COMP2022|2922
Models of Computation

Deterministic Finite Automata (DFA)

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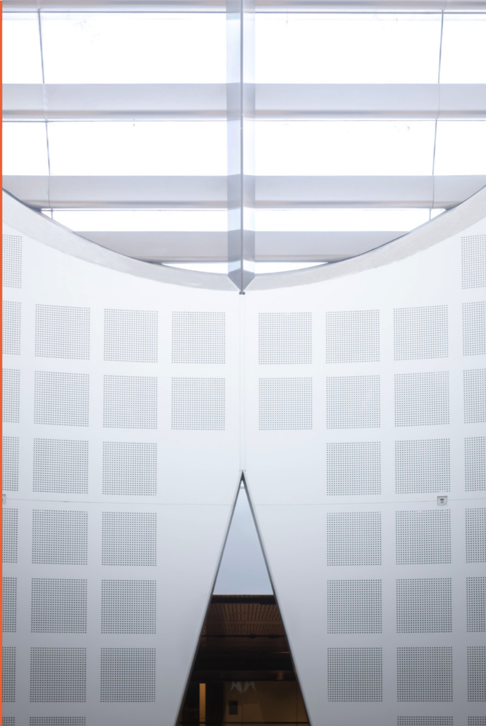
THE UNIVERSITY OF SYDNEY



Reminder...

- If you don't understand what something means, read its **Definition** again and look at **Examples**.
- If you don't understand why something is useful, read the **Theorems** and **Facts** about it.
- If you don't understand why something is relevant, look at its **Applications**.

These will be clearly marked in the lecture notes (sometimes definitions will be marked like this, in red)

How can we specify decision problems = languages?

1. In English.
2. In mathematics/set-theoretic notation, and recursive definitions.
3. By regular expressions (Lecture 1)
4. By automata (Today)
5. By context-free grammars
6. By Turing-machines

Automata in a nutshell

An automaton is a character-processing program that:

- Takes a string as input.
- Can only use variables with finite domains — we use one variable `state` taking finitely many values.
- Can read the input string character by character: `get_char()`
- Can test for end of input: `end_of_input()`
- Must decide to "Accept" or "Reject" the input string.

Why study such a simple model of computation?

1. It is part of computing culture.
   - first appeared in McCulloch and Pitt's model of a neural network (1943)
   - then formalised by Kleene (American mathematician) as a model of stimulus and response
2. It has numerous practical applications.
   - scanner (aka lexical analyser)
   - pattern matching
   - communication protocols with bounded memory
   - circuits with feedback
   - finite-state reactive systems
   - finite-state controllers
   - non-player characters in computer games
   - ...
3. It is simple to implement/code.

Program representation of automata

What strings over alphabet $\Sigma = \{a, b\}$ does the following code accept?

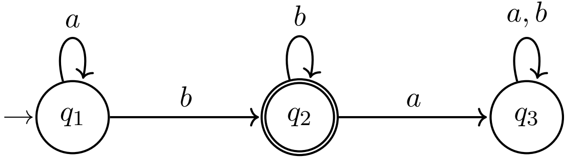
```
state = 1
while not end_of_input():
  x = get_char()
  if state==1 and x=="b" then state=2
  else
  if state==2 and x=="a" then state=3
if state==2 return "Accept"
else return "Reject"
```

1. All strings that match the regular expression $(a \mid b)^*$.
2. All strings that do contain a b but not an a.
3. All strings that match the regular expression a^*bb^*.

Graphical representation of automata

Such programs have a graphical representation as a directed edge-labeled graph:

- Vertices represent states.
- Labeled-edges $q \xrightarrow{a} q'$ represent transitions between states.
- The start state is marked with an incoming arrow.
- The final states are marked with an extra circle.



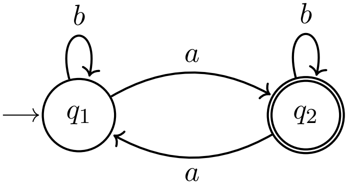
The language of a DFA

The run on input x is the path from the start state that is labeled by x.

If the run on x ends in a final state, the automaton accepts x, otherwise it rejects x.

Example

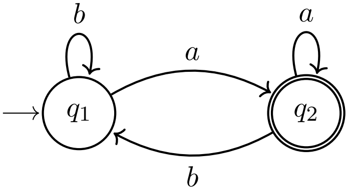
Exactly which strings does this automaton accept?



