Presentation Notes for MAT 355: Differential Geometry

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1 Introduction

The presentation will be a gentle introduction into the proof of Poincare's conjecture for dimension ≥ 6. The proof requires a working knowledge of manifolds, algebraic topology, in particular higher homotopy and homology groups, and assumes the h-cobordism theorem.

2 Poincare's Conjecture

Generalized Poincare's Conjecture Every closed n-manifold (compact manifold without boundary) that is homotopy equivalent to the n-sphere (homotopy n-sphere) in the chosen category (i.e. topological manifolds, PL manifolds, or smooth manifolds) is isomorphic in the chosen category (i.e. homeomorphic, PL-homeomorphic, or diffeomorphic) to the standard n-sphere, S^n.

For more precision, we define a manifold with boundary rigorously. The prototype of a manifold with boundary is the closed upper half-space

$$\mathcal{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n | x^n \geq 0\}$$

with the subspace topology inherited from $\mathbb{R}^n$. The points $(x^1, ..., x^n)$ where $x^n > 0$ are called the interior points of $\mathcal{H}^n$ and the points with $x^n = 0$ are called the boundary points of $\mathcal{H}^n$. These two sets are denoted by $int(\mathcal{H}^n)$ and $\partial(\mathcal{H}^n)$. The upper half plane then serves as model for manifolds with boundary. We say that a topological space M is locally $\mathcal{H}^n$ if every point $p \in M$ has a neighborhood U homeomorphic to an open subset of $\mathcal{H}^n$ (equipped with subspace topology). A topological n-manifold with boundary is a second, countable, Hausdorff topological space that is locally $\mathcal{H}^n$. For $n \geq 2$, a chart on M is defined to be a pair (U, ϕ) that consists of an open set U in M and a homeomorphism $\phi : U \to \phi(U) \in \mathcal{H}^n$ of U with an open subset $\phi(U)$ of $\mathcal{H}^n$. When $n = 1$, we have to allow two local models, the right half plane $\mathcal{H}^1$ and the left half plane $\mathcal{L}^1$. Then a chart (U, ϕ) in dimension *1* consists of an open set U in M and a homeomorphism ϕ of U with an open subset of $\mathcal{H}^1$ or $\mathcal{L}^1$. A collection $\{(U, \phi)\}$ of charts is a C^∞ atlas if for any two charts $(U\phi)$ and V, ψ, the transition map

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \subset \mathcal{H}^n$$

is a diffeomorphism. A C^∞ manifold with boundary is a topological manifold with boundary together with a maximal C^∞ atlas. A point p of M is called an interior point if in some chart (U, ϕ), the point $\phi(p)$ is an interior point of $\mathcal{H}^n$. Likewise, p is a boundary point of M if $\phi(p)$ is a boundary point of $\mathcal{H}^n$. The definition is independent of the charts since if we assume that (V, ψ) is another chart on M, then the diffeomorphism $\psi \circ \phi^{-1}$ maps $\phi(p)$ to $\psi(p)$. The result follows from the fact that if U and V are open subsets of the upper half-space $\mathcal{H}^n$ and $f : U \to V$ a diffeomorphism, then f maps interior points to interior points and boundary points to boundary points.

Poincare's Conjecture (Stephen Smale) For $n \geq 6$, any simply connected, closed n-manifold M whose homology groups $H_p(M)$ are isomorphic to $H_p(S^n)$ for all $p \in \mathbb{Z}$ is homeomorphic to S^n.

Note that the version above is a stronger theorem than GPC. But we do not that every closed topological manifold of dimension ≥ 6 is homeomorphic to a CW complex (found in Kirby-Siebenmann).

Moreover, one version of Whitehead's theorem states that a homology equivalence between simply connected CW complexes is a homotopy equivalence. (If a closed n-manifold is a homotopy sphere, then its fundamental group is obviously trivial and the homology groups are isomorphic).

Dimension 3 was proved by Grigori Perelman, dimension 4 was proved by Michael Freedman, and dimension ≥ 5 was proved by Stephen Smales, who all won the Fields Medal. It turns out that the conjecture is true in **Top** in all dimensions, true in **PL** in dimensions other than 4, unknown in dimension 4, and false generally in **Diff** with the first counterexample found in dimension **7** (by John Milnor who proved the existence of exotic 7-spheres that are homeomorphic but not diffeomorphic to the 7-sphere).

