ECS 020 Summary

1. Discrete systems and structures

Discrete: Composed of distinct, separable parts.

Structures: Objects built up from simpler objects according to some definite pattern.

Discrete Mathematics: The study of discrete, mathematical objects and structures.

2. Propositional logic

Propositional Logic: The logic of compound statements built from simpler statements using so-called Boolean connectives.

Proposition: 1) A declarative statement with some definite meaning (not vague or ambiguous), 2) having a truth value that is either true (T) or false (F), 3) it is never both, neither, or somewhere "in between".

We might know the actual truth value, and the truth value might depend on the situation or context.

3. Boolean Operators / Connectives

An operator or connective combines one or more operand expressions into a larger expression.

1) Unary operators take 1 operand (e.g., negation, -3)
2) Binary operators take 2 operands (e.g., multiplication, 3×4)

Propositional or Boolean operators operate on propositions (or their truth values) instead of on numbers

4. Negation Operator (NOT, ¬)

Unary operator. Truth table:

p	$\neg p$
T	F
F	T

5. Conjunction Operator (AND, ∧)

Binary operator. Truth table:

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

¬ and ∧ operations together are sufficient to express any Boolean truth table.

6. Disjunction Operator (OR, ∨)

Binary operator. Truth table:

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

¬ and ∨ operations together are sufficient to express any Boolean truth table.

7. Exclusive Or Operator (XOR, ⊕)

Binary operator. Truth table:

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

8. Implication Operator (→)

Binary operator. $p \rightarrow q$ means p (hypothesis / antecedent) implies q (conclusion / consequent).

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Common phrases meaning $p \rightarrow q$:

"p implies q", "q if p", "p only if q", "p is sufficient for q", "q is necessary for p"

Converse, Inverse, Contrapositive for $p \rightarrow q$:

Converse: $q \rightarrow p$

Inverse: $\neg p \rightarrow \neg q$

Contrapositive: $\neg q \rightarrow \neg p$ (same as $p \rightarrow q$)

9. Biconditional Operator (↔)

Binary operator. $p \leftrightarrow q$ means that $p \rightarrow q$ and $q \rightarrow p$. p is true if and only if (IFF) q is true (p and q have the same truth value).

p	q	$p \rightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

This truth table is the exact opposite of ⊕'s. So $p \leftrightarrow q$ means $\neg(p \oplus q)$.

$p \leftrightarrow q$ does not imply that p and q are true, or that either of them causes the other, or that they have a common cause.

10. Boolean Operations Summary

p	q	$\neg p$	$p \land q$	$p \lor q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$
F	F	T	F	F	F	T	T
F	T	T	F	T	T	T	F
T	F	F	F	T	T	F	F
T	T	F	T	T	F	T	T

Order of operation: $\neg$, $\land$, $\lor$, $\oplus$, $\to$, $\leftrightarrow$.

Ex: $p \lor \neg q \to p \land q$ means $(p \lor (\neg q)) \to (p \land q)$

Precedence of $\lor$ or $\oplus$ is ambiguous and often depends on the programming language.

11. Propositional Consistency

Two different compound propositions may be True at the same time. We call them consistent.

Use truth table to solve this kind of problem.

Ex: Among four people, P1, P2, P3, P4, at least one of is truthful, and at least one is lying. One of the truthful ones has a treasure in their pocket. They each know who has the treasure and each of them makes a statement:

S1 (by P1): I don't have the treasure.

S2 (by P2): My pockets are empty.

S3 (by P3): P1 is lying.

S4 (by P4): P1 is lying.

Where is the treasure?

P1	P2	P3	P4	Consist?	Why
T	T	T	T	NO	Violating "at least one is lying".
T	T	T	L	NO	If P4 is lying, then P1 is truthful, but P3 is truthful, then P1 is lying, this violates S3.
…	…	…	…	…	…
T	T	L	T	YES	

As a result, person 3 is lying, other people are truthful.

12. Propositional Equivalence

Two syntactically (i.e., textually) different compound propositions may be the semantically identical (i.e., have the same meaning). We call them equivalent.

1) Tautology
   A tautology is a compound proposition that is always true no matter what the truth values of its atomic propositions are!
   Ex: $p \lor \neg p = T$ always
2) Contradictions
   A contradiction is a compound proposition that is false no matter what!
   Ex: $p \land \neg p = F$ always

p	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

3) Logical Equivalence
   Compound proposition p is logically equivalent to compound proposition q, written $p \Leftrightarrow q$, if and only if the compound proposition $p \leftrightarrow q$ is a taulolgy.

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity laws
$p \lor \mathbf{T} \equiv \mathbf{T}$ $p \land \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Exclusive or: $p \oplus q \Leftrightarrow (p \lor q) \land \neg(p \land q)$
$p \oplus q \Leftrightarrow (p \land \neg q) \lor (q \land \neg p)$

Implication: $p \to q \Leftrightarrow \neg p \lor q$

p	q	$p \to q$
F	F	T
F	T	T
T	F	**F**
T	T	T

Biconditional: $p \leftrightarrow q \Leftrightarrow (p \to q) \land (q \to p)$
$p \leftrightarrow q \Leftrightarrow \neg(p \oplus q)$

13. Logical Inference

Definition: An Inference Rule is a pattern establishing that if we know that a set of antecedent statements of certain forms are all true, then we can validly deduce that a certain related consequent statement is true.

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Formal proof is based on the rules above

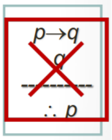
Definition: A formal proof of a conclusion C, given premises p1, p2, …, pn sequence of steps, apply inference rule to premises or previously proven statements (antecedents), and yield new true statement (the consequent)

A proof: if the premises are true, then the conclusion is true.

A fallacy is an inference rule or other proof method that is not logically valid.

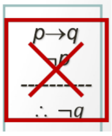
Fallacy of *affirming the consequent*:

- "$p \to q$ is true, and q is true, so p must be true." (No, because **F→T** is also true.)
- p is *sufficient* but not *necessary* for q



Fallacy of *denying the antecedent*:

- "$p \to q$ is true, and p is false, so q must be false." (No, because **F→T** is also true.)
- p is *sufficient* but not *necessary* for q



14. Propositional Functions (predicate and quantifier)

1) Propositional function $P(x)$: A statement involving the variables x.

2) A statement from $P(x_1, x_2, \dots, x_n)$ is the value of the propositional function P at the n-th tuple $(x_1, x_2, \dots, x_n)$, and P is called the predicate.

3) The domain of discourse, denote U, is the set of values x that x is allowed to take in $P(x)$.

4) The universal quantification of $P(x)$ is the proposition "$P(x)$ is true for all value of x in U".

Notation: $\forall x P(x)$, $\forall$ is called the universal quantifier.

"$\forall x P(x)$"=True, when $P(x)$ is true for every x in U.

"$\forall x P(x)$"=False, when there is an x in U for which $P(x)$ is false.

Examples:

"for all integers n, 2n is even" (True)

"for all real numbers x, $x^2 - 1 > 0$" (False, $x = 0$)

"for all CS major students S, S must take discrete math" (True)

5) The existential quantification of $P(x)$ is the proposition "There exists an element x in U such that $P(x)$ is true."

Notation: $\exists x P(x)$, $\exists$ is called the existential quantifier.

"$\exists x P(x)$"=True, when there is an x in U for which $P(x)$ is true.

"$\exists x P(x)$"=False, when $P(x)$ is false for every x in U.

Examples:

"there exists an integer n, $2 * n$ is even" (True)

"there exists a student S, S works hard" (True)

"there exists a real number x, $x^2 < 0$" (False)