HW4 #7:

Let $x^* \in S$ st $\nabla f(x^*) = 0$ and assume the conditions of the problem hold.

WTS: $f(x^*) < f(x) \ \forall x \in S \setminus \{x^*\}$.

Let $\tilde{x} \in S \setminus \{x^*\}$ and define $x(t) = (1-t)x^* + t\tilde{x}$.

(Note that $x(0) = x^*$ and $x(1) = x'$ and $x'(t) = \tilde{x}' - x^*$)

Let $\tilde{f}(t) = f(x(t))$. Then for t st $0 \leq t \leq 1$ (by FTC!):

$$\tilde{f}'(t) = \tilde{f}'(0) + \int_0^t \tilde{f}''(s)\,ds. \qquad (*)$$

But, have:

$$\tilde{f}'(t) = x'(t)^T \nabla f(x(t)) = (\tilde{x} - x^*)^T \nabla f(x(t)) =$$

$$\tilde{f}''(t) = (\tilde{x} - x^*)^T \nabla^2 f(x(t))(\tilde{x} - x^*).$$

Since $\nabla^2 f(x)$ is positive definite for all $x \in S$, by definition can conclude that $\tilde{f}''(t) > 0$ for all t st $0 \leq t \leq 1$. From $(*)$, we can then conclude:

$$\tilde{f}'(t) = (\tilde{x} - x^*)^T \underbrace{\nabla f(x^*)}_{=0} + \int_0^t \underbrace{\tilde{f}''(s)}_{>0}\,ds > 0.$$

OK, Apply FTC one more time:

$$f(\tilde{x}) = f(x(1)) = \tilde{f}(0) + \int_0^1 \underbrace{\tilde{f}'(t)}_{>0}\,dt > \tilde{f}(0) = f(x^*).$$

Since $\tilde{x}$ was arbitrary, and since $[\tilde{x}, x^*] \subseteq S \ \forall \tilde{x}$ by convexity of S, this proves the result. $\blacksquare$