Comparison Theorems for Slice Sampling

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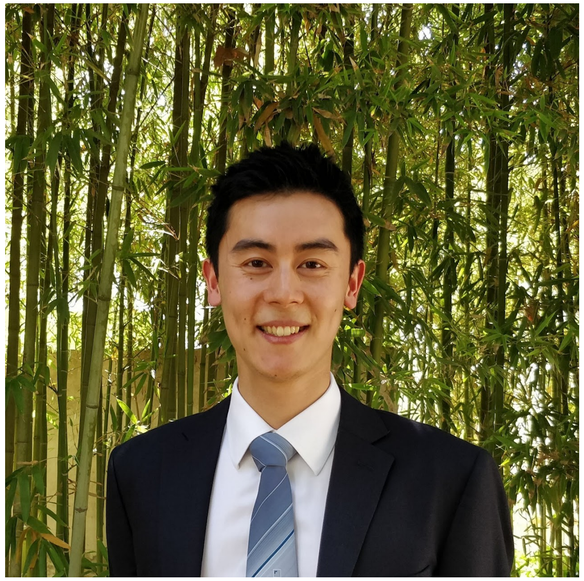


Some personal background

- Sam Power @ University of Bristol
  - Lecturer in Statistical Science here since January 2024
  - Postdoc with Christophe Andrieu, Anthony Lee here since August 2020
- Interested in { Modelling, Computation, Analysis, Probability, … }
  - often Markov Processes (Long-Time Behaviour, Comparisons)

Plan for today

- a sampling problem
- a cute reduction / trick
- a Markov chain
- a convergence analysis (not ours)
- a refined analysis (ours)
- some illustrative examples

feel free to stop me at any point

An Elementary Trick

- let $\pi \ll \mathrm{Leb}$; write $\pi\left(\mathrm{d}x\right) = \pi\left(x\right)\,\mathrm{d}x$
- observe that

$$\pi\left(x\right) = \int_{\mathbf{R}_+} \mathbf{1}\left[\pi\left(x\right) > t\right]\,\mathrm{d}t$$

- introduce

$$\Pi\left(\mathrm{d}x, \mathrm{d}t\right) = \mathbf{1}\left[\pi\left(x\right) > t\right]\,\mathrm{d}x\mathrm{d}t$$

- if you ‘solve’ Π, then you ‘solve’ π

A Trade

- if you ‘solve’ Π, then you ‘solve’ π
- there ought to be *some* ‘conservation of difficulty’, at some level
- general density on a simple set $\rightsquigarrow$ uniform density on a general set
- at least *some* tasks ought to be easier in this world

‘Vertical Likelihood’

- in some circles, this is known as the ‘vertical likelihood representation’
- one can build various methods on the back of this representation
  - { Accept / Reject, Nested Sampling, Wang-Landau, MUCA, …}
- today: ‘Markov chain Monte Carlo’ on Π

Markov chain Monte Carlo

- "target" distribution π on $\mathbf{R}^d$
- want samples from π to answer questions
- MCMC: use *iterative* strategy to obtain *approximate* samples

$$X_0 \to X_1 \to X_2 \to \cdots \to X_T \overset{d}{\approx} \pi$$

$$\frac{1}{T} \sum_{0<t\leqslant T} f(X_t) \approx \int \pi(\mathrm{d}x) f(x) =: \pi(f)$$

Coming up with a Markov chain

- $\Pi\left(\mathrm{d}x, \mathrm{d}t\right) = \mathbf{1}\left[\pi\left(x\right) > t\right]\ \mathrm{d}x\mathrm{d}t$
- (x, t) has a natural bipartite structure
- simple option: alternating conditional simulation

