THE HANNA NEUMANN CONJECTURE FOR GRAPHS OF FREE GROUPS WITH CYCLIC EDGE GROUPS

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Abstract. The Hanna Neumann Conjecture (HNC) for a free group G predicts that $\overline{\chi}(U \cap V) \leqslant \overline{\chi}(U)\overline{\chi}(V)$ for all finitely generated subgroups U and V, where $\overline{\chi}(H) = \min\{-\chi(H), 0\}$ denotes the *reduced Euler characteristic* of H. A strengthened version of the HNC was proved independently by Friedman and Mineyev in 2011. Recently, Antolín and Jaikin-Zapirain introduced the L^2-Hall property and showed that if G is a hyperbolic limit group that satisfies this property, then G satisfies the HNC. Antolín–Jaikin-Zapirain established the L^2-Hall property for free and surface groups, which Brown–Kharlampovich extended to all limit groups. In this article, we prove the L^2-Hall property for graphs of free groups with cyclic edge groups that are hyperbolic relative to virtually abelian subgroups and also give another proof of the L^2-Hall property for limit groups. As a corollary, we show that all these groups satisfy a strengthened version of the HNC.

1. Introduction

A group G has the *Howson property* if, for all finitely generated subgroups $U, V \leqslant G$, the intersection $U \cap V$ is finitely generated. The property is named after Albert G. Howson, who proved it for free groups in [How54]. Shortly thereafter, Hanna Neumann [Neu57] quantified this property by proving that

$$\mathrm{rk}(U \cap V) - 1 \leqslant 2(\mathrm{rk}(U) - 1)(\mathrm{rk}(V) - 1)$$

whenever U and V are finitely generated subgroups of a common free group, and she conjectured that the factor of 2 on the right-hand side of the inequality could be dropped. This became known as the *Hanna Neumann conjecture*, and was the beginning of a fruitful line of research concerning these type of inequalities [Dic94a, Tar96, Min11]. As we shall see, such inequalities are not limited to free groups.

Walter Neumann [Neu90] formulated a stronger version of H. Neumann's conjecture, described in Conjecture 1.1. Given a group G with a finite classifying space, we denote by $\overline{\chi}(G) = \min\{-\chi(G), 0\}$ the *reduced Euler characteristic* of G.

Conjecture 1.1 (The Geometric Hanna Neumann Conjecture (GHNC)). *Let U and V be finitely generated subgroups of a free group G. Let T be a complete set of representatives for the double (U, V)-cosets in G. Then*

$$\sum_{t \in T} \overline{\chi}(U \cap V^t) \leqslant \overline{\chi}(U)\overline{\chi}(V). \tag{1.1}$$

This was called the *Strengthened Hanna Neumann Conjecture* by Dicks [Dic94b] and the name *Geometric Hanna Neumann Conjecture* was suggested by Antolín–Jaikin-Zapirain [AJZ22] because this statement, written in terms of the reduced

Euler characteristics $\overline{\chi}$, is the natural reformulation of the conjecture for other abstract (or even profinite) groups G. Indeed, Conjecture 1.1 makes sense whenever G is a group such that all of its finitely generated subgroups are of finite type (so that $\overline{\chi}(H)$ is defined for all finitely generated $H \leqslant G$). Conjecture 1.1 was resolved independently by Friedman [Fri15] and Mineyev [Min12]. More recently, Jaikin-Zapirain [JZ17] gave an alternative proof which also applies to free pro-p groups G. Later on, groups of dimension 2 were shown to satisfy Conjecture 1.1, such as Demushkin groups by Jaikin-Zapirain–Shusterman [JZS19] and surface groups by Antolín–Jaikin-Zapirain [AJZ22]. An important aspect of the latter article is that the authors introduce the *L^2-Hall property* as an intermediate step towards establishing Conjecture 1.1 for surface groups. This opened the possibility to showing that the GHNC holds for many more classes of groups.

We briefly recall the L^2-Hall property mentioned above. Let $\mathcal{U}(G)$ denote the algebra of affiliated operators of a group G. Then G is said to have the L^2-Hall property if for all finitely generated subgroups $H \leqslant G$ there exists a finite-index subgroup $G_1 \leqslant G$ containing H such that the kernel of the corestriction map

$$H_1(H;\mathcal{U}(G)) \longrightarrow H_1(G_1;\mathcal{U}(G))$$

has zero $\mathcal{U}(G)$-dimension (see Definitions 3.2 and 3.5 for more details). This property is named L^2-Hall because of its similarity with the local retractions property which M. Hall property [Hal49] established for free groups: if F is free and H is a finitely generated subgroup, then there is a finite-index subgroup $G \leqslant F$ containing H and a retraction $G \longrightarrow H$. The local retractions property was extended to surface groups by Scott [Sco78] and subsequently to all limit groups by Wilton [Wil08].

Antolín–Jaikin-Zapirain proved that free and surface groups have the L^2-Hall property [AJZ22, Theorem 4.4] and showed that if G is a hyperbolic limit group that has the L^2-Hall property, then Conjecture 1.1 holds for G [AJZ22, Theorem 1.3]. Recently, Brown and Kharlampovich [BK23, Corollary 28] proved that the L^2-Hall property holds for limit groups and hence that Conjecture 1.1 holds for hyperbolic limit groups G.

The main result of this article establishes the L^2-Hall property for toral relatively hyperbolic graphs of free groups with cyclic edge groups (and hence Conjecture 1.1 for these groups (Corollary C)). This is a class of groups that contains not only free groups, surface groups, and some limit groups, but also groups that do not fit into these classes, like the one-relator group with presentation $\langle a, b, c \mid a^2b^2c^3\rangle$ (see Remark 1.2).

Theorem A (Theorem 4.9)**.** *Let G be a group splitting as a finite graph of finitely generated free groups with cyclic edge groups. If G is hyperbolic relative to virtually abelian subgroups, then G satisfies the L^2-Hall property.*

We can also prove the L^2-Hall property for the class of limit groups. Perhaps the most famous characterisation of this class is the one confirmed by Sela [Sel06] in his solution of Tarski's problem on classifying finitely generated groups with the same existential theory as a free group. Kharlampovich and Miasnikov also made powerful advances on the structure theory of limit groups, proving that limit groups are exactly the finitely generated subgroups of ICE groups [KM98]; this is the smallest class of groups containing all finitely generated free groups that is closed under extending centralisers (Definition 5.1). Wilton [Wil08] used this hierarchy in his proof of the local retractions property for limit groups. We build on the methods of Wilton to establish the L^2-Hall property for limit groups in our next result, giving an alternative proof of [BK23, Corollary 28]. The potential interest in revisiting the L^2-Hall property for limit groups is to give an inductive argument that could work for more general finite abelian hierarchies (see Conjecture 1.3 below).

Theorem B (Theorem 5.7)**.** *Limit groups satisfy the L^2-Hall property.*

Antolín and Jaikin-Zapirain's proof that the L^2-Hall property implies the GHNC for hyperbolic limit groups also applies to toral relatively hyperbolic graphs of free groups with cyclic edge groups and all limit groups. To see this, one needs to incorporate recent results of Minasyan [Min23] and Minasyan–Mineh [MM22] on the Wilson–Zalesskii property and double coset separability, which were not available to Antolín–Jaikin-Zapirain. We review how all these ingredients fit together in Section 6. Thus, the following is a consequence of Theorems A and B.

Corollary C (Corollary 6.5)**.** *Suppose that G is either a limit group or that it splits as a finite graph of free groups with cyclic edge groups that is hyperbolic relative to virtually abelian subgroups. Then G satisfies the Geometric Hanna Neumann conjecture.*

The proofs of Theorems A and B are inspired by Wise's proof of subgroup separability in graphs of free groups with cyclic edge group [Wis00] and Wilton's proof of the local retractions property in limit groups [Wil08], respectively. In the proof of Theorem A, we make crucial use of the following result at several points.

Theorem D (Theorem 3.17)**.** *Let G be a finitely generated locally indicable group with $\operatorname{cd}(G) = 2$ and $b_2^{(2)}(G) = 0$. Suppose that G has a finite-index subgroup that satisfies the L^2-Hall property. Then G satisfies the L^2-Hall property.*

We give one of the reasons why Theorem D (or, in fact, the stronger version that we prove in Theorem 3.17) is needed in our proof of Theorem A. Wise showed in [Wis00, Theorem 4.18] that subgroup separable (in particular, toral relatively hyperbolic) graphs of free groups with cyclic edge groups have finite-index subgroups that are fundamental groups of *clean* graphs of graphs with S^1 edge spaces (here clean means that the edge maps are embeddings). In Wise's argument, it suffices to work with clean graphs of spaces because virtually subgroup separable groups are, again, subgroup separable. However, in general, it is unclear whether the L^2-Hall property passes to finite-index overgroups. Thankfully, Theorem D implies that this is true in our setting. Note that not all subgroup separable graphs of free groups with cyclic edge groups are L^2-Hall; for instance, $F_2 \times \mathbb{Z}$ is not L^2-Hall.

Remark 1.2. There are conjectures that relate the classes of groups of Theorems A and B. Wise asked whether graphs of free groups with cyclic edge groups are virtually limit groups if and only if they do not contain $F_2 \times \mathbb{Z}$ (see [Wis18, Problem 1.5]). If this was true, then Theorem D, together with the L^2-Hall property for limit groups (as proved in [BK23] or Theorem B) would imply Theorem A. This is the case for the hyperbolic one-relator group $G = \langle a, b, c \mid a^2b^2c^3 \rangle$, which is a non-limit group that falls under the assumptions of Theorem A, while it is also virtually limit by [Wis18].

It is desirable to have a class of groups satisfying the GHNC containing both the graphs of free groups under consideration and limit groups, as this would provide a unifying framework for our results. We conclude the introduction by proposing such a class. Let $\mathcal{C}_0$ be the class of groups containing only the trivial group. Inductively, we define $\mathcal{C}_{n+1}$ to be the class of groups G such that either G is virtually in $\mathcal{C}_n$ or G has the form $H*_A$ (resp. $H *_A K$), where H (resp. H and K) belong to $\mathcal{C}_n$ and A is a finitely generated free abelian group. We say that G admits a *finite abelian hierarchy* if it lies in $\mathcal{A}_n$ for some n.

Conjecture 1.3. *Let G be a group that admits a finite abelian hierarchy. Suppose that G is torsion-free and hyperbolic relative to virtually abelian subgroups. Then G is L^2-Hall and satisfies the Geometric Hanna Neumann Conjecture.*

The inductive proof of Theorem B already suggests that the argument could carry over into Conjecture 1.3. Another evidence for this conjecture is the weaker statement proved in [AJZ22, Theorem 9.1] that, for a group G as in Conjecture 1.3, the left hand side of the inequality (1.1) is bounded.

1.1. **Organisation of the paper.** In Section 2, we recall some standard notions that will appear throughout the article, such as graphs of groups (and spaces), group homology, and in particular L^2-homology of groups. In Section 3, we introduce Antolín–Jaikin-Zapirain's L^2-Hall property and discuss both examples and non-examples of groups with this property. The main result of this section is Corollary 3.20, which states, under some additional assumptions, that if G is a torsion-free group with an L^2-Hall subgroup of finite index, then G is L^2-Hall. This result crucially gives us the flexibility to pass to finite-index subgroups in the proof of Theorem A, which is given in Section 4. We prove Theorem B in Section 5 by adequately modifying Wilton's argument on the local retractions property for these groups. Finally, in Section 6, we review the arguments of Antolín–Jaikin-Zapirain to explain how Corollary C follows from our results combined with recent advances of Minasyan and Mineh on double coset separability.

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2. Preliminaries

2.1. **Graphs of groups and spaces.** Graphs of groups were introduced as combinatorial objects in [Ser77]. In [SW79], Scott and Wall introduced graphs of spaces in order to study graphs of groups topologically. Since we will use both viewpoints in this article, we take the time to introduce them here.

Throughout this subsection, Γ denotes a directed graph, $\mathsf{Vert}(\Gamma)$ and $\mathsf{Edge}(\Gamma)$ denote the vertex and edge sets of Γ, respectively. For any edge $e \in \mathsf{Edge}(\Gamma)$, let $\mathsf{o}(e) \in \mathsf{Vert}(\Gamma)$ and $\mathsf{t}(e) \in \mathsf{Vert}(\Gamma)$ denote the origin a terminus of e.

Definition 2.1 (Graph of groups)**.** A *graph of groups* $\mathcal{G}$ consists of the following data:

(1) a connected directed graph Γ, called the *underlying graph* of $\mathcal{G}$;
(2) groups G_v and G_e for every vertex $v \in \mathsf{Vert}(\Gamma)$ and edge $e \in \mathsf{Edge}(\Gamma)$;
(3) monomorphisms $\varphi_{e,\mathsf{o}} \colon G_e \longrightarrow G_{\mathsf{o}(e)}$ and $\varphi_{e,\mathsf{t}} \colon G_e \longrightarrow G_{\mathsf{t}(e)}$ for every edge $e \in \mathsf{Edge}(\gamma)$.

The groups G_v and G_e are called the *vertex groups* and *edge groups* of $\mathcal{G}$. The monomorphisms $\varphi_{e,\mathsf{o}}$ and $\varphi_{e,\mathsf{t}}$ are called the *edge maps* of $\mathcal{G}$.

We now review two ways to look at the fundamental group of a graph of groups.

Definition 2.2 (Based fundamental group)**.** With the same notation as in Definition 2.1, let $v_0 \in \mathsf{Vert}(\Gamma)$ be a base vertex and for each $e \in \mathsf{Edge}(\gamma)$ introduce the formal symbol t_e. Let $F(\mathcal{G})$ be the group freely generated by the vertex groups G_v and the symbols t_e subject to the relations $t_e \varphi_{e,\mathsf{t}}(g) t_e^{-1} = \varphi_{e,\mathsf{o}}(g)$ for $e \in \mathsf{Edge}(\Gamma)$ and $g \in G_e$. The *fundamental group* of the graph of group based at v_0, denoted $\pi_1(\mathcal{G}, v_0)$, is the subgroup of $P(\mathcal{G})$ consisting of the elements that can be represented as words $g_0 t_{e_1}^{\varepsilon_1} g_1 \cdots t_{e_n}^{\varepsilon_n} g_n$, where $g_i \in F_{\mathsf{t}(e_i)}$, where

$$\varepsilon_i = \pm 1, \qquad \begin{cases} g_i \in F_{\mathsf{t}(e_i)} & \text{if } \varepsilon_i = 1 \\ g_i \in F_{\mathsf{o}(e_i)} & \text{if } \varepsilon_i = -1 \end{cases}, \qquad g_0, g_n \in G_{v_0}$$

and where $(e_1, \ldots, e_n)$ forms a (not necessarily directed) loop.

An equivalent way to look at the fundamental group is explained in Definition 2.3. This will be used when considering splittings of $\mathcal{G}$ over simpler subgraphs of groups as in Proposition 3.9.

Definition 2.3 (Fundamental group relative to a spanning tree)**.** With the notation of Definition 2.1, let T be a spanning tree of Γ. The *fundamental group* of $\mathcal{G}$ relative to T, denoted by $\pi_1(\mathcal{G}, T)$, is the group freely generated by the groups G_v for all $v \in \mathsf{Vert}(\Gamma)$, and the formal symbols t_e for all $e \in \mathsf{Edge}(\Gamma)$, subject to two types of relations: $t_e \phi_{e,\mathsf{o}}(x) t_e^{-1} = \phi_{e,\mathsf{t}}(x)$ for all $e \in \mathsf{Edge}(\Gamma)$ and $x \in G_e$; and $t_e = 1$ for all $e \in \mathsf{Edge}(\Gamma) \smallsetminus \mathsf{Edge}(T)$. The two definitions coincide by [Ser77, Proposition 20, Chapitre I, §5].

A graph of groups is *finite* if its underlying graph is finite. If G is isomorphic to the fundamental group of a graph of groups, we say that G *splits* as a graph of groups. In this situation, we will often abuse terminology and say that G *is* a graph of groups. If the vertex and edge groups of a graph of groups $\mathcal{G}$ lie in classes $\mathcal{C}$ and $\mathcal{D}$, respectively, then we will say that $\mathcal{G}$ (or its fundamental group) is a graph of $\mathcal{C}$ groups with $\mathcal{D}$ edge groups. We will be mostly interested in graphs of free groups with cyclic edge groups in this article. A notable subclass which will appear is the class of *generalised Baumslag–Solitar groups*, which are the groups that split as finite graphs of $\mathbb{Z}$'s with $\mathbb{Z}$ edge groups.

Definition 2.4. Let $\mathcal{G} = (G_v, G_e; \Gamma)$ be a graph of groups. A graph of groups $\mathcal{H} = (H_v, H_e; \Upsilon)$ is a *subgraph of groups* of $\mathcal{G}$ if

(1) there is an injection $\Upsilon \hookrightarrow \Gamma$ (via which we think of Υ as a subgraph of Γ),
(2) there are inclusions $f_v \colon H_v \longhookrightarrow G_v$ and $f_e \colon H_e \longhookrightarrow G_e$ for all vertices and edges of Υ (via which we think of every H_v (resp. H_e) as a subgroup of G_v (resp. G_e)),
(3) $H_e = H_{\mathsf{o}(e)} \cap G_e$ and $H_e = H_{\mathsf{t}(e)} \cap G_e$ for every $e \in \mathsf{Edge}(\Upsilon)$, and

(4) the diagrams

$$\begin{tikzcd} H_e \arrow[r, hook] \arrow[d, hook, "f_e"] & H_{\mathsf{o}(e)} \arrow[d, hook, "f_{\mathsf{o}(e)}"] \\ G_e \arrow[r, hook] & G_{\mathsf{o}(e)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H_e \arrow[r, hook] \arrow[d, hook, "f_e"] & H_{\mathsf{t}(e)} \arrow[d, hook, "f_{\mathsf{t}(e)}"] \\ G_e \arrow[r, hook] & G_{\mathsf{t}(e)} \end{tikzcd}$$

commute for all $e \in \mathsf{Edge}(\Upsilon)$, where the horizontal maps are the edge maps of the respective graphs of groups.

Lemma 2.5 ([Bas93, Corollary 1.14])**.** *If $\mathcal{G}$ is a graph of groups and $\mathcal{H}$ is a subgraph of groups, then there is a canonical injective homomorphism $\pi_1(\mathcal{H}, v) \hookrightarrow \pi_1(\mathcal{G}, v)$ for any vertex v in the underlying graph of $\mathcal{H}$.*

Finally, note that if H is an arbitrary subgroup of a graph of groups G, then H inherits a graph of groups structure, which comes from the action of H on the Bass–Serre tree associated to G (see [Ser77, Théorème 13, Chapitre I, §5]).

We will often switch between the graph of groups and graph of spaces viewpoint, the latter of which we introduce now.

Definition 2.6 (Graph of spaces)**.** A *graph of spaces* $\mathcal{X}$ consists of the following data:

(1) a connected directed graph Γ, called the *underlying graph* of $\mathcal{X}$;
(2) based connected CW-complexes (X_v, x_v) and (X_e, x_e) for every vertex $v \in \mathsf{Vert}(\Gamma)$ and edge $e \in \mathsf{Edge}(\Gamma)$;
(3) based π_1-injective continuous maps $f_{e,\mathsf{o}} \colon X_e \longrightarrow X_{\mathsf{o}(e)}$ and $f_{e,\mathsf{t}} \colon X_e \longrightarrow X_{\mathsf{t}(e)}$ for every edge $e \in \mathsf{Edge}(\gamma)$.

The spaces X_v and X_e are called the *vertex spaces* and the *edge spaces* of $\mathcal{X}$. The maps $f_{e,\mathsf{o}}$ and $f_{e,\mathsf{t}}$ are called the edge maps. The *geometric realisation* of $\mathcal{X}$ is the quotient of the space

$$X = \left(\bigsqcup_{v \in \mathsf{Vert}(\Gamma)} X_v \right) \sqcup \left(\bigsqcup_{e \in \mathsf{Edge}(\Gamma)} X_e \times [0,1] \right)$$

by the relations $f_{e,\mathsf{o}}(x) \sim (x, 0)$ and $\varphi_{e,1}(x) \sim (x, 1)$ for all $x \in X_e$ and all $e \in \mathsf{Vert}(E)$. The *fundamental group* of the topological space $\mathcal{X}$ based at x_{v_0}, denoted $\pi_1(\mathcal{X}, x_{v_0})$, is defined to be $\pi_1(X, x_{v_0})$. When no confusion arises, we will usually refer to the geometric realisation of $\mathcal{X}$ as a graph of spaces.

There is a correspondence between graphs of spaces and graphs of groups. If $\mathcal{G}$ is a graph of groups, then a graph of spaces $\mathcal{X}$ can be constructed as follows: For each $v \in \mathsf{Vert}(\Gamma)$ and $e \in \mathsf{Edge}(\Gamma)$, let $X_v = K(G_v, 1)$ and $X_e = K(G_e, 1)$, and let $f_{e,\mathsf{o}}$ and $f_{e,\mathsf{t}}$ be maps inducing $\varphi_{e,\mathsf{o}}$ and $\varphi_{e,\mathsf{t}}$. Then there is an isomorphism between $\pi_1(\mathcal{G}, v_0)$ and $\pi_1(\mathcal{X}, x_{v_0})$ (which depends on the choices of v_0 and x_{v_0} up to conjugation). Similarly, given a graph of spaces, we can form a graph of groups with vertex groups $\pi_1(X_v, x_v)$, edge groups $\pi_1(X_e, x_e)$, and edge maps $(f_{e,\mathsf{o}})_*$ and $(f_{e,\mathsf{t}})_*$.

We will often realise graphs of free groups with cyclic edge groups as the fundamental group of a *graphs of graphs with S^1 edge spaces*, by which we understand a graph of spaces where every vertex space is a graph and every edge space is a copy of S^1.

The notion of a precovering will appear throughout Sections 4 and 5, so we recall it here. It shows up naturally when completing a compact subspace of a covering space to a finite-sheeted covering.

Definition 2.7. A map between (the geometric realisations of) graphs of spaces $X' \longrightarrow X$ is a *precovering* if it is locally injective, all the maps $X'_e \longrightarrow X_{f(e)}$ and $X'_v \longrightarrow X_{f(v)}$ are covering maps, and all the diagrams

$$\begin{tikzcd} X'_e \arrow[r] \arrow[d] & X'_{\mathsf{o}(e)} \arrow[d] \\ X_{f(e)} \arrow[r] & X_{f(\mathsf{o}(e))} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X'_e \arrow[r] \arrow[d] & X'_{\mathsf{t}(e)} \arrow[d] \\ X_{f(e)} \arrow[r] & X_{f(\mathsf{t}(e))} \end{tikzcd}$$

commute. The domain X' is called a *precover*.

A precovering $X' \longrightarrow X$ is a covering if and only if all the elevations of edge maps of X to X' are edge maps of X'. The fact that covering maps induce injections on fundamental groups also applies to precoverings.

Lemma 2.8 ([Wil08, Proposition 2.19])**.** *A precovering $X' \longrightarrow X$ induces an injection $\pi_1(X') \longrightarrow \pi_1(X)$.*

We will also require subgraphs of spaces, which induce subgraphs of groups in the sense of Definition 2.4.

Definition 2.9. Let $\mathcal{X} = (X_v, X_e; \Gamma)$ be a graph of spaces. A graph of spaces $\mathcal{Y} = (Y_v, Y_e; \Upsilon)$ is a *subgraph of spaces* of $\mathcal{X}$ if

(1) there is an injection $\Upsilon \hookrightarrow \Gamma$ (via which we think of Υ as a subgraph of Γ),
(2) there are π_1-injective inclusions $f_v \colon Y_v \hookrightarrow X_v$ and $f_e \colon Y_e \hookrightarrow X_e$ for all edges and vertices of Υ (via which we think of every Y_v (resp. Y_e) as a subspace of X_v (resp. X_e)),
(3) $\pi_1(Y_e) = \pi_1(Y_{\mathsf{o}(e)}) \cap \pi_1(X_e)$ and $\pi_1(Y_e) = \pi_1(Y_{\mathsf{t}(e)}) \cap \pi_1(X_e)$ for every edge $e \in \mathsf{Edge}(\Upsilon)$, and
(4) the diagrams

$$\begin{tikzcd} Y_e \arrow[r, hook] \arrow[d, hook, "f_e"] & Y_{\mathsf{o}(e)} \arrow[d, hook, "f_{\mathsf{o}(e)}"] \\ X_e \arrow[r, hook] & X_{\mathsf{o}(e)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y_e \arrow[r, hook] \arrow[d, hook, "f_e"] & Y_{\mathsf{t}(e)} \arrow[d, hook, "f_{\mathsf{t}(e)}"] \\ X_e \arrow[r, hook] & X_{\mathsf{t}(e)} \end{tikzcd}$$

commute for all $e \in \mathsf{Edge}(\Upsilon)$, where the horizontal maps are edge maps in the corresponding graphs of spaces.

We close by remarking that if X decomposes as a graph of spaces and $Y \longrightarrow X$ is a covering space, then Y inherits a graph of spaces structure where every vertex (resp. edge) space of Y covers some vertex (resp. edge) space of X.

2.2. **Homology of groups.** Unless stated otherwise, all modules will are assumed to be left modules. Let R be a ring. Given a right R-module M and a left R-module N, we can define the abelian group $\operatorname{Tor}_i^R(M,N)$. By definition, $\operatorname{Tor}_0^R(M,N) \cong M \otimes_R N$ as an abelian group. In general, the functors $\operatorname{Tor}_n^R(M,-)$ are the derived functors of $M \otimes_R -$. More concretely, we choose a projective resolution $P_\bullet \longrightarrow N \longrightarrow 0$ and define $\operatorname{Tor}_n^M(M,N) := H_n(M \otimes_R P_\bullet)$.

Let S be another ring. If M is additionally an (S,R)-bimodule, then $\operatorname{Tor}_n^R(M,N)$ is naturally a left S-module for all n. Similarly, if N is an (R,S)-bimodule, then $\operatorname{Tor}_n^R(M,N)$ is naturally a right S-module. A standard tool we will use is the long exact sequence in Tor associated to a short exact sequence of modules. Let $0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$ be a short exact sequence of R-modules and let M be an (S,R)-bimodule. Then there is a long exact sequence of left S-modules of the form

$$\begin{tikzcd} & & \cdots \arrow[r] & \operatorname{Tor}_{n+1}^R(M,N_3) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & \operatorname{Tor}_n^R(M,N_1) \arrow[r] & \operatorname{Tor}_n^R(M,N_2) \arrow[r] & \operatorname{Tor}_n^R(M,N_3) \arrow[r] & \cdots . \end{tikzcd}$$

A standard reference for this material is [Wei94, Chapters 2 and 3].

Let G be a group and let M be an $R[G]$-module. As in [Bro94, Chapter III, Section 2], the n-dimensional homology of G with coefficients in M is given by

$$H_n(G;M) := \operatorname{Tor}_n^{R[G]}(R,M)$$

where R denotes the trivial right $R[G]$-module. Chiswell's Mayer–Vietoris exact sequence will be a very useful tools when establishing the L^2-Hall property for certain graphs of groups.

Theorem 2.10 ([Chi76, Theorem 2])**.** *Let R be a ring, let $\mathcal{G}$ be a graph of groups with underlying graph Γ and $G = \pi_1(\mathcal{G})$, and let M be an $R[G]$-module. Then there is a long exact sequence*

$$\begin{tikzcd} & & \cdots \arrow[r] & H_{n+1}(G;M) \arrow[dll, rounded corners] \\ & \bigoplus_{e \in \mathsf{Edge}(\Gamma)} H_n(G_e;M) \arrow[r] & \bigoplus_{v \in \mathsf{Vert}(\Gamma)} H_n(G_v;M) \arrow[r] & H_n(G;M) \arrow[r] & \cdots . \end{tikzcd}$$

Given a field K and a group G, denote by I_G the augmentation ideal of the group ring $K[G]$ (in practice, this notation will present no ambiguity as the choice of coefficient field K will be clear from the context). Given a subgroup $H \leqslant G$, we denote by I_H^G the left $K[G]$-submodule of I_G generated by I_H. In addition, even if H is not normal in G, we will write $K[G/H]$ to refer to the left $K[G]$-module of left cosets of H in G. The following canonical isomorphisms will be useful later.

Lemma 2.11 ([JZ23], Lemma 2.1)**.** *Let $T \leqslant H \leqslant G$ be subgroups. Then the following holds.*

(a) The canonical map $K[G] \otimes_{K[H]} I_H \longrightarrow I_H^G$ that sends $a \otimes b$ to $a \cdot b$ for all $a \in K[G]$ and $b \in I_H$ is an isomorphism of left $K[G]$-modules.
(b) The canonical map $K[G] \otimes_{K[H]} \left(I_H/I_T^H\right) \longrightarrow I_H^G/I_T^G$ that sends $a \otimes (b + I_T^H)$ to $ab + I_T^G$ for all $a \in K[G]$ and $b \in I_H$ is an isomorphism of left $K[G]$-modules.
(c) The kernel of the canonical map of $K[G]$-modules $K[G/T] \longrightarrow K[G/H]$ is naturally isomorphic to I_H^G/I_T^G.

2.3. **Hughes-free division rings and L^2-Betti numbers.** Let G be a locally indicable group and let K be a field. An embedding $\varphi\colon K[G] \hookrightarrow \mathcal{D}$ of the group algebra $K[G]$ into a division ring $\mathcal{D}$ is called *Hughes-free* if the following conditions hold.

(1) The image $\varphi(K[G])$ generates $\mathcal{D}$ as a division ring.

(2) Let $H \leqslant G$ be a finitely generated subgroup and let $f\colon H \longrightarrow \mathbb{Z}$ be an epimorphism with kernel N, and let $t \in H$ map to a generator of $\mathbb{Z}$ under f. Let $\mathcal{D}_N$ denote the division closure of $\varphi(K[N])$. Then $\{\varphi(t^i) : i \in \mathbb{Z}\} \subseteq \mathcal{D}$ is linearly independent over $\mathcal{D}_N$.

By a theorem of Hughes, if a Hughes-free embedding of $K[G]$ exists, then it is unique up to $K[G]$-isomorphism [Hug70]. Thus, if $K[G]$ has a Hughes-free embedding, then we denote the division ring by $\mathcal{D}_{K[G]}$ and think of $K[G]$ as a subset of $\mathcal{D}_{K[G]}$. We will call $\mathcal{D}_{K[G]}$ the *Hughes-free division ring* of $K[G]$. Note that if $H \leqslant G$ is any subgroup, then the division closure of $K[H]$ in $\mathcal{D}_{K[G]}$ is isomorphic to the Hughes-free division ring $\mathcal{D}_{K[H]}$. The existence of Hughes-free division rings has been established for many classes of locally indicable groups, and in particular for all locally indicable groups when the ground field K has characteristic zero.

Proposition 2.12. *Let G be locally indicable. Then $\mathcal{D}_{K[G]}$ exists if*

(a) the field K has characteristic zero (in this case, $\mathcal{D}_{K[G]}$ is Hughes-free); or
(b) if G residually (locally indicable and amenable) or virtually compact special.

Proof. If K is of characteristic zero, then the existence of $\mathcal{D}_{K[G]}$ is a consequence of the resolution of the Atiyah conjecture for locally indicable groups [JZLÁ20, Corollary 1.4]. If K is of arbitrary characteristic, then $\mathcal{D}_{K[G]}$ exists for G residually (locally indicable and amenable) by [JZ21, Corollary 1.3] and for G virtually compact special by [FSP23, Theorem 1.2]. □

The groups we will be working with in this article are locally indicable and virtually compact special, so we will always assume that any group algebra $K[G]$ has a Hughes-free division ring $\mathcal{D}_{K[G]}$.

Lemma 2.13. *Let G be a graph of free groups with cyclic edge groups. If K is a field of characteristic zero, then $\mathcal{D}_{K[G]}$ exists. If we assume that G is subgroup separable, then $\mathcal{D}_{K[G]}$ exists for arbitrary K.*

Proof. Graphs of groups are the direct limits of their finite subgraphs of groups, and direct limits of Hughes-free division rings are Hughes-free. Thus it suffices to prove the lemma assuming that G is a finite graph of groups. First note that G is locally indicable, a fact which follows easily from [How82, Theorem 4.2]. Thus if K is of characteristic zero, then $\mathcal{D}_{K[G]}$ exists by Proposition 2.12(1). If G is subgroup separable, then G is virtually compact special [MM22, Corollary 2.3] and thus $\mathcal{D}_{K[G]}$ exists by Proposition 2.12(2). □

Remark 2.14. Let G be a graph of free groups with cyclic edge groups. It is known that $K[G]$ embeds in a division ring by [FSP23, Theorem 1.3], but it is not known whether the embedding is Hughes-free. Jaikin-Zapirain conjectures that Hughes-free embeddings of $K[G]$ exist for all locally indicable groups G and all fields K [JZ21, Conjecture 1].

Hughes-free division rings provide powerful homological invariants. Recall that modules over a division ring are automatically free modules and that they have a well-defined dimension. Thus, if M is a $K[G]$-module, we can define its $\mathcal{D}_{K[G]}$-dimension by

$$\dim_{\mathcal{D}_{K[G]}} M := \dim_{\mathcal{D}_{K[G]}}(\mathcal{D}_{K[G]} \otimes_{K[G]} M)$$

and more generally $\mathcal{D}_{K[G]}$-Betti numbers by

$$\beta_n^{K[G]}(M) := \dim_{\mathcal{D}_{K[G]}} \operatorname{Tor}_n^{K[G]}(\mathcal{D}_{K[G]}, M). \tag{2.1}$$

We will not need these $\mathcal{D}_{K[G]}$-Betti numbers of general $K[G]$-modules until Section 6. Note that $\beta_0^{k[G]}(M) = \dim_{\mathcal{D}_{K[G]}} M$. When $K = \mathbb{C}$, we will write $\beta_n^{(2)}(M)$ instead of $\beta_n^{K[G]}(M)$. Setting K to be the trivial $K[G]$-module, we obtain homological numerical invariants of the group G:

$$b_n^{K[G]}(G) := \beta_n^{K[G]}(K) = \dim_{\mathcal{D}_{K[G]}} \operatorname{Tor}_n^{K[G]}(\mathcal{D}_{K[G]}, K). \tag{2.2}$$

We will refer to these as the $\mathcal{D}_{K[G]}$-Betti numbers of G.

The properties listed in the following proposition will be used throughout the article. We emphasize point (1) below, which states that when $K = \mathbb{C}$, the $\mathcal{D}_{K[G]}$-Betti numbers coincide with the L^2-Betti numbers of G.

Proposition 2.15. *Let G be a locally indicable group and let K be a field such that $\mathcal{D}_{K[G]}$ exists.*

(1) If $K = \mathbb{C}$, then $b_n^{K[G]}(G) = b_n^{(2)}(G)$ for all n.
(2) If G is nontrivial, then $b_0^{K[G]}(G) = 0$, otherwise $b_0^{K[G]}(G) = 1$.
(3) If G is of finite type, then $\chi(G) = \sum_{i=0}^{\infty}(-1)^i b_i^{K[G]}(G)$.
(4) Let $H \leqslant G$ be a subgroup of finite index. Then $\mathcal{D}_{K[H]} \otimes_{K[H]} K[G] \cong \mathcal{D}_{K[G]}$ as $(\mathcal{D}_{K[H]}, K[G])$-bimodules. Consequently, $b_n^{K[H]}(H) = |G : H| \cdot b_n^{K[G]}(G)$ for all n.
(5) If G is free on n-generators, then $b_1^{K[G]}(G) = n - 1$ and $b_n^{K[G]}(G) = 0$ for all $n \neq 1$. If G is amenable, then $b_n^{K[G]}(G) = 0$ for all n.

Proof. Statement (1) follows from [JZLÁ20, Theorem 1.1], while (2) and (3) can be proved directly from the definitions. Statement (4) is a direct consequence of [Grä20, Corollary 8.3] (for a detailed proof see [Fis21, Lemma 6.3]). For (5), the claim about free groups can be proved easily using (2) and (3). The claim about amenable groups follows from [HK21, Theorem 3.9(6)] (only the case $K = \mathbb{Q}$ is treated there, but the case with K arbitrary has the same proof). $\square$

3. L^2-independence and the L^2-Hall property

In this section we discuss the L^2-Hall property and the concept of L^2-independent subgroups in more detail. We then study various combinatorial situations (in terms of graphs of groups) that provide L^2-independent subgroups (which we shall need in the proofs of Theorems A and B) and show in Theorem 3.17 that the L^2-Hall property passes to finite-index overgroups in our setting (as anticipated in Theorem D).

Convention 3.1. In this section, K always denotes a field. Apart from in some isolated examples, all groups appearing are assumed to be locally indicable and we assume that their group algebras over K have Hughes-free embeddings (recall that this is the case when char $K = 0$ by Proposition 2.12).

3.1. **Definitions and basic properties.**

The notion of L^2-independence and the L^2-Hall property were introduced Antolín–Jaikin-Zapirain [AJZ22] in connection with proving that surface groups satisfy the Strengthened Hanna Neumann Conjecture. We recall these definitions.

Definition 3.2. Let H be a subgroup of G. Consider the natural surjection of left $K[G]$-modules $K[G/H] \longrightarrow K$. This induces a natural map

$$\operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, K[G/H]) \longrightarrow \operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, K).$$

We say that H is $\mathcal{D}_{K[G]}$*-independent* if the map is injective. When $K = \mathbb{C}$, we will say that H is L^2*-independent* in G.

The injectivity of the above map depends on the choice of embedding of H into G. For example, the embedding $f\colon F(a,b,c) \longrightarrow G = F(x,y,z)$ defined by $f(a) = x^2$, $f(b) = y$ and $f(c) = y^x$ does not lead to an L^2-independent subgroup of G. For this reason, the following definition will be useful later.

Definition 3.3. Given a monomorphism $f\colon H \hookrightarrow G$, we will say that f is $\mathcal{D}_{K[G]}$*-injective* if $f(H)$ is $\mathcal{D}_{K[G]}$-independent in G (or L^2*-injective* when $K = \mathbb{C}$).

By [AJZ22, Proposition 4.2], H is $\mathcal{D}_{K[G]}$-independent in G if and only if the co-restriction map $H_1(H;\mathcal{D}_{K[G]}) \longrightarrow H_1(G;\mathcal{D}_{K[G]})$ is injective. So Definition 3.2 is the natural generalisation of Antolín–Jaikin-Zapirain's definition of L^2-independence [AJZ22, Section 4] for other division rings $\mathcal{D}_{K[G]}$. Working in this greater generality will uniformly include various cases of interest while adding no technical difficulty.

The augmentation ideal corresponding to a subgroup captures a lot of structure of the subgroup and hence Proposition 3.4 provides a useful reformulation of the notion of $\mathcal{D}_{K[G]}$-independence.

Proposition 3.4 ([AJZ22, Corollary 4.3])**.** *Let $H \leqslant U \leqslant G$ be finitely generated subgroups and suppose that $b_2^{K[G]}(G) = 0$. Then H is $\mathcal{D}_{K[G]}$-independent in U if and only if $b_1^{K[G]}(I_U^G/I_H^G) = 0$.*

Definition 3.5. We say that a group G is $\mathcal{D}_{K[G]}$*-Hall* or has the $\mathcal{D}_{K[G]}$*-Hall property* if for every finitely generated subgroup $H \leqslant G$ there exists a finite-index subgroup $G_1 \leqslant G$ such that H is $\mathcal{D}_{K[G]}$-independent in G_1. If $K = \mathbb{C}$, we say that G is L^2*-Hall* or has the L^2*-Hall property.*

Remark 3.6. Note that the L^2-Hall property can be defined for all groups, while the $\mathcal{D}_{K[G]}$-Hall property only makes sense for locally indicable groups for which $\mathcal{D}_{K[G]}$ exists. Indeed, if $H \leqslant G$ and $\mathcal{U}(G)$ is the algebra of affiliated operators of G, then we say that H is L^2*-independent* in G if

$$\dim_{\mathcal{U}(G)} \ker\left(H_1(H;\mathcal{U}(G)) \longrightarrow H_1(G;\mathcal{U}(G))\right) = 0 \tag{3.1}$$

and that G has the L^2*-Hall property* if every finitely generated subgroup of G is L^2-independent in a finite-index subgroup of G. These definitions agree with Definitions 3.2 and 3.5 by [AJZ22, Lemma 4.1]. We mention this because we will discuss the L^2-Hall property for some non locally indicable groups later in this section. On the other hand, an advantage of working with the $\mathcal{D}_{K[G]}$-Hall property is that the condition that $H_1(H;\mathcal{D}_{K[G]}) \longrightarrow H_1(G;\mathcal{D}_{K[G]})$ be injective is somewhat less awkward than the condition in 3.1.

The following hereditary feature of the L^2-Hall property will be useful later.

Lemma 3.7. *The $\mathcal{D}_{K[G]}$-Hall property passes to subgroups.*

Proof. Let G be a $\mathcal{D}_{K[G]}$-Hall group and let $H \leqslant G$ be a subgroup. Let $U \leqslant H$ be a finitely generated subgroup. Then there is a subgroup $G_0 \leqslant G$ of finite index such that the horizontal map in the diagram

$$\begin{tikzcd} H_1(U;\mathcal{D}_{K[G_0]}) \arrow[rr] \arrow[dr] & & H_1(G_0;\mathcal{D}_{K[G_0]}) \\ & H_1(G_0\cap H;\mathcal{D}_{K[G_0]}) \arrow[ur] & \end{tikzcd}$$

is injective. But then $H_1(U;\mathcal{D}_{K[G_0]}) \longrightarrow H_1(G_0\cap H;\mathcal{D}_{K[G_0]})$ is injective. Since extensions of division rings are faithfully flat, the commutative diagram

$$\begin{tikzcd} H_1(U;\mathcal{D}_{K[G_0]}) \arrow[r] \arrow[d, no head, "\cong"] & H_1(G_0\cap H;\mathcal{D}_{K[G_0]}) \arrow[d, no head, "\cong"] \\ \mathcal{D}_{K[G_0]} \underset{\mathcal{D}_{K[G_0\cap H]}}{\otimes} H_1(U;\mathcal{D}_{K[G_0\cap H]}) \arrow[r] & \mathcal{D}_{K[G_0]} \underset{\mathcal{D}_{K[G_0\cap H]}}{\otimes} H_1(G_0\cap H;\mathcal{D}_{K[G_0\cap H]}) \end{tikzcd}$$

implies that $H_1(U;\mathcal{D}_{k[G_0\cap H]}) \longrightarrow H_1(G_0\cap H;\mathcal{D}_{k[G_0\cap H]})$ is injective. Hence, H has the L^2-Hall property. $\square$

We now collect various instances where we understand L^2-independent subgraphs of groups. The first of such examples is Lemma 3.8 and will be useful when establishing the L^2-Hall property for graphs of free groups with cyclic edge groups.

Lemma 3.8. *Let $\mathcal{Y}$ be a subgraph of groups of $\mathcal{Z}$, and let $Z := \pi_1(\mathcal{Z})$. If*

1. *the maps $Y_v \longrightarrow Z_v$ are $\mathcal{D}_{K[Z]}$-injective for all $v \in \mathsf{Vert}(\Gamma^{\mathcal{Y}})$,*
2. *$b_1^{\mathcal{D}_{K[Z]}}(Z_e) = 0$ for all $e \in \mathsf{Edge}(\Gamma^{\mathcal{Z}})$, and*
3. *the groups Y_e and Z_e are isomorphic for all $e \in \mathsf{Edge}(\Gamma^{\mathcal{Y}})$*

then the canonical injection $\pi_1(\mathcal{Y}) \longrightarrow \pi_1(\mathcal{Z})$ is $\mathcal{D}_{K[Z]}$-injective.

Proof. We view $\pi_1(\mathcal{Y})$ as a subgroup of Z via the canonical inclusion. The subgraph of groups $\mathcal{Y}$ of $\mathcal{Z}$ induces a map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus\limits_{e\in\mathsf{Edge}(\Gamma^{\mathcal{Y}})} K[Y/Y_v] \arrow[r] \arrow[d] & \bigoplus\limits_{v\in\mathsf{Vert}(\Gamma^{\mathcal{Y}})} K[Y/Y_v] \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \bigoplus\limits_{e\in\mathsf{Edge}(\Gamma^{\mathcal{Z}})} K[Z/Z_e] \arrow[r] & \bigoplus\limits_{v\in\mathsf{Edge}(\Gamma^{\mathcal{Z}})} K[Z/Z_v] \arrow[r] & K \arrow[r] & 0\,. \end{tikzcd}$$

Since Chiswell's Mayer–Vietoris exact sequence is induced by applying a Tor functor to the short exact sequences of the above form (see the proof of [Chi76, Theorem 2]), the long exact sequences are automatically natural and thus we obtain maps between Chiswell's exact sequences for $\mathcal{Y}$ and $\mathcal{Z}$:

$$\begin{tikzcd} \bigoplus\limits_{e\in\mathsf{Edge}(\Gamma^{\mathcal{Y}})} H_1(Y_e) \arrow[r] \arrow[d, "\cong"] & \bigoplus\limits_{v\in\mathsf{Vert}(\Gamma^{\mathcal{Y}})} H_1(Y_v) \arrow[r] \arrow[d, hook] & H_1(\pi_1(\mathcal{Y})) \arrow[r] \arrow[d] & \bigoplus\limits_{ve\in\mathsf{Edge}(\Gamma^{\mathcal{Y}})} H_0(Y_e) \arrow[d, "\cong"] \\ \bigoplus\limits_{e\in\mathsf{Edge}(\Gamma^{\mathcal{Z}})} H_1(Z_e) \arrow[r] & \bigoplus\limits_{v\in\mathsf{Vert}(\Gamma^{\mathcal{Z}})} H_1(Z_v) \arrow[r] & H_1(Z) \arrow[r] & \bigoplus\limits_{e\in\mathsf{Edge}(\Gamma^{\mathcal{Z}})} H_0(Z_e)\,, \end{tikzcd}$$

where $H_i(-)$ stands for $H_i(-;\mathcal{D}_{K[Z]})$ (for $i = 0, 1$). By the Four Lemma, the map $H_1(\pi_1(\mathcal{Y})) \longrightarrow H_1(Z)$ is injective. $\square$

Our following technical proposition will be crucial to establish the L^2-Hall property for limit groups in Section 5. We also consider it to be of potential interest for proving that relatively hyperbolic groups with a finite abelian hierarchy have the L^2-Hall property (Conjecture 1.3).

Proposition 3.9. *Let $\mathcal{W}$ be a subgraph of groups of $\mathcal{Z}$ that have the same underlying graph Γ and all of whose edge groups are infinite cyclic. Let $G = \pi_1(\mathcal{Z})$ and suppose that there is a bipartite structure $\mathsf{Vert}(\Gamma) = \mathsf{Vert}_{\mathsf{o}} \sqcup \mathsf{Vert}_{\mathsf{t}}$ of Γ so that no two different edges of $\mathsf{Edge}(\Gamma)$ have the same endpoints. We moreover assume that the orientation on Γ is such that $\mathsf{o}(e) \in \mathsf{Vert}_{\mathsf{o}}$ and $\mathsf{t}(e) \in \mathsf{Vert}_{\mathsf{t}}$ for all $e \in \mathsf{Edge}(\Gamma)$. We denote by z_e a generator of the infinite cyclic group Z_e. Let T be a spanning tree of Γ. Fix a presentation of G (as described in Definition 2.3) and, for every $e \in \mathsf{Edge}(\Gamma) \setminus \mathsf{Edge}(T)$, denote by t_e the formal letter associated to e. For all $v \in \mathsf{Vert}_{\mathsf{o}}$, we consider finite subsets $\mathcal{L}_v^{(0)} \subseteq \mathcal{L}_v \subseteq Z_v$ such that*

$$\mathcal{L}_v \setminus \mathcal{L}_v^{(0)} \subseteq \bigcup_{\mathsf{o}(e)=v} \phi_{\mathsf{o},e}(z_e) \subseteq W_v \cup \left(\mathcal{L}_v \setminus \mathcal{L}_v^{(0)}\right).$$

Suppose that, for all $v \in \mathsf{Vert}_{\mathsf{o}}$, the natural map

$$W_v * \left(\coprod_{\mathcal{L}_v} \mathbb{Z}\right) \longrightarrow Z_v$$

is injective and $\mathcal{D}_{K[G]}$-injective. We fix a subset $\mathsf{Edge}(T) \subseteq E_T \subseteq \mathsf{Edge}(\Gamma)$ such that $\phi_{e,\mathsf{o}}(z_e) \in \mathcal{L}_{\mathsf{o}(e)} \setminus \mathcal{L}_{\mathsf{o}(e)}^{(0)}$ for all $e \in \mathsf{Edge}(\Gamma) \setminus E_T$. If we name $\mathcal{L}^{(0)} = \bigcup_{v \in \mathsf{Vert}(\Gamma)} \mathcal{L}_v^{(0)}$ and $\mathcal{L}^{(t)} = \{t_e : e \in \mathsf{Edge}(\Gamma) \setminus E_T\}$, then the natural map

$$\pi_1(\mathcal{W}) * \left(\coprod_{\mathcal{L}^{(0)} \bigcup \mathcal{L}^{(t)}} \mathbb{Z}\right) \longrightarrow \pi_1(\mathcal{Z}) \tag{3.2}$$

is injective and $\mathcal{D}_{K[G]}$-injective.

Proof. Recall from Lemma 2.5 that, given a subgraph of groups $\mathcal{H}$ of $\mathcal{G}$, the canonical map $\pi_1(\mathcal{H}) \longrightarrow \pi_1(\mathcal{G})$ injective. We will consider several intermediate graph of groups $\mathcal{W} \leqslant \mathcal{W}^{(1)} \leqslant \mathcal{W}^{(2)} \leqslant \mathcal{W}^{(3)} \leqslant \mathcal{W}^{(4)} \leqslant \mathcal{Z}$ to prove the claim. We will only specify their vertex and edge groups, and the corresponding edge maps will be assumed to be the restrictions of the edge maps of $\mathcal{Z}$.

The graph of groups $\mathcal{W}^{(1)}$ is defined as follows. For all $v \in \mathsf{Vert}_{\mathsf{t}}$, $W_v^{(1)} = Z_v$. For all $v \in \mathsf{Vert}_{\mathsf{t}}$ and $e \in \mathsf{Edge}(\Gamma)$, $W_v^{(1)} = W_v$ and $W_e^{(1)} = W_e$. By Lemma 3.8, the canonical map

$$\pi_1(\mathcal{W}) \longrightarrow \pi_1(\mathcal{W}^{(1)}) \tag{3.3}$$

is $\mathcal{D}_{K[G]}$-injective.

We split $\mathcal{L}_v \setminus \mathcal{L}_v^{(0)}$ as a disjoint union of $\mathcal{L}_v^{(1)}$ and $\mathcal{L}_v^{(2)}$, where $\mathcal{L}_v^{(1)}$ consists exactly of the elements $\phi_{\mathsf{o},e}(z_e) \in \mathcal{L}_v$ such that $e \in E_T$. Consider another intermediate graph of groups $\mathcal{W}^{(1)} \leqslant \mathcal{W}^{(2)} \leqslant \mathcal{Z}$ defined as follows:

- $W_v^{(2)} = W_v^{(1)} * \left(\coprod_{\mathcal{L}_v^{(1)}} \mathbb{Z}\right)$ for $v \in \mathsf{Vert}_{\mathsf{o}}$;
- $W_v^{(2)} = W_v^{(1)}$ for $v \in \mathsf{Vert}_{\mathsf{t}}$;
- $W_e^{(2)} = Z_e$ for $e \in E_T$;
- $W_e^{(2)} = W_e^{(1)}$ for $e \in \mathsf{Edge}(\Gamma) \setminus E_T$.

Letting $E^{(1)} = \{t_e : e \in E_T \smallsetminus E(T), \phi_{e,\mathsf{o}}(z_e) \in \mathcal{L}_v^{(1)}\}$, the canonical map

$$\pi_1(\mathcal{W}^{(1)}) * \left(\coprod_{E^{(1)}} \mathbb{Z}\right) \longrightarrow \pi_1(\mathcal{W}^{(2)}) \tag{3.4}$$

is an isomorphism, so $\pi_1(\mathcal{W}^{(1)}) \longrightarrow \pi_1(\mathcal{W}^{(2)})$ is $\mathcal{D}_{K[G]}$-injective.

We define $\mathcal{W}^{(2)} \leqslant \mathcal{W}^{(3)} \leqslant \mathcal{Z}$ as follows:

- $W_v^{(3)} = W_v^{(2)} * \left(\coprod_{\mathcal{L}_v^{(0)}} \mathbb{Z}\right)$ for $v \in \mathsf{Vert}_\mathsf{o}$;
- $W_v^{(3)} = W_v^{(2)}$ for $v \in \mathsf{Vert}_\mathsf{t}$;
- $W_e^{(3)} = W_v^{(2)}$ for $e \in E_T$;
- $W_e^{(3)} = W_v^{(2)}$ for $e \in \mathsf{Edge}(\Gamma) \smallsetminus E_T$.

It is immediate to see from the presentation of $\pi_1(\mathcal{W}^{(3)})$ that the canonical map

$$\pi_1(\mathcal{W}^{(2)}) * \left(\coprod_{\mathcal{L}_v^{(0)}} \mathbb{Z}\right) \longrightarrow \pi_1(\mathcal{W}^{(3)}) \tag{3.5}$$

is an isomorphism. Finally, we define $\mathcal{W}^{(3)} \leqslant \mathcal{W}^{(4)} \leqslant \mathcal{Z}$ as follows:

- $W_v^{(4)} = W_v^{(3)} * \left(\coprod_{\mathcal{L}_v^{(2)}} \mathbb{Z}\right)$ for $v \in \mathsf{Vert}_\mathsf{o}$;
- $W_v^{(4)} = W_v^{(3)}$ for $v \in \mathsf{Vert}_\mathsf{t}$;
- $W_e^{(4)} = Z_e$ for $e \in \mathsf{Edge}(\Gamma)$.

We observe that

$$\pi_1(\mathcal{W}^{(3)}) * \left(\coprod_{\mathcal{L}_v^{(t)}} \mathbb{Z}\right) \longrightarrow \pi_1(\mathcal{W}^{(4)}) \tag{3.6}$$

is an isomorphism. Observe that $\mathcal{W}^{(4)} \leqslant \mathcal{Z}$ admits the following description:

- $W_v^{(4)} = W_v * \left(\coprod_{\mathcal{L}_v} \mathbb{Z}\right)$ for $v \in \mathsf{Vert}_\mathsf{o}$;
- $W_v^{(4)} = Z_v$ for $v \in \mathsf{Vert}_\mathsf{t}$;
- $W_e^{(4)} = Z_e$ for $e \in \mathsf{Edge}(\Gamma)$.

By our assumption on (3.2) and by Lemma 3.8, the canonical map

$$\pi_1(\mathcal{W}^{(4)}) \longrightarrow \pi_1(\mathcal{Z}) \tag{3.7}$$

is $\mathcal{D}_{K[G]}$-injective. From the chain of injections and $\mathcal{D}_{K[G]}$-injections described in (3.3), (3.4), (3.5), (3.6) and (3.7); we conclude that the canonical map

$$\pi_1(\mathcal{W}) * \left(\coprod_{\mathcal{L}^{(0)} \bigcup \mathcal{L}^{(t)}} \mathbb{Z}\right) \longrightarrow \pi_1(\mathcal{Z})$$

is injective and $\mathcal{D}_{K[G]}$-injective. The proof is complete. □

3.2. **Examples.** We are already in a position to establish the $\mathcal{D}_{K[G]}$-Hall property for some classes of groups.

Example 3.10 (Amenable groups)**.** Let G be a group with the property that $b_1^{K[G]}(H) = 0$ for all subgroups $H \leqslant G$. Then G is trivially $\mathcal{D}_{K[G]}$-Hall. Since amenable groups have vanishing L^2-Betti numbers above degree 0 and amenability passes to subgroups, this shows that amenable groups are L^2-Hall. If G is amenable and $K[G]$ is a domain (which is the case for us, since we are assuming Convention 3.1), then the same reasoning shows that G is $\mathcal{D}_{K[G]}$-Hall.

There are also non-amenable groups which are L^2-Hall for the reason discussed above. As an example, let T be a Tarski monster of prime order p and let $G = T \times \mathbb{Z}$. Since all of the proper subgroups of T are isomorphic to $\mathbb{Z}/p$, it follows that every finitely generated subgroup H of G has $b_1^{(2)}(H) = 0$ and therefore G is L^2-Hall. However, T is non-amenable, and therefore so is G. Note that G is not locally indicable (or even torsion-free) and therefore $\mathcal{D}_{K[G]}$ does not exist. However, it still makes sense to discuss the L^2-Hall property for this group since L^2-invariants are defined for all groups.

Example 3.11 (Free groups)**.** Let F be a finitely generated free group and let $H \leqslant F$ be a finitely generated. A classical theorem of Marshall Hall [Hal49] states that H is a free factor in some finite-index subgroup $F' \leqslant F$. By Lemma 3.8, H is $\mathcal{D}_{K[F]}$-independent in F', showing that F is $\mathcal{D}_{K[F]}$-Hall.

Fundamental groups of closed surfaces also satisfy an analogous principle to Hall's theorem, namely that finitely generated subgroups are virtual retracts, as proved by Scott [Sco78] using hyperbolic geometry (see also [Wil07] for a more combinatorial proof). This directly implies that surface groups are subgroup separable. Moreover, Antolín–Jaikin-Zapirain proved that they are L^2-Hall in [AJZ22, Theorem 4.4] using these virtual retractions combined with other algebraic ideas (such as the theory of Demushkin groups and the cohomological goodness of surface groups). We now use Scott's argument to give a more topological proof of the L^2-Hall property for surface groups.

Proposition 3.12. *Surface groups satisfy the $\mathcal{D}_{K[G]}$-Hall property.*

Remark 3.13. The only surface that has a fundamental group with torsion is $\mathbb{R}P^2$, where $G = \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$. In this case, we may *define* $b_0^{K[G]}(G) = \frac{1}{2}$ and $b_n^{K[G]}(G) = 0$ for all $n \geqslant 1$, which is consistent with the index scaling formula Proposition 2.15(4). In this sense, $\mathbb{Z}/2$ also has the $\mathcal{D}_{K[G]}$-Hall property (and in fact so do all finite groups).

Proof of Proposition 3.12. We say that a compact connected subsurface X of a connected surface S is *incompressible* if no component of the closure of the complement $S \smallsetminus X$ is a disc. If $\pi_1(X) \neq 1$, then X is incompressible if and only if the induced map $\pi_1(X) \longrightarrow \pi_1(S)$ is injective.

Let G be the fundamental group of a closed connected surface Σ with $\chi(\Sigma) \leqslant 0$ (the case when $\chi(\Sigma) > 0$ is trivial). Let $H \leqslant G$ be a non-trivial finitely generated subgroup. Let $\Sigma' \longrightarrow \Sigma$ be the covering space corresponding to H. Then Σ' is a (possibly non-compact) surface with fundamental group H. Let Σ_c be a compact core for Σ', that is, $\Sigma_c \subseteq \Sigma'$ is a compact, connected, incompressible subsurface such that the natural map $\pi_1(\Sigma_c) \longrightarrow \pi_1(\Sigma')$ is an isomorphism. The existence of Σ_c is ensured by [Sco78, Lemma 1.5]. Scott also showed in [Sco78, Lemma 1.4 and Theorem 3.3] that there is a commutative diagram

$$\begin{tikzcd} & \widehat{\Sigma} \arrow[dr] & \\ \Sigma_c \arrow[ur, hook] \arrow[rr] & & \Sigma \end{tikzcd}$$

where $\widehat{\Sigma} \longrightarrow \Sigma$ is an intermediate finite-sheeted covering into which Σ_c projects homeomorphically. Since $\pi_1\Sigma_c \cong H \neq 1$, the boundary $\partial\Sigma_c$ is incompressible in Σ_c. Consequently, Σ_c is an incompressible subsurface of $\widehat{\Sigma}$ and every connected component $\widehat{\Sigma}_i$ of the closure of the complement $\widehat{\Sigma} \smallsetminus \Sigma_c$ has the property that its boundary is incompressible. It follows that $\widehat{\Sigma}$ admits a decomposition as a finite graph of spaces where the vertex spaces are $\{\widehat{\Sigma}_i, \Sigma_c\}$, various of which are glued along some of their boundary components (so the edge spaces are circles and the edge maps are π_1-injective). This produces a splitting for the fundamental group $\pi_1\widehat{\Sigma}$ where one vertex is $\pi_1(\Sigma_c)$, the other vertices are $\pi_1\widehat{\Sigma}_i$ and the edge groups are infinite cyclic. By Lemma 3.8, the group $H = \pi_1(\Sigma_c)$ is $\mathcal{D}_{K[G]}$-independent in $\pi_1(\widehat{\Sigma})$, and therefore G is $\mathcal{D}_{K[G]}$-Hall. □

The ideas of Proposition 3.12, such as the construction of a compact core for a subgroup H and the reconstruction of H from cyclic splittings, motivates the strategy that we follow in Theorem 4.9 for more general graphs of free groups with cyclic edges. We can now explain the simpler case when the edge groups are trivial (i.e. the case of free products), which generalises the proof that free groups are $\mathcal{D}_{K[G]}$-Hall.

Proposition 3.14. *The class of finitely generated subgroup separable $\mathcal{D}_{K[G]}$-Hall groups is closed under free products.*

Proof. Let A and B be finitely generated subgroup separable groups with the $\mathcal{D}_{K[G]}$-Hall property. Let X_A and X_B be classifying spaces for A and B, respectively, and let X be the space obtained from X_A, X_B, and an edge $I = [0,1]$ by gluing the point $0 \in I$ to a basepoint in X_A and the point $1 \in I$ to a basepoint in X_B. Then X is a classifying space for $A * B$, and has a natural graph of spaces structure, where the underlying graph is an edge.

Let $H \leqslant A * B$ be a finitely generated subgroup and let $Y \longrightarrow X$ be the covering space corresponding to H. Let Z be a finite core for Y, i.e. $Z \hookrightarrow Y$ induces a π_1-isomorphism, the underlying graph of Z is finite, and $Z_v = Y_v$ for all vertices v in the underlying graph of Z. Denote the fundamental groups of the A-vertices (i.e. those vertex spaces in Z covering X_A) by X_{A_i}, where $\pi_1(X_{A_i}) = A_i \leqslant A$. Similarly, denote the B-vertices by X_{B_j}, where $\pi_1(X_{B_j}) = B_j \leqslant B$. For each i (resp. j), let $A_i' \leqslant A$ (resp. $B_j' \leqslant B$) be a finite-index subgroup containing A_i (resp. B_j) such that $A_i \hookrightarrow A_i'$ (resp. $B_j \hookrightarrow B_j'$) is $\mathcal{D}_{K[G]}$-injective.

Let $X_{A_i} \longrightarrow X_{A_i'}$ be the covering map associated to $A_i \leqslant A_i'$ and let $P_i \subseteq X_{A_i}$ be the set of points that are the endpoints of edges Z. By subgroup separability of A, we may find a finite-index subgroup A_i'' such that $A_i \leqslant A_i'' \leqslant A_i'$ and such that the induced covering map $X_{A_i} \longrightarrow X_{A_i''}$ is injective on P_i. Note that A_i is still $\mathcal{D}_{K[G]}$-injective in A_i''. A similar discussion applies to the B-vertices, where we obtain new groups B_j'' and spaces $X_{B_j''}$ satisfying the analogous conditions.

Let $\overline{Z}$ be the following graph of spaces: it has the same underlying as Z, the vertex spaces X_{A_i} (resp. B_j) are replaced with $X_{A_i''}$ (resp. $X_{B_j''}$), and there is an edge joining the points $x \in X_{A_i''}$ and $y \in X_{B_j''}$ if and only if they are the images of points x' and y' under the coverings $X_{A_i} \longrightarrow X_{A_i''}$ and $X_{B_j} \longrightarrow X_{B_j''}$, respectively, and x' and y' were joined by an edge in Z. From the construction, the covering spaces of the vertices induce a map of graphs of spaces $Z \longrightarrow \overline{Z}$ (which is an isomorphism on underlying graphs). Then $\pi_1(Z) \hookrightarrow \pi_1(\overline{Z})$ is $\mathcal{D}_{K[G]}$-injective by Lemma 3.8.

The process of completing $\overline{Z}$ to a finite-sheeted cover $\widehat{Z}$ of X is standard. This is detailed, for instance, in [Wil07, Theorem 3.2]. For this, one adds various disjoint copies of the vertices X_A and X_B to the precover $\overline{Z}$ until the resulting space satisfies *Stallings' principle* (see [Wil07, Proposition 3.1]). Then, certain pairs of the hanging elevations of edge maps can be glued together along additional trivial edge spaces to produce the finite-sheeted cover $\overline{Z} \longrightarrow X$. As before, the inclusion $\overline{Z} \hookrightarrow \widehat{Z}$ induces a $\mathcal{D}_{K[G]}$-injection on fundamental groups, which proves the claim. □

It is natural to ask whether subgroup separability is needed in Proposition 3.14, but it is unclear to the authors if, for instance, the free product of finitely generated and residually finite L^2-Hall groups is L^2-Hall. For non-residually finite groups we notice the following.

Remark 3.15. The L^2-Hall property is not closed under free products in general. Let A be an infinite, simple, amenable group (finitely generated examples of such groups exist by [JM13]). Then A has the L^2-Hall property but $A * A$ does not. To see this, let $F \leqslant A * A$ be a free subgroup of rank $d(F) > 2$. Then $b_1^{(2)}(F) > 1 = b_1^{(2)}(A * A)$ and hence F is not L^2-independent in $A * A$. Moreover, A is simple and therefore $A * A$ has no nontrivial finite-index subgroups. We conclude that $A * A$ does not have the L^2-Hall property.

We conclude with some non-examples.

Example 3.16. Fundamental groups of hyperbolic 3-manifolds and (nonabelian free)-by-cyclic groups are examples of groups G with $b_1^{K[G]}(G) = 0$ that contain nonabelian free subgroups. Consequently, they are not $\mathcal{D}_{K[G]}$-Hall. For a similar reason, non-solvable generalised Baumslag–Solitar groups are not $\mathcal{D}_{K[G]}$-Hall.

3.3. **Passing to finite-index overgroups.** In this subsection we prove Theorem 3.17. This will be crucial when establishing the L^2-Hall property for graphs of free groups with cyclic edge groups. Theorem D from the introduction will follow from Corollary 3.20 and Lemma 3.21.

Theorem 3.17. *Let G be a finitely generated and suppose that $G_1 \leqslant G$ is a finite-index subgroup and that $H \leqslant G$ is a finitely generated subgroup such that $b_2^{K[H]}(H) = 0$. Then the following hold.*

(1) If H is $\mathcal{D}_{K[G]}$-independent in G, then $H \cap G_1$ is $\mathcal{D}_{K[G]}$-independent in G_1.
(2) If there exists a finite-index subgroup $H_0 \leqslant H$ such that H_0 is $\mathcal{D}_{K[G]}$-independent in G_1, then there exists a finite-index subgroup $G_0 \leqslant G$ containing H as a $\mathcal{D}_{K[G]}$-independent subgroup.

The first statement (a) is essentially [AJZ22, Proposition 5.2], whose argument is followed to additionally prove statement (b). We first prove the following simple lemma.

Lemma 3.18. *Let G be a finitely generated group and suppose that $H \leqslant T \leqslant G$ are subgroups such that $|T : H| < \infty$. If H is $\mathcal{D}_{K[G]}$-independent in G, then T is $\mathcal{D}_{K[G]}$-independent in G.*

Proof. Consider the short exact sequence of $K[G]$-modules

$$0 \longrightarrow I_T^G / I_H^G \longrightarrow I_G / I_H^G \longrightarrow I_G / I_T^G \longrightarrow 0.$$

The induced long exact sequence in $\operatorname{Tor}_\bullet^{K[G]}(\mathcal{D}_{K[G]}, -)$ contains the following sequence of $\mathcal{D}_{K[G]}$-modules:

$$\operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, I_G/I_H^G) \longrightarrow \operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, I_G/I_T^G) \longrightarrow \operatorname{Tor}_0^{K[G]}(\mathcal{D}_{K[G]}, I_T^G/I_H^G).$$

By Proposition 3.4 and the assumption that H is $\mathcal{D}_{K[G]}$-independent in G, it follows that the left-most term $\operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, I_G/I_H^G)$ is zero. Moreover, since H is finite index in T, it is not hard to see that I_T^G/I_H^G is finitely generated as K-vector space, so $\operatorname{Tor}_0^{K[G]}(\mathcal{D}_{K[G]}, I_T^G/I_H^G) = 0$. It follows directly from the short exact sequence above that $\operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, I_G/I_T^G) = 0$. This implies, again by Proposition 3.4, that T is $\mathcal{D}_{K[G]}$-independent in G. ☐

We are now ready to explain the proof of Theorem 3.17.

Proof of Theorem 3.17. Let $H_1 = H \cap G_1$. We begin by proving statement (1). By Proposition 3.4, it is enough to show that

$$\operatorname{Tor}_1^{K[G_1]}(\mathcal{D}_{K[G_1]}, I_{G_1}/I_{H_1}^{G_1}) = 0.$$

Claim 3.19. *As subsets of $K[G]$, we have the equality $I_{H_1}^{G_1} = I_{G_1} \cap I_H^G$.*

Proof. Consider the following commutative diagram of natural maps

$$\begin{tikzcd} K[G_1] \arrow[r, hook, "\iota_1"] \arrow[d, "p_{H_1}^{G_1}"] & K[G] \arrow[d, "p_H^G"] \\ K[G_1/H_1] \arrow[r, hook, "\iota_2"] & K[G/H]. \end{tikzcd}$$

The horizontal arrows ι_1 and ι_2 are injective. It is clear that $I_{H_1}^{G_1} \subseteq I_{G_1} \cap I_H^G$. For the reverse inclusion, we will use the above diagram. If $x \in I_{G_1} \cap I_H^G$, then $x \in K[G_1]$ and x belongs to the kernel of $p_H^G \circ \iota_1$. By the commutativity of the diagram and the injectivity of ι_2, the element x must belong to the kernel of $p_{H_1}^{G_1}$, which equals $I_{H_1}^{G_1}$ by Lemma 2.11. ◇

Claim 3.19 implies that the natural map of $K[G_1]$-modules $I_{G_1}/I_{H_1}^{G_1} \longrightarrow I_G/I_H^G$ is injective. Furthermore, since $I_{G_1}/I_{H_1}^{G_1}$ (resp. I_G/I_H^G) is the kernel of the augmentation $K[G_1/H_1] \longrightarrow K$ (resp. $K[G/H] \longrightarrow K$) by Lemma 2.11, there is an exact sequence of $K[G_1]$-modules of the form

$$0 \longrightarrow I_{G_1}/I_{H_1}^{G_1} \longrightarrow I_G/I_H^G \longrightarrow K[G/H]/K[G_1/H_1] \longrightarrow 0.$$

Let $T \subseteq G_1$ be a set of representatives for the double (G_1, H)-cosets in G such that $1 \in T$. Denote by M_t the $K[G_1]$-module $K[G_1/(H^t \cap G_1)]$. Then $K[G/H] \cong \oplus_{t \in T} M_t$ as $K[G_1]$-modules. Let $\mathcal{D} = \mathcal{D}_{K[G_1]}$. Notice that $\operatorname{Tor}_2^{K[G_1]}(\mathcal{D}, M_t) = 0$ for all $t \in T$. The reason is that, by expression (2.2) and Shapiro's Lemma, its $\mathcal{D}$-dimension equals

$$b_2^{K[H^t \cap G_1]}(H^t \cap G_1) = b_2^{K[H]}(H) \cdot |H^t : H^t \cap G_1| = 0. \tag{3.8}$$

We have used the fact that $H^t \cap G_1$ is finite index in $H^t \cong H$, as well as the multiplicativity of $\mathcal{D}_{K[G]}$-Betti numbers (Proposition 2.15(4)). By the additivity of the $\mathcal{D}$-dimension function, it follows from Equation (3.8) that the $K[G_1]$-module $N = K[G/H]/K[G_1/H_1] \cong \oplus_{t \in T \smallsetminus \{1\}} M_t$ has $\operatorname{Tor}_2^{K[G_1]}(\mathcal{D}, N) = 0$.

The long exact sequence in Tor gives us an exact sequence of $\mathcal{D}$-modules of the form

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{Tor}_2^{K[G_1]}(\mathcal{D}, N) \arrow[dll, rounded corners] \\ \mathrm{Tor}_1^{K[G_1]}(\mathcal{D}, I_{G_1}/I_{H_1}^{G_1}) \arrow[r] & \mathrm{Tor}_1^{K[G_1]}(\mathcal{D}, I_G/I_H^G) \arrow[r] & \mathrm{Tor}_1^{K[G_1]}(\mathcal{D}, N). \end{tikzcd} \tag{3.9}$$

We have already proved that $\mathrm{Tor}_2^{K[G_1]}(\mathcal{D}, N) = 0$. So statement (1) will follow from diagram (3.9) if we prove that $\mathrm{Tor}_1^{K[G_1]}(\mathcal{D}, I_G/I_H^G) = 0$. We know that $\mathcal{D}_{K[G]} \cong \mathcal{D} \otimes_{K[G_1]} K[G]$ as $(\mathcal{D}, K[G])$-bimodules by Proposition 2.15(4). By Shapiro's lemma,

$$\begin{aligned} \mathrm{Tor}_1^{K[G_1]}(\mathcal{D}, I_G/I_H^G) &\cong \mathrm{Tor}_1^{K[G]}(\mathcal{D} \otimes_{K[G_1]} K[G], I_G/I_H^G) \\ &\cong \mathrm{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, I_G/I_H^G) = 0, \end{aligned}$$

where the vanishing of the last module follows from Proposition 3.4 and the assumption that H is $\mathcal{D}_{K[G]}$-independent in G. This completes the proof of the first statement (1).

We now prove (2). The subgroups $H_0 \leqslant H \cap G_1 \leqslant G_1$ have the property that $|H \cap G_1 : H_0| < \infty$ and that H_0 is $\mathcal{D}_{K[G]}$-independent in G_1. By Lemma 3.18, $H \cap G_1$ is $\mathcal{D}_{K[G]}$-independent in G_1. Let $G_2 \trianglelefteqslant G_1$ be a normal subgroup of finite index; since $b_2^{K[H]}(H) = 0$, it follows from part (1) that $H \cap G_2$ is $\mathcal{D}_{K[G]}$-independent in G_2. Thus, by replacing G_1 by G_2, we may assume that G_1 is normal in G and that $H \cap G_1$ is $\mathcal{D}_{K[G]}$-independent in G_1.

We claim that H is $\mathcal{D}_{K[G]}$-independent in $G_0 = \langle G_1, H \rangle = G_1 \cdot H$. For this, we first observe that $T = \{1\}$ is a set of representatives of the double (G_1, H)-cosets in G_0. So the argument given in (1) shows that the canonical map

$$I_{G_1}/I_{H \cap G_1}^{G_1} \longrightarrow I_{G_0}/I_H^{G_0}$$

is an isomorphism of $K[G_1]$-modules. Moreover, using that $H \cap G_1$ is $\mathcal{D}_{K[G]}$-independent in G_1 and arguing as in part (1), we deduce that

$$\mathrm{Tor}_1^{K[G_0]}(\mathcal{D}_{K[G_0]}, I_{G_0}/I_H^{G_0}) \cong \mathrm{Tor}_1^{K[G_1]}(\mathcal{D}_{K[G_1]}, I_{G_1}/I_{H \cap G_1}^{G_1}) = 0.$$

Thus H is $\mathcal{D}_{K[G]}$-independent in G_0 by Proposition 3.4. $\square$

The following is a direct consequence of Theorem 3.17.

Corollary 3.20. *Let G be a finitely generated group and suppose that all finitely generated subgroups $H \leqslant G$ have the property that $b_2^{K[H]}(H) = 0$ and that there exist finite-index subgroups $H_1 \leqslant H$ and $G_1 \leqslant G$ such that H_1 is $\mathcal{D}_{K[G]}$-independent in G_1. Then G is $\mathcal{D}_{K[G]}$-Hall.*

Corollary 3.20 offers a more flexible reformulation of the $\mathcal{D}_{K[G]}$-hall property which will be used in Theorem 4.9 to establish the $\mathcal{D}_{K[G]}$-hall property for various graphs of free groups and cyclic edge groups. Moreover, the local condition on the vanishing of $b_2^{K[H]}(H) = 0$ for all $H \leqslant G$ can be condensed for certain groups of cohomological dimension 2 using the following lemma.

Lemma 3.21. *Let G be a group of cohomological dimension $\mathrm{cd}_K(G) = n$ with $b_n^{K[G]}(G) = 0$. Then $b_n^{K[H]}(H) = 0$ for every subgroup $H \leqslant G$.*

Proof. Note that the natural map

$$\mathcal{D}_{K[H]} \otimes_{K[H]} F \longrightarrow \mathcal{D}_{K[G]} \otimes_{K[G]} F \tag{3.10}$$

is injective, where F is a free left $K[G]$-module. To see this, it is enough to prove the claim when $F = K[G]$. Let T be a right transversal for H in G. The Hughes-freeness of $\mathcal{D}_{K[G]}$ implies that the map $\oplus_{t\in T}\mathcal{D}_{K[H]} \cdot t \to \mathcal{D}_{K[G]}$ induced by the inclusions $\mathcal{D}_{K[H]} \cdot t \hookrightarrow \mathcal{D}_{K[G]}$ is injective [Grä20, Corollary 8.3]. The map of (3.10) when $F = K[G]$ equals the composition

$$\mathcal{D}_{K[H]} \otimes_{K[H]} K[G] \xrightarrow{\cong} \mathcal{D}_{K[H]} \otimes_{K[H]} \left(\bigoplus_{t\in T} K[H]\cdot t\right) \xrightarrow{\cong} \bigoplus_{t\in T} \mathcal{D}_{K[H]} \cdot t \hookrightarrow \mathcal{D}_{K[G]}$$

and is therefore injective, as desired.

The claim now follows easily. Let $0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow K \longrightarrow 0$ be a free resolution of the trivial $K[G]$-module K. This resolution exists because G has a classifying space of dimension at most n (we do not claim the modules F_i to be finitely generated). If $b_n^{K[H]}(H) \neq 0$, then there is a non-trivial element z in the kernel of $\mathcal{D}_{K[H]} \otimes_{K[H]} F_n \longrightarrow \mathcal{D}_{K[H]} \otimes_{K[H]} F_{n-1}$. Then z is also a nonzero element of the kernel of $\mathcal{D}_{K[G]} \otimes_{K[G]} F_n \longrightarrow \mathcal{D}_{K[G]} \otimes_{K[G]} F_{n-1}$ and therefore $b_n^{K[G]}(G) \neq 0$. $\square$

While we only have a conjectural characterisation of which general graphs of free groups with cyclic edge are L^2-Hall, the case of an amalgam is entirely understood.

Corollary 3.22. *If G is an amalgam of free groups over a cyclic subgroup, then G has the L^2-Hall property if and only if it does not contain a subgroup isomorphic to $F_2 \times \mathbb{Z}$.*

Proof. First note that $F_2 \times \mathbb{Z}$ is not L^2-Hall and so it cannot be a subgroup of an L^2-Hall group by Lemma 3.7. Conversely, assume that G does not contain a copy of $F_2 \times \mathbb{Z}$. Then [Wis18, Theorem 1.2] implies that G has a finite-index subgroup that is a limit group. Limit groups are L^2-Hall by [BK23] and have vanishing second L^2-Betti number by [BK17]. Thus, G is L^2-Hall by Corollary 3.20. $\square$

4. Graphs of free groups with cyclic edge groups

Remark 4.1. For this section and the next, we will focus on the L^2-Hall property. Indeed, for graphs of free groups with cyclic edge groups and for limit groups, the L^2-Betti numbers and the $\mathcal{D}_{K[G]}$-Betti numbers coincide. For this reason, and for simplicity, this and the following sections are written in terms of L^2-homology.

Throughout this section, G will denote the fundamental group of a finite graph of free groups with cyclic edge groups $(G_v, G_e; \Gamma)$ and X will denote the geometric realisation of a corresponding graph of spaces $\mathcal{X} = (X_v, X_e; \Gamma)$ with S^1 edge spaces such that $G = \pi_1(X)$. The attaching maps $X_e \to X_v$ are always assumed to be immersions. We will prove the L^2-Hall property for some of these groups in Theorem 4.9. Our strategy is as follows.

- We allow ourselves to work with clean graphs of groups by Theorem 4.7 below and Theorem 3.17(2).
- Given a finitely generated subgroup H of G, in order to craft $G_1 \leqslant G$ of finite index and an L^2-injective map $H \longrightarrow G_1$, we will use the cyclic splitting of G and the L^2-injectivity criteria for graphs of groups developed in the previous section (such as Lemma 3.8). The construction of G_1 uses

Wise's argument on the subgroup separability of some graphs of free groups with cyclic edge groups [Wis00].

- However, this geometric construction will not be directly applicable to H and G, but only to further finite-index subgroup $H_0 \leqslant H$ and $G_0 \leqslant G$, so we will also require Theorem 3.17(2) to reach the same conclusion about H and G.

We now proceed with the construction. The following definition was introduced by Wise [Wis00, Definition 4.16].

Definition 4.2 (The weighted graph Φ_X associated to X)**.** A *weighted graph* is a directed graph whose edges have two integer labels (one on each endpoint). A weighted graph Γ is *balanced* if whenever $\sigma\colon S^1 \to \Phi_X$ is an oriented combinatorial loop (which means that S^1 is given a graph structure by subdivision which makes σ into a map of graphs), the product of the outgoing weights divided by the product of the incoming weights on S^1 equals ± 1 (where the weights on S^1 are induced by σ). Moreover, Γ is *solvable* if it can be oriented so that every vertex has at most one outgoing edge and the weight of every incoming edge is ± 1.

We are going to associate to $X = (X_v, X_e; \Gamma)$ a weighted graph Φ_X that is defined as follows. Fix an orientation for every simple closed combinatorial loop of all vertex spaces X_v and fix an orientation of S^1 (say counter-clockwise, viewed as a subset of $\mathbb{C}$). The set of edges of Φ_X equals the set of edges of Γ. We identify the endpoint v of e to the endpoint v' of e' if and only if the images of the attaching maps $X_e \longrightarrow X_v$ and $X_{e'} \longrightarrow X_{v'}$ are equal. If $X_e \longrightarrow X_v$ represents the nth power of a primitive element in $\pi_1(X_v)$, then if the attaching map respects orientations we put a weight of $|n|$ on the end e of v; otherwise we put a weight of $-|n|$.

Definition 4.3. A connected weighted graph Γ determines a graph of spaces X_Γ. as follows. For each vertex (resp. edge) of Γ there is a vertex (resp. edge) space homeomorphic to S^1, all oriented counter-clockwise. The edge spaces are attached to the vertex spaces by degree n covers, where n is the weight on the corresponding end of the edge (we take $n < 0$ to mean that the covering map is of degree $|n|$ in the usual sense and it reverses orientations). We call $\pi_1(X_\Gamma)$ the *generalised Baumslag–Solitar group associated to* Γ and X_Γ the *generalised Baumslag–Solitar complex associated to* Γ.

