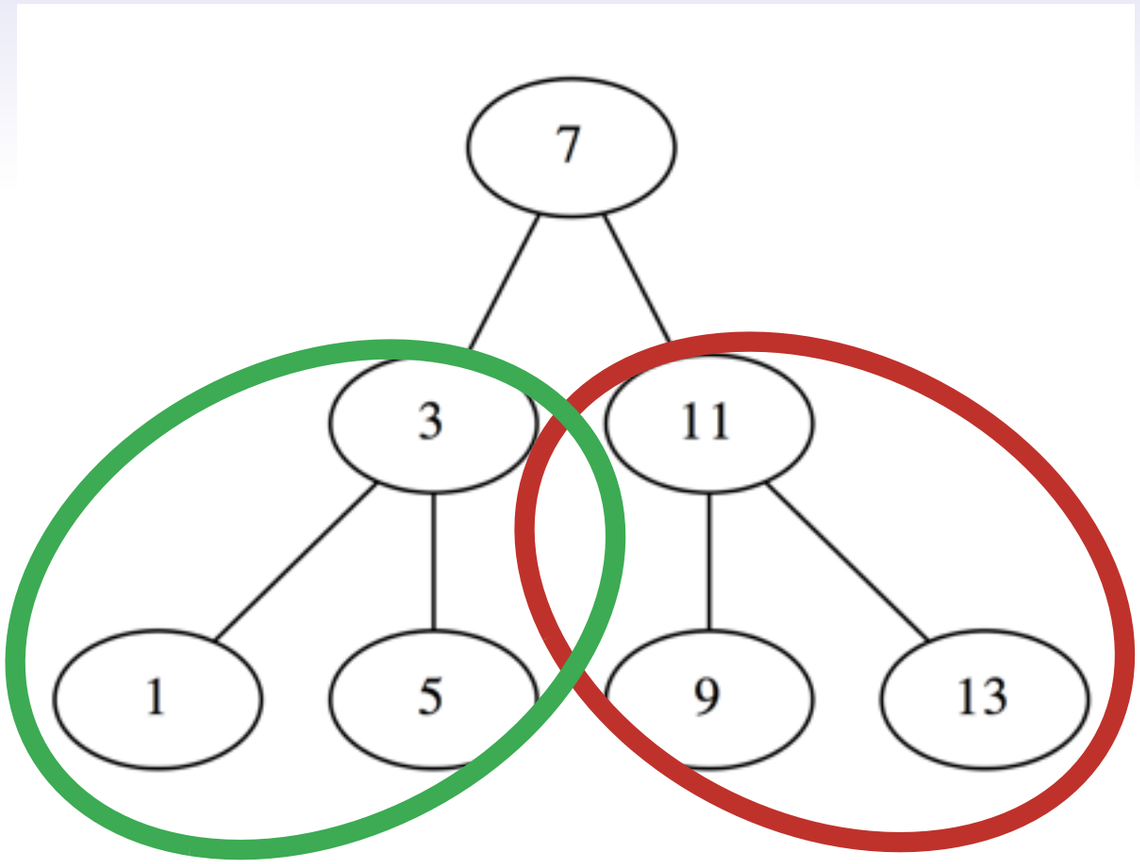
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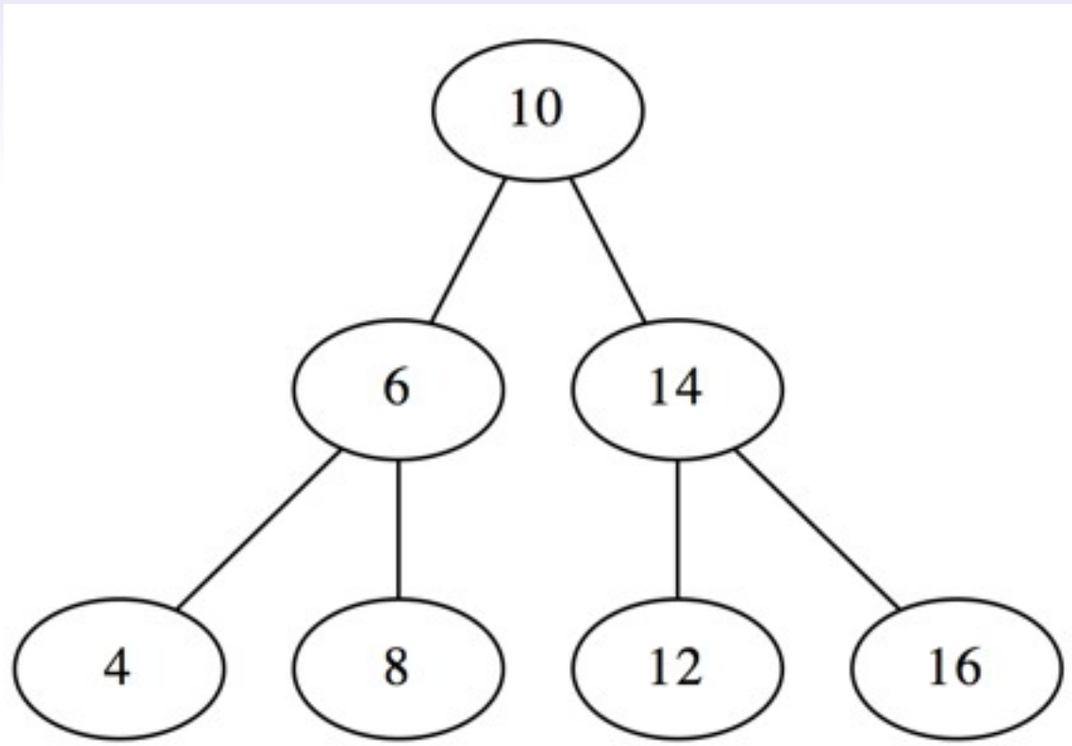
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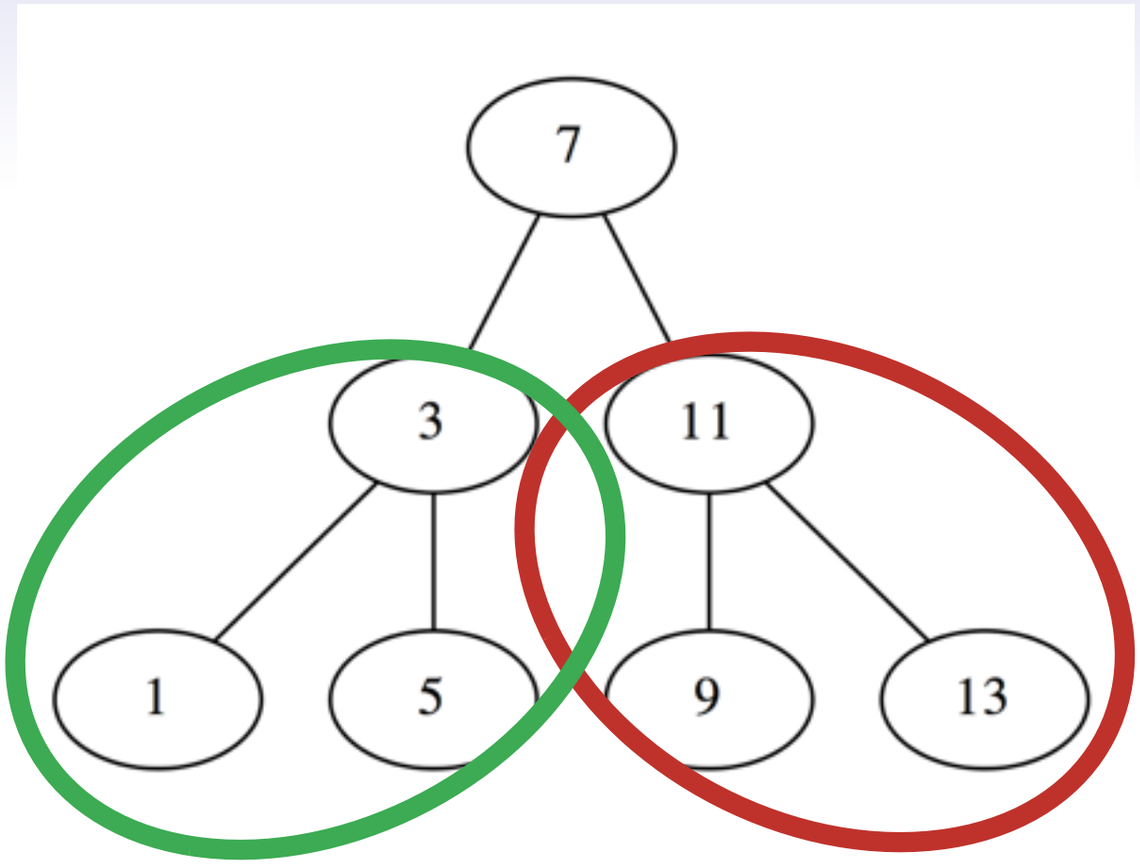
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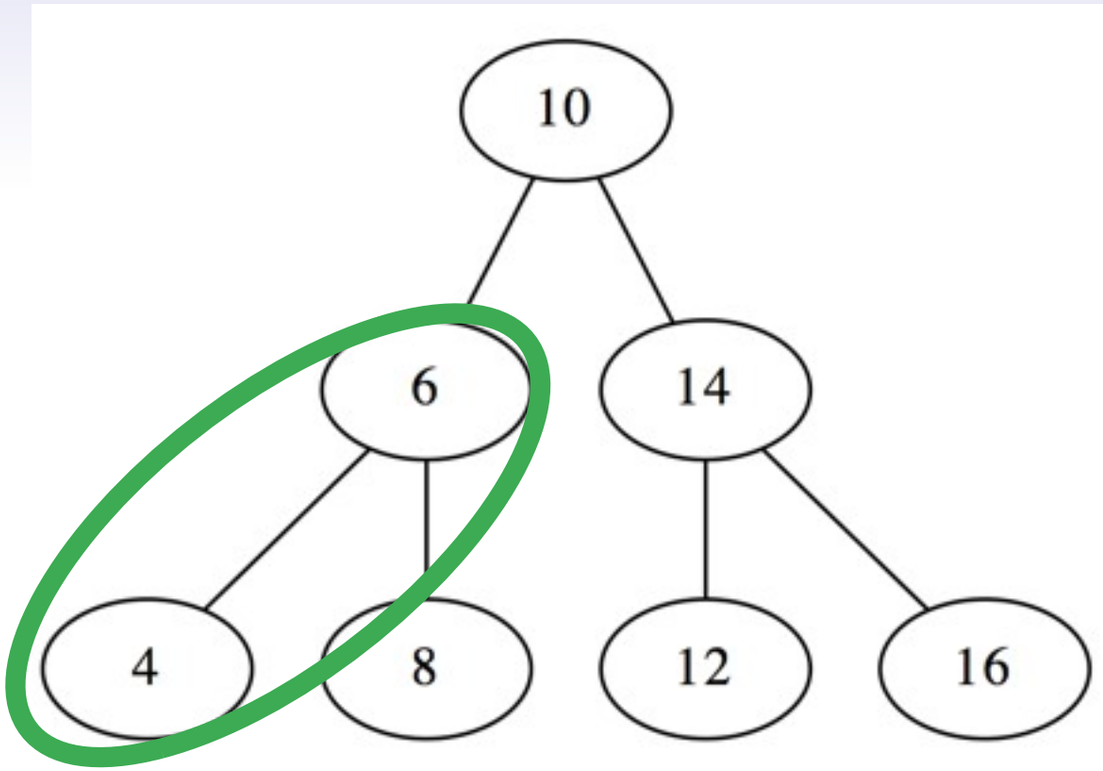
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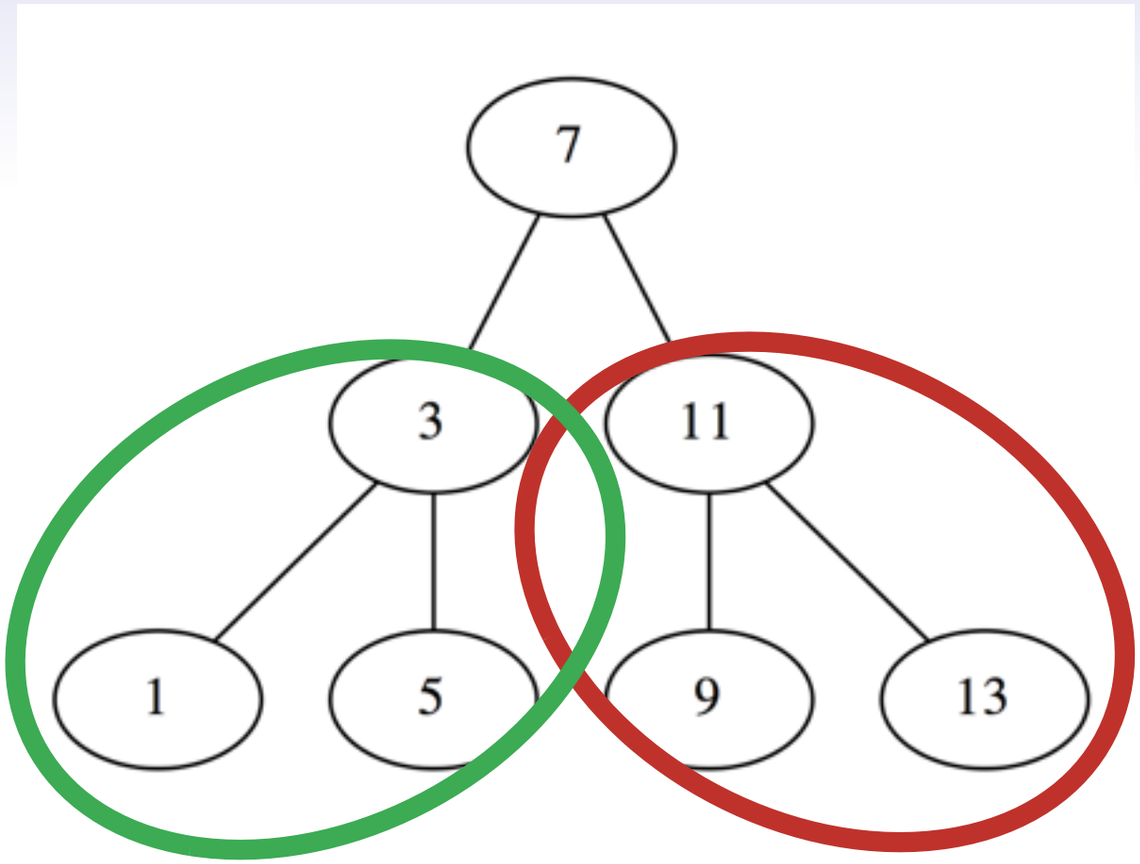
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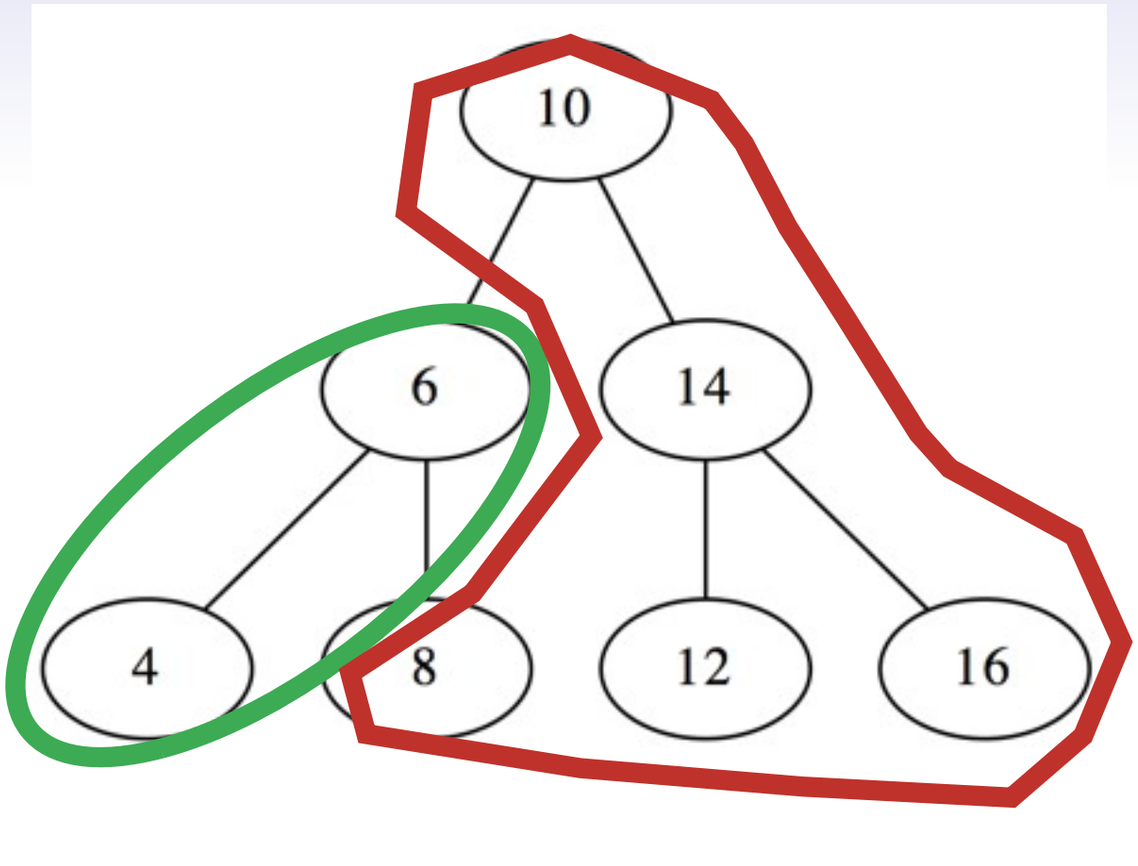
