Weak Poincaré Inequalities

Markov chain convergence

$(P, \mu) : \mu_0 P^n \longrightarrow \mu$

L^2 convergence

Let $f \in L^2(\mu)$, $\mu(f) = 0$

$\|P^n f\|_2^2 \longrightarrow 0$ as $n \to \infty$

$(P^n f)(x) = \mathbb{E}[f(X_n) | X_0 = x]$

$\|f\|_2^2 = \mu(f^2)$

Best case: $\|P^n f\|_2^2 \leq (1-\gamma)^n \|f\|_2^2$

exponential decay

Grönwall's inequality: $\frac{dx}{dt} \leq -\lambda x$

$\Longrightarrow x(t) \leq x(0) e^{-\lambda t}$

This decay holds if

$\forall f \quad \|f\|^2 - \|Pf\|^2 \geq \gamma \|f\|^2 \quad (*)$

For a MK P w/ inv. meas μ.

$\rightarrow \mathcal{E}(P, f) = \langle f, (I-P) f \rangle_{L^2(\mu)}$

$(*)$: $\mathcal{E}(P^*P, f) \geq \gamma \|f\|_2$

"Poincaré Inequality"

$$\mu(x)P(x,y) = \mu(y)P^*(y,x)$$

What if no PI?

(can only take $\gamma = 0$)

might want $\|P^n f\|^2 \leq \gamma(n)\|f\|^2$

If $\gamma(n_0) < 1$

$\|P^{n_0} f\|^2 \leq \gamma(n_0)\|f\|^2$

$\Longrightarrow$ can't be the case

- could do:

let $\Phi : L^2(\mu) \longrightarrow \mathbb{R}$

which is "norm-like" square

$\Phi(f) \geq 0$

$\Phi(cf) = c^2 \Phi(f)$ "sieve"

$\Phi(Pf) \leq \Phi(f)$

for us: usually $\|\cdot\|_{osc}^2$

Goal: $\|P^n f\|_2^2 \leq \gamma(n)\Phi(f)$

Rmk: Possible to hold in

: Possible ... in nontrivial ways.

Weak Poincaré Inequality

PI for P: $\mathcal{E}(P, f) \geq \gamma \, |f|_2^2$

WPI for P:

for $u > 0$,

- $\mathcal{E}(P, f) \geq u \cdot |f|_2^2 - K(u) \cdot \Phi(f)$
- for $r > 0$, $|f|_2^2 \leq \alpha(r) \, \mathcal{E}(P, f) + r \, \Phi(f)$
- for $s > 0$, $|f|_2^2 \leq s \cdot \mathcal{E}(P, f) + \beta(s) \, \Phi(f)$

- $\mathcal{E}(P, f) \geq u |f|^2 - K(u) \, \Phi(f)$

$$\frac{\mathcal{E}(P, f)}{\Phi(f)} \geq u \frac{|f|^2}{\Phi(f)} - K(u)$$

$$\sup_u \{ uv - K(u) \} =: K^*(v)$$

$$\Longrightarrow \quad \frac{\mathcal{E}(P,f)}{\Phi(f)} \geq K^*\left(\frac{|f|^2}{\Phi(f)}\right)$$

If $$\frac{\mathcal{E}(P^*P,f)}{\Phi(f)} \geq K^*\left(\frac{|f|_2^2}{\Phi(f)}\right)$$

then $$\|Pf\|_2^2 \leq \Phi(f)\left(\mathrm{id}-K^*\right)\left(\frac{|f|_2^2}{\Phi(f)}\right)$$

iterating this + using $\Phi(Pf) \leq \Phi(f)$,

$$\|P^n f\|_2^2 \leq \Phi(f)\left(\mathrm{id}-K^*\right)^{\circ n}\left(\frac{|f|_2^2}{\Phi(f)}\right)$$

⋮ basic calculus comparisons

$$\leq \Phi(f) \cdot \underbrace{\gamma(n)}_{\text{rate of decay}}$$

$\approx$ decay of K^* at 0

If convergence is fast

α is small

β is small $\{K(u) = u\beta(\frac{1}{u})\}$

K is small

K^* is big

$$V_n = \|P^n f\|_2^2 / \mathcal{E}(f)$$

$$V_n \leq V_{n-1} - K^*(V_{n-1})$$

pretend this is an ODE

$$F(x) = \int_x^1 \frac{dv}{K^*(v)} \cdots$$

$$\|f\|_{osc} = \max f - \min f$$

Suppose that P is μ-rev.,
and $\forall f$, $\|P^n f\|_2^2 \leq \gamma(n) \mathcal{E}(f)$ explicit

Then, $\exists$ α (or β, or K, ...)
s.t. P^*P satisfies a WPI..
