Contractivity of Markov Processes

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Lecture 3

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Advanced Examples

- In the previous examples, the natural metric was sufficient.
- In the absence of uniform convexity, etc., we will have to work a bit harder.
- Often things work one way in the tails, and another way in the bulk.
- We want to interpolate between these two regimes gracefully.

Multiscale Contractions and Hybrid Metrics

- Let $\{\mathsf{d}_i : i \in [N]\}$ be a collection of metrics. Suppose that there exists a matrix $A \in \mathbb{R}_+^{N \times N}$ such that for each pair $x, y \in E$, there exists a coupling of $P(x, \cdot)$ and $P(y, \cdot)$ such that

$$i \in [N] \implies \mathbb{E}\left[\mathsf{d}_i(X_1, Y_1) \mid (X_0, Y_0) = (x, y)\right] \leqslant \sum_{j \in [N]} A_{i,j} \cdot \mathsf{d}_j(x, y).$$

- If A has spectral radius $\rho(A) < 1$, then there exists $v \in \mathbb{R}_+^N$ such that P has positive curvature $\kappa \geqslant 1 - \rho(A)$ with respect to the metric

$$\mathsf{d}^v(x, y) := \sum_{i \in [N]} v_i \cdot \mathsf{d}_i(x, y).$$

- Proof: Perron-Frobenius says that A has a positive leading eigenvector v; try it.

Program

- We will treat some interesting cases where obtaining contraction with a standard metric is not feasible, but where 'multiscale' contraction with respect to a collection of standard metrics is feasible.
- In today's examples, $N = 2$ will generally suffice
  - (though one construction arguably takes $N = \infty$ in some sense)
- We will design our metrics so that e.g. d_1 contracts well in the bulk, and d_2 contracts well in the tails.

Example 1: Warped Euclidean Metric

- Consider again the diffusion process

$$\mathrm{d}X_t = b\left(X_t\right)\,\mathrm{d}t + \mathrm{d}\,W_t,$$

  and assume that b satisfies

$$\|x-y\| = r \quad \Longrightarrow \quad \langle b\left(x\right) - b\left(y\right), x-y\rangle \leqslant -\kappa\left(r\right)\cdot\frac{\|x-y\|^2}{2}$$

  for some $\kappa : [0, \infty) \to (-\infty, \infty)$ which is continuous, bounded at 0, and satisfies $\liminf_{r\to\infty} \kappa(r) > 0$.
- 'distant contractivity', 'strongly convex in the tails', . . .

Example 1: Warped Euclidean Metric

- In this case, synchronous coupling will **not** give contraction.
  - Definitely not in the almost-sure sense, actually not even on average.
- Try the opposite thing: 'reflection coupling'.
  - Roughly: use opposite Brownian motion in direction of $(x - y)$.
- In original metric, this will *also* not give contraction; makes things *worse*.
- The trick: work with *a function of* the original metric.
  - If $f(0) = 0$, f is increasing, concave, then $f \circ \mathrm{d}$ is also a metric.
  - Concave $f \rightsquigarrow$ short-range repulsion 'loses to' reflected diffusion.

Example 1: Warped Euclidean Metric

- Some Ito calculus gives that if $R_t = \|X_t - Y_t\|_2$, then

$$\mathrm{d}f\left(R_t\right) \leqslant 2 \cdot f'\left(R_t\right) \cdot \mathrm{d}W + \left\{-\frac{1}{2} \cdot f'\left(R_t\right) \cdot R_t \cdot \kappa\left(R_t\right) + 2 \cdot f''\left(R_t\right)\right\} \mathrm{d}t,$$

- So, sufficient to find an increasing, concave f so that

$$-\frac{1}{2} \cdot f'(r) \cdot r \cdot \kappa(r) + 2 \cdot f''(r) \leqslant -c \cdot f(r)$$

- From here, explicitly construct such an f for suitable c.
- Roughly, solve the differential inequality with $c = 0$, use to build ansatz for $c > 0$, solve for small r, large r separately (is $\kappa > 0$ or < 0?).

Example 2: Discrete + Weighted Discrete

- Suppose that there exists a Lyapunov function $V : \mathcal{X} \to [0, \infty)$ and constants $\gamma \in (0, 1)$, $K > 0$ such that there holds the drift condition

$$PV(x) \leqslant (1-\gamma) \cdot V(x) + K.$$

- Suppose also that there exists $R > 2 \cdot K \cdot \gamma^{-1}$, $\alpha \in (0, 1)$, and $\nu \in \mathcal{P}(\mathcal{X})$ such that for $x \in \mathcal{C} := \{x : V(x) \leqslant R\}$, there holds the minorisation condition

$$P(x, \mathrm{d}y) \geqslant \alpha \cdot \nu(\mathrm{d}y).$$

- In the tails, you drift inwards. In the bulk, you couple uniformly easily.
- ‘Meyn and Tweedie’-style condition.

