Piecewise-Linear Knot Theory and First and Second Tait Conjectures

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I pledge my honor that I have not violated the university Honor Code.

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0 Introduction

In this paper, following the work of Lickorish, we define a knot as a subset of a piecewise-linear 3-sphere that is a piecewise linear, simple closed curve whereby piecewise linear, a curve consists of finitely many line segments that are straight in the linear structure of S^3. Granting knots a piecewise linear structure prevents them from having pathological behaviors and we call such well behaved knots tame knots. One can equivalently define tame knots as smooth embeddings of a unit circle S^1 into a unit 3-sphere S^3 and proceed in the category of smooth manifolds instead of piecewise linear manifolds but this may lead to wild knots with infinitely many kinks. This paper has two objectives: 1) to offer a self-contained introduction of piecewise-linear knot theory for readers who are familiar with basic point-set topology and algebra and 2) to present complete proofs of the first and second Tait conjectures for alternating knots.

Section 1 of the paper will introduce the readers to basic homology theory covering from simplicial homology and singular homology to Euler characteristic and Mayer-Vietoris sequence. The themes in homology theory will recur throughout the paper, underpinning results both in piecewise linear topology and knot theory. Section 1 will closely follow [1]. Section 2 will introduce the readers to piecewise-linear topology which sets the foundation for piecewise-linear knot theory. In particular, the concepts of triangulation, regular neighborhoods, and PL homeomorphism will play an important role in defining equivalence classes on links and knots and giving a linking number a homological interpretation. Although section 2 follows the developments in [3] and [4], I have recast several definitions and arguments and supplied original proofs to present a clearer, self-contained exposition. Section 3 will introduce the readers to basic piecewise linear knot theory with a focus and knot diagrams and Jones polynomial. The paper will eventually prove the first Tait conjecture which claims that a reduced alternating diagram of a knot has minimal crossing and the second Tait conjecture which claims that any reduced alternating diagrams of a knot must have the same writhe. Section 3 will closely follow [3].

1 Homology Theory

1.1 CW-Complexes

A **CW-complex (cell complex)** X is a topological space obtained inductively by gluing together topological open balls of different dimensions, called cells, in the following manner:

1. X^0, referred to as the 0-skeleton of the complex, is a set of discrete points or *0-cells*.
2. Suppose a $(n-1)$-skeleton, X^{n-1} is given. A n-skeleton, X^n is constructed from X^{n-1} by gluing n-dimensional cells e^n_α through $\phi_\alpha : \partial B^n = S^{n-1} \to X^{n-1}$, α is an index set. In other words, X^n is the quotient space $X^{n-1} \bigsqcup_\alpha D^n_\alpha / \sim$ under the equivalence relation $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$.

The process can be stopped after finite steps such that $X = \bigcup_n X^n$ or can be continued indefinitely. If $X = \bigcup_n X^n$, the space is given a weak topology (i.e. $A \subset X$ is open $\Leftrightarrow$ $A \cap X^n$ is open in X^n for all n). A 0-dimensional CW complex is given a discrete topology. If a n-dimensional CW complex is constructed by gluing one or more copies of n-dimensional cells to a k-dimensional CW complex ($k < n$), then the CW complex is given a quotient topology defined by the gluing maps relative to the disjoint union topology of the space before identification.

Any gluing map $\phi_\alpha : \partial B^n \to X^{n-1}$ can be extended to a **characteristic map** $\Phi_\alpha : D^n_\alpha \to X$ such that $\Phi_\alpha|_{\mathring{D}^n_\alpha}$ is a homeomorphism onto e^n_α. In other words, the characteristic map is the composition $D^n_\alpha \hookrightarrow X^{n-1} \bigsqcup_\alpha D^n_\alpha \to X^n \hookrightarrow X$. We define a subcomplex A of X as a closed subspace $A \subset X$ that has a cell complex structure. A pair (X,A) is called a CW pair. One can easily verify that each n-skeleton of a cell complex X is a subcomplex. If (X,A) is a CW pair, then one can verify that the quotient space X/A has a cell complex structure: the cells of X/A are the cells of $X - A$ with one new 0-cell. If a cell e^n_α of $X - A$ was glued by the map $\phi_\alpha : S^{n-1} \to X^{n-1}$, the corresponding cell in X/A inherits the following composition as its gluing map: $S^{n-1} \to X^{n-1} \to X^{n-1}/A^{n-1}$. If X, Y are two cell complexes, the product space $X \times Y$ also gives a cell complex whose cells are products $e^m_\alpha \times e^n_\beta$, homeomorphic to $m+n$-dimensional open balls. Now, we define two important operations on topological spaces, suspension and join.

1.1.1 Suspension

For a topological space X, define the **suspension** SX to be the quotient space of $X \times I$ by identifying $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. If $X = S^1$, SX will give the double cone which is a union of two copies of the cone, CX pointing opposite directions. If X is a cell complex, then both CX and SX, being quotients of the product cell complex $X \times I$, possess a cell structure.

1.1.2 Join

Given X and Y, define the **join** $X \star Y$ to be the union of all line segments joining a point of X to a point of Y. One can notice then that $CX = X \star \{p\}$ and $SX = X \star \{p,q\}$ where p,q are vertices of the double cone. One can show that $X \star Y$ is the quotient space of $X \times [0,1] \times Y$ under the relations $(x,y_1,0) \sim (x,y_2,0), (x_1,y,1) \sim (x_2,y,1)$. This intuitively represents collapsing $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y. Every point of $X \star Y$ can be represented as a linear combination $t_1 x + (1-t_1)y, 0 \leqslant t_i \leqslant 1$ where $0x + 1y = y, 1x + 0y = x$. Likewise, every point of $X_1 \star ... \star X_n$ can be represented as a linear combination $t_1 x_1 + ... + t_n x_n$ with $0 \leqslant t_i \leqslant 1$, $\sum_i t_i = 1$, and $0t_i$ can be removed from the combination. If X_i is a point, then the join $\star X_i$ is a *n-1* dimensional simplex (high dimensional analog of triangles) which we will define rigorously later. If X and Y are cell complexes, then one can give $X \star Y$ a cell structure where $X, Y \subset X$ are subcomplexes and the remaining cells are given by product cells of $X \times Y \times (0,1)$.

1.2 Simplicial Complexes and Δ-Complexes

While CW-complexes convey a variety of topological information, a similar notion of simplicial and Δ complexes capture combinatorial features that often make concrete computations simpler and more convenient than using CW-complexes. A ***n*-simplex**, σ is the convex hull of n affinely independent points, $\{v_0, ..., v_n\}$ (i.e. each point $x \in \sigma$ can be uniquely expressed as $\sum_{i=0}^n t_i v_i$ where $\sum_{i=0}^n t_i = 1$. In this case, $(t_1, \ldots, t_n)$ is called the **barycentric** coordinate of x). This condition is equivalent to the linear independence of vectors $\{v_1 - v_0, ..., v_n - v_0\}$ where v_0 is a reference point. The points v_i's are referred to as vertices of the simplex and σ is said to be spanned by the vertices v_i's, which we often denote as $\sigma = v_0 v_1 ... v_n$. A face τ of σ is a simplex spanned by a nonempty subset of vertices of σ, and we write $\tau < \sigma$.

An **orientation of a *p*-simplex** (not to be confused with orientation on a face induced by a boundary operator which will appear later when defining homology groups) is the ordering of the vertices such that two orderings return the same orientation if and only if they differ by an even permutation. In other words, switching the order of two vertices changes the orientation. The orientation of a simplex $v_0...v_n$ is then the strict total ordering of vertices $v_0 < \cdots < v_n$ and hence it naturally determines the orientation of its faces. A standard n-simplex, denoted Δ^n is a n-simplex whose vertices are standard unit vectors along the coordinate axes of the Euclidean space and whose orientation is canonically determined by the Euclidean space. One can always construct a canonical linear homeomorphism from Δ^n to an arbitrary n-simplex $v_0...v_n$ that preserves the orientation by mapping the barycentric coordinate of a point in Δ^n, $(t_0, ... t_n)$ to $\sum_i t_i v_i$.

A set K of simplices in $\mathbb{R}^n$ is a **simplicial complex** if it satisfies the following conditions:

1. if $\sigma \in K$ and $\tau < \sigma$, then $\tau \in K$
2. if $\sigma, \tau \in K$ and $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau < \sigma$ and $\sigma \cap \tau < \tau$
3. for every $x \in \sigma \in K$, there exists a neighborhood of x that intersects with only finitely many $\tau \in K$ (K is then referred as locally finite).

Simplicial complexes K and L are defined to be isomorphic, denoted $K \cong L$, if there exists a face preserving bijection. We call the subset $|K| = \bigcup\{\sigma \in K\} \subset \mathbb{R}^n$ the polyhedron of K. Equivalently, we can define a polyhedron to be a subspace P of $\mathbb{R}^n$ such that for every point $a \in P$, there exists a cone $N = a \star L$ where L is a compact subset of P and N is a neighborhood of a in P (i.e. there exists an open neighborhood of a, U in $\mathbb{R}^n$ such that $U \cap P \subset N$). One can show that then P is a realization of a simplicial complex. Analogous to cell complexes, we give the simplicial complex a weak topology, such that a subset $A \subset |K|$ is closed if and only if for all $\sigma \in K$, $A \cap \sigma$ is closed.

Property (3) shows that the weak topology then coincides with the subspace topology of $|K|$. To add more details, suppose we are given a set X and maps $f_i : (X_i, \tau_i) \to X$ where τ_i is the topology of X_i. The final topology on X with respect to $\{f_i\}$ is defined as

$$\tau_{\text{final}} = \left\{ U \subset X | f_i^{-1}(U) \in \tau_i \text{ for every } i \right\}.$$

This makes the final topology the finest topology which makes each f_i continuous. Now, let $X = |K| \subset \mathbb{R}^n$ and f_i the inclusions $j_\sigma : \sigma \to |K|$. Since every simplex σ inherits a subspace topology τ_σ from $\mathbb{R}^n$, the final topology coincides with the weak topology, i.e.

$$\tau_{\text{weak}} = \left\{ U \subset |K| \big| U \cap \sigma \text{ is open in } \sigma \text{ for all } \sigma \in K \right\}.$$

Hence the weak topology is the finest topology that makes the inclusion maps j_σ continuous. In other words, if any other topology τ makes every j_σ continuous, then $\tau \subseteq \tau_{\text{weak}}$. Now consider the subspace topology on $|K|$, denote it τ_{sub}. If $U \in \tau_{\text{sub}}, U = V \cap |K|$ for some V open in $\mathbb{R}^n$. Then for any arbitrary simplex σ, $j_\sigma^{-1}(U) = U \cap \sigma = (V \cap |K|) \cap \sigma = V \cap \sigma$, and $V \cap \sigma$ is open in σ. Hence, each j_σ is continuous, implying that $\tau_{\text{sub}} \subseteq \tau_{\text{weak}}$. To prove that they are actually equal, suppose $U \in \tau_{\text{weak}}$, i.e. for every $\sigma \in K, U \cap \sigma$ is open in σ. Let $C = |K| \backslash U$. C is closed in the subspace topology on $|K|$ if and only if $C = D \cap |K|$ for some closed set $D \subset \mathbb{R}^n$. It suffices to prove then that C contains all its limit points (relative to $\mathbb{R}^n$, we denote the closure of C in $\mathbb{R}^n$ as $\overline{C}$) that are also in $|K|$. Since $U \cap \sigma$ is open in σ, $C \cap \sigma = \sigma \backslash (U \cap \sigma)$ is closed in σ for every $\sigma \in K$ and hence $C \cap \sigma$ is closed in $\mathbb{R}^n$. Now let x in $\overline{C} \cap |K|$ be arbitrary . Since K is locally finite, there exists a open neighborhood of x, say N_x in $\mathbb{R}^n$ such that N_x intersects only finitely many simplices, say $\sigma_1, ..., \sigma_k$. Moreover, any such neighborhood of x must contain points from C since x is in the closure of C. If y is an arbitrary point in $N_x \cap C$, y must belong to at least one of the simplices $\sigma_1, \ldots, \sigma_k$. Hence $y \in N_x \cap C \cap \bigcup_i \sigma_i = \bigcup_i (N_x \cap C \cap \sigma_i)$. Let $C' = \bigcup_i (C \cap \sigma_i)$. Since each $C \cap \sigma_i$ is closed in $\mathbb{R}^n$, and C' is a finite union, C' is closed in $\mathbb{R}^n$. Since N_x contains points of C, and such points must lie in C', x must be a limit point of C', i.e. $x \in C'$. Therefore, $x \in \bigcup_i (C \cap \sigma_i)$. Since there exists j such that $x \in C \cap \sigma_j$, $x \in C$ implies that $\overline{C} \cap |K| = C$, proving C is closed in the subspace topology on $|K|$. This proves $\tau_{\text{weak}} \subseteq \tau_{\text{sub}}$.

A simplicial complex L is a subcomplex of K, denoted $L < K$, if $L \subseteq K$. $K^{(p)} = \{\sigma \in K : dim\sigma \leqslant p\}$ is referred to as the p-skeleton of K. Given a simplex σ, its boundary subcomplex $\dot{\sigma}$ is defined as the collection of proper faces of σ (i.e. $\dot{\sigma} = \{\tau < \sigma : \tau \neq \sigma\}$) and its interior is defined as $\mathring{\sigma} = \sigma - |\dot{\sigma}|$. By convention, for $\sigma \in K^0, \mathring{\sigma} = \sigma$. If L is a subcomplex of K, $|L| = A$ where $A \subset |K|$, then we will denote $L = K|A$.

A simplicial complex K_1 is called a subdivision of a complex K, denoted $K_1 \prec K$, if $|K_1| = |K|$ and every simplex τ of K_1 is contained in some simplex σ of K. If σ is a simplex, $\dot{\sigma}$ is the boundary complex defined as above, and $L \prec \dot{\sigma}$, and $x \in \mathring{\sigma}$, then the complex $K = L \cup \{xw_0w_1...w_k : w_0w_1...w_k \in L\}$ is called a subdivision obtained by **starring σ at x over L**. A **derived subdivision** of a complex K is obtained inductively as follows. Assume that the *(p-1)*-skeleton $K^{(p-1)}$ has a subdivision L and let σ be a p-simplex of K. Pick a point $\hat{\sigma}$ in $\mathring{\sigma}$ and star σ at $\hat{\sigma}$ over $L||\dot{\sigma}|$, that is the subcomplex of L whose realization is the boundary of σ. This yields a subdivision of the p-skeleton and repeating this process yields a derived subdivision. If L is a subcomplex of K and we inductively pick points $\hat{\sigma} \in \mathring{\sigma}$ for $\sigma \notin L$, then the resulting subdivision is called the **derived subdivision of K mod L**. If we pick $\hat{\sigma}$ at every step to be the the barycenter $\beta(\sigma)$ of $\sigma = v_0...v_p$, that is $\beta(\sigma) = \sum \frac{1}{p+1} v_i$, then the derived subdivision is known as the **first barycentric subdivision**, denote K^1. Notice one can compute the nth barycentric subdivision by reiterating the process on the derived subdivision. We denote the nth barycentric subdivision as K^n.

A diameter of a simplex is defined to be the maximum distance between any two of the points contained in the simplex, defined by using the metric of the ambient Euclidean space. The diameter of any simplex $\sigma = v_0,...,v_n$ is equal to the maximum distance between any two vertices of σ since for any arbitrary point $\sum_i t_i v_i, \sum t_i = 1$ and $v \in \sigma$,

$$|v - \sum_i t_i v_i| = |\sum_i t_i(v - v_i)| \leqslant \sum_i t_i |v - v_i| \leqslant \sum_i t_i \max |v - v_i| = max|v - v_i|$$

and we may repeat the same for $v \in \sigma$. The mesh of a simplicial complex K is the maximum diameter of simplexes in K. By the same logic as above, mesh $K = |v_i - v_j|$ where v_i, v_j are vertices of some simplex $\sigma \in K$.

Lemma 1.2.1 The diameter of each simplex of the barycentric subdivision of $v_0...v_n$ is at most $n/(n+1)$ times the diameter of $v_0,...,v_n$.

Proof: Let $w_0,...,w_n$ be a simplex in the subdivision of $v_0,...,v_n$. We prove the lemma by showing that the distance between any two vertices w_i, w_j is at most $n/(n+1)$ times the diameter of $v_0,...,v_n$. Consider two cases. First, suppose that neither w_i or w_j is the barycenter b of $v_0,...,v_n$. Then, both w_i and w_j must belong in a face of $v_0,...,v_n$. Since the face is a lower dimensional simplex, the result follows from induction. Next, suppose that w_i is a barycenter b of $v_0,...,v_n$. Then we may assume w_j to be the vertex v_k by the inequality above. Now, suppose b_i is a barycenter of $v_0,...,\hat{v}_i,...,v_n$. Then, $b = \frac{1}{(n+1)}v_i + \frac{n}{n+1}b_i$. This implies that b lies on the line segment between v_i and b_i and the distance from b to v_i is the length of $v_i b_i$ times $n/(n+1)$. Therefore, the distance from b to v_i is bounded by $n/(n+1)$ times the diameter of $v_0,...,v_n$. Essentially, the theorem implies that if we repeat the barycentric subdivision, we can produce arbitrary small simplices since $(n/(n+1))^r \to 0$ as $r \to \infty$.

A **Δ-complex** structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ where n depends on α and the collection meets the following conditions:

1. $\sigma_\alpha|_{\mathring{\Delta}^n}$ is an injection, and for every point p in X, there exists a unique α such that $p \in \sigma_\alpha(\mathring{\Delta}^n)$
2. If $\tau < \Delta^n$, then $\sigma_\alpha|_\tau$ is identical to one of the maps $\sigma_\beta : \Delta^{n-1} \to X$ after identifying τ with Δ^{n-1} via a canonical linear homeomorphism that preserves the orientation of the simplex.
3. $A \subset X$ is open if and only if for every σ_α, $\sigma_\alpha^{-1}(A)$ is open in Δ^n.

One can now notice that every Δ-complex is a CW-complex, and every simplicial complex is a Δ-complex. More precisely, Δ-complex is a CW complex with a restriction that each cell e_α^n has a characteristic map $\sigma_\alpha : \Delta^n \to X$ whose restriction to each $(n-1)$ dimensional face $\tau < \Delta^n$ is a distinguished characteristic map for e_β^{n-1} of X, β is some index for a $n-1$ cell of X.