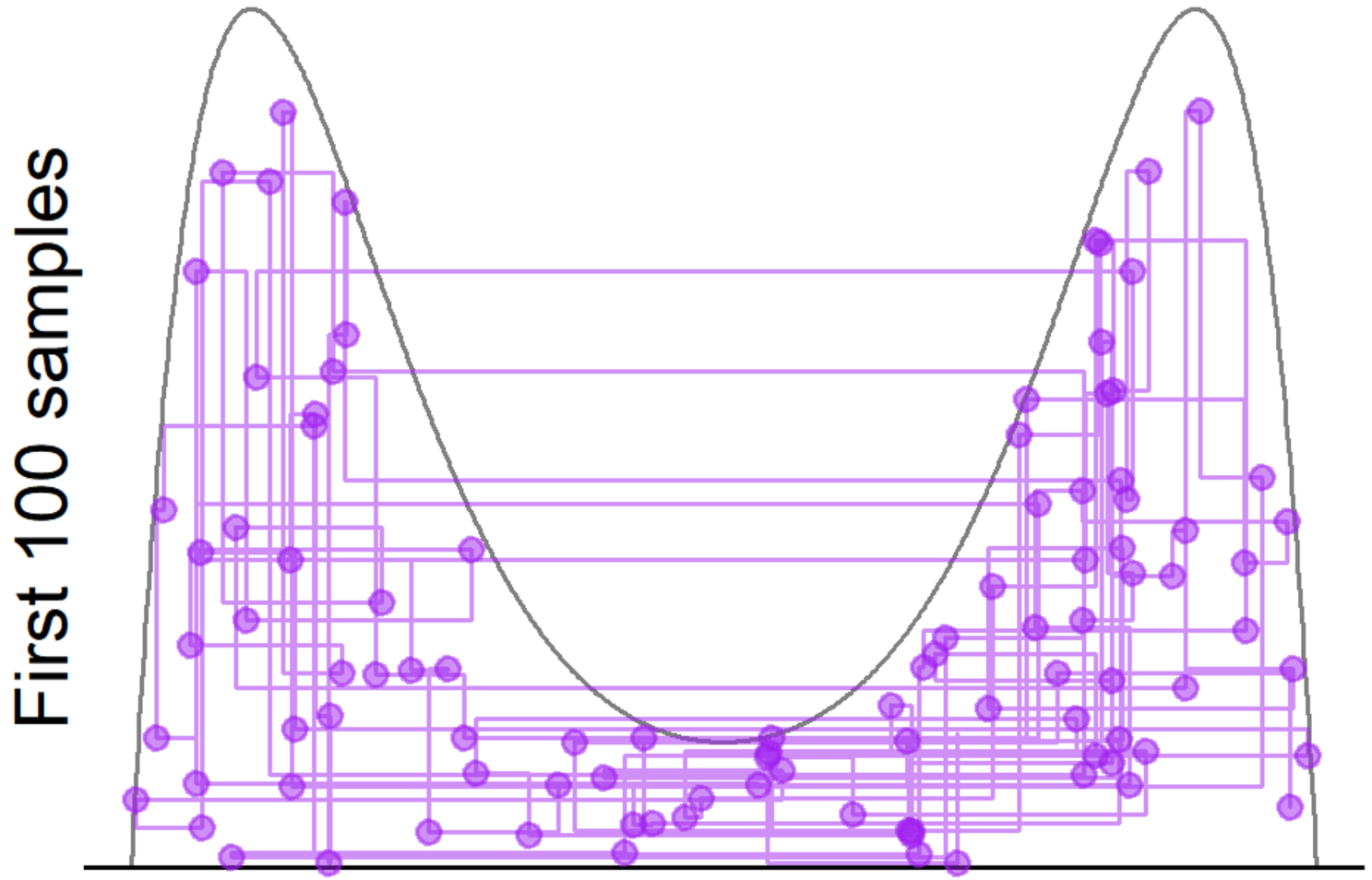
$$\left[T_n \mid X_{n-1} = x\right] \sim \Pi\left(\mathrm{d}t \mid X = x\right)$$

$$\left[X_n \mid T_n = t\right] \sim \Pi\left(\mathrm{d}x \mid T = t\right)$$

- ‘two-component Gibbs sampler’

What do these conditionals look like?

- $\Pi\left(\mathrm{d}t \mid X = x\right)$ is just Unif $\left(\mathrm{d}t; \left[0, \pi(x)\right]\right)$
- $\Pi\left(\mathrm{d}x \mid T = t\right)$ is just Unif $\left(\mathrm{d}x; \mathsf{G}(t)\right)$ …
  - … where $\mathsf{G}(t) = \{x : \pi(x) > t\}$
  - also, will need $m(t) = \mathrm{Vol}\left(\mathsf{G}(t)\right)$, $\nu_t = \text{Unif}\left(\mathrm{d}x; \mathsf{G}(t)\right)$
  - (time to get the chalk out!)



Neal’s ‘Slice Sampler’

Some Connections

- the use of the ‘vertical likelihood’ representation to construct a Markov process has antecedents
  - ‘data augmentation’, imputation strategies for missing data problems
  - ‘random cluster model’ for Ising / Percolation
    - { Fortuin-Kasteleyn / Swendsen-Wang / Edwards-Sokal / … }
    - other { dual / loop-type / … } representations of spin systems

Long-Time Behaviour of Slice Sampling

- under mild conditions, gives an ergodic, π-invariant Markov chain
  - ⇝ fit for purpose in MCMC
- under still mild conditions, is even *exponentially* convergent
  - ⇝ bonus results, e.g. Markov chain CLT
  - surprisingly hard to break

Detailed Study of Dynamics

- for accurate quantitative understanding, need to study the chain carefully
  - we care about the chain $\left(X_n\right)$,
  - but we implement the chain $\left(X_n, T_n\right)$,
  - and we might even prefer to look at $\left(T_n\right)$, which is univariate
- duality / interleaving: all three processes converge equivalently

