Proof of Tarski's Theorem on the Model-Completeness and Completeness of Real Closed Fields

Sam Park

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Abstract

This is a summary of materials in model theory I studied in 2024 summer with Professor John P. Burgess. It will review the basic pillars of model theory, including Gödel's completeness theorem, Löwenheim-Skolem theorem, elementary extension, model consistency and completeness. In the process, the paper will cover how different results in model theory can be applied to abstract algebra. Finally, the paper will cover the model theoretic proofs for Tarski's theorem on the model-completeness and completeness of real closed fields. The entire work is based on Abraham Robinson's *Introduction to Model Theory and to the Metamathematics of Algebra* Chapter 1-4 while I consulted David Marker's *Model Theory* and Chang and Keisler's *Model Theory* for edification. This paper aims to clarify ambiguities in Robinson's work by supplementing missing gaps in his proofs and correcting the mistakes he made.

1 Understanding the Relationship Between Deductive and Descriptive Concepts

A formal language, call it L, consist of atomic symbols that include 1) **object symbols**, denoted by lowercase letters, *a, b, ...,* 2) **variables** denoted by *u, v, x, ...,* 3) **relative symbols**, classified into disjoint classes, R_n, $n \in \mathbb{Z}_{\geqslant 0}$, where n stands for the number of placeholders each relative symbol in the class contains, and a relative symbol in R_n is denoted by an italic uppercase letter with n empty spaces in parentheses, separated by commas (i.e. A(,,...)), 4) **connectives** that consist of $\sim$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), and $\equiv$ (equivalence), 5) **quantifiers** that consist of the universal quantifier $\forall$ and the existential quantifier $\exists$, 6) **separation symbols**, denoted by two brackets $[$ and $]$.

Using the atomic symbols enumerated above, an **atomic formula** is obtained by filling in the empty spaces of a relative symbol with object symbols or variables (i.e. *A(a,x,c)* where *A* is a 3-place relative symbol). Then well-formed formulae (wff) are defined inductively, following the rules below.

1. An atomic formula square bracketed is a wff.

2. Let X and Y be a wff. Then $[\sim X]$, $[X \wedge Y]$, $[X \vee Y]$, $[X \supset Y]$, and $[X \equiv Y]$ are all well-formed formulae.

3. If X is a wff, then $[(\forall x)X]$ and $[(\exists x)X]$ are both wff given that X does not contain $(\forall x)$ nor $(\exists x)$.

One may now assume that every language contains at least one wff. A wff is a **sentence** if every variable it contains is quantified by a quantifier (i.e. Every variable contained in the wff appears inside the scope of a quantifier that quantifies that variable). Every other wff is called a **predicate**, and any variable that is not quantified is called **free**. This completes the description of a language in the Lower predicate calculus. Hereinafter, this paper will use simplified rules for writing wff, adopted from *Robinson, 1963 (pp. 5)* for economy.

For any language L, we now define a set of sentences in L called a theory of L, whose elements are called theorems. Given any sentences *X, Y, Z* in L, the followings, called the rules of deduction, are all elements of the theory of L (*Robinson, pp.6*).

$$[X \supset [Y \supset X]]$$
$$[X \supset [X \supset Y]] \supset [X \supset Y]$$
$$[[X \supset Y] \supset [[Y \supset Z] \supset [X \supset Z]]]$$
$$[[X \wedge Y] \supset X]$$
$$[[X \wedge Y] \supset Y]$$
$$[[X \supset Y] \supset [[X \supset Z] \supset [X \supset [Y \vee Z]]]]$$
$$[X \supset [X \vee Y]]$$
$$[Y \supset [X \vee Y]]$$
$$[[X \supset Z] \supset [[Y \supset Z] \supset [[X \vee Y] \supset Z]]]$$
$$[[X \equiv Y] \supset [X \supset Y]]$$
$$[[X \equiv Y] \supset [Y \supset X]]$$
$$[[X \supset Y] \supset [[Y \supset X] \supset [X \equiv Y]]]$$
$$[[X \supset Y] \supset [[\sim Y] \supset [\sim X]]]$$
$$[X \supset [\sim [\sim X]]]$$
$$[[\sim [\sim X]] \supset X]$$

Moreover, (1) if X and $[X \supset Y]$ are theorems, then Y is a theorem (known as the rule of *modus ponens*), (2) if $[X \supset Y(a)]$ is a theorem and X does not contain a, then $[X \supset [(\forall z)Y(z)]]$ is a theorem, (3) if $[Y(a) \supset X]$ is a theorem and X does not contain a, then $[[(\exists z)Y(z)] \supset X]$ is a theorem (rule (3) is often referred to as the rule of existential generalization). One can replace a formula in a theorem consisting of a relative symbol of order 0 with an arbitrary sentence and still obtain a theorem. Last but not least, if (*i*) and (*ii*) are both theorems, then (*iii*) is a theorem.

$$(i) \, [[[\ldots [X_1 \wedge X_2] \wedge X_3] \wedge \ldots \wedge X_n] \supset Y_m], \quad m = 1,2,\ldots,k$$
$$(ii) \, [[[\ldots [Y_1 \wedge Y_2] Y_3] \wedge \ldots \wedge Y_k] \supset Z]$$
$$(iii) \, [[[\ldots [X_1 \wedge X_2] \wedge X_3] \wedge \ldots \wedge X_n] \supset Z]$$

It is the result of induction on complexity that every sentence X is equivalent to a sentence X' where X' is in prenex normal form, that is, it is of form $Q_1 x_1 Q_2 x_2 ... Q_n x_n M$ where, Q_k's are quantifiers and M is a quantifier-free formula, called the matrix of X' (i.e. $X \equiv X'$ is a theorem). Now, let K be a set of sentences in L. A sentence Y is deducible from K if there is a finite sequence of sentences, say $X_1, X_2, ..., X_n$ in K such that $[[[...[X_1 \wedge X_2] \wedge X_3]... \wedge X_n] \supset Y]$ is a theorem. In symbols, we write $K \vdash Y$. Hereinafter, the set of sentences that are deducible from K will be expressed as *S(K)*. A set of sentences K is contradictory if every sentence in L is contained in *S(K)*. If a set of sentences is not contradictory, it is consistent.

Theorem 1.1 A set of sentences K is contradictory if and only if $[X \wedge [\sim X]] \in S(K)$. *Proof*: ($\Rightarrow$) is trivial. ($\Leftarrow$) follows *ex qualso quodlibet*. Since there is a finite sequence of sentences in K and whose conjunction, call it S, is such that $[S \supset [X \wedge [\sim X]]$ is a theorem, and both $[[X \wedge \sim X] \supset X]$ and $[X \wedge \sim X] \supset \sim X]$ are theorems, $[S \supset X]$ and $[S \supset \sim X]$ are both theorems. By rules of deduction, for any arbitrary sentence Y in L, $[X \supset [X \vee Y]]$ and $[\sim X \supset [\sim X \vee Y]]$ are theorems. Hence $[S \supset [X \vee Y]]$ and $[S \supset [\sim X \vee Y]]$ are theorems. But since $[S \supset \sim X]$ is a theorem, $S \supset Y$ is a theorem. This completes the proof.

If the previous section covered the syntactic perspective, one may also offer a semantic understanding of a language. A *structure M* is a set of individuals denoted by lowercase italics *a,b,...* and of relations of some order n, denoted the same way as relative symbols aforementioned (e.g. $B(,)$ is a relation of order 2), such that for every relation of order n (e.g. $B(,)$) and n-ple of individuals (e.g. *a,b*) defined in M, the instance $B(a,b)$ is either true or false in M. One can then construct a one-to-one correspondence C between a language L and a structure M by mapping individuals of M to the set of object symbols in L and the relations of M to relative symbols in L whose orders are the same. If every object symbols and relative symbols in a wff X are mapped into by C, then X is defined in M under C. Every atomic formula defined in M that does not contain a variable can then be expressed as

a relation, followed by n-ple of individuals in M, whose truth value determines whether the sentence holds or does not hold in M. The rules for determining whether an arbitrary sentence defined in M is true or not true is offered in *Robinson, pp. 9*. If all sentences of a set K is true in a structure M under C, M is called the model of K under C. As a side note, we make some remarks on quantifiers. Let $Y = Y(x)$ in which z is not quantified and there is no other free variable. Then $[(\forall z)Y(z)]$ holds in the structure M if and only if $Y(a)$ holds in M for all object symbols $a \in L$ that corresponds to objects in M. Existential quantification is treated similarly. Given a finite set of sentences K then, the rules can be used to determine whether any $Y \in K$ holds or does not hold in a structure M. However, if the set of sentences is infinite, the argument gets more subtle. The complication is that the cardinal number of the set of symbols of L may smaller than the number of objects in the structure M such that there is no correspondence. In this case we must consider a denumerably infinite language that possesses enough symbols to allow the construction of a correspondence or embed L in a larger language L'. In Boolos, Jeffrey, and Burgess, one can find another way of defining the correspondence between a structure and a language. In particular, let $\mathcal{M} \vDash S$ mean that the sentence S is true in the structure $\mathcal{M}$. Then $\mathcal{M} \vDash \forall x F(x)$ if and only if for all m in the domain of the structure, $M \vDash F[m]$ by which, the authors mean that one can extend the language L to $L \cup \{c\}$ and if m is in the domain of $\mathcal{M}$ such that we extend the interpretation of $\mathcal{M}$ to assign c the element m (call such structure $\mathcal{M}_m^c$), then

$$\mathcal{M} \vDash F[m] \text{ iff } \mathcal{M}_m^c \vDash F(c).$$

Define K again to be a set of sentences from the language L that is well defined in structure M with correspondence C. Then we claim without proof that every theorem of the language L contained K must hold in the structure M. Obviously, a contradictory set K cannot possess a model since there is no model that satisfies $Z = [Y \wedge [\sim Y]]$.

Theorem 1.2a (Gödel's Completeness Theorem) If a sentence X of L is true in every model in which it is defined, then X is a theorem.

Theorem 1.2b Let K be a set of sentences. If Y is a sentence such that it is both defined and true in any structure that makes up a model of K, then Y is deducible from K. In particular, there exists a finite subset of K from which one can deduce Y.

Theorem 1.2c *If a set of sentences K is consistent, it must possess a model.*

Before we proceed with the proof, notice that $1.2a$ and $1.2b$ both reduces to proving $1.2c$. For instance, suppose X in $1.2a$ is not a theorem. Then the set consisting of one sentence $[\sim X]$ cannot be contradictory since if it were contradictory, then $[[\sim X] \supset Y]$ wold be a theorem for any Y, and $[[\sim X] \supset X]$, $[[\sim [\sim X]] \vee X]$ and $[X \vee X]$ and hence X would be theorems in L which is a contradiction. Hence, $[\sim X]$ is consistent and from 1.2c, we may conclude that it has a model, say M. But then X is defined but does not hold in this model M which is a contradiction. Hence, by contradiction, it proves $1.2a$.

Proof: To prove the theorem, we will first assume that the set K only consists of sentences that are either atomic formulae of order 0 or built up from such atomic formulae by means of connectives. Define S to be the set of all relative symbols that appear in K. Now, whether an element A in S is true under some structure M can be expressed through a valuation function, $\varphi(A)$ which returns a 0 if A is true in M and 1 otherwise. By identifying 0 with "true", 1 with "false", one may then evaluate the truth value of any sentence in K using the standard truth table in propositional logic. If K has a single sentence, one can assign an appropriate value $\varphi(A) = 0$ or $= 1$ to its relative symbols so that the sentence as a whole holds in M (such an assignment always exists since the sentence is assumed to be consistent and if a sentence X is a contradiction in propositional logic, then by the rules of deduction, any sentence in L is deducible from X, making it inconsistent in lower predicate calculus as we defined above). If K consist of finite sentences, one can take a conjunction of all such sentences and assign an appropriate value to its relative symbols so that the conjunction holds in M.

Now, consider the case where the cardinal of K is arbitrarily transfinite. Consider S, the set of relative symbols in K, as an abstract set of characters (which may be countable or non-countable) and define a *partial valuation* $\varphi : V \to \{0, 1\}$ where V is a subset of S. The valuation φ is total if

the domain V equals the entire set S, and the domain of φ is denoted by D_φ. If U is any subset of S, then $\varphi|U$ will denote the restriction of the valuation φ to $D_\varphi \cap U$. First, we proceed to prove the following lemma:

Special Valuation Lemma: Let $\phi = \{\varphi_v\}$ be a set of partial valuations of S, $I = \{v\}$ is the index set, such that every finite subset U of S is included in the domain of some partial valuation $\varphi \in \phi$. Then there exists a total valuation ψ such that for every finite subset $U \subset S$, there exists a partial valuation $\varphi_v \in \phi$ that includes U in its domain and $\psi|U = \varphi_v|U$.

Proof: We will define a partial valuation ψ of S to be admissible if for every finite subset $U \subset S$, there exists $\varphi_v \in \phi$ such that $\psi|U = \varphi_v|U \cap D_\psi$. Given partial valuations φ and ψ, ψ is an extension of φ if $D_\varphi \subset D_\psi$ and $\psi|D_\varphi = \varphi$ (and this will be denoted in symbols $\psi \succ \varphi$). Define Ψ to be the set of admissible partial valuations of S. Note that Ψ is not empty for it includes the empty partial valuation whose domain is $\varnothing$ and moreover, the elements of Ψ are partially ordered by the relation of extension, $\succ$. Consider any nonempty totally ordered subset $\Psi' \subset \Psi$, then define a new valuation ψ' whose domain is the union of the domains of $\psi_\mu \in \Psi'$ and whose value for any argument is the joint value of all ψ_μ defined for that argument and one can find that ψ' is the upper bound of Ψ'. Since our choice of Ψ' was arbitrary, by Zorn's lemma, Ψ must contain at least one maximal element, call it ψ_0. We now claim that ψ_0 is the total valuation of S we are looking for. Now we proceed by proof by contradiction. If $S - D_{\psi_0}$ is empty, then ψ_0 is a total valuation and for every finite subset U of S, there exists a $\varphi_v \in \Phi$ that coincides with ψ_0 on U since ψ_0 is admissible, which then completes the proof of the lemma. Hence, suppose that $S - D_{\psi_0}$ is not empty, and let $A \in S - D_{\psi_0}$. Define a partial valuation $\psi_1 : D_{\psi_0} \cup \{A\} \to \{0,1\}$ such that $\psi_1 = \psi_0$ on D_{ψ_0} and $\psi_1(A) = 0$. Since ψ_0 is a maximal element of Ψ, ψ_1 cannnot be admissible, that is, there exists a finite subset V of S such that the conditions $V \subset D_{\varphi_v}$ and $\psi_1|V = \varphi_v|D_{\psi_1} \cup V$ are not met by any partial valuation $\varphi_v \in \phi$. Likewise, define a partial valuation $\psi_2 : D_{\psi_0} \cup \{A\} \to \{0,1\}$ such that $\psi_2 = \psi_0$ on D_{ψ_0} and $\psi_2(A) = 1$. For the same reasons as above, ψ_2 cannot be admissible, and hence, there exists a finite subset W of S such that the conditions $W \subset D_{\varphi_v}$ and $\psi_2|W = \varphi_v|D_{\psi_2} \cup W$ are not satifised by any partial valuation $\varphi_v \in \phi$. V and W must then contain A which both follow from the fact that ψ_0 is adimssible. Now define $U := V \cup W$. Since ψ_0 is an admissible valuation and U is a union of finite sets and hence finite itself, there exists $\varphi_\mu \in \Phi$ such that $U \subset D_{\varphi_\mu}$ and $\psi_0|U = \varphi_\mu|U \cap D_{\varphi_\mu}$. Since $A \in U$, either $\varphi_\mu(A) = 0$ or $= 1$. But if $\varphi_\mu(A) = 0$, then $\psi_1|V = \varphi_\mu|V \cap D_{\psi_1}$ and $V \subset D_{\varphi_\mu}$, and if $\varphi_\mu(A) = 1$, then $\psi_2|W = \varphi_\mu|W \cap D_{\psi_2}$ and $W \subset D_{\varphi_\mu}$. These both contradict the fact that ψ_1 and ψ_2 are not admissible valuations, proving that $S - D_{\psi_0}$ must be empty.

Since we have finished the proof for the special valuation lemma, we return to the proof of Gödel's completeness theorem. We will define K to be the transfinite set of sentences without object symbols or quantifiers that is, K can contain relative symbols of only order 0, and we will define S to be the set of such relative symbols that appear in K. Now define S' to be a finite subset of S and K' to be the set of sentences whose relative symbols belong only to S'. Although the set K' itself may be transfinite in cardinality, since the number of relative symbols that appear in the set is finite, and each sentence is finite in length, all the sentences in K' must be logically equivalent in the sense of propositional calculus, to some sentence of a finite subset of K'. We can now conclude that there must be a total valuation φ_v of S' such that φ_v attributes the value 0 to all elements of S' using the truth value evaluation. One can assign the value of φ_v arbitrarily for elements of S' that are not contained in any of the sentences in S'. Now if we define $\Phi = \{\varphi_v\}$ to be the set of all total valuations of S' (or equivalently, all partial valuations of S) obtained as above, by the *special valuation lemma*, there is a total valuation of S, denote it as ψ_0. Now pick any arbitrary sentence X in K and define V to be the set of all relative symbols that appear in X. Then, there exists a partial valuation $\varphi_v \in \Phi$ such that $V \subset D_{\varphi_v}$ and $\psi_0 = \varphi_v$ on V. If we define S' to be the domain of φ_v, then X must belong to the set K' that corresponds to S'. Since $\varphi_v(X) = 0$, $\psi_0(X) = 0$. This proves that Gödel's completeness theorem holds when the set K consists only of sentences built up from relative symbols of order 0.

