

Standard form of an LP

($\frac{10}{3/27}$)

①

Recall, LP in standard form:

To start solving LPs, we convert them into standard form.
Standard form is a better fit.

minimize $c^T x$ for objective and a budget.

subject to $Ax = b$

$x \geq 0$

We will usually put problems in standard form before applying algorithms for solving LPs. Computer can do it for us, but good to know how to do these transformations by hand.

Any LP can be converted to standard form.

Let's look at the rules:

1) Convert maximization to minimization:

- Just replace $c^T x$ with $-c^T x$.

2) Flip inequality constraints:

- Replace a " \leq " constraint with a " \geq " constraint by multiplying both sides by -1 .

3) Convert to " ≥ 0 " constraints:

- Replace a constraint like " $x \geq 5$ " with a new equality constraint and inequality constraint:
 $x' = x - 5$ and $x' \geq 0$

(2)

4) Eliminate "free variables":

- All variables x_i must be included in the vector inequality $x \geq 0$.
- A variable w/ no inequality constraint applied to it is called a free variable.
- Replace a free variable x_i with one new equality constraint and two new inequality constraints:

$$\left\{ \begin{array}{l} x'_i - x''_i = 0 \\ x'_i \geq 0 \\ x''_i \geq 0 \end{array} \right.$$

5) Convert general inequality constraints to inequality constraints by introducing slack variables:

- A constraint of the form:

$$a_i^T x = \sum_j a_{ij} x_j \leq b_i$$

can be converted to an equality constraint by introducing a new slack variable $s_i \geq 0$: so that:

$$\sum_j a_{ij} x_j + s_i = b_i$$

Comments:

- The way of applying these rules ~~in~~ sequence to reduce an LP to standard form ~~is~~ is nonunique! (Similar to Gaussian elimination.)
- There is also some redundancy among these rules.
- LP ~~standard~~ could be reduced to one of several different standard forms

(3)

- When reducing to standard form, we only ever add new variables. For this reason, we can think of reduction to standard form as a process where we lift the LP to an equivalent LP (in standard form) in a higher-dimensional space.

Example: Let's reduce the following LP to standard form:

$$\text{maximize } -5x_1 - 3x_2 + 7x_3$$

$$\text{subject to } 2x_1 + 4x_2 + 6x_3 = 7$$

$$3x_1 - 5x_2 + 3x_3 \leq 5$$

$$-4x_1 - 9x_2 + 4x_3 \leq -4$$

$$x_1 \geq -2$$

$$0 \leq x_2 \leq 4$$

We'll apply the following operations:

1) convert to minimization:

$$-5x_1 - 3x_2 + 7x_3 \rightarrow 5x_1 + 3x_2 - 7x_3$$

2) notice that x_3 is free: ~~so we add constraints~~

$$\left\{ \begin{array}{l} x_3 = x_4 - x_5 \\ x_4 \geq 0 \\ x_5 \geq 0 \end{array} \right.$$

3) flip the inequality " $x_2 \leq 4$ " to get " $-x_2 \geq -4$ ", introduce new inequality constraint " $x_6 = 4 - x_2$ " to get " $x_6 \geq 0$ " and " $x_6 = 4 - x_2$ "

4) likewise, convert " $x_1 \geq -2$ " to " $x_7 \geq 0$ " and " $x_7 = x_1 + 2$ "

5) introduce slack variables :

$$\left\{ \begin{array}{l} 3x_1 - 5x_2 + 3x_3 \leq 5 \\ -4x_1 - 9x_2 + 4x_3 \leq -4 \end{array} \right.$$

↓

$$\left\{ \begin{array}{l} 3x_1 - 5x_2 + 3x_3 + s_1 = 5 \\ -4x_1 - 9x_2 + 4x_3 + s_2 = -4 \\ s_1 \geq 0 \\ s_2 \geq 0 \end{array} \right.$$

Altogether, we get the LP:

$$\begin{aligned} \text{minimize } & 5x_1 + 3x_2 - 7x_3 \\ & 2x_1 + 4x_2 + 6x_3 = 7 \\ & 3x_1 - 5x_2 + 3x_3 + s_1 = 5 \\ & -4x_1 - 9x_2 + 4x_3 + s_2 = -4 \\ & x_3 - x_4 + x_5 = 0 \\ & x_2 + x_6 = 4 \\ & -x_1 + x_7 = 2 \\ & x_i \geq 0, \quad i = 1, \dots, 7 \\ & s_i \geq 0, \quad i = 1, 2 \end{aligned}$$

Comments:

- The equality constraints are now six in total.
- Ambient dimension of the LP is 9.
- Only three variables affect the cost function.
- Compare with Griva — they get a different form!

