There are 52 cards in a poker.

4 suits: diamonds ♦ × 13 spades ♠ × 13

clubs ♣ × 13 hearts ♥ × 13

kinds: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A (13 in total)

Poker Problem

1. Straight Flush

Contains 5 cards of sequential rank, all of the same suit.

2	3	4	5	6
3	4	5	6	7
4	5	6	7	8
5	6	7	8	9
6	7	8	9	10
7	8	9	10	J
8	9	10	J	Q
9	10	J	Q	K
10	J	Q	K	A
A	2	3	4	5

can be: clubs, diamonds, spades, hearts

There are 10 possible combanitions, each combanition has 4 possible suits.

As a result, there can be 10 × 4 possible straight flush.

$\binom{10}{1}\cdot\binom{4}{1} = 10 \times 4 = 40$

$$\boxed{\mathbb{P}(\text{Straight flush}) = \frac{\binom{10}{1}\cdot\binom{4}{1}}{\binom{52}{5}}}$$

2. Four of a Kind

Contains 4 cards of one rank and one card of another.

$\underbrace{\_\ \_\ \_\ \_}_{\text{4 of one kind}}$ | $\underbrace{\_}_{\text{another kind}}$ → First 4 $\_\ \_\ \_\ \_$ → Last $\_$ 52 − 4 = 48 possible outcomes

13

A	A	A	A
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
⋮	⋮	⋮	⋮
Q	Q	Q	Q
K	K	K	K

For each possible combanition, all 4 suits are contained.

There are 13 possible outcomes for the first 4 cards.

and 48 possible outcomes for the other card.

As a result, there can be 13 × 48 possible four of a kind.

$$\boxed{\mathbb{P}(\text{Four of a kind}) = \frac{\binom{13}{1}\cdot\binom{48}{1}}{\binom{52}{5}}}$$

3. Full House

Contains three card of one rank and two cards of another rank.

$\underbrace{\_\ \_\ \_}_{\text{3 of one kind}}$ | $\underbrace{\_\ \_}_{\text{2 of one kind}}$ → First 3 $\_\ \_\ \_$

13

A	A	A
2	2	2
3	3	3
4	4	4
⋮	⋮	⋮
Q	Q	Q
K	K	K

Each combanition, the first one can have 4 possible suits, second one can have 3 possible suits, third one can have 2 possible suits.

So it can be A♦ A♣ A♠ for the first combanition.

A♦ A♣ A♥

A♦ A♠ A♥

A♣ A♠ A♥

4

As a result, there are $\underline{13 \times 4}$ possible outcomes for the first 3 cards.

↓ Last 2

$\_\ \_$

Can be any two cards from the rest card, but cannot have the same rank as first 3.

If $\underline{A}\ \underline{A}\ \underline{A}$, then $\_\ \_$. Each combination, the first one can have 4 possible suits, second one can have 3 possible suits.

12

2	2
3	3
4	4
⋮	⋮
Q	Q
K	K

So it can be 2♣ 2♠ for the first combanition.

2♣ 2♦

2♣ 2♥

2♠ 2♦

2♠ 2♥

2♦ 2♥

6

As a result, there are $\underline{12 \times 6}$ possible outcomes for last 2 cards.

$$\boxed{\mathbb{P}(\text{Full House}) = \frac{\binom{13}{1}\cdot\binom{4}{3} \times \binom{12}{1}\cdot\binom{4}{2}}{\binom{52}{5}}}$$

4. <u>Flush.</u>

Contains 5 cards all the same suit, not all of sequential rank.

_ _ _ _ _ There are 4 suits: spades ♠ ×13, diamonds ♦ ×13, clubs ♣ ×13, hearts ♥ ×13

For each suit, we choose 5 cards. So there are $\binom{13}{5}$ outcomes for each suit.

As a result, there are $4\times\binom{13}{5}$ possible outcomes in total.

$$\boxed{\mathbb{P}(\text{Flush}) = \frac{\binom{4}{1}\cdot\binom{13}{5}}{\binom{52}{5}}}$$

* this contains outcomes that all 5 cards are consequtive.

For flush does not contain all 5 cards are consequtive. We can have:

In each suit, we have 10 combanitions that have 5 consequitive rankes.

$$10\begin{cases} A & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 9 & 10 & J & Q & K \end{cases}$$

As a result, there are $\binom{13}{5}-10$ outcomes to avoid consequtive ranks in each suit.

$$\boxed{\mathbb{P}(\text{Flush}) = \frac{\binom{4}{1}\cdot\left[\binom{13}{5}-\binom{10}{1}\right]}{\binom{52}{5}}}$$

5. <u>Straight</u>

Contains 5 cards that are ranked concequtively, not all same suit.