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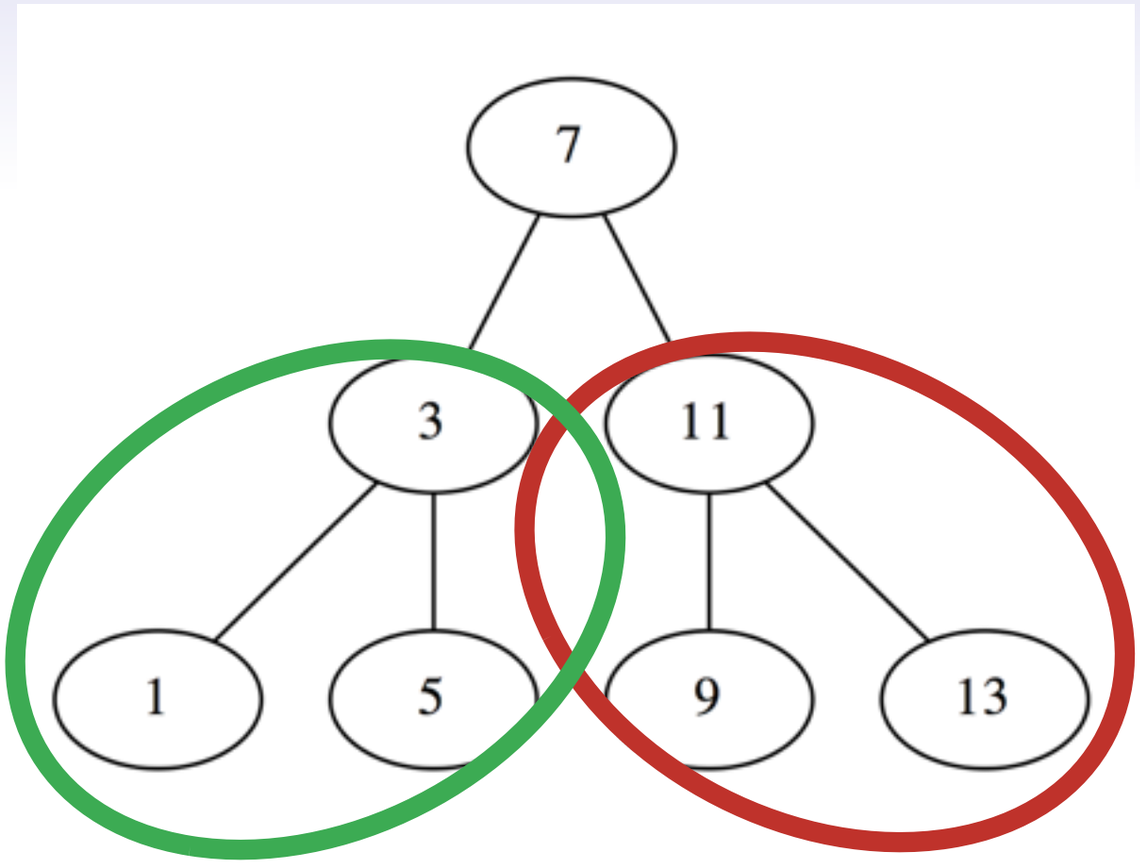
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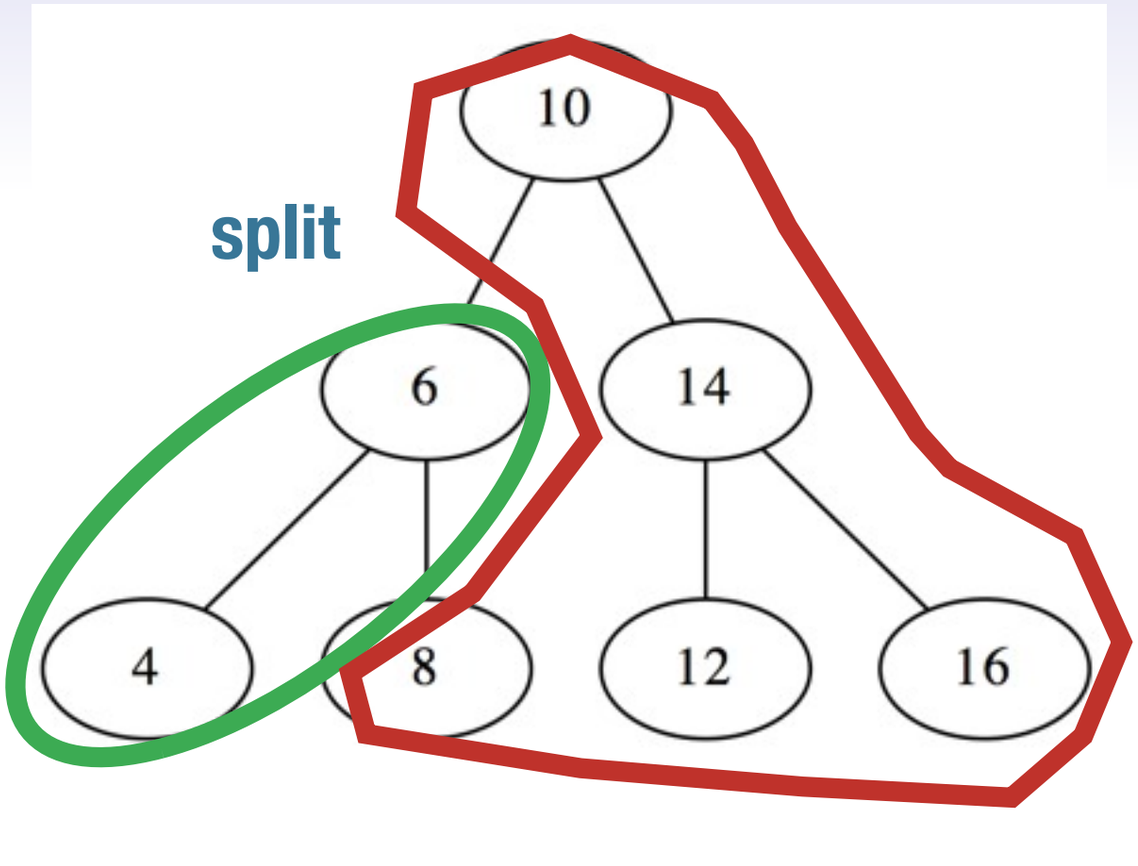
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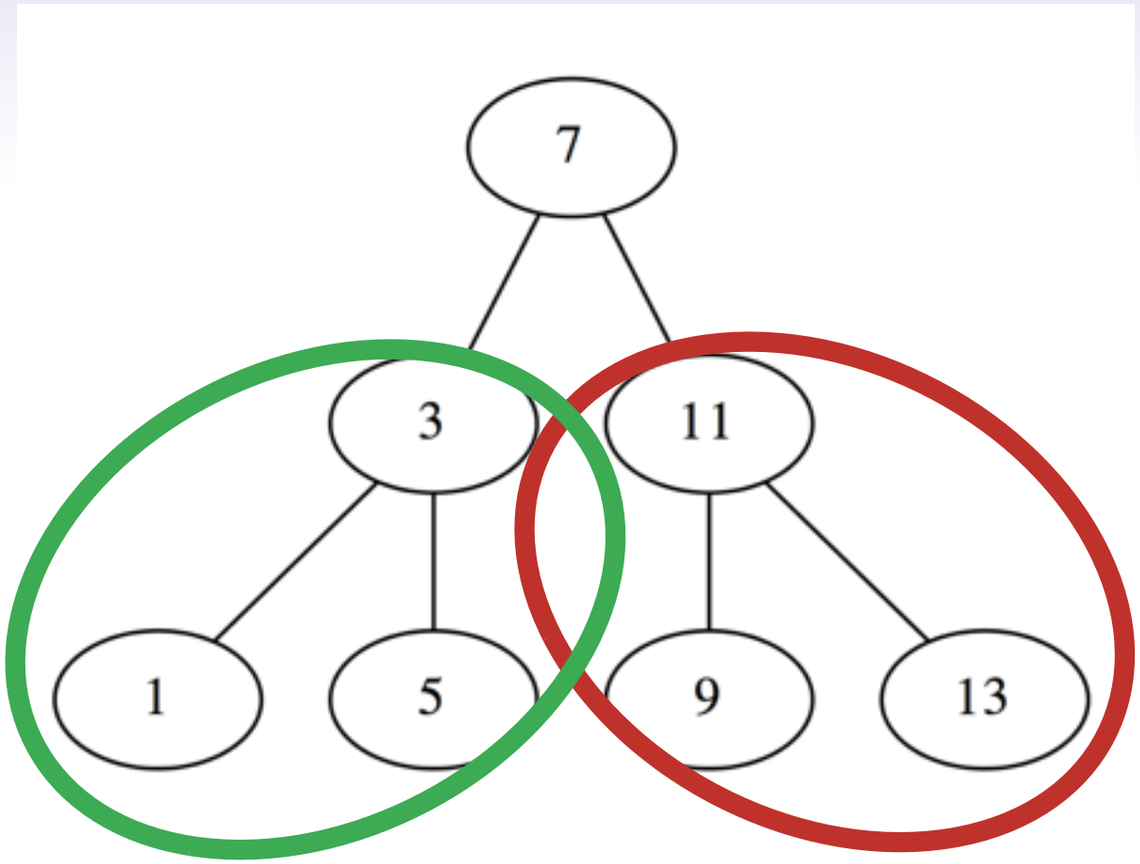
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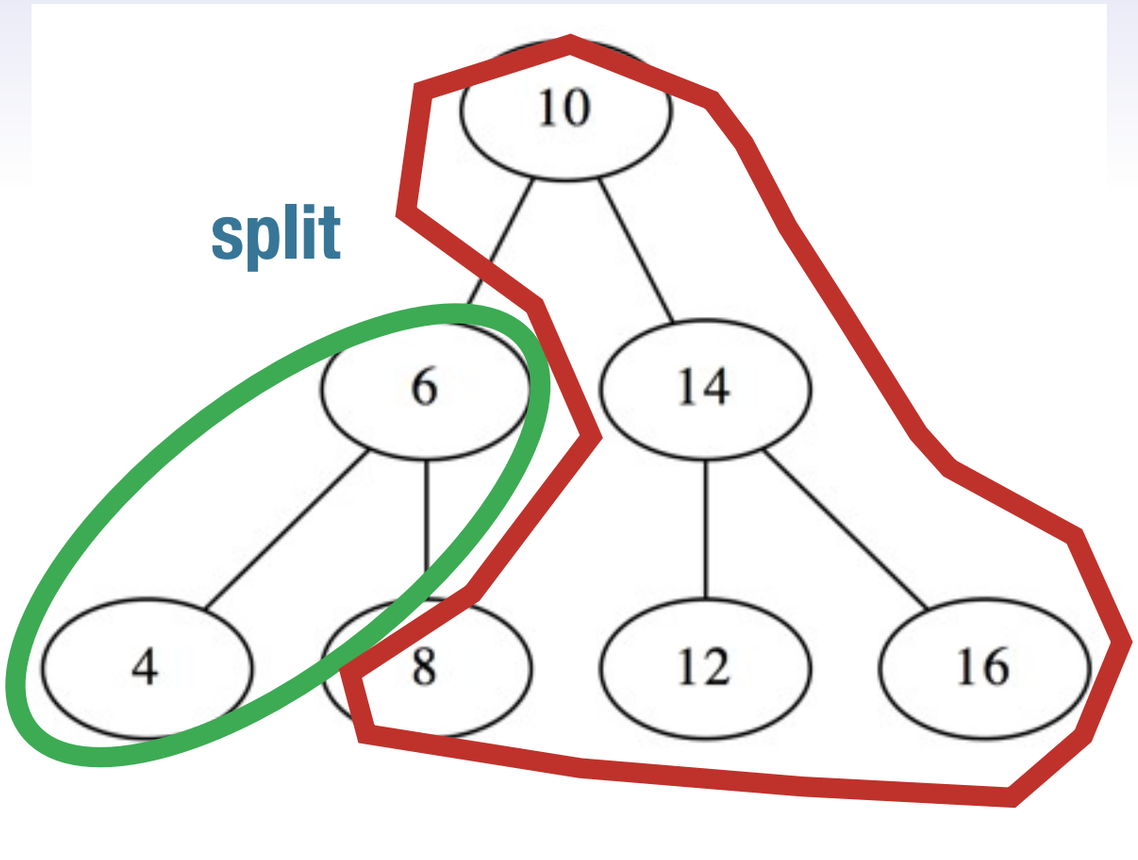
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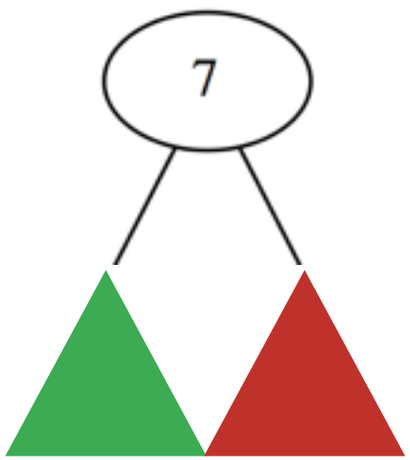
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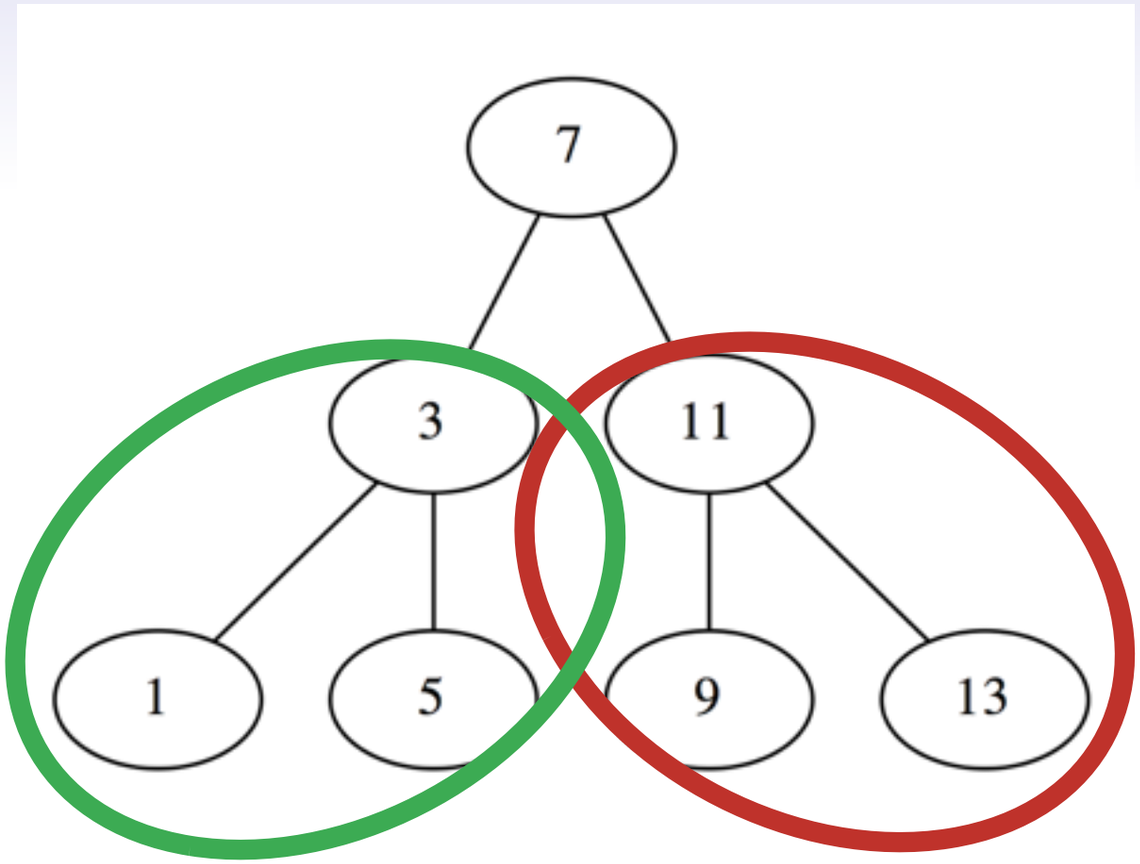