Features of the Height Process

- the chain $\left(T_n\right)$ depends on π only through the mass function $m\left(t\right)$:

$$P\left(t, t'\right) = \frac{m\left(t''\right)}{t''} - \int_{t''}^{\infty} \frac{m\left(s\right)}{s^2}\,\mathrm{d}s, \qquad t'' = \max\left\{t, t'\right\}$$

- so, we can always study a ‘nicer’ $\widetilde{\pi}$ for which

$$m_{\pi} = m_{\widetilde{\pi}}$$

- WLOG { centered, isotropic, spherically-symmetric, unimodal, … }

The Art of WLOG

- fix a π (hence m), pick a dimension $d \in \mathbf{N}$, and set

$$\phi(r) = \log\left(\frac{1}{m^{-1}\left(r^d\right)}\right),$$

  which is increasing in r

- then, with $\widetilde{\pi}(x) \propto \exp\left(-\phi\left(\|x\|\right)\right)$, we have $m_\pi = m_{\widetilde{\pi}}$
  - $\approx$ symmetric rearrangement; very regular

Convergence by Coupling

- given positions X, X', sample heights by $U \sim \text{Unif}\left(\left[0,1\right]\right)$, then

$$T = U \cdot \pi(X), \quad T' = U \cdot \pi(X')$$

  - match relative heights

- given heights T, T', sample positions by $S \sim \text{Unif}\left(\mathbf{S}^{d-1}\right)$, then

$$X = \phi^{-1}\left(\log\left(\frac{1}{T}\right)\right)^{1/d} \cdot S, \quad X' = \phi^{-1}\left(\log\left(\frac{1}{T'}\right)\right)^{1/d} \cdot S$$

  - match direction and magnitude

Convergence Rates

- for convex ϕ, chain contracts at rate $\leq \frac{d}{d+1}$
- for $\phi(r) = \frac{d+m}{2} \cdot \log\left(1+r^2\right)$, chain contracts at rate $\leq \frac{d \cdot \left(d+m\right)}{\left(d+1\right) \cdot \left(d+m-1\right)}$
- other specific cases can be analysed
- byproduct of coupling construction: chain is *stochastically monotone*

Some General Comments

- increasing dimension slows down convergence
- heavy tails slow down convergence a bit
  - … but not *too* dramatically
- very hard to make the convergence awful
  - by “WLOG”, always equivalent to a ‘pretty reasonable’ π
- impression: method seems robust, potentially algorithmically appealing

Partial Recap

- sampling problem
- vertical likelihood representation
- Slice Sampling
- practical algorithm
  - … ?

Implementing Slice Sampling

$$\text{“}\left[X_n \mid T_n = t\right] \sim \Pi\left(\mathrm{d}x \mid T = t\right)\text{”}$$

– Sam Power, Slide 11

Life on the Slice

- recall that $\Pi\left(\mathrm{d}x \mid T = t\right)$ is $\nu_t = \mathsf{Unif}\left(\mathrm{d}x; \mathsf{G}\left(t\right)\right)$
- if $\mathsf{G}(t)$ is a { ball, box, simplex, … }, then sampling from ν_t is fine
- if not, then we have a new problem

Hybrid Slice Sampling

- instead of
  - "$\left[X_n \mid T_n = t\right] \sim \nu_t$"
- select a ν_t-invariant Markov kernel H_t, and do
  - "$\left[X_n \mid T_n = t, X_{n-1} = x\right] \sim H_t\left(x, \mathrm{d}x'\right)$"

Towards Comparison

- let $U\left(x, \mathrm{d}x'\right)$ denote the “Ideal” Slice Sampling kernel
- let $H\left(x, \mathrm{d}x'\right)$ denote the “Hybrid” Slice Sampling kernel
- how do these kernels *compare* in terms of efficient convergence?

Dirichlet Form Analysis of Markov Chains

- if P is a μ-reversible Markov kernel, then define

$$\mathscr{E}_P\left(f\right) = \int \mu\left(\mathrm{d}x\right) \cdot P\left(x, \mathrm{d}y\right) \cdot \frac{\left[f\left(x\right) - f\left(y\right)\right]^2}{2}$$

  - note that taking $P = \mu$ gives $\mathscr{E}_\mu\left(f\right) = \mathrm{var}_\mu\left(f\right)$

- if $\mathscr{E}_P\left(f\right) \geq \gamma \cdot \mathscr{E}_\mu\left(f\right)$ for all f, then "spectral gap" $\geq \gamma$

  - "energy $\approx$ entropy dissipation $\gtrsim$ entropy"

  - weakened / generalised version: $\mathscr{E}_\mu\left(f\right) \leq \alpha \cdot \mathscr{E}_P\left(f\right) + \beta \cdot \Phi\left(f\right)$

Comparisons for HSS

- in complete generality, $\mathscr{E}_H \leq \mathscr{E}_U$
  - the ideal Slice Sampler should be preferred when possible
- we want to examine when $\mathscr{E}_H \gtrsim \mathscr{E}_U$
  - the practical algorithm is almost as good as the theoretical ideal

when should this be true?