Now suppose the set K consists of relative symbols of arbitrary order, yet without quantifiers or variables. Define S to be the set of all atomic formulae in K. Construct a one-to-one correspondence $C{:}S \hookrightarrow S'$ where S' is the set of relative symbols of order 0 in a given language L. (There are two disclaimers to note: first, if L does not contain enough relative symbols of order 0 to match S', one

can always replace the language with a more extensive one. Robinson means that when one replaces each atomic formula in K with a new 0-ary symbol to reduce K to propositional calculus, there might not be enough such unary symbols available in K and this can be solved by extending the language L. Second, the mapping must be injective, that is any atomic formula that contain different relative symbols or the same relative symbol with different object symbols must be mapped to a different element). Define K' to be the set of all sentences of form X', $X' = C(X), X \in K$. I will demonstrate that K' must be consistent using proof by contradiction. Suppose K' is contradictory. Then there must exist a finite sequence of sentences $X_1', ..., X_n' \in K'$ such that $[[X_1' \wedge X_2' \wedge ... \wedge X_n'] \supset A' \cap \sim A']$ is a theorem of L (A' is an element of S and $C(A) = A'$). Since one can replace a relative symbol of order 0 with an arbitrary sentence and still obtain a theorem, $[[X_1 \wedge X_2 \wedge ... \wedge X_n] \supset A \wedge \sim A]$ is also a theorem of L, where $X_k = C^{-1}(X_k')$. However, this is a contradiction since we have assumed K to be consistent. Since K' only contains relative symbols of order 0, from the previous result, K' possesses a model, call it M'. Now define a new structure M such that $R(a_1, ..., a_n)$ holds in M if and only if $C(R(a_1, ..., a_n))$ holds in M'. It follows that M must be a model of K.

Last but not least, assume the set K of any arbitrary sentences with restrictions. Suppose that K contains at least one sentence with a quantifier, since this is the only remaining case to get proved. For every sentence X in K, let X' be its prenex normal form. Then $X \equiv X'$ is a theorem of L for all $X \in K$ and therefore, every model of K' must be a model of K. By rules of deduction and modus ponens, $X \subset X'$ is a theorem for all $X \in K$, and it follows that if K is consistent, K' must also be consistent (This can easily be demonstrated by proving its contrapositive. Suppose K' is contradictory. Then there exists a finite sequence of sentences in K', $X_1', ..., X_n'$ such that $X_1' \wedge, ..., \wedge X_n' \supset A \wedge \sim A$ is a theorem. But since $X_k' \supset X_k, (1 \leqslant k \leqslant n)$ is a theorem, by rules of deduction, $X_1 \wedge, ..., \wedge X_n \supset A \wedge \sim A$ must be a theorem). Hence we may safely assume that K only consists of prenex normal forms and proceed with our proof. We will now define a sequence of sets of sentences, $K_0, K_1, ..., K_n$ and a sequence of sets of object symbols, $P_0, P_1, ..., P_n$ as follows:

1. $K_0 = K, P_0 = \{set\ of\ all\ object\ symbols\ in\ K\}$.

2. Let K_1 contain all sentences of K_0 and if any sentence $X \in K$ begins with an existential quantifier, obtain X^* by removing the existential quantifier and replacing the corresponding variable by an object symbol that does not appear elsewhere and include X^* in K_1. P_1 contains both P_0 and newly introduced object symbols.

3. Let K_2 contain all sentences of K_1 and if any sentence $X \in K_1$ begins with a universal quantifier, construct a set of sentences $X_\mu, I = \{\mu\}$ whose element is obtained by removing the universal quantifier and replacing the corresponding variable with every element of P_1. Then $K_2 = K_1 \cup X_\mu$ and define $P_2 = P_1$.

4. In general, if m is odd, K_m and P_m are obtained using step 2 and if m is even, K_m and P_m are obtained using step 3. Now define $K' = \cup_n \{K_n\}$ and $P' = \cup_n \{P_n\}$. We claim that P' will consist of the individual objects of the model of K.

We proceed to show that K' must be consistent. Since K' is contradictory if and only if some finite subset of K' is contradictory which again may happen if and only if K_n for some n is contradictory, it will suffice to prove that all K_n must be consistent. Suppose there exists a smallest n, $n = m$ such that K_m is contradictory. Then m is either odd or even.

1. Suppose m is even. Then there exist sentences $Y_1, ..., Y_k \in K_m$ and $Z_1, ..., Z_l \in K_m - K_{m-1}$ such that the set $\{Y_1, ..., Y_k, Z_1, ..., Z_l\}$ is contradictory. By our definition of K_m, there exist $V_1, ... V_l \in K_{m-1}$ such that $V_i = [(\forall x S_i(x))], 1 \leqslant i \leqslant l$ and $Z_i = S_i(a_i)$. If $\{Y_1, ..., Y_k, Z_1, ..., Z_l\}$ is contradictory, then a sentence $W = [[A] \wedge \sim [A]]$ is deducible from $\{Y_1, ..., Y_k, Z_1, ..., Z_l\}$. But this is equivalent to $\{Y_1, ..., Y_k, Z_1, ..., Z_l\} \supset W$ being deducible from $\{Z_1, ..., Z_l\}$. Since $[V_i \supset Z_i]$ are theorems, it follows that $\{Y_1, ..., Y_k, V_1, ..., V_l\} \vdash W$. This is a contradiction since $\{Y_1, ..., Y_k, V_1, ..., V_l\}$ is a subset of K_{m-1} which was initially assumed to be consistent.

2. Suppose that m is odd. Then there exist sentences $Y_1, ..., Y_k \in K_m$ and $Z_1, ..., Z_l \in K_m - K_{m-1}$ such that the set $\{Y_1, ..., Y_k, Z_1, ..., Z_l\}$ is contradictory. There then exist $V_1, ... V_l \in K_{m-1}$ such that $V_i = [(\exists x S_i(x))], 1 \leqslant i \leqslant l, Z_i = S_i(a_i), a_i \in P_m - P_{m-1}$. Define W again as in step 1, and W must be deducible from $\{Y_1, ..., Y_k, Z_1, ..., Z_l\}$. This is equivalent to $[Y_1 \wedge ... \wedge Y_k \wedge Z_2 \wedge ... Z_l \supset$

$W] = U$ being deducible from Z_1. Then $Z_1 \supset U$ and $V_1 \supset U$ are both theorems (the latter by the rule of existential generalization). It then follows that $\{Y_1, ..., Y_k, V_1, Z_2, ..., Z_l\}$ is contradictory. Repeatedly applying the same procedure to $Z_2, ..., Z_l$, we get that $\{Y_1, ..., Y_k, V_1, ..., V_l\}$ is contradictory. But this is again a contradiction since the set is a subset of K_{m-1} which was initially assumed to be consistent. Therefore, we can conclude that K' must be consistent.

Define K^* to be the set of sentences of K' that do not have any quantifiers. Since $K^* \subset K'$, both K^* and K' have the same relative symbols, and the set of object symbols in K^* is identical to P', by our previous result, K^* has a model M whose set of individual objects is P' and whose set of relations are identical to the set of relative symbols that appear in P'. We claim that M is the model of K. First, we prove that M is a model of K' using strong induction on the number of quantifiers. When a sentence has not quantifier, such a sentence is already an element of K^* and therefore holds under M. Now suppose the assertion holds for every sentence in K' with $k < n$ quantifiers and consider sentence $X \in K'$ that has exactly n quantifiers. X can either begin with an existential quantifier or a universal quantifier.

1. Suppose that X begins with an existential quantifier (i.e. $X = [(\exists y)S(y)]$). Then X must belong to all K_m from some m onward, where m must be an even integer. But then, K_{m+1} contains a sentence $S(a)$ which is true under M by induction. Hence, $[(\exists y)S(y)]$ must hold in M.

2. Suppose that X begins with a universal quantifier (i.e. $X = [(\forall y)S(y)]$). We must show that $S(y)$ holds in M for all $a \in P'$. For every element $a \in P'$, there exists an odd integer m such that $X \in K_m$ and $a \in P_m$. Then $S(a) \in K_m + 1$ and hence $S(a)$ must hold under M by induction. This completes the proof of Gödel's completeness theorem.

Corollary 1.1: Löwenheim Skolem Theorem. *If a consistent set of sentences K is finite in cardinality, its model is either finite or countable. If K is transfinite, then K has a model whose cardinal does not exceed that of K.*

Proof: If K is finite, then all K_n and P_n are finite and hence P' is at most countable. On the other hand, if K is infinite, say of cardinal k, then all K_n and P_n can have at most k elements and hence P' can have at most $\aleph_0 k = k$ elements. We have proved the Löwenheim Skolem theorem.

A set of sentences K is a *T-system* if it contains all the sentences that are deducible from the set (i.e. $S(K) = K$). A set of structures V is a variety of K (or V is just a variety) if V consists of all structures that are models of K and we denote this in symbols, $K \rightarrow V$. We prove the following theorem on varieties of structures.

Theorem 1.3 Compactness Theorem for the Varieties of Structures: *Let $\{V_v\}$ be a set of varieties of structures whose finite intersection of any of its elements is non-empty. Then the intersection $\bigcap_v \{V_v\}$ is non-empty.*

Proof: Since $V_v \in \{V_v\}$ are varieties, there exist sets of sentences K_v such that $K_v \rightarrow V_v$. Define $K := \bigcup_v \{K_v\}$. Every model of K must be a model of every K_v and hence must belong to every V_v. Hence it is sufficient to show that K possesses a model, which by Gödel's completeness theorem, holds as long as K is consistent (ie. every finite subset of K is consistent). Consider an arbitrary finite subset, $\{X_1, ..., X_n\}$ of K. There then exist sets K_v such that $X_i \in K_i, 1 \leqslant i \leqslant n$. Define M to be an element of $V_1 \cap ... \cap V_n$ where $K_i \rightarrow V_i, 1 \leqslant i \leqslant n$. Such a M must exist by the assumption of the theorem and hence the sentences X_i must hold in M, demonstrating that $\{X_1, ..., X_n\}$ must be consistent. This proves that K must be consistent.

Corollary 1.2. Suppose K is a set of sentences and assume any finite subset of K has a model. Then K must have a model.

Proof: If every finite subset of K possesses a model, then every finite subset of K is consistent. (This follows from taking the contrapositive of the statement that any contradictory set of sentences cannot have a model). Then it follows from definition that K is consistent and hence by theorem 1.2.c, K must possess a model.

2 Specifying Axiomatic Systems and Algebraic Theories

Two structures are *similar* if they contain the same relations. If M and M' are similar structures, M is called a partial structure or a substructure of M' and M' is called a extension of M if the set of individuals of M is a subset of the set of individuals of M' and for any n-place relation A and n individuals $a_1, ..., a_n$ that belong to M, $A(a_1, ..., a_n)$ holds in M' if and only if it holds in M. We denote this writing $M \subset M'$. Now define M to be any structure. Consider the set of all atomic formulae $A(a_1, ..., a_n)$ that is true in M, where A is a relation and a_i is an individual in M. Such a set is called the *positive diagram* of M and is denoted as $D^+(M)$. The set of all formulae $\sim A(a_1, ..., a_n)$ that hold in M is called the *negative diagram* of M and is denoted as $D^-(M)$. The *diagram* of M is then $D(M) = D^+(M) \cup D^-(M)$ (here, we are implicitly expanding the language L so it includes constant symbols for each individual in the domain of the structure). Recall that an equivalence relation is a binary relation that is reflexive, transitive, symmetric, and equality is an equivalence relation that is substitutive (i.e. substitutive means if E is an equivalence relation such that

$$(\forall x_1) \ldots (\forall x_n)(\forall y_1) \ldots (\forall y_1)[E(x_1, y_1) \wedge \cdots \wedge E(x_n y_n) \wedge A(x_1, \ldots, x_n) \supset A(y_1, \ldots, y_n)]$$

Suppose that $M \subset M'$. Let E be an equality relation. If there does not exist an individual $a \in M' - M$ such that there exists $b \in M$ and $E(a, b)$, then the relation of inclusion between the structures M and M' is called normal.

Theorem 2.1. Let M and M' be two similar structures. M' is an extension of M (i.e. $M \subset M'$) if and only if M' is a model of $D(M)$, the diagram of M.

Proof: Since M and M' are similar structures, they contain the same relations. Suppose $M \subset M'$, then the elements of M are all contained in elements of M'. Then every sentence in $D^+(M)$ holds in the structure M' by the definition of extension, and likewise $D^-(M)$ holds in the structure of M'. To show the other direction, let L be the common language shared between M and M'. Expand the language L by defining
$$L_M = L \cup \{c_a \,:\, a \in |M|\}.$$

At the same time, we expand the structure M to assign an interpretation to each added constant symbol so every individual in the domain corresponds to a constant symbol. Now consider the following function
$$f : |M| \to |M'| \quad f(a) := (c_a)^{M'}.$$

Since M' is a model of $D(M)$, the function is injective since $a \neq b$ will imply $c_a \neq c_b$ in $D(M)$ and hence $f(a) \neq f(b)$ in $|M'|$. Moreover, f preserves both relations and functions. For instance, suppose $M \vDash R(\bar{a})$ for $\bar{a} \in |M|^n$, then $R(c_{\bar{a}}) \in D(M)$ an hence $M' \vDash R(f(\bar{a}))$. Otherwise, $\sim R(c_{\bar{a}}) \in D(M)$ and hence $M' \vDash \neg R(f(\bar{a}))$. Preservation of functions and constants are shown similarly. Hence $f : M \hookrightarrow M'$ is an injective homomorphism such that

$$f[M] \subseteq |M'|$$

and $f[M]$ is isomorphic to M. By identifying M with $f[M]$ inside M', one proves that M' is an extension of M.

A one-to-one correspondence C that maps the individuals and relations of a structure M respectively to the individuals and relations of a structure M' is called an *isomorphism* if for every n-place relation R in M, R holds between certain n individuals $a_1, ..., a_n$ of M if and only if $R'(C(a_1), ..., C(a_m))$ holds in M', $R' = C(R)$. The concept of isomorphism as defined in first order logic represents the notion of isomorphism as defined in abstract algebra. The notion of homomorphism and automorphism are defined in a similar manner.

3 Axioms of Algebraic Structures

We start off by axiomatizing formally the notion of equality (Remember that equality is a distinct concept from the relation of equivalence. Any relation that satisfies the condition of reflexivity,

symmetry, and transitivity is called the relation of equivalence, which the relation of equality must include the condition of substitutivity as well). Regarding the relation $E(x,y)$ to represent equality, the relation E must satisfy the following conditions (*Robinson, pp.23*).

$$(\forall x)\, E(x,x)$$
$$(\forall x)(\forall y)\,[E(x,y) \supset E(y,x)]$$
$$(\forall x)(\forall y)(\forall z)\,[E(x,y) \wedge E(y,z) \supset E(x,z)]$$
$$(\forall x_1)\ldots(\forall x_n)(\forall y_1)\ldots(\forall y_n)[E(x_1,y_1) \wedge E(x_2,y_2) \wedge \ldots \wedge E(x_n,y_n) \wedge$$
$$A(x_1,\ldots,x_n) \supset A(y_1,\ldots,y_n)]$$

where the first three axioms respectively represent the reflexivity, symmetry, and transitivity of the relation E, and the last axiom represents the condition of substitutivity. We will refer to the four axioms above as the *axiom of equality*.

Next, we axiomatize the notion of a group, using the relation $E(x,y)$ as conditioned above (reading x equals y), and $S(x,y,z)$ (reading z is the product of x and y) *(Robinson, pp. 26)*.

$$(\forall x)(\forall y)(\exists z)\, S(x,y,z)$$
$$(\forall x)(\forall y)(\forall z)(\forall w)\,[S(x,y,z) \wedge S(x,y,w) \supset E(z,w)]$$
$$(\forall x)(\forall y)(\forall z)(\forall t)(\forall u)(\forall v)\,[S(x,y,z) \wedge S(z,t,u) \wedge S(y,t,v) \supset S(x,v,u)]$$
$$(\exists x)(\forall y)(\exists z)\,[S(x,y,y) \wedge S(z,y,x)]$$

Each axiom above respectively ensures the closure property, uniqueness of the product, the associativity of the product, and the existence of a left unit and a left inverse. The union of the axioms of equality with the four sentences above make K_G, the axioms of a group. But one can also obtain an alternative to K_G by introducing a constant e and replacing the last sentence above with $(\forall y)(\exists z)[S(e,y,y) \wedge S(z,y,e)]$. The resulting set of axioms will be denoted by K_G'. Extending the logic, one can then define two different sets of axioms for an abelian group, an ordered abelian group (introduce a two-place relation $Q(x,y)$ that reads x is smaller than y), a general ring (using two three-place relations S(x,y,z) and P(x,y,z), the former representing the sum operation and the latter the multiplication operation), a commutative ring. The details are provided in *Robinson, pp.26-27*. We provide an example of how to axiomatize division ring (skew field):

$$(\exists x)(\exists y)[\sim E(x,y)]$$
$$(\forall x)(\forall y)(\exists z)[S(x,x,x) \vee P(x,z,y)]$$
$$(\forall x)(\forall y)(\exists z)[S(x,x,x) \vee P(z,x,y)]$$

The first axiom requires that the structure contain at least two individuals (which will correspond to 0 and 1), the second and third axioms require that $xz = y$ and $zx = y$ can be solved for y, given $x \neq 0$. If we require that multiplication is commutative, a skew field becomes a (commutative) field.

