Spectral and Metric Ergodicity in Markov Chains

Motivational Problem

‘Geometric Methods in Probability’
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Setup / Notation

- Let $\mathcal{X}$ be a well-behaved state space
- Let π be a probability measure on $\mathcal{X}$
- Let P be a Markov kernel on $\mathcal{X}$ which is π-reversible, i.e.

$$\pi(x) \cdot P\left(x, y\right) = \pi\left(y\right) \cdot P\left(y, x\right)$$

  and '*reasonably ergodic*' (details to follow).

- Mostly fine to think of everything as finite.

General Questions

- When I am studying Markov chains, I am usually interested in …
  - quantifying convergence to equilibrium
  - convergence of averages along paths
  - concentration phenomena
  - non-asymptotic results
- There are *many* approaches to each of these questions

"Geometric Methods in Probability"

… *Geometric* Ergodicity?

Metric Ergodicity

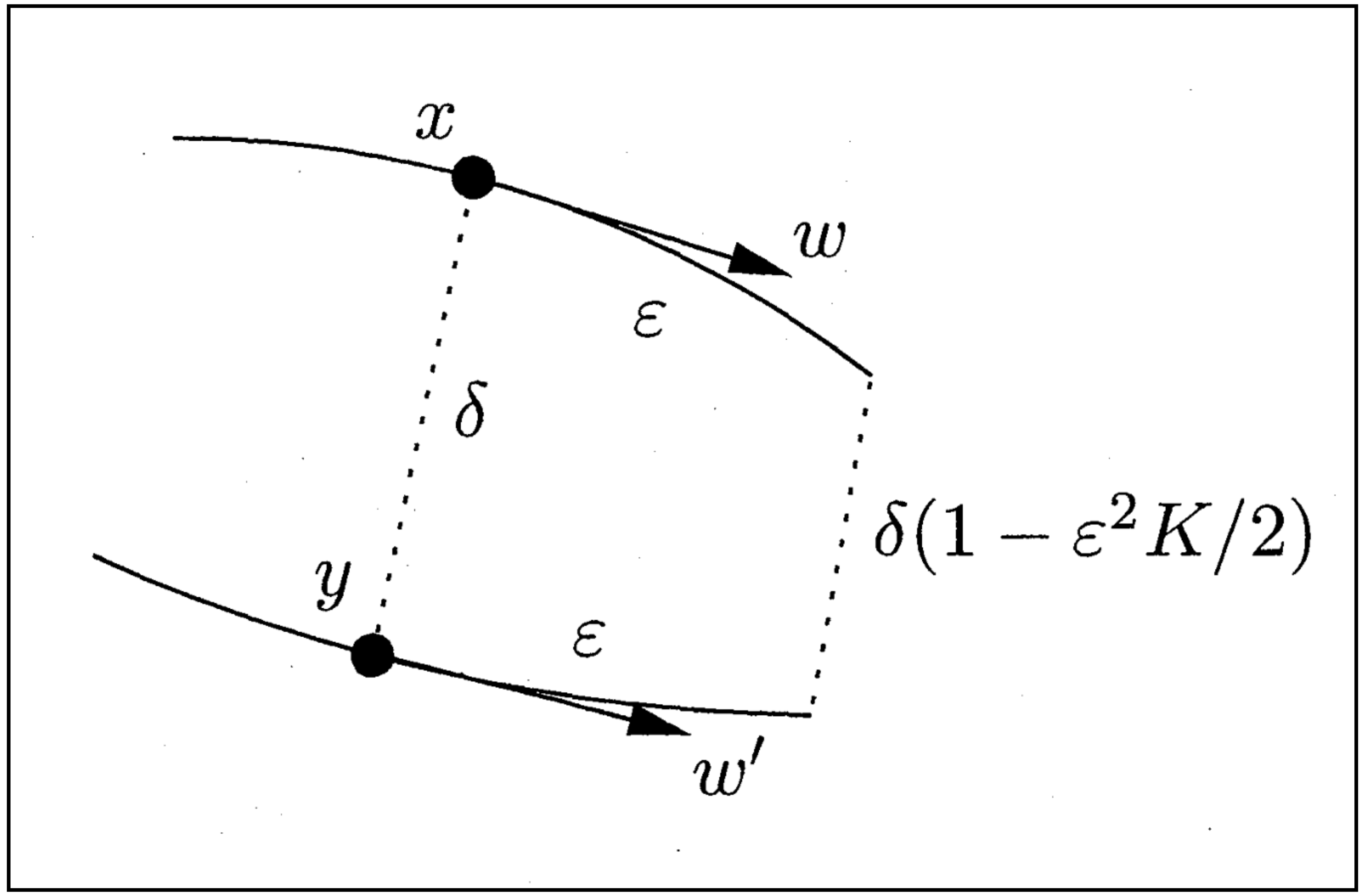
- One nice notion of convergence is ‘*metric*’ or ‘*transport*’ convergence
  - Given a metric d on $\mathscr{X}$, say that P has *curvature* $\kappa_{\mathsf{d}} \in (0,1]$ if