1.3 Homology Theory

1.3.1 Simplicial Homology

Assume that a space X has a Δ-complex structure. Define $\Delta_n(X)$ to be the free abelian group whose generators are the characteristic maps $\sigma_\alpha : \Delta^n \to X$ defined as above. Then, every element of $\Delta_n(X)$, called n-chain, can be represented as a finite sum $\sum_\alpha n_\alpha \sigma_\alpha, n_\alpha \in \mathbb{Z}$. Next define a **boundary homomorphism** $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ as follows:

$$\partial_n(\sigma_\alpha) = \sum_i (-1^i)\sigma_\alpha|_{v_0,...,\hat{v}_i,...,v_n},$$

where $\sigma_\alpha|_{v_0,...,\hat{v}_i,...,v_n}$ is the restriction of the characteristic map σ_α to the span of sequence of vertices $v_0...v_n$ after deleting the vertex v_i from the sequence. By construction, the composition map $\partial_{n-1} \circ \partial_n : \Delta_n(X) \to \Delta_{n-2}(X)$ is a zero map. This gives a sequence of abelian groups and group homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where ∂_0 is the zero map to the zero group and $Im\partial_{n+1} \subset Ker\partial_n$ since $\partial_n\partial_{n+1} = 0$. In general, any such sequence with the property $\partial_n \circ \partial_{n+1} = 0$ for all n is called a chain complex. The nth homology group of the chain complex, denoted H_n is then defined to be the quotient group $Ker\partial_n/Im\partial_{n+1}$. We refer to the elements of $Ker\partial_n$ as cycles, elements of $Im\partial_{n+1}$ as boundaries, and elements of H_n as homology classes. If the chain complex was generated by a Δ-complex, the homology group $Ker\partial_n/Im\partial_{n+1}$, denoted $H_n^{\Delta}(X)$, is called the nth simplicial homology group. Next, we explain the notion of singular homology that is defined for spaces that are not Δ-complexes and prove that for Δ-complexes, the notion of singular and simplicial homology coincide.

1.3.2 Singular Homology

A **singular n-simplex** in a space X is a continuous map $\sigma : \Delta^n \to X$. $C_n(X)$ is the free abelian group whose generators are the set of all singular n-simplices in X. Next, a boundary map is defined in the same manner as simplicial homology:

$$\partial_n(\sigma) = \sum_i (-1^i)\sigma|_{v_0,\ldots,\hat{v}_i,\ldots,v_n},$$

assuming as above that there is canonical homeomorphism between $v_0,\ldots,\hat{v}_i,\ldots,v_n$ and Δ^{n-1} that preserves the ordering of vertices such that $\sigma|_{v_0,\ldots,\hat{v}_i,\ldots,v_n}$ is a map from $\Delta^{n-1} \to X$, and hence a $n-1$ chain. By the same logic as simplicial homology, $C_n(X)$ generates a chain complex and one can equivalently defined the singular homology group, $H_n(X) = ker\partial_n/Im\partial_{n+1}$. Without proof, we mention that the 0th singular homology group of X, $H_0(X) = \mathbb{Z}^{\#\text{path components}}$ and if X is path connected, $H_1(X)$ is the abelianization of $\pi_1(X)$.

If X is a point p, then $H_n(X) = 0$ whenever $n > 0$ and $\mathbb{Z}$ when $n = 0$. This follows since for each n there is a unique simplex $\sigma_n : \Delta^n \to p$ and $\partial(\sigma_n) = \sum_i(-1)^i\sigma_{n-1}$, which is 0 if n is odd and σ_{n-1} if n is even. This yields the chain complex below and the theorem follows.

$$\cdots \to \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Let X and Y be two topological spaces and $f : X \to Y$ is a continuous map. $f : X \to Y$ induces a homomorphism $f_\# : C_n(X) \to C_n(Y), \sigma \mapsto f \circ \sigma$ which can be extended linearly such that $f_\#(\sum_i n_i\sigma_i) = \sum_i n_i f_\#(\sigma_i) = \sum_i n_i f \circ \sigma_i$ (hereinafter, we denote $f \circ \sigma = f\sigma$). Since

$$f_\#\partial(\sigma) = f_\#(\sum_i(-1)^i\sigma|_{v_0,\ldots,\hat{v}_i,\ldots,v_n}) = \sum_i(-1)^i f\sigma|_{v_0,\ldots,\hat{v}_i,\ldots,v_n} = \partial f_\#(\sigma),$$

it follows that $f_\#\partial = \partial f_\#$. The induced homomorphism then defines a commutative diagram between two chain complexes as follows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial^X} & C_n(X) & \xrightarrow{\partial^X} & C_{n-1}(X) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \\
\cdots & \longrightarrow & C_{n+1}(Y) & \xrightarrow{\partial^Y} & C_n(Y) & \xrightarrow{\partial^Y} & C_{n-1}(Y) & \longrightarrow & \cdots
\end{array}$$

Since $f_\#\partial = \partial f_\#$, $f_\#$ maps cycles to cycles and boundaries to boundaries. Hence, it induces a homomorphism $f_* : H_n(X) \to H_n(Y), \sigma + Im(\partial^X_{n+1}) \mapsto f_\#(\sigma) + Im(\partial^Y_{n+1})$, for $\sigma \in Ker(\partial^X_n)$. One can easily notice that $(fg)_* = f_*g_*$ and the induced homomorphism of the identity map from X to Y is an identity map. To check that the homology group gives a topological invariant, we first prove a lemma.

Lemma.1.3.2.1 The product $\Delta^n \times \{1\}$ can be subdivided into $(n+1)$ simplices:

Proof: Let $\Delta^n \times \{0\} = v_0,...,v_n, \Delta^n \times \{1\} = w_0,...,w_n$, such that v_i and w_i have the same image under the projection map $\Delta^n \times I \to \Delta^n$. Interpolate $v_0,...v_n$ to $w_0,...,w_n$ by sequentially moving one vertex v_i up to w_i in decreasing order of i from n to 0. In the ith step, $v_0,...,v_i,w_{i+1},...,w_n$ interpolates to $w_0,...,v_{i-1},w_i,...,w_n$. The region between these two n-simplices is a *(n+1)* simplex spanned by the set of vertices $\{v_0,...,v_i,w_i,...,w_n\}$ whose lower face is the n-simplex $v_0,...,v_i,w_{i+1},...,w_n$ and upper face is $w_0,...,v_{i-1},w_i,...,w_n$. Repeating the step, $\Delta^n \times I$ decomposes into a union of $(n+1)$-simplices that intersect the next in an n-dimensional face.

Theorem 1.3.2.1 If $f,g : X \to Y$ are homotopic to each other, then they induce the same homomorphisms f_* and g_*. As an immediate corollary, if X and Y are homotopy equivalent by maps $f : X \to Y, h : Y \to X$, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism and vice versa for h_*.

Proof: Let $F : X \times I \to Y$ define a homotopy between f and g such that $F(x,0) = f(x), F(x,1) = g(x)$. Define a prism operator $P : C_n(X) \to C_{n+1}(Y)$ as follows:

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathbb{1})|_{v_0,...,v_i,w_i,...,w_n}$$

(refer to the proof of lemma above for notation). We claim that $\partial P = g_\# - f_\# - P\partial$. For $\sigma : \Delta^n \to X$,

$$\partial P(\sigma) = \sum_{j\leqslant i}(-1)^i(-1)^j F \circ (\sigma \times \mathbb{1})|_{v_0,...,\hat{v}_j,...,v_i,w_i,...,w_n} + \sum_{j\geqslant i}(-1)^i(-1)^{j+1} F \circ (\sigma \times \mathbb{1})|_{v_0,...,v_i,...,\hat{w}_j,...,w_n}.$$

When $i = j$, the two terms cancel out except when $i = j = 0$, the term $F \circ (\sigma \times \mathbb{1}|_{\hat{v_0},w_0,...,w_n})$ remains, and when $i = j = 1$, the term $-F \circ (\sigma \times \mathbb{1}|_{v_0,...,v_n,\hat{w_n}})$ remains. Each term is respectively to $g_\#(\sigma)$ and $-f_\#(\sigma)$. When $i \neq j$, one can check that the sum of terms is equal to $-P\partial(\sigma)$ after comparing the computations. Hence it follows that $\partial P = g_\# - f_\# - P\partial$. If $\tau \in C_n(X)$ is cycle, then $g_\#(\tau) - f_\#(\tau) = \partial P(\tau)$ since $\partial P(\tau) = 0$. It follows that $g_\#(\tau) - f_\#(\tau)$ is a boundary, being in the image of a boundary map. We conclude that g_* equals f_* on the homology class of τ. The theorem then implies the corollary since if X and Y are homotopy equivalent (i.e $(h \circ f) \cong \mathbb{1}_X$), then $(h \circ f)* = h_* \circ f_* = \mathbb{1}$ and by the same logic, $f_* \circ h_* = \mathbb{1}$.

For a chain complex, one can construct an augmented chain complex as follows:

$$\cdots C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $\varepsilon : C_0(X) \to \mathbb{Z}, \sum_i n_i\sigma_i \mapsto \sum_i n_i$. Then the homology groups of the augmented chain complex is called the reduced homology group, denoted $\tilde{H}_n(X)$ ($\tilde{H}_{-1}(X)$ is a trivial group). One can verify that $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$.

1.3.3 Exact Sequences

A sequence of homomorphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

is called an exact sequence if for all n, $Ker\alpha_n = Im\alpha_{n+1}$. If α_n's are abelian groups, an exact sequence makes up a chain complex whose homology groups are all trivial. The following algebraic properties hold for exact sequences.

1. α is injective if and only if $0 \to A \xrightarrow{\alpha} B$ is exact.

2. α is surjective if and only if $A \xrightarrow{\alpha} B \to 0$ is exact.

3. α is an isomorphism if and only if $0 \to A \xrightarrow{\alpha} B \to 0$ is exact.

4. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact if and only if α is injective, β is surjective, and $C \cong B/Im\alpha$.

An exact sequence of the form $0 \to A \to B \to C \to 0$ is called a short exact sequence.

Theorem 1.3.3.1 Suppose X is a topological space with a nonempty closed subspace A such that there exists a neighborhood of A that deformation retracts to A. Then there exists an exact sequence

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \widetilde{H}_{n-1}(X) \longrightarrow \cdots \longrightarrow \widetilde{H}_0(X/A) \longrightarrow 0$$

where i is the inclusion map and j is the quotient map (such a pair *(X,A)* is also called a good pair).

We postpone the proof for this theorem, and proceed to considering a more general sequence of homology groups. Given a space X and $A \subset X$, we define the quotient group $C_n(X,A) := C_n(X)/C_n(A)$. Every chain $\sigma : \Delta^n \to A$ is trivial in $C_n(X,A)$ by construction and the quotient boundary map $\partial : C_n(X,A) \to C_{n-1}(X,A)$ is naturally induced by the boundary maps $\partial : C_n(X) \to C_{n-1}(X)$ and $\partial|_{C_n(A)}$. The following sequence

$$\cdots \longrightarrow C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \longrightarrow \cdots$$

is a chain complex since $\partial_{n-1} \circ \partial_n = 0$. The homology groups defined on the chain complex above, denoted $H_n(X,A)$, are called relative homology groups. The elements (cosets) of $H_n(X,A)$ are represented by chains $\alpha \in C_n(X)$ whose boundary $\partial(\alpha)$ is in $C_{n-1}(A)$. Such chains, denote them α's are called relative cycles. Then $[\alpha]$ is trivial in $H_n(X,A)$ if and only if $\alpha = \partial\beta + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$. We can prove that the relative homology groups for any pair of spaces $X,A)$ can be turned into a exact sequence

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots \longrightarrow H_0(X,A) \to 0.$$

Briefly note that the sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \longrightarrow 0$$

where i is an inclusion map, j is a quotient map is an exact sequence by property 4 of exact sequences above. Now we prove a more general version of the theorem below.

Theorem 1.3.3.2 A **short exact sequence of chain complexes** refers to a commutative diagram whose columns are short exact sequences and rows are chain complexes as follows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \longrightarrow & \cdots \\
 & & \Big\downarrow i & & \Big\downarrow i & & \Big\downarrow i & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & \cdots \\
 & & \Big\downarrow j & & \Big\downarrow j & & \Big\downarrow j & & \\
\cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then we can deduce from the short exact sequence of chain complexes, a long exact sequence of homology groups as below:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

Proof: Since the diagram is commutative, the maps i, j induce a homomorphism i_* and j_* between homology groups. We define the boundary map $\partial : H_n(C) \to H_{n-1}(A)$ for $[c] \in H_n(C)$ where c is a cycle in C_n. Since j is surjective, there exists $b \in B_n$ such that $j(b) = c$. Then $\partial(b) \in Ker(j)$ since $j(\partial b) = \partial j(b) = \partial(c) = 0)$. Since $Ker j = Im i$, there exists $a \in A_{n-1}$ such that $i(a) = \partial(b)$. Moreover $\partial(a) = 0$ since $i(\partial(a) = \partial i(a) = \partial\partial(b) = 0$. We now define $\partial : H_n(C) \to H_{n-1}(A), [c] \mapsto [a]$. The function is well defined since a is uniquely determined once ∂b is determined and had we picked b' instead of b, then from, $j(b') = j(b)$, it follows that $b' - b \in Ker j = Im i$. Hence, there exists a' such that $b' - b = i(a')$. Since replacing b with $b + (i(a'))$ amounts to changing a to $a + \partial a'$ (note $i(a + \partial a') = i(a) + i(\partial a') = \partial(b + i(a')))$, the function is still well defined. Last but not least, had we chosen c' instead of c such that $[c'] = [c]$, then from $c' = j(b')$ for some $b' \in B_n$, we have $c + \partial c' = c + \partial j(b') = c + j(\partial b') = j(b + \partial b')$, and notice that b is replaced by $b + \partial b'$ which does not the change the value of a. The map $\partial : H_n(C) \to H_{n-1}(A)$ defined as above is a homomorphism since i and j are homomorphisms. It remains to show that the so constructed sequence below is indeed exact.

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \longrightarrow \cdots$$

This reduces to proving the six following claims.

1. $Im i_* \subset Ker j_*$: This is trivial since $j \circ i = 0$
2. $Ker j_* \subset Im i_*$: An element in $Ker j_*$ can be represented by a cycle b in B_n such that $j(b)$ is a boundary (i.e. there exists $c' \in C_{n+1}$ such that $j(b) = \partial c'$. Then there exists $b' \in B_{n+1}$ such that $c' = j(b')$ since j is surjective. From $\partial j(b') = \partial c' = j(b)$, we conclude that $j(b - \partial b') = j(b) - \partial j(b') = 0$. It follows that there exists $a \in A_n$ such that $i(a) = b - \partial b'$. Since $i(\partial a) = \partial i(a) = \partial(b - \partial b') = \partial b = 0$ and i is an injective map, a must be a cycle. Hence, $i_*[a] = [b - \partial b'] = [b]$, which shows that $Ker j_* \subset Im i_*$.
3. $Im j_* \subset Ker\partial$: If $[b] \in H_n(B)$, then $b \in B_n$ and $\partial b = 0$. Since $j_*[b] = [j(b)]$ and $\partial b = 0$, $\partial[j(b)] = 0$.
4. $Ker\partial \subset Im j_*$: If $[c] \in Ker\partial$, then there exists $a' \in A_n$ such that $\partial[c] = [a] \in H_{n-1}(A)$ where $a = \partial a'$ for some $a' \in A_n$. Since $\partial(b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = 0$, $b - i(a')$ is a cycle. From $j(b - i(a')) = j(b) - ji(a') = j(b) = c$, we conclude that $\partial[j(b)] = [a]$.
5. $Im\partial \subset Ker i_*$: Suppose $[c] \in H_n(C)$. Since $i_*\partial([c]) = [\partial b] = 0$, we conclude that $i_*\partial = 0$.
6. $Ker i_* \subset Im\partial$: Suppose $a \in A_{n-1}$ is a cycle. Then there exists $b \in B_n$ such that $i(a) = \partial b$. Since $\partial j(b) = j(\partial b) = ji(a) = 0$, $j(b)$ is a cycle and ∂ maps $[j(b)]$ to $[a]$.

Corollary 1.3.3.1 There is a long exact sequence of homology groups as follows:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots \to H_0(X,A) \to 0$$

Let $(X,A),(Y,B)$ be pairs of spaces. Then the map $f : X \times Y$ such that $f(A) \subset B$ induces a homomorphism $f_\# : C_n(X,A) \to C_n(Y,B)$. The relation $f_\#\partial = \partial f_\#$ equally holds and it induces a homomorphism $f_* : H_n(X,A) \to H_n(Y,B)$. Moreover, if $f,g : (X,A) \to (Y,B)$ are homotopic to each other along maps from (X,A) to (Y,B), then $f_* = g_*$. Proof can be found in [2].

1.3.4 Excision Theorem

Theorem 1.3.4.1 Suppose $Z \subset A \subset X$ are three topological spaces such that $\bar{Z} \subset \mathring{A}$. Then the inclusion map $X - Z, A - Z) \hookrightarrow (X,A)$ induces an isomorphism $H_n(X - Z, A - Z)$ for all n. The theorem can be equivalent reformulated such that for subspaces $A,B \subset X$ whose interiors make up an open cover of X, the inclusion $(B, A \cap B) \hookrightarrow (X,A)$ induces an isomorphism between $H_n(B, A \cap B)$ and $H_n(X,A)$ for all n. The equivalence between two formulations follows from defining $B = X - Z$ and $Z = X - B$.

Lemma 1.3.4.1 Let $\mathcal{U} = \{U_j\}$ be a collection of subspaces of X such that their interiors make an open cover of X. Define $C_n^{\mathcal{U}}(X)$ to be a subgroup of $C_n(X)$ consisting of chains $\sigma_i : \Delta^n \to X$ whose image is wholly contained in some element of U. Since the boundary map takes $\partial : C_n^U(X)$ to $C_{n-1}^{\mathcal{U}}(X)$, the groups $C_n^{\mathcal{U}}$ make up a chain complex as well. The homology groups of this chain complex is denoted $H_n^{\mathcal{U}}(X)$. The inclusion $i : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence, that is, there exists a chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ such that $i\rho$ and ρi are chain homotopic to the identity map. Therefore, i induces an isomorphism between $H_n^{\mathcal{U}}(X)$ and $H_n(X)$ for all n.