3 Prerequisites

We assume a working knowledge of homotopy and isotopy.

Definition 3.1. Path Homotopy Let X be a topological space. A path is a continuous function $f : [0,1] \to X$. Let $I = [0,1]$ be an index set, such that $t \in I$ is an index for a family of path functions $f_t : [0,1] \to X$. A path homotopy is then a continuous function $F : [0,1] \times [0,1] \to X$ where $F(s,t) = f_t(s)$ is continuous (both along s and t) and $f_t(0) = x_0, f_t(1) = x_1$ are fixed. The notion captures the intuition behind continuously deforming two paths with fixed starting and ending point.

Defintion 3.2 This definition extends to the idea of a homotopy between two continuous functions f and g that share the same domain and codomain. A homotopy between continuous maps $f, g : X \to Y$ is a continuous function $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x), H(x,1) = g(x)$ for all $x \in X$. If there is a homotopy between f and g, we write $f \simeq g$.

Definition 3.3 An isotopy is a homotopy between topological (or smooth) embeddings. A diffeotopy is a homotopy $F : M \times I \to M$ such that M is a manifold and each $f_t$is a diffeomorphism. From now on, we will denote two diffeomorphic spaces X and Y as $X \cong Y$.

Definition 3.4 Two topological spaces X, Y are **homotopy equivalent** if there exists continuous maps $f : X \to Y, g : Y \to X$ such that $f \circ g \simeq \mathbb{1}_Y$ and $g \circ f \simeq \mathbb{1}_X$. We then write $X \simeq Y$. This notion is slightly weaker than homeomorphism (counterexamples: a line and a point is homotopy equivalent but not homeomorphic (one direction is trivial, the other direction, just consider $f_t = (1-t)x$) / disk with a hole and a circle are homotopy equivalent but not homeomorphic). Intuitively, homotopy equivalence captures the idea of deforming / squashing a space. Homotopy equivalence is yet a powerful tool for classifying surfaces (preserves homotopy groups and homology groups).

Lemma 3.1 The connected components of a locally path-connected space are the same as its path-connected components.

Proof: We use the fact that the path connected components of a locally path connected space X are always open in X. Since path connectededness implies connectedness, one direction is obvious. It remains to check that every connected component is path connected. BWOC, suppose a connected component is not path connected. Then, the connected component is covered by path connected components, each of which is open in X. This leads to a contradiction.

Definition 3.5 The n-th homotopy group of a space X with basepoint x_0, denoted $\pi_n(X, x_0)$ refers to the group whose elements are based homotopy classes of maps $f : S^n \to X$. Equivalently, we may define $\pi_n(X, x_0)$ to be the group of homotopy classes of maps $f : [0,1]^n \to X$ such that $f(\partial[0,1]^n)$ gets mapped to the same point.

Extending the definition, $\pi_0(X, x_0)$ will then be the set of path connected component (in particular, if the space is locally path connected, then it is the set of all connected components).A space X is simply connected if $\pi_1(X), \pi_0(X)$ is trivial and n-connected if $\pi_i(X)$ is trivial for all $i \leq n$. Higher fundamental groups are algebraic groups but the proofs are non-trivial. We take this for granted.

Definition 3.6 A relative nth homotopy group of a space X with subspace X' and basepoint $x \in X'$ consists of elements that are equivalence classes of maps $f : D^n \to X$ under based homotopy such that $f(\partial D^n) = f(S^{n-1}) \subset X'$ and the base point $x_0 = f(y)$ for some fixed $y \in S^{n-1}$. The nth relative homotopy group is denoted $\pi_n(X, X')$ or $\pi_n(X, X', x_0)$. It trivially follows that the relative homotopy group $\pi_n(X, x_0, x_0)$ is equivalent to $\pi_n(X, x_0)$ since a map that identifies $\partial D^n \equiv S^{n-1}$ to a point is essentially a map with domain S^n.

