MATH-UA 252/MA-UY 3204 - Fall 2022 - Homework #4

Problem 1. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a stationary point $x^* \in \mathbb{R}^2$ such that x^* satisfies the second-order necessary conditions for optimality but which is *not* a local optimum.

Problem 2. A *convex set* $A \subseteq \mathbb{R}^n$ is a set such that:

$$(1-\alpha)x + \alpha x' \in A, \qquad \forall x, x' \in A, \qquad \forall \alpha \in [0,1]. \tag{1}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Prove that for any $y \in \mathbb{R}$, the set:

$$L(y) = \{x \in \mathbb{R}^n : f(x) \leq y\} \tag{2}$$

is a convex set. (Note: the empty set is convex.)

Problem 3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and $C^1(\mathbb{R}^n)$ (it is *continuously differentiable*). Consider the set:

$$H(x) = \left\{x' \in \mathbb{R}^n : \nabla f(x)^\top (x' - x) \leq 0\right\}. \tag{3}$$

Fix $x \in \mathbb{R}^n$ and let $y = f(x)$. Prove that $L(y) \subseteq H(x)$.

Problem 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a *strictly* convex function (see notes for definition). Prove in this case that $L(f(x)) \cap H(x) = \{x\}$ holds for each $x \in \mathbb{R}^n$. That is, the set of values for which f does not exceed $f(x)$ and the halfspace supported by $\nabla f(x)$ at x intersect in exactly one point: x.

Problem 5. Let f be $C^2(\mathbb{R}^n)$ and have a positive definite Hessian. Let $x \in \mathbb{R}^n$, and let $p \in \mathbb{R}^n$ be a descent direction at x. Prove that there exists $\alpha > 0$ such that $f(x + \alpha p) < f(x)$.

Problem 6. The *spectral theorem for real matrices* says that if $A \in \mathbb{R}^{n \times n}$ is symmetric, then:

1. The eigenvalues of A are real.
2. The normalized eigenvectors of A form an orthonormal basis for $\mathbb{R}^n$.

Let $q_1, \ldots, q_n$ be the normalized eigenvectors of a symmetric matrix A. That is, there exist eigenvalues λ_i $(i = 1, \ldots, n)$, not necessarily distinct, such that:

$$Aq_i = \lambda_i q_i, \qquad i = 1, \ldots, n. \tag{4}$$

First, show that a consequence of this is that we have an orthogonal eigenvalue decomposition of A:

$$A = Q\Lambda Q^\top, \qquad Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \tag{5}$$

where $Q^{-1} = Q^\top$ (i.e., Q is an orthogonal matrix), and Λ is diagonal. Second, show that we can write the eigenvalue decomposition in outer product form, i.e.:

$$A = \sum_{i=1}^n \lambda_i q_i q_i^\top. \tag{6}$$

Finally, prove that $\lambda_i > 0$ $(i = 1, \ldots, n)$ if and only if A is positive definite.

Problem 7. Let S be a convex set, let $f : \mathbb{R}^n \to \mathbb{R}$ be in $C^2(S)$, and let $x^* \in S$ be a local minimum of f. Show that if $\nabla^2 f(x)$ is positive definite for all $x \in S$ that x^* is a global minimum of f over S. *Hint*: pick another point $x' \in S$, and parametrize the straight line from x to x'. Apply the fundamental theorem of calculus twice.

Problem 8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2, let $b \in \mathbb{R}^n$, and let $A \in \mathbb{R}^{n \times n}$ be invertible. Show that the Newton step for $g(x) = f(Ax + b)$ is the same as the Newton step for $f(x)$.