Proof: We construct a subdivision operator $S : C_n(X) \to C_n(X)$ that is chain homotopic to the identity map in a linear setting. Let Y be a convex set in a Euclidean space and define $LC_n(Y) \subset C_n(Y)$ to be a subgroup of linear maps $\Delta^n \to Y$. Since the boundary map restricts to $LC_n(Y) \to LC_{n-1}(Y)$, the groups generate a subcomplex. We will designate a linear map $\lambda : \Delta^n \to Y$ by $[w_0, ..., w_n]$ where each w_i is the image of the vertex v_i of Δ^n under λ. Moreover, we will deal with the augmented complex $LC(Y)$ such that $LC_{-1}(Y) = \mathbb{Z}$ (assuming that $\partial w_0 = \varnothing$ for all 0 simplices w_0). For each $b \in Y$, define a homomorphism $b : LC_n(Y) \to LC_{n+1}(Y), [w_0, ..., w_n] \mapsto [b, w_0, ..., w_n]$. Then $\partial b([w_0, ..., w_n]) = [w_0, ..., w_n] - b(\partial[w_0, ...w_n])$. Hence for all $\alpha \in LC_n(Y), \partial b(\alpha) = \alpha - b(\partial\alpha)$ which implies that $\partial b + b\partial = \mathbb{1}$. This means that b is a chain homotopy between the identity map and the zero map on $LC(Y)$ (which in turn also shows that the complex has trivial homology group). Now, we define the subdivision homomorphism $S : LC_n(Y) \to LC_n(Y)$ inductively as follows. Suppose $\lambda = [w_0, ..., w_n]$ is a generator of $LC_n(Y)$. Denote b_λ to be the image under λ of the barycenter of Δ^n. Define $S(\lambda) = b_\lambda(S\partial\lambda)$, where b_λ is the homomorphism $LC_{n-1}(Y) \to LC_n(Y)$ defined as above and the base case begins with $S : LC_{-1}(Y) \to LC_{-1}(Y)$ is an identity map (this automatically implies that S is an identity map on $LC_0(Y)$ as well). We check that $\partial S = S\partial$. Since $S = \mathbb{1}$ on $LC_{-1}(Y)$ and $LC_0(Y)$, the result is trivial. We then proceed by induction. For general $LC_n(Y)$,

$$\partial S(\lambda) = \partial(b_\lambda(S\partial\lambda)) = S\partial(\lambda) - b_\lambda(\partial S\partial\lambda) = S\partial\lambda - b_\lambda S(\partial\partial\lambda) = S\partial\lambda$$

where the second equality follows from $\partial b_\lambda = \mathbb{1} = b_\lambda\partial$ and the third inequality follows from induction on n. Now we define a chain homotopy $T : LC_n(Y) \to LC_{n+1}(Y)$ between S and the identity. Define T inductively by $T = 0$ when $n = -1$ and $T(\lambda) = b_\lambda(\lambda - T\partial\lambda)$ when $n \geqslant 0$. For $LC_{-1}(Y)$, $T = 0$ and $S = \mathbb{1}$ which makes $\partial T + T\partial = \mathbb{1} - S$ trivial. For $n \geqslant 0$,

$$\partial T\lambda = \partial(b_\lambda(\lambda - T\partial\lambda)) = \lambda - T\partial\lambda - b_\lambda(\partial(\lambda - T\partial\lambda)) = \lambda - T\partial\lambda - b_\lambda(S\partial\lambda + T\partial\partial\lambda) = \lambda - T\partial\lambda - S\lambda$$

where the second equality follows from $\partial b_\lambda = \mathbb{1} - b_\lambda\partial$, the third equality follows from induction, and the last equality from $\partial\partial = 0$ and the definition of S. Hence in a linear setting, we have constructed S that is chain homotopic to the identity map by T.

Next extend this to general chains. Let $l_n : \Delta^n \to \Delta^n$ be the identity map and define $S : C_n(X) \to C_n(X)$ by $S\sigma = \sigma_\# S(l_n)$ where $S(l_n)$ follows the definition above (l_n is trivially a linear map) and $\sigma_\#$ is a chain map between $C_n(\Delta^n)$ and $C_n(X)$. We will just write $S\sigma = \sigma_\# S\Delta^n$ instead to follow Hatcher's notational convention. We verify that $\partial S = S\partial$ to prove that S is a chain map:

$$\begin{aligned}\partial S\sigma = \partial\sigma_\# S\Delta^n = \sigma_\#\partial S\Delta^n = \partial_\# S\partial\Delta^n = \sigma_\# S(\sum_i(-1)^i\Delta_i^n)\\ = \sum_i(-1)^i\sigma_\# S\Delta_i^n = \sum_i(-1)^i S(\sigma|_{\Delta_i^n}) = S(\sum_i(-1)^i S(\sigma|_{\Delta_i^n})) = S(\sum_i(-1)^i\sigma|_{\Delta_i^n}) = S(\partial\sigma)\end{aligned}$$

where Δ_i^n refers to the ith face of Δ^n. Likewise, one can define a chain homotopy $T : C_n(X) \to C_n(X+1)$ by $T\sigma = \sigma_\# T\ l_n$ where recall l_n is the identity map on Δ^n. Again, to follow Hatcher's notational convention, we write this as $T\sigma = \sigma_\# T\Delta^n$. This entails $\partial T + T\partial = \mathbb{1} - S$ from

$$\partial T_\sigma = \partial\sigma_\# T\Delta^n = \sigma_\# T\Delta^n = \sigma_\# T\Delta^n = \sigma_\#(\Delta^n - S\Delta^n - T\partial\Delta^n - T\partial\Delta^n) = \sigma - S\sigma - \sigma_\# T\partial\Delta^n = \sigma - S\sigma - T(\partial\sigma)$$

where the third equality follows from induction and the last equality follows from $\partial S = S\partial$.

If we iterate S m times, a chain homotopy between S^m and $\mathbb{1}$ is given by $D_m = \sum_{0\leqslant i<m} TS^i$. This follows from

$$\begin{aligned}\partial D_m + D_m\partial = \sum_{0\leqslant i<m}(\partial TS^i + TS^i\partial) = \sum_{0\leqslant i<m}(\partial TS^i + T\partial S^i) = \sum_{0\leqslant i<m}(\partial T + T\partial)S^i \\ = \sum_{0\leqslant i<m}(\mathbb{1}-S)S^i = \sum_{0\leqslant i<m}(S^i - S^{i+1}) = \mathbb{1} - S^m.\end{aligned}$$

Notice that for each singular n-simplex $\sigma : \Delta^n \to X$, there exists $m \in \mathbb{N}$ such that $S^m(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$. This follows since Δ^n is compact and hence, the diameter of the simplices of $S^m(\Delta^n)$ will be less than a Lebesgue number of the cover of Δ^n by the open sets $\sigma^{-1}(\mathring{U}_j)$ for large enough m. To view it differently, $S^m(l_n)$ where l_n is the identity map on Δ^n can be thought of as a barycentric subdivision of the standard n-simplex or to be more rigorous a linear combination of n-simplices in the barycentric subdivision with varying signs. Since we have shown that repeated barycentric subdivision decreases the diameter of the simplex, the Lebesgue argument follows. Define $m(\sigma)$ to be the least m such that $S^m\sigma \subset C_n^{\mathcal{U}}(X)$. Let $D : C_n(X) \to C_{n+1}(X)$ by $D\sigma = D_{m(\sigma)}\sigma$. We want to show that there exists a chain map $\rho : C_n(X) \to C_n(X)$ with the image in $C_n^{\mathcal{U}}(X)$ such that $\partial D + D\partial = \mathbb{1} - \rho$. We just have to let $\rho = \mathbb{1} - \partial D - D\partial$ and show that ρ is a chain. This follows from $\partial\rho(\sigma) = \partial\sigma - \partial^2 D\sigma - \partial D\partial\sigma = \partial\sigma - \partial D\partial\sigma$ and $\rho(\partial\sigma) = \partial\sigma - \partial D\partial\sigma - D\partial^2\sigma = \partial\sigma - \partial D\partial\sigma$. If we compute $\rho(\sigma)$ as in the following terms:

$$\rho(\sigma) = \sigma - \partial D\sigma - D(\partial\sigma) = \sigma - \partial D_{m(\sigma)}\sigma - D(\partial\sigma) = S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$$

we find out that the term $S^{m(\sigma)}(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$ by the way we defined $m(\sigma)$. The remaining terms are $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ are linear combinations of terms $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ where σ_j is the restriction of σ to the face of the standard complex. Since $m(\sigma_j) \leqslant m(\sigma)$, the remaining terms consist of terms $TS^i(\sigma_j)$ such that $i \geqslant m(\sigma_j)$. These terms are in $C_n^{\mathcal{U}}(X)$ since T maps $C_{n-1}^{\mathcal{U}}(X)$ to $C_n^{\mathcal{U}}(X)$. Hence the chain homotopy equation $\partial D + D\partial = \mathbb{1} - \rho$ entails that $\partial D + D\partial = \mathbb{1} - i\rho$ where ρ is viewed as a map $C_n(X) \to C_n^{\mathcal{U}}(X)$ and $i : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is the inclusion map. If $\sigma \in C_n^{\mathcal{U}}(X)$, then $m(\sigma) = 0$ and hence the summation for $D\sigma$ is empty. This proves that ρ is a chain homotopy inverse for i. This ends the proof for lemma 1.3.4.1.

We continue the proof for theorem 1.3.4.1, in particular, the second formulation. Let $U = \{A, B\}$ be the cover of X. Define $C_n(A+B)$ to be the sums of chains in A and B. From the proof of lemma 1.3.4.1, we know that $\partial D + D\partial = \mathbb{1} - i\rho$ where $\rho i = \mathbb{1}$. All the maps in this formula take chains in A to chains in A. Therefore, they induce quotient maps when we factor out chains in A. The quotient maps also satisfy the formulas and therefore the inclusion map $C_n(A+B)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$ induces an isomorphism on the homology groups. The map $C_n(B)/C_n(A\cap B) \to C_n(A+B)/C_n(A)$ also induces an isomorphism on homology groups since both quotient groups are free with each of their basis being singular n-simplices in B that do not lie in A. This shows that there is an isomorphism between $H_n(A\cap B)$ and $H_n(X,A)$.

Finally, we prove theorem 1.3.3.1. Recall that we should show that if (X,A) are good pairs, then the quotient map $q : (X,A) \to (X/A, A/A)$ induces an isomorphism $H_n(X,A) \to H_n(X/A, A/A)$ which is isomorphic to $\tilde{H}_n(X,A)$ for all n. Let V be the neighborhood of A to deformation retracts onto A. Now, consider the following diagram which we claim to be commutative.

$$\begin{tikzcd}
H_n(X,A) \arrow[r] \arrow[d, "q_*"'] & H_n(X,V) \arrow[r] \arrow[d, "q_*"'] & H_n(X-A, V-A) \arrow[d, "q_*"'] \\
H_n\big(X/A, A/A\big) \arrow[r] & H_n\big(X/A, V/A\big) \arrow[r] & H_n\big(X/A - A/A,\ V/A - A/A\big)
\end{tikzcd}$$

First consider the top left arrow. The map is an isomorphism since the deformation retraction of V to A implies that the pairs (V,A) and (A,A) are homotopic equivalent and hence the group $H_n(V,A)$ is 0. The rest follows from the property of the exact sequence. Since there is also a deformation retraction of V/A to A/A, the bottom left horizontal arrow is also an isomorphism for the same reason. The rest of the horizontal maps are isomorphisms from the excision theorem. Since the right vertical maps q_* is also an isomorphism from q being a homeomorphism away from A, the left vertical map must also be an isomorphism.

Corollary 1.3.4.1 $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_k(S^n) = 0$ when $k \neq n$.

Proof: Consider a pair of spaces $(X,A) = (D^n, S^{n-1})$. Then the quotient space $X/A = S^n$. Construct the long exact sequence of homology groups for (X,A) by theorem 1.3.3.1 since (X,A) is a good pair:

$$\longrightarrow \tilde{H}_n(S^{n-1}) \longrightarrow \tilde{H}_n(D^n) \longrightarrow \tilde{H}_n(S^n) \longrightarrow \tilde{H}_{n-1}(S^{n-1}) \longrightarrow \cdots \longrightarrow \tilde{H}_0(S^n) \longrightarrow 0$$

Notice that $\tilde{H}_i(D^n)$ is 0 since D^n is contractible. The exact sequence property then implies that the maps of the form $\tilde{H}_k(S^n) \longrightarrow \tilde{H}_{k-1}(S^{n-1})$ are isomorphisms for all positive k. Now $\tilde{H}_0(S^0) = \mathbb{Z}$ and $\tilde{H}_k(S^0) = 0$ whenever $k \neq 0$. Hence, by induction it follows that $\tilde{H}_n(S^n) = \mathbb{Z}$ and $\tilde{H}_k(S^n) = 0$ whenever $k \neq n$.

1.3.5 Equivalence Between Simplicial and Singular Homology

Before we proceed, we introduce an important lemma (omitting the proof) that will be use to prove a theorem below.

Lemma 1.3.5.1 Consider the commutative diagram of abelian groups below. If the two horizontal rows are exact sequences and $\alpha, \beta, \delta, \varepsilon$ are all isomorphisms, then γ is also an isomorphism.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "y"] & D \arrow[r, "\ell"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "\ell'"] & E'
\end{tikzcd}$$

Now suppose X is a Δ-complex such that $A \subset X$ is a Δ-subcomplex (i.e. X,A is a Δ-complex pair. Then we can define $H_n^{\Delta}(X,A)$ the same way we defined for singular homology by constructing a relative chain between $\Delta_n(X,A) = \Delta_n(X)/\Delta_n(A)$.

Theorem 1.3.5.1 $H_n^{\Delta}(X)$ and $H_n(X)$ are isomorphic.

Proof: Recall that X^k denotes the k-skeleton of X that consists of all simplices with dimension less or equal to k. Consider the following diagram of two exact sequences:

$$\begin{tikzcd}
H_{n+1}^{\Delta}(X^k,X^{k-1}) \arrow[r] \arrow[d] & H_n^{\Delta}(X^{k-1}) \arrow[r] \arrow[d] & H_n^{\Delta}(X^k) \arrow[r] \arrow[d] & H_n^{\Delta}\big(X^k,X^{k-1}\big) \arrow[r] \arrow[d] & H_{n-1}^{\Delta}(X^{k-1}) \arrow[d] \\
H_{n+1}(X^i,X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n\big(X^k,X^{k-1}\big) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

We show that the diagram is commutative. First, consider the first and fourth vertical maps. If $n \neq k$, then $\Delta_n(X^k,X^{k-1}) = 0$. If $n = k$, then $\Delta_n(X^k,X^{k-1})$ is a free abelian group generated by k-simplices of X. It then follows trivially that $H_n^{\Delta}(X^k,X^{k-1}$ is 0 when $n \neq k$ and the free abelian group if $n = k$. Next, consider $H_n(X^k,X^{k-1})$. Let $\Phi : \bigsqcup_{\alpha}(\Delta_{\alpha}^k, \partial\Delta_{\alpha}^k)$ be formed by every characteristic map $\Delta^k \to X$. Φ induces a homeomorphism between $\bigsqcup_{\alpha}\Delta_{\alpha}^k/\bigsqcup_{\alpha}\partial\Delta_{\alpha}^k$ and X^k/X^{k-1}. Hence, Φ induces isomorphisms on singular homology groups. It follows that $H_n(X^k,X^{k-1}) = 0$ when $n \neq k$ and $H_n(X^k,X^{k-1})$ is a free abelian group when $n = k$, generated by the relative cycles that are the images of the characteristic maps of all the k-simplices. Since we can show that the second and fifth vertical maps are isomorphism using proof by induction, the result follows from lemma 1.3.5.1.

Whenever X is a Δ-complex with finitely many n-simplices (i.e. $\sigma : \Delta^n \to X$), then $\Delta_n(X)$ is a finitely generated group. Since the subgroup of a finitely generated abelian group is finitely generated, the subgroup of cycles is also finitely generated, and hence, $H_n^{\Delta}(X)$ is finitely generated. The fundamental theorem of finitely generated abelian group tells that the homology group $H_n(X) \cong \mathbb{Z}^k \oplus \bigoplus_{i=1}^{r} \mathbb{Z}_{k_i}$. k is then called the nth Betti number of X and the orders of the finite cyclic summands are called the torsion coefficients.

1.3.6 Degree and Cellular Homology

Consider a map $f : S^n \to S^n$. f induces a homomorphism $f_* : \tilde{H}_n(S^n) \cong \mathbb{Z} \to \tilde{H}_n(S^n) \cong \mathbb{Z})$ that maps $\alpha \to d\alpha$ for some integer d. The integer d is called the degree of f, denoted $deg(f)$. The following are important properties of degree.

1. $deg(\mathbb{1}) = 1$
2. If f is not surjective, then $deg(f) = 0$. If $x_0 \notin f(S^n)$, then f can be decomposed into $S^n \to S^n - \{x_0\} \hookrightarrow S^n$. Since $H_n(S^n - \{x_0\}) = 0$ from the contractibility of $S^n - \{x_0\}, f_* = 0$.
3. If f and g are homotopic, then $deg(f) = deg(g)$.
4. $deg(f \circ g) = deg(f)deg(g)$. As a corollary, whenever $f \circ g \simeq \mathbb{1}$, then $deg(f) = \pm 1$.
5. Whenever f is a reflection of S^n, then $deg(f) = -1$. Here, reflection refers to a map that changes the sign of only one coordinate. (i.e. $r : S^1 \to S^1, (x_1, x_2) \mapsto (x_1, -x_2)$). We omit the proof.
6. If $f : S^n \to S^n$ is an antipodal map, then f has degree $(-1)^{n+1}$. This follows from the fact that an antipodal map is a composition of $n+1$ reflection maps.
7. If $f : S^n \to S^n$ has no fixed points, then its degree is $(-1)^{n+1}$. This follows from the fact that one can homotope f to the antipodal map although we omit the details.