Example 2: Discrete + Weighted Discrete

- Write d_1 for the discrete metric, and d_2 for the V-weighted discrete metric.
- A case-by-case analysis shows that

$$\mathbb{E}\left[\mathsf{d}_1(X_1, Y_1) \mid (X_0, Y_0) = (x, y)\right] \leqslant (1-\alpha) \cdot \mathsf{d}_1(x, y) + \frac{\alpha}{R} \cdot \mathsf{d}_2(x, y) \tag{1}$$

$$\mathbb{E}\left[\mathsf{d}_2(X_1, Y_1) \mid (X_0, Y_0) = (x, y)\right] \leqslant 2 \cdot K \cdot \mathsf{d}_1(x, y) + (1-\gamma) \cdot \mathsf{d}_2(x, y) \tag{2}$$

- Conclude by multiscale contraction lemma, studying eigenvalues of 2×2 matrix.

Example 3: Euclidean + Discrete

- Let P be a Markov kernel on a metric space (E, d) of finite diameter D.
- Suppose that there exist $r > 0$, $\alpha \in (0, 1)$, $\kappa \in (0, 1)$ such that

$$\begin{aligned} \mathsf{d}(x, y) \leqslant r &\implies \mathrm{TV}(\delta_x P, \delta_y P) \leqslant 1 - \alpha \\ \mathsf{d}(x, y) > r &\implies \mathcal{T}_{1,\mathsf{d}}(\delta_x P, \delta_y P) \leqslant (1 - \kappa) \cdot \mathsf{d}(x, y) . \end{aligned}$$

- When far apart, chains contract well spatially. When close, chains can coalesce exactly with good probability.

Example 3: Euclidean + Discrete

- Write d_1 for the discrete metric, and d_2 for the original metric d.
- Again, a case-by-case analysis shows that

$$\mathbb{E}\left[d_1(X_1, Y_1) \mid (X_0, Y_0) = (x, y)\right] \leqslant (1-\alpha)\cdot d_1(x, y) + \frac{\alpha}{r}\cdot d_2(x, y) \qquad (3)$$

$$\mathbb{E}\left[d_2(X_1, Y_1) \mid (X_0, Y_0) = (x, y)\right] \leqslant D\cdot d_1(x, y) + (1-\kappa)\cdot d_2(x, y) \qquad (4)$$

- Again, conclude by multiscale contraction lemma.
- Can replace ‘finite diameter’ with ‘bounded moves’ condition.

Example 4: Modified Euclidean + V-Discrete

- Return to the diffusion process

$$\mathrm{d}X_t = b\left(X_t\right)\,\mathrm{d}t + \mathrm{d}\,W_t\,,$$

now assuming that b satisfies

$$\|x-y\| = r \qquad \Longrightarrow \qquad \left\langle b\left(x\right) - b\left(y\right), x-y\right\rangle \leqslant -\kappa\left(r\right)\cdot\frac{\|x-y\|^2}{2}$$

for some positive $\kappa : [0,\infty) \to (-\infty,\infty)$.

- Note that there is **no** assumption of contractivity, only ‘bounded repulsion’.
- Assume also ‘inward drift’ in the form

$$\mathcal{L}\,V \leqslant -\gamma\,V + K$$

for positive, coercive V, $\gamma > 0$, $K > 0$.

Example 4: Modified Euclidean + V-Discrete

- In the tails, the drift condition will contract a weighted discrete metric.
- In the bulk, if we use a reflection coupling, then the bounded repulsion condition will allow us to contract a modified Euclidean metric.
- Take d_1 as a modified Euclidean metric (design warping carefully).
- Take d_2 as the V-weighted discrete metric.
- Careful analysis (again, splitting based on value of r to design f) shows that $\mathrm{d}_1, \mathrm{d}_2$ can be made to jointly contract.

Some Other Examples

- Discrete-Weighted Discrete Contractivity for Feynman-Kac Semigroups.
- Discrete-Euclidean Contractivity for Switched Markov Processes, PDMPs.
- Modified Euclidean Contractivity for Particle Systems, Propagation of Chaos.
- . . .

Closing Thoughts

- Many Markov chains have positive curvature in some metric.
- Knowing that a Markov chain has positive curvature in some metric tells you a lot about the convergence behaviour of the chain.
- These insights are enabled by Optimal Transport theory.
- Establishing that ‘nice’ Markov chains have positive curvature can be very clean.
- Establishing that ‘tough’ Markov chains have positive curvature requires more care and creativity, but can be highly rewarding.
- For many relevant Markov chains, good curvature estimates are not yet known.
- There is much work yet to be done!