Every continuous map between topological spaces $f : X \to Y$ induces a map $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$. $\pi_n(f)$ essentially sends the image of S^n in X to Y via composition with f. Induced maps are well defined since if $H : S^n \times [0,1] \to X$ defines a homotopy between images in X, then $f \circ H : S^n \times [0,1] \to X \to Y$ defines a homotopy between images in Y. We omit the details.

Definition 3.6 A map $f : X \to Y$ is n-connected if the induced maps $\pi_i(f)$ is an isomorphism for all $i < n$ and $\pi_n(f)$ is surjective.

Suppose $X \subset Y$. Then the inclusion map $f : X \to Y$ is 1-connected if $\pi_0(f)$ is an isomorphism (i.e. the numbers of path components of X and Y are same) and $\pi_1(f)$ is a surjection (there are no new elements of fundamental group added from $X - Y$).

4 h-cobordism

Definition 4.1 An n-dimensional cobordism is a 5-tuple, $(W; M_0, f_0, M_1, f_1)$, where W is a compact n-dimensional manifold, M_0, M_1 are $n-1$-dimensional closed manifolds such that $\partial W = \partial_0 W \sqcup \partial_1 W$ and there exist diffeomorphisms $f_i : M_i \to \partial_i W$ for $i = 0, 1$. A compact manifold is a manifold that is compact (we are assuming that a manifold possesses an inherent topology). A closed manifold is a manifold without boundary that is compact. Conventionally, we write cobordisms as a triple $(W; \partial_0 W, \partial_1 W)$ and treat each f_i as an identity map.

Definition 4.2 A cobordism $(W; \partial_0 W, \partial_1 W)$ is an h-cobordism if the inclusion map $\partial_0 W \to W$ and $\partial_1 W \to W$ are both homotopy equivalences.

Definition 4.3 If $(W; M_0, f_0, M_1, f_1)$ and $(W'; M_0, f_0', M_1', f_1')$ have the same dimension, they are diffeomorphic relative to M_0 if there is a diffeomorphism $F : W \to W'$ such that $F \circ f_0 = f_0'$.

Definition 4.4 An h-cobordism $(W; \partial_0 W, \partial_1 W)$ is called trivial if it is diffeomorphic to $(\partial_0 W \times [0,1]; \partial_0 W \times \{0\}, \partial_0 W \times \{1\})$ relative to $\partial_0 W$.

Theorem 4.1 ***h*-Cobordism Theorem** Any h-cobordism $(W; M_0, f_0, M_1, f_1)$ where M_0 is simply connected, closed manifold and $\dim(W) \geq 6$ is trivial.

5 Handles

A diffeomorphism class refers to an equivalence class of manifolds, $[M]_{diff} = \{N | N \cong M\}$. These structures are called **handles**.

Definition 5.1 An n-dimensional handle of index q is a structure diffeomorphic to $D^q \times D^{n-q}$. We will refer to this as an n, q handle or if the dimension is clear, simply a q-handle. We impose certain rules on how a handle can be embedded in a topological space.

We note that (n, q) handles are n-manifolds with boundary since it is the product of q-manifolds and $n - q$ manifolds, each with a boundary. It is a classic result that the boundary of the product manifold $M \times N$ is then $(\partial M \times N) \cup (M \times \partial N)$.

We define one more concept. A manifold with corners is a Hausdorff, second countable topological object that is homeomorphic such that every point has a neighborhood homeomorphic to an open set of $[0,\infty)^k \times \mathbb{R}^{n-k}$ for some $1 \leq k \leq n$. In fact, every manifold with corner is homeomorphic to a manifold with boundary.

Definition 5.2 The core of an (n,q) handle is $D^q \times \{0\}$. The cocore of an $(n,q)-$handle is $\{0\} \times D^{n-q}$. The boundary of the core is then $S^{q-1} \times \{0\}$ and the boundary of the cocore is $\{0\} \times S^{n-q-1}$.

Definition 5.3 The transverse sphere of a handle (ϕ^q) (this is a q handle assuming that the dimension is clear) is the boundary of the cocore, $\{0\} \times S^{n-q-1}$.