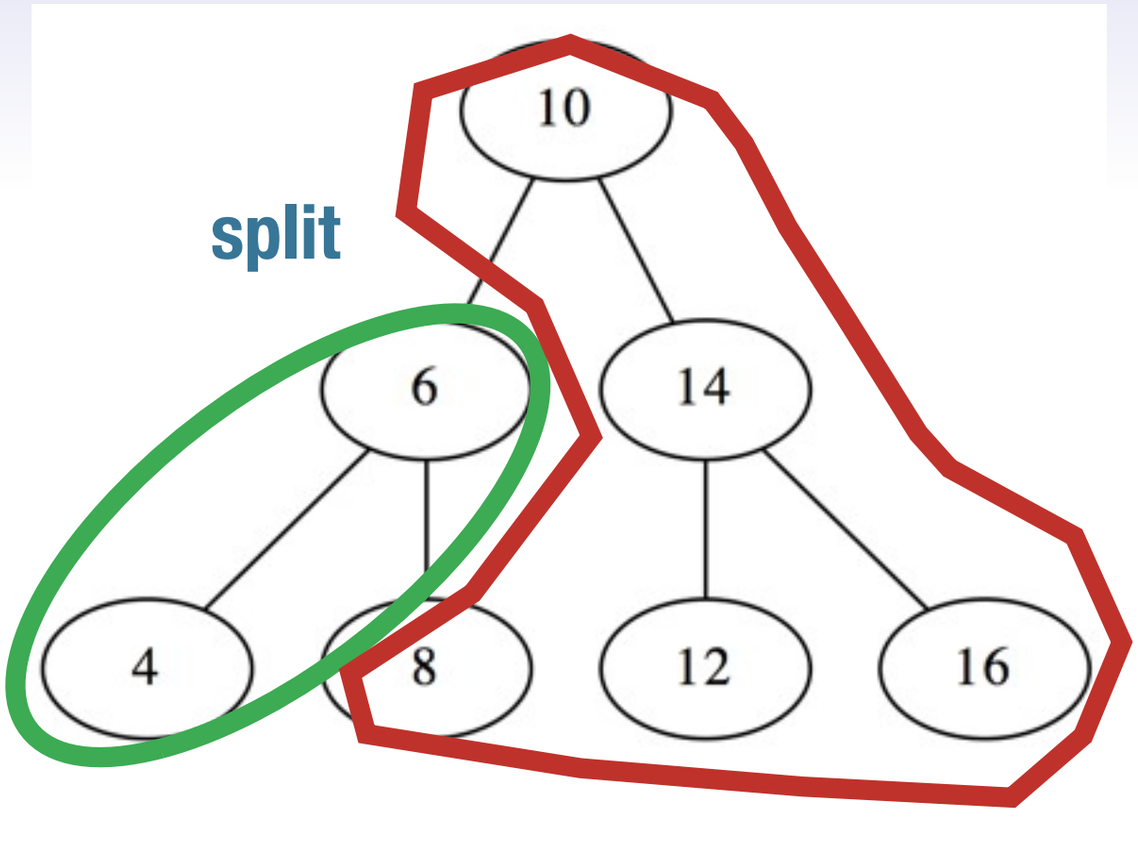

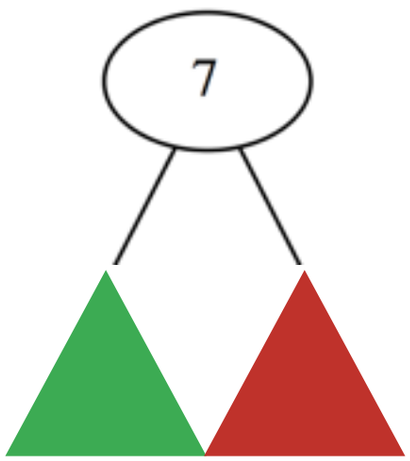

span O(depth t * depth t')