Next, we formalize the concept of the characteristic of a field. Define predicates $S_n(x,y)$ recursively as follows:

$$S_0(x,y) = S(y,y,y)$$
$$S_n(x,y) = (\exists z)[S_{n-1}(x,z) \wedge S(z,x,y)], n = 1,2,\ldots$$

The predicate $S_n(x,y)$ states that $y = nx$. Using the relation, S, one can then define the sentence X_p that states that $nx = 0$ for all x, that is, the structure is of characteristic n.

$$X_n = (\forall x)(\forall y)[S_n(x,y) \supset S(y,y,y)], n = 1,2,3,\ldots$$

A set of axioms for the concept of a commutative field of characteristic p, for instance, can then be expressed by taking the union of K_{CF}, the set of axioms of a commutative field, with X_p. In order to express the notion of a commutative field with 0 characteristic, one can add to K_{CF}, the infinite set of sentences $\{\sim X_2, \sim X_3, \sim X_5, \ldots, \sim X_p, \ldots\}$ where p varies over all prime integers. We will denote the axioms for a commutative field of characteristic 0 as K_F^0).

We next formalize the concept of an Archimedean ordered field. Recalling that the predicate $Q(x,y)$ reads x is smaller than or equal to y, we introduce a new relation $Q_n(x,y) = (\exists z)[S_n(x,z) \wedge Q(y,z)]$ that reads $nx \geqslant y$. Then the axioms for an Archimedean ordered field can be expressed by incorporating Archimedes' axiom as written below into the set of axioms for an ordered field:

$$(\forall x)(\forall y)[E(0,x) \vee \sim Q(0,x) \vee Q_1(x,y) \vee Q_2(x,y) \vee ... \vee Q_n(x,y) \vee ...]$$

Next, we define how to axiomatize the concept of an algebraically closed field. We begin with defining the predicates $P_n(x,y), n = 0,1,2...$ that reads, $y = x^n$.

$$P_0(x,y) = E(y,1)$$
$$P_n(x,y) = (\exists z)[P_{n-1}(x,z) \wedge P(x,y,z)], n = 1,2,...$$

The equation, $x_0 + x_1 y + ... + x_n y^n = z$ can then be expressed as

$$T_n(x_0,...,x_n,y,z) = (\exists u_0)(\exists u_1)...(\exists u_n)(\exists v_0)(\exists v_1)...(\exists v_n)(\exists w_1)...(\exists w_{n-1})$$
$$[P_0(y,u_0) \wedge P_1(y,u_1) \wedge ...P_n(y,u_n) \wedge P(x_0,u_0,v_0) \wedge P(x_1,u_1,v_1) \wedge ... \wedge P_n(x_n,u_n,v_n) \wedge$$
$$S(v_0,v_1,w_1) \wedge S(w_1,v_2,w_2) \wedge ... \wedge S(w_{n-2},v_{n-1},w_{n-1}) \wedge S(w_{n-1},v_n,z)].$$

The axioms for an algebraically closed field is then obtained by taking the union of the set of axioms for a commutative field, K_{CF} with the sequence of axioms

$$(\forall x_0)(\forall x_1)...(\forall x_{n-1})(\exists y)T_n(x_0,x_1,...,x_{n-1},1,y,0), n = 2,3,....$$

A *formally real field* is defined as a commutative field such that the sum of the squares is always non-zero unless all the bases of the squares, and hence the squares themselves, are equal to 0. A field is *real closed* if the field is both formally real and it possesses no algebraic extension that is formally real. It is known that an ordered field F is real-closed if every monic polynomial with odd degree in $F[x]$ possesses a root in F and every positive element of F has a square root in F. In a real closed ordered field, every non-negative element must possess a square root and vice versa. Hence, if $\bar{K}_{OF}$ denotes the set of axioms for the concept of a real-closed ordered field, the predicate $Q(x,y)$ in $\bar{K}_{OF}$ may be replaced by the predicate $(\exists z)(\exists w)[(S(x,z,y) \wedge P(w,w,z)]$. The axioms for the real-ordered field is important since it will be the basis for proving Tarski's theorem later in the paper.

4 Embedding Theorems and Normal Chains

A *vocabulary* of a set of sentences K, is the set of all individuals and relations that occur inside K, and a wff X is well defined in K, if the vocabulary of X is a subset of the vocabulary of K. Two sets of sentences K and K' are *related* if there is a one-to-one correspondence between the individuals a of K and the individuals a' of K', between the relations A of K and certain predicates R' defined in K', and between the relations A' of K' and certain predicates R defined in K, and the two following conditions are met.

1. For any sentence $X \in K$, if its individuals are replaced by the corresponding individuals in K' and the relations are replaced by the corresponding predicates in K', then the result, call it K' is deducible from K'. The same holds in the opposite direction, that is for any sentence $K' \in K'$ whose individuals and relations are replaced by the individuals and predicates in K.

2. Let $A(x_1,....,x_n)$ be an arbitrary relation defined in K. Let $R'(x_1,...,x_n)$ be its corresponding predicate defined in K', and $R(x_1,...,x_n)$ the predicate obtained by replacing the individuals and relations that appear in R' with the corresponding individuals and relations of K'. Then $K \vdash (\forall x_1)...(\forall x_n)[A(x_1,...,x_n) \equiv R(x_1,...,x_n)]$.

We provide some example. Consider the set of axioms for the ordered abelian group K'_{OAG} that uses relations E,S,Q and e where e denotes the identity element of a group. Suppose we want to obtain

another set of axioms K^* that is related to K'_{OAG}. Recall the axioms that involved the two place relation $Q(x,y)$ as follows:

$$(\forall x)(\forall y)(\forall z)(\forall w)[E(x,y) \wedge E(z,w) \wedge Q(x,z) \supset Q(y,w)]$$
$$(\forall x)(\forall y)(\forall z)[Q(x,y) \wedge Q(y,z) \supset Q(x,z)]$$
$$(\forall x)(\forall y)[Q(x,y) \wedge Q(y,x)]$$
$$(\forall x)(\forall y)[[Q(x,y) \wedge Q(y,x)] \equiv E(x,y)]$$
$$(\forall x)(\forall y)(\forall z)(\forall v)(\forall w)[S(x,y,z) \wedge S(x,v,w) \wedge Q(y,v) \supset Q(z,w)]$$

To get K^*, we delete those axioms above and instead introduce a new unary relation $P(x)$ that satisfies the following sentences:

$$(\forall x)(\forall y)[E(x,y) \wedge P(x) \supset P(y)]$$
$$(\forall x)(\forall y)[S(x,y,0) \supset P(x) \vee P(y)]$$
$$(\forall x)(\forall y)[S(x,y,0) \wedge P(x) \wedge P(y) \supset E(x,0)]$$
$$(\forall x)(\forall y)(\forall z)[S(x,y,z) \wedge P(x) \wedge P(y) \supset P(z)]$$

where the relation P means that $x \geqslant 0$. Adding those axioms to those of the abelian group K'_{AG} gives the set K^*. To show that K^* is related to K'_{OAG}, we must show the one-to-one correspondence between the predicates of one axiom and the relations of the other axiom. Such correspondence will map E,S,e to themselves and $Q(x,y)$ is mapped to the predicate

$$(\exists z)[S(x,z,y) \wedge P(z)]$$

while $P(x)$ is mapped to $Q(e,x)$. One can then check that the two conditions are satisfied. For instance consider $Q(x,y) \in K'_{OAG}$. $Q(x,y)$ is mapped to the predicate $(\exists z)[S(x,y,z) \wedge P(z)]$ and this predicate is in turn mapped to $[(\exists z)[S(x,z,y) \wedge Q(e,z)]]$. One can check that the two sentences turn out to be equivalent in K'_{OAG}. As a side note, if K and K' are related, and you obtain a sentence X and X'' be applying the conversion procedure twice on X, we know that $X \equiv X''$ is deducible from K. In addition, by definition, X is deducible from K if and only if X' is deducible from K'. Hence two related sets are syntactically equivalent in the sense that they are either both consistent or contradictory.

Let M be a structure and K a set of sentences such that the vocabulary of K and the individuals and relations of M need not coincide (although they may overlap). We will say a structure M can be *embedded* in a model M' of K, if M' is a model of K and M is a substructure of M' (i.e. M' is an extension of M) if we ignore the relations of M' that do not appear in M.

Theorem 4.1 *Let K be a set of sentences. Then a structure M can be embedded in a model of K if and only if every finite substructure of M can be embedded in a model of K.*

Proof: Recall that a structure is an extension of a structure M if and only if it is a model of the diagram $D(M)$. In other words, as long as $K \cup D(M)$ is consistent, it would have a model that both extends M and makes K true. For sake of proof by contradiction, suppose $K \cup D(M)$ is inconsistent. Then there is a finite subset of $K \cup D(M)$, denote it, $K \cup D^*$ that is contradictory (D^* is a finite subset of $D(M)$). Since D^* contains only a finite number of individuals and relations of M, and hence possess a model M^* that is also a finite substructure of M. But $D^* \subset D(M^*)$ and therefore, $K \cup D(M^*)$ is also contradictory. This contradicts our assumption that every finite substructure of M possesses an extension that satisfies K. This proves that $K \cup D(M)$ is consistent.

Theorem 4.2 *If R be a ring. If every finitely generated subring of R can be embedded in a division ring, then R can be embedded in a division ring.*

Proof: Consider any finite subset R^* of R and consider the intersection of all subrings of R that contain R^*. Call such an intersection, R_1. Then R_1 is a finitely generated ring and by the premise of the theorem, can be embedded in a division ring. Since R^* is a subset of R_1, R^* can also be embedded in a division ring. Since every finite substructure (R^* is arbitrary) can be embedded in a division ring, the structure of R can be embedded in a division ring.

A sequence of sentences $Y_n, n = 1, 2, ...$ in a language L is said to constitute an *increasing chain* if $Y_n \supset Y_m$ is a theorem for all $n \geq m$. A sequence constitutes a *strictly increasing chain* if $Y_n \supset Y_m$ is a theorem only if $n \geq m$.

Theorem 4.3 *Let K be a set of sentences, $\{Y_1, Y_2, ...$ whose elements make up a strictly increasing chain. Then there does not exist any sentence Y in L such that $K \vdash Y$ and $Y \vdash K$ (i.e. K and Y are equivalent).*

Proof: Suppose there exists a sentence Y in language L that is equivalent to K. Y must be deducible from a finite set of K, call it $K' = \{Y_i, Y_j, ...Y_l\}, i < j < l$. Then we have $Y_l \supset Y_i, Y_l \supset Y_i$, etc as theorems in L. As a result, Y must be deducible from Y_l alone, that is, $Y_l \supset Y$ is a theorem. We also have that $Y \vdash K$ which implies that $Y \supset Y_{l+1}$ is a theorem of L. This results in a contradiction since $Y_l \supset Y_{l+1}$ is a theorem contrary to the premise of the theorem.

Corollary 4.1: *The notion of a commutative field of characteristic 0 cannot be expressed by a finite number of axioms within first order logic.*

Proof: Suppose there is a finite set of axioms that express the notion of a commutative field with characteristic 0. Take their conjunction and denote it as sentence Y. Let Y_1 be the conjunction of the axioms that expresses the notion of a commutative field, K_{CF}. Recalling that the sentence X_n expresses $y = nx$, define $Y_2 = Y_1 \wedge \sim X_2$. $Y_3 = Y_2 \wedge \sim X_3$, and $Y_n = Y_{n-1} \wedge X_{p_n-1}$, p_n is the nth prime number. The sequence $Y_1, Y_2, ...$ is a strictly increasing sequence. Define K to be the set of sentences $\{Y_1, Y_2, ...\}$. Since K constitutes an increasing chain of sentences and the models of K are commutative fields of characteristic 0, there is no finite set of sentences equivalent to K_{CF}^0.

Theorem 4.4 *Let X be a sentence constructed in terms of equality, addition, and multiplication that holds true in all commutative fields of characteristic 0. Then there exists a prime number p_0 such that X holds in all commutative fields of characteristic $p > p_0$.*

Proof: Since $K_0^{CF} \vdash X$, X must be deducible from a finite subset K' of K_0^{CF}, more specifically, the union of K_{CF} and some finite subset of the set $\{\sim X_2, \sim X_3, ...\}$. Let p_0 be the highest subscript that belongs to K'. Then all fields with characteristic $p > p_0$ satisfies X.

We provide some application of the theorem above. Let $q(x_1, ..., x_n) \in \mathbb{Z}[x^1, ..., x^n]$ and let $q^{(p)}(x_1, ..., x_n)$ denote the same polynomial but coefficients in the prime field of characteristic p, $\mathbb{F}_p$. Suppose $q_1(x_1, ..., x_n) = 0, q_2(x_1, ..., x_n) = 0, ..., q_k(x_1, ..., x_n) = 0$ does not have a solution in any extension of $\mathbb{Q}$, then there exists a positive integer p_0 such that the set

$$\{q_1^{(p)}(x_1, ..., x_n) = 0, ..., q_k^{(p)}(x_1, ..., x_n) = 0\}$$

does not possess a solution in any field of characteristic p where $p > p_0$.

Proof: Any polynomial equation with variables can be expressed in first order logic, without using any object symbols. For instance, the polynomial $y = 3x^2$ can be expressed as $Q(x, y) = (\exists z)[P(x, x, z) \wedge S_3(x, y)]$ and if we want to interpret Q in $\mathbb{F}_2$, then we can add the predicate X_2 to denote that every coefficient has characteristic 2. The hypothesis of the statement can be then be written as
$$X = (\forall x_n)...(\forall x_n)[\sim [Q_{q1}(x_1, ..., x_n, 0) \wedge \cdots \wedge Q_{qk}(x_1, ..., x_n, 0)]]$$

holds in fields of characteristic 0. Then by theorem 4.4, there exists p_0 such that X holds in any field of characteristic $p > p_0$ which proves the theorem.

Theorem 4.5. Suppose R is an integral domain with unit 1, such that every element of R is contained in a finite number of its prime ideals. Let X be a sentence formulated in terms of equality, addition, multiplication, and individuals of R. Suppose X is true in all fields that are extensions of R (i.e. X is deducible from the axioms of R). Then X holds in all fields that are extensions of the quotient ring R/J for all but finitely many prime ideal J in R.

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Proof: Recall that the relations E, S, and P respectively stand for equality, addition, and multiplication. Start by defining $D(R)$ to be the diagram of R. By the premise of the theorem, X is true in all models of $K_{CF} \cup D(R)$, that is, X is deducible from $K_{CF} \cup D(R)$ (recall that whenever M' is an extension of M, then M' must be a model of $D(M)$). Then, $\{Y_1, ..., Y_k\} \supset X$ is deducible from K_{CF} where $\{Y_1, ..., Y_k\}$ is a finite subset of $D(R)$. Recalling the the diagram is a set of atomic formulae and the negation of atomic formulae that hold in a structure, $\{Y_1, ..., Y_k\}$ must be one of six different forms:

$$E(a,b), S(a,b,c), P(a,b,c), \sim E(a,b), \sim S(a,b,c), \sim P(a,b,c)$$

Note that any sentence of form $\sim S(a,b,c)$ may be replaced by the conjunction $S(a,b,d) \wedge \sim E(c,d)$ and $\sim P(a,b,c)$ can be replaced in a similar manner. Moreover, $\sim E(a,b)$ can be replaced by the conjunction $\sim E(c,0) \wedge S(a,c,b)$. Hence the six forms of sentences reduces to one of the following four forms:

$$E(a,b), S(a,b,c), P(a,b,c), \sim E(c,0)$$

The number of sentences in the set $\{Y_1, ..., Y_k\}$ must be finite, and hence the number of individuals c_i such that $\sim E(c_i, 0)$ appears among the Y_i must also be finite. By the premise of the theorem, c_i can be included in only finitely many prime ideals of R. Now consider any prime ideal J that does not contain c_i. We claim that the sentence X must hold in an extension field F of R/J for such J. If we can show that $Y_1, ...Y_k$ hold in F, since $Y_1 \wedge .... \wedge Y_k \supset X$ is deducible from K_{CF}, we can immediately show that X must hold in F. Moreover, $Y_1, ...Y_k$ can hold in F if and only if they hold in R/J since the sentences do not contain any variables or quantifiers and F is an extension of R/J. But the quotient ring R/J is the image of the canonical ring homomorphism: $\varphi : R \to R/J$ and therefore, if the relations E,S,P hold in R, it must hold in R/J. Any sentence Y_j of the form $\sim E(c_i, 0)$ must also hold in R/J since if $E(c_i, 0)$ in R/J, $c_i \in J$, but this is a contradiction since we have picked J such that it does not contain c_i. This proves that the sentence X must hold in R/J and consequently in the field F.

