Diversity-Coverage Index (DCI)

1. DCI Formulation

Definition. Let $d \in \mathscr{D}$ index a quality dimension with levels $\mathscr{L}_d = \{0,\dots,K_d-1\}$. Let $K_d := |\mathscr{L}_d|$, $\Delta^{K_d-1} = \{u \in \mathbb{R}_+^{K_d} : \sum_{\ell=0}^{K_d-1} u(\ell) = 1\}$, and let L_d be the level r.v. on $\mathscr{L}_d$. P denotes the persona random variable with weights $\pi = (\pi_a)$, where $\sum_a \pi_a = 1$ ($\Pr[P=a] = \pi_a$). For persona a, let $p_{a,d} \in \Delta^{K_d-1}$ be the empirical distribution of optima (argmax over cases) on dimension d, i.e. $p_{a,d}(\ell) = \Pr[L_d = \ell \mid P = a]$. The persona-marginal probability is defined as $q_d(\ell) = \sum_a \pi_a p_{a,d}(\ell) = \Pr[L_d = \ell]$. If available, $b_d \in \Delta^{K_d-1}$ serves as a persona-agnostic baseline on $\mathscr{L}_d$. Unless otherwise stated, we use equal persona weights ($\pi = 1/|\mathscr{P}|$) and set $\lambda = 0.5$. Let CDF $F_u(\ell)$ be $\sum_{j=0}^{\ell} u(j)$.

Persona signal via normalized mutual information. Write

$$H(L_d) = H(q_d), \qquad H(L_d \mid P) = \sum_a \pi_a H(p_{a,d}), \qquad H(P) = -\sum_a \pi_a \log \pi_a.$$

Define the symmetric normalized mutual information:

$$I_d \equiv \mathrm{NMI}_d = \frac{2I(P;L_d)}{H(L_d)+H(P)} = \frac{2\,[H(L_d)-H(L_d \mid P)]}{H(L_d)+H(P)} \in [0,1].$$

Large I_d means within-persona coherence (low $H(L_d \mid P)$) and across-persona disagreement (high $H(L_d)$).

Distance from a generic judge (persona-averaged). If a baseline b_d exists, measure the gap between each persona's optima distribution and the generic judge, then average across personas:

$$J_{a,d} = \mathrm{JS}\big(p_{a,d} \,\|\, b_d\big) \in [0,1], \qquad S_{a,d} = \widetilde{W}_1\big(p_{a,d}, b_d\big) = \frac{W_1(p_{a,d}, b_d)}{K_d - 1} \in [0,1].$$

Here JS is Jensen–Shannon divergence with base-2 logs (hence ≤ 1). For the Earth Mover's distance on the unit grid $\mathscr{L}_d = \{0,\dots,K_d-1\}$,

$$W_1(u,v) = \sum_{\ell=0}^{K_d-1} \big|F_u(\ell) - F_v(\ell)\big|,$$

and we normalize by the maximal possible value, $K_d - 1$:

$$\widetilde{W}_1(u,v) = \frac{W_1(u,v)}{K_d-1} \in [0,1].$$

Aggregate over personas with weights π_a to define the distinctness term:

$$\bar{J}_d = \sum_a \pi_a J_{a,d}, \qquad \bar{S}_d = \sum_a \pi_a S_{a,d}, \qquad D_d = \tfrac{1}{2}\big(\bar{J}_d + \bar{S}_d\big) \in [0,1].$$

DCI per dimension and overall. For $\lambda \in [0,1]$, define

$$\mathrm{DCI}_d(\lambda) = \lambda\, I_d + (1-\lambda)\, D_d \in [0,1].$$

If no baseline exists, set $\lambda = 1$ so $\mathrm{DCI}_d = I_d$. The overall index averages across dimensions:

$$\mathrm{DCI}(\lambda) = \frac{1}{|\mathscr{D}|} \sum_{d\in\mathscr{D}} \mathrm{DCI}_d(\lambda) \in [0,1].$$