Span O((log n)^3) if balanced

traversing entire tree multiplies span by log(n)

```
fun mergesort (t : tree) : tree =
  case t of
       Empty => Empty
     | Node (l , x , r) =>
                  (insert(x,
                          merge (mergesort l ,
                                 mergesort r))
```

splitting is given

Span O((log n)^3) if balanced

traversing entire tree multiplies span by log(n)

```
fun mergesort (t : tree) : tree =
  case t of
       Empty => Empty
     | Node (l , x , r) =>
                  (insert(x,
                          merge (mergesort l ,
                                 mergesort r))
```

splitting is given

O((log n)^2) if recursive results are balanced

Span O((log n)^3) if balanced

traversing entire tree multiplies span by log(n)

```
fun mergesort (t : tree) : tree =
  case t of
       Empty => Empty
     | Node (l , x , r) =>
        rebalance (insert(x,
                          merge (mergesort l ,
                                 mergesort r))
```

splitting is given

O((log n)^2) if recursive results are balanced

```
fun mergesort (t : tree) : tree =
  case t of
       Empty => Empty
     | Node (l , x , r) =>
        rebalance (insert(x,
                          merge (mergesort l ,
                                 mergesort r))
```

```
fun mergesort (t : tree) : tree =
  case t of
       Empty => Empty
     | Node (l , x , r) =>
        rebalance (insert(x,
                          merge (mergesort l ,
                                 mergesort r))
```

recurs on an unbalanced tree!

why doesn't it ruin the work and span?

```
fun mergesort (t : tree) : tree =
  case t of
       Empty => Empty
     | Node (l , x , r) =>
        rebalance (insert(x,
                          merge (mergesort l ,
                                 mergesort r))
```

recurs on an unbalanced tree!
why doesn't it ruin the work and span?

rebalance: n work and $(\log n)^2$ span **if** depth(t) = c*log(n)

```
fun mergesort (t : tree) : tree =
  case t of
       Empty => Empty
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        rebalance (insert(x,
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```

recurs on an unbalanced tree!
why doesn't it ruin the work and span?

rebalance: n work and $(\log n)^2$ span **if** depth(t) = c*log(n)

depth(merge(t,t')) ≤ depth(t) + depth(t')

```
fun mergesort (t : tree) : tree =
  case t of
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        rebalance (insert(x,
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```

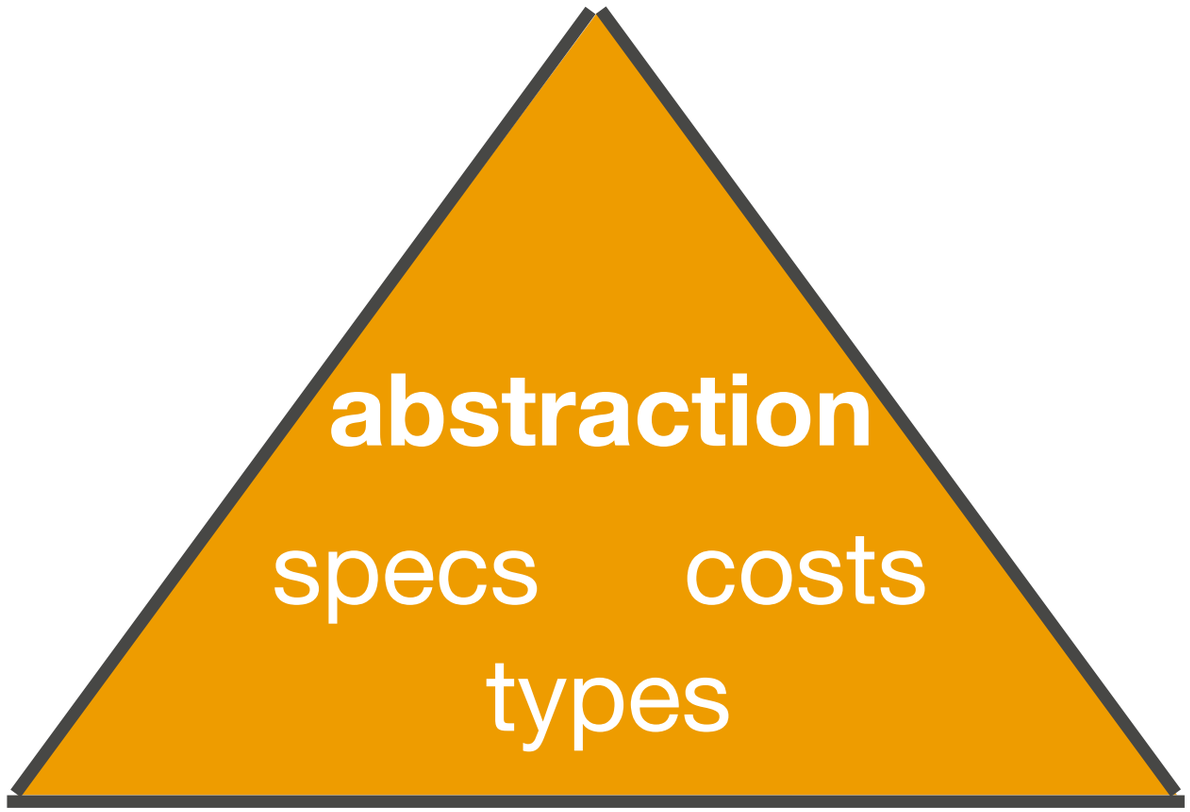
recurs on an unbalanced tree!
why doesn't it ruin the work and span?

rebalance: n work and $(\log n)^2$ span **if** depth(t) = c*log(n)

depth(merge(t,t')) ≤ depth(t) + depth(t')

Parallelizability: $n/(\log n)^2$ better than log n

write code



reason about behavior

reason about efficiency

Data structures matter

- Lists are bad for parallelism
- Trees are better
- Sequences are even better

n-way parallelism

`Seq.map f <x1,…,xn> = <f x1,…,f xn>`

Work is the sum of the works of `f xi`

Span is the max of the works of `f xi`

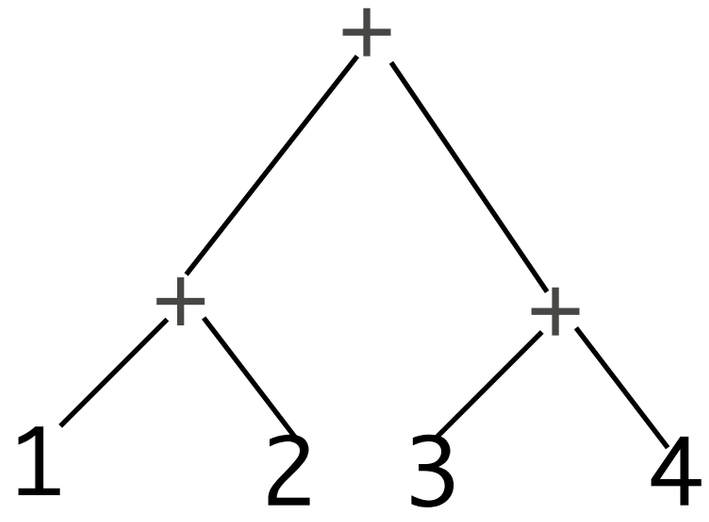
N-way parallelism

`Seq.reduce` $\oplus$ `0` $\langle x_1,\ldots,x_n\rangle = x_1 \oplus x_2 \oplus \ldots \oplus x_n$

Work is the sum of the works of $\oplus$'s

Span is given by balanced tree (log):

`reduce + 0 <1,2,3,4> =`



n-body simulation

[initial conditions from Princeton COS 126]

n-body simulation

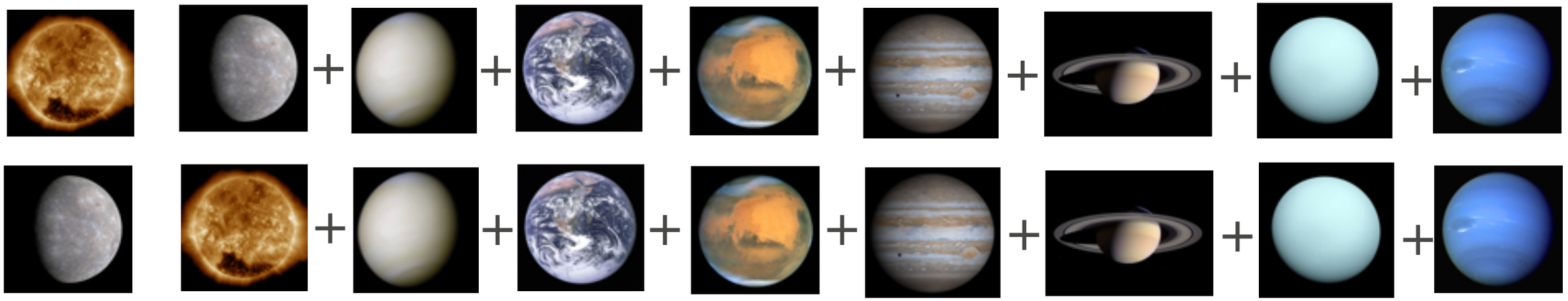
force on each body is sum of forces due to all other bodies

