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Programming assignment #4 (bonus)

In this homework, we'll explore implementing the sine function, $y = \sin(x)$.

Bonus problem 1. Some values of sin and cos should be known to you (e.g., $\sin(k\pi) = 0$ for $k \in \mathbb{Z}$, $\sin(\pi/2) = 1$, …). Recall that $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$. Using **only** known values of sin and cos, program a piecewise Hermite approximation of sin and cos of arbitrary degree. That is, write a function `y = sin(x, p)` which will approximate $y = \sin(x, p)$ for any value of x, where p is the degree of the Hermite interpolant. Here are some hints:

- Use the fact that sin and cos are 2π-periodic.
- Note the simple relationships among the derivatives of sin and cos.
- You should be able to divide $[0, 2\pi]$ into at least 8 equal subintervals. Dividing it into fewer subintervals might be dangerous.
- Although Method I and Method II (see WHW#4) will work for low degree, you will run into trouble for higher degrees. Use the method of interpolation based on Newton's interpolating polynomial (this will require you to use *generalized divided differences*).

Bonus problem 2. Make plots of the errors of your interpolants and explain their behavior (include an extra short file with a written explanation—do not answer in your Python file using comments). Do this for varying p (you must choose the range yourself). When you plot the error, it is recommended to plot the absolute relative error using a `semilogy` plot. This will give a sense of the order of accuracy over the interval $[0, 2\pi]$.

(*Hint*: when you make these plots, use `x = np.linspace(0, 2*np.pi, N)` to get the x values. When you make the plots, keep increasing N until the plots are sufficiently sampled—that is, increasing N further should not visibly alter the plots.).

Bonus problem 3. Let $h_p(x)$ be the piecewise Hermite interpolant programmed in Problem 1. When evaluating the error, it is also helpful to estimate quantity:

$$\frac{\|h_p - f\|}{\|f\|}, \qquad f(x) = \sin(x). \tag{1}$$

for difference choices of p. As we have seen, there are different norms we could use, e.g.:

$$\|f\|_{1,[0,2\pi]} = \int_0^{2\pi} |f(x)| dx, \tag{2}$$

$$\|f\|_{2,[0,2\pi]} = \sqrt{\int_0^{2\pi} |f(x)|^2 dx}, \tag{3}$$

$$\|f\|_{\infty,[0,2\pi]} = \max_{0 \le x \le 2\pi} |f(x)| \tag{4}$$

It might be a little unclear how to evaluate these norms using Python. In this problem, you will learn how to do so in order to evaluate (1) for the L^1 and L^2 norms. (*Note*: Evaluating $\|f\|$ is simple for the function $f(x) = \sin(x)$. The challenging part is dealing with the numerator, $\|h_p - f\|$.)

For $\|g\|_{1,[0,2\pi]}$ and $\|g\|_{2,[0,2\pi]}$, consider the following idea:

1. Use the composite trapezoid rule with equally spaced nodes to evaluate the integrals in (2) or (3).

2. Do this for $N + 1$ nodes and $2N + 1$ nodes, getting two different approximate values of $\|h_p - f\| \,/\, \|f\|$, call them E_N and E_{2N}.

3. Check the value of $|E_N - E_{2N}|$. If it is close to machine precision (approximately 10^{-15} for double precision), then declare victory and use the value of E_N as your approximation.

4. Otherwise, increase the value of N and try again (e.g., set $N \leftarrow 2N$ and start over from Step 2).

This "meta-algorithm" will work fine for the composite trapezoid rule, but you are free to experiment with other choices of numerical integration.

Bonus problem 4. The L^∞ norm is trickier (or, at least, it requires more algorithmic savvy). For $\|g\|_{\infty,[0,2\pi]}$, you must evaluate:

$$\max_{0 \le x \le 2\pi} |h_p(x) - \sin(x)|. \tag{5}$$

Note that $h_p(x)$ is a piecewise polynomial; hence, $|h_p(x) - \sin(x)|$ will be defined piecewise. In order to evaluate (5), it is enough to maximum $|h_p(x) - \sin(x)|$ over each subinterval. It may be the case that over a single subinterval, $|h_p(x) - \sin(x)|$ has numerous local minima and maxima, which can be found by locating the zeros of $h_p'(x) - \frac{d}{dx}\sin(x)$ (note that there is no $| \cdot |$ in this expression—why?).

(*Hint*: please only attempt this problem if you have finished everything else in this course, feel like you have studied for the final and are confident, and are bored and looking to amuse yourself... Better yet, go outside and enjoy the sun!)