Suppose for $f : S^n \to S^n$, there exists some $y \in S^n$ such that $f^{-1}(y) = \{x_1, ..., x_m\}$ is a finite set (if f is smooth, the existence of y follows from Sard's theorem and extends to smooth, compact, connected manifolds, but we omit the details). Let $U_1, ..., U_m$ be disjoint neighborhoods of each point in $f^{-1}(y)$ such that every U_i is mapped into a neighborhood V of y. Then $f(U_i - x_i) \subset V - y$ for every i. Consider the following diagram:

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\approx"'] \arrow[d, "k_i"] & H_n(V, V - y) \arrow[d, "\approx"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[u, "j"] \arrow[ul, "\approx"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\approx"']
\end{tikzcd}$$

The k_i and p_i are homomorphisms induced by the inclusion map. The two isomorphisms in the upper half follow from the excision theorem and the two isomorphisms in the lower half of the diagram follows from the exact sequence of pairs. It follows that the diagram is commutative and hence $H_n(U_i, U_i - x_i) \cong H_n(V, V - y) \cong \mathbb{Z}$. f_* at the very top of the diagram hence is a homomorphism from the ring of integers to the ring of integers and we define the degree of f_*, the multiplicative factor, to be the local degree of f at x_i, denoted $\deg f|x_i$. For example, if f is a homeomorphism, then for any choice of y, there is only one x_i. All the maps in the diagram become isomorphisms and $\deg f|x_i = \deg(f)$ $= \pm 1$. On the other hand, if f maps each U_i homeomorphically to V, then for each i, $\mathrm{def} f|x_i = \pm 1$ (the two degrees define the only isomorphisms between $\mathbb{Z}$ and $\mathbb{Z}$.

Theorem 1.3.6.1 $\deg f = \sum_i \deg f|x_i$.

Proof: Consider the diagram above one again. $H_n(S^n, S^n - f^{-1}(y))$ is a direct sum of the groups $H_n(U_i, U_i - x_i) \cong \mathbb{Z}$ since

$$H_n(S^n, S^n - f^{-1}(y)) \cong H_n(\bigsqcup_i (U_i, U_i - x_i))$$

by excision theorem and the homology group of disjoint union is the direct sum of homology groups. Since the upper triangle of the diagram commutes, the k_i defines the inclusion of the ith summand and p_i defines the projection of the direct sum back to its ith summand. Since the lower triangle is also commutative, $p_i j(1) = 1$ and hence $j(1) = (1, ..., 1) = \sum_i k_i(1)$. Since f_* takes $k_i(1)$ to deg $f|x_i$, $\sum_i k_i(1) = j(1)$ is mapped to $sum_i \deg f|x_i$. By the commutativity of the lower square, $\deg f = \sum_i \deg f|x_i$.

Now, we cover new tools to compute the homology group of CW complexes involving the notion of degree. First, we prove some basic properties about CW complexes. Recall that if X is a CW complex, X^n refers to the n-skeleton of X.

Lemma 1.3.6.1 If X is CW complex, then the following properties hold.

1. $H_k(X^n, X^{n-1})$ is zero if $k \neq n$ and a free abelian group whose generators correspond to n-cells of X.
2. $H_k(X^n) = 0$ if $k > n$. If X is finite dimensional, then $H_k(X) = 0$ for $k > n$.
3. The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism between $H^k(X^n)$ and $H_k(X)$ if $k < n$.

Proof: (X^n, X^{n-1}) is a good pair and the quotient space X^n/X^{n-1} is essentially a wedge sum of n spheres, one for each n-cell. Recall that for good pairs (X,A), a quotient map induces an isomorphism between $H_n(X,A)$ and $\tilde{H}_n(X/A)$ for all n. In this case, there is a isomorphism between $H_k(X^n, X^{n-1})$ and the reduced homology group of wedge sum of n-spheres. The reduced homology group of the wedge sum of spheres is the direct sum of reduced homology group of n spheres. More precisely, $\tilde{H}_k(\bigvee_\alpha S^n_\alpha) \cong \bigoplus_\alpha \tilde{H}_k(S^n_\alpha)$. If $k = n$, $\tilde{H}_k(S^n_\alpha) = \mathbb{Z}$ $k \neq n$, and otherwise 0. This proves the first claim. To prove the second claim, consider the long exact sequence of (X^n, X^{n-1}).

$$\cdots \longrightarrow H_{k+1}(X^n, X^{n-1}) \longrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow H_k(X^n, X^{n-1)} \longrightarrow \cdots$$

If k does not equal n or *n-1*, then $H_{k+1}(X^n, X^{n-1})$ and $H_k(X^n, X^{n-1})$ are zero form the first claim. By the properties of exact sequence, $H_k(X^{n-1} \approx H_k(X^n)$ for $k \neq n, n-1$. If $k > n$, $H_k(X^n) \approx H_k(X^{n-1}) \approx \cdots \approx H_k(X^0) = 0$. This proves the second claim. For the third claim, first suppose X is finite dimensional. Then whenever $k < n$, $H_k(X^n) \approx H_k(X^{n+1}) \approx \cdots \approx H_k(X^{n+m})$ for all non-negative m, which proves the claim. For infinite dimensional X, we use the fact that a singular chain in X has a compact image and hence each the image of each chain can intersect only finitely many cells of X. We omit the details.

Now a cellular homology group is defined from a chain complex below:

$$\cdots \longrightarrow H_{n_1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots.$$

where the boundary map d_n is constructed as follows. Consider the long exact sequence $0 \to H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1})$ and the sequence $0 \to H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X^{n-1}, X^{n-2})$. The boundary map d_n is then defined to be the composition $j_{n-1} \circ \partial_n$. Then the composition of maps $d_n \circ d_{n+1}$ is zero and the chain complex above is well defined. Since from lemma 1.3.6.1, $H_n(X^n X^{n-1})$ is a free abelian group with generators corresponding to n-cells of X, the elements of the chain $H_n(X^n, X^{n-1})$ are linear combinations of n-cells of X. The homology groups of the chain complex above is called the **celluar homology groups** of X and denoted $H_n^{CW}(X)$.

Theorem 1.3.6.2 $H_n^{CW}(X)$ is isomorphic to $H_n(X)$.

Proof: Consider the diagram below. $H_n(X)$ is isomorphic to $H_n(X^n)/Im\partial_{n+1}$ by the property of exact sequence. Since j_n is injective, it maps $Im\partial_{n+1}$ onto $Im(j_n\partial_{n+1}) = Im(d_{n+1})$ isomorphically and maps $H_n(X^n)$ onto $Im\, j_n = Ker\partial_n$ isomorphically. Since j_{n-1} is also injective, $Ker\ \partial_n = Ker\ d_n$. Hence $j_n : H_n(X^n) \to H_n(X^n, X^{n-1})$ induces an isomorphism between $H_n(X^n)/Im\partial_{n+1}$ and $Ker\ d_n/Im\ d_{n+1}$.

$$
\begin{tikzcd}
 & 0 \arrow[dr] & & H_n(X^{n+1}) \approx H_n(X) & \\
 & & H_n(X^n) \arrow[ur] \arrow[d, "j_n"] & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1},X^n) \arrow[r, "d_{n+1}"] \arrow[ur, "\partial_{n+1}"] & H_n(X^n,X^{n-1}) \arrow[r, "d_n"] \arrow[d, "\partial_n"] & H_{n-1}(X^{n-1},X^{n-2}) \arrow[r] & \cdots \\
 & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
 & 0 \arrow[ur] & & &
\end{tikzcd}
$$

Corollary 1.3.6.1

1. If X is a CW complex with no n-cells, then $H_n(X) = 0$. This follows from the fact that both $H_{n+1}(X^{n+1}, X^n)$ and $H_n(X^n, X^{n-1})$ are 0.
2. If X is a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements. This follows from the fact that $H_n(X^n, X^{n-1})$ is a free Abelian group generated by k elements, and since the subgroup $Ker\ d_n$ can be generated by at most k elements, and so can the quotient $Ker\ d_n/Im\ d_{n+1}$.
3. If X is a CW complex such that no two of its cells are in adjacent dimensions, then $H_n(X)$ is a free abelian group whose basis corresponds one-to-one with the n-cells of X.

The third corollary for instance automatically implies that $\mathbb{C}P^n$ has a homology group

$$H_i(\mathbb{C}P^n) \approx \begin{cases} \mathbb{Z} \text{ for } i = 0, 2, \cdots, 2n \\ 0 \text{ otherwise} \end{cases}$$

since it has a CW structure of one cell in each even dimension $2k \leqslant 2n$.

Cellular homology provides an efficient algorithm to study the topological properties of cell complexes. For instance, if M_g is a closed, orientable surface with genus g, then its CW structure consists of one 0-cell, $2g$ 1-cells, and one 2-cell attached by the gluing scheme $[a_1, b_1] \cdots [a_g, b_g]$. If one considers the cellular chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

both d_1 and d_2 are zero maps. The homology groups of M_g, $H_n(M_g)$ are $\mathbb{Z}$ when $n = 0, 2$ and $\mathbb{Z}^{2g}$ when $n = 1$.

1.3.7 Euler Characteristic and Mayer Vietoris Sequence

Recall that when X is a Δ-complex structure with finitely many n-simplices, then $\Delta_n(X)$ is a finitely generated free Abelian group. By the fundamental theory of finitely generated abelian group,

$$H_n(X) \cong \mathbb{Z}^k \oplus \bigoplus_{i=1}^{r} \mathbb{Z}_{k_i}$$

where k is referred to as the nth Betti number, β_n. The **Euler characteristic** of X, denoted $\chi(X)$is then defined as $\sum_n(-1)^n\beta_n$. β_n is also called the rank of $H_n(X)$. If X is a finite CW complex, then $\chi(X) = \sum_n(-1)^n c_n$ where c_n denotes the number of n cells in X. In particular, the Euler characteristic is a homotopy invariant. We omit the proof. Recall M_g is a closed, orientable surface with genus g. The Euler characteristic of $\chi(M_g)$ is $2-2g$ and hence the genus of a closed, orientable surface M can also be computed in terms of Euler characteristic, i.e. $2g = 2-\chi(M)$.

Theorem 1.3.8.1 Suppose X is a topological space such that there exists a pair of subspaces $A,B \subseteq X$ such that the union of the interiors of A and B equals X. Then there exists an exact sequence of the form

$$\cdots \longrightarrow H_n(A\cap B) \xrightarrow{\Phi} H_n(A)\oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B) \longrightarrow \cdots \longrightarrow H_0(X) \longrightarrow 0$$

The exact sequence is called the **Mayer Vietoris sequence**.

Proof: Let $C_n(A+B)$ denote the subgroup of $C_n(X)$ that consists of sums of chains in A and chains in B. Consider the following chain complex:

$$0 \to C_n(A\cap B) \xrightarrow{\varphi} C_n(A)\oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$$

Recall that $C_n^{\mathcal{U}}(X)$ is a subgroup of $C_n(X)$ consisting of $\sum_i \sigma_i$ such that σ_i has an image belonging to some set in $\mathcal{U}$. Let $\varphi = (i_{A*}, -i_{B*})$, that is a pair of inclusion induced maps such that $\varphi(\sigma) = (\sigma,-\sigma)$ and let ψ be the sum of inclusion induced maps $j_{1*}+j_{2*}$ such that $(\sigma,\tau)\mapsto\sigma+\tau$. φ is injective sine both i_{A*} and i_{B*} are injective and ψ is trivially surjective. Moreover $Im\varphi \subset Ker\psi$ since $\psi\circ\varphi = 0$ since $\sigma\mapsto(\sigma,-\sigma)\mapsto\sigma+(-\sigma)$ and $Ker\psi\subset Im\varphi$ since for any pair $(\sigma,\tau)\in C_n(A)\oplus C_n(B)$ such that $x+y=0$, then $\sigma = -\tau$ and hence σ is a chain in both A and B (i.e. $\sigma\in C_n(A\cap B)$). This implies that $(\sigma,\tau) = (\sigma,-\sigma)\in Im\varphi$. Hence, the short exact sequence above is well defined and the Mayer Vietoris sequence is the long exact sequence of homology groups associated with the short exact sequence. We can also obtain the Mayer Vietoris sequence for reduced homology groups, by augmenting the short exact sequence as follows:

$$\begin{tikzcd}
0 \arrow[r] & C_0(A\cap B) \arrow[r, "\varphi"] \arrow[d, "\varepsilon"] & C_0(A)\oplus C_0(B) \arrow[r, "\psi"] \arrow[d, "\varepsilon\oplus\varepsilon"] & C_0(A+B) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\varphi"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\psi"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

2 Piecewise Linear Topology

2.1 Simplicial and PL Maps

Let K a simplicial complex K and $\sigma\in K$, the star of σ in K is defined to be the subcomplex $St(\sigma,K) = \{\tau\in K | \exists\ \eta\in K : \sigma,\tau<\eta\}$. In other words, $St(\sigma,K)$ is the closure of the following set:

$$\{\tau\in K : \sigma<\tau\}.$$

A link of σ in K is defined to be a subcomplex $Lk(\sigma,K) = \{\tau\in St(\sigma,K) : \tau\cap\sigma = \varnothing$. We denote $st(\sigma,K) := |St(\sigma,K)|$ and $lk(\sigma,K) := |Lk(\sigma,K)|$. One can easily notice that $st(\sigma,K) = v\star lk(\sigma,K)$. Moreover, $Lk(\sigma,K)$ consists of all simplices $\tau\in K, \tau\cap\sigma=\varnothing$ such that $\sigma\star\tau$ is a simplex in K. In fact, $Lk(\sigma,K)$ may be equivalently defined as

$$Lk(\sigma,K) = \{\tau\in K \Big| \sigma\cap\tau = \varnothing \text{ and } \sigma\star\tau\in K\}.$$

If $\sigma\cap\tau=\varnothing, \sigma\star\tau\in K$, then both σ and τ is a face of $\sigma\star\tau$ which itself is a simplex. On the other hand, suppose $\sigma\cap\tau=\varnothing$ and $\exists\ \eta\in K$ such that $\sigma<\eta, \tau<\eta$. If $V(\sigma),V(\tau),V(\eta)$ denotes the vertex sets of σ,τ,η, then $V(\sigma)\subset V(\eta)$ and $V(\tau)\subset V(\eta)$. Hence $V(\sigma)\sqcup V(\tau)\subset V(\eta)$. Then $V(\sigma)\sqcup V(\tau)$ spans a simplex by the definition of a simplicial complex. Call such simplex ω. One can easily notice that $\omega = \sigma\star\tau$, proving that two definitions are actually equivalent.

The open star of σ in K is defined as $\mathring{st}(\sigma,K) := st(\sigma,K) - lk(\sigma,K)$. Consistent with the definition above, $st(v,K)$ (or $st(v)$ abbreviating K) where v is a vertex denotes the set of all simplices σ such that there exists some simplex η that has both v and σ as its faces, and $\mathring{st}(v)$ will specifically denote the union

$$\mathring{st}(v) = st(v) - lk(v) = \bigcup_{v\in\sigma,\mathring{\sigma}\in K} \mathring{\sigma}.$$

where the link of a vertex $v \in K$, denoted $Lk(v,K)$ will be similarly defined as a set containing every simplex $\sigma \in X$ such that $v \notin \sigma$ and there exists a simplex η in X that contains v as a vertex and σ as a face and $lk(v) = |Lk(v)|$. In other words, for every $\sigma \in Lk(v,K)$, $v \star \sigma \in K$.

Theorem 2.1.1 Let v be a vertex of a simplicial complex K. Then $\mathring{st}(v)$ is an open set in $|K|$, containing v where v is the only vertex of K that is contained in $\mathring{st}(v)$. Moreover, $\{\mathring{st}(v)\}_{v\in K^0}$ is an open cover of $|K|$.

Proof: We prove that $|K| - \mathring{st}(v)$ is closed in $|K|$. If $v \notin \sigma$, then v does not belong in the faces of σ. Hence, whenever $\mathring{\sigma} \subset |K| - \mathring{st}(v)$, $\sigma \subset |K| - \mathring{st}(v)$. Since σ is compact, σ is closed. From

$$|K| - \mathring{st}(v) = \bigcup_{\mathring{\sigma}\subset|K|-\mathring{st}(v)} \sigma,$$

we conclude that $|K| - \mathring{st}(v)$ is an arbitrary union of closed sets and hence is itself closed. It follows that v is the only vertex in $\mathring{st}(v)$ since the only open simplex containing the vertex is by construction, the vertex v itself (viewed as a 0-skeleton), and the collection of $\{\mathring{st}(v) : v \in K^0\}$ form an open cover since every p in K must belong in $\mathring{\sigma}$ for some σ. Pick any vertex v of σ and $p \in \mathring{st}(v)$.

Now suppose L is another simplicial complex. A map $f : |K| \to |L|$ is a **simplicial map** if it satisfies the following conditions:

1. For every vertex v of K, $f(v)$ is a vertex of L.
2. For every $\sigma = v_0,...,v_k \in K$, the vertices $f(v_0),...,f(v_k)$ are all contained in some simplex of L
3. For every $\sigma = v_0,...,v_k \in K$ and $p = \sum_{i=0}^k a_i v_i \in \mathring{\sigma}$, $f(p) = \sum_{i=0}^k a_i f(v_i)$.

The first condition says that a simplicial map must map a 0-skeleton to a 0-skeleton and the second and third condition says that a simplicial map must map a simplex in K onto a simplex in K linearly. Notation-wise, we will write $f : K \to L$ since the map depends on the simplicial structure as well. Every simplicial map is continuous by the pasting lemma and the behavior of the map is completely determined by its image of the 0-skeleton of K. This implies that a vertex map $K^0 \to L^0$ can be uniquely extended to a simplicial map if the vertex map satisfies the second condition. On the other hand, if a polyhedron P is defined instead as a subset of $\mathbb{R}^n$ such that every point admits a cone neighborhood N ($N = a \star L$ where L is compact in $\mathbb{R}^n$ whose topology is induced from the metric $d(x,y) = \|x-y\|_\infty$ that contains an open neighborhood of the point), then a map between two polyhedra, P and Q, $f : P \to Q$ is a **simplicial map** if every point $a \in P$ has a cone neighborhood N such that $f(\lambda a + \mu x) = \lambda f(a) + \mu f(x)$, where $x \in L, \lambda, \mu \geqslant 0$, and $\lambda + \mu = 1$. In fact, a compact polyhedron so defined must be a finite union of simplices and a general polyhedron is a locally finite union of simplices. Proof can be found in [3].

