MCMC with Expensive Likelihoods

- Bayesian problems where $p(y|\theta)$ is expensive
  - PDE-based inverse problems
  - MJPs, SDEs, ODEs
  - large data
- $\exists$ "surrogate"/"approximate" likelihood $\hat{p}(y|\theta)$

notation: $\pi(x)$ real target

$\hat{\pi}(x)$ surrogate target

$\beta(r) = \min(1, r)$

ideal: MCMC targeting π, low cost/accepted step of reasonable size

Alg 1 MH(π, q)

at x, $y \sim q(x \to y)$

$$\alpha = \beta\left(\frac{\pi(y)\, q(y \to x)}{\pi(x)\, q(x \to y)}\right)$$

wp α, move to y

else stay at x

$\Longrightarrow$ π-rev MC

π-rev MC $\Longleftarrow$

Alg 2 DAMH$(\pi, q, \hat{\pi})$

at x, $y \sim q(x \to y)$

$$\alpha_1 = \beta\left(\frac{\hat{\pi}(y)\, q(y \to x)}{\hat{\pi}(x)\, q(x \to y)}\right)$$

wp α_1 proceed to stage 2

else stay at x

stage 2: $\alpha_2 = \beta\left(\frac{\pi(y) / \hat{\pi}(y)}{\pi(x) / \hat{\pi}(x)}\right)$

wp α_2, move to y

else stay at x

Alg 3 $STM(\pi, \hat{q}, \hat{\pi})$ ($\hat{q}$ is $\hat{\pi}$-reversible)

at x, $y \sim \hat{q}(x \to y)$

$$\alpha = \beta\left(\frac{\pi(y)/\hat{\pi}(y)}{\pi(x)/\hat{\pi}(x)}\right)$$

wp α, move to y

else stay at x

move $\sim k^{\frac{1}{2}}$ as far

cost/iteration $\sim k \cdot cost(\hat{\pi}) + cost(\pi)$

- Alg 2 is a case of this where $\hat{q} = MH(\hat{\pi}, q)$
- generalisation $\hat{q} = MH(\hat{\pi}, q)^k$: for moderate k, (hope) $MH(\hat{\pi}, q)^k \approx MH(\pi, q)^k$

if $\pi, \hat{\pi}$ are p.m.s, $\hat{q}$ is $\hat{\pi}$-rev,
then $STM(\pi, \hat{\pi}, \hat{q})$ is π-rev

$\Longrightarrow$ recursive implementation of STM

π_0 cheapest surrogate

π_L finest model/true model

$\Longrightarrow$ hierarchy of models $\{\pi_\ell\}_{\ell=0}^{L}$

want a π_L-reversible MK

want, $\forall l$, MK q_l which is π_l-reversible

define $q_0 = \mathrm{RWMH}(\pi_0)$

for $l = 1, \dots, L$,

define $q_l = \mathrm{STM}(\pi_l, \pi_{l-1}, q_{l-1}^{N_{l-1}''})$ $N_{l-1} \geq 1$

$\Rightarrow$ q_L is π_L-reversible

Pf lemma from p5 + recursion

- c.f. Multigrid, MGMC (Goodman-Sokal, Liu-Sabatti)
- c.f. tempered transitions

Convergence

① if $c_1 \leq \frac{\pi(x)}{\hat{\pi}(x)} \leq c_2$, pay a constant price

② if tails of $\log \pi(x) - \log \hat{\pi}(x)$ are sub-exponential, then converge at polynomial rate

③ in STM, as $k \to \infty$, behave like IMH

④ variance bounding, geometric ergodicity.

$\exists$ optimal scaling

. Multilevel MCMC ⇝ Multilevel DA-MCMC + AEM refining $\hat{\pi}$