

MATH-UA 253/MA-UY 3204 - Fall 2022 - Homework #4

Problem 1. Give an example of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and a stationary point $x^* \in \mathbb{R}^2$ such that x^* satisfies the second-order necessary conditions for optimality but which is *not* a local optimum.

Problem 2. A *convex set* $A \subseteq \mathbb{R}^n$ is a set such that:

$$(1 - \alpha)x + \alpha x' \in A, \quad \forall x, x' \in A, \quad \forall \alpha \in [0, 1]. \quad (1)$$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function. Prove that for any $y \in \mathbb{R}$, the set:

$$L(y) = \{x \in \mathbb{R}^n : f(x) \leq y\} \quad (2)$$

is a convex set. (Note: the empty set is convex.)

Problem 3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and $C^1(\mathbb{R}^n)$ (it is *continuously differentiable*). Consider the set:

$$H(x) = \{x' \in \mathbb{R}^n : \nabla f(x)^\top (x' - x) \leq 0\}. \quad (3)$$

Fix $x \in \mathbb{R}^n$ and let $y = f(x)$. Prove that $L(y) \subseteq H(x)$.

Problem 4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a *strictly* convex function (see notes for definition). Prove in this case that:

$$L(f(x)) \cap \{x' \in \mathbb{R}^n : \nabla f(x)^\top (x' - x) = 0\} = \{x\} \quad (4)$$

holds for each $x \in \mathbb{R}^n$. That is, the set of values for which f does not exceed $f(x)$ and the halfplane with normal $\nabla f(x)$ at x intersect in exactly one point: x .

Problem 5. Let f be $C^2(\mathbb{R}^n)$ and have a positive definite Hessian. Let $x \in \mathbb{R}^n$, and let $p \in \mathbb{R}^n$ be a descent direction at x . Prove that there exists $\alpha > 0$ such that $f(x + \alpha p) < f(x)$.

Problem 6. The *spectral theorem for real matrices* says that if $A \in \mathbb{R}^{n \times n}$ is symmetric, then:

1. The eigenvalues of A are real.
2. The normalized eigenvectors of A form an orthonormal basis for \mathbb{R}^n .

Let q_1, \dots, q_n be the normalized eigenvectors of a symmetric matrix A . That is, there exist eigenvalues λ_i ($i = 1, \dots, n$), not necessarily distinct, such that:

$$Aq_i = \lambda_i q_i, \quad i = 1, \dots, n. \quad (5)$$

First, show that a consequence of this is that we have an orthogonal eigenvalue decomposition of A :

$$A = Q\Lambda Q^\top, \quad Q = [q_1 \quad \cdots \quad q_n], \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad (6)$$

where $Q^{-1} = Q^\top$ (i.e., Q is an orthogonal matrix), and Λ is diagonal. Second, show that we can write the eigenvalue decomposition in outer product form, i.e.:

$$A = \sum_{i=1}^n \lambda_i q_i q_i^\top. \quad (7)$$

Finally, prove that $\lambda_i > 0$ ($i = 1, \dots, n$) if and only if A is positive definite.

Problem 7. Let S be a convex set, let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be in $C^2(S)$, and let $x^* \in S$ be a local minimum of f . Show that if $\nabla^2 f(x)$ is positive definite for all $x \in S$ that x^* is a global minimum of f over S . *Hint:* pick another point $x' \in S$, and parametrize the straight line from x to x' . Apply the fundamental theorem of calculus twice.

Problem 8. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be C^2 , let $b \in \mathbb{R}^n$, and let $A \in \mathbb{R}^{n \times n}$ be invertible. Show that the Newton step for $g(x) = f(Ax + b)$ is the same as the Newton step for $f(x)$.