If $f|_\sigma$ is injective for every $\sigma \in K$, then the simplicial map is called nondegenerate (i.e. it defines a linear bijection between simplices of K and L). If there are subdivisions K' of K and L' of L such that $f : K' \to L'$ is a simplicial map, then $f : |K| \to |L|$ is called **piecewise linear** (or PL) map. Lastly, if there is a homeomorphism f between topological spaces $|K|$ and $|L|$ such that both f and f^{-1} are PL maps, f is called a **PL homeomorphism** and $|K|$ and $|L|$

are said to be PL homeomorphic.

Again, let K, L be simplicial complexes and suppose $f : |K| \to |L|$ is continuous. A simplicial map $\varphi : K \to L$ is defined to be a simplicial approximation to f if $f(\mathring{st}(v)) \subset \mathring{st}(\varphi(v))$ for every $v \in K^0$.

Theorem 2.1.2 Suppose $\varphi : K \to L$ gives a simplicial approximation to $f : |K| \to |L|$. Then for all $p \in |K|$, $f(p)$ and $\varphi(p)$ are contained in the same simplex in L.

Proof: Let $p \in |K|$. Then there exists $\sigma = v_0, ..., v_r \in K$ such that $p \in \mathring{\sigma}$ and for all $j \in \{0, 1.., r\}$, $f(p) \in f(\mathring{\sigma}) \subset f(\mathring{st}(v_j)) \subset \mathring{st}(\varphi(v_j))$. Likewise, if $f(p) \in \mathring{\tau}, \tau \in L$, for all j, $\mathring{\tau} \cap \mathring{st}(\varphi(v_j)) \neq \varnothing$. This implies that $\mathring{\tau} \subset \mathring{st}(\varphi(v_j))$ for all j by the definition of star. We conclude that $\varphi(v_j)$ is a vertex of τ for all j since φ is a simplicial map. From $p = \sum_{j=0}^{r} a_j v_j, \varphi(p) = \sum_{j=0}^{r} a_j \varphi(v_j) \in \tau$ and τ is a simplex that contains both $f(p)$ and $\varphi(p)$.

Corollary 2.1.1 If $\varphi : K \to L$ is a simplicial approximation to $f : |K| \to |L|$, $d(f, \varphi) := \sup_{p \in |K|} |f(p) - \varphi(p)| \leqslant \text{mesh } L$.

Lemma 2.1.1 If $f : K \to L$ is a simplicial map and φ is a simplicial approximation to f, then $\varphi = f$.

Proof: For every vertex $v \in K$, $f(v) \in f(\mathring{st}(v)) \subset \mathring{st}(\varphi(v))$. Since f is a simplicial map, $f(v)$ is a vertex in L. From Theorem 2.1.1, we conclude that $f(v)\varphi(v)$. Since f and φ agree on all vertices and both are simplicial maps, they agree everywhere on K.

Theorem 2.1.3 Let φ be a simplicial approximation to $f : |K| \to |L|$. Suppose K_1 is a subcomplex of K and $f|_{|K_1|}$ is a simplicial map. Then there is a homotopy between f and φ that is relative to $|K_1|$.

Proof: We claim that the function $F : |K| \times I \to |L|$ given by

$$F(p,t) = t\varphi(p) + (1-t)f(p)$$

gives the desired homotopy. From theorem 2.1.2, we know that both $f(p)$ and $\varphi(p)$ belong in a common simplex whose convexity implies that the line segment joining the image points also lie in the simplex, and hence the codomain of F is well defined. Moreover F is continuous and for all $p \in |K|$, $F(p,0) = f(p), F(p,1) = \varphi(p)$.Since $f(\mathring{st}_{K_1}(v)) \subset f(\mathring{st}_K(v)) \subset \mathring{st}(\varphi(v))$ for all $v \in K_1$, $\varphi|_{|K_1|}$ is a simplicial approximation to $f|_{|K_1|}$. From lemma 2.1.1, we conclude that f agrees with φ on $|K_1|$.

Theorem 2.1.4 Suppose $f : |K| \to |L|$ is continuous and $\varphi : K^{(0)} \to L^{(0)}$. Then φ can be extended to a simplicial approximation to f if for all $v \in K^{(0)}$, $f(\mathring{st}(v)) \subset \mathring{st}(\varphi(v))$.

Proof: It suffices to show that whenever $\sigma = v_0, ..., v_r \in K$, $\varphi(v_0), \varphi(v_1), ..., \varphi(v_r)$ is a complex in L. Since for all $j \in \{0, 1, ..., r\}$ $f((s)) \subset f(\mathring{st}(v_j)) \subset \mathring{st}(\varphi(v_j)))$, $\bigcap_j \mathring{st}(\varphi(v_j)) \neq \varnothing$. Hence, there exists a simplex $\tau \in L$ such that $\mathring{\tau} \subset \mathring{st}(\varphi(v_j))$ for all j. Since $\varphi(v_j)$ must be a vertex of τ for all j, φ is indeed a simplicial map.

Theorem 2.1.5 Suppose $f : |K| \to |L|$ be a continuous map. Let $\{K'_n\}_{n \in \mathbb{N}}$ be a sequence of subdivisions $K'_n \prec K$ where $\lim_{n \to \infty} \text{mesh } K_n = 0$. Then for large enough n, there exists a simplicial map $\varphi : K_n \to L$ that is a simplicial approximation to f.

Proof: By theorem 2.1.1, $\{\mathring{st}(w) : w \in L^{(0)}\}$ is an open cover of $|L|$. Moreover, $\{f^{-1}(\mathring{st}(w)) : w \in L^{(0)}\}$ is an open cover of $|K|$ since f is continuous. Pick $\delta > 0$ such that any ball of radius δ lies in an open set in the cover (this follows by Lebesgue's number lemma since $|K|$ is a compact metric space). Then pick large enough n such that mesh $K_n < \delta/2$. Then for all $\sigma \in K_n$, the diameter of σ is less than or equal to $\delta/2$. Hence, for every vertex v of K_n, $\mathring{st}(v) \subset B_\delta(v)$. Since there exists $w \in L^{(0)}$ such that $B_\delta(v) \in f^{-1}(\mathring{st}(w))$, for each $v \in$, there exists $w \in L^{(0)}$ such that $\mathring{st}(v) \subset f^{-1}(\mathring{st}(w)$. Now, for each vertex v in K_n, define $\varphi(v)$ to be any such vertex w of L. Then $\mathring{st}(v) \subset f^{-1}(\mathring{st}(\varphi(v)))$ which implies that for all $v \in K_n^{(0)}$, $f(\mathring{st}(v)) \subset \mathring{st}(\varphi(v)))$. By theorem 2.1.4, φ can be extended to a simplicial approximation to *f*.

Corollary 2.1.2 (*Simplicial Approximation Theorem*): Suppose K and L are simplicial complexes and $f : |K| \to |L|$ is a continuous function. Then there exists $K' \prec K$ and a simplicial map $g : K' \to L$ that is homotopic to f. The theorem implies that a continuous map from a space to a space can be approximated by a linear map after decomposing the spaces into small enough simplices. (the simplicial approximation theorem follows from theorem 2.1.5 and a straight-line homotopy.)

2.2 Combinatorial Manifolds and Triangulation

A **combinatorial n-manifold** is a complex K for which the link of each p-simplex σ is PL homeomorphic to either the boundary of an $(n-p)$ simplex or to an $(n-p-1)$ simplex. The link $Lk(\sigma, K)$ may be interpreted to represent the boundary of the neighborhood of σ. The boundary of an $(n-p)$ simplex is topologically a unit $n-p-1$ sphere S^{n-p-1} whereas an $n-p-1$ simplex is topologically a unit $n-p-1$ ball B^{n-p-1}. Heuristically, if the link $Lk(\sigma, K)$ looks like a sphere, the simplex σ may be understood as an "interior" simplex whereas if $Lk(\sigma, K)$ looks like a ball, σ may be understood to form the "boundary" of the combinatorial manifold.

Now let X be a topological space. A **triangulation** of X is defined to be a homeomorphism $t : |K| \to X$ where K is a simplicial complex (we will often use the word triangulation to denote $|K|$ instead). We say two triangulations $t : |K| \to X, t' : |K'| \to X$ are equivalent if there exists a *PL* homeomorphism $p : |K| \to |K'|$ such that $t' \circ h = t$. In general, a ***PL n*-manifold** is a topological n-manifold M with a triangulation $t : |K| \to M$ such that K is a combinatorial n-manifold. In this case, we say t defines a **PL triangulation** of M or a **PL structure** on M. M is a ***PL* n-ball** if K can be chosen to be an n-simplex. Likewise M is a ***PL* n-sphere** if K can be chosen to the boundary subcomplex of an *n+1*-simplex (both a complex consisting of one simplex and a complex consisting of a boundary subcomplex of a simplex are by definition a combinatorial manifold).

Definition 2.2.1 From now on, B^n will denote an equivalence class of compact polyhedra that are PL homeomorphic to $[-1,1]^n$. Likewise, S^n will denote an equivalence class of compact polyhedra that are PL homeomorphic to $\partial([-1,1]^n)$.

Definition 2.2.2 In this paper, we follow [3] and define **PL n-manifold** as a polyhedron M such that each point $x \in M$ has a neighborhood contained in M that is PL homeomorphic to a polyhedron open in $\mathbb{R}^n$.

With such an approach above, one no longer has to construct an explicit triangulation for a PL manifold that is not a polyhedron, subsuming every operation under the category of polyhedra. In practice, both B^n and S^n will be treated as a polyhedron rather than as an equivalence class. In dimension 2 and 3, every polyhedron that is homeomorphic to one another is PL homeomorphic by the Hauptvermutung for $n \leqslant 3$ (for more details, read [12]). Note that every n simplex is PL homeomorphic to $[-1,1]^n$. Proof can be found in [3].

We briefly note that one can reverse the process and give a *PL* structure to a topological n-manifold (M, Σ), where Σ is the atlas on M. We say M has a *PL* structure if for any two charts U, V of Σ and for any pair of maps $(\phi_U : U \to \mathbb{R}^n, \phi_V : V \to \mathbb{R}^n)$, the transition map $\phi_V \circ (\phi_U)^{-1} : \phi_U(U \cap V) \to \phi_V(U \cap V)$ is a *PL* homeomorphism. We note that $\mathbb{R}^n$ admits triangulations (for instance, subdividing the resulting cubes of the integer grid although we do not specify one method) and cite the work of Cairns and Whitehead that every smooth manifold admits a PL structure to note in particular that every open set of $\mathbb{R}^n$ also admits a triangulation. Indeed, if M has a PL structure, then M is a PL manifold, i.e. there is a triangulation map between M and a combinatorial manifold and the category of PL manifolds turn out to be isomorphic to the category of combinatorial manifolds. The proof is beyond the scope of the paper.

2.3 Neighborhood in PL Topology

2.3.1 Simplicial Neighborhood and Derived Neighborhood

Let L be a subcomplex of a complex K. The simplicial neighborhood of L in K, denoted $N(L,K)$ is defined as follows:

$$N(L,K) = \{\sigma : \sigma \in K, \sigma < \tau, \tau \cap |L| \neq \varnothing\} = \bigcup \{St(v,K) : v \in L^{(0)}\}.$$

The simplicial neighborhood is essentially the smallest subcomplex of K that is also a topological neighborhood of L. We say that L is full in K and denote it $L \lhd K$ if for any simplex $\sigma \in K$, whenever all of the vertices of σ belong in L, then σ is in L.

Theorem 2.3.1 Let L be a subcomplex of K. Then the followings are equivalent.

1. $L \lhd K$.
2. For every simplex $\sigma \in K$, $\sigma \cap |L|$ is a face of σ or empty.
3. Let $\sigma \in K - L$. Then $\sigma \cap |L| \neq |\dot{\sigma}|$.

Proof: Suppose 2 is true. Let σ be an arbitrary simplex of K such that its vertices all belong in L. Then σ must meet L in its face and in particular, its vertices must belong in a face of σ. This is only possible if the face of σ is trivial, i.e. it is σ itself. This proves $2 \Rightarrow 1$. Now suppose 1 is true. For contradiction, suppose there exists a simplex $\sigma \in K - L$ such that $\sigma \cap |L| = |\dot{\sigma}|$. This means that all of the vertices of σ belong in L and hence $\sigma \in L$ since $L \lhd K$. This is a contradiction proving $1 \Rightarrow 3$. Now suppose L is not full in K, i.e. there exists a set of vertices in $K \cap L$ such that the simplex spanned by that set, call it σ, is not in L. In particular then one proceed by induction on the dimension of the faces of σ until one finds a face τ whose boundary is contained in L. This proves $3 \Rightarrow 1$. Now suppose L is full in K and suppose the intersection between a simplex σ of K and L is not empty. Iff $\sigma = v_1 ... v_n$, and if L contains $v_{i_l}, ..., v_{i_k}$, then the simplex $v_{i_1} ... v_{i_k}$ which is a face of σ must belong in L. Since $v_{i_1} ... v_{i_k}$ is precisely where σ meets L, $1 \Rightarrow 2$.

Theorem 2.3.2 Let L be a subcomplex of K. Then there exists a subdivision $K' \prec K$ such that $L \lhd K'$. Moreover, if $L \lhd K$ and $(K', L') \prec (K, L)$, then $L' \lhd K'$.

Proof: For every simplex $\sigma \in K - L$ that meets L in its boundary ($K - L$ should not be confused with $|K| - |L|$), star σ at a point $\hat{\sigma} \in \mathring{\sigma}$ over L to get the subdivision K'. If $\sigma \in K' - L$ and $\dot{\sigma} \subset L$, then $\dot{\sigma} \subset K$ implies $\sigma \in K$. But this contradicts $\sigma \in K'$ since σ should have been starred by construction. The rest follows from theorem 2.3.1 part 2. Similarly, the second part of the theorem can be proved from theorem 2.3.1. part 3.

Now we define $C(L,K) = \{\sigma \in K : \sigma \cap |L| = \varnothing\}$ to be the simplicial complement of L in K. This is essentially the set of all simplices in K that does not touch L. One can observe that $K = N(L,K) \cup C(L,K)$. Define $\dot{N}(L,K) = N(L,K) \cap C(L,K) = \{\sigma \in N(L,K) : \sigma \cap |L| = \varnothing\}$. Since $C(L,K) \lhd K = N(L,K) \cup C(L,K)$ trivially, $\dot{N}(L,K) \lhd C(L,K)$.

Suppose K' is a subdivision of K mod $L \cup C(L,K)$. Then $N(L,K')$ is called a derived neighborhood of L in K. Essentially, one is subdividing the simplicial complex K by only dividing the simplices of K that do not belong in $L \cup C(L,K)$. Those simplices are precisely ones that do intersect L but are not fully contained in L.

Theorem 2.3.3 Let K' be a subdivision defined as above. Then $L \lhd K'$.

Proof: We first prove that L is indeed a subcomplex of K'. The subdivision of K mod $L \cup C(L,K)$ leaves all simplices $\sigma \in L$ untouched by construction. Hence the set of simplices constituting L is still a subset of K' and since L was originally a subcomplex of K, L still remains closed under taking faces in K'. Hence L is a subcomplex of K'. Now it remains to check that L is full in K', that is whenever $\sigma \in K'$ has all its vertices also contained in $L^{(0)}$, then $\sigma \in L$. If $\sigma \in K' \cap L$ and its vertices belong in L, the result holds trivially. If $\sigma \in K' \cap C(L,K)$, none of its vertices can belong in L. Hence it only remains to check $\sigma \in K'$ such that $\sigma \notin L$ and $\sigma \notin C(L,K)$. In such a case, there exists a simplex $\tau \in K - (C(L,K) \cup L)$ such that σ^0 (set of vertices of σ) consists of a new point in the interior of τ (or in the interior of the faces of τ) and vertices of $C(L,K) \cup L$. Since such a new point cannot belong in L, it is impossible for σ^0 to belong in $L^{(0)}$. This proves the theorem.

Now if K_1' and K_2' are different subdivisions of K mod $L \cup C(L,K)$, they are canonically isomorphic via the map $\phi : K_1 \to K_2$ such that $\phi|_{L\cup C(K,L)} = \mathbb{1}$. Hence we will not differentiate $N(L,K_1')$ and $N(L,K_2')$. The boundary of $N(L,K')$ is defined to be the subcomplex

$$\dot{N}(L,K') = \{\sigma \in N(L,K') : \sigma \cap |L| = \varnothing\}.$$

Theorem 2.3.4 Assume $L \lhd K$ and $(K_1,L_1) \prec (K,L)$. Then there exist derived neighborhoods $N(L,K')$ and $N(L_1,K_1')$ such that $|N(L,K')| = |N(L_1,K_1')|$.

We define an ε-neighborhood of L in K. Given that $L \lhd K$, define a simplicial map $f : K \to [0,1]$ that extends the vertex map

$$f(v) = \begin{cases} 0 & \text{if } v \in L, \\ 1 & \text{if } v \notin L. \end{cases}$$

Now pick $\sigma \in K$ such that $\sigma \notin L \cup C(L,K)$ and pick $\hat{\sigma} \in \mathring{\sigma} \cap f^{-1}(\varepsilon)$. If K' is the derived subdivision of K mod $L \cup C(L,K)$ with respect to $\hat{\sigma}$, then $N(L,K')$ is the ε neighborhood of L in K, also denoted $N_\varepsilon(L,K)$. Now proceeding to the proof, pick $\varepsilon > 0$ such that $f^{-1}((0,\varepsilon))$ does not contain any vertex of K or K_1. For each $\sigma \in K$ that intersects $|L|$, pick $\hat{\sigma} \in \mathring{\sigma} \cap f^{-1}(\varepsilon)$. The derived subdivision K' gives the desired derived neighborhood. Likewise, repeat the same for every simplex $\sigma \in K_1$ that intersects $|L_1|$.

2.3.2 Regular Neighborhood

Now we assume $L \lhd K$ (one can always choose a derived subdivision of K mod L to make L full in the new subdivision). A **regular neighborhood** of Y in X is defined to be the polyhedron $|N(L,K')|$.

The concept of a regular neighborhood is integral in PL topology and hence we provide some intuition on what it represents. Suppose L is a smaller shape (simplical complex) embedded inside a larger shape (simplicial complex) K. We subdivide the simplicial complex K and collect all the small enough simplices σ in the derived subidivison of K that intersect with $|L|$. The union of the realizations of the collection of such simplices make up the regular neighborhood $N(L,K')$.