A Natural Approach

- to say that H gives a good Markov chain, we are arguing that
  - U gives a good Markov chain, and
  - H is a good approximation of U
- in principle, H could fail to approximate U well, but still work well
  - our analysis would fail to capture this (∃ examples)

Approximate … how?

- HSS replaces "$X \sim \nu_t$" by "$X \sim H_t\left(x, \mathrm{d}x'\right)$"
- to study how H approximates U …
  - … first study how H_t approximates ν_t
- assumption: for each t, there exists $\sigma_t > 0$ such that $\mathscr{E}_{H_t} \geq \sigma_t \cdot \mathscr{E}_{\nu_t}$
  - "on-slice kernel" mixes to "uniform on-slice" at rate σ_t

A Generic Result

- suppose that
  - for each t, $\mathscr{E}_{H_t} \geq \sigma_t \cdot \mathscr{E}_{\nu_t}$, and
  - $\sigma_H := \inf_{t \in \mathsf{T}} \sigma_t > 0$
- it then follows that $\mathscr{E}_H \geq \sigma_H \cdot \mathscr{E}_U$
- moreover, if $\mathscr{E}_U \geq \gamma_U \cdot \mathscr{E}_\pi$, then it further follows that $\mathscr{E}_H \geq \sigma_H \cdot \gamma_H \cdot \mathscr{E}_\pi$
  - interpretation: HSS is at most a factor σ_H ‘worse’ than ideal SS

Some Remarks

- this particular analysis is quite worst-case, but easy to state
- our methods also provide more refined estimates
  - can accommodate inhomogeneous mixing rates
  - can accommodate on-slice kernels with slower-than-exponential mixing

A Key Decomposition

- key decomposition formula:

$$\mathscr{E}_H(f) = \int \mathscr{E}_{H_t}(f) \cdot m(t)\ \mathrm{d}t$$

- taking $H_t = \nu_t$ yields

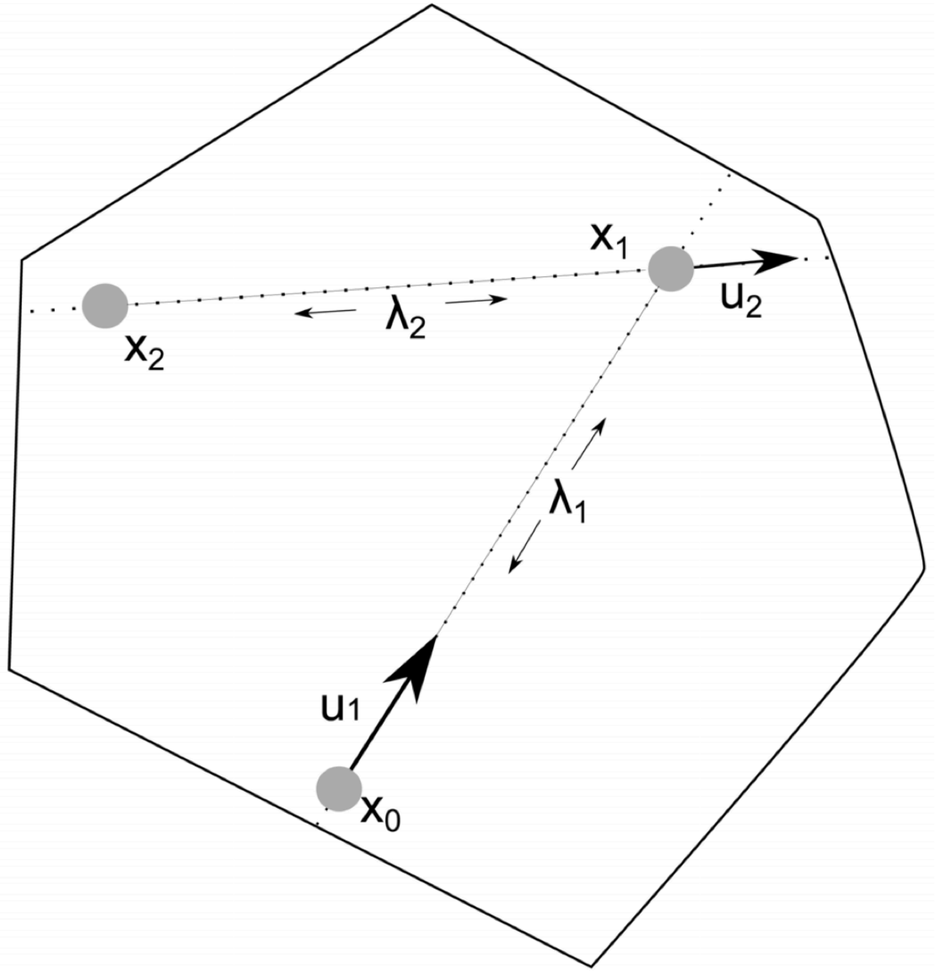
$$\mathscr{E}_U(f) = \int \mathscr{E}_{\nu_t}(f) \cdot m(t)\ \mathrm{d}t$$

- easy to combine with estimates like

$$\mathscr{E}_{\nu_t}(f) \leq \alpha \cdot \mathscr{E}_{H_t}(f) + \beta \cdot \Phi(f)$$

Case Study: Hit-and-Run on the Slice

- low-tech method for sampling uniform distributions on convex body G
- at $X_{n-1} = x$,
  - sample $U_n \sim \text{Unif}\left(\mathbb{S}^{d-1}\right)$
  - look at $L_n = \left(x + U_n \mathbf{R}\right) \cap \text{G}$
  - sample $X_n \sim \text{Unif}\left(L_n\right)$
- easy to implement, well-studied, …



(diagram from “optGpSampler” paper)

Convergence of Hit-and-Run

- let $\mathsf{G} \subset \mathbf{R}^d$ be convex,
  - containing a ball of radius r_{G}, contained in a ball of radius R_{G} (chalk!)
  - write $\kappa_{\mathsf{G}} := R_{\mathsf{G}}/r_{\mathsf{G}} \geqslant 1.$
- Lovász-Vempala prove that for some universal $c > 0$, it holds that

$$\gamma_{\mathrm{H\&R}} \geqslant c \cdot d^{-2} \cdot \kappa_{\mathsf{G}}^{-2}.$$

- high dimension is hard, inhomogeneity of scales is hard

Hit-and-Run Hybrid Slice Sampling

- if π has convex super-level sets, then results of LV give us a bound

$$\sigma_t \geqslant c \cdot d^{-2} \cdot \kappa_{\mathsf{G}(t)}^{-2}$$

- interpretation: life is good if super-level sets $\mathsf{G}(t)$ are well-conditioned
  - (if not: worse, though not disastrous)

Well-Conditioned Level Sets

- let $V : \mathbf{R}^d \to \mathbf{R}$ be m-strongly convex and L-smooth
  - i.e. eigs $\left(\text{Hess} V(x)\right) \in [m, L]$
  - write $\kappa_V = L/m \geqslant 1$
- let density π have the form $\pi(x) = \text{decreasing}\left(V(x)\right)$
- then for all t, it holds that $\kappa_{\text{G}(t)} \leqslant \sqrt{\kappa_V}$.

Some Applications

- if π has this form, then $\gamma_H \gtrsim d^{-2} \cdot \kappa_V^{-1} \cdot \gamma_U$
  - H&R-HSS is only worse than ideal SS by factor $d^2 \cdot \kappa_V$
- if e.g. $\pi \propto \exp(-V)$,
  - combine with works on ideal SS, ⇝ decorrelation time of $\lesssim d^3 \cdot \kappa_V$
- if e.g. π is multivariate Student-t, then $\kappa_V = 1, \sigma_H \gtrsim d^{-2}$
  - combine with earlier work, ⇝ decorrelation time of $\lesssim d^4$

Some Recap

- slice sampling performs well in theory, and in practice (when possible)
- hybrid slice sampling performs well in practice, is *typically* possible,
  - … and we provide here some theory to support this
- comparison principles: i) is U good?, ii) is H similar enough to U?
- generally, $H \preceq U$,
  - … but if $H_t \geq \sigma_H \cdot \nu_t$, then $H \geq \sigma_H \cdot U$.

A Glimpse at the Paper

- today: exponential rates, Hit-and-Run on the slice
- in the paper: slower-than-exponential rates, other examples of on-slice kernels, stepping-out and shrinkage, ‘generalised’ slice sampling with different reference measures, ….
- theoretical framework is very robust to which on-slice kernels are used
- actually, theoretical framework is much more general than slice sampling
  - “Markov chain comparison”, “weak Poincaré inequalities”, …

Finale

- Vertical Likelihood Representation ⇝ Slice Sampling Markov chain
- Practical considerations ⇝ *Hybrid* Slice Sampling Algorithm
- Dirichlet Forms ⇝ Comparison Theory for Markov Chains
- Mixing of On-Slice Kernels ⇝ Complexity Bounds for HSS

—Bonus Material—

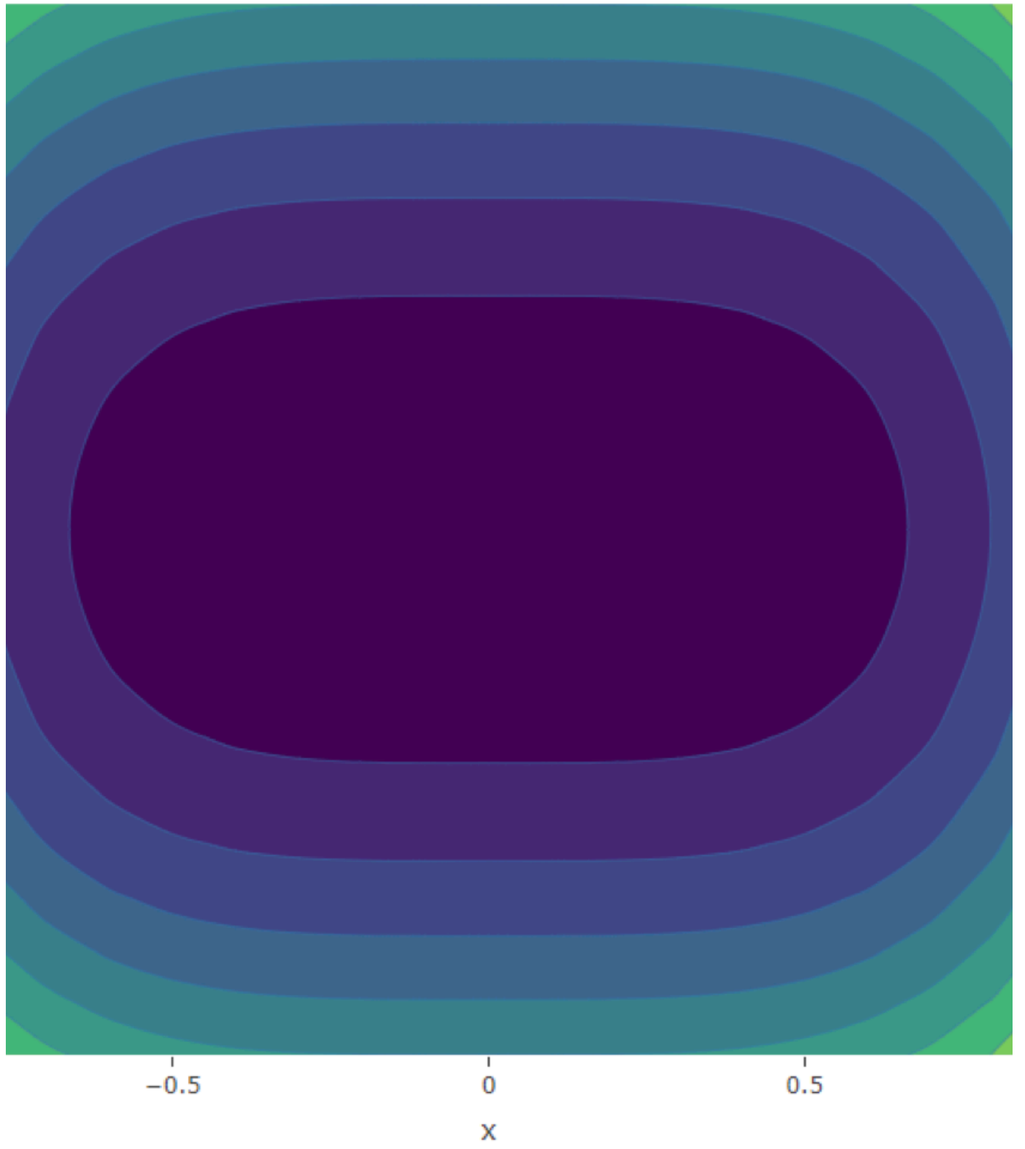
Advanced Applications

- let $1 \leqslant p_2 \leqslant p_1$, $1 \leqslant q_1 \leqslant q_2$, and suppose that

$$\|x\| \sim 0^+ \implies \|x\|^{p_1} \lesssim V(x) \lesssim \|x\|^{p_2}$$

$$\|x\| \sim \infty \implies \|x\|^{q_1} \lesssim V(x) \lesssim \|x\|^{q_2}$$

- if $p_1 = p_2$, $q_1 \neq q_2$, then convergence rate decays quasi-exponentially
- if $p_1 \neq p_2$, then convergence rate decays only polynomially
- message: in this case, bulk behaviour matters more than tail behaviour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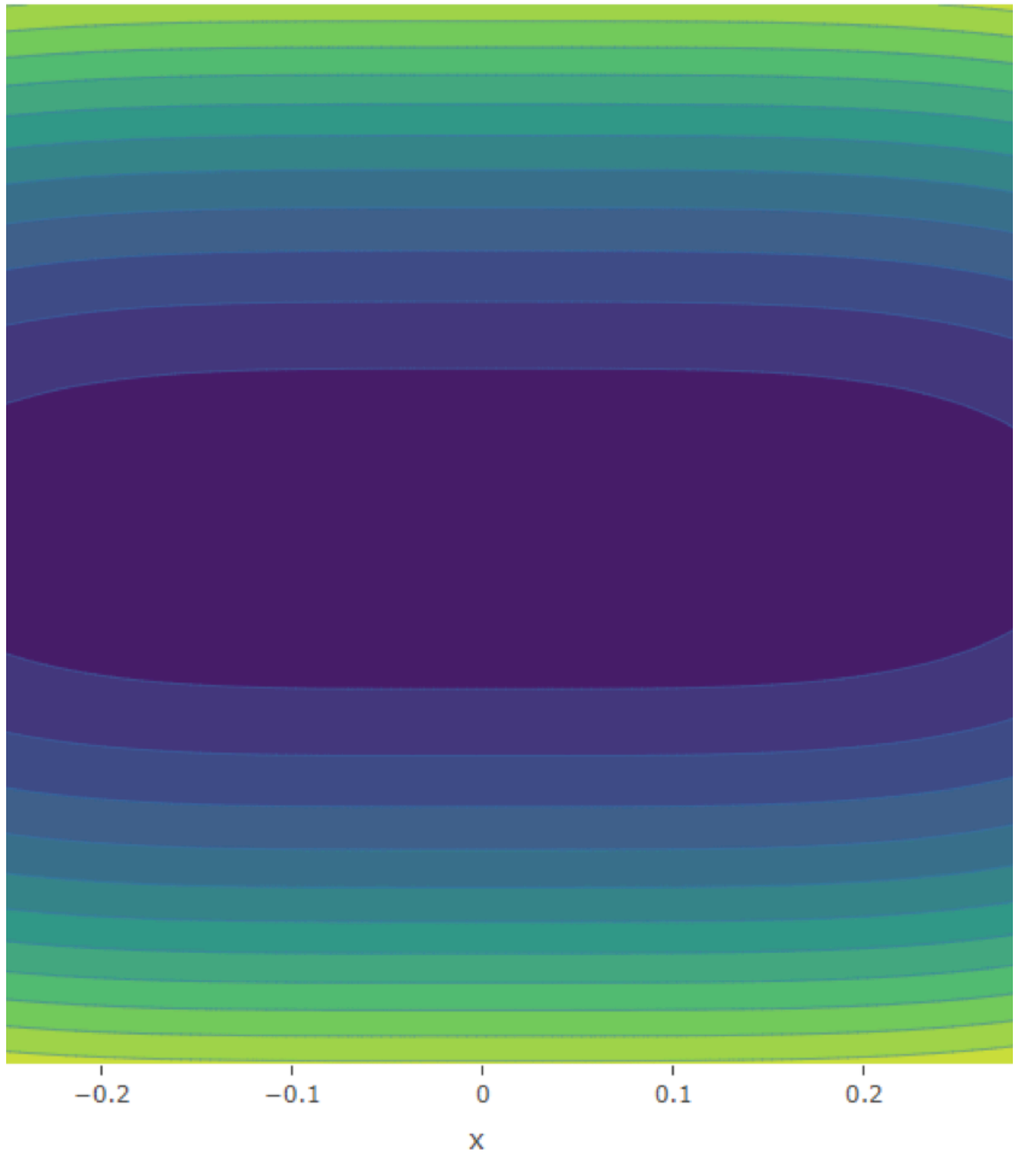
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x

$$\kappa_{\mathsf{G}}(t) \leq \begin{cases} c_{\kappa}^{-} \cdot \left(\log\left(\frac{1}{t}\right)\right)^{\theta} & 0 < t \leq \exp(-1); \\ c_{\kappa}^{+} \cdot \left(\log\left(\frac{1}{t}\right)\right)^{-\vartheta} & \exp(-1) \leq t < 1; \end{cases}$$

with $\theta = \frac{1}{q_1} - \frac{1}{q_2}$, $\vartheta = \frac{1}{p_2} - \frac{1}{p_1}$, *and such that the mass function satisfies*

$$m(t) \leq c_m \cdot \left(\log\left(\frac{1}{t}\right)\right)^{d/r}$$

with $r = q_1$. *By application of Proposition 40, we see that for* $p_1 = p_2$, *there holds a WPI with*

$$\beta(s) \leq c^{(1)} \cdot \exp\left(-c^{(2)} \cdot s^{\frac{q_1 \cdot q_1}{q_2 - q_1}}\right),$$

whereas for $p_1 > p_2$, *one instead obtains a WPI with*

$$\beta(s) \leq c^{(3)} \cdot s^{-\left(1+\frac{d}{q_1}\right) \cdot \frac{p_1 \cdot p_2}{p_1 - p_2}}.$$

