r-stacked Billera-Lee polytopes and regular triangulations

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1 Introduction

The goal of this project is to produce examples to support help us support a positive answer to a question by S. Murai and E. Nevo. In short P is a simplicial polytope with $g_r = 0$, where $(g_0, g_1, \dots g_{\lfloor d/2 \rfloor})$ is the simplicial g-vector, then the stacked triangulation of P (the they showed exists) is regular, that is, it is the projection of the lower pointing faces of a polytope Q with $\dim Q = \dim P + 1$ that projects to P.

Regular triangulations are very nice and have been widely studied as a combinatorial structure that comes with a polytope. They always exist and can be parametrized as the vertices of a very nice polytope called the **Secondary polytope** of P or the **GKZ polytope** of P. Their combinatorial structure is fascinating and understanding it reveals a lot of information about the polytope.

2 Basic definitions

A polytope P is the convex hull of finitely many points in $\mathbb{R}^d$. The dimension of P is the dimension of the smallest affine space that contains it. This affine space is called the affine hull of P. From now on assume that P is d-dimensional. A **face** F of P is a set of the form $P \cap H$, where H is a hyperplane that does not intersect the interior of P.

Notice that a face of polytope is itself a polytope and has therefore a dimension. 0-dimensional faces are called vertices. Notice that a face F is the convex hull of the vertices it contains. A k-dimensional polytope is a **simplex** if it has $k+1$ vertices. A polytope is **simplicial** if all its faces are simplices.

Let $f_{i-1}(P)$ denote the number of $i-1$ dimensional faces of P. The empty set is a face of P whose dimension is -1 by convention. This numbers allow us to construct the **f-vector** of P, that is, the vector $(f_{-1}, f_0, \dots, f_{d-1})$. One of the main goals in the theory of polytopes is to classify all possible f-vectors of all polytopes. This is in general VERY hard. Little is known for general polytopes of dimension four or more. However, f-vectors of simplicial polytopes are completely classified using techniques from commutative algebra and toric varieties. We will now explain this classification that is relevant for this project.

From now on P denotes a simplicial d-polytope where $d \geq 4$ is an integer. We define the h**-vector** of P to be the vector $(h_0, \dots, h_d)$ such that

$$\sum_{j=0}^{d} f_{j-1}(x-1)^{d-j} = \sum_{j=0}^{d} h_j x^{d-j}$$

The entries of the h-vector are known to be positive and symmetric, that is $h_i = h_{d-i}$ the case $i = 0$ of this relations is just the Euler-Characteristic of a sphere (the boundary of the polytope is homemorphic to $\mathbb{S}^{d-1}$), so this relation can be thought as a generalisation of this famous topological fact. Note also it is easy to recover the f-vector of a polytope if we know the h-vector. This fact is known as the **Dehn-Sommerville relations**.

An **abstract simplicial complex** Δ on a finite vertex set V is a subset of $\mathcal{P}(V)$ that contains all one element subsets and such that if $A \in \Delta$ and $B \subseteq A$ then $B \in \Delta$. The elements of Δ are called faces. A **geometric simplicial complex** Δ in $\mathbb{R}^d$ is a finite collection of simplices such that if Γ is a simplex in Δ and F is a face of Γ then $F \in \Delta$. A geometric simplicial complex Δ induces an abstract simplicial complex $\hat{\Delta}$: the vertex set of $\hat{\Delta}$ is the set of all vertices of the simplices in Δ the faces of $\hat{\Delta}$ are the sets of vertices that form a simplex of Δ. $\hat{\Delta}$ is said to be a **realisation** of Δ. It is easy to show that if two geometric simplicial complexes induce the same abstract simplicial complex (possibly after renaming some vertices), they are homeomorphic (via a piecewise linear map). Thus we can recover the topology of Δ from $\hat{\Delta}$, so from now on we make no distinction between abstract and geometric simplicial complexes. The **dimension** of a simplicial complex is the maximal dimension of one face, in the abstract case the dimension can be defined combinatorial as the size of largest element minus one. A simplicial complex is **pure** if all it's maximal faces (ordered by inclusion) have the same dimension.

Notice that a d-polytope P in $\mathbb{R}^d$ is simplicial if and only if its boundary ∂P is a simplicial complex. The boundary complex has many interesting properties that we will state now.

