Vanishing of the second L^2-Betti number for (p, q, r)-complexes

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June 2020

A thesis submitted to McGill University in partial fulfilment of the requirements of the degree of Master of Science (M.Sc.)

Abstract

We describe the thin cactus property and the energy criterion, two conditions on 2-complexes developed by Wise [1] that imply the vanishing of the second L^2-Betti number. We then apply them to certain subclasses of $(4, 4, 3)$- and $(3, 9, 3)$-complexes.

Abrégé

Nous décrivons le «thin cactus property» et le «energy criterion,» deux conditions sur les complexes 2-dimensionels développées par Wise [1] qui impliquent que le deuxième nombre Betti-L^2 est zéro. Nous appliquons ces conditions sur certaines sous-classes de complexes $(4, 4, 3)$ et $(3, 9, 3)$.

Acknowledgements

I thank my supervisor Dani Wise for introducing me to mathematical research and for the countless hours of conversation over the past year. I thank my mother and brother, Ersy and Ben, for their support and company during the pandemic. I thank NSERC for their financial support that allowed me to write this thesis. Finally, I thank Laura Rivera for her endless support, and for the beautiful figures in this text.

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1 Introduction

L^2-homology has many interesting group theoretic applications. For example, it is related to deficiency and algebraic fibring. There is a list of some of the main applications of L^2-invariants in the introduction of [2]. We are mainly interested in the vanishing of the second L^2-Betti number, which is linked to nonpositive immersions, coherence, and local-indicability (see section 2.4). We are thus interested in finding classes of 2-complexes with vanishing $H_2^{(2)}$ and are guided by the following conjecture of Wise [1].

Conjecture 1.1. *Let $\widetilde{X}$ be an angled 2-complex with nonpositive sectional curvature. Then $H_2^{(2)}(\widetilde{X}) = 0$.*

Throughout the text, we will often let $\widetilde{X}$ denote an arbitrary 2-complex, since we usually have in mind the case where $\widetilde{X}$ is a universal cover. The following results of Wise, from [3] and [1], respectively, provide evidence for Conjecture 1.1.

Theorem 1.2. *If $\widetilde{X}$ is a 2-complex satisfying one of the following conditions, then $\widetilde{X}$ has nonpositive sectional curvature:*

(i) *X is a $(p, 3, p-3)$-complex where $p \geq 6$,*
(ii) *X is a $(p, 4, p-2)$-complex where $p \geq 4$,*
(iii) *X is a $(p, 5, p-1)$-complex where $p \geq 4$.*

Theorem 1.3. *If $\widetilde{X}$ is a 2-complex satisfying one of the following conditions, then $H_2^{(2)}(\widetilde{X}) = 0$:*

(i) *X is a $(p, 3, p-4)$-complex where $p \geq 6$,*
(ii) *X is a $(p, 4, p-2)$-complex where $p \geq 4$,*
(iii) *X is a $(p, 5, p-1)$-complex where $p \geq 4$.*

The main example of 2-complexes with nonpositive sectional curvature are those that are nonpositively curved and locally embed in $\mathbb{R}^3$. Roughly, a (p, q, r)-complex is a 2-complex whose 2-cells have at least p-sides, the girth of every link is at least q, and the valence of all vertices in every link is at most r. We refer the reader to section 2 for more precise definitions.

It seems to be a significant challenge to prove that $(p, 3, p-3)$-complexes have vanishing $H_2^{(2)}$, and we do not do so here. However, we do provide new examples of (p, q, r)-complexes with nonpositive sectional curvature and vanishing second L^2-Betti number in Propositions 3.5 and 4.5.

The thesis is organized as follows. In section 2 we provide some of the necessary theoretical background for the rest of the work. In section 3, we describe the *thin cactus property*, a condition developed by Wise in [1] that implies the vanishing of $H_2^{(2)}$, and apply it to a subclass of $(4, 4, 3)$-complexes in Proposition 3.5. Finally, in section 4, we describe the *energy criterion*, Wise's second condition on 2-complexes implying $H_2^{(2)} = 0$. In Proposition 4.5, we apply the energy criterion to show that certain $(3, 9, 3)$-complexes have vanishing $H_2^{(2)}$.

2 Background

2.1 L^2-homology

We collect the essential facts (for our purposes) about L^2-homology. A comprehensive treatment of the theory can be found in [2]. Let G be a discrete group, and let

$$\ell^2(G) \;=\; \left\{ \sum_{g \in G} z_g g : z_g \in \mathbb{C}, \sum_{g \in G} |z_g|^2 < \infty \right\}$$

be the set of all square-summable formal sums over G with complex coefficients. The inner-product $\left\langle \sum_{g \in G} z_g g, \sum_{g \in G} w_g g \right\rangle \;=\; \sum_{g \in G} z_g \overline{w_g}$ gives $\ell^2(G)$ a Hilbert space structure.

A linear operator $T : \ell^2(G) \to \ell^2(G)$ is *bounded* if $\|T\|_{op} = \inf\{c \in \mathbb{R} : \|T(v)\| \leq c\|v\| \text{ for all } v \in \ell^2(G)\} < \infty$. There is a natural linear action of G on $\ell^2(G)$ given by $g \cdot \sum_{s \in G} z_s s = \sum_{s \in G} z_s gs$. A linear operator T is *G-equivariant* if $T(g \cdot v) = g \cdot T(v)$ for all $g \in G$. The *group von Neumann algebra* $\mathcal{N}(G)$ of G is then defined to be the space of bounded, G-equivariant linear operators on $\ell^2(G)$.

Recall that an action of G on a cell-complex X is *cellular* if every $g \in G$ maps open cells homeomorphically to open cells, and is *cocompact* if the quotient space X/G is compact. Define a *G-basis* for the n-skeleton X^n to be a collection of open n-cells of X such that every open n-cell is in the G-orbit of some cell in the G-basis.

Suppose that X is a cell complex with a cocompact and cellular G-action. Then $C_*(X)$ is the $\mathbb{Z}G$-chain complex whose chain module $C_n(X)$ is a $\mathbb{Z}G$-module with a finite G-basis of n-cells, for every $n \in \mathbb{N}$. The cellular L^2-chain complex is

$$C_*^{(2)}(X) \;=\; \ell^2(G) \otimes_{\mathbb{Z}G} C_*(X).$$

Hence, we will usually represent n-cycles as a formal sum $\sum_{d \in X^n} z_d d$, where $\sum_{d \in X^n} |z_d|^2 < \infty$. Moreover, we will refer to z_d as the *coordinate* of the n-cell d. The L^2-boundary maps are $\partial_n^{(2)} = \mathrm{id}_{\ell^2(G)} \otimes_{\mathbb{Z}G} \partial_n$, where $\partial_n : C_n(X) \to C_{n-1}$ is the usual boundary map. The L^2-homology groups are then $H_n^{(2)}(X) = \ker \partial_n^{(2)} / \mathrm{cl}(\mathrm{im}\, \partial_{n+1}^{(2)})$. We quotient by the closure of $\mathrm{im}\, \partial_{n+1}^{(2)}$ (taken with respect to the Hilbert space topology) because it gives $H_n^{(2)}(X)$ the structure of a *Hilbert $\mathcal{N}(G)$-module*; the *n-th L^2-Betti number* $b_n^{(2)}(X)$ is then the *von Neumann dimension* of $H_n^{(2)}(X)$. We will not define Hilbert $\mathcal{N}(G)$-modules or their associated von Neumann dimension since their definitions are quite technical and we will not need them in what follows. The facts about L^2-Betti numbers we will require are recorded in the following proposition.

Proposition 2.1. *Let G be a discrete group, let X be a cell complex, and let $G \curvearrowright X$ be a cellular cocompact action. Then*

(i) $H_n^{(2)}(X) = 0$ *if and only if* $b_n^{(2)}(X) = 0$,

(ii) $\chi(X/G) = \sum_{i=0}^{\dim(X)} (-1)^i b_i^{(2)}(X)$, *and*

(iii) $b_0^{(2)}(X) = 1/|G|$ *if G is finite, and* $b_0^{(2)}(X) = 0$ *if G is infinite.*

Proofs of these statements can be found in [2] (Theorem 1.12(a) and Theorem 1.35(2),(8)).

2.2 (p, q, r)-complexes

Definition 2.2 (combinatorial map and cell complex)**.** A map $f : A \to B$ between cell complexes is *combinatorial* if it maps open cells homeomorphically to open cells. Combinatorial cell-complexes are defined by induction on their dimension as follows. Declare every 0-dimensional cell complex to be combinatorial. An n-dimensional cell complex X is combinatorial if its $(n-1)$-skeleton is combinatorial, and the attaching maps of n-cells d^n factor as

$\partial d^n \to S^{n-1} \xrightarrow{f} X^{n-1}$, where S^{n-1} has a combinatorial cell structure and f is combinatorial. An arbitrary cell complex X is *combinatorial* if each of its n-skeleta X^n are combinatorial.

For example, the n-sphere with its cell-structure consisting of a 0-cell and an n-cell is *not* a combinatorial cell-complex.

To define (p, q, r)-complexes, a class of combinatorial 2-complexes that will be the focal point of this study, we need the following notions. Let X be a combinatorial cell-complex. Intuitively, the *link* of a vertex $v \in X^0$, denoted $\text{link}(v)$ is obtained by intersecting an ϵ-sphere centred at v with X. In the case where X is 2-dimensional, we can define $\text{link}(v)$ more precisely: $\text{link}(v)$ is a graph whose vertices are in one-to-one correspondence with the edges of X adjacent to v; two vertices are joined by an edge for every time the corresponding edges are traversed consecutively in the attaching map of a 2-cell.

A 1-cell in X has *perimeter* q if it is traversed q times by the attaching map of 2-cells. The *girth* of a graph G is the length of the shortest cycle in G.

Definition 2.3 ((p, q, r)-complexes)**.** A combinatorial 2-complex X is said to be a (p, q, r)-complex if every 2-cell has attaching map of length $\geq p$, every 0-cell has link of girth $\geq q$, and every 1-cell has perimeter $\leq r$.