Theorem 2.3.5 Let N_1,N_2 be regular neighborhoods of Y in X. Then there is a PL homeomorphism $h : X \to X$ such that $h|_Y = \mathbb{1}$ and $h(N_1) = h(N_2)$. Moreover, if Y is compact, then we may choose the function h so that $h = \mathbb{1}$ outside some compact subset of X.

Proof: Let $N_1 = |N(L_1,K_1')|, N_2 = |N(L_2,K_2')|$ where for $i = 1,2$, $L_i < K_i$ respectively triangulates $Y \subset X$ and $L_i \lhd K_i$. Pick a triangulation $L_0 < K_0$ of $Y \subset X$ where K_0 subdivides both K_1 and K_2. Then by theorem 2.3.1, there exist derived subdivisions K_i'' of K' mod $L_i \cup C(L_i, K_i)$ for $i = 0, 1, 2$, such that

$$|N(L_0, K_0'')| = |N(L_1, K_1'')| = |N(L_2, K_2'')|.$$

For $i = 1, 2$, we may pick canonical isomorphisms $\phi_i : K_i' \to K_i''$ such that $\phi_i|_{L_i \cup C(L_i, K_i)} = \mathbb{1}_{L_i \cup C(L_i, K_i)}$ and maps $N(L_i, K_i')$ to $N(L_i, K_i'')$. Then $\phi_2^{-1} \circ \phi_1 : X \to X$ is a PL homeomorphism that is an identity map on $Y \cup [C(L_1, K_1) \cap C(L_2, K_2)]$ and maps N_1 to N_2.

We mention the following two theorems without proofs (proofs can be found in [3] and [4]).

Theorem 2.3.6 If X is a subpolyhedron of a PL manifold M, a regular neighborhood N of X contained in M is itself a PL manifold. If X belongs to the interior of M and there exists some triangulation $L < K$ of $X \subset M$ such that $N = |N(L, K')|$, then $\partial N = |\dot{N}(L, K')|$.

Theorem 2.3.7 (Simplicial Neighborhood Theorem) If X is a subpolyhedron in the interior of a PL manifold M and N is a neighborhood of X in int M, then N is a regular neighborhood of X if and only if N is a PL manifold with boundary and there exist triangulations $L, J < K$ respectively of $X, \partial N \subset N$ such that $L \lhd K, K = N(L, K)$, and $J = \dot{N}(L, K)$.

2.4 Isotopies in PL Category

Let X be a subpolyhedron of a polyhedron M. An **isotopy of X in M** is a level preserving, closed, PL embedding (a topological embedding that is piecewise linear) $F : X \times [0,1] \to M \times [0,1]$. A map $G : A \times [0,1] \to B \times [0,1]$ is level-preserving if whenever $G(a,t) = (b,s)$, $s = t$. One can interpret level preserving then as mapping the slice $A \times \{t\}$ into the slice $B \times \{t\}$ for every $t \in [0,1]$. The isoptopy of X in M is then a continuous deformation of the subpolyhedron X contained in M such that F is a PL homeomorphism onto its image and if we denote $F(x,t) = (f_t(x), t)$, each $f_t : X \to M$ must be a PL embedding since F is level preserving. An **isotopy of M** is defined as a level preserving PL homeomorphism $H : M \times [0,1] \to M \times [0,1]$ such that H_0 is an identity map. Again, since H is level preserving, $H(y,t) = (h_t(y), y)$ for all $t \in [0,1]$ where each $h_t : M \to M$ is a PL homeomorphism and an isotopy H of M represents a continuous family of PL self-homeomorphisms $\{h_t\}$ such that h_0 is an identity map. In this case, we say h_1 is ambient isotopic to the identity map. An isotopy F of $X \subset M$ is said to be **ambient** if there exists an isotopy H of M such that the diagram below commutes:

$$\begin{tikzcd}
X \times I \arrow[rr, "F_0 \times Id"] \arrow[dr, "F"'] & & M \times I \arrow[dl, "H"] \\
& M \times I &
\end{tikzcd}$$

In particular, $F_0 \times Id$ maps (x,t) to $(f_0(x), t)$ which represents the initial position of X. H then applies the isotopy of the ambient space, mapping $(f_0(x), t)$ to $(f_t(x), t)$, i.e. the commutativity of the diagram implies that $f_t(x) = h_t(f_0(x))$. The intuition is that the isotopy of X in M is ambient if a movement of X in M can be realized instead by moving M (the ambient space) itself. An isotopy / ambient isotopy F is said to fix a subset $V \subset X$ if $F|_{V \times I} = F_0 \times id|_I$ on $V \times I$. In this case, we write F is mod V. The proof for the following theorem can be found in [3].

Theorem 2.4.1 (Regular Neighborhood Theorem) Suppose X is a subpolyhedron that is in the interior of a PL manifold M. Let N_1 and N_2 be regular neighborhoods of X in the interior of M. Then there exists an isotopy of M that is fixed on X and fixed outside an arbitrary open neighborhood of $N_1 \cup N_2$. Moreover the isotopy maps N_1 to N_2.

2.5 Orientation of PL Manifolds

Let $r_n : [-1,1]^n \to [-1,1]^n$ be defined as follows:

$$r_n(x_1,...,x_n) = (-x_1,x_2,...,x_n).$$

Theorem 2.5.1 Let I denote [-1,1]. Then $r_n|_{\partial I^n}$ is not homotopic to the identity map.

Proof: Recall from definition 2.2.1 that S^{n-1} denotes a compact polyhedron PL homeomorphic to $\partial([-1,1]^n)$. W.L.O.G, let $S^{n-1} = \partial([-1,1]^n)$. Let $\hat{S}^{n-1}$ denote the usual unit $n-1$ sphere. Recall from section 1.3.6 that the degree of an identity map is 1 whereas the degree of a reflection map is -1. Since homotopic maps have the same degree, the reflection map is not homotopic to the identity map. Now, fix a homeomorphism $\phi : \partial I^n \to \hat{S}^{n-1}, x \mapsto \frac{x}{\|x\|}$. Then $\hat{r}_n = \phi \circ r_n \circ \phi^{-1}$. Any homotopy $H : \partial I^n \times I \to \partial I^n$ between two maps f_0, f_1 on ∂I^n corresponds to a homotopy $\hat{H} : \hat{S}^{n-1} \times I \to \hat{S}^{n-1}$ between two maps $\hat{f}_0, \hat{f}_1$ on $\hat{S}^{n-1}$ by $\tilde{H}(y,t) = \phi(H(\phi^{-1}(y),t))$; that is, f_0 is homotopic to f_1 if and only if $\hat{f}_0$ is homotopic to $\hat{f}_1$. Since $\hat{r}_n$ is not homotopic to the identity map on $\hat{S}^{n-1}$, r^n is not homotopic to the identity map on ∂I^n.

Theorem 2.5.2 Let B_1^n, B_2^n be PL balls and $h_0, h_1 : B_1^n \to B_2^n$ be PL homeomorphisms which agree on $\partial B_1^n = S_1^{n-1}$. Then h_0 and h_1 are ambient isotopic mod S_1^{n-1}.

Proof: It suffices to prove the theorem for PL homeomorphisms $h_0, h_1 : [0,1]^n \to [0,1]^n$ (recall definition 2.1.1 and note that $[0,1]^n$ is PL homeomorphic to $[-1,1]^n$). Let I denote $[0,1]$. Define the homeomorphism H of $I^n \times I$ as follows:

$$H_0 = h_0, H_1 = h_1, H|_{\partial I^n \times I} = (h_0|_{\partial I^n}) \times Id$$

Now let $x_0 = (0,...,0,\frac{1}{2}) \in I^n \times I$. Consider any arbitrary point $p \in (I^n \times [0,1]) \backslash \{x_0\}$. Then there is a unique ray that starts at x_0 and passes through p. Let b be the unique point that such a ray intersects $\partial(I^n \times [0,1])$. Then p can be written uniquely as $p = (1-s)x_0 + sb$ for some $s \in (0,1]$. Let $H(p) = (1-s)x_0 + sH(b)$. One can check that such H is the desired ambient isotopy between h_0 and h_1 mod ∂I^n.

Lemma 2.5.1 Let $p, q \in int\ I^n$. Then there exists an isotopy of I^n mod ∂I^n that maps p to q.

Proof: Connect p and q by a straight line segment $\gamma(t) = p + t(q-p), t \in [0,1]$. Pick $\varepsilon > 0$ so small that

$$N = \{x \in I^n | \inf_{y \in \gamma([0,1])} \|x-y\|_\infty \leqslant \varepsilon\} \subset Int\ I^n.$$

Then subdivide I^n so that N is a subpolyhedron of the resulting triangulation. Define $v(x)$ by $v(x) = (1 - \|x - p\|_\infty/\varepsilon)(q-p)$ for $x \in N$ and $v(x) = 0$ for $x \in I^n \backslash N$. Then $H_t(x) = x + tv(x)$ for $t \in [0,1]$ is a PL homeomorphism that is an identity map on ∂I^n. Moreover the family $\{H_t\}$ is a PL isotopy and hence H_1 is a PL homeomorphism. Define $h = H_1$. Then $h(p) = q$. The rest of the proof follows from theorem 2.5.2.

Theorem 2.5.3 Let $h : \partial I^n \to \partial I^n$ be a PL homeomorphism where I^n denotes the unit n-cube, $[-1,1]^n$. Then h is ambient isotopic to either the identity map or the reflection map r^n. (For proof of this theorem, see [3]).

Theorem 2.5.4 (Disc Theorem) Suppose M is a connected PL n-manifold and $h_1, h_2 : I^n \to int\ M$ are PL embeddings. Then h_1 is ambient isotopic to either h_2 or $h_2 \circ r_n$. (For proof of this theorem, see [3]).

The disk theorem implies that if M is a connected PL n-manifold, there can be either one or two ambient isotopy classes of PL embeddings $I^n \to int\ M$. If $n > 1$, a PL n-manifold M is **orientable** if there are two ambient isotopy classes. If there is only one ambient isotopy class, M is said to be **non-orientable**. An **orientation** of an orientable manifold M is then defined as a choice of an ambient isotopy class. When an embedding $h : I^n \to int\ M$ belongs in the chosen class, h is said to orient M. Now suppose that h orients M. If $g : M \to M$ is a PL homeomorphism, then g is said to be **orientation-preserving** if $g \circ h$ is isotopic to h. Otherwise, g is said to be **orientation-reversing**.

Corollary 2.5.1 A self-homeomorphism f of ∂I^n is isotopic to the identity map if and only if f preserves orientation.

3 Piecewise Linear Knot Theory

3.1 Links and Knots

Recall definition 2.1.1. S^3 is a boundary of Δ^3 that is a simplicial subcomplex of a triangulated B^4.

Definition 3.1.1. A **link** L of m components is defined as a subset of $\mathbb{R}^3$ or S^3 that consists of m disjoint, piecewise linear, simple closed curves. A link of one component is called a **knot**.

A topological curve is a continuous function $\gamma : I \to X$. A curve γ is closed if $I = [a,b]$ and $\gamma(a) = \gamma(b)$. A piecewise linear curve means that the curve is the union of finitely many straight line segments, where by straight, it means straight in the affine structure defined from the triangulation of $\mathbb{R}^3$ or S^3. In practice, knots and links will be drawn as well rounded curves. While a knot and a link can be defined in a smooth category as smooth embeddings $S^1 \to S^3$ where S^1 and S^3 in this context represent the unit 1-sphere and the unit 3-sphere, the piecewise linear structure of the curve prevents the link from having pathological behaviors such as having infinite kinks that turn smaller as they converge to a point. Such ill behaved knots are called **wild knots** while well behaved knots are called **tame knots**.

Definition 3.1.2. Links L_1 and L_2 in S^3 are defined to be **equivalent** if there exists an orientation preserving PL homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = h(L_2)$.

In particular, from corollary 2.5.1, an orientation preserving PL homeomorphism of S^3 is isotopic to the identity map, that is there exists a homotopy $h_t : S^3 \to S^3$ such that $h_0 = \mathbb{1}$, $h_1 = h$, and $(x,t) \mapsto (h_t x, t)$ is a level-preserving PL homeomorphism from $S^3 \times [0,1]$ to itself. This intuitively implies that the entirety of S^3 can be continuously deformed to distort L_1 to L_2. To be more precise, define $F_0 : L_1 \to S^3$ such that $F_0(x) = (x)$ and define $F : L_1 \times I \to S^3 \times I$ by restricting the isotopy H to $L_1 \times I$, that is

$$F(x,t) = H(h_0(x),t) = H(x,t) = (h_t(x),t) \text{ for } x \in L_1.$$

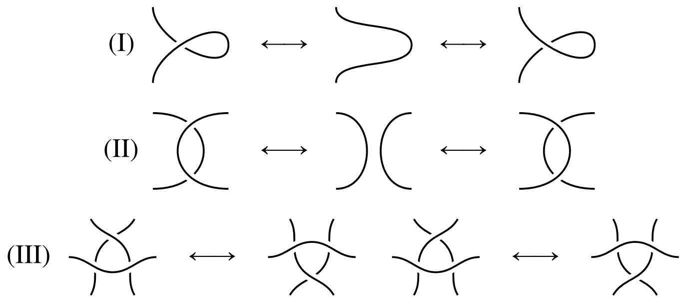
Then since $H \circ (F_0 \times Id)$ maps $(x,t) \in L_1 \times I$ to $H(f_0(x),t) = H(x,t)$ and F maps $(x,t) \in L_1 \times I$ to $(h_t(x),t) = H(x,t)$, it follows that $H \circ (F_0 \times Id) = F$. Hence $F_0 = h_0|_{L_1} : L_1 \to S^3$ and $F_1 = h_1|_{L_1} : L_1 \to S^3$ are ambient isotopic.

3.2 Link Diagram and Reidemeister Moves

To define a link diagram, we will from now on assume that L is a PL embedding of S^1 in $\mathbb{R}^3$. Without proof, we claim that if L_1, L_2 are equivalent in $\mathbb{R}^3$, then one can change L_1 into L_2 by finding a planar triangle that intersects the link in one edge of the triangle and replace the edge with two other edges of the triangle, and repeating such moves finitely many times (proof can be found in [10]). Using this fact, one can reposition a link L in $\mathbb{R}^3$ such that if it gets projected onto $\mathbb{R}^2$ by the canonical projection map, the projections of any two line segments of the link intersect at most one point, none of the intersection is an end point for the image of two disjoint line segments, and no point belongs to the projection of three line segments. Then the image of the projection map alongside information at the intersections that shows which line segment goes over the other in $\mathbb{R}^3$ is called the **link diagram** of L (One may also project S^3 to S^2

but the construction of the projection map involves subtle geometric intricacies. We omit the details). The ”over and under” information is conventionally expressed using breaks in the segments that go under.

It is known that any diagrams of two equivalent links can be related by a sequence of what are called Reidemeister moves or orientation preserving homeomorphisms between planes. The three Reidemeister moves are shown below. Note that the third Reidemeister move preserves the crossing information of the knot diagram. Moreover, any orientation preserving homeomorphism between planes must also preserve the crossing information of the knot diagram.



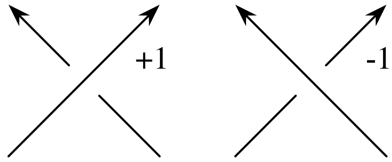
Now we will assume that both S^3 and $\mathbb{R}^3$ are oriented. Then there are 2^n different ways to orient the components of an n-link. If a knot K is oriented, the reverse of K, rK is a knot with opposite orientation. If $\rho : S^3 \to S^3$ is an orientation reversing homeomorphism, then $\rho(K)$, denoted $\bar{K}$, is called the obverse or reflection of K. A knot K is called an unknot if it bounds an PL embedded 2 ball in $\mathbb{R}^3$ or S^3. Using the triangle moves elaborated above, one can use the 2-simplices of the triangulation of the disk to change the unknot to a boundary of a 2-simplex linearly embedded in S^3 (more precisely, a linear embedding of a simplex in $\mathbb{R}^4$ such that the image of the embedding is restricted to S^3).

Let K_1 and K_2 be two different knots in S^3. Treating each knot as lying in distinct two copies of S^3, one can remove from each 3-sphere a 3-ball that intersects the knot in an unknotted spanning arc (this means that the ball arc pair is PL homeomorphic to the product of the interval with a disc-point pair), and identify together the resulting boundary spheres and their intersections with the knots (in a way that orientations align). Here, a spanning arc is a string inside the 3-ball whose two endpoints lie on the boundary sphere and whose interior stays strictly in the interior of the 3-ball. Hence, each arc meets the boundary of its ball at just its end points. The resulting new knot is denoted $K_1 + K_2$. One can check that the addition of oriented knots is well defined up to equivalence (details are provided in [3]).

Definition 3.2.1 A knot K is called a prime knot if

1. K is not an unknot.
2. If $K = K_1 + K_2$, then either K_1 or K_2 is an unknot.

The **crossing number** of a knot K refers to the minimal number of crossings for a diagram of K. An **unknotting number** $u(K)$ of a knot K refers to the minimum number of changes in crossing information of an arbitrary diagram of K (e.g. from ”over” to ”under”) to get a diagram of an unknot. As a point traverses along the diagram of a knot K and it alternates between ”under” and ”over” at each crossing point, the knot K is called an alternating knot. For every crossing of an oriented knot diagram, one give a sign ± 1 by the following figure below:



It is not difficult to notice that changing the orientation of a knot will not affect the sign of the crossing.

Definition 3.2.2. Suppose that L is a link with two components L_1 and L_2. Then the linking number, denoted $lk(L_1,L_2)$ refers to the half of the sum of the signs in a diagram of L for crossings one of whose strand belongs to L_1 and the other strand belongs to L_2.

By definition, $lk(L_1,L_2) = lk(L_2,L_1)$. The linking number of a link also embodies a relationship with homology theory. Let K be a tame knot in S^3 and pick a triangulation T (here T is a simplicial complex) of S^3 such that K is a 1-subcomplex. Then the simplicial neighborhood

$$N(K,T) = \bigcup\{\sigma \in T^2 | \sigma < \tau, \tau \cap |K|\}$$

is also a regular neighborhood of K. Define the **exterior** X of K in S^3 as the closure of $S^3 - N$ and hence X is a connected 3-manifold with a boundary that is a triangulated torus (for more details, see [3]). This X has the same homotopy type as $S^3 - K$, where $X \cap N = \partial X = \partial N$ and $X \cup N = S^3$.