6 Attaching Handles

Definition 6.1 Given an n-dimensional manifold M with boundary ∂M and a smooth embedding (a topological embedding that is an immersion) $\phi^q : S^{q-1} \times D^{n-q} \to \partial M$, one can attach a q-handle to M. This yields a new manifold $M + (\phi^q) = M \cup_{\phi^q} D^q \times D^{n-q}$.

We prove that $M + \phi^q$ is a manifold with boundary. Since M and $D^q \times D^{n-q}$ are both n-manifolds, any point that does not belong in $\phi^q(S^{q-1} \times D^{n-q})$ has a neighborhood that is homeomorphic to an open set in $\mathcal{H}^n$. Now, consider the image of ϕ^q. Since $S^{q-1} \times D^{n-q}$ is a subset of ∂q-handle, and ϕ^q is an embedding of the subset of the boundary of the handle to the boundary of M, any point in the topological interior of image has a neighborhood homeomorphic to one copy of $\mathcal{H}^n$ and another copy of $\mathcal{H}^n$ (this follows from the fact that the basis of $\mathcal{H}^n$ consists of $B(p,r) \cap \mathcal{H}^n$ and if the intersection is nonempty, the basis element is homeomorphic to the half plane itself). Hence, the embedding will glue these two copies of the upper half plane along $x^n = 0$. The resulting space is $\mathbb{R}^n$ and hence $M + (\phi)^q$ is a topological manifold. (Indeed, it is a smooth manifold but we omit the proof).

It follows that the boundary of $M + \phi^q$ can be obtained from the boundary of M, removing the interior of the image of ϕ^q, and adding the boundary of the handle that is not embedded into M. Recall that $\partial\phi^q = \partial(D^q \times D^{n-q}) = S^{q-1} \times D^{n-q} \cup D^q \times S^{n-q-1}$. Hence $D^q \times S^{n-q-1}$ is a part of the boundary of the new manifold $M + \phi^p$. To make this rigorous, the attaching map uses only

$$S^{q-1} \times D^{n-q} = S^{q-1} \times (intD^{n-q)} \cup S^{q-1} \times S^{n-q-1}).$$

Since the interior of a product space is the product of the interiors, the interior of $S^{q-1} \times D^{n-q}$ is $S^{q-1} \times intD^{n-q}$. Let A be the image of the embedding. Since ϕ^q is an embedding,

$$Int_{\partial M} A = \phi^q(S^{q-1} \times intD^{n-q})$$

Then,

$$\partial H - S^{q-1} \times intD^{n-q} = S^{q-1} \times D^{n-q} \cup D^q \times S^{n-q-1} - S^{q-1} \times intD^{n-q}$$

is mapped to the boundary of the new manifold. In particular, $D^q \times S^{n-q-1}$ will be part of the boundary of the new manifold.

Definition 6.2 Let W be an $(n-1)$ manifold. Then an embedding $\phi : S^{q-1} \times D^{n-q} \to W$ is said to be trivial if ϕ is a composition of two embeddings, $f \circ g$ where $f : D^{n-1} \to W$ and $g : S^{q-1} \times D^{n-q} \to D^{n-1}$. This means that any trivial embedding sends $S^{q-1} \times D^{n-q}$.

7 Handlebody Decomposition

Recall that a cobordism is essentially a collection of manifolds, $(W, \partial_0 W, \partial_1 W)$ such that $\partial W = \partial_0 W \sqcup \partial_1 W$. We aim to trivialize W (recall that this is showing that W is diffeomorphic to $\partial W_0 \times [0,1]$ relative to $\partial_0 W$.

Definition 7.1 A handlebody decomposition of a manifold W with $\partial W = \partial_0 W \sqcup \partial_1 W$ (relative to ∂_W) refers to a manifold W' that is diffeomorphic to W and

$$W' = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + \cdots + (\phi_r^{q_r}).$$