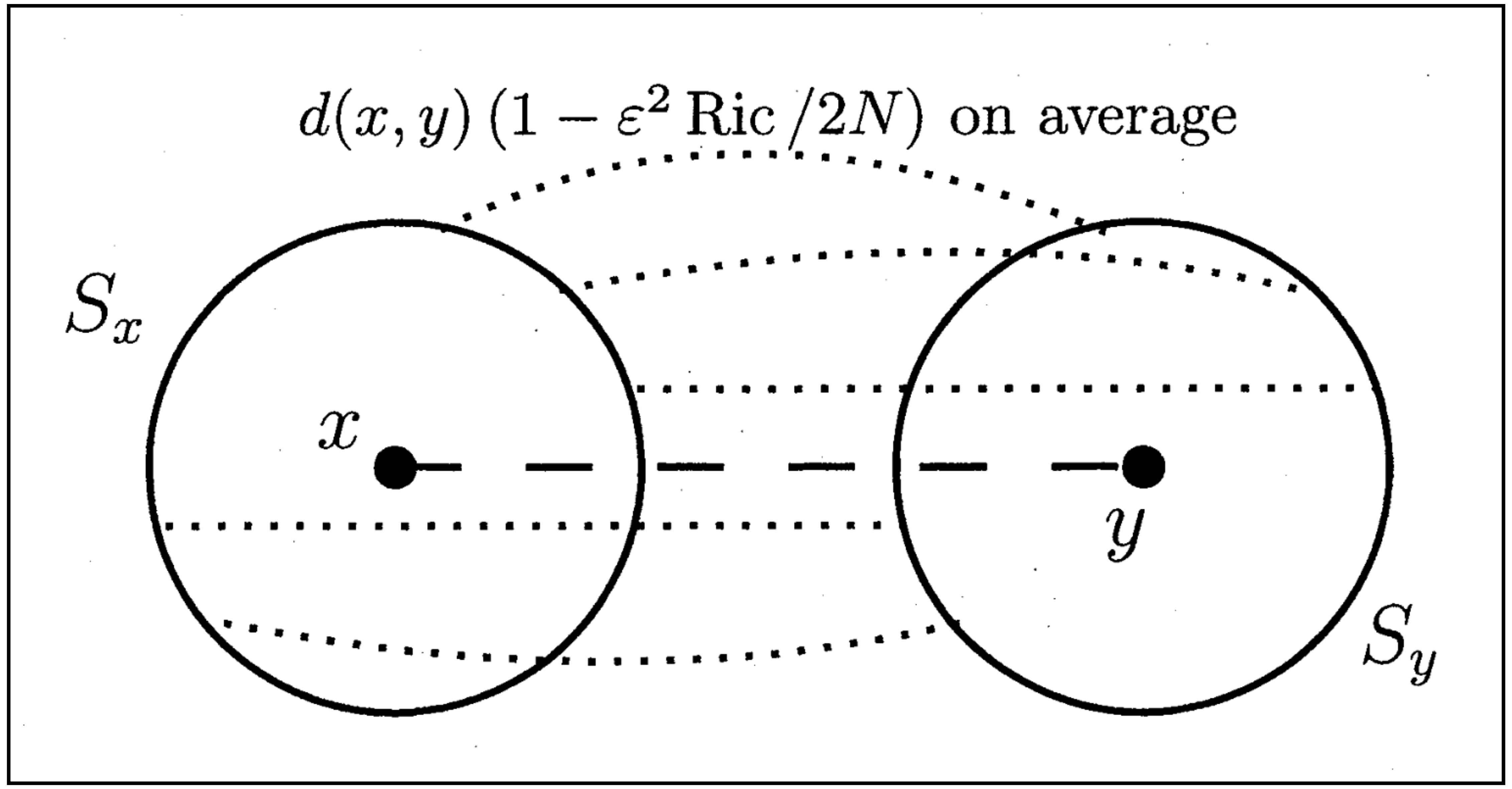
$$\mathbf{E}\left[\mathsf{d}\left(X_1, Y_1\right) \mid X_0 = x, Y_0 = y\right] \leq \left(1 - \kappa_{\mathsf{d}}\right) \cdot \mathsf{d}\left(x, y\right)$$