Corollary 4.2 Define R to be an integral domain such that all of its elements are contained at most in a finite number of prime ideals of R. If a polynomial $p(x_1, ..., x_n)$ is irreducible in all extension fields of R, then it is irreducible in all extensions of the integral domain R/J for all but finitely many J's where J is a prime ideal of R.

Although infinitary conjunction and disjunction do not belong to the language of first order logic, we can study its properties and acquire meaningful results for the study of model theory. Let $Y = X_1 \wedge X_2 \wedge ...$ be an infinite conjunction in some extended language L that do not belong to Lower Predicate Calculus while each X_i belong to L. We say, Y is effectively finite if there exists a partial conjunction

$$Y_m = X_1 \wedge X_2 \wedge ... \wedge X_m$$

such that $Y_m \supset Y$ holds in all models in which Y is defined.

Theorem 4.6. *Y is equivalent to a sentence Z of L if and only if Y is effectively finite.*

Proof: $\Leftarrow$ is trivial. To prove the opposite direction, we prove its contraposition. Consider the sequence of partial conjunctions $Y_1, Y_2, ...$. This sequence constitutes an increasing chain. Suppose that for every positive integer m, there exists a positive integer n larger than m such that $Y_n \supset Y_m$ is not a theorem. If this is the case, we can choose a strictly increasing chain, denote it

$$...Y_{k_n} \supset ...Y_{k_3} \supset ... \supset Y_{k_1}$$

The set K of the elements of this chain holds if and only if Y holds. This is a contradiction since there is no sentence is equivalent to a strictly increasing chain.

Let $Y = [X_1 \vee X_2 \vee ... \vee X_m...]$ be an infinite disjunction. Y is effectively finite if for some positive integer m, $Y \supset Y_m$ holds in all models in which it is defined. If such is the case, $Y \supset Y_n$ is also a theorem for all $n \geqslant m$ and $Y_k \supset Y_n$ is a theorem for all positive integer k and $n \geqslant m$. The infinite disjunction Y is effectively finite if and only if there exists $m \in \mathbb{Z}_{>0}$ such that $Y_n \supset Y_m$ is a theorem for all $n \in \mathbb{Z}_{>0}$.

Define Z to be the expression

$$\forall u \forall v \forall w ... \forall z [Z_1(u,v,w,...z) \vee Z_2(u,v,w,...z) \vee ... \vee Z_n(u,v,w,...z) \vee ...]$$

where Z_n are wff. Let $W_m(u,v,w,...z) = [Z_1(u,v,w,...z) \vee ... \vee Z_m(u,v,w,...,z)]$ be called the partial disjunction and define $Z_m^* = \forall u \forall v \forall w ... \forall z W_m(u,v,w,...,z)$. Z is effectively finite if there exists $m \in \mathbb{Z}_{>0}$ such that $Z \supset Z_m^*$ holds in all structures in which Z is defined.

Theorem 4.7. *A sentence Z as defined above is effectively finite if and only if there exists a set of sentences K whose vocabulary is a subset of that of Z and Z holds in a structure M if and only if M is a model of K.*

Proof: $\Rightarrow$ is trivial. Now, suppose a set K exists as specified in the theorem and for sake of contradiction, suppose that Z is not effectively finite. Then for every $m \in \mathbb{Z}_{>0}$, there exists a structure M such that $Z \supset Z_m^*$ does not hold. Such a structure satisfies Z and therefore must be a model of K as assumed in the theorem. Therefore, $K \cup \{\sim Z_m^*\}$ is consistent for all $m \in \mathbb{Z}_{>0}$. Now substitute individuals $a,b,c,...,d$ that do not appear in Z for the variables $u,v,w,...z$ in Z. $K \cup \{\sim Z_1(a,b,c...d), \sim Z_2(a,b,c,...d),..., \sim Z_n(a,b,c,...d),...\}$ is consistent. Let M^* be the model of such a sentence. It must satisfy K and cannot satisfy Z. This contradicts our premise since Z does not hold in the model of K. This concludes that Z must be effectively finite.

Theorem 4.8 *The set of axioms for the notion of an Archimedean ordered field, K_{AF} cannot be replaced by a set of sentences K in L defined in terms of the vocabulary of K_{AF}.*

Proof: Define V to be the conjunction of the set of axioms for an ordered field, and define Z $= \forall x \forall y [V \wedge E(0,x)] \vee [V \wedge \sim Q(0,x)] \vee [V \wedge Q_1(x,y)] \vee [V \wedge Q_2(x,y)] \vee ...]$. It is not difficult to notice that Z constitutes the axioms for the notion of an Archimedean ordered field. If there is a set K that is equivalent to Z, then Z must be effectively finite, that is, there exists $m \in \mathbb{Z}_{>0}$ such that $Z \supset Z_m^*$ holds in all models that define Z. Yet this is not true in an Archimedean ordered field (the sentence when $x = 1$ for instance, implies that for all elements y in an Archimedean ordered field, $m - 1 > y$ which is not true).

Now we cover how model theory can be used to study group theory. Let K_G be the set of axioms for a group. Suppose we want to define the notion of a subgroup and a quotient group in first order logic. If G is a group, J is a subgroup of G, and H is a normal subgroup of J, define relations $R(x)$ and $T(x)$ to represent membership in J and H. The concept that the elements that satisfy R make up a subgroup can then be expressed as follows:

$$\forall x \forall y \forall z [R(x) \wedge R(y) \wedge S(x,y,z) \supset R(z)]$$
$$R(e)$$
$$\forall x \forall y [R(x) \wedge S(x,y,e) \supset R(y)]$$

We denote the union of the group axioms and the sentences listed above as K_{GR}. We can similarly define the set of sentences K_{GT}. Now the fact that H is a normal subgroup of J may be expressed as follows:

$$\forall x [T(x) \supset R(x)]$$
$$\forall x \forall y \forall z \forall t \forall u [S(x,y,e) \wedge S(x,z,t) \wedge S(t,y,u) \wedge R(x) \wedge T(z) \supset T(u)]$$

The last element expresses that $\forall g \in H \; \forall k \in J, gkg^{-1} \in J$. Now suppose we want to express the fact that the sentence X is satisfied by a quotient group, J/H. We define a new relation $E^*(x,y) = \exists z [T(z) \wedge S(x,z,y)]$ to represent that the elements x,y belong in the same coset. Moreover, addition in quotient ring can be expressed as $S^*(x,y,z) = \exists w \exists t [S(x,y,w) \wedge S(w,t,z) \vee T(t)]$. We may then replace E and S that appear in the sentence X with E^* and S^* and replace E^* and S^* again with the right hand sides of the equations above to get the sentence X^*. More specifically, define

$$E^*(x,y) = (\exists z)[T(z) \wedge P(z,y,x)].$$

This means that x and y represent the same element in the normal subgroup if there exists an element $z \in H$ such that $yx^{-1} = z^{-1}$. Moreover let $S^*(x,y,z)$ to hold if $(\exists z)[S(x,y,w) \wedge E^*(w,z)]$ holds. Now we have X holds in J/H if and only if X^* holds in G.

A *normal chain* of a group G is a finite sequence of subgroups of G, $\{G_1, G_2, ...G_k\}$ such that $G = G_0 \supset G_1 \supset ... \supset G_k = \{e\}$ and G_j is a normal subgroup of G_{j-1}. A *quasi-elementary property* of a group is a property expressed by a set of sentences K in first-order logic involving E,S, and e such that whenever the property holds in a group G, then it holds in all its subgroup. Let Π be a set of quasi-elementary properties $\{P_l, ..., P_k\}$. A group G is of type Π if G possesses a normal chain of length k such that the quotient group G_{j-1}/G_j has the property P_j. Now consider a subgroup H of G and the sequence of the subgroups of H, $\{H_0, H_1, ...H_k\}$ where $H_j = G_j \cap H$. The sequence is a normal chain since H_j is a normal subgroup of H_{j-1}. Moreover, notice that $H_{j-1}/H_j \cong$ subgroup of G_{j-1}/G_j (*Proof*: Consider the canonical map $\varphi : G_{j-1} \to G_{j-1}/G_j$. Then H_{j-1} contains the kernel of φ and therefore, φ maps H_{j-1} to a subgroup of G_{j-1}/G_j. Restrict φ to H_{j-1}. Then $\varphi|_{H_{j-1}} : H_{j-1} \to G_{j-1}/G_j$ and its image is the subgroup $\varphi(H_{j-1}) \leqslant G_{j-1}/G_j$. The kernel of this restricted map is $H_{j-1} \cap G_j = H_j$ and hence by first isomorphism theorem, $H_{j-1}/H_j \cong \phi(H_{j-1}) \leqslant G_{j-1}/G_j)$. As a result, P_j is satisfied by H_{j-1}/H_j. This demonstrates that H must also be of type Π.

Now we introduce k one-place relations $R_1, ...R_k$ to construct the set of sentences K_π that satisfies the following conditions.

1. contains K_G, the set of axioms that define a group

2. contains sentences that that equality is substitutive with respect to all the R_j

3. the set G_j of elements that satisfy the relation R_j constitute a subgroup of the group described by K_G for every j

4. contains sentences that state that G_j is a normal subgroup of G_{j-1}

5. contains sentences which state that G_{j-1}/G_j satisfies the property P_j

6. contains the sentence that states that G_k only contains e

From what we have covered so far, it is possible to write K_π in the language of first order logic and every model of K_π is a group of type Π.

Theorem 4.9 A a group G is of type Π if and only if $D(G) \cup K_\pi$ is consistent.

Proof: If the group G is of type Π, then there is a normal chain $\{G_0, ..., G_k\}$. Suppose we introduce new relations R_j into G such that $R_j(a)$ holds for $a \in G$ if and only if $a \in G_j, j = 1, \cdots, k$. Then G becomes a model of $D(G)$ and K_π. Now we show the other direction. Suppose $D(G) \cup K_\pi$ is consistent, let G' be the model. Since G' is an extension of G since it is a model of $D(G)$ and a group of type Π since it is a model of K_π, it follows that G is also of type Π (since every subgroup H of a type Π group is also type Π.)

Theorem 4.10 If every finitely generated subgroup of G is type Π, then G is of type Π.

Proof. Construct K_π. If every finite subset of $D(G) \cup K_\pi$ is consistent, then by compactness, $D(G) \cup K_\pi$ will be consistent. Now every finite subset of $D(G) \cup K_\pi$ will be consistent if for every finite subset of G, call it G', $D(G)' \cup K_\pi$ is consistent where $D(G')$ is the diagram from restricting the elements of G to G'. But then as usual, one can generate a subgroup generated by elements in G', call this subgroup G'' and by the hypothesis of the theorem, $D(G'') \cup K_\pi$ is consistent. Since $D(G') \subset D(G'')$, it follows that $D(G') \cup K_\pi$ is also consistent.

We seem some applications in algebra. A group G is solvable of rank k if there exists a normal chain, length k such that the quotient groups are all Abelian. We may then say that G is of type $\Pi = \{P_1, ..., P_k\}$ where $P_1 = \cdots = P_k = P = (\forall x)(\forall y)(\forall z)[S(x,y,z) \supset S(y,x,z)]$. It then follows from our theorems above that if every finitely generated subgroup of G is solvable of rank k, then G itself

is also solvable of rank k. We provide another example. Let $\Pi = \{P_1, P_2\}$ where P_2 is the commutative property as above and P_1 is the property that for some positive integer n, G, the group of type Π does not include more than n different elements. Once one checks that these properties are quasi-elementary, one can conclude that if every finitely generated subgroup of G contains a normal subgroup of index less or equal to n, then G must also contain a normal subgroup of index less than or equal to n.

5 Skolem Functions

Consider any sentence of a language L in a prenex normal form. For instance, let sentence

$$Y = (\forall x)(\exists y)(\forall z)(\forall u)(\exists v)(\forall w)(\exists t)\, Q(x, y, z, u, v, w, t)$$

where Q is the matrix of the sentence. For all individuals a, b, c, d in the model M of Y, there exists functions $\varphi(x), \psi(x, z, u), \chi(a, b, c, d)$ such that the sentence $Q(a, \varphi(a), b, c, \psi(a, b, c), d, \chi(a, b, c, d))$ is also true in M (although strictly speaking the newly introduced function symbols, called Skolem functors or Herbrand functors, do not belong to the language L yet). In general, one can replace the variables that belong to the existential quantifier by functions of variables that belong to the universal quantifiers that precede the existential quantifier. A sentence so obtained is called the *open form*.

Now suppose M is a model of a sentence X and M' is an extension of the model M. Suppose we want to express the fact that X holds in M, and yet in terms of the semantic interpretation of X under M'. We introduce a new relation R that does not belong to M such that for $a \in \{$domain of $M\}$ $R(a)$ holds if and only if a belongs to M. Let the new structure that contains the relation R be called M'_R. Now for a sentence Y that is true in M, we want to define a related sentence Y_R that is true in M'_R. The mapping of Y to Y_R is called relativization with respect to R and indeed, such a map is well defined not only for sentences but for all wff of a language L. We denote the mapping as $\rho([Y]) = Y_R$ and define it as follows:

If X is an atomic formula, then $\rho([X]) = [X]$,

$$\rho([\sim X]) = [\sim \rho(X)]$$
$$\rho([X \vee Y]) = [\rho(X) \vee \rho(Y)]$$
$$\rho([X \wedge Y]) = [\rho(X) \wedge \rho(Y)]$$
$$\rho([X \supset Y]) = [\rho(X) \supset \rho(Y)]$$
$$\rho([X \equiv Y]) = [\rho(X) \equiv \rho(Y)]$$

If X is a well formed formula that does not contain another quantifier bounding y, then

$$\rho(((\forall y)X)) = (\forall y)\,[[R(y)] \supset \rho(X)]$$
$$\rho(((\exists y)X)) = (\exists y)\,[[R(y)] \wedge \rho(X)]$$

Theorem 5.1. *X holds in M if and only if X_R holds in M'_R.*

Proof: If X is of order 1, the theorem obviously holds. For the rest, we will proceed by induction on the order of sentences. Suppose the theorem holds for sentences Y and Z. Consider the sentence, $X = Y \wedge Z$. If X holds in M, then both Y and Z must hold in M and thereby $\rho(Y)$ and $\rho(Z)$ must also hold in M'_R. As a result, $\rho([Y \wedge Z]) = [\rho(Y) \wedge \rho(Z)]$ must hold in M'_R. On the other hand, if X dose not hold in M, then either Y or Z does not hold in M. By induction, either $\rho(Y)$ or $\rho(Z)$ does not hold in M'_R and hence, $\rho(X)$ does not hold in M'_R. The proofs for negation, disjunction, implication, and equivalence are similar. Suppose $X = [\forall y Z(y)]$. If X is true in M, then for all elements a of M, $Z(a)$ is true in M and hence by induction $\rho(Z(a))$ holds in M'_R. Consequently, $R(a) \supset \rho(Z(a))$ holds in M'_R for all elements a of M'_R. In other words, $\rho(X) = (\forall y)[[R(y)] \supset \rho(Z(y))]$ holds in M'_R. Now, suppose X does not hold in M. Then there exists an individual a of M that does not satisfy Z and hence for such a, $[R(a) \supset \rho(Z(y))]$ does not hold in M'_R. As a result, $(\forall y)[[R(y)] \supset \rho(Z(y))]$ does not hold in M'_R. The proof for existential quantification is similar. This completes the proof of the theorem. When we transform a set of sentences K that does not contain the relation R into $K' = \{\rho(X) | X \in K\}$, we might encounter a problem if no elements of M', the model of K' satisfy the relation R. Hence we will add to the sentence, $\exists x R(x)$ into K' if K has no individuals or $R(a)$ for individuals a that appear in K.

6 Extension of Models

A set of sentences K is called *complete* if for every sentence X defined in K, either X or $\sim X$ is deducible from K. Let M and M' be two similar structures, that is they share the same relations and let A be a subset of the set of individuals that both M and M' share (A can also be an empty set). M is *elementarily equivalent* to M' with respect to A if any sentence X defined in the vocabulary of M and M' and whose individuals belong entirely to A either hold or does not hold in both M and M'. To put it differently, let $S(M,A)$ be the set of all sentences that only contain individuals in A and hold in M. Then two similar structures M and M' are elementarily equivalent if $S(M,A) = S(M',A)$. If M and M' are similar structures such that M' is an extension of M and A is the set of individuals in M, then M is an elementary extension M' if M and M' are elementarily equivalent with respect to A. We denote M' is an elementary extension of M as $MeeM'$. Last but not least, two structures M and M' are elementarily equivalence if A is an empty set. This intuitively means that any sentence that does not contain a free variable or a constant symbol have the same truth value in both M and M'. Hence, if we have two ordered structures, $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$, the two structures are not elementarily equivalence since the sentence

$$(\forall z)(\exists x)(x^2 = z)$$

does not hold in $\mathbb{Q}$ but holds in $\mathbb{R}$.