We furthermore require that the image of $\phi_i^{q_i}$ is contained inside $\partial_1(\partial_0 W \times [0,1] + (\phi_1^{q_1} + \cdots + \phi_{i-1}^{q_{i-1}}))$. We check that this construction is well defined using induction. Consider the base manifold $X_0 = \partial_0 W \times [0,1]$. We may write ∂X_0 as $\partial_0 X_0 \sqcup \partial_1 X_0$ where $\partial X_0 = \partial_0 W \times \{0\}$ and $\partial_1 X_0 = \partial X_0 \backslash \partial_0 X_0$. Since $\partial_0 W$ is a manifold without boundary, $\partial X_0 = \partial_0 W \times \{0\} \sqcup \partial_0 W \times \{1\}$ each of which is diffeomorphic to $\partial_0 W$. This makes X_0 a cobordism. Now, moving to the inductive step, suppose X_i is a cobordism whose boundary is decomposed as above. Let

$$X_{i+1} = X_i \cup_{\phi_{i+1}^{q_{i+1}}} (D^{q_{i+1}} \times D^{n-q_{i+1}})$$

such that $im\phi_{i+1}^{q_{i+1}} \subset \partial_1 X_i$. Since X_{i+1} is a manifold with boundary and $\partial_0 X_{i+1} = \partial_0 W \times \{0\}$ is unaffected by the gluing, it only remains to verify $\partial_1 X_{i+1}$ is a manifold, which trivially follows from induction.

Lemma 7.1 If W is a compact manifold with dimension larger or equal to 6 and $\partial W = \partial_0 W \sqcup \partial_1 W$, then there is a handlebody decomposition of W relatively to $\partial_0 W$. The proof requires Morse theory which is beyond the scope of the paper.

Lemma 7.2 Let $(W, \partial_0 W, \partial_1 W)$ be an oriented, compact h-cobordism of dimension at least 6 with $\partial_0 W$ simply connected. Then for any $2 \leq q \leq n-3$, we have handles such that

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1})$$

The handlebody decomposition of the form above is called a normal form.

The rest of the proof follows by using a representative matrix of an h-cobordism which is a $p_q \times p_q$ matrix describing the action of the boundary operator on the basis $\{[\phi_i^{q+1}]\}_{1 \leq i \leq p_{q+1}}$ in terms of the basis $\{[\phi_i^{q+1}]\}_{1 \leq i \leq p_q}$. These are very technical notions and hence the details of the proof of the h-cobordism theorem (and even the sketch) is way beyond the scope of the presentation. For those who are interested, we direct you to Mackie-Mason's "The h-Cobordism Theorem".

8 Poincare Conjecture

Theorem (Poincare's Conjecture): For $n \geq 6$, any simply connected, closed n-manifold M whose homology groups $H_p(M)$ are isomorphic to $H_p(S^n)$ for all $p \in \mathbb{Z}$ is homeomorphic to S^n.

Theorem h-cobordism theorem Any h-cobordism $(W; \partial_0 W, \partial_1 W)$ with dimension of $W \geq 6$ and $\partial_0 W$ simply connected is diffeomorphic relative to $\partial_0 W$ to the trivial h-cobordism $(\partial_0 W \times [0,1]; \partial_0 W \times \{0\}, \partial_0 W \times \{1\})$.

We show in this presentation how Poincare's conjecture can be derived from the h-cobordism theorem. This is a remarkable moment where differential geometry intersects with topology.

Lemma 9.1 For $n \geq 6$, let M be a simply connected n-manifold with $H_j(M)$ isomorphic to $H_j(S^n)$ for all $j \in \mathbb{N}$. Take two disjoint disks $D_i^n \subset M$ for $i = 0, 1$. Let $N = M - int(D_0^n) - int(D_1^n)$. Then the inclusion of the boundary spheres $S_i^{n-1} \to N$ is a homotopy equivalence for $i =$ 0,1. (Here, a disk the embedding of a closed ball in the manifold. Existence follows from the very definition of a manifold).