n-body simulation

force on each body is sum of forces due to all other bodies

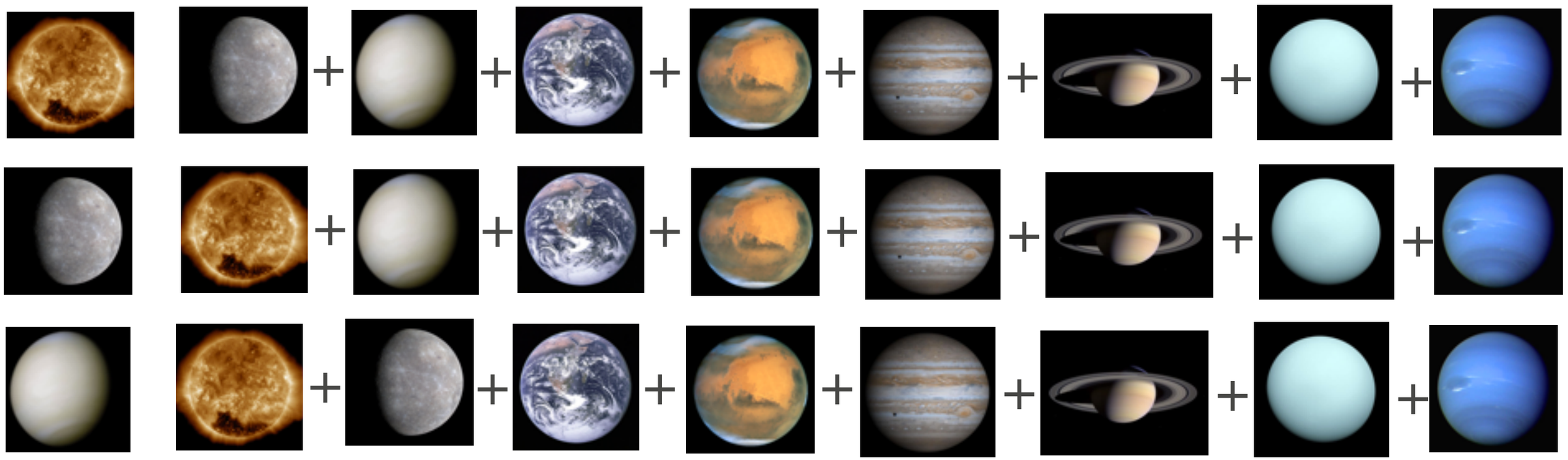
n-body simulation

force on each body is sum of forces due to all other bodies



n-body simulation

force on each body is sum of forces due to all other bodies



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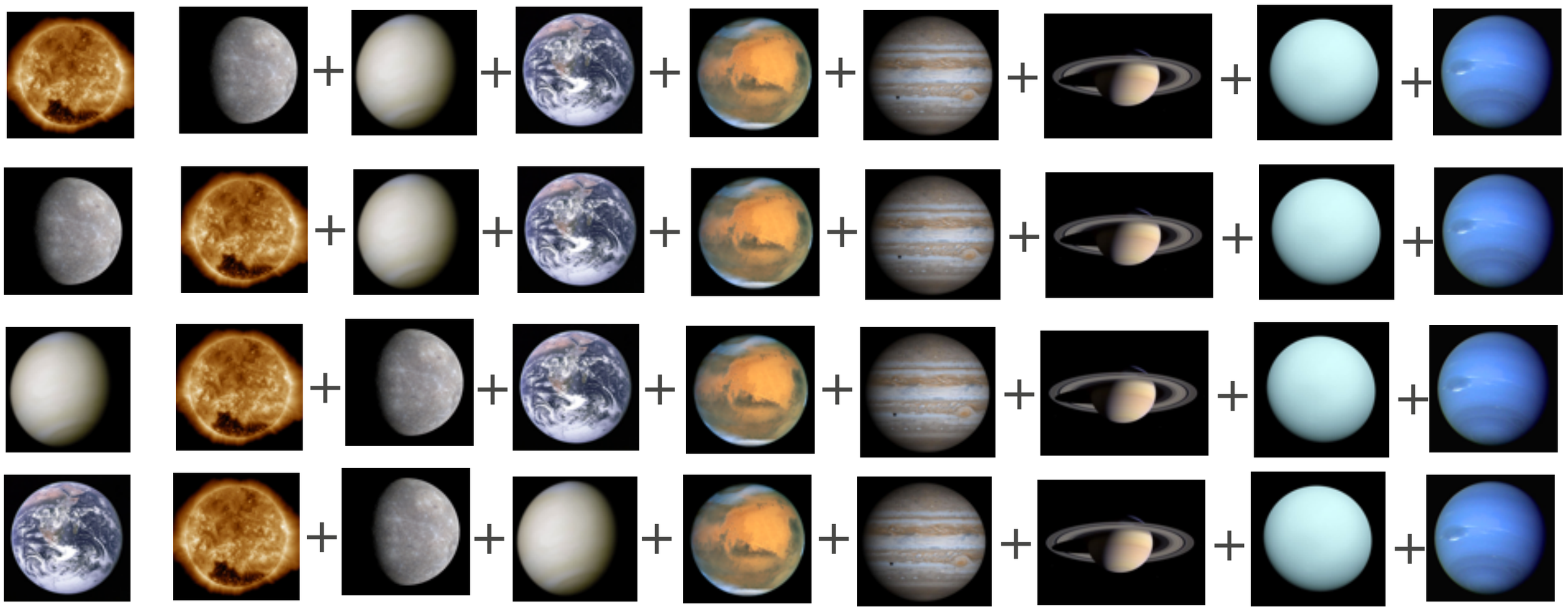

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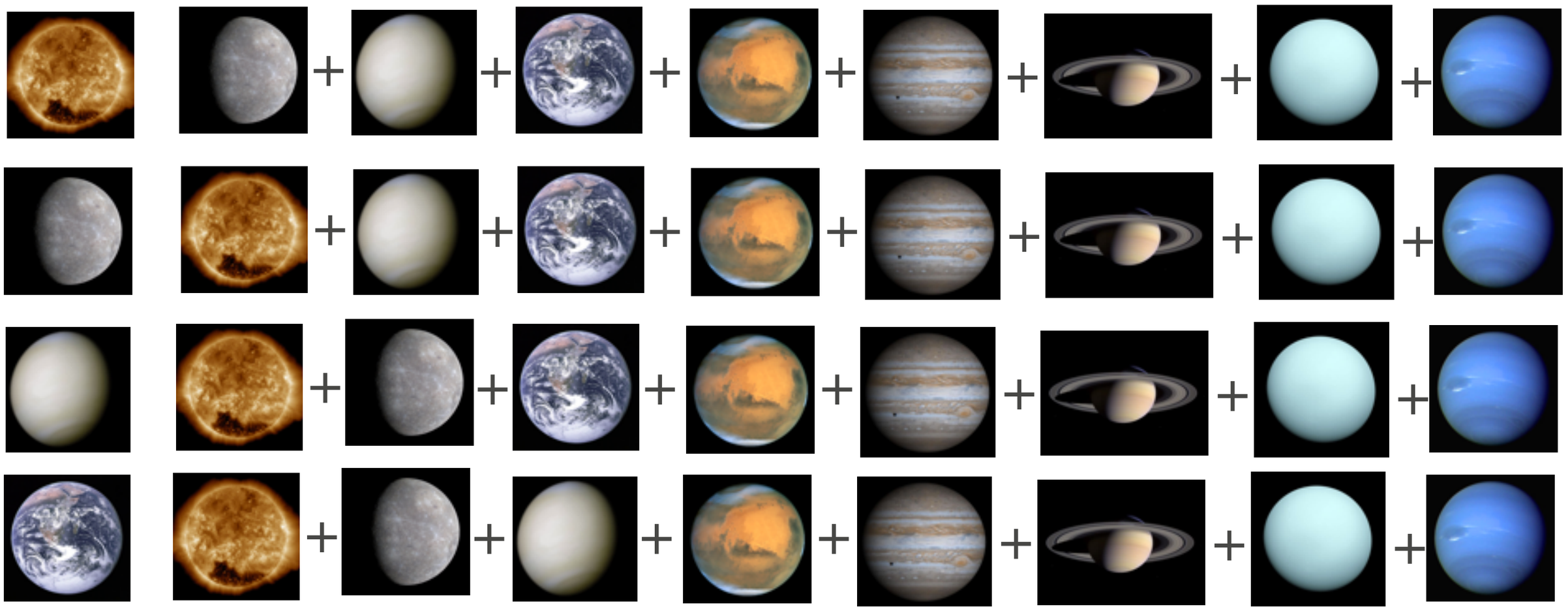

- .
- .
- .

n-body simulation

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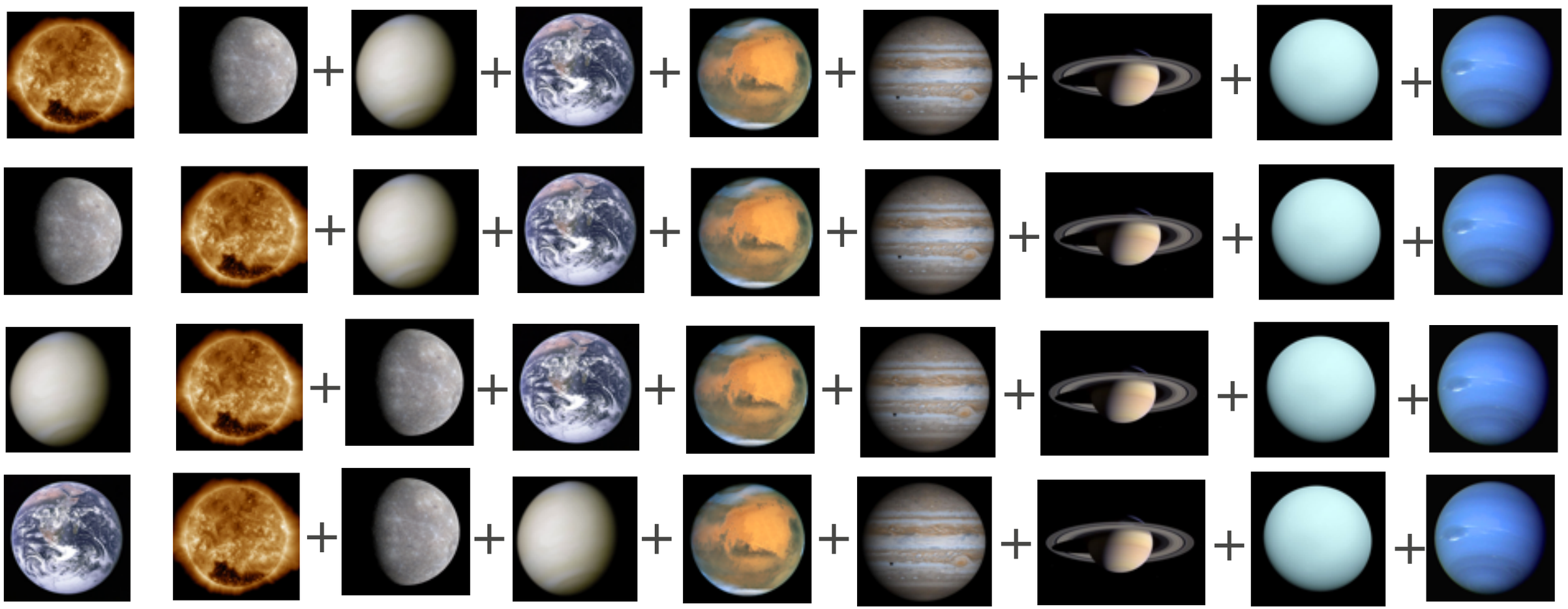