under a suitable coupling of $P(x, \cdot\ )$ and $P\left(y, \cdot\ \right)$

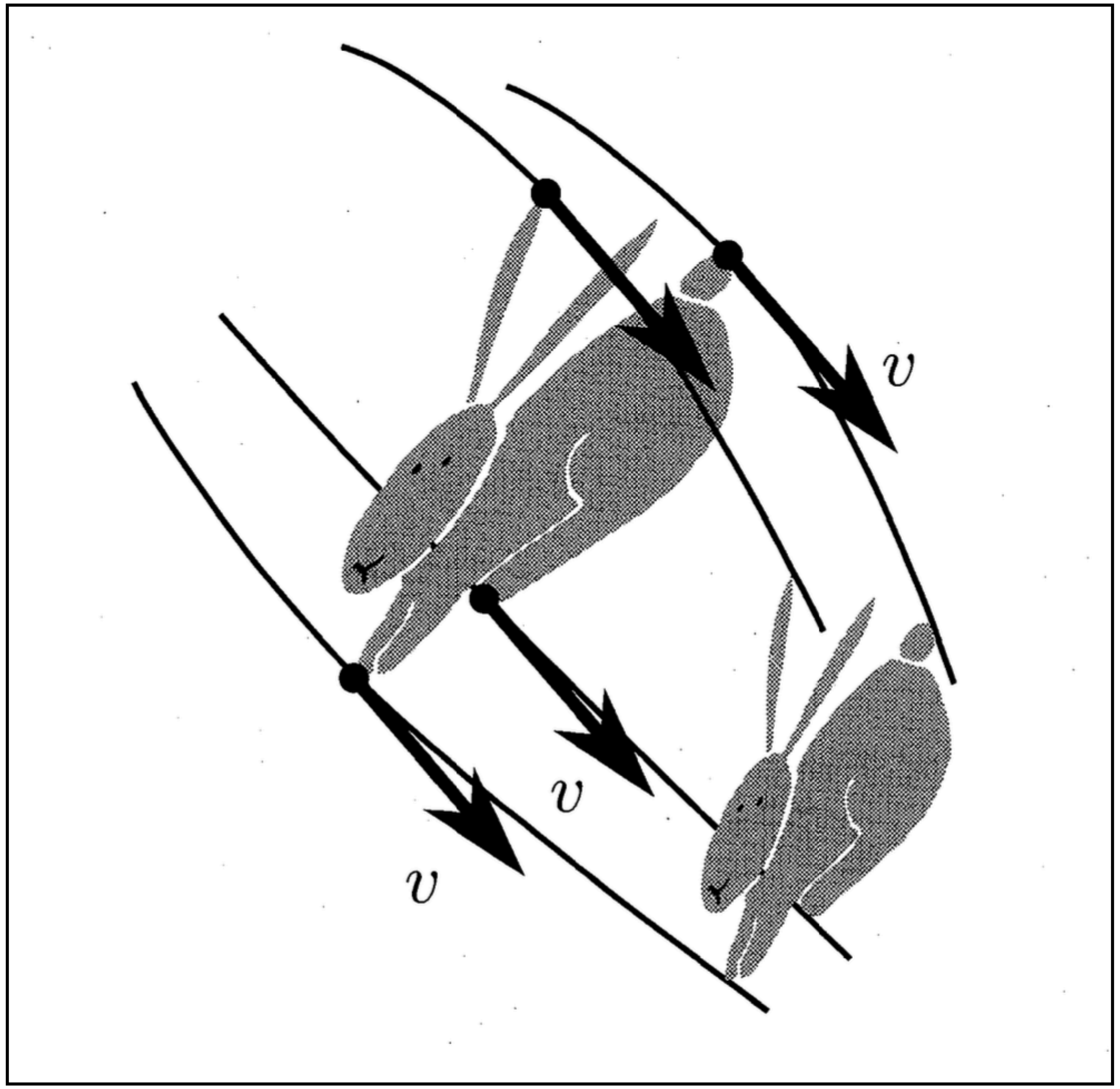
  - ‘dynamics are *contractive on average* under the semigroup’