Theorem 6.1 *Let M be a substructure of M' and A be the set of individuals in M. Then M' is an elementary extension of M if and only if for every sentence X beginning with an existential quantifier that is defined in M and holds in M', there exists an element $a \in A$ such that $Z(a)$ holds in M'. Intuitively, one can check that the extension is elementary by confirming that no existential truth in M' requires a new element outside of M.*

Proof: ($\Rightarrow$) holds by the definition of elementary extension. To prove the other direction, we only have to show that for any sentence X that is defined in M and holds in M' must hold in M. We first begin by transforming sentence X to its prenex normal form. If X is an atomic formula, since M' is an extension of M, X either holds or does not hold both in M and M'. If the sentence X is a sentence that does not contain any quantifier, its truth value is uniquely determined by the truth table of propositional calculus and therefore the sentence again equally holds in both structures or does not hold. Now for sake of contradiction, suppose that there are sentences in prenex normal form (with at least one quantifier) that hold in M' but not in M. Let X be such a sentence with the least number of quantifiers. If X begins with a universal quantifier (i.e. $X = \forall y Z(y)$, the sentence $Z(a)$ does not hold for some a in M. Then $Z(a)$ cannot hold in M' since the number of quantifiers is less than that of X and thereby $\forall y Z(y)$ no longer holds in M', contradicting our assumption. Now suppose, X begins with an existential quantifier, that is, $X = \exists y Z(y)$. Since the sentence holds in M', by the premise of the theorem, there is an individual $a \in A$ such that $Z(a)$ holds in M'. Since $Z(a)$ contains less quantifiers, $Z(a)$ must hold in M which means that $\exists y Z(y)$ must again hold in M. This completes the proof of the theorem.

We mention that if $MeeM'$ and $M'eeM''$, then $MeeM''$. If we suppose for contradiction that the sentence X holds in M but not in M', then both X and $\sim X$ are true in M'' which is a contradiction. We define a set of sentences $\Phi = \{M_v\}$ to be *monotonic* if for any elements M_v, M_μ in Φ, either M_μ extends M_v or M_v extends M_μ (we write $M_v \subset M_\mu$ or $M_\mu \subset M_v$). We define the union of the monotonic set, $M = \bigcup_v \{M_v\}$ to be the structure that contains all the individuals of $M_v \in \Phi$ and the relation $R(a_1, ..., a_n)$ holds if and only if it holds for any M_v that contains the individuals $a_1, ..., a_n$.

Theorem 6.2 If Φ' is a monotonic set of structures such that for any M_μ, M_v in Φ', either $M_\mu ee M_v$ or $M_v ee M_\mu$, then $M = \bigcup_v \{M_v\}$ is an elementary extension of every M_v in Φ.

Proof: Define $S(M_v)$ to be the set of all sentences that are defined and true in M_v. Using the notion that we employed previously, $S(M_v) = S(M_v, A_v)$, where A_v is the set of all individuals in M_v. Define S to be the union $\bigcup_v \{S(M_v)\}$. Since $S(M_v)$ makes up a monotonic set, every finite subset $S' \subset S$ is contained in at least one M_v. By compactness theorem, S is consistent and hence possesses a model M'. Since for every M_v, $S(M_v)$ contains the diagram $D(M_v)$ and $S(M_v) \subset S$, M' is an extension of M_v by theorem 2.1. Also since M' satisfies every sentence in $S(M_v)$, M' is an elementary extension

of M_v. To show that $M = \bigcup_v \{M_v\}$ is an elementary extension of M_v, it suffices to show that M' is an elementary extension of M. Let X be a sentence defined in M such that it is true in M' that begins with an existential quantifier. The number of individuals that appear in $X = \exists y Z(y)$ is finite and hence X must be all contained in some M_v. Since $a \in A_v, Z(a)$ must hold in M_v and hence in M'. But A is contained in M so X must hold in M. Therefore, M' is an elementary extension of M, which finishes our proof.

Theorem 6.3 *Define M and M' to be two similar structures and A, the set of individuals common to both M and M'. If M and M' are elementarily equivalent with respect to A, then there is a structure M^* that is an elementary extension of both M and M'.*

Proof: Define $S(M)$ and $S(M')$ to be the set of every sentence that is true in the structure M and M' respectively. We first show that the union of the sets $T := S(M) \cup S(M')$ is consistent. For sake of contradiction, suppose T is inconsistent. Then there exists some finite subset T' that is inconsistent. Since both $S(M)$ and $S(M')$ is consistent, T' must contain both elements of $S(M)$ and $S(M')$. Without loss of generality, we may assume that T' consists of two sentences $X \in S(M)$ and $X' \in S(M')$ where X and X' are obtained by taking the conjunction of all sentences in each set. Now denote $a_1, \ldots, a_k$ to be the individuals in X that do not belong in M' and vice versa for $b_1, \ldots, b_j$. Then we may write

$$X = Y(a_1, \ldots, a_k) \text{ and } X' = Y'(b_1, \ldots, b_j).$$

If $Y(a_1, \ldots, a_k) \wedge Y'(b_1, \ldots, b_j)$ is contradictory, then

$$(\exists x_1) \ldots (\exists x_k)(\exists y_1) \ldots (\exists y_j)[Y(x_1, \ldots, x_k) \wedge Y'(y_1, \ldots, y_j)]$$

must also be contradictory (minor detail but the variables x_i's and y_i's are assumed not to appear anywhere else). Because $b_1, \ldots, b_j$ do not appear in $Y(a_1, \ldots, a_k)$, the y_i's also do not appear in $Y(x_1, \ldots, x_k)$ and hence the sentence above can be written as

$$[(\exists x_1) \ldots (\exists x_k)Y(x_1, \ldots, x_k)] \wedge [(\exists y_1) \ldots (\exists y_i)Y'(y_1, \ldots, y_j)].$$

Each conjunct in the sentence above is well defined in both M and M' and hence each belongs in $S(M, A)$ and $S(M', A)$ (essentially since we replaced each constant with a variable and an existential quantifier). By assumption $S(M, A) = S(M', A)$ and hence the sentence above holds in both M and M' and is consistent. This in turn implies that T is consistent and has a model, call it M^*. Because M^* is an elementary extension of M' (S(M') is true in M^*), this proves the theorem.

Theorem 6.5. (Upper Löwenheim-Skolem Theorem) Give a structure M of cardinal α, we can find an elementary extension of M with any cardinal $\alpha' > \alpha$.

Proof: We first provide the construction. Pick. a set of individuals B such that its elements do not belong in $|M|$ (we will call $|M|, A$ for convenience) such that the cardinality of B is $\alpha' - \alpha$. Moreover fix an element $a \in A$. This will define the domain of the new structure M'. Moreover, for any relations R contain in M, we will define $R(a'_1, \ldots, a'_n)$ to hold in M' if and only if either all a'_i's are elements of A that satisfies the relation R, or whenever $a'_i \notin A$, we replace it with $a \in A$ and see if they satisfy the relation R. We will show that such defined structure M' is indeed an elementary extension of M. It will suffice to show that any sentence defined in M that holds in M' also holds in M. We only have to consider prenex normal form. First replace any instance of elements of B with the fixed element a. In particular, if the sentence does not contain any elements in B, the sentence will stay the same. Call this mapping by replacing instances $X' \to X$. It only suffices to show that X holds in M if X' holds in M'. If X' is free of quantifier, this. is obvious from the way we constructed M'. When sentence X' has a quantifier, we use induction on the number of quantifiers. Suppose $X' = [(\exists y)Z'(y)]$ is true in M'. Then $X' \to X$ where $X = [(\exists y)Z(y)]$ where $Z' \to Z$. Suppose for some $b' \in A \cup B$, $Z'(b')$ holds. Then $Z'(b') \to Z(b)$ where $b \in A$. Hence, $Z(b)$ holds in M and $[(\exists y)Z(y)]$ holds in M. If $X' = [(\forall y)Z'(y)]$ holds in M', one can also apply the same logic and attain $[(\forall y)Z(y)]$ holds in M.

Theorem 6.4 *Let M be a structure containing the equality relation, that has infinite cardinality α and the number of its relations do not exceed the number of the individuals. Then there exists an*

elementary extension M' of M of cardinal α that contains at least one element that does not belong in M. Moreover, for any cardinal $\alpha' > \alpha$, there exists an elementary extension M' of M of cardinal α'.

Proof: Pick an individual c not contained in M. Let the set of sentences K express the fact that c is not contained in M, that is, $K = \{\sim E(c,b_v)|b_v \in \{\text{domain of } M\}\}$. Define $H = S(M) \cup K$. H must be consistent. Suppose, for sake of contradiction, that H is inconsistent. Then there is a finite subset, $H' = K' \cup S'(K' \subset K, S' \subset S(M))$ of H that is contradictory. Note that K' cannot be empty since any finite subset of $S(M)$ is consistent. Write $K' = \{\sim E(c,b_1),..., \sim E(c,b_k)\}, k \geqslant 1$ and write X to be the conjunction of all elements in S'. If H' is inconsistent, then the conjunction of all its elements, that is, $Y = X \wedge \sim E(c,b_1) \wedge ... \wedge \sim E(c,b_k)$ must also be inconsistent. As a result, $Y \supset Z$ is a theorem, where $Z = [A \wedge \sim A], A$ is some relation in the language L. By existential generalization, $Y' = \exists z[X \wedge \sim E(z,b_1) \wedge ... \wedge \sim E(z,b_k)]$. But then Y' must be inconsistent and the same applies to its equivalent form, $Y'' = X \wedge [(\exists z)[\sim E(z,b_1) \wedge ... \wedge \sim E(z,b_k)]]$. However, Y'' holds in M since both conjuncts are true under M and this results in a contradiction. Hence, we may conclude that H is consistent and possesses a model, call it M'. By the Löwenheim-Skolem theorem, the cardinal of M' need not exceed the number of sentences in H. The set $S(M)$ has cardinal α (see proof on *Robinson, pp. 61*, we use the fact that the number of relations do not exceed α) and the set K contains exactly α elements and hence H contains exactly $\alpha + \alpha = \alpha$ elements. Since M' contains at least α elements and therefore, M' has cardinality α and contains an element c not contained in M. Since M' satisfies $S(M)$, we have an elementary extension of M with cardinality K.

To show that there is an elementary extension of cardinality $\alpha' > \alpha$, pick a set of constants $C = \{c_\mu\}$, cardinality α', whose elements are not contained in M. We define two sets of sentences, $K_1 = \{\sim E(c_\mu,b_v)\}, K_2 = \{\sim E(c_\mu,c_v)\}$ where c_μ, c_v, b_v respectively varies across elements of C, C, and M. Now define the set $H = S(M) \cup K_1 \cup K_2$. Since the cardinal of K_1 is α', that of K_2) is also α', the cardinal of H is $\alpha + \alpha' + \alpha' = \alpha'$ (recall from above that $S(M)$ has cardinal α'). We claim that H is consistent. For suppose that H is inconsistent. Resembling the proof in the paragraph above, there exists a sentence $X \in S(M)$ and finite subsets $K_1' \subset K_1, K\_2 \subset K_2$ such that $\{X\} \cup K_1', \cup K_2'$ is inconsistent. Take the conjunction of all the elements of K_1' and K_2' and we get a sentence of the form $Y(b_1,...,b_j,c_1,...,c_l)$ where the individuals displayed are all that appear in the sentence. Then $Z = X \wedge Y(b_1,...,b_j,c_1,...,c_l)$ is inconsistent. By existential generalization, $X \wedge \exists z_1...\exists z_j Y(b_1,...,b_j,z_1,...z_l)$ is inconsistent. Yet it is not difficult to find that both conjuncts are true in M and hence we get a contradiction. Hence H is consistent and has a model M'. Again, M' has cardinal $\geqslant \alpha'$ since it contains the set C, but the Löwenheim-Skolem theorem tell us that M' need not have more than α' individuals. This proves that M' has cardinal α'.

7   Prefix Problem

Consider the set of all wff X in langauge L of prenex normal form. We will such a set or class of sentences as N. A *block* of quantifiers in the prefix of $X \in N$ is a sequence of consecutive quantifiers of same type (existential or universal) that cannot be extended to left or right. For example, the sentence $\forall x \forall y \exists z \exists w \forall t \forall v Q(x,y,z,w,t)$, Q is the matrix, has three blocks of quantifiers, namely, $\forall x \forall y, \exists z \exists w, \forall t \forall v$. A wff $X \in N$ is said to belong to the class B_n if it contains n or less quantifiers. The example provided above belongs to $B_n, n \geqslant 3$ then since it contains 3 blocks of quantifiers. We also divide the class B_n into two subclasses A_n and E_n. $X \in B_n$ belongs to A_n if X either contains less than n blocks of quantifiers (i.e. $X \in B_{n-1}$ or X contains exactly n blocks of quantifiers whose first block contains universal quantifiers. $X \in B_n$ belongs to $X \in E_n$ if again it contains less than n blocks of quantifiers or it contains exactly n blocks of quantifiers whose first block contains existential quantifiers. $B_n = A_n \cup E_n, n \in \mathbb{Z}_{\geqslant 0}$ trivially.

A sentence X is persistent under extension if whenever a structure M is a model of X, all extensions of M are models of X. X is persistent under restriction if whenever a structure M is a model of X, all its substructures that contain the individuals of X are models of X. If a sentence is both persistent under extension and restriction, it will be called invariant. From now on, we will use the expression "X is persistent" to mean "X is persistent under extension" while keeping the expression

"*X* is persistent under restriction." we can extend the definitions above a step further. Given a set of axioms *K*, *X* is persistent (under extension) with respect to *K* if whenever a model *M* of *K* is a model of *X*, all the extensions of *M* that are models of *K* are models of *X*. If *X* is persistent under restriction with respect to *K*, whenever a model *M* of *K* is a model of *X*, then all substructures of *M* that are models of *K* and contain the individuals of *X* are models of *X*. If *X* is both persistent under extension and restriction with respect to *K*, it will be called invariant with respect to *K*.

We can define the notion of persistence for a set of sentences *K* instead of a single sentence *X* in a similar manner above. Now, we define the notion of persistent for a predicate. Suppose $Q(x_1, ..., x_n)$ is a predicate whose vocabulary is defined in *M*. We define the *n-dimensional Cartesian space* M^n as the set of *n*-tuples $a_1, ..., a_n$ whose *coordinates* $a_i, i = 1, ..., n$ are individuals of *M*. The predicate *Q* is said to hold at a point $P \in M_n, P = (a_1, ..., a_n)$ if $Q(a_1, ..., a_n)$ is true in *M*. We define *Q* to be persistent (under extension) at a point $P \in M^n$ if ($Q(a_1, ..., a_n)$ is true in *M* and in all extensions of *M*. *Q* is persistent in *M* if it is persistent at all the points of M^n at which it holds. *Q* is persistent if it is persistent at all points at which it holds, in all structures that defines *Q*. We define persistence under restriction in a similar manner as above. For instance, *Q* is persistent under restriction in *M* if it is persistent under restriction for all points of *M* at which it holds. If *Q* is both persistent and persistent under restriction, it is invariant.

If we add the condition that *M* is a model of a set of axioms *K*, we can define persistence for a predicate with respect to a set of axioms *K*. Using the long list of definitions provided so far, one can notice with ease that if sentences *X* and *Y* are persistent under restriction with respect to a set of axioms *K*, then both $X \wedge Y$ and $X \vee Y$ are persistent under restriction with respect to *K*. Moreover, if *X* is not persistent with respect to a set of axioms *K*, then $\sim X$ is not persistent under restriction with respect to *K*. For suppose that there are two models of *K*, M_1 and M_2 such that M_1 is a substructure of M_2, and *X* holds in M_1 but not in M_2. Then $\sim X$ holds in M_2 but not in M_1. Conversely, if we assume that $\sim X$ is not persistent under restriction with respect to *K*, then there are models M_1 and M_2 of *K* such that $\sim X$ holds in M_2 but not in M_1. Then *X* is true in M_1 but not in M_2. In other words, *X* is not persistent with respect to *K*. This lets us conclude that *X* is persistent under extension with respect to *K* if and only if $\sim X$ is persistent under restriction with respect to *K*. Moreover, *X* is invariant with respect to *K* if and only if $\sim X$ is invariant with respect to *K*. We can extend the results above to predicates as well applying the same logic.

We define a set of sentences *H* to be **conjunctive** whenever for any X_1 and X_2 are in *H*, there exists X_3 in *H* such that $X_1 \wedge X_2 \equiv X_3$ is a theorem. *H* is conjunctive relative to a set of sentences *K* if $K \vdash X_1 \wedge X_2 \equiv X_3$. A set of sentences *H* is disjunctive if whenever X_1 and X_2 are in *H*, there exists $X_3 \in H$ such that $X_1 \vee X_2 \equiv X_3$ is a theorem and disjunctive relative to *K* if $K \vdash X_1 \vee X_2 \equiv X_3$. Next, we define a set of well formed formulae, *H* to be conjunctive if for any wff X_1 and X_2 in *H*, there exists $X_3 \in H$ such that $\forall y_1 ... \forall y_k [X_1 \wedge X_2 \equiv X_3]$ where $y_1, ..., y_k$ are all the free variables that appear in $X_1 \wedge X_2 \equiv X_3$. We define disjunctive set of wff in a similar manner. If a set of sentences *H* is persistent or persistent under restriction relative to a set of axioms *K*, then *H* is conjunctive and disjunctive relative to *K*. This follows from a remark above that if two sentences *X* and *Y* are persistent with respect to *K*, then $X \wedge Y$ and $X \vee Y$ are persistent with respect to *K*. If *H* is the set of predicates with free variables $x_1, ..., x_n$ that are persistent relative to *K*, *H* is both conjunctive and disjunctive. Now we claim that for all integers *n*, the classes A_n and E_n are both conjunctive and disjunctive. This can be proven by removing the overlapping variables of both conjuncts (disjuncts) and interchange the order of quantifiers. The details are provided in *Robinson, pp.68-69*. A sentence that does not contain any quantifier is called *quantifier-free*. A quantifier-free sentence is invariant.