Proof: Before proceeding to the proof, we note that the homology groups of spheres are as follows:

$$H_k(S^n) = \begin{cases} \mathbb{Z} \text{ if } k = 0, n \neq 0 \\ \mathbb{Z} \text{ if } k = n, n \neq 0 \\ \mathbb{Z} \oplus \mathbb{Z} \text{ if } k = 0, n = 0 \\ 0 \text{ otherwise} \end{cases}$$

We also claim a sublemma as follows: If $f : (Y, B) \to (Z, C)$ is a map of pairs such that $f : Y \to Z, f|_B : B \to C$ are both homotopy equivalences, then $H_n(Y, B)$ and $H_n(Z, C)$ are isomorphic to each other. We first show that a relative homology group

$$H_j(M - int(D_0^n) - int(D_1^n), S_0^{n-1}) = 0$$

for all j. By excision theorem, if we pick a small enough open ball U in $intD_0^n$ such that $\bar{U} \subset intD_0^n$, then $H_j(M - int(D_1^n), D_0^n)$ is isomorphic to $H_j(M - int(D_1^n) - U, D_0^n - U)$. Since $M - int(D_1^n) - U$ deformation retracts to $M - int(D_1^n) - D_0^n$ and the deformation map restricted to $D_0^n - U$ is also a deformation retraction to S_0^{n-1}, we conclude that $H_j(M - int(D_0^n) - int(D_1^n), S_0^{n-1})$ is isomorphic to $H_j(M - int(D_1^n), D_0^n)$. (this follows from the fact that a deformation retraction is a homotopy equivalence and the sublemma above). Hence, it suffices to prove that $H_j(M - int(D_1^n), D_0^n) = 0$ for all j.

Now consider the long exact sequence of a pair $(M - int(D_1^n), D_0^n)$ as follows:

$$\begin{aligned} \cdots \longrightarrow H_j\big(D_0^n\big) \longrightarrow H_j\big(M - \mathrm{int}(D_1^n)\big) \longrightarrow H_j\big(M - \mathrm{int}(D_1^n), D_0^n\big) \\ \longrightarrow H_{j-1}\big(D_0^n\big) \longrightarrow H_{j-1}\big(M - \mathrm{int}(D_1^n)\big) \longrightarrow H_{j-1}\big(M - \mathrm{int}(D_1^n), D_0^n\big) \longrightarrow \cdots \end{aligned}$$

Since all disks are homotopic to points, $H_j(D_0^n) = 0$ for all j. Hence this sequence leads to

$$\begin{aligned} \cdots \longrightarrow 0 \longrightarrow H_j\big(M - \mathrm{int}(D_1^n)\big) \longrightarrow H_j\big(M - \mathrm{int}(D_1^n), D_0^n\big) \longrightarrow 0 \\ \longrightarrow H_{j-1}\big(M - \mathrm{int}(D_1^n)\big) \longrightarrow H_{j-1}\big(M - \mathrm{int}(D_1^n), D_0^n\big) \longrightarrow 0 \longrightarrow \cdots \end{aligned}$$

Any two terms bracketed by trivial groups are isomorphic to each other by the properties of long exact sequence. That is, $H_n(M - int(D_1^n), D_0^n)$ is isomorphic to $H_n(M - int(D_1^n)$. Hence, we consider another exact sequence from $(M, M - int(D_1^n))$.

$$\begin{aligned} \cdots \to H_j\big(M - \mathrm{int}(D_1^n)\big) \to H_j(M) \to H_j\big(M,\, M - \mathrm{int}(D_1^n)\big) \\ \to H_{j-1}\big(M - \mathrm{int}(D_1^n)\big) \to H_{j-1}(M) \to H_{j-1}\big(M,\, M - \mathrm{int}(D_1^n)\big) \to \cdots \end{aligned}$$

Now by excision theorem and the sublemma, we may excise $M - D_1^n$ from $H_j(M, M - int(D_1^m))$ in a similar manner as above. This gives a sequence

$$\begin{aligned} \cdots \to H_j\big(M - \mathrm{int}(D_1^n)\big) \to H_j(M) \to H_j\big(D_1^n, S_1^{n-1}\big) \to H_{j-1}\big(M - \mathrm{int}(D_1^n)\big) \\ \to H_{j-1}(M) \to H_{j-1}\big(D_1^n, S_1^{n-1}\big) \to \cdots \end{aligned}$$