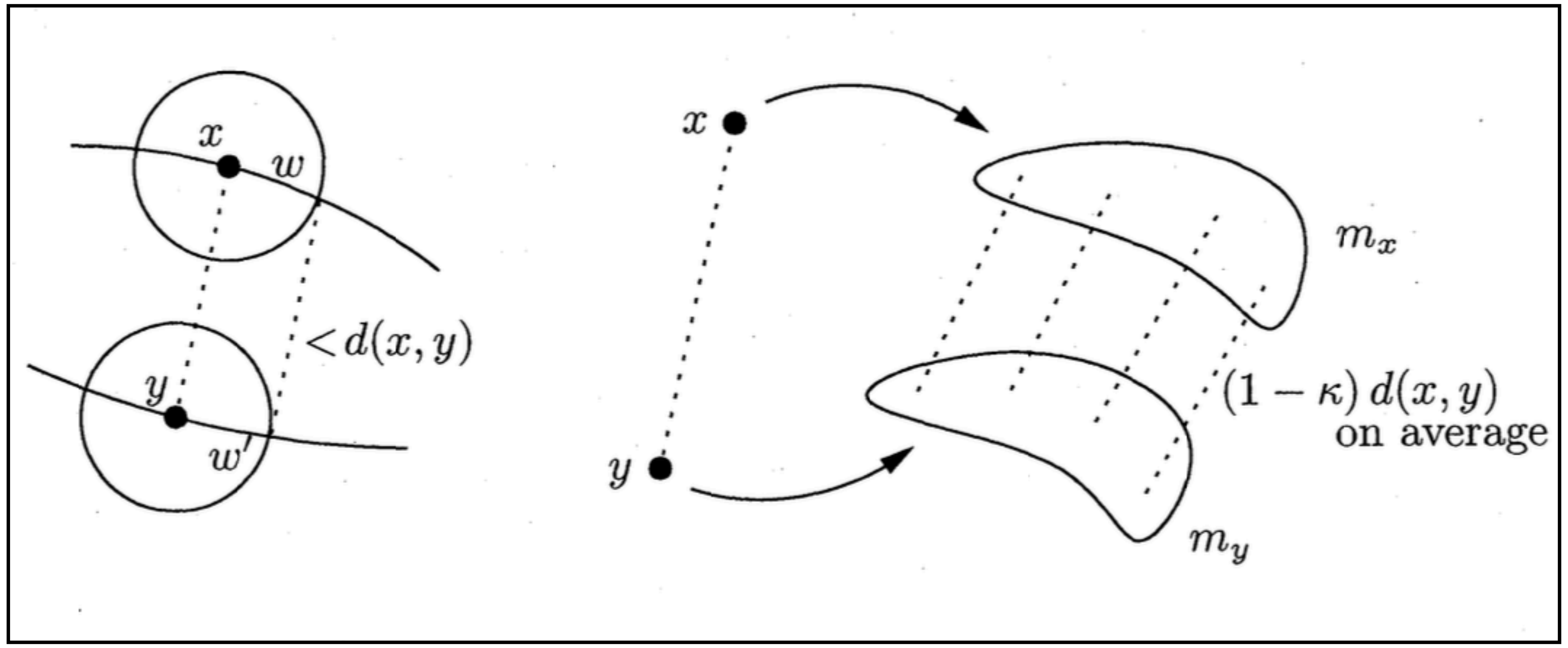
Visualising Curvature

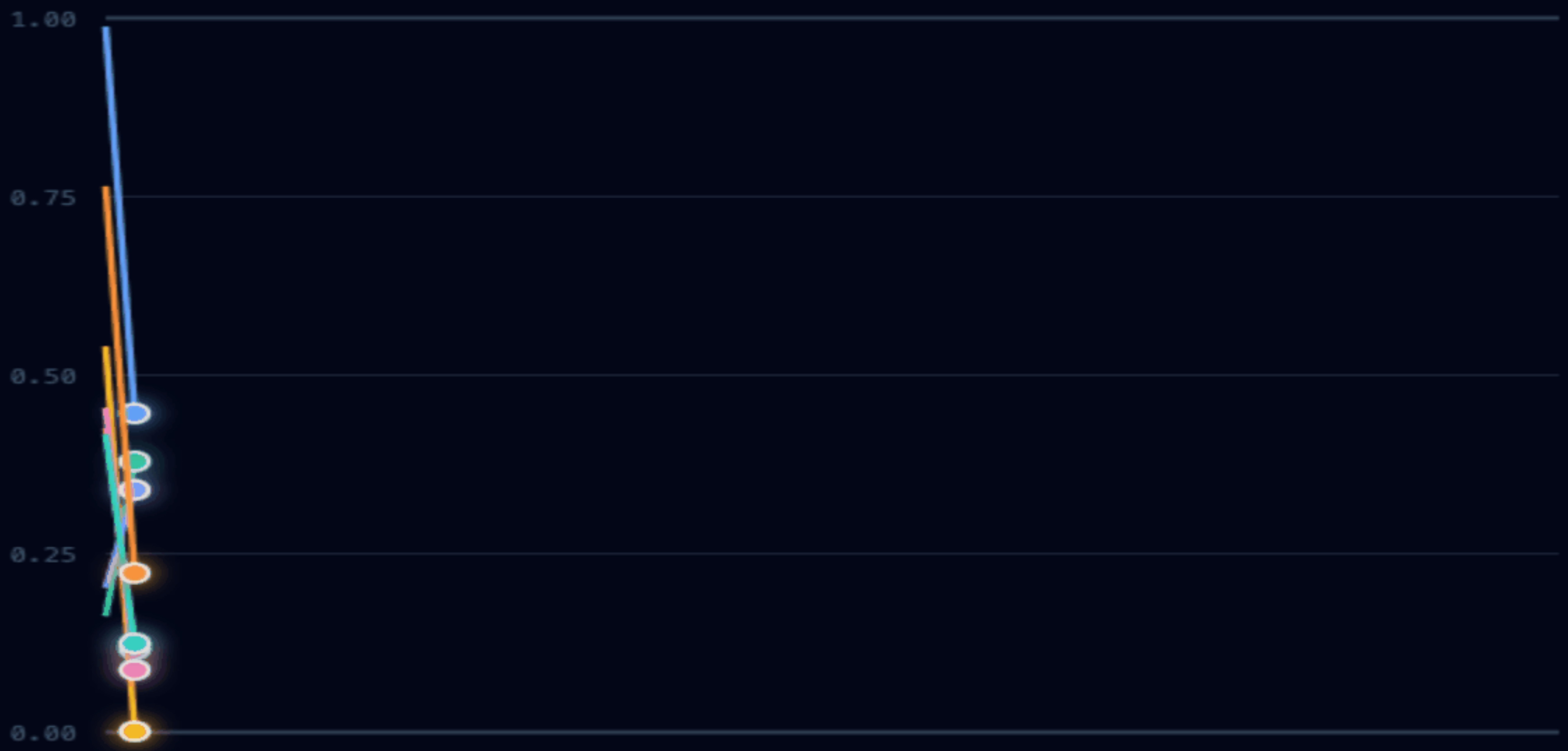




Visualising Curvature



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Metric Ergodicity

- Some consequences:
  - Exponential convergence to equilibrium in ‘transport distance’
  - Path-space concentration inequalities for d-Lipschitz functions
  - (Much more!)