2.3 Nonpositive sectional curvature

A combinatorial 2-complex X is an *angled 2-complex* if for every 2-cell D, every corner c of D is assigned a positive real number $\angle(c)$ called an *angle*. The *curvature of a 2-cell* D with n sides is

$$\kappa(D) \;=\; \sum_{c \in \text{corners}(D)} \angle(c) - (n-2)\pi.$$

This formula is motivated by remarking that the sum of the interior angles of an n-gon is $(n-2)\pi$ in flat space (e.g. $\mathbb{E}^2$), greater that $(n-2)\pi$ in positively curved space (e.g. $\mathbb{S}^2$), and less than $(n-2)\pi$ in negatively curved space (e.g. $\mathbb{H}^2$).

An angled 2-complex X has *nonpositive sectional curvature* if $\kappa(D) \leq 0$ for every 2-cell of X, and if for every $v \in X^0$, for each non-empty, connected subgraph $\Gamma \subseteq \text{link}(v)$ such that every vertex of Γ has valence ≥ 2, we have

$$2\pi - \pi\chi(\Gamma) - \sum_{e \in \mathrm{edges}(\Gamma)} \angle(c_e) \ \leq \ 0,$$

where c_e is the corner associated to the edge e of $\mathrm{link}(v)$.

2.4 Local indicability, coherence, and nonpositive immersions

A group G is *locally indicable* if for every finitely generated subgroup H of G, there is a non-trivial homomoprhism $H \to \mathbb{Z}$. If every finitely generated subgroup of G is finitely presented, then G is *coherent*.

One of the interests in studying L^2-homology comes from its relationship to the nonpositive immersions property. An *immersion* is a locally injective combinatorial map. A combinatorial 2-complex X has the *nonpositive immersions property* if for every immersion $Y \to X$ with Y a compact, connected 2-complex, either Y is simply connected, or $\chi(Y) \leq 0$. The vanishing of $H_2^{(2)}$ is related to the nonpositive immersions property via the following result from [1].

Lemma 2.4. *Let $\mathcal{K}$ be a class of compact 2-complexes such that if $X \in \mathcal{K}$ and $Y \to X$ is an immersion with Y compact and connected, then $Y \in \mathcal{K}$. If $b_2^{(2)}(\widetilde{X}) = 0$ for every $X \in \mathcal{K}$, where $\pi_1 X \curvearrowright \widetilde{X}$ is the group action, then every $X \in \mathcal{K}$ has the nonpositive immersions property.*

Proof. Let $X \in \mathcal{K}$. By Proposition 2.1,

$$\chi(X) \ = \ b_0^{(2)}(\widetilde{X}) - b_1^{(2)}(\widetilde{X}) + b_2^{(2)}(\widetilde{X}) \ = \ b_0^{(2)}(\widetilde{X}) - b_1^{(2)}(\widetilde{X}) \leq b_0^{(2)}(\widetilde{X}) \ = \ \frac{1}{|\pi_1 X|}.$$

Then, either $\pi_1(X)$ is trivial, or $|\pi_1 X| > 1$, implying that $\chi(X) < 1 \Rightarrow \chi(X) \leq 0$.

Let $Y \to X$ be an immersion with Y compact and connected. Then $Y \in \mathcal{K}$, and by the same argument as above, either $\pi_1 Y = 1$ or $\chi(Y) \leq 0$. Therefore, X has the nonpositive sectional curvature property. $\square$

In Section 4 we will describe a class of 2-complexes whose universal covers have vanishing $H_2^{(2)}$ and are closed under immersions, namely the class of 2-complexes satisfying the *energy* *criterion*. The algebraic significance of a 2-complex X having nonpositive immersions is that it implies $\pi_1 X$ is locally indicable, and it is conjectured that it also implies $\pi_1 X$ is coherent [4].

2.5 Triangles of groups

We will use triangles of groups in Example 3.6, this subsection will not be needed otherwise. The reader is refered to [5] for a more thorough exposition. A *triangle of groups* is a commutative diagram

$$\begin{tikzcd}
 & & V_2 & & \\
 & E_1 \arrow[ur] \arrow[ddl] & & E_3 \arrow[ul] \arrow[ddr] & \\
 & & F \arrow[ul] \arrow[ur] \arrow[d] & & \\
V_3 & & E_2 \arrow[ll] \arrow[rr] & & V_1
\end{tikzcd}$$

of groups, where all homomorphisms are injections. The groups V_i are the *vertex groups*, E_i are the *edge groups*, and F is the *face group* of the triangle of groups. For each pairwise distinct triple $i, j, k \in \{1, 2, 3\}$, there is a homomorphism $\phi_k : E_i *_F E_j \to V_k$. The length of the shortest word in $\ker(\phi_k)$ is always even, for if $w \in \ker(\phi_k)$ has length $2n+1$, we can conjugate w by its rightmost letter to obtain an element of length $2n$ in $\ker(\phi_k)$. Hence, the *angle at* V_k, denoted $\angle(V_k)$, is π/n, where $2n$ is the length of the shortest word in $\ker(\phi_k)$. The triangle of groups is *nonpositively curved* if $\sum_{k=1}^{3} \angle(V_k) \leq \pi$.

Let G be the colimit of a nonpositively curved triangle of groups. We build a triangle-complex L on which G acts as follows. First, apply the $K(\pi, 1)$ functor to the triangle of groups to obtain the same diagram, except the injective group homomorphisms become injective continuous maps between cell complexes. Next, take $K(E_i, 1) \times [0, 1]$ for each i, and identify $K(E_i, 1) \times \{0\}$ with its image in $K(V_j, 1)$ and $K(E_i, 1) \times \{1\}$ with its image in $K(V_k, 1)$ (where i, j, k are pairwise distinct). Finally, let $\triangle$ be a Euclidean triangle with unit-length edges, add $K(F, 1) \times \triangle$ to the complex built thus far, and identify $K(F, 1) \times \partial\triangle$ naturally with subcomplexes of the vertex and edge spaces. The resulting space K has fundamental group isomorphic to G, and therefore G acts freely on the universal cover $\widetilde{K}$. Composing the natural projection $r : K \to \triangle$ with the covering map $\phi : \widetilde{K} \to K$, we have a map $f = r \circ \phi : \widetilde{K} \to \triangle$. Define $L = \widetilde{K}/\sim$, where $x \sim y$ if and only if x and y lie in the same component of $f^{-1}(\triangle)$. Hence, L is a triangle-complex with an induced G-action. It turns out that L is simply connected, and $L/G = \triangle$.

The link of a vertex v in L lying over V_k is encoded by the maps

$$V_k/E_j \leftarrow V_k/F \to V_k/E_i.$$

The edges of $\mathrm{link}(v)$ correspond to the cosets in V_k/F, and the vertices correspond to the cosets in V_k/E_i and V_k/E_j. Vertices v_1E_i and v_2E_j are joined by an edge g if and only if $gF \subseteq v_1E_i \cap v_2E_j$. The following lemma gives an equivalent description of the links when the face-group is trivial.

Lemma 2.5. *Let V be a vertex-group of a nonpositively curved triangle of groups with trivial face-group, and let E_1 and E_2 be its adjacent edge-groups. Let V act on a finite bipartite graph Γ, such that the following conditions hold:*

(i) *there is an edge e of Γ such that $V \cdot e = \Gamma$,*

(ii) *if $g \in V$ and $g \cdot e = e$, then $g = 1$, and*

(iii) *the vertices of Γ belong to the two classes Γ_1 and Γ_2. For every $g \in G$, $g \cdot \Gamma_i = \Gamma_i$. If $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$ be the endpoints of the edge $g \cdot e$, then $\mathrm{Stab}(v_i) = gE_ig^{-1}$.*

Then $\mathrm{link}(v)$ is isomorphic to Γ for every vertex v over V.

Proof. We will define a map f from the vertices of Γ to $\mathrm{link}(v)$ and then prove that it is a graph isomorphism. For a vertex $v \in \Gamma_i$ that is the endpoint of an edge $g \cdot e$, let $f(v_i) = gE_i$. To check that f is well-defined, suppose that $v \in \Gamma$ is the endpoint of $g_1 \cdot e$ and $g_2 \cdot e$. Since $g_2g_1^{-1} \cdot (g_1 \cdot e) = g_2 \cdot e$, then $g_2g_1^{-1} \in \mathrm{Stab}(v) = g_1E_ig_1^{-1} \Rightarrow g_2 \in g_1E_i \Rightarrow g_1E_i = g_2E_i$. Thus, f is well-defined.

Now, f is clearly surjective. To check injectivity, let $v_1, v_2 \in \Gamma_i$ be vertices that are endpoints of $g_1 \cdot e$ and $g_2 \cdot e$ such that $g_1E_i = g_2E_i$. Then $g_2g_1^{-1} \in g_2E_ig_1^{-1} = g_1E_ig_1^{-1} = \mathrm{Stab}(v_1)$. Since $g_2g_1^{-1} \cdot (g_1 \cdot e) = g_2 \cdot e$, we must have that v_1 is an endpoint of both $g_1 \cdot e$ and $g_2 \cdot e$, and therefore $v_1 = v_2$.

Finally, we verify that f is a graph isomorphism. Let $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$ be endpoints of an edge $g \cdot e$. Then $g \cdot E_1$ and $g \cdot E_2$ are endpoints of the edge g in $\mathrm{link}(v)$. Property

(ii) implies that there are $|V|$ edges in Γ. By definition, there are also $|V|$ edges in $\mathrm{link}(v)$. Therefore, f is a graph isomorphism. □

3 The Thin Cactus Property

In this section, we describe the first of two conditions on combinatorial 2-complexes developed by Wise in [1] implying the vanishing of $H_2^{(2)}$.

3.1 Statement and proof

Definition 3.1 (Dual 1-skeleton, cactus-dual, thin cactus property)**.** Let $\widetilde{X}$ be a combinatorial 2-complex. The *dual* 1*-skeleton* Γ *of* $\widetilde{X}$ is the graph containing a *face-vertex* for every 2-cell and an *edge vertex* for every 1-cell of $\widetilde{X}$. A face-vertex and an edge-vertex are joined by an edge if and only if the corresponding edge is traversed by the attaching map of the corresponding face in $\widetilde{X}$. Note that there is a natural embedding $\Gamma \hookrightarrow \widetilde{X}$, so it is often useful to view Γ as a subset of $\widetilde{X}$.