1. All strings
2. All strings that end in an a.
3. All strings with an odd number of a's.
4. All strings that do not contain an a.

Example

Exactly which strings does this automaton accept?



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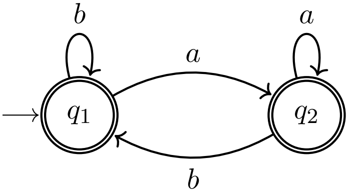

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Definition of DFA

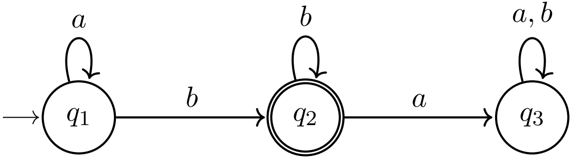
Definition
A deterministic finite automaton (DFA) M consists of 5 items

$$(Q, \Sigma, \delta, q_0, F)$$

where

1. Q is a finite set of states,
2. Σ is the alphabet (aka input alphabet),
3. $\delta : Q \times \Sigma \to Q$ is the transition function,
   - If $\delta(q, a) = q'$ we write $q \xrightarrow{a} q'$, called a transition.
4. $q_0 \in Q$ is the start state (aka initial state), and
5. $F \subseteq Q$ is the set of final states (aka accepting states).

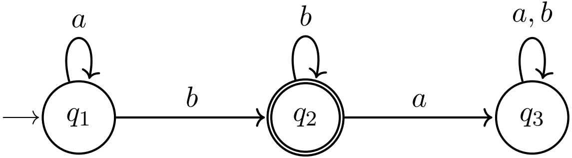
Example



1. the set Q of states is
2. the alphabet Σ is
3. the start state is
4. the set F of final states is
5. the transitions are[1]

[1] Convention: we need not draw the 'rejecting sink'.

Example




```
Sigma = a b
Q = q1 q2 q3
start = q1
F = q2
q1 a q1
q1 b q2
q2 a q3
q2 b q2
q3 a q3
q3 b q3
```

Regular languages

The language of a DFA M (aka, language recognised by M) is the set of strings that it accepts (no more, no less). It is written $L(M)$.[1]

The languages of DFAs are so important, we give them a name:

Definition

A language $L \subseteq \Sigma^*$ is called regular if $L = L(M)$ for some DFA M.[2]

[1] A more formal (mathy) definition is given in the "extra slides".

[2] This means that M must accept all strings in L and reject all strings (in Σ^*) that are not in L.

Designing automata tips (i)

Imagine you are the automaton reading the string symbol by symbol:

- what information about the string read so far do you need to make a decision on whether to accept or reject.
- can you update this information if another input symbol arrives?
- the states will store this information.

Designing automata

Draw an automaton for the language of strings over $\Sigma = \{a, b\}$ that contain aab as a substring.

Designing automata tips (ii)

Build automata out of other automata using closure properties of the regular languages.

If you have DFA for L_1, L_2 then you can get DFA for

1. $\Sigma^* \setminus L_1$
2. $L_1 \cup L_2$ (and thus, by DeMorgan!, also $L_1 \cap L_2$)
3. $L_1 L_2$
4. $(L_1)^*$

Let's see how to do complement (union is in the tutorial, concatenation and star are trickier and we will do it in a future lecture).

Regular languages closed under complementation

Theorem
If L is regular, then $\Sigma^* \setminus L$ is regular.

Idea
Swap final and non-final states in a DFA for L

Regular languages closed under complementation

Theorem
If L is regular, then $\Sigma^* \setminus L$ is regular.

Construction: "swap final and non-final states"

- Given DFA $M = (Q, \Sigma, \delta, q_0, F)$ recognising L
- We build a DFA M' recognising $\Sigma^* \setminus L$ as follows:
  Define $M' = (Q, \Sigma, \delta, q_0, F')$ where $F' = Q \setminus F$.

Why is this correct?

Regular languages closed under complementation

Theorem

If L is regular, then $\Sigma^* \setminus L$ is regular.

Example

Give a DFA for the language of strings over $\{a, b\}$ that do **not** contain aab as a substring.

Important questions about DFA

- Which languages can be described by DFAs? All languages?
  - No! (we will come back to this in a future lecture)
- There are natural decision problems associated with DFAs. Are there algorithms that solve them?

1. Membership problem

   Input: DFA M, string w.
   Output: decide if $w \in L(M)$.

2. Non-emptiness problem (tutorial)

   Input: DFA M.
   Output: decide if $L(M) \neq \emptyset$.