Theorem 3.2.1 Suppose K is an oriented knot in oriented S^3, and X is the exterior of K. Then $H_1(X)$ is isomorphic to $\mathbb{Z}$ that is generated by the equivalence class of a simple closed curve in ∂N that bounds a disc in N that meets K at one point.

Proof: Since all links are piecewise linearly embedded in S^3, we may assume that X and N are simplicial subcomplexes of some triangulation of S^3 and compute the simplicial homology groups. Since $\mathring{X} \cup \mathring{N} = S^3$, we can construct a Mayer-Vietoris sequence as follows:

$$\cdots \longrightarrow H_3(X) \oplus H_3(N) \longrightarrow H_3(S^3) \longrightarrow \cdots H_1(X \cap N) \longrightarrow H_1(X) \oplus H_1(N) \longrightarrow H_1(S^3) \longrightarrow \cdots$$

Since any connected, triangulated 3-manifold with non-empty boundary deformation retracts to some 2-dimensional subcomplex, X has a zero 3-dimensional homology group (see [13]; in fact, every PL manifold M^{n+1} with boundary can be collapsed to some n-dimensional polyhedron). Moreover, $X \cap N \cong T^2 = S^1 \times S^1$ and $H_3(T^2) \cong 0$. This follows trivially from Meyer Vietoris sequence since a torus is the union of two open cylinder both of which are homotopy equivalent to S^1 which in turn induces a exact sequence $H_3(S^1) \oplus H_3(S^1) \to H_3(T^2) \to H_2(S^1 \sqcup S^1)$. Similarly, $H_3(D^2 \times S^1) \cong 0$. Extending the same logic, one can compute every group in the exact sequence above except $H_2(X)$ and $H_1(X)$. Now, the groups $H_3(S^3)$ and $H_2(X \cap N) \cong H_2(T^2)$ are both $\mathbb{Z}$. Notice that the representation of the generator (that is, the chain consisting of the sum across all 3-simplices of S^3 coherently oriented) of $H_3(S^3)$ is pulled back to the sum of coherently oriented 2-simplexes in X and 2-simplices in N which maps by the boundary map to the sum of 2-simplices in $X \cap N$. Then the sum of 2-simplices get pull back again to a sum of coherently oriented 2-simplices in $X \cap N$. This shows that the generator of $H_3(S^3)$ is sent to a generator of $H_2(X \cap N)$ and hence the map $H_3(S^3) \to H_2(X \cap N)$ is an isomorphism implying $H_2(X \cap N \cong 0$. Then by exactness and from $H^2(S^3) = 0$, we conclude that $H_2(X) \oplus H_2(N) = 0$.

Moreover, since $H_2(S^3)$ and $H_1(S^3)$ are both 0, $H_1(X \cap N) \to H_1(X) \oplus H_1(N)$ is an isomorphism. Since $H_1(X \cap N) \cong H_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, $H_1(X) \cong \mathbb{Z}$. Now suppose μ is a simple closed curve in $X \cap N$ that bounds a disc in N as given in the premise. W.L.O.G., suppose μ is oriented so it encircles K with a right hand screw. Since μ is not a multiple of other element in $H_1(X \cap N)$, it is indivisible and it represents 0 in $H_1(N)$. By the isomorphism $H_1(X \cap N) \to H_1(N)$, $[\mu]$ gets mapped to $(1,0) \in H_1(X) \oplus H_1(N)$.

Corollary 3.2.1 If C is an oriented simple closed curve in X, then the homology class $[C] \in H_1(X)$ is $\mathrm{lk}(C,K)$.

The theorem above shows that a unique element of $H_1(X \cap N)$ must get mapped to $(0,1)$ where 1 is represented by K. Since $(0,1)$ is indivisible, its class in $H_1(X) \oplus H_1(N)$ must be represented by a simple closed curve in $X \cap N$. We define such a simple closed curve, denoted μ, as a **meridian** of K.

3.3 The Jones Polynomial

Now, we discuss a way to associate every knot a Laurent polynomial using the knot diagram. The polynomial will serve as an invariant as we will later show that it remains the same up to Reidemeister moves on the diagram. This in turn implies that knots with distinct polynomials cannot be equivalent. In this paper, we first introduce a slightly different polynomial known as the Kauffman bracket polynomial and subsequently define the Jones polynomial.

Definition 3.3.1 The Kauffman bracket is a function that maps a knot diagram D in $\mathbb{R}^2$ (or $S^2 = \mathbb{R}^2 \cup \infty$) to a Laurent polynomial $\langle D \rangle$ in $\mathbb{Z}[A^{-1}, A]$ such that it satisfies the following rules:

1. $\langle \bigcirc \rangle = 1$
2. $\langle D \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle$
3. $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

We clarify some notations above. First, $\bigcirc$ is a diagram of an unknot that has no crossing and $D \sqcup \bigcirc$ is a diagram consisting of D and the closed curve $\bigcirc$ that do not cross each other. The brackets in the third equation refer to three knot diagrams that are equal to each other except near one point where their configuration differ as described in the drawings. The third condition implies that the bracket polynomial of a knot diagram with n polynomials can be expressed as a sum of 2^n diagrams, each without any crossing. Then the first and condition implies that the bracket polynomial equals $A^{2n} + A^{-2n}$. The Kauffman bracket also extends for link diagrams. Hence if link L consists of n components each without crossing, by 1 and 2, the bracket polynomial for its diagram equals $-A^{-2} - A^2)^{c-1}$. This naturally motivates to define the bracket polynomial of an empty diagram as $-A^{-2} - A^2)^{c-1}$. To check whether the bracket polynomial is well defined, one must verify that the orientation preserving homeomorphism between diagrams do not affect the polynomial and the polynomial is invariant under Reidemeister moves.

Theorem 3.3.1 A type I Reidemeister move on a diagram changes its bracket polynomial as follows:

$$\langle \circlearrowleft \rangle = -A^3\langle \frown \rangle, \qquad \langle \circlearrowright \rangle = -A^{-3}\langle \frown \rangle$$

Proof: The proof follows from applying condition 3 and subsequently condition 2 above. One then gets

$$\langle \circlearrowleft \rangle = (A(-A)^{-2} - A^2) + A^{-1}\langle \frown \rangle = -A^3\langle \frown \rangle$$

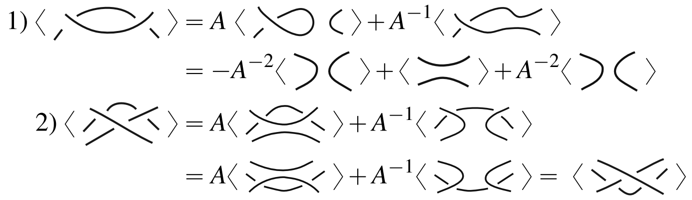
The second equality also follows similarly.

If in condition 3, the crossing of the diagram on the left hand side was switched, then A and A^{-1} would be interchanged since the diagram is rotated by $\pi/2$. Recall that $\bar{D}$ is the diagram of the image of an orientation reversing homeomorphism which indicates that the overs and unders at the crossings get reversed. Hence $\langle \bar{D} \rangle = \overline{\langle D \rangle}$ where $\overline{\langle D \rangle}$ denotes the involution on $\mathbb{Z}[A^{-1}, A]$. Using this algorithm, one can easily compute the Kauffman bracket of a diagram of a Hopf link and a trefoil knot to be respectively $(-A^4 - A^{-4})$ and $(A^{-7} - A^{-3} - A^5)$.

Figure 1: Side-by-side diagrams of the Hopf link (left) and the Trefoil knot (right).

Theorem 3.3.2 Type 2 and type 3 Reidemeister moves do not change the bracket of the diagram D.

Proof: The theorem is proved by applying the rules of Kauffman bracket.



Definition 3.3.2 If D is a diagram of an oriented link, the **writhe** of D, denoted $w(D)$ is the sum of the sign of crossings, as defined in section 3. While the first Reidemeister move changes the writhe by $pm1$, the writhe of a link does not change under second and third Reidemeister moves. This indicates that both the Kauffman bracket polynomial and the writhe of an oriented diagram can serve as a good invariant for identifying links and knots, motivating the definition of the Jones invariant.

Theorem 3.3.3 Let L be an oriented link and let D be the diagram of L. Then

$$(-A)^{-3w(D)}\langle D\rangle$$

is an invariant of L up to equivalence.

Proof: From theorem 3.3.2, we know that type 2 and type 3 Reidemeister moves do not affect that bracket polynomial of D and the writhe of D. Moreover, theorem 3.3.1 indicates that if the crossing has a positive sign, then the type 1 Reidemeister move multiplies the bracket by $-A^3$ while the drop in writhe offsets the factor with $-A^{-3}$. The same logic follows when the crossing has a negative sign. Since the Reidemeister moves do not affect the Jones invariant and every equivalent diagram is related by Reidemeister moves, the theorem follows.

Definition 3.3.3 Let L be an oriented link and D be its link diagram. Then the **Jones polynomial** of L is a Laurent polynomial in an indeterminate $t^{1/2}$ defined as follows:

$$V(L) = \left((-A)^{-3w(D)}\langle D\rangle\right)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

By theorem 3.3.3, the Jones polynomial is well defined. Now we prove some properties of the Jones polynomial.

Theorem 3.3.4 The Jones polynomial is a function

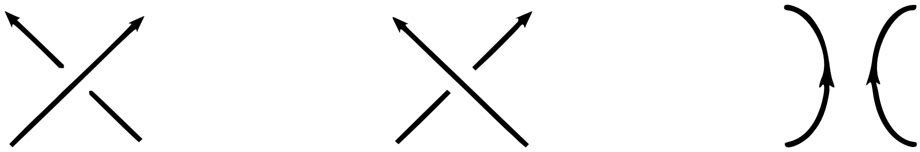
$$V : \{\text{ Oriented links embedded in } S^3\} \to \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

that satisfies the following properties.

1. $V(\text{unknot})$ equals 1.

2. Suppose three oriented links L_1, L_2, L_3 are same everywhere but in the neighborhood of a point as below. Then

$$t^{-1}V(L_1) = tV(L_2) + (t^{-1/2} - t^{1/2})V(L_3) = 0$$



(a) L_1 (b) L_2 (c) L_3

Figure 2: Three diagrams side by side: L_1, L_2, and L_3. Diagram is cited from [1].

Proof: The first property is trivial since a diagram with no crossing has writhe 0 and $\langle \bigcirc \rangle = 1$. Now we prove the second property. Recall from Kauffman bracket rules,

$$\langle \times \rangle = A\langle \,)\,(\, \rangle + A^{-1}\langle \asymp \rangle$$
$$\langle \times \rangle = A^{-1}\langle \,)\,(\, \rangle + A\langle \asymp \rangle.$$

Multiply both sides of the first equation by A and both sides of the second equation by A^{-1}, then subtract the second to get

$$A\langle \times \rangle - A^{-1}\langle \times \rangle = (A^2 - A^{-2})\langle \,)\,(\, \rangle.$$

Since $w(L_1) - 1 = w(L_3) = w(L_2) + 1$,

$$-A^4 V(L_1) + A^{-4} V(L_2) = (A^2 - A^{-2}) V(L_3).$$

Substituting $t^{1/2}$ for A^{-2} returns

$$t^{-1} V(L_1) = tV(L_2) + (t^{-1/2} - t^{1/2}) V(L_3) = 0.$$

Theorem 3.3.5 Suppose L' is a link composed of a link $\{L\}$ and a trivial unknotted component that is unlinked with L. Then $V(L') = (-t^{-1/2} - t^{1/2}) V(L)$.

Proof: Let D' be a diagram of L', and D be the diagram of L. Then $D' = D \sqcup \bigcirc$. This implies that

$$\langle D' \rangle = (-A^{-2} - A^2)\langle D \rangle.$$

The Jones polynomial $V(L)$ is then obtained from the Kauffman bracket $\langle D \rangle$ as follows:

$$\begin{aligned} V(L) &= (-A)^{-3w(D')}\langle D' \rangle \\ &= (-A)^{-3w(D)}[(-A^{-2} - A^2)\langle D \rangle] \\ &= (-A^{-2} - A^2) \cdot (-A)^{-3w(D)}\langle D \rangle \end{aligned}$$

Substituting $t^{1/2} = A^{-2}$ proves the theorem.

In general, theorem 3.3.4 provides an algorithm to compute the Jones polynomial of any oriented link. This is because any link, by changing the crossings of a diagram, can be changed to an unlink of n unknots whose Jones polynomial is then computed as $(-t^{-1/2} - t^{1/2})^{n-1}$ by theorem 3.3.5. If K_1, K_2 are knots, then the Jones polynomial of the knot sum $K_1 + K_2$ is trivially the product of the Jones polynomial, i.e. $V(K_1 + K_2) = V(K_1)V(K_2)$.

3.4 Alternating Links and Alternating Digrams

In this section, we first examine the geometric properties of an alternating link and define prerequisite notions. We omit the proofs for it requires substantial knowledge of geometric topology. Moreover, we assume that every embedding of a 2 sphere in S^3 is piecewise linear. We first state a classic theorem without proof.

Theorem 3.4.1 (Schönflies Theorem) Suppose $e : S^2 \to S^3$ is a PL embedding. Then $S^3 - e(S^2)$ consists of two components, each of whose closure is a PL 3-ball.

Definition 3.4.1 A link $L \subset S^3$ that has at least two components is called a **split link** if there exists S^2 in $S^3 - L$ that separates S^3 into two balls, each of which contains a component of L.

Definition 3.4.2 Let D be a link diagram. D is a **split diagram** if there exists a simple closed curve in $S^2 - D$ that separates S^2 into two discs, each of which contains a part of D.

Theorem 3.4.2 Let L be a link with an alternating diagram denoted D. Then L is a split link if and only if D is a split diagram.

We note that the definition of prime knots may be extended to links. A link $L \subset S^3$ that is not an unknot is **prime** if every 2-sphere that intersects L transversely at two points, bounds a 3-ball that intersects L in one unknotted arc (See section 2 for definition of unknotted arc). Two submanifolds of a smooth manifold is defined to intersect transversely if at each point of their intersection, the tangent spaces of each submanifold generates the tangent space of the ambient manifold at the point. A diagram $D \subset S^2$ of a link that is not an unknot is a **prime diagram** if any simple closed curve in S^2 that intersects D transversely at two points bounds a disc which intersects D in a diagram U of the unknotted ball-arc pair described above. We say that D is strongly prime if U is a trivial diagram with no crossing.

Theorem 3.4.3 Let L be a link with an alternating diagram denoted D. Then L is a prime link if and only if D is a prime diagram.

Now we are in a good position to finally prove the first two of Tait conjectures. Suppose D is an n-crossing diagram of a link whose crossings are labeled $1,2,3,...,n$. We define the state for D as a function $s : \{1,2,3,...,n\} \to \{-1,1\}$. Trivially, there exist 2^n states for a given D. Now, given D and a state s, define sD to be the diagram constructed from D by replacing each crossing of the diagram with two segments with two segments that do not intersect each other. Indeed, there are two ways to replace the crossing as specified below (the left arrow is called the positive way and the right arrow is called the negative way).

The diagram sD now has no crossing at all and hence is a disjoint union of simple closed curves. Let $|sD|$ denote the number of such curves. We claim the following theorem.

Theorem 3.4.4 Let D be a link diagram with n crossings. Then the Kauffman bracket for D is given by

$$\langle D \rangle = \sum_{s} \left(A^{\sum_{i=1}^{n} s(i)} (-A^{-2} - A^{2})^{|sD|-1} \right).$$

The summation is taken over all states of D, $f : \{1,2,3,\ldots,n\} \to \{-1,1\}$.

Proof: The proof of the theorem directly follows from the fact that it satisfies the definition of a Kauffman bracket in definition 3.3.1.

Now, define s_+ and s_- so that for all i, $s_+(i) = 1$ and $s_-(i) = -1$. We say that a diagram D is plus adequate if $|s_+D| > |sD|$ for all s that satisfies $\sum_{i=1}^{n} s(i) = n-2$. Likewise a diagram D is minus adequate if $|s_D| > |sD|$ for all

s that satisfies $\sum_{i=1}^{n} s(i) = 2-n$. If both hold, D is called adequate. While the definition is a little convoluted, we introduce an algorithm to check whether a diagram is adequate. Suppose we change D to s_+D by replacing all the crossings in the positive way. Then D is plus adequate if the two segments of s_+D that replaced the crossing no longer belong to the same component of s_+D. Likewise, D is minus adequate if the two segments of s_-D that replaced the crossing no longer belong to the same component of s_-D. The algorithm introduces us to the next theorem.

Theorem 3.4.5 A reduced alternating link diagram is adequate. The term reduced means that there is no crossing of the form (nor the reflection of such form) featured in the figure below. Such a crossing is also called a nugatory or removable crossing since if we extend the crossing arcs to carve the plane into four quadrants, the complement of the diagram in the plane features in diagonally opposite quadrants near the crossing. Then in practice, one can remove the crossing by rotating half of the link.

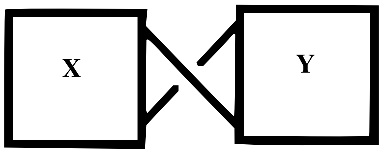


Figure 3: Nugatory Crossing; Diagram is cited from [1].

Proof: Color the complementary regions of the link diagram as black and white like a chessboard. Since the link diagram is alternating, the components of S_+D make up the boundaries of the regions of either color, say black where the corners of the regions are smoothed out. Likewise, the components of S_-D make up the boundaries of the white regions. Since there is no nugatory crossing and hence no region abuts itself, D is an adequate diagram. Here, reduced means that there is no crossing of the form (see the figure below) (in which the squares labeled X and Y contain the whole diagram away form the crossing). Such a crossing is called a nugatory or removable crossing. It is a crossing at which one region of the complement of the diagram in the plane features twice, appearing near the crossing in a pair of diagonally opposite quadrants. In practice such a crossing could be removed by rotating half of the link.

Let P be a Laurent polynomial. Then $M(P), m(P)$ will respectively denote the maximum and minimum powers of the indeterminate of P. We introduce how to determine $M\langle D\rangle$ and $m\langle D\rangle$ where D is a link diagram.