Let's rewrite this LP in matrix form; define: (5)

$$c = \begin{bmatrix} 5 & 3 & -7 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \in \mathbb{R}^9$$

Since $x = [x_1, \dots, x_7, s_1, s_2]^T$ it may be the solution we want to

$$A = \begin{bmatrix} 2 & 4 & 6 & & & & & & \\ 3 & -5 & 3 & & & & & & \\ -4 & -9 & 4 & & & & & & \\ & 1 & -1 & 1 & & & & & \\ & 1 & & & 1 & & & & \\ -2 & & & & & 1 & & & \\ & & & & & & 1 & & \end{bmatrix} \in \mathbb{R}^{6 \times 9} \quad b = \begin{bmatrix} 7 \\ 5 \\ -4 \\ 0 \\ 4 \\ 2 \end{bmatrix} \in \mathbb{R}^6$$

$$x = [x_1, \dots, x_7, s_1, s_2]^T \in \mathbb{R}^9$$

This gives:

$$\begin{aligned} & \text{minimize } c^T x \\ & \text{subject to } Ax = b \\ & \quad x \geq 0 \end{aligned}$$

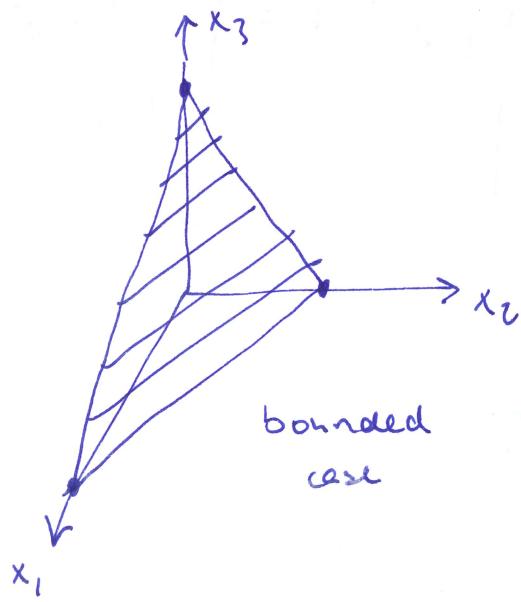
as desired.

We can deduce some properties of the LP from the matrix A.

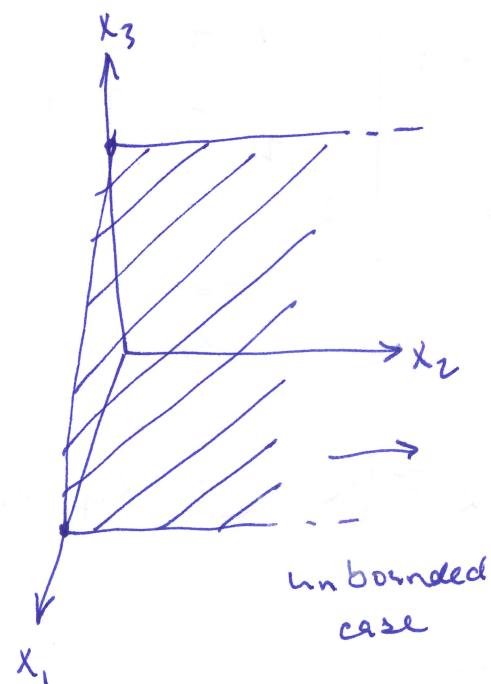
If the dimension of the column space of A is n, then we can solve $Ax=b$ for x. Then there are two cases: $x = A^{-1}b \geq 0$ or not. If so, we have the solution to our LP since the

feasible set is just $\{A^{-1}b\}$. Otherwise, the constraint set is (6)
inconsistent (i.e. no point satisfies all constraints simultaneously).

Otherwise, if $m \leq n$ and if A is full rank, then the dimension of the feasible set is just m . E.g. if $n=3$ and $m=2$, $Ax=b$ and $x \geq 0$ define a polygonal set ~~with~~ in the nonnegative octant:



bounded
case



unbounded
case