A *cactus-dual* is a sub-tree T of Γ such that the inclusion $T \hookrightarrow \Gamma$ is a local-isomorphism at every edge-vertex (see Fig. 1). We say that T is *thin* if for every face-vertex v_d in T, its adjacent edge-vertices $u_{e_1}, \ldots, u_{e_n}$ satisfy

$$\sum_{i \neq k} \frac{1}{\deg(u_{e_i}) - 1} \geq 1 \tag{1}$$

for each $k \in \{1, \ldots, n\}$. We emphasize that this condition must hold at every face-vertex in T. Note that we take $1/0 = \infty$, so (1) holds trivially if $\deg(u_{e_i}) = 1$ for some $i \neq k$.

Finally, a complex $\widetilde{X}$ is said to have the *thin cactus property* provided that every face vertex of the dual 1-skeleton lies in some thin cactus-dual. We use the word "thin" because, intuitively, thin cacti do not have too many face-vertices adjacent to an edge-vertex. It turns out that condition (1) is exactly what is needed to ensure the vanishing of $H_2^{(2)}$.

To show that combinatorial 2-complexes with the thin cactus property have vanishing $H_2^{(2)}$, we first need the following lemma.

Lemma 3.2. *Let $z_1, \ldots, z_n, z \in \mathbb{C}$ and suppose that the z_i satisfy the constraint $\sum_{i=1}^{n} z_i = z$. Then $\sum_{i=1}^{n} |z_i|^2 \geq |z|^2/n$, and equality is reached only when $z_i = z/n$ for each i.*

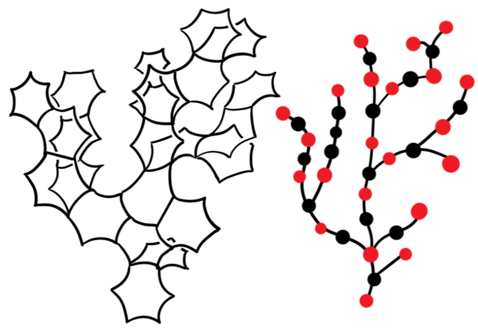

Figure 1: A cell complex and a cactus-dual. Note that the cactus-dual does not need to contain every face-vertex of the dual 1-skeleton.

Proof. Let $(z_1, \ldots, z_n), (1, \ldots, 1) \in \mathbb{C}^n$. Then, with the standard inner product on $\mathbb{C}^n$, the Cauchy-Schwartz inequality implies that $\sum_{i=1}^n |z_i|^2 \geq |z|^2/n$. Moreover, equality is reached if and only if the two vectors are linearly dependent, or equivalently, if and only if $z_1 = \ldots z_n = z/n$. $\square$

The proof of the following theorem follows closely the one originally given by Wise in [1].

Theorem 3.3. *Let $\widetilde{X}$ be a combinatorial 2-complex having the thin cactus property. Then $H_2^{(2)}(\widetilde{X}) = 0$.*

Proof. Let $Z = \sum_{d \in \widetilde{X}^2} z_d d \in H_2^{(2)}(\widetilde{X})$ be an L^2 2-cycle, and suppose for a contradiction $Z \neq 0$. Let $d_0 \in \widetilde{X}^2$ be a 2-cell with non-zero coordinate z_{d_0}, and let T be a thin cactus containing the face-vertex v_{d_0} corresponding to d_0. We then choose v_{d_0} as a base-point for T; by declaring that the edges of T have length $1/2$, every face-vertex has integral distance from v_{d_0}. This choice of edge length endows T with a natural metric d_T. Define $\widetilde{X}^2_n \subseteq \widetilde{X}^2$ to be the set of 2-cells whose associated face-vertices have distance n from v_{d_0}.

By induction, we will show that $\sum_{d \in \widetilde{X}^2_{n+1}} |z_d|^2 \geq \sum_{d \in \widetilde{X}^2_n} |z_d|^2$. For the base case, let $u_1, \ldots, u_m$ be v_{d_0}'s adjacent edge vertices. For each u_i, let v_{ij}, $j = 1, \ldots, k_i$ be its adjacent face-vertices such that $\mathrm{d}_T(v_{d_0}, v_{ij}) = 1$. Note that $k_i = \deg(u_i) - 1$. Since the coordinates $z_{d_{ij}}$ (corresponding to the face-vertices v_{ij}) are subject to the cycle condition $\sum_{i=1}^{k_i} (\pm z_{d_{ij}}) = z_{d_0}$, Lemma 3.2 implies that the minimum of $\sum_{i=1}^{k_i} \left| z_{d_{ij}} \right|^2$ occurs at $\pm z_{d_{ij}} = z_{d_0}/k_i$, and is thus equal to $|z_{d_0}|^2/k_i$. Hence,

$$\sum_{d\in\widetilde{X}_1^2}|z_d|^2=\sum_{i=1}^{m}\sum_{j=1}^{k_i}\left|z_{d_{ij}}\right|^2\geq\sum_{i=1}^{m}\frac{|z_{d_0}|^2}{k_i}=|z_{d_0}|^2\sum_{i=1}^{m}\frac{1}{\deg(u_{e_i})-1}\geq|z_{d_0}|^2=\sum_{d\in\widetilde{X}_0^2}|z_d|^2,$$

where the last inequality follows from (1). This proves the base case.

Now, let $d\in\widetilde{X}_n^2$ be an arbitrary 2-cell, let v_d be the corresponding face-vertex, and let $u_1^d,\dots,u_{l_d}^d$ be the edge-vertices adjacent to v_d such that $\mathrm{d}_T(v_{d_0},u_i^d)=n+1/2$. Moreover, let $v_{i1}^d,\dots,v_{ik_i^d}^d$ be the face-vertices adjacent to u_i^d such that $\mathrm{d}_T(v_{d_0},v_{ij}^d)=n+1$. Note that $k_i^d=\deg(u_i^d)-1$. Let c_{ij} be the 2-cell corresponding to the face-vertex v_{ij}^d. Since the coordinates $z_{c_{ij}}$ satisfy the cycle condition $\sum_{j=1}^{k_i^d}(\pm z_{c_{ij}})=z_d$, the minimum of $\sum_{j=1}^{k_i^d}\left|z_{d_{ij}}\right|^2$ occurs at $\pm z_{d_{ij}}=z_d/k_i^d$, and is therefore equal to $|z_d|^2/k_i^d$ by Lemma 3.2. Hence,

$$\sum_{d\in\widetilde{X}_{n+1}^2}|z_d|^2=\sum_{d\in\widetilde{X}_n^2}\sum_{i=1}^{l_d}\sum_{j=1}^{k_i^d}\left|z_{d_{ij}}\right|^2\geq\sum_{d\in\widetilde{X}_n^2}\sum_{i=1}^{l_d}\frac{|z_d|^2}{k_i^d}=\sum_{d\in\widetilde{X}_n^2}\sum_{i=1}^{l_d}\frac{|z_d|^2}{\deg(u_i^d)-1}\geq\sum_{d\in\widetilde{X}_n^2}|z_d|^2,$$

which proves the inductive step.

From the induction, we have that $\sum_{d\in\widetilde{X}_n^2}z_d^2\geq z_{d_0}^2>0$. Therefore,

$$\sum_{d\in\widetilde{X}^2}|z_d|^2=\sum_{n=0}^{\infty}\sum_{d\in\widetilde{X}_n^2}|z_d|^2\geq\sum_{n=0}^{\infty}|z_{d_0}|^2=\infty,$$

contradicting the assumption that Z is an L^2 cycle. We conclude that $H_2^{(2)}(\widetilde{X})=0$. $\square$

3.2 Application to $(4,4,3)$-complexes

We will now apply Theorem 3.3 to show that a special class of $(4,4,3)$-complexes have vanishing $H_2^{(2)}$. Before proving Proposition 3.5, we need the following definitions.

Definition 3.4 (Alternating vertices, routine)**.** Let $\widetilde{X}$ be a combinatorial 2-complex, and let $\widetilde{X}^1$ be given the graph metric d, with each edge having length 1. Suppose that a condition $\mathcal{P}$ on the links of 0-cells of $\widetilde{X}$ holds for every $v\in\widetilde{X}^0$ such that $\mathrm{d}(b,v)=2n-1$ for some $n\in\mathbb{N}$ and some base-point $b\in\widetilde{X}^0$. Then $\mathcal{P}$ is said to hold at *alternating vertices* of $\widetilde{X}^0$.

$\widetilde{X}$ is *routine* if every 1-cell is traversed by the attaching map of at least two 2-cells.

Proposition 3.5. *Let* $\widetilde{X}$ *be a routine* $(4,4,3)$*-complex such that every* 2*-cell is a square and suppose that* $\mathrm{girth}(\mathrm{link}(v))\geq 6$ *at alternating vertices* v *of* $\widetilde{X}^0$*. Then* $H_2^{(2)}(\widetilde{X})=0$.

Proof. Let $d \in \widetilde{X}^2$ be an arbitrary square and let b be any of its vertices. Since $\widetilde{X}$ is routine $(4,4,3)$-complex, we can apply Proposition 10.2. of [1] to obtain a cellular map $g : \widetilde{X} \to \mathbb{R}_{\geq 0}$ such that every vertex is mapped to an integer, b is the only vertex mapping to 0, and every 2-cell is mapped to $[n, n+1]$ for some $n \in \mathbb{Z}_{\geq 0}$. Moreover, the boundary of each square is of the form $\alpha\epsilon_1\gamma\epsilon_2$, where either

(i) $\alpha, \epsilon_1, \epsilon_2$, and γ are edges, or

(ii) α is a vertex, ϵ_1 and ϵ_2 are edges, and γ is a concatenation of two edges,

and in both cases $g(\alpha) = \{n\}$, $g(\epsilon_i) = [n, n+1]$, $g(\gamma) = \{n+1\}$. Both cases are shown in Fig. 2. Define *type (i)* [*type (ii)*] *squares* to be squares with boundary of type (i) [type (ii)]. Proposition 10.2 of [1] also implies that if d' is a square mapping to $[n, n+1]$, so that γ maps to $n+1$, then every square other than d' intersecting an edge of γ maps to $[n+1, n+2]$. We will use this property later in the proof.

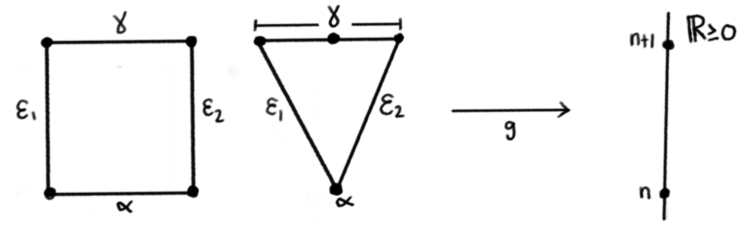

Figure 2: The two types of squares and their mappings to $\mathbb{R}_{\geq 0}$.

Let Γ be the dual 1-skeleton and let v_d be the face-vertex in Γ corresponding to d. We will inductively construct a thin cactus T containing v_d by defining an ascending sequence of finite subgraphs $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k \subseteq T_{k+1} \subseteq \cdots \subseteq \Gamma$, and then letting $T = \bigcup_{k \in \mathbb{N}} T_k$.

Let $T_0 := \{v_d\}$. Since b is the only vertex mapping to 0, the square d is of type (ii). Then, let γ_1 and γ_2 be the consecutive edges of d such that $g(\gamma_i) = \{1\}$, and let ϵ be one of the edges of d such that $g(\epsilon) = [0, 1]$. Let $u_{\gamma_1}, u_{\gamma_2}$, and u_ϵ be the corresponding edge-vertices in Γ; add them to T_1, as well as the edges joining them to v_d. Since $T \hookrightarrow \Gamma$ is required to be an isomorphism at edge-vertices, add all the face vertices and adjacent to $u_{\gamma_1}, u_{\gamma_2}$, and u_ϵ, and the edges joining them, to T_1. This concludes the construction of T_1, which is depicted in Fig. 3.

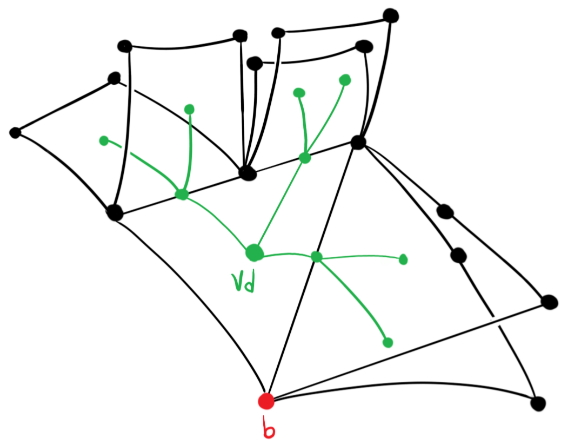

Figure 3: T_1, the first stage of the cactus-dual T.

Now suppose we have defined defined T_j for $j \leq k$. Let s be a square mapping to $[n, n+1]$, and such that $v_s \in T_k \setminus T_{k-1}$ is the corresponding face-vertex. First, suppose s is a type (i) square; let e_1 be the edge of s mapping to $n+1$, and let e_2 be the edge mapping to $[n, n+1]$ with bottom endpoint having link of girth 6. Then let the corresponding edge vertices be u_{e_1} and u_{e_2}; add them to T_{k+1}, along with the edges joining them to v_s. Moreover, add all the face vertices in Γ that are adjacent to u_{e_1} and u_{e_2}, as well as the edges joining them to u_{e_1} and u_{e_2}.

Now suppose that s is a type (ii) square. Let e_1 and e_2 be the edges mapping to $n+1$; add the corresponding edge-vertices u_{e_1}, u_{e_2} to T_{k+1}, as well as the face-vertices adjacent to them. Finally, add the edges joining u_{e_1} and u_{e_2} to v_s, as well as the edges joining the other face-vertices to u_{e_1} and u_{e_2}. The local construction of T_{k+1} is shown in Fig. 4. Part of the cactus-dual is shown in Fig. 5.

Suppose that T is not a tree. Then there is a non-trivial, non-backtracking cycle $\alpha \subseteq T$ such that α traverses the vertices and edges of T at most once. The map $g : \widetilde{X} \to \mathbb{R}_{\geq 0}$ induces a map g' on the vertices of T as follows. If v is a face-vertex [edge-vertex] corresponding to a square [an edge] mapping to $[n, n+1]$, then $g'(v) = n + 1/2$. If V is an edge-vertex corresponding to an edge mapping to $\{n\}$, then $g'(v) = n$. Since α is compact, there is at least one vertex v in α such that $g'(v) \geq g'(v')$ for all other vertices $v' \in \alpha$. Suppose $g'(v)$ is an integer; therefore, v is an edge-vertex corresponding to an edge mapping to $\{n\}$ for some $n \in \mathbb{N}$. Since v has valence 2, it is adjacent to two face-vertices mapping to $n - 1/2$.

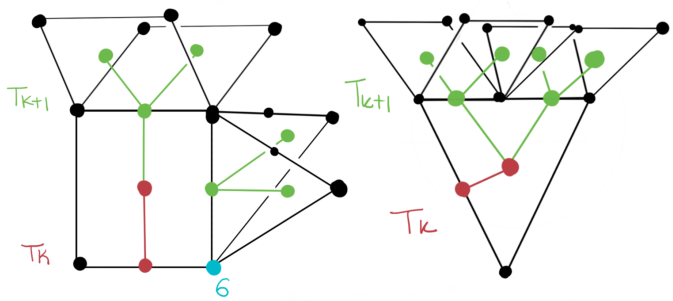

Figure 4: Building the $(k+1)$th stage at squares of type (i) and (ii). The link of blue vertex in the left figure has girth 6.

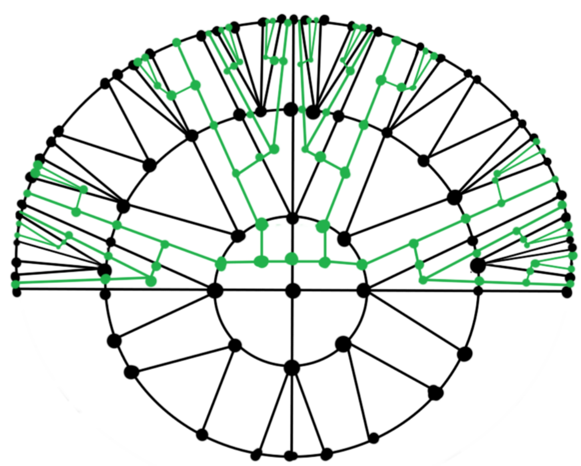

Figure 5: A subtree of the cactus-dual T.

However, Proposition 10.2 of [1] states that no two squares mapping to $[n-1, n]$ can meet along an edge mapping to $\{n\}$. Hence, $g'(v) = n+1/2$ for some $n \in \mathbb{N}$. If v is an edge-vertex, then it is adjacent to a face-vertex u such that $g'(u) = n + 1/2$, so we may assume that v is a face-vertex to begin with.

First, suppose that v corresponds to a type (i) square s. Since $g'(v)$ is maximal, α has one vertical edge and one horizontal edge at v. Let v' be the other endpoint of the horizontal edge. By construction, v' is the edge-vertex of an edge whose lower vertex x has a link of girth ≥ 6. Therefore, the squares meeting s along the edge corresponding to v' are of type (ii). Let u be the face-vertex adjacent to v' corresponding to a type (ii) square. Note that $g'(u) = n + 1/2$, so the next edge-vertex u' in α must also map to $n + 1/2$. Hence, the next vertex w must correspond to a type (i) square, and $g'(w) = n + 1/2$, as shown in Fig. 6.

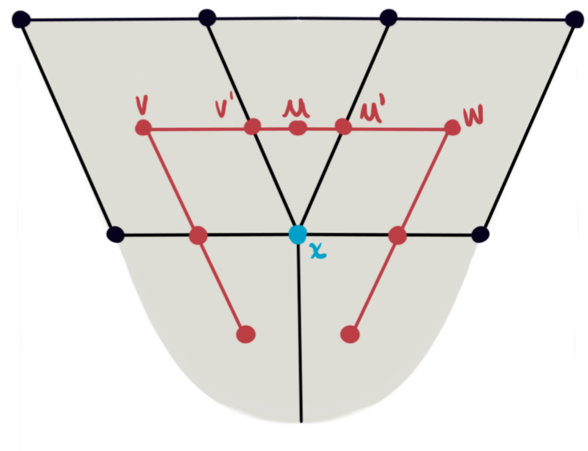

Figure 6: α is the red graph, and the blue vertex should have a link of girth 6. The outline of the bottom two 2-cells are not drawn since they may be type (i) or type (ii) 2-cells.

Note that $g(x) = n$. Since all squares are either of type (i) or type (ii), there can only be one 1-cell at x mapping to $[n-1, n]$. Hence, $\mathrm{link}(x)$ has girth at most 5, contradicting $\mathrm{link}(x) \geq 6$. We conclude that v cannot be the face-vertex of a type (i) square. However, if v is the face-vertex of a type (ii) square, then v plays the role of u in the preceding discussion. Thus, all possible cases for v are ruled out, so there cannot be any non-trivial loops in T. Therefore, T is a tree.

Since every face-vertex of T has degree 3, and every edge has perimeter at most 3, 1 holds and T is thin. Since the 2-cell d was arbitrarily chosen, $\widetilde{X}$ has the thin cactus property, and therefore $H_2^{(2)}(\widetilde{X}) = 0$. □

Example 3.6. We will construct $(4, 4, 3)$-complex $\widetilde{X}$ with alternating links of girth 6 using the following triangle of groups.

$$\begin{tikzcd}
 & & \mathbb{Z}_2 \times \mathbb{Z}_2 & & \\
 & \mathbb{Z}_2 \arrow[ur] \arrow[dl] & \{1\} \arrow[l] \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[ul] \arrow[dr] & \\
G_1 & & \mathbb{Z}_3 \arrow[ll] \arrow[rr] & & G_2
\end{tikzcd}$$

The group G_1 is isomorphic to the group of orientation preserving isometries of the 3-dimensional cube, so $G_1 \cong S_4$. It is generated by the rotations a and b of order 3 and 2, respectively, shown in Fig. 7. The inclusions $\mathbb{Z}_3 \to G_1$ and $\mathbb{Z}_3 \to G_1$ are then given by $1 \mapsto a$ and $1 \mapsto b$, respectively. Let Γ_1 be the graph obtained by subdividing every edge of the

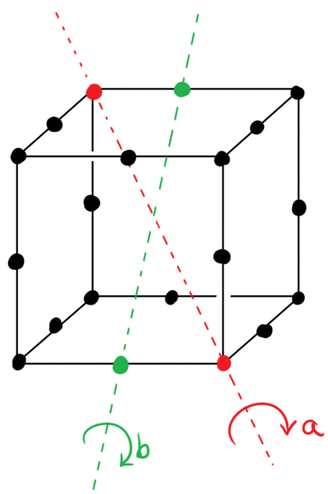

Figure 7: Rotations generating G_1.