A Word on Metrics

- Lots of metrics are worth considering in this context
    - { Discrete, Cayley, Graph, Hamming, Weighted Discrete, Euclidean, ‘Warped’ Euclidean, ‘Hybrid’ / ‘Multiscale’, … }
    - The freedom to choose your own metric can be *essential*
        - (This point will return)

Spectral Ergodicity

- A fairly robust notion of convergence is '*spectral*' or 'L^2' convergence
  - Given $f : \mathscr{X} \to \mathbf{R}$, define $Pf(x) = \mathbf{E}\left[f\left(X_1\right) \mid X_0 = x\right]$
  - Jensen's inequality shows that if $\mathbf{E}_\pi\left[f(X)\right] = 0$, then

$$\mathbf{E}_\pi\left[\left|Pf(X)\right|^2\right] \leq \mathbf{E}_\pi\left[\left|f(X)\right|^2\right]$$

  - 'variance is *non-expansive* under the semigroup'

Spectral Gap

- In general, for centred f, it holds that

$$\mathbf{E}_\pi\left[\left|Pf(X)\right|^2\right] \leq \mathbf{E}_\pi\left[\left|f(X)\right|^2\right]$$

- Say that P has a *spectral gap* of $\gamma \in (0,1]$ if one even has

$$\mathbf{E}_\pi\left[\left|Pf(X)\right|^2\right] \leq \left(1-\gamma\right) \cdot \mathbf{E}_\pi\left[\left|f(X)\right|^2\right]$$

- ‘variance is *contractive* under the semigroup’

Spectral Ergodicity

- Some consequences:
  - Exponential convergence to equilibrium in ‘chi-squared divergence’
  - Path-space concentration inequalities for bounded functions
  - (Much more!)

Comparing the Two

- Roughly speaking,
    - Positive curvature <u>implies</u> many other forms of convergence
    - Positive spectral gap <u>is implied by</u> many other forms of convergence
    - It is useful to be able to translate between the two

$$\kappa_{\mathsf{d}} > 0 \implies \gamma > 0$$

- The following is known:

  If P ‘has finite d-variance’,

  then it follows that

$$\gamma \geq \kappa_{\mathsf{d}}$$

- ‘Metric Ergodicity' **implies** ‘Spectral Ergodicity’, *quantitatively*.

Sketch Proof

- $\kappa \geq 0$ is equivalent to $\| Pf \|_{\mathrm{Lip}} \leq (1 - \kappa) \cdot \| f \|_{\mathrm{Lip}}$
- Unfolding this, see that $P^n f$ is quickly 'very Lipschitz'
- Infer that $\mathbf{E}_\pi \left[\left| P^n f(X) \right|^2 \right] \lesssim (1 - \kappa)^n$
- Using reversibility, control spectrum of P

Reversing the Implication?

- There are many varied ways to prove positive spectral gap.
- For certain questions, positive curvature is particularly useful.
- Is there a chance to go

    from ‘Spectral Ergodicity’

