r-stacked Billera-Lee polytopes and regular triangulations

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1 Introduction

The goal of this project is to produce examples to support help us support a positive answer to a question by S. Murai and E. Nevo. In short P is a simplicial polytope with $g_r = 0$, where $(g_0, g_1, \dots g_{\lfloor d/2 \rfloor})$ is the simplicial g-vector, then the stacked triangulation of P (the they showed exists) is regular, that is, it is the projection of the lower pointing faces of a polytope Q with $\dim Q = \dim P + 1$ that projects to P.

Regular triangulations are very nice and have been widely studied as a combinatorial structure that comes with a polytope. They always exist and can be parametrized as the vertices of a very nice polytope called the **Secondary polytope** of P or the **GKZ polytope** of P. Their combinatorial structure is fascinating and understanding it reveals a lot of information about the polytope.

2 Basic definitions

A polytope P is the convex hull of finitely many points in $\mathbb{R}^d$. The dimension of P is the dimension of the smallest affine space that contains it. This affine space is called the affine hull of P. From now on assume that P is d-dimensional. A **face** F of P is a set of the form $P \cap H$, where H is a hyperplane that does not intersect the interior of P.

Notice that a face of polytope is itself a polytope and has therefore a dimension. 0-dimensional faces are called vertices. Notice that a face F is the convex hull of the vertices it contains. A k-dimensional polytope is a **simplex** if it has $k+1$ vertices. A polytope is **simplicial** if all its faces are simplices.

Let $f_{i-1}(P)$ denote the number of $i-1$ dimensional faces of P. The empty set is a face of P whose dimension is -1 by convention. This numbers allow us to construct the **f-vector** of P, that is, the vector $(f_{-1}, f_0, \dots, f_{d-1})$. One of the main goals in the theory of polytopes is to classify all possible f-vectors of all polytopes. This is in general VERY hard. Little is known for general polytopes of dimension four or more. However, f-vectors of simplicial polytopes are completely classified using techniques from commutative algebra and toric varieties. We will now explain this classification that is relevant for this project.

From now on P denotes a simplicial d-polytope where $d \geq 4$ is an integer. We define the h**-vector** of P to be the vector $(h_0, \ldots, h_d)$ such that

$$\sum_{j=0}^{d} f_{j-1}(x-1)^{d-j} = \sum_{j=0}^{d} h_j x^{d-j}$$

The entries of the h-vector are known to be positive and symmetric, that is $h_i = h_{d-i}$ the case $i = 0$ of this relations is just the Euler-Characteristic of a sphere (the boundary of the polytope is homemorphic to $\mathbb{S}^{d-1}$), so this relation can be thought as a generalisation of this famous topological fact. Note also it is easy to recover the f-vector of a polytope if we know the h-vector. This fact is known as the **Dehn-Sommerville relations**.

An **abstract simplicial complex** Δ on a finite vertex set V is a subset of $\mathcal{P}(V)$ that contains all one element subsets and such that if $A \in \Delta$ and $B \subseteq A$ then $B \in \Delta$. The elements of Δ are called faces. A **geometric simplicial complex** Δ in $\mathbb{R}^d$ is a finite collection of simplices such that if Γ is a simplex in Δ and F is a face of Γ then $F \in \Delta$. A geometric simplicial complex Δ induces an abstract simplicial complex $\hat{\Delta}$: the vertex set of $\hat{\Delta}$ is the set of all vertices of the simplices in Δ the faces of $\hat{\Delta}$ are the sets of vertices that form a simplex of Δ. $\hat{\Delta}$ is said to be a **realisation** of Δ. It is easy to show that if two geometric simplicial complexes induce the same abstract simplicial complex (possibly after renaming some vertices), they are homeomorphic (via a piecewise linear map). Thus we can recover the topology of Δ from $\hat{\Delta}$, so from now on we make no distinction between abstract and geometric simplicial complexes. The **dimension** of a simplicial complex is the maximal dimension of one face, in the abstract case the dimension can be defined combinatorial as the size of largest element minus one. A simplicial complex is **pure** if all it's maximal faces (ordered by inclusion) have the same dimension.

A **shelling** of a pure d dimensional simplicial complex Δ is an ordering $F_1, \ldots, F_k$ of the facets of Δ such that for $1 < j \leq k$ the if for every $1 \leq s < j$ there exists $1 \leq s' < j$ and $x_{s'} \in F_j$ such that $F_s \cap F_j \subseteq F_{s'} \cap F_j = F_j \backslash \{x_{s'}\}$, that is the complex with vertices in F_j generated by $F_s \cap F_j$ for $s < j$ is pure $d-1$ dimensional. A complex is **shellable** if it has a shelling. Shellable complexes are nicely constructed by induction and it is very easy to keep track of the homotopy type of a shellable complex in terms of the shelling: it is homotopic to a wedge of d-spheres where the number of spheres is give by the number of facets of the shelling that don't introduce new faces of dimension smaller than d. One nice property that we are going to use is that if Δ is a shellable sub complex of the boundary of a simplicial $(d+1)$-polytope then it is homeomorphic to a d-ball whose boundary (ridges only contained in one facet) generate a complex homeomorphic to a $(d-1)$-sphere.

One good approach to study some properties of simplicial complexes is to study them locally for this we define that **star** of a face F of Δ, denoted by $\mathrm{st}(F)$ to be the complex generated by all the facets of Δ that contain F. Define the **link** of F, denoted by $\mathrm{lk}(F)$, to be the set of all faces of $\mathrm{st}(F)$ that are disjoint to F.

Notice that a d-polytope P in $\mathbb{R}^d$ is simplicial if and only if its boundary ∂P is a simplicial complex. The boundary complex has many interesting properties that we will state now.

A **line shelling** of a simplicial polytope P consists of an ordering $F_1, \dots F_k$ of the facets of P that is obtained as follows: pick a point x in the interior of the polytope and a generic directed line ℓ through x (i.e it does not pass through the intersection of the affine span of any pair of facets). Then order the facets in the order they are intersected by ℓ starting at x and going in the direction of ℓ. Once Once we pass all the intersections we go to the $-\infty$ of ℓ and start listing the facets as they are intersected. It is easy to check that a line shelling is indeed a shelling, so boundary complexes of simplicial polytopes are always shellable.

A triangulation of P is a geometric simplicial complex Σ whose vertex set is the set of vertices of P and such that the union of all simplices of σ is P. Triangulations always exist and endow P with a rich combinatorial structure. Among all triangulations there is a very special kind of triangulations called regular triangulations. They are obtained as follows: denote by $V(P)$ the set of vertices of P. A function $\omega : V(P) \to \mathbb{R}_{\geq 0}$ is generic if the convex hull of the vertices $(v, \omega(v))$ is a simplicial $(d+1)$-polytope in $\mathbb{R}^{d+1}$ with no facet orthogonal to $\mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^{d+1}$. Denote the polytope by $\omega_*(P)$. Genericity of ω guarantees that we can split the facets of $\omega_*(P)$ in two, the lower pointing facets, whose outer pointing normal vector has negative $d+1$ coordinate, and the upper pointing facets, whose outer pointing normal vector has positive $d+1$ coordinate. Projecting the lower pointing facets gives a triangulation of P. All the triangulations obtained in this way are called **regular**.

For a pair of integers $n > k$, there exists unique $i < k$ and $a_k \geq a_{k-1} \cdots \geq a_i \geq i > 0$ such that

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i}$$

This decomposition allows us to define

$$\partial^k n = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-1}$$

And M**-sequence** is a sequence $(a_0, a_1, \dots a_n)$ such that $a_0 = 1$ and $\partial^k(a_{k+1}) \leq a_k$ for $1 \leq k < n$. It is a theorem of Macaulay that a sequence is an M sequence if and only if there exists a finite family A of monomials in variables $x_1, \dots, x_{a_1}$ such that if $m | m'$ and $m' \in A$ then $m \in A$ and a_j is the number of monomials of degree j in A.

3 The generalised lower bound theorem

In 1971 Peter McMullen and Walkup published a conjecture that completely characterised the possible f-vectors of simplicial polytopes. It became a theorem in the late 70's after the work of Billera-Lee (sufficiency of the conditions) and Stanley(necessity).

Theorem 3.1. *A vector $(g_0, g_1, \dots, g_{\lfloor \frac{d}{2} \rfloor})$ is the g-vector of a simplicial d-polytope if and only if it is an M-sequence.*

This theorem gives a fast way of checking wether an integer vector is the f-vector of a polytope. To show the sufficiency of this theorem Billera and Lee constructed a polytope for every integer vector satisfying those inequalities. One of the goals of this project is to construct Billera-Lee polytopes on Sage.

In the same paper, McMullen and Walkup also posted a conjecture on when a g_k is equal to 0. The conjecture was verified in the case $k = 2$ and remained open until 2012 when Murai and Nevo finally posted a solution.

