Midterm #2 for MATH-UA.0123-001

Problem 1. Let $f(x,y) = \cos(x^2 + 2y)$, and let $\boldsymbol{u} = (\cos(\theta), \sin(\theta))$. Compute $D_{\boldsymbol{u}} f(\sqrt{\frac{\pi}{2}}, -\frac{\pi}{2})$ for $\theta = \pi/4$.

Problem 2. Consider the sphere of radius r whose center is the origin. Show that the normal line for each point on the sphere passes through the origin. *Hint*: start by writing down a level set function $f(x,y,z)$ such that $f(x,y,z) = 0$ is the sphere of interest.

Problem 3. Use the method of Lagrange multipliers to prove that the rectangle of maximum area with a given perimeter p is a square.

Problem 4. Consider the rectangle $R = [0,\pi] \times [0,\pi]$, and the integral:

$$I = \iint_R \cos(x)\sin(y)dA.$$

Use the midpoint rule with $m = n = 2$ (that is, divide R into *four* equal squares *total*) to approximate I. Next, evaluate I by doing the double integral. What is the error in the midpoint rule approximation?

Problem 5. Evaluate $\iiint_E \sqrt{x^2 + y^2}$ where E is the region that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$, and between the planes $z = z_0$ and $z = z_1$, where $z_0 < z_1$.

Problem 6. Set up and evaluate a triple integral in spherical coordinates to find the volume of a sphere of radius r.

Problem 7. Find the average distance between the origin and a point in a spherical shell (centered about the origin) with inner and outer radii $r_0 < r_1$.

Problem 8 (*bonus*). Observe that if $r_0 = r_1$ in Problem 7, the average distance is obviously just r_0, since the average distance to the origin of a point on a sphere is just the radius of the sphere itself. Prove this by taking the limit as $r_1 \to r_0$ of the result of Problem 7.

Problem 9. Find the volume of the solid E that lies below the cone $z = -\sqrt{x^2 + y^2}$ and above the sphere $x^2 + y^2 + z^2 = r^2$.

Problem 10. Consider the vector field $\boldsymbol{F}(x,y) = (-y, x)$. Is this a gradient vector field? Why or why not? Let C be a circular arc of radius r subtending an angle θ. Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Problem 11. Let $\boldsymbol{F}(x,y) = (xy^2, x^2 y)$, and let C be the unit circle. What is $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$?

Problem 12 (*bonus*). A *spherical rectangle* is a set $[\rho_0, \rho_1] \times [\theta_0, \theta_1] \times [\phi_0, \phi_1] \subseteq \mathbb{R}^3$, parametrized using spherical coordinates. Assuming that $\rho_0 \le \rho_1, \theta_0 \le \theta_1$, and $\phi_0 \le \phi_1$, draw or name as many different kinds of shapes that you can think of which are actually just spherical rectangles. (If your drawings suck, you won't get any points.)