Lemma 4.4. *Let X be a graph of free groups with cyclic edge groups and let Γ be a component of the weighted graph Γ. Let G be the generalised Baumslag–Solitar group associated to Γ. Then the natural map $G \longrightarrow \pi_1(X)$ is π_1-injective.*

Proof. Fix normal forms for elements in G and $\pi_1(X)$. It is then not hard to see that elements of G in normal form are sent to elements of $\pi_1(X)$ in normal form. □

Example 4.5. The Baumslag–Solitar group $\mathrm{BS}(m, n)$ is the fundamental group of a graph of spaces of the form $(S^1, S^1; \Gamma)$ where Γ is a single loop and the two attaching maps are degree m and n covering maps $S^1 \longrightarrow S^1$. In this case Φ_X is a loop with one vertex and one edge, where the ends of the edge are labeled by m and n. Note that Φ_X is balanced if and only if $m = \pm n$ and it is solvable if and only if $m = \pm 1$ or $n = \pm 1$.

Many properties of graphs of free groups with cyclic edge groups can be characterised by the properties of Φ_X. The following definition and result are due to Wise.

Definition 4.6 (Clean graph of spaces)**.** A graph of spaces is *clean* if every edge map is a topological embedding.

Theorem 4.7 ([Wis00, Theorems 4.18 and 5.1])**.** *Let $G = \pi_1(X)$ be a finitely generated graph of free groups with cyclic edge groups. The following are equivalent:*

(1) G is subgroup separable;
(2) Φ_X is balanced;
(3) the generalised Baumslag–Solitar groups associated to the components of Φ_X are all subgroup separable;
(4) X has a finite clean cover.

We highlight the following recent result of Abgrall and Munro that confirms a conjecture of Wise [Wis00, Conjecture 6.2] and gives an easily computable criterion for when a graph of free groups with cyclic edge groups is residually finite.

Theorem 4.8 ([AM])**.** *Let $G = \pi_1(X)$ be a finitely generated graph of free groups with cyclic edge groups. The following are equivalent:*

(1) G is residually finite;
(2) every component of Φ_X is balanced or solvable;
(3) the generalised Baumslag–Solitar groups associated to the components of Φ_X are all residually finite.

The main goal of this section is to establish the following theorem.

Theorem 4.9. *Let G split as a finitely generated graph of free groups with cyclic edge groups and let $G = \pi_1(X)$, where X is as above. If Φ_X is balanced and solvable, then G has the L^2-Hall property. Equivalently, if G is hyperbolic relative to virtually abelian subgroups, then G has the L^2-Hall property.*

Remark 4.10. The condition that every component of Φ_X be solvable is necessary, since otherwise G would contain a non-solvable Baumslag–Solitar subgroup. Such groups do not have the L^2-Hall property since they contain nonabelian free subgroups but have vanishing first L^2-Betti number. On the other hand, there are graphs of free groups with cyclic edge groups where Φ_X is unbalanced yet G is still L^2-Hall (e.g. $\mathrm{BS}(1, n)$ for $n \neq \pm 1$).

Motivated by this remark, we make the following conjecture, which is formally similar to Theorems 4.7 and 4.8.

Conjecture 4.11. *Let $G = \pi_1(X)$ be a finitely generated graph of free groups with cyclic edge groups. The following are equivalent:*

(1) G has the L^2-Hall property;
(2) every component of Φ_X is solvable;
(3) the generalised Baumslag–Solitar groups associated to the components of Φ_X all have the L^2-Hall property.

4.1. **The proof of Theorem A.** We prove Theorem 4.9 above (which is Theorem A from the introduction). We make a few simplifying reductions that we hope will make the visualisation of the objects easier as well as put us in the context of the proof of [Wis00, Theorem 5.2].

Claim 4.12. *It is enough to consider the case where all the edge groups are infinite cyclic.*

Proof. A balanced graph of free groups with cyclic edge groups is the free product of balanced graphs of free groups all whose edge groups are infinite cyclic (all of which are subgroup separable by Theorem 4.7). By Proposition 3.14, it is enough to prove that each free factor is L^2-Hall, hence the claim. $\diamond$

Claim 4.13. *It is enough to prove Theorem 4.9 in the case where X is clean.*

Proof. By [Wis00, Lemma 4.4 and Theorem 4.18], X has a clean finite-sheeted covering $X_c \longrightarrow X$. By Corollary 3.20, the L^2-Hall property passes to finite-index overgroups, so it is enough to prove Theorem 4.9 for $\pi_1(X_c)$. $\diamond$

Definition 4.14. Let X be a clean graph of free groups with cyclic edge groups. An *immersed Klein bottle* in X is a subcomplex $K \subseteq X$ that corresponds to a loop in Φ_X whose associated generalised Baumslag–Solitar group is a Klein bottle group. Similarly, an immersed torus is a subcomplex corresponding to a loop in Φ_X whose associated generalised Baumslag–Solitar group is $\mathbb{Z}^2$.

Note that an immersed Klein bottle K is indeed the image of a Klein bottle surface S under a cellular immersion, where S is the generalised Baumslag–Solitar complex associated to the loop corresponding to K. Similarly, if T is an immersed torus, then there is a topological torus S and a cellular immersion $S \longrightarrow T$.

Claim 4.15. *It is enough to prove Theorem 4.9 in the case where X is clean and does not contain any immersed Klein bottles.*

Proof. By Claim 4.13, we may assume that X is clean. Let $K_1, \dots, K_n$ denote the immersed Klein bottles in X. By subgroup separability of $G = \pi_1(X)$, there is a finite-sheeted regular cover $p_1 \colon X_1 \longrightarrow X$ where the components of $p^{-1}(K_1)$ are all immersed tori. Assume now that we have constructed some finite-sheeted regular cover $p_i \colon X_i \longrightarrow X$ so that for each $j \leqslant i$ every component of $p_i^{-1}(K_j)$ is an immersed torus. Let K be an immersed Klein bottle component of $p_i^{-1}(K_{i+1})$. Again we may pass to a further finite-sheeted cover $q_{i+1} \colon X_{i+1} \longrightarrow X_i$ such that the composition

$$p_{i+1} \colon X_{i+1} \longrightarrow X_i \longrightarrow X$$

is regular and every component of $q_{i+1}^{-1}(K)$ is an immersed torus. But then every component of $p_{i+1}^{-1}(K_{i+1})$ is an immersed torus by regularity of the cover. By Corollary 3.20, it is enough to show that $\pi_1(X_n)$ has the L^2-Hall property. $\diamond$

Proof (of Theorem 4.9). By the claims above, we assume without loss of generality that G is the fundamental group of a clean graph of spaces $X = (X_v, X_e; \Gamma)$, where the vertex spaces X_v are graphs and the edge spaces X_e are circles. Moreover, we assume that X does not contain any immersed Klein bottles.

Let H be a finitely generated subgroup of G. Following the proof of [Wis00, Theorem 5.2], we will show that there is a finite-index subgroup $H_1 \leqslant H$ that is L^2-independent in a finite-index subgroup $G_1 \leqslant G$. By Theorem 3.17, this is enough to prove the theorem. We break up our proof into steps in a similar way to the proof of [Wis00, Theorem 5.2].

Let $Y \longrightarrow X$ be the covering space corresponding to H. Note that Y has a natural decomposition into a clean graph of spaces $(Y_v, Y_e; \Gamma_Y)$, where each of the vertex spaces are graphs and each of the edge spaces are homeomorphic to either S^1 or $\mathbb{R}$.

Step 1 (The subcomplex)**.** Denote the underlying graph of Y by $\Gamma(Y)$. Since H is finitely generated, there is a finite connected subgraph Υ of $\Gamma(Y)$ such that the inclusion $Y_\Upsilon \hookrightarrow Y$ of the restricted graph of spaces Y_Υ is a π_1-isomorphism.

Step 2 (Pruning)**.** For each vertex space Y_v of Y_Υ, let Z_v be the smallest connected subgraph containing the images of all the edge spaces of Y_Υ and such that $Z_v \hookrightarrow Y_v$ induces a π_1-isomorphism. Let $Z \subseteq Y_\Upsilon$ be the union of the spaces Z_v and the edge spaces Y_e of Y_v. Note that Z is connected and has a natural graph of spaces structure $(Z_v, Z_e = Y_e; \Upsilon)$ such that the inclusion $Z \hookrightarrow Y_\Upsilon$ still induces a π_1-isomorphism. The resulting vertex spaces of Z_v have a compact core with pairs of infinite rays attached to them coming from the attaching maps of non-compact edge spaces in Y (see Figure 1).

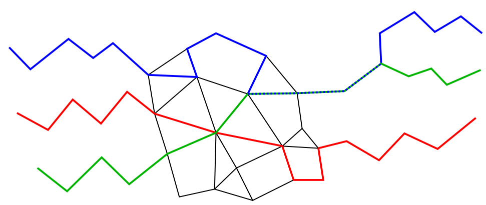

FIGURE 1. A vertex space of Z. The coloured lines represent attaching maps of non-compact edge spaces, each of which being homeomorphic to $\mathbb{R}$.

Step 3 (L^2-independence of periphery closing)**.** This is the main step of the proof. Let e be an edge of Υ and let $I = [0,1]$ be the closed unit interval. If $Z_e \cong S^1$, then we call $Z_e \times I$ a *cylinder*; if $Z_e \cong \mathbb{R}$, then $Z_e \times I$ is a *strip*. If two strips in Z have a non-compact intersection, then the periodicity of the attaching maps implies that their intersection must in fact be homeomorphic to $\mathbb{R}$.

Note that $\mathbb{Z}$ acts on each of the strips by covering translations (where the covering refers to a strip in Z covering a cylinder in X). As in [Wis00, Theorem 5.2, Step 3], choose n large enough so that, for any edge e corresponding to a strip, all of the vertices of $Z_e \times I$ with valence at least 3 (the valence is counted in the vertex graphs adjacent to the strip) are a distance less than n apart. Then quotient the strips of Z by the action of $n\mathbb{Z}$ to form a new complex $A = Z/\sim$. Now A is a clean compact graph of graphs with S^1 edge spaces. A typical vertex space is shown in Figure 2.

We need to introduce a definition based on one given by Hsu and Wise [HW10, Definition 9.1]. Declare two strips to be *equivalent* if their intersection is a copy of $\mathbb{R}$; this rule generates an equivalence relation on the set of strips in Z. The *periphery* containing a strip S is the union of the strips in the equivalence class of S.

Fix a periphery P. By the assumption that X contains no immersed Klein bottles, there is a compact subset $K \subseteq Z$ such that $P \smallsetminus K$ is homeomorphic to two disjoint copies of $\mathbb{R}_{>0} \times \Omega$, where Ω is some finite graph. The effect of quotienting by the action of $n\mathbb{Z}$ can then be rephrased as follows: choose K compact and sufficiently large so that K is a compact core for Z and all the vertices in $P \smallsetminus K$ are of degree 2. We also require, for every strip $S \subseteq P$, that $K \cap S$ be a fundamental domain for the action of $n\mathbb{Z}$ on S. Denote by R_1 and R_2 the copies of $\mathbb{R}_{>0} \times \Omega$ in $P \smallsetminus K$. We then form quotient of the complex $Z \smallsetminus (R_1 \sqcup R_2)$ by identifying the two

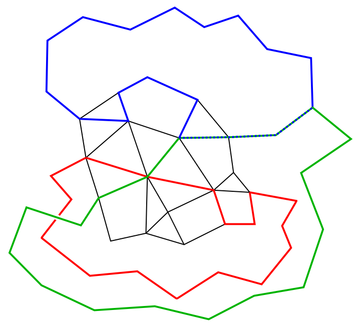

FIGURE 2. A vertex space of A. The vertex space is obtained from that in Figure 1 by quotienting the coloured lines by the action of $n\mathbb{Z}$.

copies of $\partial R_1 \cong \partial R_2 \cong \Omega$ (see Figure 3). Performing this process for each periphery yields the complex A.

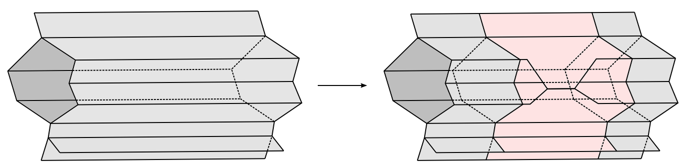

FIGURE 3. Part of a periphery P is shown on the right. The pink region represents $K \cap P$, where $K \subseteq Z$ is as above. The horizontal lines are contained in vertex spaces of Z. Outside of K, the horizontal lines do not intersect since all of the vertices there are of degree 2. However, they may have a compact intersection inside K as shown in the figure. In this figure, the graph $\Omega \cong \partial R_1 \cong \partial R_2$ is a cycle with two finite trees hanging off it. On the left is part of a copy of $\mathbb{R} \times \Omega$. The whole diagram represents an immersion $\mathbb{R} \times \Omega \longrightarrow P$, which is an isomorphism outside of a compact set.

Claim 4.16. *The injections $\partial R_i \hookrightarrow P$ are π_1-injective.*

Proof. First note that there is a cellular immersion $\mathbb{R} \times \Omega \longrightarrow P$, which is an isomorphism outside of a compact set (see Figure 3). The immersion fits into the commutative diagram

$$\begin{tikzcd} \mathbb{R} \times \Omega \arrow[r] \arrow[d] & P \arrow[d] \\ S_0 \arrow[r] & T_0 \end{tikzcd}$$

where the vertical maps are covering spaces, S_0 is the graph of spaces of a generalised Baumslag–Solitar group, and T_0 is its image in X. Covering maps are π_1-injective and so is the map $S_0 \longrightarrow T_0$ by Lemma 4.4. Hence, $\mathbb{R} \times \Omega \longrightarrow P$ is π_1-injective. Since $\partial R_i \hookrightarrow \mathbb{R} \times \Omega$ is a π_1-isomorphism for $i = 1, 2$, it follows that the maps $\partial R_i \hookrightarrow P$ are π_1-injective. $\diamond$

Claim 4.17. *The quotient map $q\colon Z \longrightarrow A$ is π_1-injective and $q_*(\pi_1(Z))$ is L^2-independent in $\pi_1(A)$.*

Proof. Denote the peripheries of Z by $P_1, \dots, P_n$ and for each $i = 1, \dots, n$ let Ω_i be the finite graph such that there is a immersion $\mathbb{R} \times \Omega_i \longrightarrow P_i$. Since the peripheries are subgraphs of spaces of Z, the groups $\pi_1(\Omega_i) \cong \pi_1(\partial R_i)$ embed in $\pi_1(Z)$ by the previous claim. The quotient map induces an inclusion

$$q_*\colon \pi_1(Z) \longrightarrow \pi_1(Z)*_{\pi_1(\Omega_1),\dots,\pi_1(\Omega_n)} \cong \pi_1(A),$$

where $\pi_1(Z)*_{\pi_1(\Omega_1),\dots,\pi_1(\Omega_n)}$ denotes the multiple HNN extension of $\pi_1(Z)$ over the subgroups $\pi_1(\Omega_i)$.

The assumption that Φ_X is balanced and solvable implies that G does not contain any non-abelian generalised Baumslag–Solitar subgroups. Hence, $\pi_1(\Omega_i)$ is either trivial or isomorphic to $\mathbb{Z}$. To see this, note that every periphery P_i covers a generalised Baumslag–Solitar subcomplex $V_i \subseteq X$, and therefore $\pi_1(V_i)$ cannot contain a nonabelian free subgroup. So from the fact that $\pi_1(\Omega_i) \leqslant \pi_1(V_i)$ we deduce that $\pi_1(\Omega_i)$ is either trivial or $\mathbb{Z}$. The vertex group of a multiple HNN extension along trivial or infinite-cyclic subgroups is L^2-independent by Lemma 3.8. $\diamond$

Step 4 (L^2-independence of vertex completion)**.** As remarked in the previous step, A is a clean compact graph of graphs with S^1 edge spaces. Moreover, since the $\mathbb{Z}$ action on the strips was by covering translations, it follows that there is a natural quotient map $A \longrightarrow X$ whose restriction to every vertex space of A is an immersion. By adding edges to the vertex spaces of A, we can complete them to coverings of the corresponding vertex spaces of X. Denote the complex obtained from A in this way by B and note that $\pi_1(B) \cong \pi_1(A) * F$, where F is a free group. Then $\pi_1(A)$ (and therefore H) is L^2-independent in $\pi_1(B)$.

Step 5 (Passing to finite-index and completing to a cover)**.** In [Wis00, Theorem 5.2, Steps 5 and 6], Wise shows how to pass to a finite-sheeted cover $B_1 \longrightarrow B$ which can be completed to a finite-sheeted cover $X_1 \longrightarrow X$ by attaching cylinders to B_1.