⋮

Parallelizability: $n^2/\log(n)$

Math is the code

```
fun accelerations (bodies : body Seq.seq) : vec Seq.seq =
    Seq.map (fn b1 =>
                 sum bodies (fn b2 => acc0n (b1 , b2)))
            bodies
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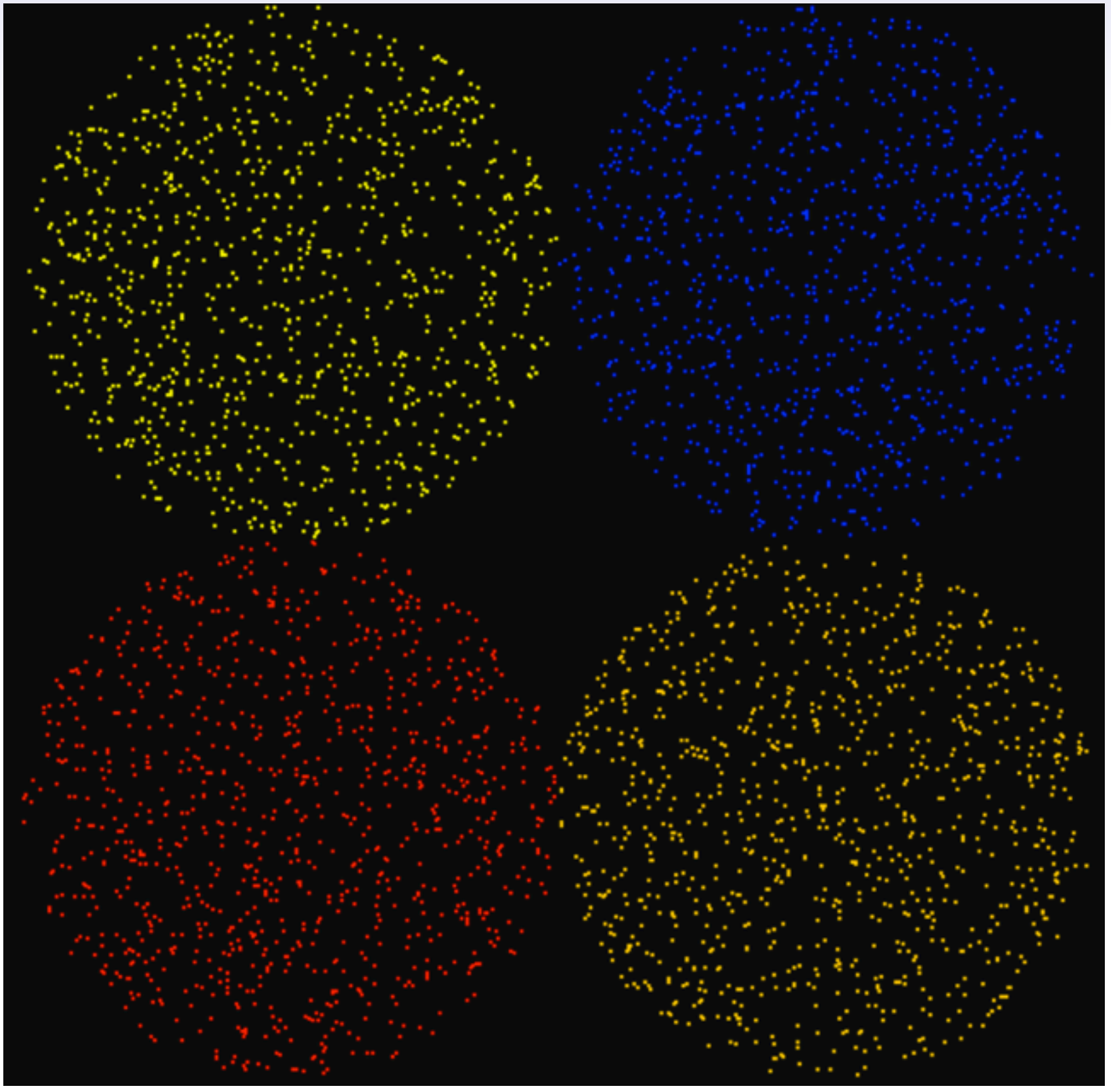
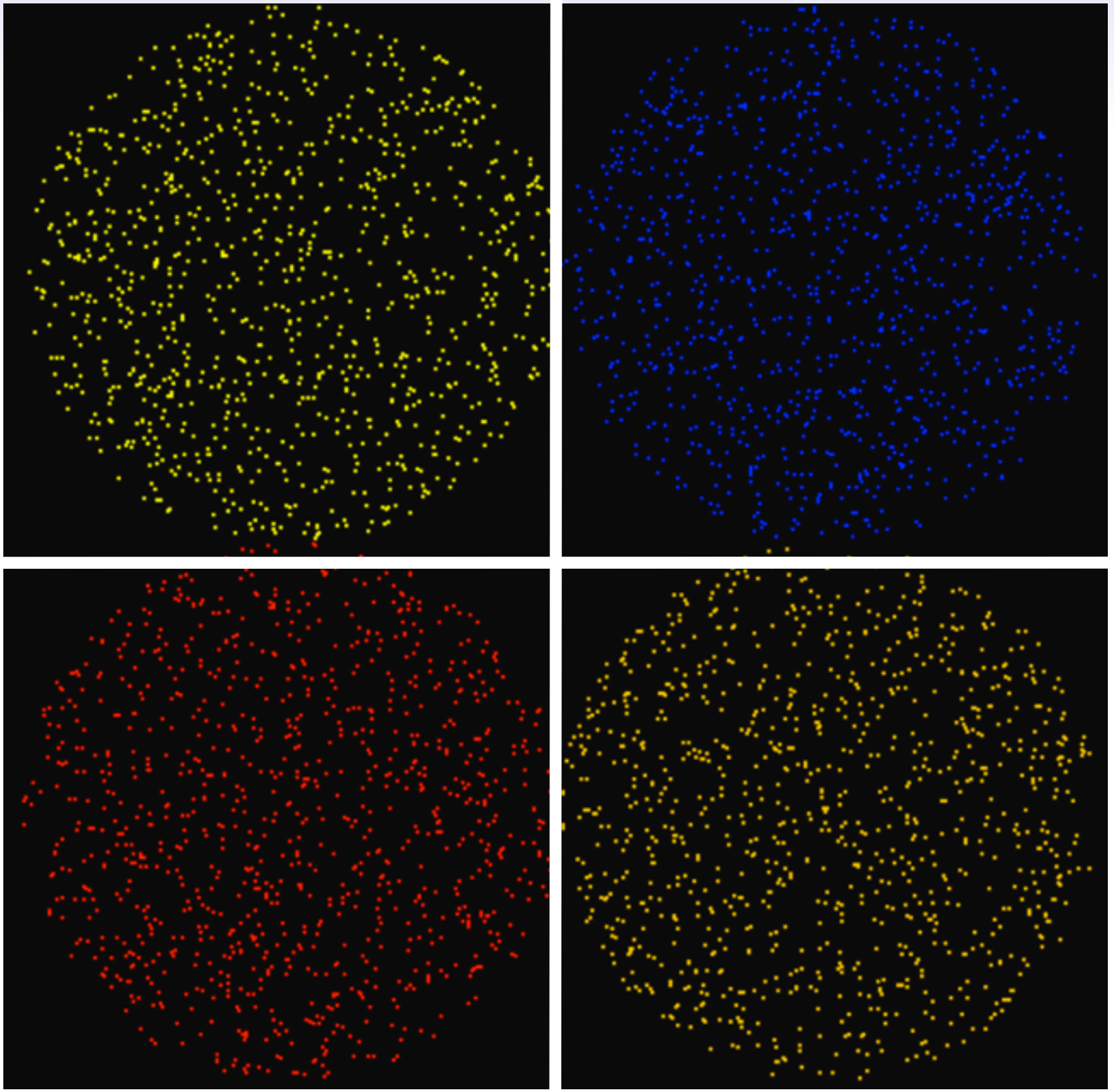
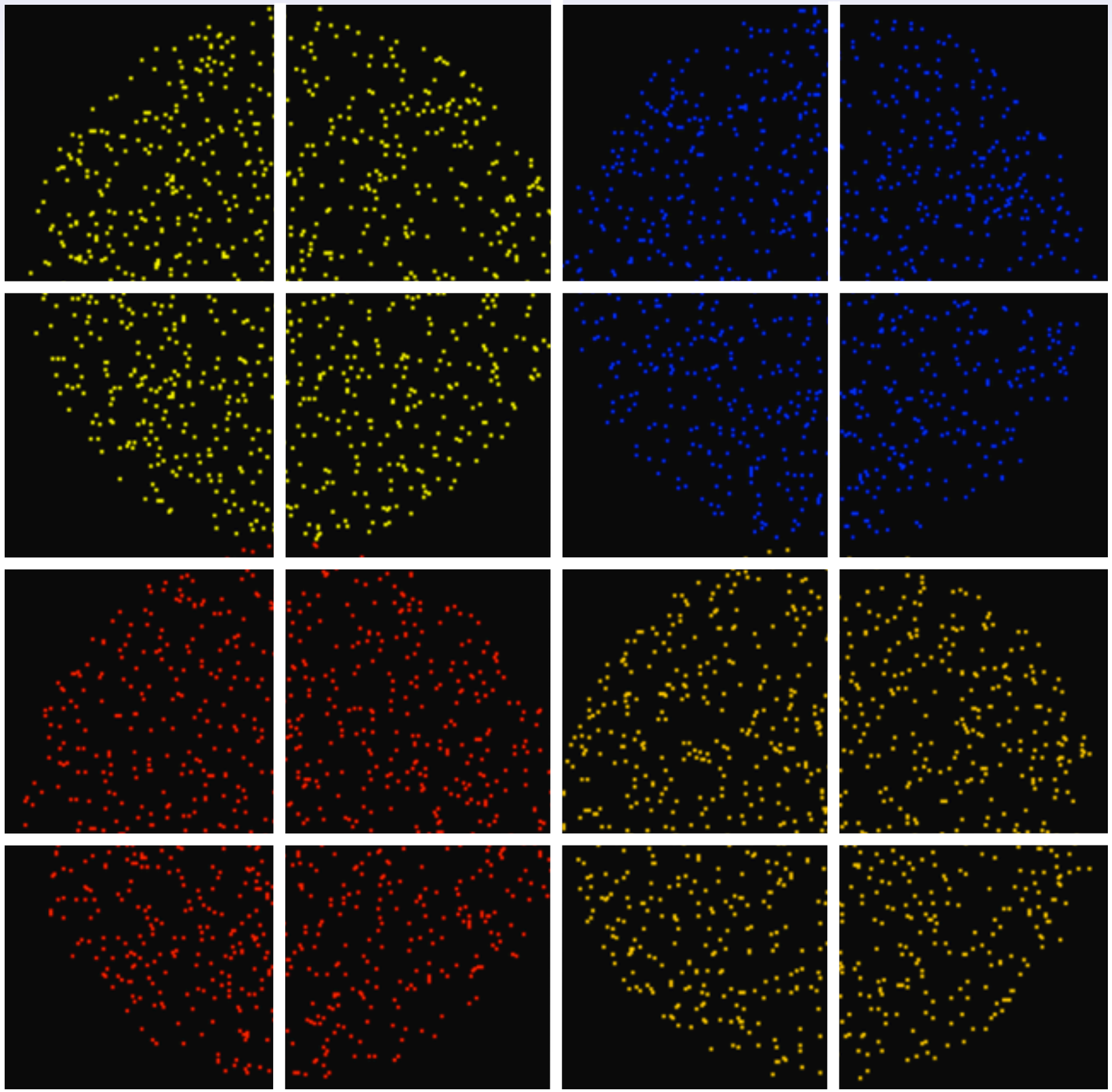
the acceleration on b1 due to b2 (Gm_2/d^2)