1-skeleton of a 3-cube. Then the action of G_1 on Γ_1 satisfies the conditions of Lemma 2.5. Hence, the links of vertices over G_1 are isomorphic to Γ_1.

Let H be the subgroup of the isometry group of the plane generated by the rotations c and d shown in Fig. 8. Then, H acts on the graph Γ_2' formed by tiling the plane by hexagons with every edge subdivided, and this action satisfies the conditions of Lemma 2.5. To get a finite graph, first note that H is a *crystallographic group*, that is, a discrete subgroup of the isometry group of the Euclidean plane with compact fundamental domain. By a theorem of Zassenhaus ([6], Theorem 2.2), the group N generated by the two blue translations in Fig. 8

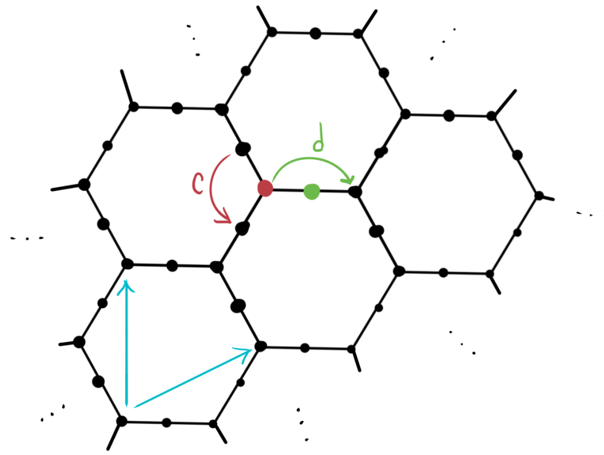

Figure 8: Rotations generating H. The $2\pi/3$-rotation c is about the red vertex, and the π-rotation d is about the green vertex. The blue arrows represent the generators of $\mathbb{Z}^2 \leq H$.

is a normal, finite index subgroup of H isomorphic to $\mathbb{Z}^2$. It is not hard to check that the subgroup $(n\mathbb{Z})^2 \leq \mathbb{Z}^2 \cong N \leq H$ is also normal in H. Hence, $G_2 = H/(n\mathbb{Z})^2$ is a finite group acting on the finite graph $\Gamma_2 = \Gamma_2'/(n\mathbb{Z})^2$ shown in Fig. 9. Note that for $n \geq 3$, Γ_2 has girth 12. Moreover, one can check that the action of G_2 on Γ_2 satisfies the hypotheses of Lemma 2.5, and therefore the links of vertices above G_2 are isomorphic to Γ_2.

The last thing to check is that the triangle of groups is indeed nonpositively curved. The angle at the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-vertex is $\pi/2$, since the shortest non-trivial word in the kernel $\mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is $\alpha\beta\alpha^{-1}\beta^{-1}$, where α and β each generate a copy of $\mathbb{Z}_2$. In view of the fact that G_1 acts freely on the edges of Γ_1, and since Γ_1 has girth 8, the shortest non-trivial word in the kernel of $\mathbb{Z}_2 * \mathbb{Z}_3 \to G_1$ has length 8, and therefore $\angle(G_1) = \pi/4$. A similar argument shows that $\angle(G_2) = \pi/6$. The triangle of groups is nonpositively curved, since $\pi/2 + \pi/4 + \pi/6 = 11\pi/12 < \pi$.

After identifying the four triangles above $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-vertices to a square, L becomes a square complex with links of girths 4 and 6, alternating around the boundary of the square. Proposition 3.5 then implies that $b_2^{(2)}(L) = 0$.

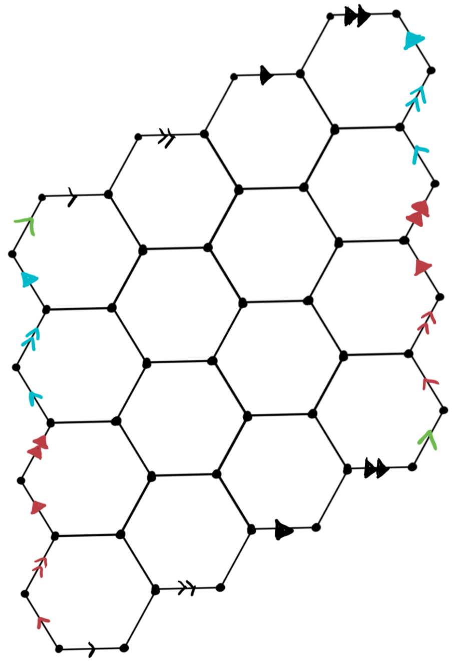

Figure 9: Depiction of Γ_2 without subdivided edges in the case $n = 4$. Pairs of edges are identified according to the labels.

3.3 Obstructions to the thin cactus property

We now describe a situation where 2-complexes do not have the thin cactus property. Let Y be a combinatorial 2-complex. As usual, each edge in the dual Γ has length $1/2$. Let $v \in \Gamma$ be a base face-vertex. Let $B(v, n)$ denote the set of face-vertices in Γ whose distance from v is less than or equal to n.

Lemma 3.7. *Let $\widetilde{X}$ be a combinatorial 2-complex such that every 1-cell has perimeter ≥ 3. If $Y \subseteq \widetilde{X}$ is a subcomplex with dual graph Γ such that every 1-cell of Y has perimeter ≥ 2 and $|B(v, n)|$ grows subexponentially for some $v \in \Gamma$, then $\widetilde{X}$ does not have the thin cactus property.*

Proof. If $\widetilde{X}$ has the thin cactus property; then there is a thin cactus-dual T based at v. Since the 1-cells of $\widetilde{X}$ have perimeter ≥ 3, all edge-vertices have degree ≥ 3. Let v' be an arbitrary face-vertex with degree d and adjacent edge-vertices $u_1, \ldots, u_d$. We see that $d \geq 3$

since condition (1) implies

$$1 \ \leq \ \sum_{i \neq l} \frac{1}{\deg(u_i) - 1} \ \leq \ (d-1)\frac{1}{3-1}$$

Let v'' be a face-vertex in $T \cap \Gamma$ whose distance from v in T is n. Since T is a tree and the edges of Y have perimeter ≥ 2, there are at least $(\deg(v'') - 1) \geq 2$ face-vertices in $T \cap \Gamma$ whose distance from v is $n+1$. Hence, there are at least 2^n face-vertices in $T \cap \Gamma$ whose distance from v in T is n. Therefore, $B(v, n) \geq 2^n$, contradicting the that $B(v, n)$ grows subexponentially. □

We can use Lemma 3.7 to show that 2-complexes containing "Euclidean planes" do not have the thin cactus property. More precisely, we have the following obstruction to the thin cactus property, which arose frequently in the examples we studied.

Proposition 3.8. *A 2-complex $\widetilde{X}$ does not have the thin cactus property if the following conditions hold:*

(i) *there is a subcomplex $Y \subseteq \widetilde{X}$ such that $f : Y \to \mathbb{E}^2$ is a homeomorphism,*

(ii) *there is a global upper bound $N \in \mathbb{N}$ on the length of the attaching maps of 2-cells in Y,*

(iii) *$\Gamma_Y^0 \hookrightarrow Y \to \mathbb{E}^2$ is a quasi-isometry, where Γ_Y is the dual 1-skeleton of Y, and Γ_Y^0 has metric induced by the graph metric on Γ_Y.*

Proof. We will show that the growth of Γ_Y is quadratic, and therefore Lemma 3.7 implies that $\widetilde{X}$ does not have the thin cactus property. Since the inclusion $\mathbb{Z}^2 \hookrightarrow \mathbb{E}^2$ is a quasi-isometry, there is a quasi-isometry $g : \Gamma_Y^0 \to \mathbb{Z}^2$ with constants $A \geq 1, B \geq 0$ such that

$$\frac{1}{A} \cdot \mathrm{d}_\Gamma(x, y) - B \ \leq \ \mathrm{d}_{\mathbb{Z}^2}(g(x), g(y)) \ \leq \ A \cdot \mathrm{d}_\Gamma(x, y) + B.$$

Let $v \in \Gamma_Y^0$ be a base-point, and let $x \in B_{\Gamma_Y^0}(v, n)$. Then $\mathrm{d}_{\mathbb{Z}^2}(g(x), g(v)) \leq A \cdot \mathrm{d}_\Gamma(x, v) + B \leq An + B$, so $g(B_{\Gamma_Y^0}(v, n)) \subseteq B_{\mathbb{Z}^2}(g(v), An + B)$.

We now bound the number of points in the preimage of a point $u \in \mathbb{Z}^2$. Let $x, y \in g^{-1}(u)$, then $\frac{1}{A} \cdot \mathrm{d}_\Gamma(x, y) - B \leq \mathrm{d}_{\mathbb{Z}^2}(g(x), g(y)) = \mathrm{d}_{\mathbb{Z}^2}(u, u) = 0$, so $\mathrm{d}_\Gamma(x, y) \leq AB$ and therefore $\mathrm{diam}(g^{-1}(u)) \leq AB$. Since the valences of vertices in Γ_Y are globally bounded above by N, the number of vertices in $g^{-1}(u)$ is bounded above by a constant $J = J(AB)$ depending only on AB, and not on v. Hence,

$$\begin{aligned} |B_\Gamma(v,n)| &\leq \left|g^{-1}(g(B_\Gamma(v,n)))\right| \\ &= \sum_{u \in g(B_\Gamma(v,n))} \left|g^{-1}(u)\right| \\ &\leq J\ |g(B_\Gamma(v,n))| \\ &\leq J\ |B_{\mathbb{Z}^2}(g(v), An+B)|. \end{aligned}$$

Since $|B_{\mathbb{Z}^2}(g(v), An+B)| \sim O(n^2)$, we have $|B_\Gamma(v,n)| \sim O(n^2)$ as desired. $\square$

Example 3.9 (Gromov (m,n)-polyhedra)**.** We give an example of a cell complex that does not have the thin cactus property, but has nonpositive sectional curvature. A *space of Gromov* (m,n)*-polyhedra* is a simply-connected combinatorial 2-complex such that every 2-cell is an m-gon and the link of every 0-cell is a complete graph on n vertices. Moreover, we require that the boundary of every 2-cell be homeomorphic to S^1. Haglund [7] and Ballmann and Brin [8] independently showed that there are uncountably many non-homeomorphic spaces of Gromov (m,n)-polyhedra for all $m \geq 6$ and $n \geq 3$.

