PDMPs with ODE Dynamics

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Overview

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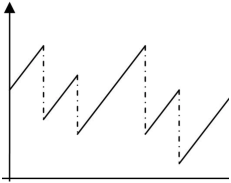

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    - Dictates how often events happen (inhomogeneous Poisson process)
  - Transition dynamics $Q(z \to dz')$
    - Dictates what happens at events (Markov jump kernel)

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- **Question**:

$$\text{Given target measure } \mu, \text{ vector field } \phi, \tag{1}$$
$$\text{how can I build } (\lambda, Q) \text{ to sample } \mu\,? \tag{2}$$

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  - Jump by flipping i^{th} velocity

Aside on Reversibility, Symmetry

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  - Much MCMC work built on reversible methods
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  - To design algorithms, *locality* is the important part
- Symmetry
  - Existing PDMPs are highly symmetric (BPS, ZZ)
  - A priori, not necessary to have symmetry
  - Want to be able to use *all* ODEs!

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  - 'in at z forwards in time = out at z backwards in time'

Choice of Event Rate (1)

- Consider 'probability current'

$$r(z,\tau) \triangleq \underbrace{\langle \nabla H(z), \phi(z,\tau) \rangle}_{\text{Energy Gain}} \quad - \quad \underbrace{\mathrm{div}_z \phi(z,\tau)}_{\text{Compressibility Penalty}} \tag{4}$$

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  - $\rightsquigarrow$ Choose $q^\tau(z \to dz')$ to be J^τ-reversible

Putting together the ingredients

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and for $\tau \in \{\pm 1\}$, Q^τ *is* J^τ*-reversible*

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- Comments on proof

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- Key point: Different event types correspond to *decompositions* of r

Split PDMPs (2)

- $z = (z_1, \cdots, z_D)$, $\tau = (\tau_1, \cdots, \tau_D) \in \{\pm 1\}^D$
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- Events of type j happen at rate $\lambda_j(z, \tau)$
  - and then jump according to $Q_j^{\tau}(z \to dz') \cdot \delta(\mathcal{F}_j(\tau), d\tau')$

Making Split-PDMPs work (1)

- Define

$$\lambda_j^0(z, \tau) = (r_j(z, \tau))_+ \tag{10}$$

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Given a fixed splitting, all trajectorially-reversible, $\tilde{\mu}$-stationary Split PDMPs take this form.

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- Choosing ϕ: some room for creativity here.

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- *Speculation*: Better dynamics $\phi \rightsquigarrow$ opportunities
- *Curiosity*: Other types of augmentation?

Thank you!