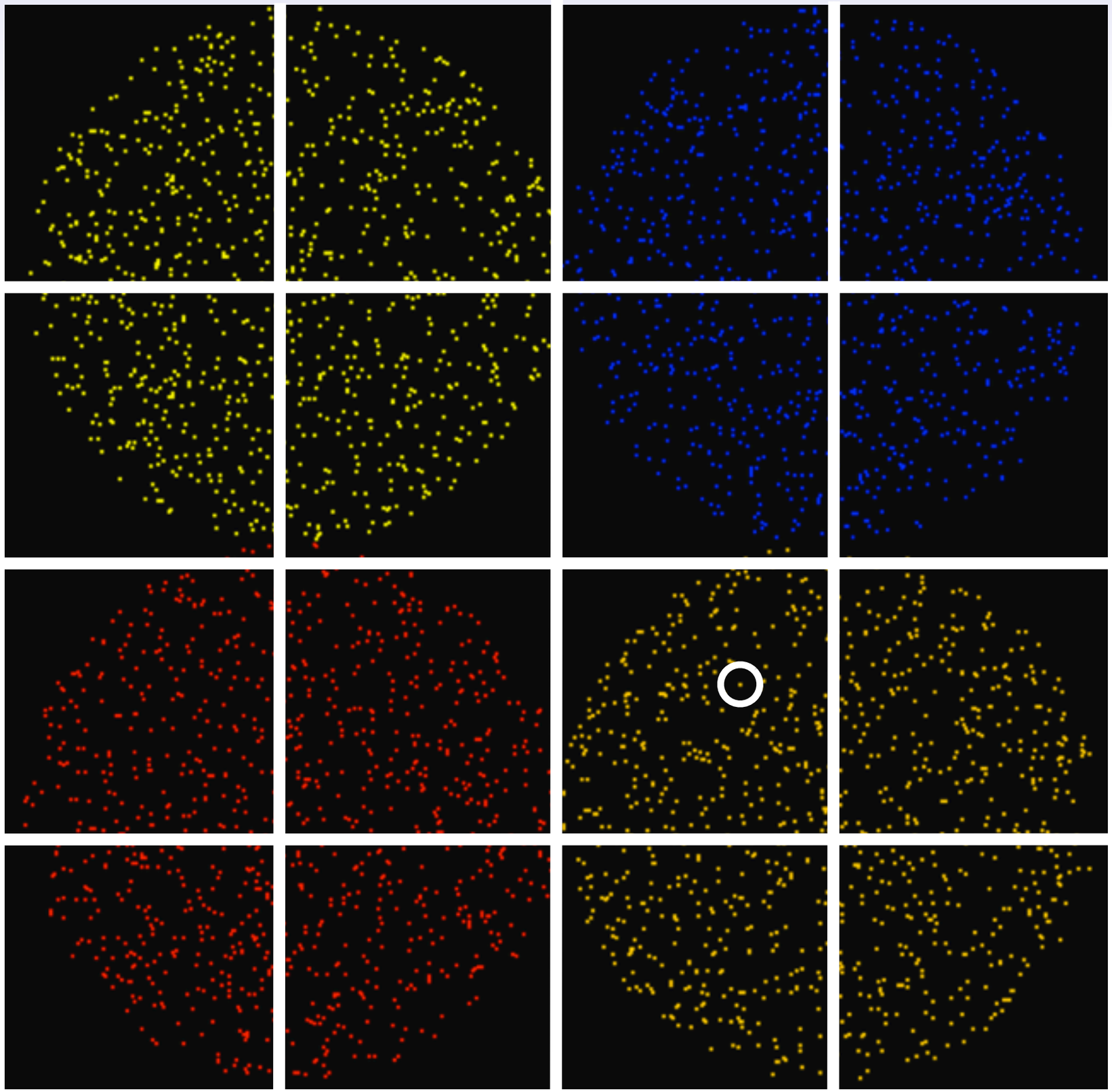
Barnes-Hut tree method

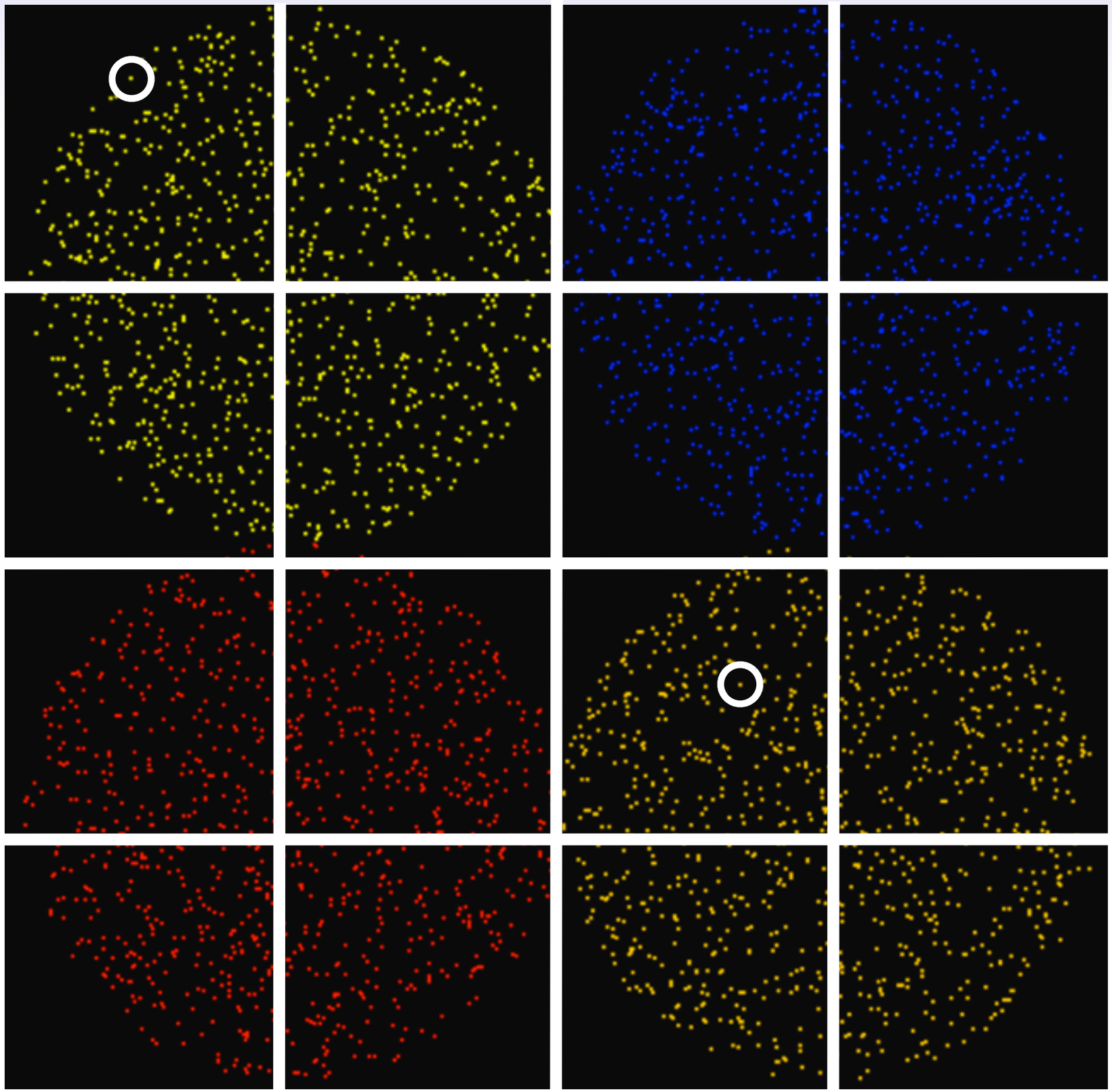
- $O(n^2)$ work is too high for some simulations
- Can approximate and still get useful results

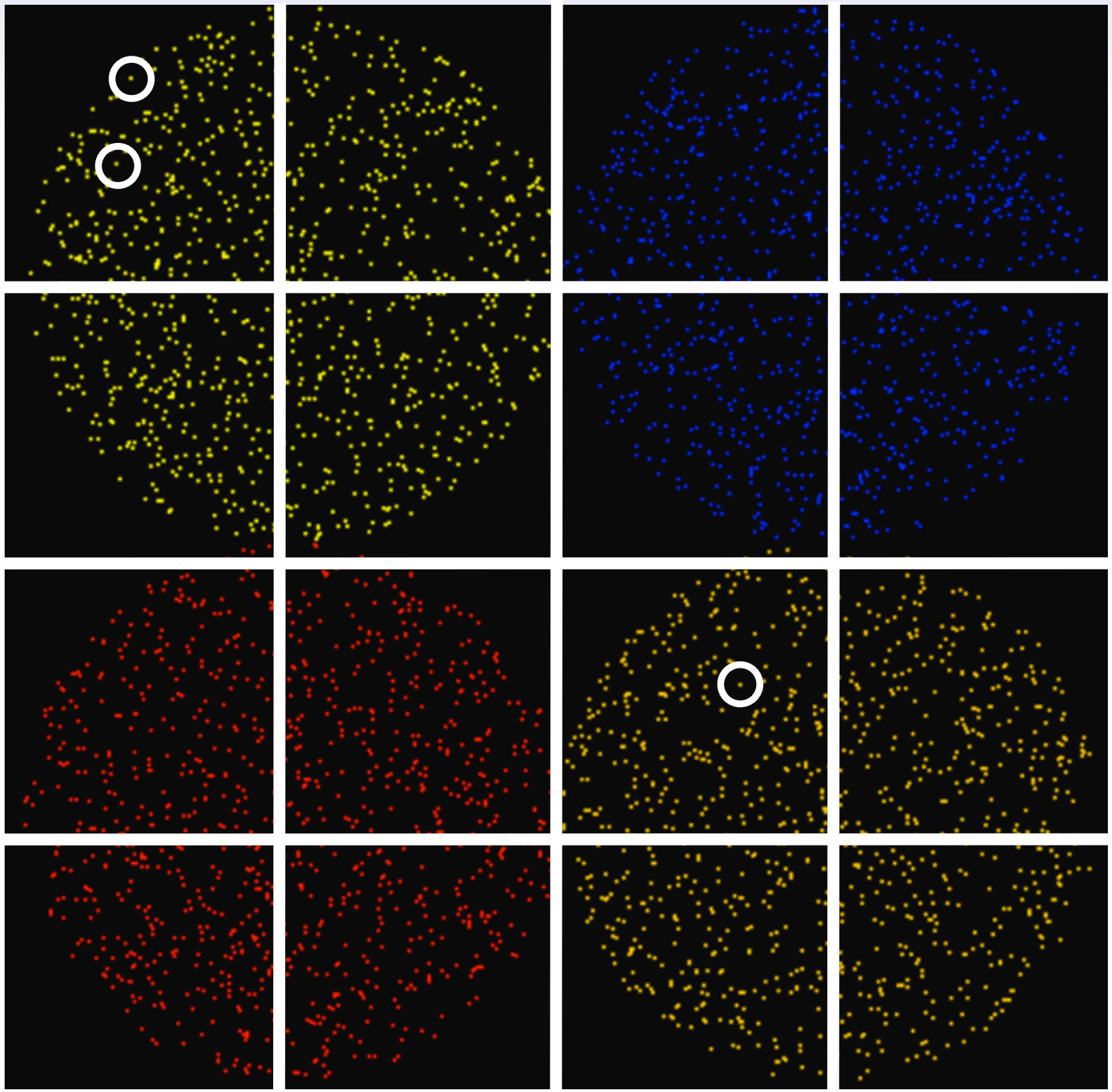


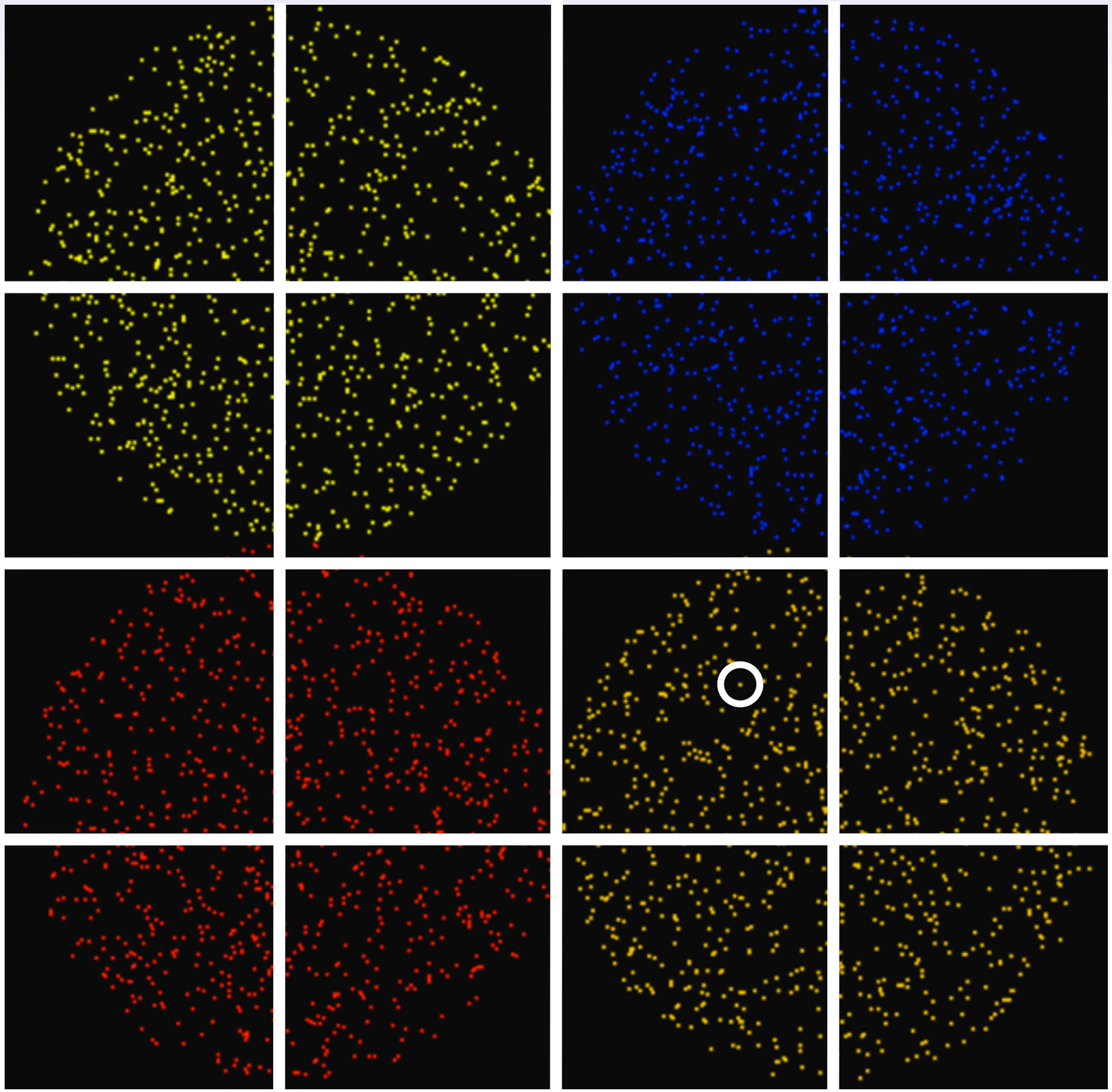


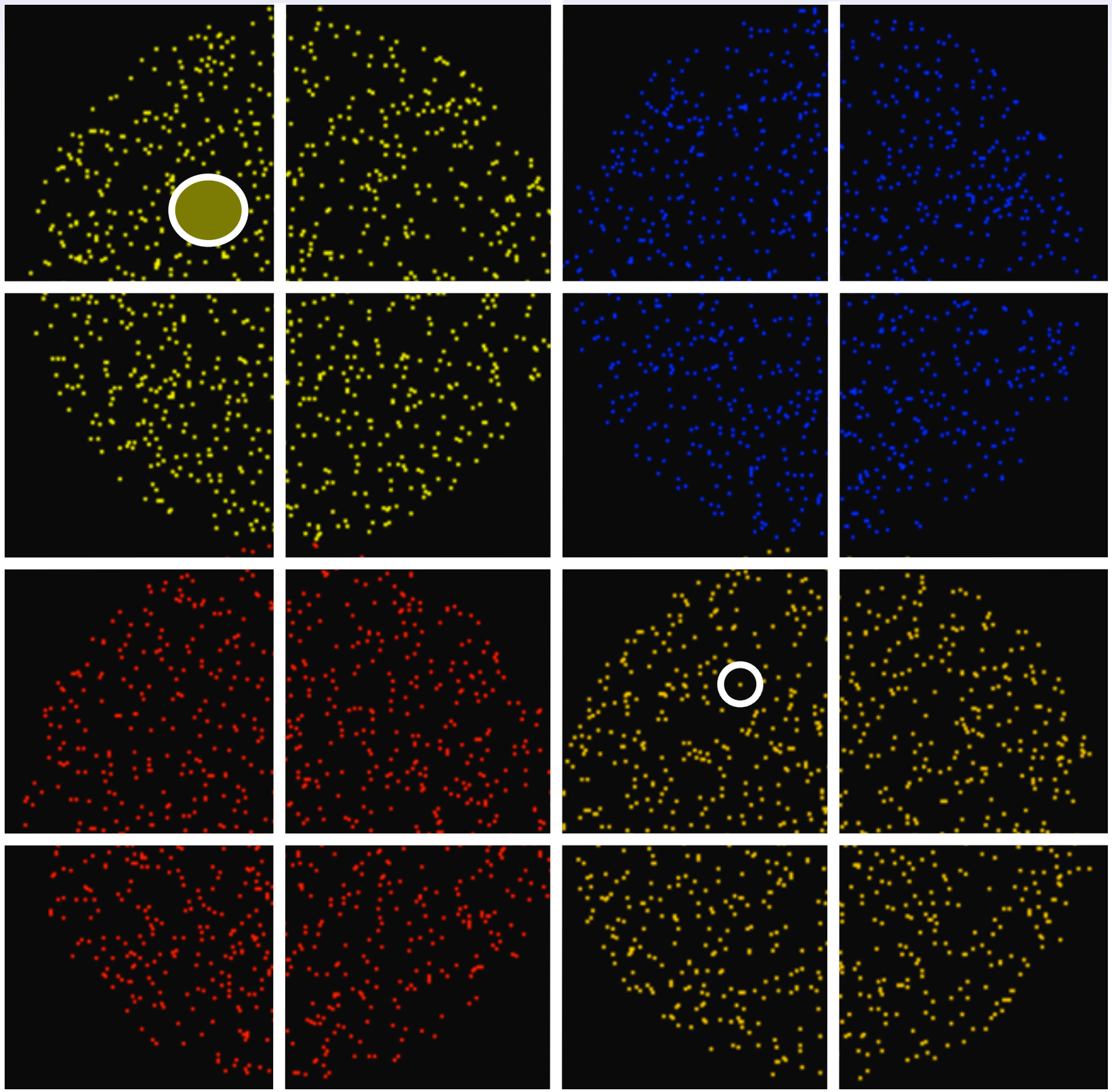




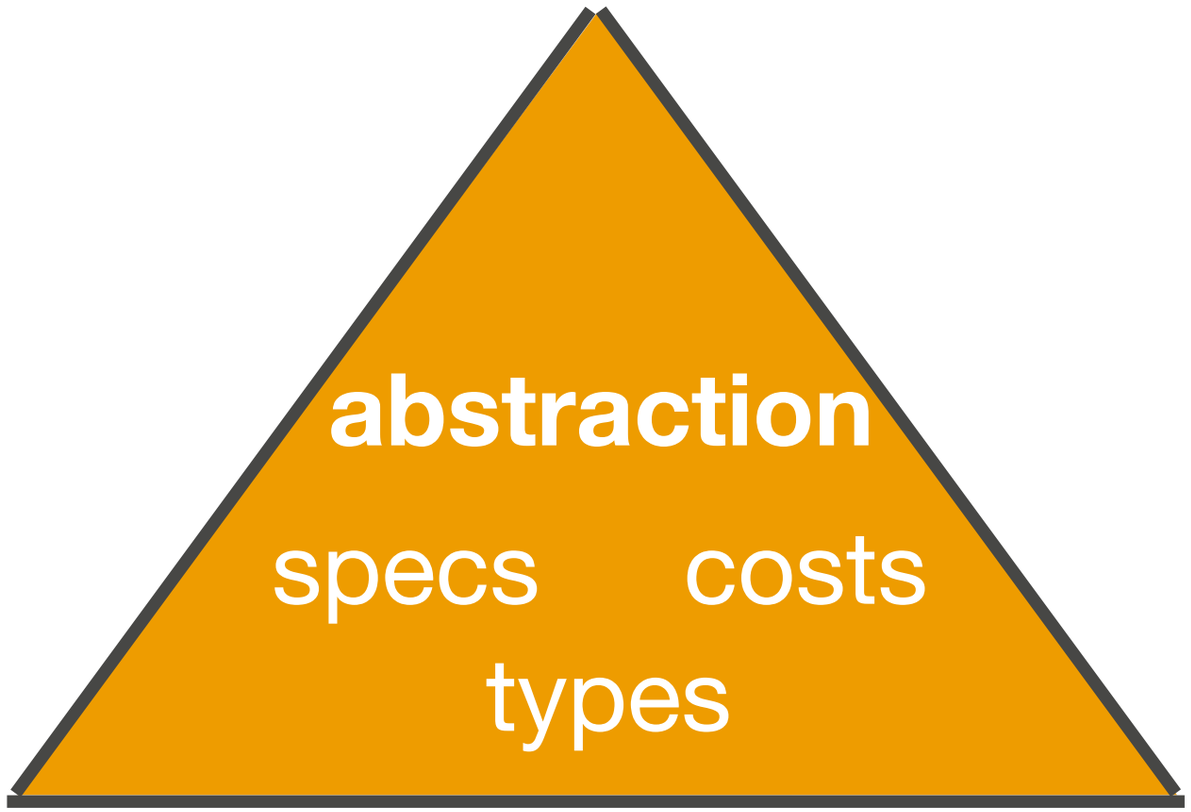








write code



reason about behavior

reason about efficiency

Conclusion

Now