For $n \geq 3$, a space $\widetilde{X}$ of Gromov (m,n)-polyhedra is an example of a $(m,3,n-1)$-complex. Hence, if $m \geq n+2 \geq 7$, then $b_2^{(2)}(\widetilde{X}) = 0$. Now, let $\widetilde{X}$ be a space of Gromov $(6,4)$-polyhedra. If we assign angles of $2\pi/3$ to every corner, $\widetilde{X}$ has nonpositive sectional curvature, so we suspect that $b_2^{(2)}(\widetilde{X}) = 0$; however, attempts at proving this have failed thus far. We can rule out the possibility of $\widetilde{X}$ having the thin cactus property as follows. Since the links of all vertices in $\widetilde{X}^0$ are complete graphs on 4 vertices, they all have girth 3; therefore, d lies in a subcomplex $Y \subseteq \widetilde{X}$ whose links are circles of length 3. Then Y can be identified with the Euclidean plane tiled by regular hexagons. This tiling clearly satisfies the conditions of Proposition 3.8, so $\widetilde{X}$ does not have the thin cactus property. Fig. 10 illustrates the failure of the thin cactus due to the Euclidean plane tiled by hexagons.

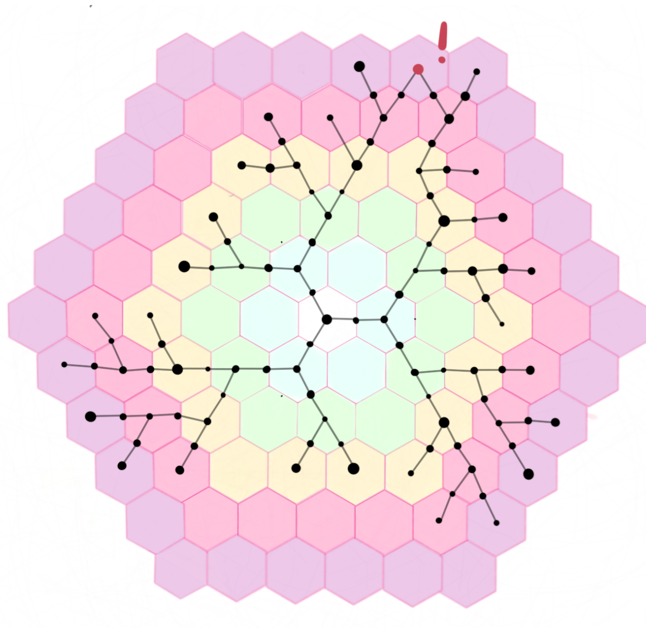

Figure 10: Failure of the thin cactus property. The white hexagon is d, face-vertices corresponding to hexagons of the same colour have the same distance from v_d in Y.

4 The Energy Criterion

4.1 Statement and proof of the energy criterion

We will now describe and give a proof of Wise's second criterion implying the vanishing of $H_2^{(2)}$, the *energy criterion* [1]. Let $\widetilde{X}$ be a 2-complex such that the lengths of the attaching maps of each 2-cell are globally bounded by some $N \in \mathbb{N}$. We assign real numbers to the corners of every 2-cell called *coefficients*, and assume that they are uniformly bounded by a constant K. A *grading* of $\widetilde{X}$ is a continuous cellular map $f : (\widetilde{X}, \widetilde{X}^0) \to (\mathbb{R}, \mathbb{Z})$ such that 2-cells are mapped to intervals $[n, n+1]$. (Note that we used a grading g to organize the construction of the thin cactus in Proposition 3.5.) The grading is said to be *admissible* with respect to the choice of coefficients if conditions A and B below are met.

A. For a 2-cell d mapping to $[n, n+1]$, we say that a corner of d is an *upper corner* [*lower corner*] if it is at a 0-cell mapping to $n+1$ [to n]. Let $\mathrm{Up}(d)$ [$\mathrm{Low}(d)$] be the set of upper [lower] corners of d. Let w_c denote the coefficient at a corner of a 2-cell. We require

$$\sum_{l \in \mathrm{Low}(d)} w_l \geq 0 \tag{2}$$

and

$$\sum_{u \in \mathrm{Up}(d)} w_u + \sum_{l \in \mathrm{Low}(d)} w_l \leq 0 \tag{3}$$

for every 2-cell in $\widetilde{X}^2$.

B. Let $v \in \widetilde{X}^0$ be a vertex. For every edge $e \in \mathrm{link}(v)$, let w_e be the coefficient of the corner corresponding to e. Our second condition is that

$$\sum_{e \in \mathrm{link}(v)^1} w_e |z_e|^2 \geq 0 \tag{4}$$

for every 1-cycle $Z = \sum_{e \in \mathrm{link}(v)^1} z_e e \in H_1^{(2)}(\mathrm{link}(v))$. Note that this condition is unambiguous since $\mathrm{link}(v)$ is 1-dimensional and therefore $H_1^{(2)}(\mathrm{link}(v))$ can be viewed as a subspace of $C_1^{(2)}(\mathrm{link}(v))$. We emphasize that this condition must hold at every vertex $v \in \widetilde{X}^0$.

Given a grading and choice of coefficients, we say that a 2-cell d is *visible if*

$$\sum_{u \in \mathrm{Up}(d)} w_u < 0. \tag{5}$$

Let $\widetilde{X}$ be a 2-complex. If, for each non-zero L^2 2-cycle $Z = \sum_{d \in \widetilde{X}^2_d} z_d d \in H_2^{(2)}(\widetilde{X})$, there exists a 2-cell d with non-zero coordinate and a choice of uniformly bounded coefficients w_c at corners of 2-cells and an admissible grading $f : (\widetilde{X}, \widetilde{X}^0) \to (\mathbb{R}, \mathbb{Z})$ for which d is visible, then $\widetilde{X}$ satisfies the *energy criterion*.

We will prove that locally finite 2-complexes $\widetilde{X}$ satisfying the energy criterion have vanishing of $H_2^{(2)}(\widetilde{X})$. While the thin cactus property is a global property of combinatorial 2-complexes, the energy criterion is a local test, and because of this is often easier to use. The main difficulty is finding the appropriate grading and choice of coefficients.

In [1], Wise does not assume local-finiteness, but uses excision for L^2-homology. Since we are mostly interested in locally finite complexes, we avoid introducing excision and use the following lemma instead.

Lemma 4.1. *Let $\widetilde{X}$ be a locally finite 2-complex and let $Z = \sum_{d \in \widetilde{X}^2} z_d d \in H_2^{(2)}(\widetilde{X})$. Then $\overline{Z} = \sum_{d \in \widetilde{X}_v^2} z_d e_d \in H_1^{(2)}(\mathrm{link}(v))$ for every $v \in \tilde{X}^2$, where $\widetilde{X}_v^2 \subseteq \widetilde{X}^2$ is the set of 2-cells whose attaching map includes v, and e_d is the edge of* link(v) *corresponding to $d \in \widetilde{X}_v^2$.*

Proof. Let $v \in \widetilde{X}^0$ and choose orientations on all 1-cells and 2-cells of $\widetilde{X}$. Let $e' \in \widetilde{X}^1$ be an edge at v. Since Z is a 2-cycle, $\sum_{d \in \widetilde{X}_{e'}^2} (-1)^{i_d} z_d = 0$, where $\widetilde{X}_{e'}^2 \subseteq \widetilde{X}^2$ is the set of 2-cells whose attaching map includes e', and $i_d = 0$ or 1, depending on whether d is oriented consistently with e'. Hence $\overline{Z}$ satisfies the cycle condition at $u_{e'}$, the vertex in link(v) corresponding to e', where the edges of link(v) are oriented according to the orientation on the 2-cells of $\widetilde{X}$. Since e' was arbitrary, $\overline{Z}$ satisfies the cycle condition at all of its vertices. The p-summability of $\overline{Z}$ follows immediately from that of Z, so $\overline{Z} = \sum_{d \in \widetilde{X}_v^2} z_d e_d \in H_1^{(2)}(\mathrm{link}(v))$. □

We closely follow Wise for the proof of the following theorem, which is a special case of Theorem 8.1 in [1].

