

# MATH-UA 252/MA-UY 3204 - Fall 2022 - Homework #4

**Problem 1.** Give an example of a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  and a stationary point  $x^* \in \mathbb{R}^2$  such that  $x^*$  satisfies the second-order necessary conditions for optimality but which is *not* a local optimum.

**Problem 2.** A *convex set*  $A \subseteq \mathbb{R}^n$  is a set such that:

$$(1 - \alpha)x + \alpha x' \in A, \quad \forall x, x' \in A, \quad \forall \alpha \in [0, 1]. \quad (1)$$

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function. Prove that for any  $y \in \mathbb{R}$ , the set:

$$L(y) = \{x \in \mathbb{R}^n : f(x) \leq y\} \quad (2)$$

is a convex set. (Note: the empty set is convex.)

**Problem 3.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be convex and  $C^1(\mathbb{R}^n)$  (it is *continuously differentiable*). Consider the set:

$$H(x) = \{x' \in \mathbb{R}^n : \nabla f(x)^\top (x' - x) \leq 0\}. \quad (3)$$

Fix  $x \in \mathbb{R}^n$  and let  $y = f(x)$ . Prove that  $L(y) \subseteq H(x)$ .

**Problem 4.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a *strictly* convex function (see notes for definition). Prove in this case that:

$$L(f(x)) \cap \{x' \in \mathbb{R}^n : \nabla f(x)^\top (x' - x) = 0\} = \{x\} \quad (4)$$

holds for each  $x \in \mathbb{R}^n$ . That is, the set of values for which  $f$  does not exceed  $f(x)$  and the halfplane with normal  $\nabla f(x)$  at  $x$  intersect in exactly one point:  $x$ .

**Problem 5.** Let  $f$  be  $C^2(\mathbb{R}^n)$  and have a positive definite Hessian. Let  $x \in \mathbb{R}^n$ , and let  $p \in \mathbb{R}^n$  be a descent direction at  $x$ . Prove that there exists  $\alpha > 0$  such that  $f(x + \alpha p) < f(x)$ .

**Problem 6.** The *spectral theorem for real matrices* says that if  $A \in \mathbb{R}^{n \times n}$  is symmetric, then:

1. The eigenvalues of  $A$  are real.
2. The normalized eigenvectors of  $A$  form an orthonormal basis for  $\mathbb{R}^n$ .

Let  $q_1, \dots, q_n$  be the normalized eigenvectors of a symmetric matrix  $A$ . That is, there exist eigenvalues  $\lambda_i$  ( $i = 1, \dots, n$ ), not necessarily distinct, such that:

$$Aq_i = \lambda_i q_i, \quad i = 1, \dots, n. \quad (5)$$

First, show that a consequence of this is that we have an orthogonal eigenvalue decomposition of  $A$ :

$$A = Q\Lambda Q^\top, \quad Q = [q_1 \quad \dots \quad q_n], \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad (6)$$

where  $Q^{-1} = Q^\top$  (i.e.,  $Q$  is an orthogonal matrix), and  $\Lambda$  is diagonal. Second, show that we can write the eigenvalue decomposition in outer product form, i.e.:

$$A = \sum_{i=1}^n \lambda_i q_i q_i^\top. \quad (7)$$

Finally, prove that  $\lambda_i > 0$  ( $i = 1, \dots, n$ ) if and only if  $A$  is positive definite.

**Problem 7.** Let  $S$  be a convex set, let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be in  $C^2(S)$ , and let  $x^* \in S$  be a local minimum of  $f$ . Show that if  $\nabla^2 f(x)$  is positive definite for all  $x \in S$  that  $x^*$  is a global minimum of  $f$  over  $S$ . *Hint:* pick another point  $x' \in S$ , and parametrize the straight line from  $x$  to  $x'$ . Apply the fundamental theorem of calculus twice.

**Problem 8.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be  $C^2$ , let  $b \in \mathbb{R}^n$ , and let  $A \in \mathbb{R}^{n \times n}$  be invertible. Show that the Newton step for  $g(x) = f(Ax + b)$  is the same as the Newton step for  $f(x)$ .