Now we consider the set of universal sentences, that is sentences that belong in the class A_1. Suppose $X \in A_1$ is true in a structure *M* and defined in a substructure M' of *M*. Since any individuals that belong to M' must be satisfied by the matrix of the sentence *X*, *X* must also hold in M', that is any sentence $X \in A_1$, *X* is persistent under restriction. The converse is not true since any theorems are persistent under restriction and need not belong to A_1. Now we prove the following theorem.

Theorem 7.1 *Let the sentences X and Y be defined in a set of axioms K such that if X holds in a model M of K, the sentence Y holds in all substructures M′ of M that are models of K. This can*

happen if and only if there exists a sentence $Z \in A_1$, defined in K such that $K \vdash X \supset Z$ and $K \vdash Z \supset Y$.

Proof: We first prove ($\Leftarrow$) direction. Suppose that there exists $Z \in A_1$ defined in K such that $X \supset Z$ and $Z \supset Y$ are deducible from K. Since X and $X \supset Z$ are true in M of K, Z is also true in M. Since universal sentences are persistent under restriction, Z is true in a substructure M' of M that is a model of K. Since $Z \sup Y$ and Z are both true in M', Y must be true in M', which completes the proof.

Now consider the ($\Rightarrow$) direction. Define H to be the set of all universal sentences W that are defined in K and $K \vdash X \supset W$. Then H is both conjunctive and disjunctive. Define the set $J = K \cup H \cup \{\sim Y\}$. We claim that J is inconsistent. For sake of contradiction, J is consistent and it has a model M'. Then M' cannot have an extension M that is a model of K and makes X true since this contradicts the assumption every substructure of a model M of K and X that satisfies K must make Y true. If we define D' to be the diagram of M', we may conclude that $K \cup D' \vdash \sim X$. Then there exists a finite subset D^* of D' such that $K \cup D^* \vdash \sim X$. Define $Q(a_1, ... a_k)$ to be the conjunction of all elements in set D^* where we only displayed all the individuals that do not appear in K. If D^* is empty, we many introduce an arbitrary element of D' to make it non-empty. Then $K \vdash Q(a_1, ..., a_k) \supset \sim X$ and by rules of deduction, $K \vdash X \supset \sim Q(a_1, ..., a_k)$. Since the individuals $a_1, ..., a_k$ do not appear in K and X is defined in K, $a_1, ..., a_k$ do not appear in X. By rules of deduction, we may conclude that $K \vdash X \supset [(\forall z_1)...(\forall z_k)[\sim Q(z_1, ..., z_k)]]$. $[(\forall z_1)...(\forall z_k)[\sim Q(z_1, ..., z_k)]]$ is an element of H and hence denote it as W. W holds in M'. But $\sim W \equiv (\exists z_1)...(\exists z_k)Q(z_1, ..., z_k)$ and $\sim W$ must hold in M' since $Q(a_1, ..., a_k)$ was defined to be a conjunction of the elements of the diagram of M'. This is a contradiction since W and $\sim W$ both hold in M' and we may conclude that J must be inconsistent. Then there exists a finite subset H^* of H such that $K \cup H^* \cup \{\sim Y\}$ is contradictory. Define Z to be a universal sentence equivalent to the conjunction of elements of H^* (since H is conjunctive). Then $Z \in H$ and $K \vdash \sim [Z \vee \sim Y]$ which entails $K \vdash Z \supset Y$. This completes the proof.

If X coincides with Y, then $K \vdash Z$. In other words, if X is a sentence defined in a set of sentences K and X is persistent under restriction relative to K, there exists a universal sentence Z such that $K \vdash X \equiv Z$. Moreover, if X and Y are two sentences such that whenever X is true in M, then Y is true in a substructure M' of M that defines Y, then there exists a universal sentence Z whose vocabularies are contained either in X or Y such that $X \vdash Z$ and $Z \vdash Y$. To prove this, we define a set of sentences K to consist of a finite number of theorems of L containing all relations and individuals of X and Y and no others. For instance if a relation $A(x_1, ..., x_n)$ is in X or Y, we can include in K the sentence $\forall x_1 ... \forall x_n [A(x_1, ..., x_n) \vee \sim A(x_1, ..., x_n)]$. Then any structure M is a model of K given that all the vocabularies of X and Y are contained in it. The proof then reduces to the theorem above. As a corollary, if X is persistent under restriction, there exists a universal sentence Z whose vocabularies are contained in X such that $X \equiv X$.

A sentence that belongs to E_1 is called an existential sentence and it is persistent under extension. Now let sentences X and Y be defined in a set of axioms K such that whenever X is true in a model M of K, Y holds in all extensions of M that are models of K. Then there is an existential sentence Z such that $K \vdash X \supset Z$ and $K \vdash Z \supset X$. To prove the above, replace the sentence X and Y respectively with $\sim Y$ and $\sim X$. By the results above, there exists a universal sentence Z' that such that $K \vdash \sim Y \supset Z'$ and $K \vdash Z' \supset \sim X$. Then $K \vdash \sim Z' \supset Y$ and $K \vdash X \supset \sim Z'$. But $\sim Z'$ is equivalent to an existential sentence, which completes the proof. As a corollary, if X is a sentence defined in K persistent relative to K, there exists an existential sentence Z defined in K such that $K \vdash X \equiv Z$.

Theorem 7.2 *Let H be a set of sentences and X a sentence that holds in every extension M' of M that defines X. Then there exists a finite subset H' of H such that for any extension M' of a model M of H', X holds in M' given that X is defined in it.*

Proof: Define H_R to be set of sentences obtained after relativizing H with respect to a relation R not contained in either H or X. Since X holds in every extension of models of H, X holds in all models of H_R. Then X is deducible from a finite subset, call it H_1 of H_R. Define H_2 to be the finite subset of H whose transformations under ρ are included in H_1. For any individual a such that $R(a) \in H_1$, add to H_2 a sentence of H containing a and call the resulting set H'. Then X holds in all extensions of models of H' that defines X proving the theorem.

Theorem 7.3 *Suppose that the sentence X holds in all substructures M′ of the models M of a set of sentences H (X must be defined in M′. Then there is a finite subset H′ of H such that X holds in all substructures M′ of models of H′ where M′ defines X.*

Proof: Let X_R be the result of relativizing X. Then $H \cup \exists x R(x) \wedge \sim X_R$ is contradictory. For if it was consistent, $\sim X_R$ is true in a model M of H that contains some element a that satisfies $R(x)$. Then $\sim X$ would hold in the substructure $M′$ of M obtained by restricting the relations of M to the elements satisfying R. This contradicts the assumption and therefore, $H \cup \exists x R(x) \wedge \sim X_R$ must be contradictory. Then there exists a finite subset $H′$ of H such that $H′ \vdash [\exists x R(x)] \supset X_R$. This proves the theorem.

Theorem 7.4 *Let H be a set of sentences defined in a set of sentences K such that H is persistent under restriction relative to K. Then there exists a set of sentences J defined in K and all elements of class A_1 such that $K \cup H \vdash J$ and $K \cup J \supset H$.*

Proof: Let J be the set of all sentences defined in K where $K \cup H \vdash J$. Let $X \in H$. X must hold in any substructure $M′$ that is a model of $K \cup H$ where $M′$ is also a model of K. By the previous theorem, there exists a finite subset $H′$ of H such that X is true in all models of K that are substructures of models of $K \cup H′$. Define Z to be the conjunction of all elements in $H′$. Then there exists a universal sentence W defined in K such that $K \vdash Z \supset W$ and $K \vdash W \supset X$. Since Z was defined to be the conjunction of elements of H, $K \cup H \vdash W$ and W is in J. Hence if $K \vdash W \supset X$, then $K \cup J \vdash X$, which proves the theorem.

Theorem 7.5 *Let $Q(x_1, ..., x_n)$ be a predicate defined in a set of sentences K that is persistent relative to K. Then there exists an existential predicate $P(x_1, ..., x_n) \in E_1$ defined in K such that $K \vdash (\forall x_1)...(\forall x_n)[Q(x_1, ..., x_n) \equiv P(x_1, ..., x_n)]$.*

Proof: Let $a_1 \ldots a_n$ be a set of distinct individuals which do not appear in K and add to K a single theorem Y that contains individuals $a_1, ..., a_n$ (e.g. for any relation A, let the theorem $Y = [A(a_1, ..., a_1) \vee \sim A(a_1, ..., a_1)] \wedge [A(a_2, ..., a_2) \vee \sim A(a_2, ..., a_2] \wedge \cdots \wedge A(a_n, ..., a_n) \vee \sim A(a_n, ..., a_n)]$. Define $K′ = K \cup \{Y\}$, then $X = Q(a_1, ..., a_n)$ is defined in $K′$ and is persistent relative to both K and $K′$. Then there exists a sentence $Z \in E_1$ defined in $K′$, i.e. $Z = P(a_1, ..., a_n)$ where we only displayed the individuals not in K such that $K′ \vdash Q(a_1, ..., a_n) \equiv P(a_1, ..., a_n)$. Since $K′ = K \cup Y$, $K \vdash Y \supset [Q(a_1, ..., a_n) \equiv P(a_1, ..., a_n)]$. Since Y is a theorem, $K \vdash Q(a_1, ..., a_n) \equiv P(a_1, ..., a_n)$ and by rules of deduction, $K \vdash (\forall x_1)...(\forall x_n)[Q(x_1, ..., x_n) \equiv P(x_1, ..., x_n)]$. This completes the proof of the theorem.

8 Obstructions to Elementary Extension

We begin by recalling the definition of an elementary extension. A structure M is an elementary extension of $M′$ if M is a substructure of $M′$ and any sentence X whose individuals all belong to M either holds both in M and $M′$ or does not hold both in M and $M′$. Using the same notation M and $M′$ to represent a structure and its extension, $M′$ is said to *obstruct* M if there does not exist any extension of $M′$ that is an elementary extension of M. A structure M is obstructed by an extension $M′$ if and only if there exists a sentence $X \in A_1$ such that X holds in M but not in $M′$. To prove the $\Leftarrow$ statement, first define $H = S(M) \cup D(M′)$ where $S(M)$ is the set of all sentences that is true in M and $D(M′)$ is the diagram of $M′$. We claim that H is contradictory. For suppose that H is consistent and possesses a model M^*. Since M^* is a model of $D(M′)$, it is an extension of $M′$ but this contradicts the assumption of the theorem since there is $M′$ obstructs M. Hence H is inconsistent, and there is a finite subset $J \subset D(M′)$ such that $S(M) \cup J$ is inconsistent. Define $Y(a_1, ..., a_n)$ as the conjunction of all elements of J, where $a_1, ..., a_n$ are individuals in Y that do not belong to M. Then $S(M) \vdash \sim Y(a_1, ..., a_n)$. Since the individuals do not appear in $S(M)$, by rules of deduction, $S(M) \vdash (\forall x_1)...(\forall x_n)[\sim Y(x_1, ..., x_n)]$. Since $X = \forall x_1)...(\forall x_n)[\sim Y(x_1, ..., x_n)]$ is deducible from $S(M)$, X holds in M and it also belongs to $S(M)$. Now, $Y(a_1, ..., a_n)$ holds in both M and $M′$ since it is the conjunction of the elements of $D(M)$ and the sentence $(\exists x_1)...(\exists x_n)Y(x_1, ..., x_n)$ also holds

in both M and M'. But $\exists x_1)\ldots(\exists x_n)Y(x_1,...,x_n) = \sim X$ and therefore, X holds in M but not in M'. Now, we prove the other direction. Let $X = \forall x_1 \ldots \forall x_n Y(x_1,...,x_n)$ be the sentence in class A_1 as described in the statement, such that the matrix Y does not contain more quantifiers. Then $\sim X \equiv Z$, where $Z = \exists x_1 \ldots \exists x_n[\sim Y(x_1,...,x_n)]$ and Z must hold in M'. Since Z is an existential sentence, it holds in all extensions of M' but cannot hold in M since X is true in M. Hence, it follows that there cannot exist any extension of M' that is an elementary extension of M. This completes the proof the statement. Now suppose M is a structure and K' is a set of sentences. We say K' obstructs M if every model M' of K' that is an extension of M obstructs M.

Theorem 8.1 *A set of sentences K' obstructs the structure M if and only if there exists a sentence $X \in A_1$ such that X is defined and holds in M and $K' \vdash \sim X$.*

Proof: We prove the $\Leftarrow$ direction first. Let M' be an extension of M that is a model of K'. Since $\sim X$ is deducible from K', $\sim X$ holds in M'. On the other hand, X holds in M. Then by the previous statement, no extension of M' can be an elementary extension of M, proving the first direction. Now, we prove the other direction. Suppose for contradiction that there does not exist any sentence X as specified in the theorem. Define H to be the set of all universal sentences that hold in M. Then $K' \cup H$ must be consistent for if K' is inconsistent, the theorem is vacuously true, and if K' is consistent and $K\ cupH$ is inconsistent, $K' \vdash \sim [X_1 \wedge \ldots \wedge X_n]$ for some elements $X_1,...,X_n$ of H' but since there exists $Z \in A_1$ such that $Z \equiv [X_1 \wedge \ldots \wedge X_n]$, the sentence $simZ$ is deducible from K' which then satisfies the theorem. Then $K' \cup H$ is consistent and has a model, say M' which is an extension of M since the diagram of M is included in H (remember that the class A_1 also includes all atomic formulae). Now suppose that M' obstructs M. Then there exists a sentence $X \in A_1$ that is true in M but not in M'. Since X is true in M, $X \in H$ and hence must hold in M' which yields a contradiction. This completes the proof of the theorem.

A set of sentences K' obstructs a set of sentences K if there exists a model M of K such that K' obstructs M. It can be shown that K' obstructs K if and only if there exists a sentence $Y \in A_2$ such that Y is defined in K' and $K' \vdash Y$ while $K \cup \{\sim Y\}$ is consistent. The proof is provided on *Robinson, pp. 77* (there is an error on page 77. The part, $(\forall y_1)\ldots(\forall y_m)[\sim Z(a_1,\ldots,a_k,y_1,\ldots,y_m)]$ which is $\sim X$, should read, $\sim [(\forall y_1)\ldots(\forall y_m)[\sim Z(a_1,\ldots,a_k,y_1,\ldots,y_m)]]$ which is $\sim X$.

Say X is any sentence in A_2 and define $\{M_v\}$ to be a monotonic set of models of X. Suppose that $X = \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m Z(x_1,\ldots,x_n,y_1,\ldots,y_m)$ where Z has not quantifiers. Using Skolem functors, we can construct its open form as

$$Z(x_1,\ldots,x_n,\varphi_1(x_1,\ldots,x_n),\ldots,\varphi_m(x_1,\ldots,x_n)))$$

Now for every sequence of individuals $a_1,...,a_n$ in M, we can always find M_v which contains all the elements in the sequence. We can then select the values of the Skolem functors in M as any possible set of values of the Skolem functions in M_v. Since M_v is a model of X, such values always exists. This shows that any sentence in A_2 must hold in M that is the union of M_v's. A sequence of structures $\{M_n\}, n = 1,2,\ldots$ is said to be increasing M_k extends M_n for $n \leqslant k$. A sentence X (or a set of sentences, X) is called σ-*persistent* if whenever X holds in all elements of an increasing sequence $\{M_n\}$, it holds in $M = \bigcup_n \{M_n\}$. Any sentence $X \in A_2$ is σ-*persistent* for instance. We can also define σ-*persistent* with respect to a set of sentences K. A sentence X (or a set of sentences X) is σ-*persistent* with respect to K if the structures that consist the increasing sequence are models of K and X and their union is a model of K, then their union is also a model of X. Now follows an important theorem about σ-*persistent* and its relationship to the class A_2 of sentences.

Theorem 8.2 *If a set of sentences H is σ-persistent with respect to a set of sentences K_0, there is a set H' of sentences in A_2 such that H is equivalent to H' with respect to K_0, that is $H \cup K_0 \vdash H'$ and $H' \cup K_0 \vdash H$ (proof in Robinson, pp 79).*

Corollary 8.1 *If a sentence X is σ-persistent with respect to a set of sentences K_0, there exists $X^* \in A_2$ such that $K_0 \vdash X \equiv X^*$.*

Proof: Suppose H in the theorem above contains only one sentence X. Then according to theo-

rem, there is a set of sentences $H' \subset A_2$ such that $\{X\} \cup K_0 \vdash H'$ and $H' \cup K_0 \vdash X$. Then there must exist a finite subset $H^* \subset H'$ such that $\{X\} \cup K_0 \vdash H^*$ and $H^* \cup K_0 \vdash X$. Since A_2 is conjunctive, we can take the conjunction of all the elements of H^*, call it X^* and still get $X^* \in A_2$. We get $\{X\} \cup K_0 \vdash X^*$ and $\{X^*\} \cup K_0 \vdash X$, that is $K_0 \vdash X \equiv X^*$. If K_0 is empty moreover, we get the following corollary: if X is σ-*persistent*, then there exists $X^* \in A_2$ such that $X \equiv X^*$ is a theorem.