_ _ _ _ _ Consider ranks, there can have 10 combanitions.

$$10\begin{cases} A & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 9 & 10 & J & Q & K \end{cases}$$

Each rank can have 4 possible suits. So there are 4^5 possible combanitions for each concequtive rank combanitions. There are $4^5\cdot 10$ total outcomes.

$$\boxed{\mathbb{P}(\text{Straight}) = \frac{\binom{10}{1}\cdot 4^5}{\binom{52}{5}}}$$

* this contains 5 cards that have same suits

~~For straight does not contain 5 cards with same suits. We can have:~~

Since not allowed 4 of 5 cards have same suit, so we have to exclude the cases that there are 4 cards in the same suit. There are 4 possible outcomes for all 5 same suit in each rank combination.

So for each rank combanition, for the first 4 cards, we can only have 4×4×4×4 ways to assign suits.

For the last number, there are 4 possible suits can take. But we need to subtract the all the same suit, so there are (4^5-4) possible suits combanitions in each rank combanition.

$$\boxed{\mathbb{P}(\text{Straight}) = \frac{\binom{10}{1}\cdot(4^5-4)}{\binom{52}{5}}}$$

6. <u>Three of a kind</u>

Contains 3 same rank. Other two can be anything but not the same rank.

_ _ _ if $\xrightarrow{\quad}$ If AAA, then _ _ $\binom{12}{2}$.

$$13\begin{cases} A & A & A \\ 2 & 2 & 2 \\ \vdots & \vdots & \vdots \\ Q & Q & Q \\ K & K & K \end{cases} \qquad \binom{12}{2}\begin{cases} 2 & 3 \\ \vdots & 4 \\ \vdots & \vdots \\ & K \\ \vdots & \vdots \end{cases}$$

$$\boxed{\mathbb{P}(\text{Three of a kind}) = \frac{\binom{4}{3}\cdot\binom{13}{1}\times\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}}$$

7. <u>Two Pairs</u>

Two same rank, another two same rank.

_ _ | _ _ | _

$$13\begin{cases} A & A \\ 2 & 2 \\ \vdots & \vdots \\ Q & Q \\ K & K \end{cases}$$

Permutation $\begin{cases} \text{First two: } \binom{13}{1}\cdot\binom{4}{2} \\ \text{Second two: } \binom{12}{1}\cdot\binom{4}{2}\cdot\frac{1}{2!} \\ \text{Last one: } \binom{11}{1}\cdot\binom{4}{1} \end{cases}$

OR

Combanition $\begin{cases} \text{First Four: } \binom{13}{2}\binom{4}{2}\binom{4}{2} \\ \text{Last one: } \binom{11}{1}\binom{4}{1} \end{cases}$

$$\boxed{\begin{aligned} \mathbb{P}(\text{Two Pairs}) &= \frac{\binom{13}{1}\cdot\binom{4}{2}\times\binom{12}{1}\cdot\binom{4}{2}\times\binom{11}{1}\binom{4}{1}}{\binom{52}{5}\cdot 2!} \\ &= \frac{\binom{13}{2}\cdot\binom{4}{2}\binom{4}{2}\times\binom{11}{1}\cdot\binom{4}{1}}{\binom{52}{5}} \end{aligned}}$$

8. <u>One Pair</u>

Contains two same rank.

_ _ | _ _ _

$$13\begin{cases} A & A \\ 2 & 2 \\ \vdots & \vdots \\ Q & Q \\ K & K \end{cases}$$

Combanition $\begin{cases} \text{First two: } \binom{13}{1}\binom{4}{2} \\ \text{Last three: } \binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1} \end{cases}$

$$\boxed{\mathbb{P}(\text{One Pair}) = \frac{\binom{13}{1}\binom{4}{2}\times\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}}$$

For Permutation

$$\mathbb{P}(\text{One Pair}) = \frac{\binom{13}{1}\binom{4}{2}\times\binom{12}{1}\binom{4}{1}\times\binom{11}{1}\binom{4}{1}\times\binom{10}{1}(}{\binom{52}{5}\cdot 3!}$$

9. <u>High Card</u>

Cannot have combanitions above.

Cannot have straight, so minus $\binom{10}{1}$ for each combination.

Cannot have flush, so minus $\binom{4}{1}$ for each combination.

$[\binom{13}{5}-\binom{10}{1}]\cdot(4^5-4)$ Possible outcomes.

$$\boxed{\mathbb{P}(\text{High Card}) = \frac{[\binom{13}{5}-\binom{10}{1}]\cdot(4^5-4)}{\binom{52}{5}}}$$