Theorem 3.2. *Let P be a simplicial d-polytope and $2 \leq r \leq \lfloor \frac{d}{2} \rfloor$. The following are equivalent:*

i. $g_r(P) = 0$.

ii. There exists a triangulation Σ of P, such that $Skel_{d-r}(\Sigma) = Skel_{d-r}(\partial P)$.

The triangulation Σ was shown to be unique by McMullen and to have a specific combinatorial description by Bagchi and Datta. This triangulation is called the $(r-1)$**-stacked triangulation** of P. Once we know that $g_r = 0$, the maximal faces of the stacked triangulation are given by the set:

$$\{F \mid F \text{ is a } d\text{-simplex with vertices on } v(P) \text{ and } \mathrm{Skel}_{d-r}(F) \subseteq \mathrm{Skel}_{d-r}(\partial P)\}$$

The case $k = 2$ was proved by Barnette and a generalisation of this fact to homology spheres was done by Kalai. This case is particularly interesting. A polytope with $g_2 = 0$ is called **stacked** and can be constructed inductively by starting with a simplex and at each step "stacking" a vertex, that is, putting a new vertex outside of the polytope and very close to one facet and taking the convex hull of the polytope and that new point.

Our goal is to study regularity of $(r-1)$-stacked triangulations. In particular, we want to answer the following question due to McMullen in his 71 paper.

Question 3.3. *Are $(r-1)$-stacked triangulations regular?*

A simple inductive argument shows that in the case $r = 2$ this is always the case. There are many methods to verify wether a triangulation is regular, in particular, one such methods uses linear programming. Our goal is to experiment with this question with the aim of answering it in the affirmative. We will now proceed to describe two things:

1. How to construct examples of polytopes with a prescribed g-vector.

2. How to check regularity of a triangulation using wether certain linear program is feasible.

After describing this two things we will do some experimentation with polytopes constructed to have $g_r = 0$. Given that the $r-1$-stack triangulation is easily constructed combinatorially, we claim that this combinatorial data can be nicely interpreted in terms of the linear program that we want to show feasible. In order to support our claim we intend to experiment with several polytopes that come from our construction.

4 Billera-Lee polytopes

Billera and Lee cooked up a construction of a polytope with a given g-vector $(g_0, g_1, \dots, g_{\lfloor \frac{d}{2} \rfloor})$. We will describe the construction and develop an algorithm to construct that Billera-Lee polytope. The construction goes as follows: For a sequence $q := (q_1 < q_2 < \cdots < q_{n-1})$ of positive real numbers we define the **cyclic polytope** of dimension d with vertices of q to be $C(n, d+1)(q) = \text{conv}\left(\{v_i := (q_i, q_i^2, \dots q_i^{d+1}) \mid 1 \leq i \leq n-1\} \cup \{v_0 := (0, 0, \dots, 0)\}\right)$. The combinatorial structure (i.e the face lattice) of this polytope is independent of q. However the line shillings may vary according to the choice of q. We will find a truncation of a line shelling of $C(n, d+1)(q)$ that produces a combinatorial d-ball whose boundary is a (polytopal) sphere with the right g-vector by choosing q to be an appropriate sequence of rational numbers and a suitable direction for the shelling. We start with a technical lemma:

Lemma 4.1 (Gale's evenness condition)**.** *$C(n, d+1)(q)$ is a simplicial polytope whose facets are described by subsequences such that every segment of consecutive indices in the sequence is even.*

We will use Gale's evenness condition and select some of the facets (that are all simplices) that form a nice shellable sub-complex that will be the initial segment of the line shelling we are looking for.

As Billera-Lee did, we first construct this complex combinatorially. It will be constructed in the star of the first or the edge between the first two vertices according to the parity of d. We do it by constructing a bijection between a nice set of monomials and the $\text{st}(u)$. Then we pick initial segments that realise the g-vector in each dimension and get a shellable complex.

More precisely, consider $\text{lk}(u)$ where $u = \{v_0\}$ if d is odd and $u = \{v_0, v_1\}$ if d is even. We need to take an even number of elements to complete a facet that has d-vertices. Facets in $\text{lk}(u)$ correspond to sequences $u_{i_1}, u_{i_2}, \dots u_{i_\mu}$, where $\mu = \lfloor \frac{(d-1)}{2} \rfloor$ and $i_j + 2 \leq i_{j+1}$ and $i_\mu < n-1$. Then we throw in the vertices u_{i_j+1} for every j.

We now define a grading for this facets that agrees the grading on the set of monomials on g_1 variables.

5 Linear programming and regular triangulations