Now $H_j(D_1^n, S_1^{n-1})$ is isomorphic to $H_j(S^n)$ (proof can be found in Hatcher). But by our hypothesis, we know that $H_j(M)$ also is the same since M has homology groups isomorphic to those of S^n. Hence whenever $j \neq n$ and $j \neq n-1$, we have a short exact sequence

$$\cdots \to 0 \to H_j(M - int(D_1^n)) \to 0$$

where $H_j(M - int(D_1^n))$ then just becomes 0. (when $j = 0$, there is some complication. we can solve this complication by considering the reduced homology group instead). For other cases,

$$\cdots \to 0 \to H_n\big(M - \mathrm{int}(D_1^n)\big) \to \mathbb{Z} \to \mathbb{Z} \to H_{n-1}\big(M - \mathrm{int}(D_1^n)\big) \to 0 \to \cdots$$

The map $\mathbb{Z} \to \mathbb{Z}$ is essentially a map between $H_n(M) \to H_n(M, M - int(D_1^n))$ which is an isomorphism since M is a n-manifold whose homology groups are isomorphic to S^n (the rest follows from the theorem on good pair which is beyond the scope of the presentation). Therefore, $H_j(M - int$ is 0 when $j = n, n-1$ by the property of exact sequences. Hence we have $H_j(M - int(D_1^n)) = H_j(N, S_0^n) = 0$ for all j. Using the same logic, we get $H_j(N, S_1^n) = 0$. We also claim that N is simply connected while the proof of this is beyond the scope of the presentation. (this follows from the fact that *(M, W)* is a CW complex with no m cells for $m \leq n-1$ which turns out to be $n-1$ connected). By Whitehead's theorem, if a continuous mapping f between CW complexes X and Y induces isomorphisms on all homotopy groups, then f is a homotopy equivalence. This completes the proof.

Lemma 9.2 Any homeomorphism $h : S^k \to S^k$ can be extended to a homeomorphism $H : D^{k+1} \to D^{k+1}$.

Proof: We may treat D^{k+1} as the product $S^k \times [0,1]$ with $S^k \times \{0\}$ identified to a single point. (this is the same as saying that the closed ball is equivalent to the cone of the boundary via the projection map). Then define H by $H(x,t) = (t \cdot h(x), t))$. H is a homeomorphism since h is a homeomorphism. (product of homeomorphism is a homeomorphism). This is also called the Alexander's trick.

Theorem 9.1 Poincare's Conjecture for dimension ≥ 6

Proof: Let D_i^n be two disjoint disks embedded in M, i = *0,1*. Let $(N = M - int(D_0^n) - int(D_1^n)$. Then $N; S_0^{n-1}, S_1^{n-1})$ is a cobordism since $\partial M = \partial D_0^n \sqcup \partial D_1^n = S_0^{n-1} \sqcup S_1^{n-1}$ since M is a closed manifold and moreover a h-cobordism by lemma 9.1. Since S^{n-1} is simply connected, by h-cobordism theorem, $(N; S_0^{n-1}, S_1^{n-1}) \cong (S_0^{n-1} \times [0,1]; S_0^{n-1} \times \{0\}, S_0^{n-1} \times \{1\})$. More precisely, $F : N \to S_0^{n-1} \times [0,1]$ is a diffeomorphism where F restricted to S_0^{n-1} is an identity map and also defines some diffeomorphism $S_0^{n-1} \to S_0^{n-1} \times \{1\}$. Now, extend this to a homeomorphism $F : D_0^n \to D_1^n$ using Alexander's trick. Now we may reassemble M by attaching D_0^n to the bottom $S^{n-1} \times \{0\}$ and D_1^n to the top $S^{n-1} \times \{1\}$ using the identifications $\partial D_0^n = S^{n-1} \times \{0\}$ and $\partial D_1^n = S^{n-1} \times \{1\}$ each glued by the identity and homeomorphism. This gives a bijection

$$h : N \cup_{\partial D_0} D_0^n \cup_{\partial D_1} D_1^n \to D_0^n \times \{0\} \cup \partial D_0^n \times [0,1] \cup D_0^n \times \{1\}.$$

But the target space is just a decomposition of the n-sphere into two caps and an equatorial cylinder. Hence this proves the Poincare conjecture.

9 Citation

[1] Allen Hatcher, Algebraic Topology

[2] Andrew R. Mackie-Mason, The h-Cobordism Theorem

[3] John M. Lee, Introduction to Smooth Manifolds

[4] John Milnor, Morse Theory