Theorem 4.2. *If $\widetilde{X}$ is a locally finite 2-complex satisfying the energy criterion, then $H_2^{(2)}(\widetilde{X}) = 0$.*

Proof. Let $Z = \sum_{d \in \widetilde{X}^2} z_d d \in H_2^{(2)}(\widetilde{X})$, and assume, for a contradiction, that $Z \neq 0$. Then let d be some 2-cell for which the coordinate z_d is non-zero, and choose coefficients w_c on all corners c, as well as an admissible grading f for which d is visible. Define

$$A_n := \sum_{d \in \widetilde{X}_n^2} \sum_{u \in \mathrm{Up}(d)} w_u |z_d|^2,$$

$$B_n := \sum_{d \in \widetilde{X}_n^2} \sum_{l \in \mathrm{Low}(d)} w_l |z_d|^2,$$

where $\widetilde{X}_n^2$ denotes the set of 2-cells mapping to $[n, n+1]$. By (2), we have that $B_n \geq 0$. Since the lengths of the attaching maps are globally bounded above by some N and the coefficients are bounded by K, we also have that $|\mathrm{Low}(d)| \leq N$ and $B_n = \sum_{d \in \widetilde{X}_n^2} \sum_{l \in \mathrm{Low}(d)} w_l |z_d|^2 \leq KN \sum_{d \in \widetilde{X}_n^2} |z_d|^2$. Hence,

$$0 \leq \sum_{n \in \mathbb{N}} B_n \leq KN \sum_{n \in \mathbb{Z}} \sum_{d \in \widetilde{X}_n^2} |z_d|^2 = KN \sum_{d \in \widetilde{X}^2} |z_d|^2 < \infty. \tag{6}$$

By Lemma 4.1, Z induces L^2 1-cycles on all of the links. Hence, (4) implies that $B_{n+1} + A_n \geq 0$ and (3) implies that $A_n + B_n \leq 0$. Hence, $B_{n+1} \geq -A_n \geq B_n$ for all $n \in \mathbb{N}$. If $d \in \widetilde{X}_m^2$, then visibility of d ensures that $A_m < 0$. Hence, $\sum_{n\in\mathbb{N}} B_n \geq \sum_{n\geq m} B_{n+1} \geq -\sum_{n\geq m} A_m = \infty$, contradicting (6). We conclude that $Z = 0$. $\square$

4.2 Application to triangle complexes

We describe a grading on combinatorial 2-complexes that will be used in Proposition 4.5. Let $\widetilde{X}$ be a combinatorial 2-complex, and let $\mathrm{d} : \widetilde{X}^0 \times \widetilde{X}^0 \to \mathbb{R}_{\geq 0}$ be the metric on the 0-skeleton of $\widetilde{X}$ induced by the graph metric on $\widetilde{X}^1$, where every 1-cell has length 1. For a base vertex $b \in \widetilde{X}^0$, let $f : \widetilde{X}^0 \to \mathbb{R}_{\geq 0}, x \mapsto \mathrm{d}(x, b)$. We can then extend f to a grading on $\widetilde{X}^1$ as follows. Let e be an edge in $\widetilde{X}^1$. If the endpoints of e map to n and $n+1$, then declare $f(e) = [n, n+1]$. If both endpoints map to n, then $f(e) = \{n\}$. We now want an extension of f to all of $\widetilde{X}$. This is not always possible, but in the case where $\widetilde{X}$ is a $(3,6)$-complex (all 2-cells have attaching maps of length ≥ 3, and all links have girth ≥ 6), we have Proposition 4.4, a result from [1], to help us. First, some definitions.

Definition 4.3. Let $v \in \widetilde{X}^0$ be a 0-cell, and let $f : \widetilde{X}^1 \to \mathbb{R}_{\geq 0}$ be the map described above. Let e be a 1-cell.

(i) e is a *level edge* if $f(e) = \{n\}$ for some $n \in \mathbb{N}$.

(ii) e is *ascending* [*descending*] at v if $f(v) = n$ and $f(e) = [n, n+1]$ [$f(e) = [n-1, n]$].

(iii) e is a *top edge* [*bottom edge*] of a 2-cell d if $f(e) = \{n\}$ or $f(e) = [n, n+1]$, and for every edge e' in the attaching map of d, $f(e') = \{k\}$ or $f(e') = [k, k+1]$ with $k \geq n$ [$k \leq n+1$].

Proposition 4.4. *Let $\widetilde{X}$ be a $(3,6)$-complex. Let $f : \widetilde{X}^1 \to \mathbb{R}$ be the map described above. Let $v \in \widetilde{X}^0$.*

(i) *There are either one or two descending edges at v (or one if $v = b$).*

(ii) *If there are two descending edges, then these two edges bound the top of a 2-cell.*

(iii) *Every level edge at v bounds the top of a 2-cell together with a descending edge at v.*

(iv) *If e is a top 1-cell of some 2-cell, then e is a bottom 1-cell of every other 2-cell whose boundary contains e.*

Proposition 4.4 implies that every 2-cell in a $(3,6)$-complex is of one of the forms shown in Fig. 11. We can then subdivide the 2-cells as shown in Fig. 12 and extend f to obtain a grading $f : \widetilde{X} \to \mathbb{R}_{\geq 0}$.

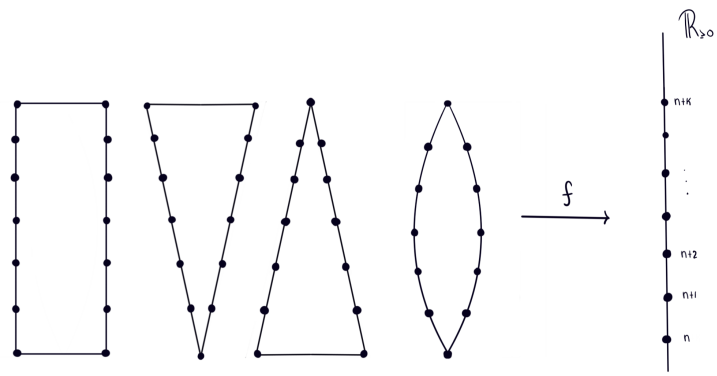

Figure 11: The four types of 2-cells under the grading $f : \widetilde{X}^1 \to \mathbb{R}_{\geq 0}$.

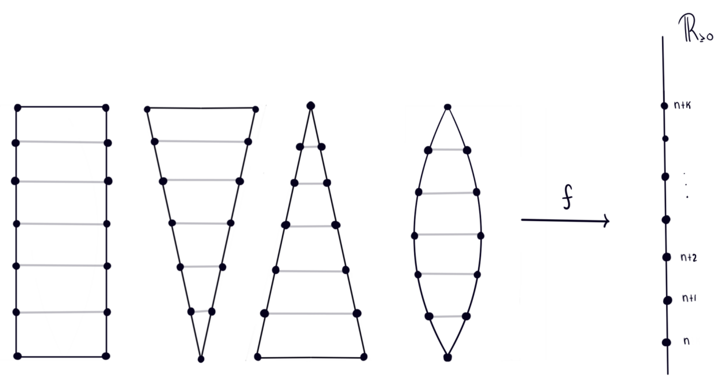

Figure 12: The four types of subdivided 2-cells under the extended grading $f : \widetilde{X} \to \mathbb{R}_{\geq 0}$.

In [1], Wise conjectures that $H_2^{(2)}(\widetilde{X}) = 0$ for $\widetilde{X}$ a $(3,6,3)$-complex, where every triangle has at least one edge with perimeter 2. These complexes contain Euclidean planes tiled by equilateral triangles, and therefore do not have the thin cactus property by Proposition 3.8. It seems that complexes with Euclidean planes also fail the energy test, though it is not as clear why this is the case due to the many choices of gradings on $\widetilde{X}$. We were not able to prove Wise's conjecture, however we were able to prove it for the weaker case where $\widetilde{X}$ is a $(3,9,3)$-complex.

Proposition 4.5. *Let $\tilde{X}$ be a $(3,9,3)$-complex. Suppose each triangle has at least one edge with perimeter 2. Then $\widetilde{X}$ satisfies the energy criterion.*

Proof. Give $\widetilde{X}$ the grading $f : \widetilde{X} \to \mathbb{R}_{\geq 0}$ described above. We begin by assigning coefficients to the corners of 2-cells with attaching maps of length ≥ 4. Subdivide every level edge of $\widetilde{X}$ by adding a 0-cell at its centre. For each 2-cell with m sides (before subdividing), assign coefficients as follows. Let d be a 2-cell and c on of its corners. If c is

(i) the unique bottom corner of d or the middle corner of a bottom level edge, we assign a coefficient of 1;

(ii) one of the top corners of d, but not the middle corner of a level edge, we assign a coefficient of 0;

(iii) any other corner, we assign a coefficient of $-\frac{1}{m-2}$.

These coefficient choices are shown in Fig. 13.

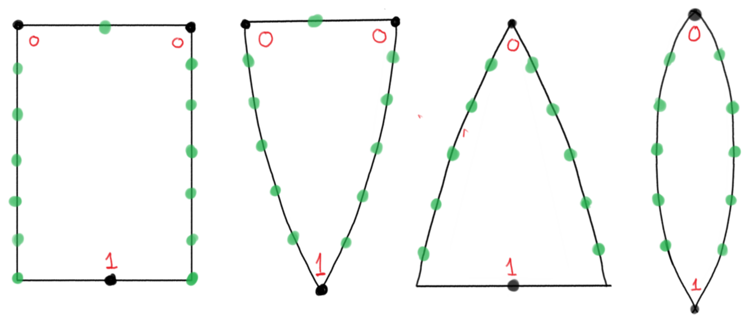

Figure 13: Coefficients of 0 and 1 are assigned as shown above. Corners at green vertices have coefficient $-\frac{1}{m-2}$.

To assign coefficients to corners after subdividing the 2-cells, we partition the coefficients at each corner so that the sum of the coefficients in every 2-cell is exactly 0 (or negative in one 2-cell). We illustrate this process with a pair of examples in Fig. 14. It is clear that (2) and (3) are satisfied for each 2-cell in the subdivision. Moreover, every 2-cell in the subdivision is visible.

Now, we assign coefficients to triangles. Let d be a triangle. If d's level edge is its bottom edge, we assign coefficients of $-\frac{1}{2}$ to the bottom corners, 1 to the middle corner of the level edge, and 0 to the top corner. Now suppose d is a triangle whose level edge is its top edge.

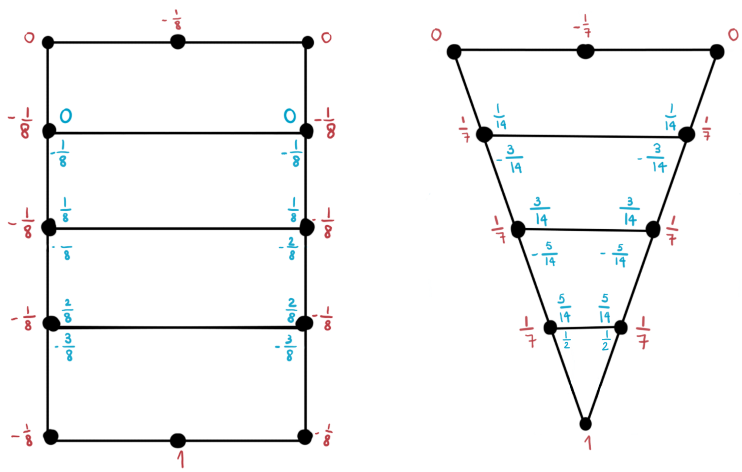

Figure 14: Examples showing how to assign coefficients to subdivided 2-cells.

