Standard form of an LP (10/27)

Recall, LP in standard form:

$$\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{subject to} \quad Ax = b \\ &\qquad\qquad\quad\; x \geq 0 \end{aligned}$$

We will usually put problems in standard form before applying algorithms for solving LPs. Computer can do it for us, but good to know how to do these transformations by hand.

$\boxed{\text{Any LP can be converted to standard form.}}$

Let's look at the rules:

1) Convert ~~maxim~~ maximization to minimization:
   - Just replace $c^T x$ with $-c^T x$.

2) Flip inequality constraints:
   - Replace a "$\leq$" constraint with a "$\geq$" constraint by multiplying both sides by -1.

3) Convert to "≥ 0" constraints:
   - Replace a constraint like "$x \geq 5$" with a new equality constraint and inequality constraint:

     "$x' = x - 5$" and "$x' \geq 0$"

4) Eliminate "free variables":

- All variables x_i must be included in the vector inequality $x \succeq 0$.
- A variable w/ no inequality constraint applied to it is called a <u>free variable</u>.
- Replace a free variable x_i with one new equality constraint and two new inequality constraints:

$$\begin{cases} x_i' - x_i'' = x_i \\ x_i' \geq 0 \\ x_i'' \geq 0 \end{cases}$$

5) Convert general inequality constraints to inequality constraints by introducing <u>slack variables</u>:

- A constraint of the form:

$$a_i^T x = \sum_j a_{ij} x_j \leq b_i$$

can be converted to an equality constraint by introducing a new slack variable $s_i \geq 0$: so that:

$$\sum_j a_{ij} x_j + s_i = b_i \, .$$

<u>Comments</u>:

- The way of applying these rules in sequence to reduce an LP to standard form is <u>nonunique</u>! (Similar to Gaussian elimination.)
- There is also some redundancy among these rules.
- LP could be reduced to one of several different standard forms.

- When reducing to standard form, we only ever add new variables. For this reason, we can think of reduction to standard form as a process where we lift the LP to an equivalent LP (in standard form) in a higher-dimensional space.

Example: Let's reduce the following LP to standard form:

$$
\begin{aligned}
\text{maximize} \quad & -5x_1 - 3x_2 + 7x_3 \\
\text{subject to} \quad & 2x_1 + 4x_2 + 6x_3 = 7 \\
& 3x_1 - 5x_2 + 3x_3 \leq 5 \\
& -4x_1 - 9x_2 + 4x_3 \leq -4 \\
& x_1 \geq -2 \\
& 0 \leq x_2 \leq 4
\end{aligned}
$$

We'll apply the following operations:

1) convert to minimization:

"$-5x_1 - 3x_2 + 7x_3$" $\mapsto$ "$5x_1 + 3x_2 - 7x_3$"

2) notice that x_3 is free: add constraints,

$$
\begin{cases}
x_3 = x_4 - x_5 \\
x_4 \geq 0 \\
x_5 \geq 0
\end{cases}
$$

3) flip the inequality "$x_2 \leq 4$" to get "$-x_2 \geq -4$", introduce new inequality constraint "$x_6 = 4 - x_2$" to get "$x_6 \geq 0$" and "$x_6 = 4 - x_2$".

4) likewise, convert "$x_1 \geq -2$" to "$x_7 \geq 0$" and "$x_7 = x_1 + 2$"

5) introduce slack variables :

$$\begin{cases} 3x_1 - 5x_2 + 3x_3 \le 5 \\ -4x_1 - 9x_2 + 4x_3 \le -4 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 3x_1 - 5x_2 + 3x_3 + s_1 = 5 \\ -4x_1 - 9x_2 + 4x_3 + s_2 = -4 \\ s_1 \ge 0 \\ s_2 \ge 0 \end{cases}$$

Altogether, we get the LP:

$$\begin{aligned} \text{minimize} \quad & 5x_1 + 3x_2 - 7x_3 \\ & 2x_1 + 4x_2 + 6x_3 = 7 \\ & 3x_1 - 5x_2 + 3x_3 + s_1 = 5 \\ & -4x_1 - 9x_2 + 4x_3 + s_2 = -4 \\ & x_3 - x_4 + x_5 = 0 \\ & x_2 + x_6 = 4 \\ & -x_1 + x_7 = 2 \\ & x_i \ge 0, \quad i = 1, \dots, 7 \\ & s_i \ge 0, \quad i = 1, 2 \end{aligned}$$

Comments:

- The equality constraints are now six in total.
- Ambient dimension of the LP is 9.
- Only three variables affect the cost function.
- Compare with Griva — they get a different form!

Let's rewrite this LP in matrix form; define:

$$c = [\underbrace{5\ 3\ {-7}\ 0\ 0\ 0\ 0}_{x_1, \dots, x_7}\ \underbrace{0\ 0}_{s_i}]^T \in \mathbb{R}^9$$

$$A = \begin{bmatrix} 2 & 4 & 6 & & & & & & \\ 3 & -5 & 3 & & & & & 1 & \\ -4 & -9 & 4 & & & & & & 1 \\ & & 1 & -1 & 1 & & & & \\ & 1 & & & & 1 & & & \\ -1 & & & & & & 1 & & \end{bmatrix} \in \mathbb{R}^{6\times 9} \qquad b = \begin{bmatrix} 7 \\ 5 \\ -4 \\ 0 \\ 4 \\ 2 \end{bmatrix} \in \mathbb{R}^6$$

$$\begin{array}{cc} \underbrace{\hphantom{2\ 4\ 6\ 0\ 0\ 0\ 0}}_{x_j} & \underbrace{\hphantom{1\ 1}}_{s_i} \end{array}$$

$$x = [x_1 \ \cdots \ x_7 \ s_1 \ s_2] \in \mathbb{R}^9$$

This gives:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

as desired.

We can deduce some properties of the LP from the matrix A. If the dimension of the column space of A is n, then we can solve $Ax = b$ for x. Then there are two cases: $x = A^{-1}b \geq 0$ or not. If so, we have the solution to our LP since the feasible set is just $\{A^{-1}b\}$. Otherwise, the constraint set is <u>inconsistent</u> (i.e. no point satisfies all constraints simultaneously).

Otherwise, if $m < n$ and if A is full rank, then the dimension of the feasible set is just m. E.g. if $n = 3$ and $m = 2$, $Ax = b$ and $x \geq 0$ define a polygonal set in the nonnegative octant: