

Poker Problem

There are 52 cards in a poker.

4 suits: diamonds $\diamond \times 13$ spades $\spadesuit \times 13$

clubs $\clubsuit \times 13$ hearts $\heartsuit \times 13$

kinds: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A (13 intotal)

1. Straight Flush.

Contains 5 cards of sequential rank, all of the same suit.

2	3	4	5	6
3	4	5	6	7
4	5	6	7	8
5	6	7	8	9
6	7	8	9	10
7	8	9	10	J
8	9	10	J	Q
9	10	J	Q	K
10	J	Q	K	A
A	2	3	4	5

can be: clubs, diamonds, spades, hearts

There are 10 possible combinations, each combination has 4 possible suits.

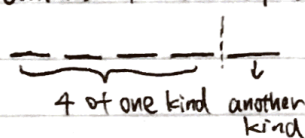
As a result, there can be 10×4 possible straight flush.

$$(10) \cdot (4) = 10 \times 4 = 40$$

$$P(\text{Straight Flush}) = \frac{(10) \cdot (4)}{\binom{52}{5}}$$

2. Four of a Kind

Contains 4 cards of one rank and one card of another.



First 4				Last
A	A	A	A	
2	2	2	2	
3	3	3	3	
4	4	4	4	
5	5	5	5	
6	6	6	6	
:	:	:	:	
Q	Q	Q	Q	
K	K	K	K	

52-4=48 possible outcomes

For each possible combination, all 4 suits are contained.

There are 13 possible outcomes for the first 4 cards.

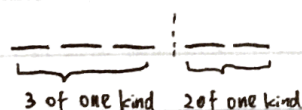
and 48 possible outcomes for the other card.

As a result, there can be 13×48 possible four of a kind.

$$P(\text{Four of a kind}) = \frac{(13) \cdot (48)}{\binom{52}{5}}$$

3. Full House

Contains three card of one rank and two cards of another rank.



↓ Last 2

First 3		
A	A	A
2	2	2
3	3	3
4	4	4
:	:	:
Q	Q	Q
K	K	K

Each combination, the first one can have 4 possible suits, second one can have 3 possible suits, third one can have 2 possible suits.

So it can be A \diamond A \clubsuit A \spadesuit for the first combination.

A \diamond A \clubsuit A \heartsuit
A \diamond A \spadesuit A \heartsuit
A \clubsuit A \spadesuit A \heartsuit
As a result, there are 13×4 possible outcomes for the first 3 cards.

Can be any two cards from

the rest card, but cannot have the same rank as first 3.

If A A A, then

2	2
3	3
4	4
:	:
Q	Q
K	K

So it can be 2 \clubsuit 2 \spadesuit for the first combination.

As a result, there are 12×6 possible outcomes for last 2 cards.

2	2
2	2
2	2
2	2
2	2
2	2

$$P(\text{Full House}) = \frac{(13) \cdot (4) \times (12) \cdot (4)}{\binom{52}{5}}$$

4. Flush.

Contains 5 cards all the same suit, not all of sequential rank.

----- There are 4 suits: spades ♠ x13, diamonds ♦ x13, clubs ♣ x13, hearts ♥ x13

For each suit, we choose 5 cards. So there are $\binom{13}{5}$ outcomes for each suit.

As a result, there are $4 \times \binom{13}{5}$ possible outcomes in total.

$$P(\text{Flush}) = \frac{\binom{4}{1} \cdot \binom{13}{5}}{\binom{52}{5}}$$

* this contains outcomes that all 5 cards are consecutive.

For flush does not contain all 5 cards are consecutive. We can have:

In each suit, we have 10 combinations that have 5 consecutive ranks.

10 $\begin{cases} A & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 9 & 10 & J & Q & K \end{cases}$

As a result, there are $\binom{13}{5} - 10$ outcomes to avoid consecutive ranks in each suit.

$$P(\text{Flush}) = \frac{\binom{4}{1} \cdot [\binom{13}{5} - 10]}{\binom{52}{5}}$$

5. Straight

Contains 5 cards that are ranked consecutively, not all same suit.

10 $\begin{cases} A & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 9 & 10 & J & Q & K \end{cases}$

Consider ranks, there can have 10 combinations.

