MATH-UA 253/MA-UY 3204 - Fall 2022 - Homework #8

Problem 1. Let $G \in \mathbb{R}^{n \times n}$ be symmetric, $c \in \mathbb{R}^n$, and $d \in \mathbb{R}$. Also, let $A \in \mathbb{R}^{m \times n}$ with $m < n$ have full row rank, and let $b \in \mathbb{R}^m$. An equality-constrained quadratic program (QP) is a minimization problem of the form:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} x^\top G x + c^\top x + d \\ \text{subject to} \quad & Ax = b. \end{aligned} \tag{1}$$

1. Prove that a set of first-order necessary conditions for optimality for a pair (x^*, λ^*) are given by:

$$\begin{bmatrix} G & -A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}. \tag{2}$$

2. Let $Z \in \mathbb{R}^{n \times (n-m)}$ be such that $\operatorname{range}(Z) = \operatorname{null}(A)$. Show that if the reduced Hessian $Z^\top G Z$ is positive definte, then (2) has a unique solution.

3. Show that if the assumptions for the previous part hold, then x^* is a strict global minimum for (1).

Problem 2. Assume G in (1) is positive definite for this problem. Show that the solution of (2) is given by:

$$\begin{aligned} x^* &= G^{-1} D^\top \left(D G^{-1} D^\top \right) D G^{-1} c + G^{-1} \left(D^\top b - c \right) \\ \lambda^* &= \left(D G^{-1} D^\top \right) \left(D G^{-1} c + b \right). \end{aligned} \tag{3}$$

(*Hint*: compute the block LU factorization of the matrix in (2).)

Problem 3. Let x be a guess for x^*, let $p = x^* - x$ (the optimal step taking us from x to x^*), let $h = Ax - b$ (the degree to which the equality constraints are violated), and let $g = c + Gx$ (the gradient of the cost function evaluated at x). Further, let $Z \in \mathbb{R}^{n \times (n-m)}$ be a basis matrix for $\operatorname{null}(A)$, and let $Y \in \mathbb{R}^{n \times m}$ be a basis matrix for $\operatorname{null}(A)^\perp$. Decompose p into its part in the Y and Z bases, i.e. let v_Y and v_Z be such that $p = Y v_Y + Z v_Z$.

1. Show that (2) can be rewritten as:

$$\begin{bmatrix} G & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} -p \\ \lambda^* \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}. \tag{4}$$

2. First, show that $A Y v_y = -h$.

3. Next, show that $Z^\top G Z v_z = -Z^\top G Y v_y - Z^\top g$.

4. Assume that $Z^\top G Z$ is positive definite. Explain how to solve for p and λ^*.

Problem 4. The methods outlined for solving an equality-constrained QP in Problems 2 and 3 are two commonly used approaches. Explain the trade-offs between these methods. What assumptions do they require? In terms of the parameters m and n, what is the performance of each method like?