Claim 4.18. *$H \cap \pi_1(B_1)$ is L^2-independent in $\pi_1(B_1)$.*

Proof. Since $\pi_1(A)$ is a free factor of $\pi_1(B) \cong \pi_1(A)*F$, we have that $\pi_1(A) \cap \pi_1(B_1)$ is a free factor of $\pi_1(B_1)$ by Kurosh's theorem [Ser77, Théorème 14, Chaptire I, §5]. So $\pi_1(A) \cap \pi_1(B_1)$ is L^2-independent in $\pi_1(B_1)$ and hence it suffices to prove that $H \cap \pi_1(B_1)$ is L^2-independent in $\pi_1(A) \cap \pi_1(B)$.

The proof of Claim 4.17 shows that $\pi_1(A)$ has a graph of groups decomposition with underlying graph a rose, where the unique vertex group is H and the edge groups are either trivial or $\mathbb{Z}$. Then $\pi_1(A) \cap \pi_1(B_1)$ also has a graph of groups decomposition with edge groups either trivial or $\mathbb{Z}$, and $H \cap \pi_1(B_1)$ is a vertex group in this decomposition. By Lemma 3.7, $H \cap \pi_1(B_1)$ is L^2-independent in $\pi_1(A) \cap \pi_1(B_1)$. $\diamond$

As mentioned above, Wise shows that we can attach cylinders to B_1 to obtain a finite-sheeted covering $X_1 \longrightarrow X$. Therefore $\pi_1(X_1)$ is a multiple HNN extension of $\pi_1(B_1)$ over cyclic subgroups, so $\pi_1(B_1)$ (an thus $H \cap \pi_1(B_1)$ as well) is L^2-independent in $\pi_1(X_1)$. In summary, $H \cap \pi_1(B_1)$ has finite index in H and $H \cap \pi_1(B_1)$ is L^2-independent in $\pi_1(X_1)$, which has finite index in $\pi_1(X)$. We conclude that G has the L^2-Hall property by Corollary 3.20. $\square$

5. L^2-Hall property for limit groups

Wilton [Wil08] proved that limit groups have the local retractions property (and hence that they are subgroup separable) using Kharlampovich and Miasnikov's [KM98] characterisation of limit groups in terms of *ICE groups* (see Definition 5.1 below). Limit groups are exactly the finitely generated groups that arise as subgroups of ICE groups. Since the local retractions property passes to subgroups, Wilton only needs to deal with ICE groups in [Wil08]. Analogously, we certified that the L^2-Hall property passes to subgroups in Lemma 3.7, so it also sufficient to deal with ICE groups in order to prove Theorem B. Our argument is different to that of [BK23] and we expect it to be flexible enough to include more general finite abelian hierarchies of relatively hyperbolic groups as in Conjecture 1.3.

Just as we followed Wise's argument on subgroup separability of balanced graphs of free groups in the previous section, here we follow the ideas developed by Wilton [Wil07, Wil08] on subgroup separability of limit groups. For the convenience of the reader, we shall stress the differences and similarities between both arguments and introduce as little new notation as possible. For this, we summarise Wise's argument to then explain what are the difficulties involved in making this work for proving the local retractions property of ICE groups, as Wilton did, and the L^2-Hall property of ICE groups, as we want.

Definition 5.1. Let G be a group and let $Z \leqslant G$ be the centraliser of an element. The group $G *_Z (Z \times \mathbb{Z}^n)$ is an *extension of G by a centraliser*. A group is an *ICE group* (standing for "iterated centraliser extension") if it can be obtained from a finitely generated free group by a finite sequence of extensions by a centraliser.

If G is an ICE group, then it has a classifying space X that can be described as follows. If G is finitely generated and free, then take X to be a bouquet of circles. Otherwise, write $G = H *_Z (Z \times \mathbb{Z}^n)$ for simpler ICE group H. It is not hard to show that we may assume that Z is infinite cyclic (see [Wil08, Remark 1.14]). Then take X to be the graph of Y and T^{n+1} with edge group S^1, where Y is the classifying space of H constructed by induction, and S^1 maps to a loop representing the centralised element in H and to a coordinate circle in T^{n+1}. The spaces obtained in this way will be called *ICE spaces*. We refer the reader to [Wil08, Section 1.6] for a concise survey of this material. We emphasise the following important theorem of Kharlampovich and Miasnikov, which gives a powerful characterisation of limit groups.

Theorem 5.2 ([KM98])**.** *A finitely generated group G is a limit group if and only if it is a subgroup of an ICE group.*

Definition 5.3. A collection of elements $\mathcal{L}$ in a group G is *independent* if g commutes with no conjugate of h for all pairs of distinct elements $g, h \in \mathcal{L}$. We also say that a collection $\mathcal{L}$ of loops in a space X is independent if they represent a independent collection of elements of $\pi_1(X)$ in the previous sense.

5.1. **From graphs of free groups to limit groups.** We now compare the arguments of Wilton and Wise. First, we remark that, in the context of limit groups, the process of getting virtually clean covers can be directly resolved using the fact that these are CAT(0) [AB06], as observed by Wilton in [Wil07, Lemma 5.11]. In any case, this is not needed for the proof and so we will not mention it here again. Let X be an ICE space obtained by gluing a torus to a simpler ICE space Y. Let

H be a finitely generated subgroup of $\pi_1(X)$ and let $X_H \longrightarrow X$ be the covering corresponding to H.

- The problem of "pruning" in Wise's argument corresponds to taking a core X' of X_H that contains the union a compact core of the fundamental group of each vertex space of X_H and all the infinite degree elevations of edge maps (i.e. the infinite strips). Here is the first difference, an important property of those elevations was that they escape any compact subset of the free splitting of the vertex space they belong to (because they act freely on the vertex graph). This property is called "properness" (as introduced in [Wil08, Definition 2.12]) and is not satisfied by *elliptic* loops. However, as a consequence of the 2-acylindricity of the Bass–Serre tree of an ICE group, non-elliptic (i.e. *hyperbolic*) loops are proper [Wil08, Lemma 2.16].
- The second part of the argument, which consists in closing up the infinite strips, is another step towards obtaining a finite-sheeted precover from the precover X' (since, after this, the preimages of points in the edge spaces are finite). However, one still had to figure out how to complete the precover so preimages of points in the vertex spaces are finite as well. This corresponds to the problem of extending finite-sheeted covers of the edge spaces to finite-sheeted covers of the vertex spaces themselves. In this setting, this relies on a (much less sophisticated) version of omnipotence of free groups. This way, one constructs a precover $W \longrightarrow X'$. An analogous version of this principle is treated in [Wil07, Section 3.2] using similar homological assumptions on the edge spaces. This problem is resolved in [Wil08] with the notion of *disparity* (called *diversity* in [Wil07]).
- Lastly, the finite intermediate precover W is shown to admit a finite-sheeted covering $W_m \longrightarrow W$ that can be completed to a finite-sheeted covering $X_m \longrightarrow X$. This is the most intricate part in Wilton's argument and relies on the notion of *tameness*, which is proved by induction.

We want to insist on the last point. Given a complex X, Scott's criterion [Wil08, Lemma 1.3] geometrically reformulates subgroup separability of the fundamental group $\pi_1(X)$ as the ability to complete precovers of X to finite-sheeted covers. Wilton notes [Wil08, Section 3] that this property is not strong enough to serve as an induction hypothesis (for complexes that admit a finite hierarchy, such as ICE spaces), and for this he considers the aforementioned notion of tameness instead [Wil08, Definition 3.1]. Similarly, the L^2-Hall property concerns the ability to complete precovers of X to finite-sheeted covers preserving the L^2-homology in the process. Again, this is not strong enough for an inductive argument as the following simple observation shows.

Example 5.4. Consider $G = \pi_1(\Sigma_2) = \langle a, b, c, d \mid [a, b] = [c, d]\rangle$, which splits as $F(a, b) *_{[a,b]=[c,d]} F(c, d)$. We consider the L^2-independent subgroups $H \leqslant F(a, b)$ and $K \leqslant F(c, d)$ given by $H = F(a^2, b^2)$ and $K = F(c^2, d^2)$. It is clear that the induced map $H * K \longrightarrow G$ is injective, although it is not L^2-injective for the obvious reason that $b_1^{(2)}(H * K) = 3 > 2 = b_1^{(2)}(G)$.

Example 5.4 illustrates that subgraphs of groups that are L^2-injective on vertex groups need not be L^2-injective overall, and so one needs some control on the non-trivial L^2-classes that are supported on the edge subgroups. Wilton's notion of tameness [Wil08, Definition 3.1] and Example 5.4 motivates the following notion of L^2-tameness that allows us to inductively have such control.

Definition 5.5. Consider a complex X, a covering $X' \longrightarrow X$ and a finite (possibly empty) collection of independent (Definition 5.3), essential loops $\mathcal{L} = \{\delta_i \colon C_i \longrightarrow X\}$. The cover X' is *L^2-tame over $\mathcal{L}$* if the following holds. Let $\Delta \subseteq X'$ be a finite subcomplex and let $\{\delta'_j \colon C'_j \longrightarrow X'\}$ be a finite collection of (pairwise non-isomorphic) infinite degree elevations where each δ'_j is an elevation of some $\delta_i \in \mathcal{L}$. Then for all sufficiently large positive integers d, there exists an intermediate finite-sheeted covering $X' \longrightarrow \widehat{X} \longrightarrow X$ that satisfies the following.

1. Every δ'_j descends to an elevation $\widehat{\delta}_j$ of degree d;
2. The $\widehat{\delta}_j$ are pairwise non-isomorphic;
3. The subcomplex Δ injects into $\widehat{X}$;
4. The induced map
$$\pi_1(X') * \left(\coprod_{\mathcal{L}} \mathbb{Z} \right) \longrightarrow \pi_1(\widehat{X}),$$
defined by mapping each $\mathbb{Z}$ labelled by i to the class of the image of $\widehat{\delta}_i$, is injective and L^2-injective.

Remark 5.6. The L^2-tameness of all coverings $X' \longrightarrow X$ with finitely generated $\pi_1(X')$ (and empty $\mathcal{L}$) implies the L^2-Hall property for $\pi_1(X)$.

As anticipated, the idea is that the strengthened version described in Definition 5.5 (with additional prescribed data relative to $\mathcal{L}$) admits a proof by induction and avoids bad embeddings as the one described in Example 5.4.

5.2. The proof of Theorem B.

By the previous discussion, the following theorem implies Theorem B from the introduction, and its proof will occupy the remainder of this section.

Theorem 5.7. *Let X be an ICE space, let $H \leqslant \pi_1(X)$ be a finitely generated subgroup and let $X_H \longrightarrow X$ be the corresponding covering. Suppose that $\mathcal{L}$ is a (possibly empty) finite set of independent loops each generating a maximal abelian subgroup of $\pi_1(X)$. Then X_H is L^2-tame over $\mathcal{L}$.*

Proof. We proceed by induction on the complexity of the ICE space. The base of the induction is the case when X is a graph, which is trivial. Now assume X is an ICE space that decomposes as a graph of spaces with two vertices (a lower complexity ICE space Y and a torus T^n) and one edge space homeomorphic to S^1. This naturally induces a graph of spaces structure for X_H, whose underlying graph we denote by $\Gamma(X_H)$. Each vertex space of this splitting is either a covering space of Y or a covering space of the torus T^n.

Denote by $\{\delta_i \colon D_i \longrightarrow X\}$ and $\{\varepsilon_i \colon E_i \longrightarrow X\}$ the hyperbolic and elliptic loops of $\mathcal{L}$, respectively, relative to the splitting of X.

Step 1 (The precovers X' and X'')**.** Let $\Delta \subseteq X_H$ be a finite subcomplex and let $\{\delta^H_j\}$ and $\{\varepsilon'_j\}$ denote fixed sets of infinite degree elevations of hyperbolic and elliptic loops, respectively, in $\mathcal{L}$. We begin by taking a subcomplex $X' \subseteq X_H$ that satisfies the following conditions:

(1) The subcomplex X' is a core for H, i.e. X' is a subgraph of spaces with finite underlying graph such that the induced map $\pi_1(X') \longrightarrow \pi_1(H)$ is an isomorphism.
(2) The subcomplex Δ is contained in X'.
(3) The image of each ε'_j is contained in X'.
(4) Each infinite-degree elevation $\delta^H_j \colon \mathbb{R} \longrightarrow X_H$ restricts to a (possibly non-full) elevation $\delta'_j \colon D'_j \longrightarrow X'$, where $D'_j \subseteq \mathbb{R}$ is a finite union of compact intervals.

The subscripts i and j are different, indicating that there may be several elevations δ^H_j for each δ_i, and likewise for ε_i. We will keep completing the precover X' further to get some more intermediate precovers $X' \subseteq X'' \subseteq \overline{X} \subseteq X_H$ of X.

From [Wil08, Lemma 2.24], we can enlarge X' to $X'' \subseteq X_H$ so that X'' still enjoys properties (1)—(4) listed above while additionally satisfying that the corresponding elevations $\delta''_j \colon D''_j \longrightarrow X''$ are disparate (in the sense of [Wil08, Definition 2.2]). In particular, the induced map

$$\pi_1(X'') \longrightarrow \pi_1(X_H) \tag{5.1}$$

is an isomorphism.

Step 2 (The precover $\overline{X}$)**.** We shall not need the definition of disparity but, instead, we will explain how this condition is used to extend the precover X'' further. Recall that each D''_j is the union of finitely many compact intervals and that D''_j fits in the following commutative diagram:

$$\begin{tikzcd} X'' \arrow[d, hook] & D''_j \arrow[l, "\delta''_j"'] \arrow[d, hook] \\ X_H \arrow[d] & \widetilde{D_i} \arrow[l, "\delta^H_j"'] \arrow[d] \\ X & D_i . \arrow[l, "\delta_i"'] \end{tikzcd}$$

For all sufficiently large positive integers d, there exists $\overline{D}_j \cong S^1$ such that $D''_j \longrightarrow D_i$ factors through an embedding $D''_j \hookrightarrow \overline{D}_j$ and a d-sheeted covering map $\overline{D}_j \longrightarrow D_i$. By [Wil08, Lemma 2.23], we can extend X'' to a pre-cover $\overline{X}$ such that each δ''_j extends to a full elevation $\overline{\delta}_j \colon \overline{D}_j \longrightarrow \overline{X}$ and the diagram

$$\begin{tikzcd} X'' \arrow[d] & D''_j \arrow[l, "\delta''_j"'] \arrow[d, hook] & \\ \overline{X} \arrow[d] & \overline{D}_j \arrow[l, "\overline{\delta}_j"'] \arrow[r, "\cong"] \arrow[d] & S^1 \arrow[d, "\deg\ d"] \\ X & D_i \arrow[l, "\delta_i"'] \arrow[r, "\cong"] & S^1 \end{tikzcd}$$

commutes. By possibly enlarging Δ, we can assume that the images of the $\overline{\delta}_j$ are contained in Δ.

In the construction of [Wil08, Lemma 2.23], one first enlarges X'' to X''' by adding some simply connected vertex spaces of X_H to obtain $X'' \hookrightarrow X''' \hookrightarrow X_H$.

In particular, the induced map $\pi_1(X'') \longrightarrow \pi_1(X''')$ is an isomorphism. Then, one considers a collection of pairs $(\phi^H_{k,\mathsf{o}}, \phi^H_{k,\mathsf{t}})$ of edge maps $\phi^H_{k,\mathsf{o}} \colon \mathbb{R}_\mathsf{o} \longrightarrow X_H$ and $\phi^H_{k,\mathsf{t}} \colon \mathbb{R}_\mathsf{t} \longrightarrow X_H$ which are elevations of the incident and terminal edge maps of some edge space S^1_k of X. Furthermore, these pairs $(\phi^H_{k,\mathsf{o}}, \phi^H_{k,\mathsf{t}})$ will have the property that these are not edge maps of X''' (such elevations are usually called *hanging elevations* of the precover X''', as in [Wil08, Remark 2.18]). For each k, we denoted by $\mathbb{R}_\mathsf{o}$ and $\mathbb{R}_\mathsf{t}$ the domains of $\phi^H_{k,\mathsf{o}}$ and $\phi^H_{k,\mathsf{t}}$, respectively (which are the universal covers of S^1_k). We fix a deck transformation $\tau \colon \mathbb{R}_\mathsf{o} \longrightarrow \mathbb{R}_\mathsf{t}$ so that the natural diagram

$$\begin{tikzcd} \mathbb{R}_\mathsf{o} \arrow[rr, "\tau"] \arrow[dr] & & \mathbb{R}_\mathsf{t} \arrow[dl] \\ & S^1_k & \end{tikzcd}$$

commutes. Then $\overline{X}$ is constructed from X''' by adding the edge space $\mathbb{R}_\mathsf{o}$ with incident and terminal edge maps given by $\phi^H_{k,\mathsf{o}}$ and $\phi^H_{k,\mathsf{t}} \circ \tau$ for each k.