2. Lemma of DCI

Lemma 1 (Shuffle Sanity). *Shuffling persona labels across i.i.d. cases removes persona effects on optimal levels, preserving the marginal π independently of $\mathscr{L}_d$ but making $\tilde{P} \perp L_d$. Thus $I(\tilde{P}; L_d) = 0$ and $I_d = 0$, so $\mathrm{DCI}_d(\lambda) = (1-\lambda)D_d$. Moreover, the plug-in estimator satisfies $\widehat{\mathrm{NMI}}_d \xrightarrow{p} 0$ as $n \to \infty$, where n is the total number of persona–case pairs (P, L_d) in the shuffled sample.*

Lemma 2 (Bounds and extremal cases). *Under the definitions above:*

- *(a) Boundedness.* $0 \le I_d, D_d, \mathrm{DCI}_d(\lambda) \le 1$. *Interpretation: each component and the combined index are normalized to* $[0,1]$.
- *(b) No persona signal (independence). $I_d = 0$ iff $P \perp L_d$. In particular, if $p_{a,d} = q_d$ for all a (no persona effect), then $I_d = 0$. Interpretation: when all personas share the same optima distribution, personas carry no information about levels.*
- *(c) No deviation from the generic judge. If $p_{a,d} = b_d$ for all a, then $\bar{J}_d = \bar{S}_d = 0$, hence $D_d = 0$ and $\mathrm{DCI}_d(\lambda) = \lambda I_d$. Interpretation: if every persona's distribution matches the generic judge, only the NMI term can contribute to DCI.*
- *(d) Perfect separability. If L_d is a deterministic injective function of P whose image has size M and π is uniform, then $H(L_d) = H(P) = \log M$ and $I_d = 1$. More generally, if $H(L_d \,|\, P) \to 0$ and $H(L_d) \to H(P)$, then $I_d \to 1$. Interpretation: each persona deterministically picks a distinct level (zero within-persona entropy), so NMI saturates at 1.*

3. Proofs for DCI lemmas

Lemma 1. By construction $\tilde{P} \perp L_d$, so $I(\tilde{P}; L_d) = 0$ and the normalized form gives $I_d = 0$; the DCI identity follows immediately. For consistency, empirical entropies on finite alphabets are strongly consistent; thus the plug-in mutual information (and its normalized variant) converges in probability to the true value 0 under independence (assuming $H(P) + H(L_d) > 0$ so the normalization is defined).

Lemma 2. (a) By construction with base-2 logs: $I_d \in [0,1]$ for the symmetric NMI; $\bar{J}_d \in [0,1]$ since JSD ≤ 1; $\bar{S}_d \in [0,1]$ because $W_1 \leq K_d - 1$ on the unit grid, so $\widetilde{W}_1 = W_1/(K_d - 1) \in [0,1]$. Convex combinations preserve bounds.

(b) $I(P;L_d) = 0$ iff $P \perp L_d$; plugging into the NMI formula yields $I_d = 0$. If $p_{a,d} = q_d$ for all a (i.e., each persona's distribution equals the marginal), then P and L_d are independent, so $I_d = 0$.

(c) If $p_{a,d} = b_d$ for all a, then each $J_{a,d} = \mathrm{JS}(p_{a,d} \| b_d) = 0$ and $S_{a,d} = \widetilde{W}_1(p_{a,d}, b_d) = 0$, hence $\bar{J}_d = \bar{S}_d = 0$ and $D_d = 0$, so $\mathrm{DCI}_d(\lambda) = \lambda I_d$.

(d) Deterministic $L_d = f(P)$ implies $H(L_d \mid P) = 0$ and $I(P;L_d) = H(L_d)$. If f is injective with uniform π on an image of size M, then $H(L_d) = \log M = H(P)$, hence

$$I_d = \frac{2H(L_d)}{H(P) + H(L_d)} = \frac{2\log M}{\log M + \log M} = 1.$$

The limit statement follows by continuity of NMI in $(H(L_d \mid P), H(L_d), H(P))$.