We assign coefficients of 0 to d's top corners. If d's level edge has perimeter 3, we assign coefficients 1 to its bottom corner and -1 to the middle corner of its level edge. Otherwise, assign coefficients $\frac{1}{2}$ to its bottom corner and $-\frac{1}{2}$ to the middle corner of its level edge. The different cases are shown in Fig. 15.

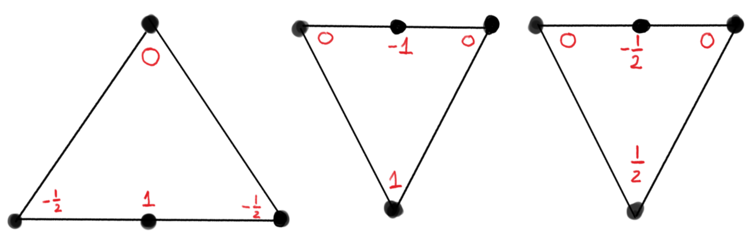

Figure 15: Coefficient choices for triangles. Note that the middle triangle has top edge of perimeter 3 and the triangle on the right has top edge of perimeter 2.

We must now check that the coefficient choices satisfy the conditions of the energy criterion. Clearly (2) and (3) are satisfied. Moreover, note that all 2-cells are visible, except the triangle with bottom level edge. However, these 2-cells are always glued to a visible 2-cell along an edge of perimeter 2. Hence, any L^2 2-cycle is non-zero on at least one visible 2-cell of $\tilde{X}$.

Finally, we verify (4). Let v be the middle vertex of a level edge. Then link(v) is of

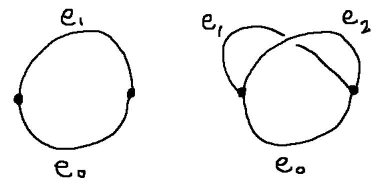

Figure 16: The left [right] graph is the link at a perimeter 2 [3] edge.

one of the two types shown in Fig. 16. Let $\sum_i z_{e_i} e_i$ be a 1-cycle on link(v), and let z_{e_0} be the coordinate on the lower edge of link(v). In the first case, the upper edge has coordinate $z_{e_1} = \pm z_{e_0}$, so the L^2 norm at v is $\sum_{e_i} w_{e_i} |z_{e_i}|^2 = 1 \cdot |z_{e_0}|^2 - 1 \cdot |z_{e_0}|^2 = 0$. In the second case, the coordinates satisfy $z_{e_0} = \pm z_{e_1} \pm z_{e_2}$. By Lemma 3.2 $|z_{e_1}|^2 + |z_{e_2}|^2 \geq z_{e_0}^2/4$. Hence, the L^2 norm satisfies

$$\sum_{e_i} w_i |z_{e_i}| = -w_{e_0} \cdot |z_{e_0}|^2 + 1 \cdot |z_{e_1}|^2 + 1 \cdot |z_{e_2}|^2 \geq -\frac{1}{2} \cdot |z_{e_0}|^2 + 1 \cdot |z_{e_1}|^2 + 1 \cdot |z_{e_2}|^2 \geq 0.$$

Now let v be a vertex that is not the middle of a level edge. Then link(v) is of one of the types shown in Fig. 17. The vertices marked 0, +, and − are vertices coming from level edges, ascending edges, and descending edges at v, respectively. Since the girth of link(v) is at least 9, all edges shown below are distinct, however the green vertices may not be. Note that all edges marked −0 have coefficient 0, and all 0+ and −+ edges have a negative coefficient. All ++ edges, as well as any edge of link(v) not shown in Fig. 17, have positive coefficient. Hence, to show that (4) holds, it suffices to show that the weighted square sum of each 0+ and −+ edge is dominated by the weighted square sum of the ++ edges above it.

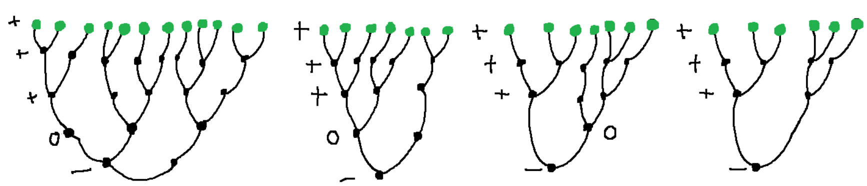

Figure 17: The different types of links.

There are four cases for the link above 0+ and −+, as shown in Fig. 17. In each case, the orange edges are ++ edges and the green edge is a 0+ or a −+ edge. Moreover, we always have that $-1/2 \leq w_0 \leq 0$, so it is enough to show that $\sum_{i \geq 1} w_i |z_{e_i}|^2 \geq |z_{e_0}|^2/2$ in each case.

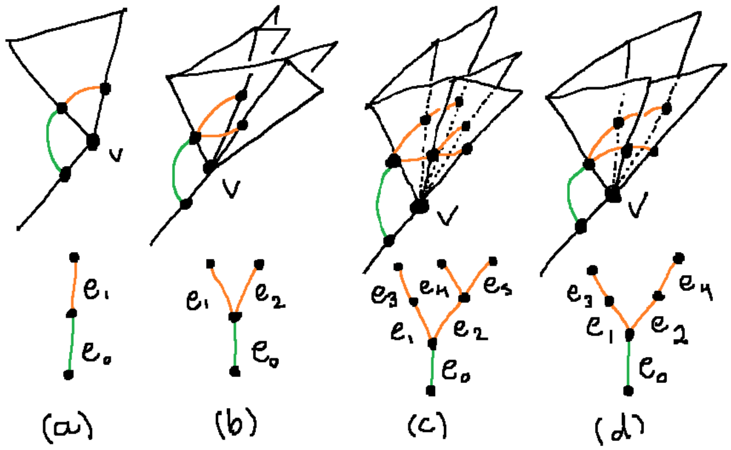

Figure 18: The different cases for the link above 0+ and −+ edges.

Case (a). The vertex connecting e_0 and e_1 is at a perimeter 2 edge, so we have $z_{e_0} = z_{e_1}$. Since e_1 is at the bottom corner of a 2-cell, we have that $w_1 \geq 1/2$. Hence, $w_1|z_{e_1}|^2 \geq |z_{e_0}|^2/2$.

Case (b). The endpoints of e_1 and e_2 are at perimeter 3 edges, which means that if the 2-cells supporting e_1 and e_2 are triangles (as drawn in Fig. 18), then the top edge must be of perimeter 2. Hence, $z_{e_1} = z_{e_2} = 1$. Note that the coordinates are subject to $z_{e_0} = \pm z_{e_1} \pm z_{e_2}$. Hence, $|z_{e_1}|^2 + |z_{e_2}|^2 \geq |z_{e_0}|^2/2$.

Case (c). The vertex joining e_1 and e_3 is supported by a perimeter 2 edge, so $z_{e_1} = z_{e_3}$. Moreover, $z_{e_0} = \pm z_{e_1} \pm z_{e_2}$, so $|z_{e_1}|^2 + |z_{e_2}|^2 \geq |z_{e_0}|^2/2$. Hence,

$$\sum_{i\geq 1} w_i|z_{e_i}|^2 \;\geq\; \frac{1}{2}\cdot|z_{e_1}|^2 + 1\cdot|z_{e_2}|^2 + \frac{1}{2}\cdot|z_{e_3}|^2 + \frac{1}{2}\cdot|z_{e_4}|^2 + \frac{1}{2}\cdot|z_{e_5}|^2 \;\geq\; |z_{e_1}|^2 + |z_{e_2}|^2 \;\geq\; |z_{e_0}|^2/2.$$

Case (d). Observe that $z_{e_1} = z_{e_3}$ and $z_{e_2} = z_{e_4}$, and that $z_{e_1}^2 + z_{e_2}^2 \geq z_{e_0}^2/2$. Hence,

$$\sum_{i\geq 1} w_i|z_{e_i}|^2 \geq \frac{1}{2}\left(|z_{e_1}|^2 + |z_{e_2}|^2 + |z_{e_3}|^2 + |z_{e_4}|^2\right) = |z_{e_1}|^2 + |z_{e_2}|^2 \geq |z_{e_0}|^2/2. \qquad \square$$

Remark 4.6. Let $\widetilde{X}$ be a $(3,9,3)$-complex satisfying the conditions of the previous proposition. Suppose that the triangles of $\widetilde{X}^2$ can be paired into subsets $P_\alpha = \{\triangle_1^\alpha, \triangle_2^\alpha\}$ such that two triangles in the same P_α meet along a 1-cell of perimeter 2. Then $\triangle_1^\alpha \cup \triangle_2^\alpha$ can be identified with a square, turning $\widetilde{X}$ into a square-complex. Under this new viewpoint, the links of $\widetilde{X}$ have girth 5, so $\widetilde{X}$ is a $(4,5,3)$-complex, which is already known to have vanishing $H_2^{(2)}$.

Remark 4.7. The proof of Proposition 4.5 cannot be directly extended to the girth 8 case, since we can no longer use the fact that the edges shown in Fig. 17 are distinct. Indeed, the link of girth 8 depicted in Fig. 19 does not satisfy (4). The blue and red numbers are the coordinates and coefficients of the edges next to them, respectively. The edges that do not have any numbers next to them all have coordinate and coefficient 1/2. The resulting weighted square-sum of the link is $-1/4$.

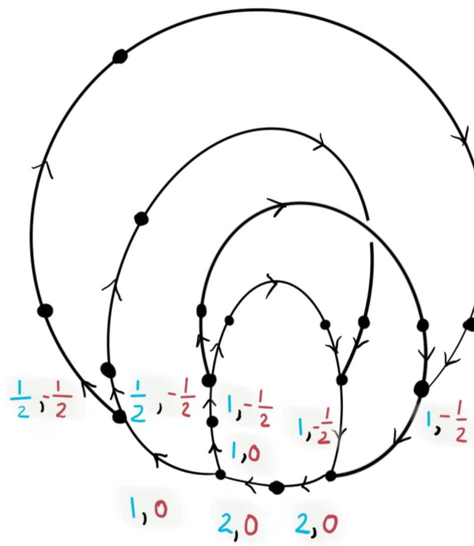

Figure 19: Example of a link failing to meet (4).

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