9 Model Consistency

Suppose K is a consistent set of sentences. A sentence X is then called *model-consistent* with K if for every model M of K, say M, $D(M) \cup K \cup \{X\}$ is consistent. That is, X is model-consistent with K if for every model M of K, there is an extension M' of M that satisfies both K and X (recall that M' is an extension of M if and only if it satisfies $D(M)$). A set of sentences H is consistent with K, if for every model M of K, $K \cup D(M) \cup H$ is consistent.

Theorem 9.1 *Say K is a set of consistent sentences in A_2. Define $X \in A_2$ as follows: $X = \forall y_1 \ldots \forall y_n Z(y_1, \ldots, y_n)$ where Z is an existential predicate. Then X is model-consistent with K if and only if for every model M of K and for every set of elements $a_1, \ldots, a_n$ in M, there is an extension M' of M that satisfies K such that $Z(a_1, .., a_n)$ holds in M'.*

Proof: First, we prove the ($\Rightarrow$) direction. Suppose a sentence X is model consistent with K. Then there exists an extension M' of M that satisfies K and X. Since $X \supset Z(a_1, ..., a_n)$ is a theorem, and the elements $a_1, ..., a_n$ is contained in M and M', $Z(a_1, ..., a_n)$ holds in M'.

Next we prove the ($\Leftarrow$) direction. Suppose that for every model M of K, there exists an extension M' of M that is a model of K such that the sentence $Z(a_1, ..., a_n)$ holds for every set of individuals $a_1, ..., a_n$ in M. Now consider any model M of K. Define S to be the set of all sentences $Z(a_1, \ldots, a_n)$ where the sequence $a_1, ..., a_n$ varies across all individuals in M. Define D to be the diagram of D. Then $K \cup D \cup S'$ is consistent where S' is any finite subset of S. To prove its consistency, write $S' = \{Z_1, ..., Z_m\}$ where $Z_j = Z(a_1^j, \ldots, a_n^h)$, $j = 1, \ldots, m$ where a_i^j are individuals contained in M. By the assumption of the theorem, the set $K \cup D \cup \{Z_1\}$ is consistent: let $M^{(1)}$ be its model and $D^{(1)}$ the diagram of $M^{(1)}$. Then by the assumption of the theorem again, $K \cup D^{(1)} \cup \{Z_2\}$ is consistent and has a model $M^{(2)}$ with diagram $D^{(2)}$. Repeating recursively, one can define a model $M^{(m)}$ of K that extends M and satisfies $\{Z_1, ..., Z_m\} = S'$. This proves that $K \cup D \cup S'$ is consistent. By compactness theorem, this shows that $K \cup D \cup S'$ is also consistent. Now define M_1 to be the model of $K \cup D \cup S$ with diagram D_1. Define S_1 to be the set of all sentences $Z(a_1, \ldots, a_n)$ with $a_1, \ldots, a_n$ in M_1. Then by the assumption of the theorem again, $K \cup D_1 \cup S_1$ is consistent and has a model M_2 with diagram D_2. In general, we can obtain a sequence of models of K, $T = \{M_1, M_2, \ldots, M_k, \ldots\}$ such that $M_k \supset M_l$ for $k \leqslant l$. Define $M^* = \cup_{\{}M_k\}$. Since K is σ-persistent, M^* is a model of K. Let $a_1, \ldots, a_n$ be the elements of M^*, then there exists an integer k such that $a_1, \ldots, a_n \in M_k$. By the way we constructed the union, $Z(a_1, \ldots, a_n)$ holds in M_{k+1} and in M_l for all $l > k$, hence in M^*. This shows that X holds in M^* that is a extension of M and a model of K. This completes the proof.

Let K be a consistent set of sentences of class A_2 and let H also be a set of sentences of class A_2, that is, $H = \{X_v\}$ where $(X_v = \forall y_1) \ldots (\forall y_n) Z_v(y_1, \ldots, y_{n_v}, Z_v$ is an existential predicate. Then H is model consistent with K if and only if for every model M of K and every Z_v and every set of elements $a_1, ..., a_{n_v}$ of M, there exists an extension M' of M that satisfies K and $Z_v(a_1, ..., a_{n_v}$. This theorem can be used to show that every field can be embedded in an algebraically closed field. Let $K = K_{CF}$ be the axioms of a commutative field. K then belongs to A_2 which is not difficult to verify. Let $H = \{Y_n\}$ be a sequence of sentences that claim that every monic polynomial of degree n possesses a root. Every sentence that makes up this sequence is also in A_2. Using the theorem above, to show that every commutative field can be embedded in an algebraically closed field, one only has to show that every monic polynomial with coefficients in a given field M possesses a root in some extension of the field. Hence the problem reduces to proving that every instance of a monic polynomial whose coefficients in M has a root in some extension of M.

10 Completeness and Model Completeness

A set of sentences K is called *complete* if for every sentence X defined in K, either X or $\sim X$ is deducible from K. If K is empty or contradictory, then it is trivially complete (if K is empty, there is no sentence defined in it).

Theorem 10.1 *If K is a non-empty, consistent set of sentences, then there exists an extension K' of K that does not contain any additional relation or individual, is consistent and complete.*

Proof: Define H to be the set of all sentences defined in K, and $J = \{K_\nu\}$ to be the set of all consistent subsets of H. Then K is an element of J since K is consistent by assumption. Now, J is partially ordered set, its order relation being inclusion, and every totally ordered subset of J, say J' has an upper bound, namely the union of the elements of J', denote it K_0: indeed K_0 is in J since if K_0 is not consistent, then some finite subset of K_0 is inconsistent which implies that some element of J' is inconsistent, yielding a contradiction. Then by Zorn's lemma, J must contain at least one maximal element, call it K^*. We claim that K^* is the extension that the theorem postulates its existence. Since K^* is a maximal element of J, it is by definition consistent and contains K. It only remains to show that K^* is complete. To prove this, it suffices to show that if a sentence $X \in H$ is not deducible from K^*, then $\sim X$ is deducible from K^*. Define $K_1 = K^* \cup \{X\}$. Since K^* is a maximal element, X is either contained in K^* or K_1 is not consistent. However, since X was assumed to be not deducible from K^*, it cannot belong in K^* and hence K_1 must be contradictory. Then there exists a finite subset of K^*, write it as $\{Y_1, \ldots, Y_m\}$ such that $\{Y_1, \ldots, Y_m, X\}$ is contradictory. Then both $\sim [Y_1 \wedge \cdots \wedge Y_m \wedge X]$ and $Y_1 \wedge \cdots \wedge Y_m \supset\sim X$ are theorems. This shows that $\sim X$ can be deduced from K^*, completing the proof of the theorem.

Consider a non-empty and consistent set of sentences, denote it K. Let X be some sentence defined in K such that either X or $\sim X$ is deducible from K. Then both sets $K_1 = K \cup \{\sim X\}$ and $K_2 = K \cup \{X\}$ is consistent. To show this, suppose K_1 is inconsistent. Then there exists a finite subset of K, write it as $\{X_1, \ldots, X_m\}$ such that $X_1 \wedge \cdots \wedge X_m \wedge X$ is inconsistent. This implies that $\sim X$ is deducible from K, that is K_2 is consistent. The same proof follows when we assume that K_2 is inconsistent. Then by the previous theorem, both K_1 and K_2 has complete extensions that are both complete extensions of K, but distinct from one another. The idea of completeness is important in math since if we construct two non-isomorphic structures that are models of the same complete set of sentences K, then every sentence that holds in one must also hold in the other. Now we prove a simple test for checking completeness.

Theorem 10.2 (Vaught's Test) *Let K be a consistent set of axioms including a relation of equality. Let α be the (finite or infinite) cardinality of K. If all the models of K are infinite (countably infinite or uncountable) and that for some infinite cardinal $\alpha' \geqslant \alpha$, all models of K are isomorphic, then K is complete.*

Proof: Let X be a sentence defined in K. Suppose, for sake of contradiction, that both X and $\sim X$ is not deducible from K. Then the sets $K_1 = K \cup \{\sim X\}$ and $K_2 = K \cup \{X\}$ are both consistent. Then both K_1 and K_2 have cardinal $\alpha + 1$ and $\alpha' \geqslant \alpha + 1$ since α' is infinite. By Löwenheim-Skolem Theorem, there exist models M_1 and M_2 respectively of K_1 and K_2 such that both models are of cardinal α'. Then M_1 and M_2 are isomorphic by the assumption of the theorem and must satisfy X and $\sim X$ simultaneously. This is a contradiction.

We look at some examples of a complete set of axioms that arise in mathematics. The first example is the set of axioms K for the concept of a densely ordered set without first or last element. K can be expressed by a finite set of sentences constructed with the relation of equality and the relation of order, respectively $E(x,y)$ and $Q(x,y)$. Since K is densely ordered, its models must be infinite. Georg Cantor proved that all countable, unbounded densely ordered sets are isomorphic (one can prove this by enumerating the elements of sets A as $a_1, a_2, \ldots$ and B as $b_1, b_2, \ldots$, pairing a_1 with b_1 and then any unpaired element in A with the smallest index with an arbitrary unpaired element in B such that the ordering is consistent, repeating the process for any unpaired element in B with the smallest index, and repeatedly alternating the process between A and B which yields a well ordered

isomorphism). Then by Vaught's test, K is complete, that is the concept of a densely ordered, unbounded set is complete. It is not difficult to show that densely ordered sets bounded from above, bounded from below, or bounded from both are also complete.

Another example is the set of sentences $\bar{K}_F^{(p)}$, that is the axioms for an algebraically closed field of characteristic p. By the Steiniz theorem, any two algebraically closed fields of the same characteristic and same uncountable cardinal are isomorphic. We can apply the Vaught's test to conclude that the concept of an algebraically closed field with a specified characteristic is complete. Once we know $\bar{K}_F^{(p)}$ is complete, then any theorem X formulated in first order logic for the complex field holds for all other algebraically closed fields of characteristic 0. Moreover, theorem X holds in all algebraically closed fields of characteristic $p > p_0$ where p_0 depends on X which follows from section 4 of the paper.

We define the notion of *model-completeness* which is modified from the notion of *completeness*. Define K to be a non-empty and consistent set of sentences. K is model-complete if for every model M of K, the set $K \cup D$ is complete, where D is the diagram of M. We can reformulate the definition of model completeness to mean for any model M of K, any extension M' that is a model of K is an elementary extension of M. These two formulations are logically equivalent. If $K \cup D$ is consistent and M and $M' \supset M$ are models of K, then M' must satisfy both K and D whereas $K \cup D$ is complete. Then for any sentence X defined and true in M, X is also defined in $K \cup D$ and is deducible from $K \cup D$. As a result, X must always hold in M' showing that M' is an elementary extension of M. To prove the other direction, suppose that for every model M of K and extension M' of M that satisfies K, M' is an elementary extension of M. Suppose, for sake of contradiction, $K \cup D$ is not complete. Then there exists a sentence X defined in $K \cup D$ such that neither X nor $\sim X$ is deducible from $K \cup D$. Either X or $\sim X$ must hold in M since X is defined in M. Without loss of generality, suppose X holds in M. If X is not deducible from $K \cup D$, then $K \cup D \cup \{\sim X\}$ is consistent and has a model M'. But M' is an extension of M and by assumption must be an elementary extension of M, implying that X must hold in M'. This yields a contradiction, completing the proof of equivalence of two definitions of model-completeness.

Next we try to find the necessary and sufficient conditions of model-completeness. In order to do so, we first define a the term *primitive*. A well-formed formula is *primitive* if it is in prenex normal form of class E_1 where its matrix consists of either an atomic formula or the negation of an atomic formula, and their conjunction. For instance, if we take a number of elements of a diagram of a model in conjunction, we can replace the individuals in the conjunction by variables and add the corresponding existential quantifiers to obtain a primitive sentence. Now we prove the following theorem.

Theorem 10.3 (Model Completeness Test). *Define K to be a non-empty, consistent set of sentences. Then K is model-complete if and only if for any two models M and M' of K such that $M' \supset M$, any primitive sentence defined in M holds in M' only if it holds in M.*

Proof: We prove ($\Rightarrow$) first. If K is model complete, then every extension M' of M, where both structures are models of K, is an elementary extension of M. Hence any sentence defined in M holds in M' if and only if it holds in M.

Next, we prove ($\Leftarrow$) direction which is more complicated. Define $S = \{(M,X)\}$ as the set of all ordered pairs where M is a model of K with diagram D and X is a sentence defined in M such that neither X nor $\sim X$ is deducible from $K \cup D$. If K is not model-complete, then S cannot be empty by definition. Moreover, S must contain pairs in which the second element is in prenex normal form since again, every sentence is logically equivalent prenex normal form that contains the same relations and individuals. Furthermore, the second member of the ordered pair must contain at least one quantifier since if it did not contain any quantifier and held in M, then it holds in all extensions of M (by the conventional truth table in propositional calculus) and hence is deducible from D, a fortiori from $K \cup D$. We can hence conclude that if S is not empty, it must contain some pair whose second member is a prenex normal form with at least one quantifier. Indeed, we can argue that the member must begin with an existential quantifier, since if (M,X) is in S, $M, \sim X$ is in S. If X

begins with a universal quantifier, then $\sim X$ is equivalent to a sentence that begins with an existential quantifier. Hence, if we define $S' = (M,X)$ to be a set of ordered pair where M is a model of K, X is a sentence in prenex normal beginning with an existential quantifier and defined in K such that neither X nor $\sim X$ is deducible from $K \cup D$, then S' is nonempty if K is not model complete. Now suppose for sake of contradiction, that K is not model complete. Let M, X_0 be an element in S' such that the number of quantifiers of X_0 is a minimum. Write $X_0 = (\exists z)Q(z)$ where Q has one less quantifier that X_0. Since neither X_0) nor $\sim X_0$ is deducible from $K \cup D$, the set $K \cup D \cup \{X_0\}$ is consistent and has a model, call it $M'.M'$ is a model of K and an extension of M where $Q(a)$ holds for some individual a in M'. Since $Q(a)$ contains less quantifier than X_0, the pair $(M', Q(a)$ does not belong in S'. Since $Q(a)$ is defined in M', either $Q(a)$ or $\sim Q(a)$ is deducible from $K \cup D'$ where $D' = D(M')$. Yet, $\sim Q(a)$ is not deducible from $K \cup D'$ since $Q(a)$ is true in M' while M' is a model of $K \cup D'$. Therefore, $Q(a)$ is deducible from $K \cup D'$. Also $Q(a) \supset [(\exists z)Q(z)]$ is a theorem and hence the sentence X_0 is deducible from $K \cup D'$. In other words, there exists a conjunction Y of elements of D', hence free of quantifiers, such that $K \vdash Y \supset X_0$. Write $Y = Y(a_1, \ldots, a_m)$ where we displayed the individuals that do not belong to M. Since all individuals of K belong in M, the individuals $a_1, \ldots, a_m$ do not appear in K. We claim that m is positive. If there were no individuals in Y that do not belong in M, then Y is just a conjunction of elements of $D = D(M)$ and hence deducible from D. But since $Y \supset X_0$ is deducible from K, X_0 is deducible from $K \cup D$ which contradicts the way we defined X_0. $K \vdash Y \supset X_0$ implies that $Z \wedge Y \supset X_0$ is a theorem for some Z that is a conjunction of finite number of elements of K. Then $y(a_1, \ldots, a_m) \supset [Z \supset X_0]$ is a theorem and by rules of deduction, $[(\exists y_1) \ldots (\exists y_m)Y(y_1, ..., y_m)] \supset [Z \supset X_0)]$ is also a theorem. If we let $X_1 = (\exists y_1) \ldots (\exists y_m)Y(y_1, ..., y_m)$, then $Z \supset [X_1 \supset X_0]$ is also a theorem, thereby $K \vdash [X_1 \supset X_0]$ since Z was defined to be a conjunction of finite number of elements of K. Now we claim that (M, X_1) belongs to S'. To check whether it (M, X_1) satisfies the conditions, first X_1 is defined in M and in prenex normal form beginning with an existential quantifier and second, X_1 is not deducible from $K \cup D$ since if it were, then X_0 is deducible from $K \cup D$ (because $K \vdash [X_1 \supset X_0]$ which is a contradiction, and $\sim X_1$ is also not deducible from $K \cup D$ since X_1 is true in M' which is a model of $K \cup D$. Notice that X_1 is primitive since its matrix consists of conjunctions of elements of diagrams some of whose individuals are replaced with variables. Hence by the assumption of the theorem, X_1 can hold in M' only if it holds in M. Since X_1 is existential, it then holds in all extensions of M and hence is deducible from D. This contradicts our conclusion above that X_1 is not deducible from $K \cup D$. This completes the proof of the model-completeness test.

We define a model M_0 of K to be a prime model of K if every model M of K has a substructure that is isomorphic to M_0. We consider some examples. If K is a set of axioms for a densely ordered set, then any countable densely ordered set is a prime model of K. If K is a set of axioms for the notion of a commutative field of characteristic 0, then its prime model is the field of rational numbers.

Theorem 10.4 *Let M_0 be a prime model of a set of sentences K. Define D_0 to be the diagram of M_0. Then any sentence X defined in K and deducible from $(K \cup D_0)$ is deducible from K only. (proof provided in Robinson, pp.95).*

Theorem 10.5 (Prime Model Test). *Let K be a model-complete set of sentences which possesses a prime model. Then K is complete.*

Proof: Let M_0 be a prime model of K, and let X be any sentence which is defined in K. Since X is defined in M_0, either X holds or does not hold in M. Suppose without loss of generality that X is true in M_0. Since K is model-complete, X is deducible from $K \cup D_0$. Since M_0 is a prime model of K, we can conclude from the previous theorem that X is deducible from K alone.