3. Equivalence problem (tutorial)

   Input: DFAs M_1, M_2.
   Output: decide if $L(M_1) = L(M_2)$.

Membership problem

Input: DFA M, string w.
Output: decide if $w \in L(M)$.

```
def membership(M,w):
  state = q_0
  while not end_of_input(w):
    x = get_char(w)
    state = δ(state,x)
  if state in F:
    return "Accept"
  else:
    return "Reject"
```

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Nondeterministic finite automata (NFA)

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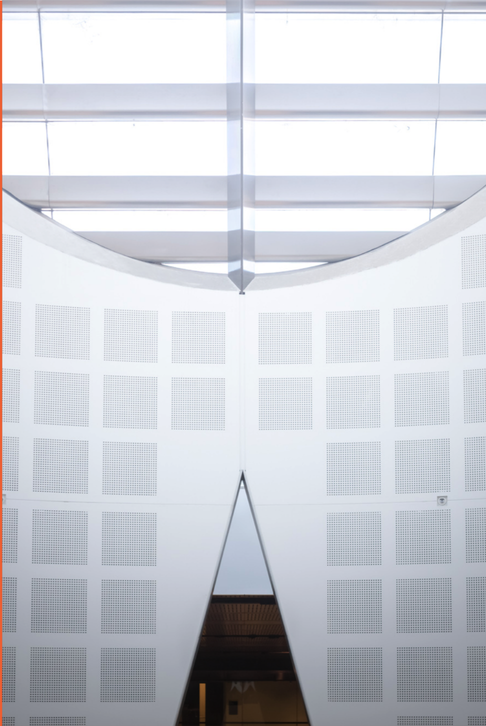
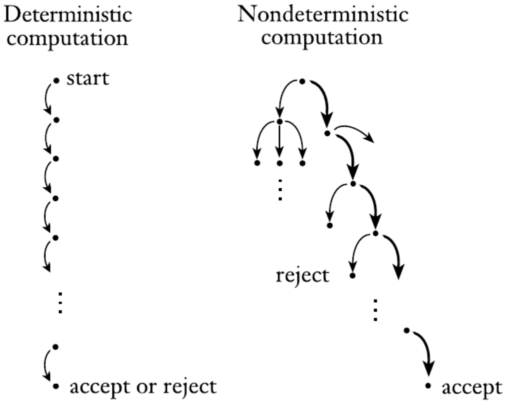


Intuitively, "nondeterminism" refers to situations in which the next state of a computation is not uniquely determined by the current state and current input.



Where does nondeterminism come from?

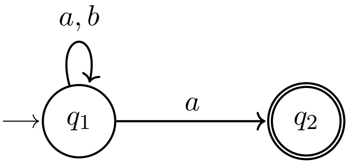
Let's introduce nondeterminism into automata.

- A nondeterministic finite automaton (NFA) is like a DFA except that states can have zero, one, or more outgoing transitions on the same input symbol.
- A string is accepted by an NFA if it labels some path from the start state to a final state.

In an NFA, a string can label zero, one, or more paths.

NFA

Exactly which strings over alphabet $\Sigma = \{a, b\}$ are accepted by this NFA?



1. all strings.
2. strings that have at least one a.
3. strings that end in an a.
4. strings that start with an a.

Pattern matching made easy!

- NFAs are good for specifying languages of the form "the string has x as a substring"
- E.g., $x = aab$

Definition of NFA

Definition

A nondeterministic finite automaton (NFA) $M = (Q, \Sigma, \delta, q_0, F)$ is the same as a DFA except that

$$\delta : Q \times \Sigma_\epsilon \to P(Q),$$

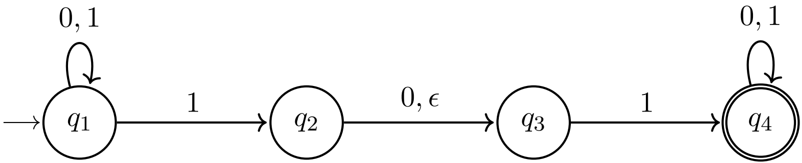
called the transition relation.

- So $\delta(q, a)$ is a set of states (empty set is allowed, multiple states are allowed)
- If $q' \in \delta(q, a)$ we write $q \xrightarrow{a} q'$, called a transition.
- Here $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$.

So, we also allow epsilon-transitions. This amounts to transitions that do not consume the next input symbol.