A triangulation of P is a geometric simplicial complex Σ whose vertex set is the set of vertices of P and such that the union of all simplices of σ is P. Triangulations always exist and endow P with a rich combinatorial structure. Among all triangulations there is a very special kind of triangulations called regular triangulations. They are obtained as follows: denote by $V(P)$ the set of vertices of P. A function $\omega : V(P) \to \mathbb{R}_{\geq 0}$ is generic if the convex hull of the vertices $(v, \omega(v))$ is a simplicial $(d+1)$-polytope in $\mathbb{R}^{d+1}$ with no facet orthogonal to $\mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^{d+1}$. Denote the polytope by $\omega_*(P)$. Genericity of ω guarantees that we can split the facets of $\omega_*(P)$ in two, the lower pointing facets, whose outer pointing normal vector has negative $d+1$ coordinate, and the upper pointing facets, whose outer pointing normal vector has positive $d+1$ coordinate. Projecting the lower pointing facets gives a triangulation of P. All the triangulations obtained in this way are called **regular**.

A **line shelling** of a simplicial polytope P consists of an ordering $F_1, \dots F_k$ of the facets of P that is obtained as follows: pick a point x in the interior of the polytope and a generic directed line ℓ through x (i.e it does not pass through the intersection of the affine span of any pair of facets). Then order the facets in the order they are intersected by ℓ starting at x and going in the direction of ℓ. Once Once we pass all the intersections we go to the $-\infty$ of ℓ and start listing the facets as they are intersected.

TO DO: put example.

For a pair of integers $n > k$, there exists unique $i < k$ and $a_k \geq a_{k-1} \cdots \geq a_i \geq i > 0$ such that

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i}$$

This decomposition allows us to define

$$\partial^k n = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-1}$$

3 The generalised lower bound theorem

In 1971 Peter McMullen and Walkup published a conjecture that completely characterised the possible f-vectors of simplicial polytopes. It became a theorem in the late 70's after the work of Billera-Lee (sufficiency of the conditions) and Stanley(necessity).

Theorem 3.1. A vector $(g_0, g_1, \ldots, g_{\lfloor \frac{d}{2} \rfloor})$ is the g-vector of a simplicial d-polytope if and only if $g_0 = 1$, $g_k \geq 0$ for $k = 1, \ldots \lfloor \frac{d}{2} \rfloor$ and $\partial^k(g_{k+1}) \leq g_k$ for $k < \lfloor \frac{d}{2} \rfloor$.

This theorem gives a fast way of checking wether an integer vector is the f-vector of a polytope. To show the sufficiency of this theorem Billera and Lee constructed a polytope for every integer vector satisfying those inequalities. One of the goals of this project is to construct Billera-Lee polytopes on Sage.

In the same paper, McMullen and Walkup also posted a conjecture on when a g_k is equal to 0. The conjecture was verified in the case $k = 2$ and remained open until 2012 when Murai and Nevo finally posted a solution.

Theorem 3.2. Let P be a simplicial d-polytope and $2 \leq r \leq \lfloor \frac{d}{2} \rfloor$. The following are equivalent:

i. $g_r(P) = 0$.

ii. There exists a triangulation Σ of P, such that $\mathrm{Skel}_{d-r}(\Sigma) = \mathrm{Skel}_{d-r}(\partial P)$.

The triangulation Σ was shown to be unique by McMullen and to have a specific combinatorial description by Bagchi and Datta. This triangulation is called the $(r-1)$**-stacked triangulation** of P. Once we know that $g_r = 0$, the maximal faces of the stacked triangulation are given by the set:

$$\{F \mid F \text{ is a } d\text{-simplex with vertices on } v(P) \text{ and } \mathrm{Skel}_{d-r}(F) \subseteq \mathrm{Skel}_{d-r}(\partial P)\}$$

The case $k = 2$ was proved by Barnette and a generalisation of this fact to homology spheres was done by Kalai. This case is particularly interesting. A polytope with $g_2 = 0$ is called **stacked** and can be constructed inductively by starting with a simplex and at each step "stacking" a vertex, that is, we put a new vertex outside of the polytope and very close to one facet and take the convex hull of the resulting polytope.