We denote by Γ the underlying graph of the splitting of $\overline{X}$. Notice that the underlying graph of X'' may be smaller. We enlarge the splitting of X'' by adding trivial vertex groups and just assume that its underlying graph is also Γ. So we view $\pi_1(X'')$ as the fundamental group of a graph of groups $\mathcal{W}$ whose underlying graph is Γ. We denote by T a spanning tree of the underlying graph $\Gamma(X''')$ of X'''. By construction, T is also a spanning tree of Γ. We name $E_T = \Gamma(X''')$.

Step 3 (The finite-sheeted precover $\widehat{X}$)**.** By [Wil08, Proposition 3.4], there exists an intermediate finite-sheeted precovering $\overline{X} \longrightarrow \widehat{X} \longrightarrow X$ satisfying the following properties for all sufficiently large positive integers d:

(1) The underlying graph of $\widehat{X}$ is Γ.
(2) The subcomplex Δ projects homeomorphically into $\widehat{X}$.
(3) Each ε'_j descends to a full elevation $\widehat{\varepsilon}_j \colon \widehat{E}_j \longrightarrow \widehat{X}$ with $\widehat{E}_j \longrightarrow E_i$ being a covering of degree d.

Since Δ injects into $\widehat{X}$, we already know that $\overline{\delta}_j$ descends to a full elevation $\widehat{\delta}_j \colon \overline{D}_j \longrightarrow \widehat{X}$. We want to apply Proposition 3.9 and prove that the natural map

$$\pi_1(X'') * \left(\coprod_{\mathcal{L}} \mathbb{Z} \right) \longrightarrow \pi_1(\widehat{X})$$

is injective and L^2-injective. Before this, we need to introduce more notation. There is a natural bipartite structure of Γ given by the bipartite structure of the splittings of ICE groups. More precisely, $\mathsf{Vert}(\Gamma)$ is split into disjoint sets Vert_o and Vert_t so that, for all $e \in \mathsf{Edge}(\Gamma)$, $\mathsf{o}(e) \in \mathsf{Vert}_\mathsf{o}$, and $X''_{\mathsf{o}(e)}$ is a covering of Y; and, similarly, $\mathsf{t}(e) \in \mathsf{Vert}_\mathsf{t}$ and $X''_{\mathsf{t}(e)}$ is a covering of the torus T^n. We denote by $\mathcal{Z}$ the graph of groups corresponding to $\pi_1(\widehat{X})$, whose underlying graph is Γ. At the end of Step 2, we defined the graphs of groups $\mathcal{W}$ and $\mathcal{Z}$, the spanning tree $T \subseteq \Gamma$ and the subgraph $E_T \subseteq \Gamma$. Denote by $\mathcal{L}^{(0)}$ the collection of elements of $\pi_1(\widehat{X})$ that are represented by the images of the elliptic loops $\{\widehat{\varepsilon}_j\}$. For each $v \in \mathsf{Vert}_\mathsf{o}$, we define $\mathcal{L}_v$ to be the subset of Z_v that contains $\mathcal{L}^{(0)}_v$ and the elements $\phi_{\mathsf{o},e}(z_e)$ such that $\phi_{\mathsf{o},e}(z_e) \notin W_v$. By construction, it is not hard to see that, up to a homotopy of $\widehat{X}$, we have that:

(a) The subset of $\pi_1(\widehat{X})$ represented by the images of $\widehat{\epsilon_j}$ is exactly $\bigcup_{v\in \mathsf{Vert}_{\mathsf{o}}} \mathcal{L}_v^{(0)}$.

(b) The subset of $\pi_1(\widehat{X})$ represented by the images of $\widehat{\delta_j}$ is

$$\{t_e : e \in \mathsf{Edge}(\Gamma) \setminus E_T\},$$

where we view $\pi_1(\widehat{X})$ as in Definition 2.3 (relative to the spanning tree T).

Before applying Proposition 3.9, we observe that we can ensure that $\overline{X}$ satisfies an additional property, on top of the three listed above. Our inductive hypothesis implies that the complex $\overline{X}_v$ is L^2-tame relative to $\mathcal{L}_v$ for each $v \in \mathsf{Vert}_{\mathsf{o}}$. Henceforth, with the same construction as in [Wil08, Proposition 3.4], and by adequately replacing the notion of tameness by our notion of L^2-tameness, we could have ensured that the finite-sheeted precover $\widehat{X}$ satisfies the following additional property:

(4) For each $v \in \mathsf{Vert}_{\mathsf{o}}$, the natural map

$$\pi_1(\overline{X}_v) * \left(\coprod_{\mathcal{L}_v} \mathbb{Z}\right) \longrightarrow \pi_1(\widehat{X}_v)$$

is injective and L^2-injective.

By applying Proposition 3.9 to the subgraph of groups $\mathcal{W} \leqslant \mathcal{Z}$ introduced above (and keeping in mind remarks (a) and (b) above), the induced map

$$\pi_1(X'') * \left(\coprod_{\mathcal{L}} \mathbb{Z}\right) \longrightarrow \pi_1(\widehat{X}), \tag{5.2}$$

is injective and L^2-injective.

Step 4 (The finite-sheeted cover $\widehat{X}^+$)**.** Finally, $\widehat{X}$ can be extended to a finite-sheeted covering $\widehat{X}^+ \longrightarrow X$ by adding additional vertex spaces glued along cylinders by [Wil08, Proposition 3.7]. Hence $\pi_1(\widehat{X})$ is the vertex group of a cyclic splitting of $\pi_1(\widehat{X}^+)$ and, by Lemma 3.8, the injective map

$$\pi_1(\widehat{X}) \longrightarrow \pi_1(\widehat{X}^+) \tag{5.3}$$

is L^2-injective.

We have gathered all the ingredients to prove that the finite-sheeted cover $\widehat{X}^+$ satisfies the forth point of the L^2-tame property, namely that the induced map

$$\pi_1(X_H) * \left(\coprod_{\mathcal{L}} \mathbb{Z}\right) \longrightarrow \pi_1(\widehat{X}^+),$$

is injective and L^2-injective. This is a direct consequence of the fact that the maps described in Equations (5.1), (5.2) and (5.3) are injective and L^2-injective. □

6. The Geometric Hanna Neumann Conjecture

The purpose of this section is to explain how to obtain the Geometric Hanna–Neumann conjecture for all the groups of Corollary C, that is, limit groups and finite graphs of free groups with cyclic edge groups that are hyperbolic relative to virtually abelian subgroups. The results that we state here are well-known. In [AJZ22, Sections 8–12], Antolín and Jaikin-Zapirain explain how the L^2-Hall property implies the geometric Hanna Neumann conjecture (GHNC) for the class of hyperbolic limit groups. Here we review their argument implementing recent work of Minasyan [Min23] and Minasyan–Mineh [MM22] that will show that the L^2-Hall property implies the GHNC for the groups of Corollary 6.5. This proves that Corollary C follows from Theorems A and B.

We retain the following convention: if G is a graph of free groups with cyclic edge groups, then $X = (X_v, X_e; \Gamma)$ will denote a corresponding graph of graphs with S^1 edge spaces such that $G = \pi_1 X$. However, in this subsection, G is not always assumed to be a graph of free groups.

Before describing under which circumstances the $\mathcal{D}_{K[G]}$-Hall property implies the GHNC, we record other properties that will be important for this. Recall that a group G is said to have the *Wilson–Zalesskii property* if G is residually finite and

$$\overline{U \cap V} = \overline{U} \cap \overline{V}$$

for all finitely generated subgroups of G, where the closures are taken in the profinite completion $\widehat{G}$ of G.

Lemma 6.1. *Let $G = \pi_1(X)$ be a finite graph of free groups with cyclic edge groups such that Φ_X is balanced and solvable (in the sense of Definition 4.2). Then G is*

(1) hyperbolic relative to virtually abelian subgroups;
(2) locally relatively quasiconvex;
(3) virtually compact special;
(4) double coset separable and therefore has the Wilson–Zalesskii property;
(5) L^2-Hall.

These conclusions also hold if G is a limit group.

Proof. (1) Let $G = \pi_1(X)$ be a graph of free groups with cyclic edge groups such that Φ_X is balanced and solvable. By the main result of Richer's masters thesis [Ric06, Main Theorem 1.2.5], G is hyperbolic relative to the generalised Baumslag–Solitar subgroups associated to the components of Φ_X (see [Ric06, Section 2.2] for this version of the statement). Since Φ_X is balanced and solvable, all of the edge weights are ± 1 and each component of Φ_X has first Betti number at most 1. Therefore, the associated generalised Baumslag–Solitar subgroups are all virtually isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$.

The case when G is a limit group follows from Dahmani's work [Dah03, Theorem 0.3].

(2) This follows from (1) and [BW13, Corollary D] in the graph of free groups case. If G is a limit group, then this is [Dah03, Proposition 4.6].

(3) This can be collected from either (1) with the main results of [HW10] and [Rey23], or from [MM22, Corollary 2.3].

(4) Minasyan–Mineh proved that limit groups and subgroup separable graphs of free groups with cyclic edge groups are double coset separable [MM22, Theorem 2.2]. Minasyan proved that double coset separability implies the Wilson–Zalesskii property [Min23, Corollary 1.2].

(5) For limit groups this follows from [BK23, Corollary 28] and for graphs of free groups this is Theorem 4.9. □

For the GHNC to hold for a class of groups, we must first ensure that the sum over the double cosets is finite. We restate [AJZ22, Theorem 9.4] in the full generality where their proof holds.

Theorem 6.2 ([AJZ22, Theorem 9.4]). *Let G be hyperbolic relative to finitely generated virtually abelian subgroups and suppose that G is locally relatively quasi-convex. Moreover, suppose that all finitely generated subgroups of G are of finite type. Let $U, V \leqslant G$ be finitely generated subgroups and let T be a complete set of (U, V)-double coset representatives. Then there are only finitely many $t \in T$ such that $U \cap V^t$ is not virtually abelian. In particular, the sum $\sum_{t \in T} \overline{\chi}(U \cap V^t)$ is finite.*

For the remainder of the section, assume that the groups appearing are locally indicable and that $\mathcal{D}_{K[G]}$ exists.

The next important step in their argument is to reformulate the Geometric Hanna Neumann Conjecture in terms of $\mathcal{D}_{K[G]}$-Betti numbers of pairs of modules. First of all, recall the $\mathcal{D}_{K[G]}$-Betti numbers of $K[G]$-modules introduced in Equation (2.1). Let M and N be two left $K[G]$-modules. Form the $K[G]$ module $M \otimes_K N$, where the G-action on simple tensors is given by the diagonal action $g \cdot (m \otimes n) := (gm) \otimes (gn)$. The n-th $\mathcal{D}_{K[G]}$-Betti number of the pair (M, N) is defined to be

$$\beta_n^{K[G]}(M, N) := \beta_n^{K[G]}(M \otimes_{\mathbb{C}} N).$$

Antolín–Jaikin-Zapirain credit this definition to Mark Shusterman (in the case $K = \mathbb{C}$). The next result is proved for limit groups in [AJZ22, Proposition 2.8]. However, the only essential property that is used is that $b_n^{(2)}(G) = 0$ for all $n \neq 1$ and any non-trivial limit group G. Henceforth, it applies in the following generality.

Proposition 6.3 ([AJZ22, Proposition 8.2]). *Let G be a group of finite type such that $b_1^{K[G]}(G) = \overline{\chi}(G)$. Then for any finitely generated subgroups $U, V \leqslant G$, we have*

$$\beta_1^{(2)}(K[G/U], K[G/V]) = \sum_{t \in U \backslash G/V} \overline{\chi}(U \cap V^t).$$

In particular, the Geometric Hanna Neumann Conjecture is equivalent to having

$$b_1^{(2)}(U)\, b_1^{(2)}(V) \leqslant \beta_1^{(2)}(K[G/U], K[G/V])$$

for all finitely generated subgroups $U, V \leqslant G$.

We mention one last key result from the article of Antolín–Jaikin-Zapirain, which is stated for hyperbolic limit groups. The proof in the general setting is nearly identical; we reproduce it here for the sake of completeness and in order to elucidate the differences in the relatively hyperbolic context.

Proposition 6.4 ([AJZ22, Proposition 11.1]). *Let G be a locally indicable group that is hyperbolic relative to virtually abelian subgroups. Additionally, assume G that is double coset separable, $\mathcal{D}_{K[G]}$-Hall, and has the Wilson–Zalesskii property. Let $U, V \leqslant G$ be finitely generated subgroups. Then there exists a normal finite-index subgroup $H \trianglelefteqslant G$ such that $\beta_1^{K[G]}(N) = 0$, where N is the kernel of the map*

$$f \colon K[G/U] \otimes_K K[G/V] \longrightarrow K[G/U] \otimes_K K[G/VH].$$

Proof. By Theorem 6.2, there are only finitely many double cosets UtV such that $U \cap V^t$ is not virtually abelian. Let $H_0 \leqslant G$ be a finite-index subgroup separating these double cosets UtV.

For each t such that UtV is not virtually abelian, let $A_t \trianglelefteqslant G$ be a finite-index normal subgroup such that $U \cap V^t$ is $\mathcal{D}_{K[G]}$-independent in $(U \cap V^t)A_t$. By [AJZ22, Corollary 10.4] (which holds for groups with the Wilson–Zalesskii property), there is a finite-index normal subgroup $H_t \trianglelefteqslant G$ such that $U \cap (H_t V)^t \leqslant (U \cap V^t)A_t$. Set $H = H_0 \cap \bigcap_t A_t$, where t runs over the double coset representatives such that UtV is not virtually abelian.

Let T be a set of (U, VH) double coset representatives containing 1, which extends to T', a set of (U, V)-double coset representatives. Since H_0 separates the non-virtually abelian (U, V)-double cosets, it follows that if $x \in T' \smallsetminus T$ then UxV is virtually abelian. Let $\pi \colon T' \longrightarrow T$ be a set-theoretic map with the property that $U\pi(t)VH = UtVH$ for all $t \in T'$. In general we have the $K[G]$-module decomposition

$$K[G/U] \otimes_K K[G/V] \cong \bigoplus_{t \in T'} K[G](U \otimes tV).$$

However, in order to analyse the kernel of f more easily, it is useful to modify the complement of $\oplus_{t \in T} K[G](U \otimes tV)$ in $K[G/U] \otimes_K K[G/V]$ and obtain the following decomposition:

$$K[G/U] \otimes_K K[G/V] \cong \bigoplus_{t \in T} K[G](U \otimes tV) \oplus \bigoplus_{t \in T' \smallsetminus T} K[G](U \otimes (tV - \pi(t)V)).$$

Let I_t denote the kernel of the map $K[G](U \otimes tV) \longrightarrow K[G](U \otimes tVH)$. Then the kernel of f has the decomposition

$$\ker f \cong \bigoplus_{t \in T} I_t \oplus \bigoplus_{t \in T' \smallsetminus T} K[G](U \otimes (tV - \pi(t)V)).$$

By exactly the same proof as in [AJZ22, Proposition 11.1], $\beta_1^{(2)}(I_t) = 0$ for each $t \in T$. On the other hand, we have isomorphisms

$$K[G](U \otimes (tV - \pi(t)V)) \cong K[G](U \otimes (tV - \pi(t)V)) \cong K[G/(U \cap V^t)],$$

so it suffices to compute $\beta_1^{K[G]}(K[G/A])$ where A is a virtually abelian group. By Shapiro's Lemma,

$$\operatorname{Tor}_1^{K[A]}(\mathcal{D}_{K[G]}, K) \cong \operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, K[G] \otimes_{K[A]} K) \cong \operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, K[G/A]),$$

and $\operatorname{Tor}_1^{K[A]}(\mathcal{D}_{K[G]}, K) = 0$ since $b_1^{K[G]}(A) = 0$. □

With all of this in place, the proof that the L^2-hall property implies the GHNC proceeds exactly as in Section 12.1 of [AJZ22]. The details of the argument are rather technical and would take us too far afield, as it relies on the theory of acceptable modules over twisted group rings (as presented in [AJZ22, Section 6]). However, with Theorem 6.2 and Propositions 6.3 and 6.4, the remaining arguments that are needed to prove that the $\mathcal{D}_{K[G]}$-Hall property implies the GHNC are exactly as in Antolín and Jaikin-Zapirain's article (i.e. there is no need to adapt them to the relatively hyperbolic case). Hence, we obtain the desired corollary.

Corollary 6.5. *Let $G = \pi_1(X)$ be a graph of free groups with cyclic edge groups such that Φ_X is balanced and solvable (equivalently, such that G is hyperbolic relative to virtually abelian subgroups) or let G be a limit group. Then G satisfies the Geometric Hanna Neumann Conjecture.*

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