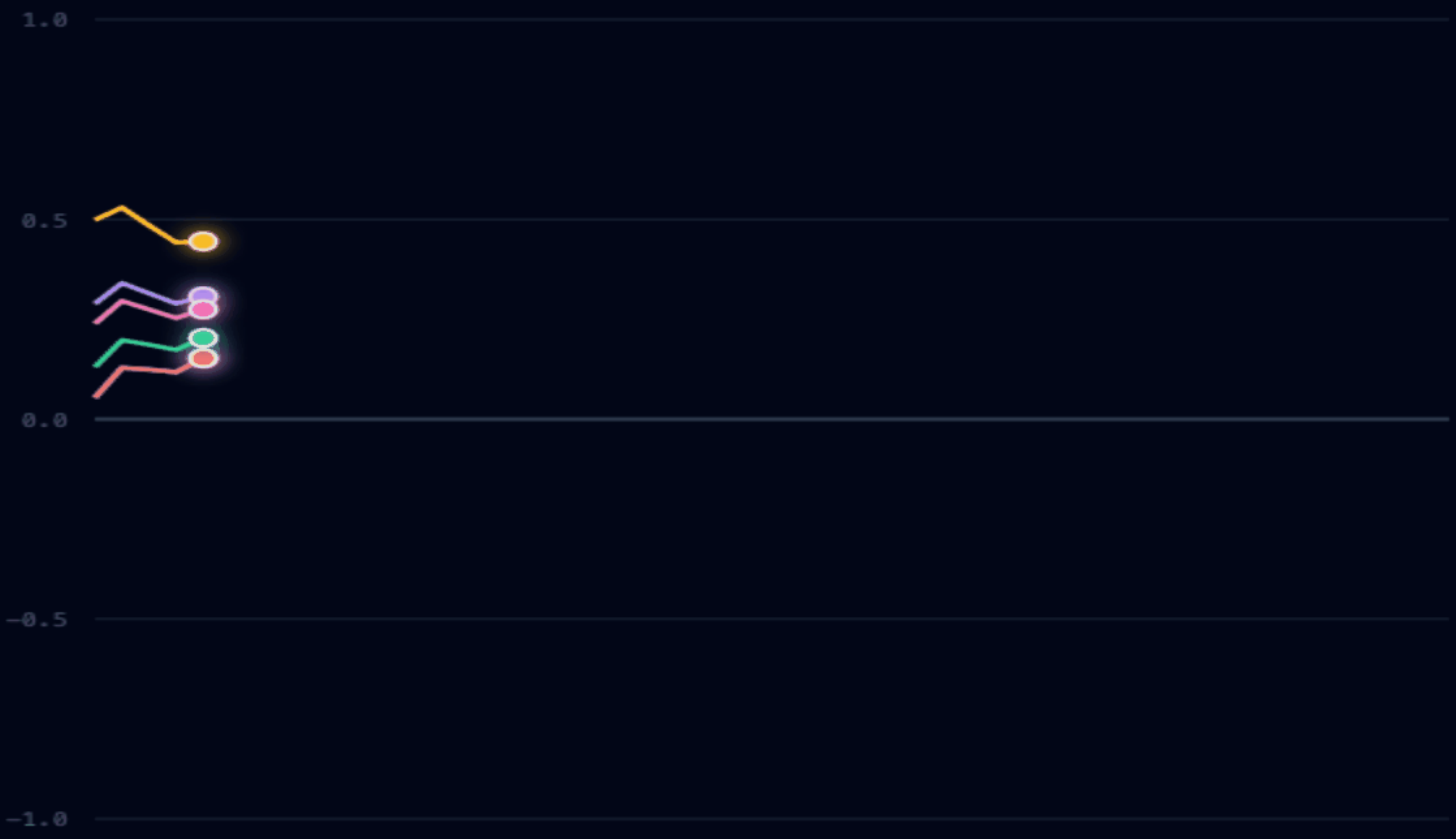
    back to ‘Metric Ergodicity’,

    quantitatively?

Some Challenges

- ‘Spectral Ergodicity’ is specified ‘*intrinsically*’ in terms of (π, P)
- ‘Metric Ergodicity’ is specified ‘*extrinsically*’: needs a metric d
- So, if we are to ‘go backwards’, we need to *select the right metric*
- Moreover, we want the metric to be ‘*nice enough*’
- *Even non-quantitatively*, not obviously possible!

A Moment of Optimism



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Some Encouragement

- For *monotone* Markov chains, the reversal **is** (quantitatively!) possible!
    - Key ingredient: monotonicity suggests a choice of metric
    - Remark: known proofs are ‘*functional*’, rather than ‘*probabilistic*’

Possible Discussion Points

- ‘The dream’ would be to, in general, ‘construct’ d such that $\kappa_{\mathsf{d}} \geq \gamma$
  - Sharp: ‘forward’ result yields that $\kappa_{\mathsf{d}} < \gamma$ is impossible
- Lots of partial results would be interesting, new, and useful, e.g.
  - confirm that $\kappa_{\mathsf{d}} > 0$ is possible in general
  - identify other ‘structured’ classes of P for which $\kappa_{\mathsf{d}} \geq \gamma$
- (also, *might not be true*! so negative results are also informative)

In a Slide

- '**Spectral Ergodicity**' is a rather **weak** form of exponential ergodicity
- '**Metric Ergodicity**' is a rather **strong** form of exponential ergodicity
- Control of κ_{d} for any (!) d implies strong control of γ
- Given control of γ, can we control κ_{d} for <u>some</u> '*well-chosen*' d?