Each rank can have 4 possible suits. So there are 4^5 possible combinations for each

consecutive rank combinations. There are $4^5 \cdot 10$ total outcomes.

$$P(\text{Straight}) = \frac{\binom{10}{1} \cdot 4^5}{\binom{52}{5}}$$

* this contains 5 cards that have same suits

For straight does not contain 5 cards with same suits. We can have:

Since not allowed 4 of 5 cards have same suit, so we have to exclude the cases that there are 4 cards in the same suit. There are 4 possible outcomes for all 4 same suit in each rank combination.

So for each rank combination, for the first 4 cards, we can only have $4 \times 4 \times 4 \times 4$ ways to assign suits.

For the last number, there are 4 possible suits can take. But we need to subtract the all the same suit, so there are $(4^5 - 4)$ possible suit combinations in each rank combination.

$$P(\text{Straight}) = \frac{\binom{10}{1} \cdot (4^5 - 4)}{\binom{52}{5}}$$

6. Three of a kind

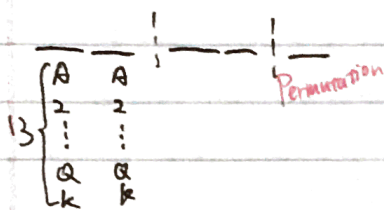
Contains 3 same rank. Other two can be anything but not the same rank.

13 $\begin{cases} A & A & A \\ 2 & 2 & 2 \\ \vdots & \vdots & \vdots \\ Q & Q & Q \\ K & K & K \end{cases}$ If AAA, then $\begin{cases} 2 & 3 \\ \vdots & \vdots \\ Q & K \end{cases}$

$$P(\text{Three of a kind}) = \frac{\binom{4}{3} \cdot \binom{13}{1} \times \binom{12}{2} \cdot \binom{4}{1} \cdot \binom{4}{1}}{\binom{52}{5}}$$

7. Two Pairs

Two same rank, another two same rank.



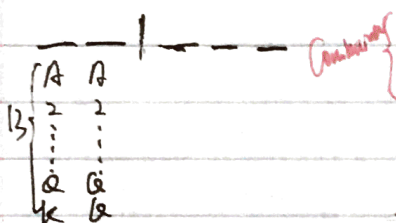
First two: $\binom{13}{1} \cdot \binom{4}{2}$
 Second two: $\binom{12}{1} \cdot \binom{4}{2} \cdot \frac{1}{2!}$
 Last one: $\binom{11}{1} \cdot \binom{4}{1}$
 OR
 First Four: $\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2}$
 Last one: $\binom{11}{1} \cdot \binom{4}{1}$

$$P(\text{Two Pairs}) = \frac{\binom{13}{1} \cdot \binom{4}{2} \times \binom{12}{1} \cdot \binom{4}{2} \times \binom{11}{1} \cdot \binom{4}{1}}{(5^2) \cdot 3!}$$

$$= \frac{\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \times \binom{11}{1} \cdot \binom{4}{1}}{(5^2)}$$

8. One Pair

Contains two same rank.



First two: $\binom{13}{1} \cdot \binom{4}{2}$
 Last three: $\binom{12}{3} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1}$

$$P(\text{One Pair}) = \frac{\binom{13}{1} \cdot \binom{4}{2} \times \binom{12}{3} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1}}{(5^2)}$$

For Permutation

$$P(\text{One Pair}) = \frac{\binom{13}{1} \cdot \binom{4}{2} \times \binom{12}{1} \cdot \binom{4}{1} \times \binom{11}{1} \cdot \binom{4}{1} \times \binom{10}{1} \cdot \binom{4}{1}}{(5^2) \cdot 3!}$$

9. High Card

Cannot have combinations above.

Cannot have straight, so minus $\binom{10}{1}$ for each combination.

Cannot have flush, so minus $\binom{4}{1}$ for each combination.

$[(\binom{13}{5}) - (\binom{10}{1})] \cdot (4^5 - 4)$ possible outcomes.

$$P(\text{High Card}) = \frac{[(\binom{13}{5}) - (\binom{10}{1})] \cdot (4^5 - 4)}{(5^2)}$$