```
signature EXTRACT_COMBINE =
sig
    structure MR : MAP_REDUCE

    structure D : DICT

    val extractcombine : ('a -> (D.Key.t * 'v) Seq.seq)
                         -> ('v * 'v -> 'v)
                         -> 'a MR.mapreducable
                         -> 'v D.dict

end
```

```
signature CLASSIFIER =
sig
    structure Dataset : MAP_REDUCE

    type category

    type document = string Seq.seq
    type labeled_document = category Seq.seq * document

    val train_classifier : labeled_document Dataset.mapreducable
                           -> document -> (category * real)
end
```

8 months ago

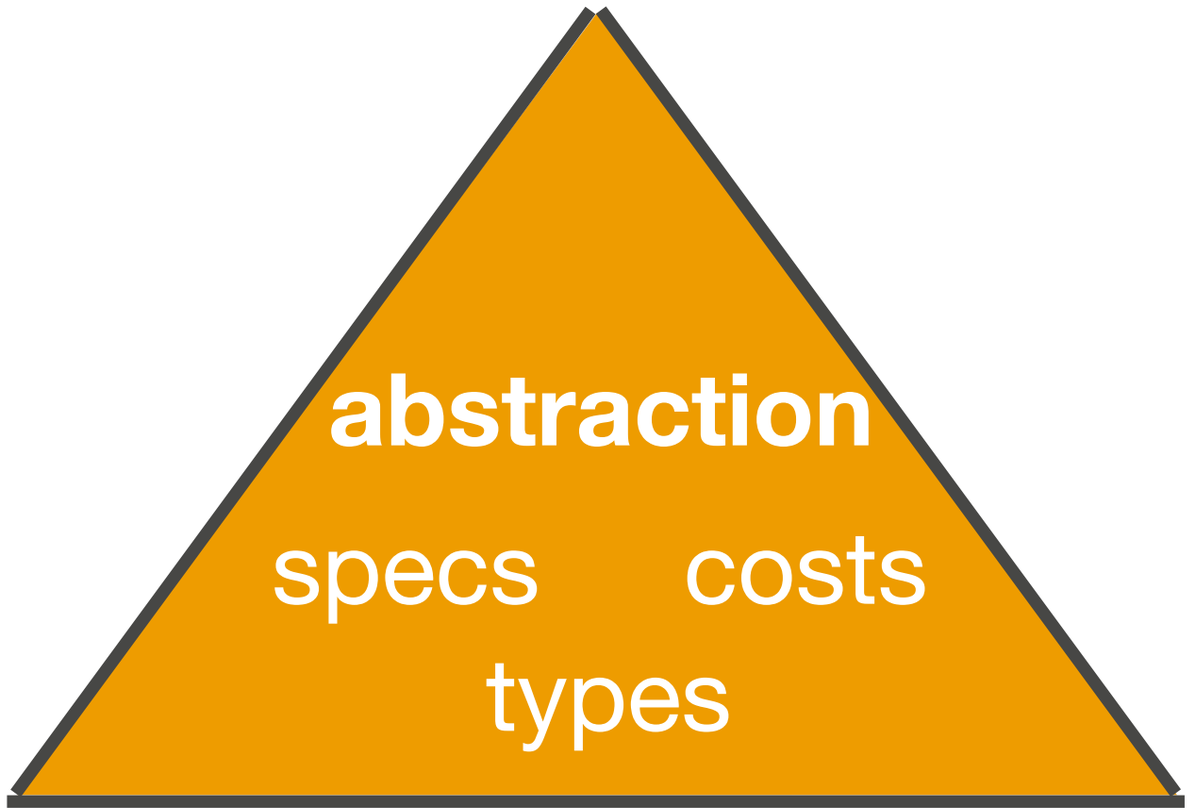
```
string suffix(int x) {

  string s;

  if (x == 1) {
     return "st";
  }
  else if (x == 2) {
     return "nd";
  }
  else  if (x == 3) {
     return "rd";
  }
  else {
     return "th";
  }

}
```

write code



reason about behavior

reason about efficiency

Code is math

Code is math

Code is art