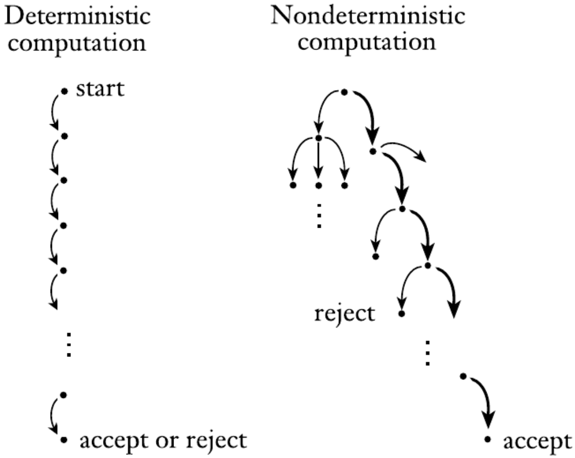
A more formal (mathy) definition is given in "extra slides"

NFA with Epsilon transitions



Draw the computation tree on the input word 011

Comparing NFAs and DFAs



Comparing NFAs and DFAs

1. In a DFA, every input has exactly one run.[3] The input string is accepted if this run is accepting.
2. In an NFA, an input may have zero, one, or more runs. The input string is accepted if at least one of its runs is accepting.
3. For every DFA M there is an NFA N such that $L(M) = L(N)$.
   - Idea: $q' = \delta(q, a)$ in M becomes $\{q'\} = \delta(q, a)$ in N.
   - You can think of a DFA as an NFA in which there is no nondeterminism.

[3] With our convention of not drawing rejecting sinks, an input of a DFA may have no runs too.

Where are we going?

- We are going to show that DFA, NFA and regular expressions specify the same set of languages!
- We will do this with a series of transformations:

1. From Regular Expressions to NFAs
2. From NFAs to NFAs without ϵ-transitions
3. From NFAs without ϵ-transitions to DFAs
4. From DFAs to Regular Expressions

From Regular Expressions to NFAs

Theorem

For every regexp R there is an NFA N such that $L(R) = L(N)$.

Since regular expressions are built recursively, this construction is also recursive.

- The base cases are $R = \emptyset, R = \epsilon, R = a$ for $a \in \Sigma$

  We must show that each of these languages is recognised by some NFA.

- The recursive cases are $R = (R_1 \mid R_2)$, $R = (R_1 R_2)$, and $R = {R_1}^*$

  We must show that if N_1, N_2 are NFAs, then there are NFAs recognising $L(N_1) \cup L(N_2)$, $L(N_1)L(N_2)$, and $L(N_1)^*$

From Regular Expressions to NFAs

The base cases are $R = \emptyset, R = \epsilon, R = a$ for $a \in \Sigma$.

NFAs are closed under union

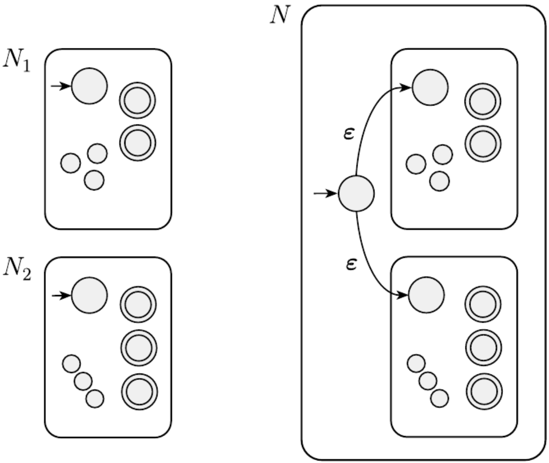
Lemma

If N_1, N_2 *are NFAs, there is an NFA* N *recognising* $L(N_1) \cup L(N_2)$

Idea: "Simulate N_1 or N_2"

- Given NFAs $N_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ construct NFA N that guesses which of N_1 or N_2 to simulate. How?
- N has states $Q_1 \cup Q_2 \cup \{q_0\}$ so that it can simulate N_1, N_2.
- N guesses from q_0 whether to go to the start state of N_1 or N_2.