Theorem 3.4.6 Suppose D is a link diagram with n crossings. Then

1. $M\langle D\rangle \leqslant n+2|s_+D|-2$. If D is plus adequate, $M\langle D\rangle = n+2|s_+D|-2$
2. $m\langle D\rangle \geqslant -n-2|s_-D|+2$. If D is minus adequate, $m\langle D\rangle = -n-2|s_-D|+2$

Proof: Let s be some state for D. Define

$$\langle D|s\rangle = A^{\sum_{i=1}^{n} s(i)}(-A^{-2}-A^{2})^{|sD|-1}.$$

One can easily verify that $\langle D\rangle = \sum_s \langle D|s\rangle$. Since $\sum_{i=1}^{n} s_+(i) = n$, $M\langle D|s_+\rangle = n+2|s_+D|-2$. Now any state s can be attained by starting with the constant state s_+, then changing the value of s_+ on integers that mark the crossings, one at a time. To put it more precisely, if we let $s = s_k, s_0 = s_+$, there exists a sequence of states $s_0, s_1, s_2, \ldots, s_{k-1}, s_k$ such that $s_{r-1}(i) = s_r(i)$ for all $i \in \{1,2,\ldots,n\}$ except for i_r where $s_{r-1}(i_r) = 1$ and $s_r(i_r) = -1$. This implies that $\sum_{i=1}^{n} s(r_i) = n-2\text{r}$ and $|s_rD| = |s_{r-1}D| \pm 1$. Then, $M\langle D|s_{r-1}\rangle - M\langle D|s_r\rangle$ is either 0 or 4. Hence, either case, $M\langle D|s_r\rangle \leqslant M\langle D|s_{r-1}\rangle$ and for all s,

$$M\langle D|s\rangle \leqslant n+2|s_+D|-2.$$

Now suppose that D is plus-adequate. Then $|s_1D| = |s_+D|-1$. In other words, $M\langle D|s_r\rangle$ decreases at the first step when one changes r from 0 to 1 and $M\langle D|s_r\rangle$ never rises afterwards. Hence $M\langle D|s\rangle < n+2|s_+D|-2$ whenever $s \neq s_+$. Since $\langle D\rangle = \sum_s \langle D|s\rangle$ and the maximal degree of $\langle D|s_+\rangle$ is never cancelled by a term from $\langle D|s\rangle$ for any s, the first statement holds. The second statement also follows similarly.

Corollary 3.4.1. Let D be an adequate diagram, then

$$M\langle D\rangle - m\langle D\rangle = 2n + 2|s_+D| + 2|s_D| - 4.$$

Proof is immediate from theorem 3.4.6. Next, we provide interpretations of $|s_+D|$ and $|s_-D|$. We define a diagram to be connected if it is not a split diagram.

Theorem 3.4.7 Suppose D is a connected link diagram with n crossings. Then

$$|s_+D| + |s_-D| \leqslant n + 2.$$

Proof: We use induction on n. When n = 0, the result holds trivially. Suppose the theorem holds when n = *n-1*. Pick a crossing of D. If we replace the crossing with two segments that do not cross, either the positive way or the negative way will result in a diagram D' that is connected since D is connected. Without loss of generality, suppose that the positive way results in a connected diagram. Then $s_+D = s_+D'$ and $|s_-D| = |s_-D'| \pm 1$. Then by induction,

$$|s_+D| + |s_D| = |s_+D'| + |s_-D'| \pm 1 \leqslant (n-1) + 2 \pm 1 \leqslant n + 2.$$

Theorem 3.4.8 Suppose D is a connected diagram with n crossings.

1. If D is an alternating diagram, then $|s_+D| + |s_-D| = n + 2$
2. If D is non-alternating and strongly prime, then

$$|s_+D| + |s_-D| < n + 2$$

Proof: 1) Suppose first that D is alternating. Recall the proof of theorem 3.4.5. If we color the complement of the link diagram in chessboard fashion, $|s_+D|$ is the number of black regions and $|s_-D|$ is the number of white regions. Hence, $|s_+D| + |s_-D|$ is the number of planar regions of the complement of the diagram. Since D is a four valent planar graph(a graph where each vertex has a degree of four) lying on S^2 and S^2 is a closed orientable surface with genus 0, the number of regions (2-cells) must be $n+2$ (since there are $2n$ 1-cells or edges and n 0-cells). Hence $|s_+D| + |s_-D| = n + 2$.

2) Now suppose that D is non-alternating and strongly prime. We use proof by induction on n starting with $n = 2$. For a two crossing non-alternating diagram with two unlinked components, the result clearly follows. Hence, suppose $n \geqslant 3$. Since D is a non-alternating diagram, there exist two consecutive crossings that are either both over-crossings or under-crossings. Denote c to be the third crossing that follows after the two such consecutive crossings. Since D is strongly prime, either we remove c the positive way or the negative way, the resulting diagram will stay connected. Now color the complementary regions of D again in the fashion of a chessboard. Form the graph Γ that consists of vertices, each of which lies in the black region and edges that join those vertices and abut at each crossing. Such a graph is often referred to as the Tait graph (see below). Since D is strongly prime, removing any of the vertices does not separate Γ. The two ways of removing c then corresponds to either removing or shrinking to a point the edge that corresponds to c. Call the new graph Γ'. Suppose deleting the interior of an edge e of Γ produces a separating vertex v (a vertex v such that deleting v and edges incident to v will disconnect the graph). Then shrinking such an edge will not produce a separating vertex since v must lie in a component of the complement of a neighborhood of e in Γ. Hence, either the negative or positive way of removing c must produce a diagram D' that is strongly prime. One can check that D' is non-alternating since it shares the two consecutive same crossings as D. Applying induction on D', we

conclude that $|s_+D'| - |s_-D'| < n+1$ and the rest of the result follows from the Euler number of the sphere as used in proof for 1.

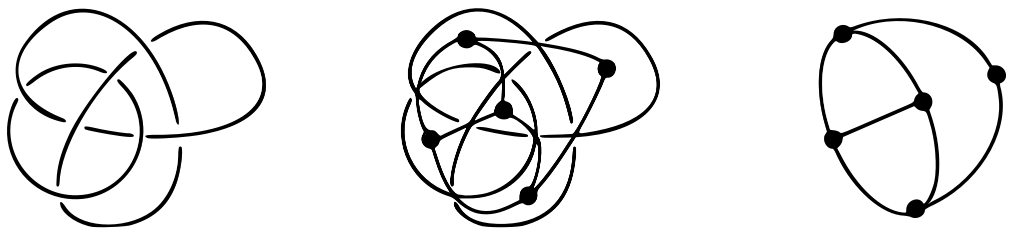


Figure 4: Tait Graph of a Nonalternating Link Diagram; Diagram is cited from [9].

3.5 Proof of First and Second Tait Conjectures

We finally prove the first two Tait conjectures in this section. Before we proceed, we must yet define a few preliminary terms. Let V be a Laurent polynomial with indeterminate t. We define the **breadth** $B(V)$ of V to be the difference $B(V) = M(V) - m(V)$.

Theorem 3.5.1 Suppose D is a connected, diagram of an oriented link L with n crossings. Let $V(L)$ be its Jones polynomial. Then the following results hold.

1. $B(V(L)) \leqslant 0$
2. If D is an alternating, reduced diagram, $B(V(L)) = n$
3. If D is a non-alternating, prime diagram, $B(V(L)) < n$

Proof: Recall that the Jones polynomial is $V(L) = (-A)^{-3w(D)}\langle D\rangle$ if we let $t = A^{-4}$. Then if $M\langle D\rangle$ and $m\langle D\rangle$ refer to the powers of A, $4B(V(L)) = B\langle D\rangle = M\langle D\rangle - m\langle D\rangle$. Then by theorem 3.4.6 and theorem 3.4.7,

$$4B(V(L)) \leqslant 2n + 2|s_+D| + 2|s_-D| - 4 \leqslant 4n.$$

This proves the first part, that $B(V(L)) \leqslant n$. Now suppose D is alternating and reduced. Then by theorem 3.4.5, an reduced alternating link diagram is adequate, and hence again by theorem 5.5, the inequality changes to equality as follows:

$$4B(V(L)) = 2n + 2|s_+D| - 2|s_+D| - 4 \leqslant 4n.$$

Then by (1) of theorem 3.4.8, $4B(V(L)) = 4n$ which proves the second part. Now suppose D is prime and non alternating. Then any diagram summand that is a nontrivial diagram of the unknot does not affect the Jones polynomial but increases the number of crossings. To be more rigorous, if $D = D'\#D_U$ is the connected-sum decomposition of the prime, non-alternating diagram, where D_U is the non-trivial diagram of an unknot, $V(D'\#D_U) = V(D') \cdot V(U) = V(D')$, but the crossing number is additive under connected sum. Hence, one can remove any unknot summand without changing the Jones polynomial and decreasing the crossings and hence we may just assume that D is a strongly prime diagram. Then by part 2 of theorem 3.4.8, part 3 of theorem 3.5.1 follows.

Corollary 3.5.1 If a link L has a connected, reduced, alternating diagram with n crossings, then it does not have any diagram with less than n crossings. Moreover, any non-alternating prime diagram for L must have more than n crossings.

Proof: From theorem 3.5.1, part 2, $B(V(L)) = n$. If L has another diagram with m crossings, then by theorem 3.5.1 part 1, $B(V(L)) = n \leqslant m$. If such diagram is non-alternating, then by theorem 3.5.1 part 3, $B(V(L)) = n < m$.

Definition 3.5.1 Let D be a diagram of a link. Define the ***r*-parallel** D^r to be the diagram that consists of each link component of D replaced by r copies of D, that are all parallel to each other and repeats the crossing behavior of the original link component. The figure below depicts a 2-parallel of a diagram.

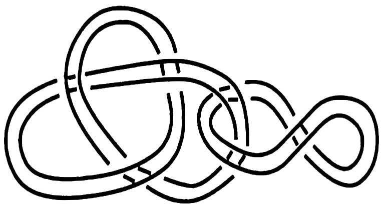


Figure 5: 2-Parallel of a Knot Diagram; Diagram is cited from [1].

Theorem 3.5.2 Let r be an arbitrary positive integer. If D is plus adequate, D^r is plus adequate. If D is minus adequate, D^r is minus adequate.

Proof: Notice that $s_+(D^r) = (s_+D)^r$ (see the figure below). If D is plus adequate, then no component of $s_+(D^r)$ crosses itself at the former crossing since it transverses parallel to the component of s_+D which also does not abut itself at the crossing. The proof for minus adequate diagrams follows in the exact same manner.

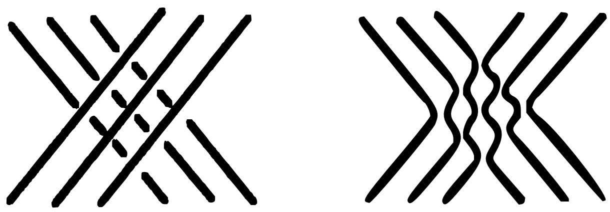


Figure 6: Changing Crossing of a r-parallel; Diagram is cited from [1].

Theorem 3.5.3 Let D and E be diagrams of the same oriented link L, let n_D and n_E represent the number of crossings of each diagram. Suppose D is a plus-adequate diagram. Then

$$n_D - w(D) \leqslant n_E - w(E)$$

where w refers the writhe of each diagram.

Proof: Let $\{L_i\}$ denote the set of components of the link L and let D_i and E_i denote the subdiagrams of D and E that correspond to L_i. Pick any non-negative integers μ_i and v_i such that $w(D_i) + \mu_i = w(E_i) + v_i$. Change each D_i to D_{*i} by adding μ_i positive kinds. Let D_* then denote the diagram attained by changing each D_i to D_{*i}. Likewise change E to E_* by adding v_i positive kinks to each E_i. Adding the kinks do not affect the plus adequacy, and D_* is plus-adequate. Moreover, $w(D_{*i}) = w(E_{*i})$ since $w(D_{*i}) = w(D_i) + \mu_i = w(E_i) + v_i = w(E_{*i})$. This implies then that $w(D_*) = w_(E_*)$ since for any oriented link diagram D with components $L_1, ..., L_m$,

$$w(D) = \sum_{i=1}^{n} \text{(sign-sum of crossings where } L_i \text{ meets itself)} + \sum_{1 \leqslant i < j \leqslant m} \text{(sign sum of crossings between } L_i \text{ and } L_j)$$

but the linking number of two components

$$lk(L_i, L_j) = \frac{1}{2}\text{(sign-sum of crossings between) } L_i \text{ and } L_j$$

is an invariant of the link itself,

$$lk_D(L_i,L_j) = lk_E(L_i,L_j) = lk_{D_*}(L_i,L_j) = lk_{E_*}(L_i,L_j) \text{ for every } i<j.$$

Now consider D_*^r and E_*^r. Notice that both are diagrams of the same link L. Now suppose we pick two distinct copies of the component L_i, say $L_i^{(a)}$ and $L_i^{(b)}$. Then each original self-crossing of L_i is duplicated into two crossings between $L_i^{(a)}$ and $L_i^{(b)}$ both of which carry the same sign as the original. Hence the sign-sum of crossings between $L_i^{(a)}$ and $L_i^{(b)}$ is $2w(D_{*i})$ and $lk(L_i^{(a)},L_i^{(b)}) = w(D_{*i})$. Since the same computation holds for every pair $a \neq b$, all pairs of the r parallels share the same linking number $w(D_{*i})$. Hence D_*^r and E_*^r have the same Jones polynomial. Moreover D_*^r and E_*^r have the same writhe since $w(D_*^r) = w(E_*^r) = r^2 w(D_*) = r^2 w(E_*)$. Hence, $\langle D_*^r\rangle = \langle E_*^r\rangle$. Then by theorem 5.5,

$$M\langle E_*^r\rangle \leqslant (n_E + \sum_i v_i)r^2 + 2(|s_+E| + \sum_i v_i)r - 2$$
$$M\langle D_*^r\rangle = (n_D + \sum_i \mu_i)r^2 + 2(|s_+D| + \sum_i \mu_i)r - 2$$

where $\sum_i v_i$ and $\sum_i \mu_i$ account for the fact that the positive way of changing the crossing of a kink increases the number of components by one and the equality for $M\langle D_*^{\rangle}$ follows from the fact that D_*^r is plus-adequate. Since the relations above hold for all r, by comparing the coefficients of r^2,

$$n_D + \sum_i \mu_i \leqslant n_E + \sum_i v_i.$$

This implies that $n_D - \sum_i w(D_i) \leqslant n_E - \sum_i w(E_i)$. Once again from the fact that the sing-sum of crossings of distinct components is solely determined by linking numbers of L, it follows that $n_D - w(D) \leqslant n_E - w(E)$ which completes the proof of the theorem.

Corollary 3.5.2 Let D and E be diagrams of the same oriented link as in theorem 3.5.3.

1. The number of negative crossings of D is less than or equal to the number of negative crossings of E.
2. A minus-adequate diagram achieves a minimal number of positive crossings.
3. An adequate diagram achieves a minimal number of crossings.
4. Two adequate diagrams of the same link share the same writhe.

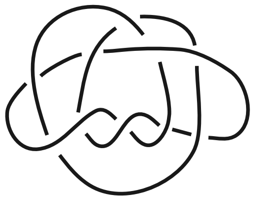
Proof: The four corollaries are essentially reformulations of theorem 3.5.3. Let $n_+(\cdot)$ and $n_-(\cdot)$ denote the number of positive and negative crossings of a diagram. and $n_-(\cdot)$ Since $n(D) - w(D) \leqslant n_E - w(E)$ and $n(D) = n_+(D) + n_-(D), w(D) = n_+(D) - n_-(D), n(D) - w(D) = 2n_-(D)$. This automatically implies that $n_-(D) \leqslant n_-(E)$, proving 1. 2 automatically follows by applying the theorem to an mirror image of the diagram. If a diagram D is adequate, then for any diagram E of the same link,

$$n(D) = n_+(D) + n_-(D) \leqslant n_+(E) + n_-(E) = n(E)$$

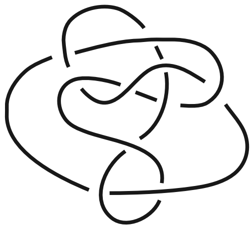
which means that the adequate diagram achieves the minimal number of crossings, proving 3. If both D and E are adequate, then $n_-(D) = n_-(E), n_+(D) = n_+(E)$, and hence

$$w(D) = n_+(D) - n_-(D) = n_+(E) - n_-(E) = w(E).$$

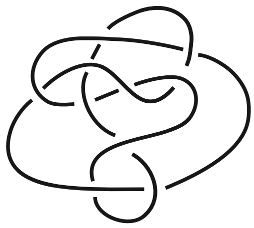
The corollary above is useful for understanding knots. For instance, consider the two knots $10_{161}, 10_{162}$ (also referred to as the Perko pair) and $\overline{10_{162}}$ shown in the figure above (the labeling of the knots was done back when the two knots 10_{161} and $\overline{10_{162}}$ were thought to be different but it was discovered later that they are equivalent). From the diagrams,



(a) 10_{161}, cite from [1]



(b) 10_{162}, cited from [1]



(c) $\overline{10_{162}}$, cited from [1]

one can observe that $w(10_{161}) = -8$ and $w(\overline{10_{162}}) = -10$. Also the diagram for 10_{161} is plus adequate while the diagram for $\overline{10_{162}}$ is minus adequate. Hence by the corollary above, the possible minimal number of positive crossing for $\overline{10_{162}}$ is 0 while any diagram of 10_{161} must have at the very least nine negative crossings. Knot theorists have classified that there are no diagrams of 10_{161} with less than ten crossings. Hence, it is impossible to display both the minimal number of negative crossings and the minimal number of positive crossings in the same diagram and hence the diagrams of 10_{161} and $\overline{10_{162}}$ each achieve the minimal number of positive and negative crossings.

We know from theorem 3.4.5 that any reduced alternating link diagram is adequate. Hence proposition 3 and 4 of corollary 3.5.2 also lead to a proof of Tait's first two conjectures which we initially claimed to show. We conclude the paper by giving the precise form of first and second Tait conjectures that we aimed to prove.

Theorem 3.5.3. (First Tait Conjecture) A reduced alternating diagram of a knot has minimal crossing.

Theorem 3.5.4 (Second Tait Conjecture) Any two reduced alternating diagrams of a knot have the same writhe.

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