Theorem 10.6 Suppose that any two models M_1, M_2 of the model-complete set of sentences K which only have individuals that appear in K in common can be embedded in a joint extension, M which satisfies K. Then K is complete.

Proof: Suppose K is not complete. Then there exists some sentence X such that $K \cup \{X\}$ and $K \cup \{\sim X\}$ are both consistent and has models respectively M_1 and M_2. We can choose M_1 and M_2 so that they only share individuals of K in common. K satisfies the condition of the theorem, there

exists a model M of K which is a joint extension of M_1 and M_2. But K is model-complete and since $\sim X$ holds in M_1, it entails that it holds in M. But since X holds in M_2 entails that X also is true in M. This yields a contradiction, showing that K is complete.

Now we cover some examples. Let M and M' be two densely unbounded ordered sets (formulated only in terms of the relations of equality and order). Define X to be any primitive sentence also formulated in terms of the equality relation E and order relation Q, consisting of individuals of M such that X is true in M'. To show that K is model complete, we have to show that this can happen only if X holds in M. Define $X = (\exists y_1)\ldots(\exists y_n)Z(y_1,\ldots,y_n)$ where Z is a matrix without any further quantifiers. Assuming that the predicate $Z(y_1,...,y_n)$ is satisfied by some set of individuals that belong to M' (i.e. satisfied by a point in space M'^n, we have to show that this predicate holds for some set of individuals in M. In mathematical language, this means that a finite system of equations and inequalities of the form $\alpha = \beta, \alpha \neq \beta, \alpha \leqslant \beta, \alpha > \beta$ where the second and fourth expressions are negations of the preceding expression and α, β denoting individuals in a structure, already possesses a solution $y_i = b_i, i = 1, 2, \ldots, n$ in M' and we have to show that it then should possess a solution in M.

We begin by reducing the list of equations and inequalities. If the system (that is, the primitive sentence) has a relation of the kind $\alpha \neq \beta$, then this relation can be replaced either by $\alpha < \beta$ or $\beta < \alpha$ depending on whether which of the two relation holds in M' for $y_i = b_i, i = 1,...,n$. If a relation of the kind, $\alpha \leqslant \beta$ occurs in the system, we can replace this by $\alpha < \beta$ or $\alpha = \beta$ depending again on which of the two relations holds in M'. Last but not least, we can get rid of all relations of the form $\alpha = \beta$ replacing the symbol on the right had side of the equation β with the symbol on the left hand side whenever if appears in the system. Hence, we can confine ourselves to a finite system of inequalities of the form, $\alpha_j < \beta_j, j = 1,...,m$ where α_j, β_j are either elements of M, a_i or unknowns y_i where it is given that the inequalities are satisfied by some $y_i = b_i$ in M'. Now we finally arrive at a situation in which all a_i are different from one another and the b_i are different from the a_i and from one another. Now order the a_i that occur in the system such that $a_1 < a_2 < \cdots < a_k, k \geqslant 0$. These elements in turn divide the elements of M' into $k+1$ intervals, call them $I'_0, I'_1, \ldots, I'_k$ that are respectively given by $y \leqslant a_1, a_1 < y \leqslant a_2, \ldots, a_k < y$. The solutions $y_i = b_i$ must be in the interior of one of the k intervals, so suppose that an interval I'_j contains a set of elements b_i of M', e.g. $b_1^{(j)} < \cdots < b_{l_j}^{(j)}$ where $a_{j-1} < b_1^{(j)}$ if I'_j possesses a left end point and $b_{l_j}^{(j)} < a_j$ if I'_j possesses a right-end point. Let the list $(c_1^{(j)}, \ldots, c_{l_j}^{(j)})$ be any set of elements in the interior of the interval I_j that belong to M such that $c_1^{(j)} < \cdots < c_{l_j}^{(j)}$. We can always come up with such a list of elements for each interval since M must be a densely ordered set. Now, map the set $S' = \{a_i, b_m^{(j)}\}, i = 1, \ldots, k, j = 0, \ldots, k, m = 1, \ldots, 1_j$ to the set $S = \{a_i, c_m^{(j)}\}$ by the correspondence $a_i \leftrightarrow a_i$ and $b_m^{(j)} \leftrightarrow c_m^{(j)}$. Then the correspondence preserves the order and the individuals c_i all belong to M. We may conclude that the concept of a densely ordered unbounded set without first or last element is model complete.

Corollary 10.1 The concept of a densely ordered set without first or last element is complete (this can be proved using Vaught's test as done previously, but it also follows from the fact that the concept is model-complete and has a prime model, which is any countable densely ordered set without first or last element).

Next, we consider a set of axioms K that represents the notion of a discretely ordered set with first but no last element, formulated with the relation $E(x,y)$ and $Q(x,y)$. We will define K to include the axioms of equivalence for the relation E, the axiom of substitutivity and the following sentences.

$$(\forall x)(\forall y)(\forall z)\left[(Q(x,y) \wedge Q(y,z)) \supset Q(x,z)\right]$$
$$(\forall x)(\forall y)\left[(Q(x,y) \wedge Q(y,x)) \supset E(x,y)\right]$$
$$(\forall x)(\forall y)\left[(Q(x,y) \vee Q(y,x))\right]$$
$$(\exists x)(\forall y)\left[Q(y,x) \supset E(x,y)\right]$$
$$(\forall x)(\exists y)(\forall z)\left[(Q(x,y) \wedge \neg E(x,y) \wedge (Q(x,z) \wedge Q(z,y)) \supset E(x,z) \vee E(z,y))\right].$$

The fourth sentence claims that there is a first element in the set (i.e. the smallest element) while the fifth sentence claims that there is an immediate successor to each element. We claim that the set K

is empty. The proof is similar to other proofs, but one must notice that K is not model-complete: for instance, the sentence $(\forall y)[Q(y,a) \supset E(a,y)]$ may hold in a model M but not in an extension M'. Hence we consider another set K' where the last two sentences are replaced with the followings (we introduce an individual a and a two-place predicate $S(x,y)$ which means that y is the successor of x).

$$(\forall y)\,[Q(y,a) \supset E(a,y)]$$
$$(\forall x)(\forall y)(\forall z)\,[(S(x,y) \wedge S(x,z) \supset Q(x,y) \wedge \neg E(x,y) \wedge E(y,z))$$
$$\wedge\,(S(x,y) \wedge Q(x,z) \wedge Q(z,y) \supset E(x,z) \vee E(z,y))]$$
$$(\forall x)(\exists y)S(x,y)$$

The proof proceeds to show that K' is both model-complete and complete (using the prime model test, where the prime model is any ordered set of cardinal ω). The details are provided in *Robinson, pp.98-100*. We continue our proof from then, that is we show that the completeness of K' entails the completeness of K. Let X be any sentence defined in K. Then by the construction of K', X is also defined in K'. Without loss of generality, assume that X is deducible from K'. Denote the last three axioms of K' (enumerated above) as Z_1, Z_2, Z_3 and denote the conjunction of the rest of the sentences as W. Since all the sentences belong to K, $K \vdash W$ and $Z_1 \wedge Z_2 \wedge Z_3 \wedge W \supset X$ is a theorem. Then equivalently, $Z_1 \supset [Z_2 \wedge Z_3 \wedge W \supset X]$ are theorems. Since a does not appear in the square brackets, by existential generalization, we can replace Z_1 with $Z_1^* = (\exists x)(\forall y)[Q(x,y) \supset E(x,y)]$ so that $Z_1^* \supset [Z_2 \wedge Z_3 \wedge W \supset X]$ and $Z_2 \wedge Z_3 \supset [Z_1^* \wedge W \supset X]$ are theorems. Now, for sake of contradiction, suppose there is a model M of K where X does not hold (i.e. X is not deducible from K). We introduce a new relation $S(x,y)$, defining $S(b,c)$ to hold between elements b,c of M if and only if

$$[Q(b,c) \wedge \neg E(b,c) \wedge ((\forall z)\,[Q(b,z) \wedge Q(z,c) \supset E(b,z) \vee E(z,c)])]$$

holds in M. Let M' be the model with the new relation S. Then Z_2 and Z_3 hold in M' and Z_1^* and W hold in M' since they hold in M. It then follows that M' is a model of K' and hence X holds in M' since X is deducible from K'. But this is a contradiction since we have defined M to be a a model where X does not hold and any sentence defined in M that holds in M' must hold in M by construction. Hence, X must also be deducible from K and K must be complete. This is the end of the proof. Now we explore how the notion of completeness and model-completeness applies to algebraically closed fields.

Let K be a set of axioms for the concept of an algebraically closed field. We claim that K is model complete. Let M be an algebraically closed field of arbitrary characteristic and M' a proper extension of M. Let X be a primitive sentence defined in M that holds in M'. We must show that X must hold in M. The axiom of an algebraically closed field is formulated in terms of the equality relation $E(x,y)$ addition relation $S(x,y)$, and multiplication relation $P(x,y)$. Hence the primitive sentence is a finite system T of equations and inequalities of the type

$$\alpha = \beta, \alpha \neq \beta, \alpha + \beta = \gamma, \alpha + \beta \neq \gamma, \alpha\beta = \gamma, \alpha\beta \neq \gamma$$

In the system, α, β, γ just refer to either individuals $a_i, i = 1, \ldots, m$ that appear in M or variables $y_i, i = 1, \ldots, m$. We are assuming that the system T is satisfied by elements $y_i = b_i$ of M'. Now consider the algebraic closure M^* of the field $M(b_1, \ldots, b_m)$. The degree of transcendence (transcendental degree) of M^* over M is at most m and the system is also satisfied in M^* since it contains all of the individuals b_i. If $b_1, \ldots, b_m$ all belong to M, then $M^* = M$ and the proof is complete. If this is not the case, there is a chain of $l + 1$ algebraically closed fields,

$$M = M_0 \supset M_1 \supset \ldots \supset M_l = M^*, l \leqslant m$$

such that each field is of transcendental degree 1 over its predecessor. Again, if T has a solution in M_0, the proof is over. If not, then there are two fields $M_i \supset M_{i+1}$ where M_i does not have a solution for T but M_{i+1} does (i.e. the primitive sentence X is true in M_{i+1} but not in M_i. Since X is then defined in M_i, we can replace M and M' with M_i and M_{i+1} where the degree of transcendence of M_{i+1} over M_i is 1 since the chain of fields will eventually show that X must also hold in M. In other words, we will consider the special case where the degree of transcendence of M' over M is 1. Then every algebraically closed field M'' that is a proper extension of M will contain a subfield

that is isomorphic to M' under a correspondence that fixes the elements of M and X will hold in M''. Hence if c is an individual that does not belong in M, then X must be deducible from the set of sentences $K' = K \cup D(M) \cup H$ where H is defined as follows: for any polynomials $p(x)$ whose coefficients are in M, pic a predicate $Q_p(x)$ defined in M that states that $p(x) = 0$ (this is well defined in M since the predicate can be formulated with the relations Q, S and E) and let H be the set of all sentences $\sim Q_p(c)$ for any polynomials $p(x)$ that is not a zero function. H in other words states that the individual c is transcendental over M. Every model of K' is then of transcendental degree 1 over M and by our construction must satisfy X. In other words, X is deducible from K' and hence from $K \cup D(M)$ and a finite subset of H, e.g. $\sim\sim Q_{p_1}(c), \ldots, \sim Q_{p_j}(c)$. This finite subset can be replaced with one sentence, $Q_{p_0}(c)$ where $p_0(x) = p_1(x) \ldots p_j(x)$. Then we have $K \cup D \vdash \sim Q_{p_0}(c) \supset X$ which implies $K \cup D \vdash [(\exists x)[\sim Q_{p_0}(x)]] \supset X$. Both K and D hold in M and therefore the sentence $(\exists x)[\sim Q_{p_0}(x)]$ holds in M. This shows that $(\exists x)[\sim Q_{p_0}(x)]$ holds in M. As a result X must hold in M which completes the proof that the concept of an algebraically closed field is model complete.

11   Tarski's Theorem

Finally we cover the proof for Tarski's theorem that the concept of a real-closed ordered field is both model-complete and complete. Let K be a set of real-closed ordered field, that is an ordered field that is both formally real and it has no algebraic extension that is formally real. Also recall that an ordered field is real-closed if every monic polynomial with odd degree whose coefficients belong to the field possesses a root in the field and every positive element of the field has a square root in the field. The set of axioms, K is formulated with the relations of equality, addition, multiplication, and order, respectively, E, S, P, Q. We will first prove that a real-closed ordered field is model-complete. To do so, we first prove an auxiliary lemma.

Lemma 11.1 *If M is a real-closed ordered field, and $M(b)$ is a simple transcendental ordered extension of M, then the order of the field $M(b)$ is determined uniquely by the set of relations $b \leqslant a_v$ or $a_v \leqslant b$ that hold in $M(b)$ where a_v varies over all elements in M.*

Proof: To prove this theorem, it will suffice to show that the sign of any nonzero element in $M(b)$ can be determined by the set of relations $b \leqslant a_v$ or $a_v \leqslant b$ (then given any two elements in $M(b)$, one can subtract one from the other and check the sign of the result to confirm the order relation). Recall in abstract algebra that if b is algebraically independent over M (which in this case it is), then the transcendental extension field is isomorphic to the field of rational functions over M with one variable. Hence for any arbitrary $t \in M(b)$,

$$t = a \frac{a_0 + a_1 b + a_2 b^2 + \cdots + b^k}{a'_0 + a'_1 b + a'_2 b^2 + \cdots + b^m}$$

where $a, a_0, a_1, \ldots, a'_1, a'_2, \ldots$ are all contained in M. Both the numerator and the denominator of t can then be decomposed into linear and quadratic factors

$$t = a \frac{\Pi(b - b_j)\Pi((b - b_j)^2 + c_j^2)}{\Pi(b - b'_j)\Pi((b - b'_j)^2 + c_j^2)}$$

where b_j, b'_j and c_j belong to M. The quadratic factors are all positive, while the sign of the linear factors is determined by the set of relations $b \leqslant a_v$ or $a_v \leqslant b$. They altogether determine the sign of t, completing the proof of the theorem.

Now let X be a primitive sentence defined in a real-closed ordered field M which is equivalent to a system T of relations of forms $\alpha = \beta, \alpha \neq \beta, \alpha \leqslant \beta, \alpha > \beta, \alpha + \beta = \gamma, \alpha + \beta \neq \gamma, \alpha\beta = \gamma, \alpha\beta \neq \gamma$ where α, β, γ are either elements a_i that belong in M or unknowns y_i. We must show that if T has a solution in some extension M' of M, it must possess a solution in M. Similar to the one we used in the proof of the model-completeness of algebraically closed fields, we may assume that the transcendental degree of M' over M is one. Let b be an arbitrary element of M' that does not belong to M. Then M' is the real closure of the field $M(b)$. Define H to be the set of sentences $\sim E(b, a_v), Q(b, a_v), Q(a_v, b)$ that hold in M and where a_v varies over all elements of

M. Let $K' = \bar{K}_{OF} \cup D \cup H$ where $D = D(M)$. If M'' is a model of K' and M^* is the real closure of $M(b)$ in M'', then M^* is isomorphic to M' by the subtheorem. Hence, X holds in M^* and M''. Then X is deducible from K' and a fortiori from $\bar{K}_{OF} \cup D$ and a finite subset of H, say $H' = \{Y_1, ..., Y_h\}$. Let $Y(b) = Y_1 \wedge \cdots \wedge Y_k$ where the the individual b which occurs in Y has been displayed (and all other individuals must belong to M). Then $\bar{K}_{OF} \cup D \vdash Y(b) \supset X$. By rules of deduction, $\bar{K}_{OF} \cup D \vdash [(\exists y)Y(y)] \supset X$. Hence the proof reduces to showing that the primitive sentence $Z = (\exists y)Y(y)$ holds in M. Let Z refer to another particular system of relations, T'. Unlike T, T' contains only a single unknown, that is y, and contains only relations of either $\alpha \leqslant \beta$ or $\alpha \neq \beta$ (rigorously speaking, it contains relations of types $\alpha \leqslant \beta$, $\alpha \geqslant \beta$, $\alpha \neq \beta$ but the first and the third is equivalent). Since T' is satisfied in M' by $y = b$, we can reduce the relations to those of the type $\alpha < \beta$]. Now we have a system of the form

1. $a_i < x, i = 1, \ldots, j$

2. $x < a_i, i = j+1, \ldots, l$

where a_i are all different from one another and follows the given order in M while x is the unknown. The system possesses a solution in M', and this can be divided into three cases. If the solution in M', lies between a_j and a_{j+1}, that is both (1) and (2) are nonempty, then $x = (a_j + a_{j+1})/2$ is a solution in M. If (2) is empty, then $x = a_j + 1$ is a solution in M'. Finally, if (1) is empty, $x = a_1 - 1$ is the required solution. This completes the proof that $\bar{K}_{OF}$ is model-complete. Every closed field has a substructure isomorphic to the field of algebraic real numbers. Hence, the prime model test shows that it must also be complete.

Theorem 11.1 *Real-closed fields are model-complete.*

Theorem 11.2 (Tarski) *Real closed-fields are complete.*