NFAs are closed under union



NFAs are closed under concatenation

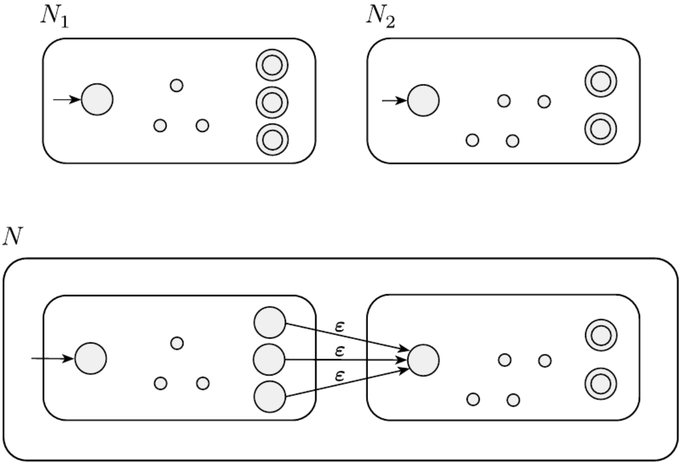
Lemma

If N_1, N_2 are NFAs, there is an NFA N recognising $L(N_1)L(N_2)$

Idea: "Simulate N_1 followed by N_2"

- Given NFAs $N_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ construct NFA N that guesses how to break the input into two pieces, the first accepted by N_1, the second by N_2. How?
- N has states $Q_1 \cup Q_2$ so that it can simulate N_1 and N_2.
- At some point when N_1 is in a final state, guess that it is time to move to the start state of N_2.

NFAs are closed under concatenation



NFAs are closed under star

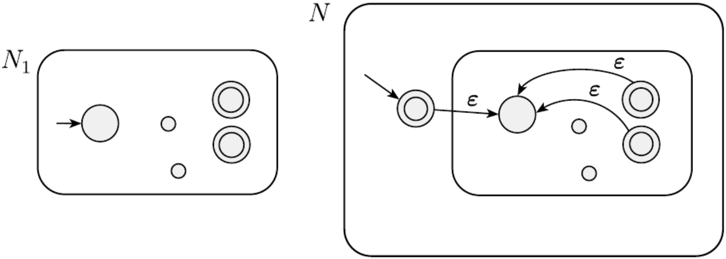
Lemma

If N_1 is an NFA, there is an NFA N recognising $L(N_1)^*$

Idea: "Repeatedly simulate N_1"

- Given NFA $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ construct NFA N that guesses how to break the input into pieces, each of which is accepted by N_1. How?
- N has states $Q \cup \{q_0\}$, and extra transitions from final states of N_1 to the initial state of N_1.
- The new state q_0 is ensure that ϵ is accepted.

NFAs are closed under star



From RE to NFA: example

Convert the regular expression $(ab \,|\, a)^*$ to an NFA.

Where are we going?

Recall:

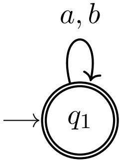
- We are going to show that DFA, NFA and regular expressions specify the same set of languages!
- We will do this with a series of transformations:

1. From Regular Expressions to NFAs (today)
2. From NFAs to NFAs without ϵ-transitions (next time)
3. From NFAs without ϵ-transitions to DFAs (next time)
4. From DFAs to Regular Expressions (next time)

Extra slides

Example

Exactly which strings does this automaton accept?



1. All strings
2. All strings that end in an a.
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The language recognised by a DFA M (math)

Definition

- A run (aka computation) of M on $w = w_1 w_2 \cdots w_n$ is a sequence of transitions $q_0 \xrightarrow{w_1} q_1 \xrightarrow{w_2} q_2 \xrightarrow{w_3} \cdots \xrightarrow{w_n} q_n$ where q_0 is the start state.
- The run is accepting if $q_n \in F$.
- If w has an accepting run then we say that M accepts w.
- The set $L(M) = \{w \in \Sigma^* : M \text{ accepts } w\}$ is the language recognised by M (aka language of M).

The language recognised by an NFA M (math)

The following definition formalises the idea that an NFA M describes the language $L(M)$ of all strings that label paths from the start state to a final state.

Definition

– A run (aka computation) of an NFA M on string w is a sequence of transitions $q_0 \xrightarrow{y_1} q_1 \xrightarrow{y_2} q_2 \ldots \xrightarrow{y_m} q_m$ such q_0 is the start state, each $y_i \in \Sigma_\epsilon$, and $w = y_1 y_2 \cdots y_m$.[4]

– The run is accepting if $q_m \in F$.

– If w has at least one accepting run, then we say that w is accepted by M.

– The language recognised by M is $L(M) = \{w \in \Sigma^* : w \text{ is accepted by } M\}$.

[4] Recall that $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$, and $\epsilon x = x\epsilon = x$ for all strings x.