Quantitative Mode Separation

Definition 33. *Fix a density function* $\varpi : \mathbb{R} \to \mathsf{T}$*, and let* $0 < t_1 \leq t_2$ *be elements of* T*. Say that* ϖ *is* (t_1, t_2)-bimodal *if*

- *for all* $t \in \mathsf{T} \setminus [t_1, t_2)$*, the super-level set* $\mathsf{G}(t)$ *consists of a single interval, and*
- *for all* $t \in [t_1, t_2)$*, the super-level set* $\mathsf{G}(t)$ *consists of a pair of disjoint sub-intervals* $\mathsf{G}(t) = \mathsf{G}_1(t) \sqcup \mathsf{G}_2(t)$ *such that commonly-labelled sub-intervals are nested, i.e. for* $i = 1, 2$ *and* $t_1 \leq s \leq t < t_2$*, there holds the inclusion* $\mathsf{G}_i(t) \subseteq \mathsf{G}_i(s)$*.*

Moreover, given a (t_1, t_2)*-bimodal density* ϖ*, define* $\delta_\varpi : \mathsf{T} \to [0, \infty)$ *by*

$$\delta_\varpi(t) = \begin{cases} \mathsf{dist}\left(\mathsf{G}_1(t), \mathsf{G}_2(t)\right) & t \in [t_1, t_2) \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathsf{dist}\left(\mathsf{G}_1(t), \mathsf{G}_2(t)\right) := \inf\left\{|x - y| : x \in \mathsf{G}_1(t), y \in \mathsf{G}_2(t)\right\}$*, and write* $\Delta_\varpi := \sup\left\{\delta_\varpi(t) : t \in \mathsf{T}\right\}$*.*

Stepping-Out and Shrinkage

Assumption 2. *Let ϖ be a (t_1, t_2)-bimodal density and $h > \Delta_\varpi$ be a stepping-out parameter.*

Under the previous assumption, for $t \in \mathsf{T}$, define the 'stepping-out and shrinkage' on-slice kernel with parameter h by

$$H_t(x, A) = \lambda(t) \cdot \nu_t(A) + \big(1 - \lambda(t)\big) \cdot \Big[\mathbf{1}_{\mathsf{G}_1(t)}(x) \cdot \nu_{t,1}(A) + \mathbf{1}_{\mathsf{G}_2(t)}(x) \cdot \nu_{t,2}(A)\Big],$$

for $x \in \mathbb{R}$, $A \in \mathcal{B}(\mathbb{R})$, where

$$\lambda(t) := \begin{cases} \frac{h - \delta_\varpi(t)}{h} \cdot \frac{m(t)}{m(t) + \delta_\varpi(t)} & t \in [t_1, t_2); \\ 1 & \text{otherwise;} \end{cases}$$

$$\nu_{t,i}(A) := \begin{cases} \frac{\nu\left(\mathsf{G}_i(t) \cap A\right)}{\nu\left(\mathsf{G}_i(t)\right)} & t \in [t_1, t_2); \\ 0 & \text{otherwise;} \end{cases} \qquad \text{for } i = 1, 2.$$

Weak Poincaré Inequalities

Definition 1. *We say that a μ-reversible, positive transition kernel P satisfies a weak Poincaré inequality (WPI) if for all $f \in \mathrm{L}_0^2(\mu)$ we have*

$$\|f\|_\mu^2 \leq s \cdot \mathcal{E}_\mu(P, f) + \beta(s) \cdot \|f\|_{\mathrm{osc}}^2, \tag{3}$$

where $\beta\colon (0,\infty) \to [0,\infty)$ is a decreasing function with $\lim_{s\to\infty} \beta(s) = 0$.

Assumption 1. *We assume that for Lebesgue-almost every $t \in \mathsf{T}$, the kernel H_t is ν_t-reversible, positive and satisfies a WPI, i.e. there is a measurable function $\beta : (0,\infty) \times \mathsf{T} \to [0,\infty)$ with $\beta(\cdot, t)$ satisfying the conditions in Definition 1 for each $t \in \mathsf{T}$, such that for each $s > 0$, $f \in \mathrm{L}^2(\nu_t)$,*

$$\mathrm{Var}_{\nu_t}(f) \leq s \cdot \mathcal{E}_{\nu_t}\left(H_t, f\right) + \beta\left(s, t\right) \cdot \|f\|_{\mathrm{osc}}^2. \tag{8}$$

Theorem 11. *Under Assumption 1, we have the following comparisons for U and H given in (6) and (7):*

For all $f \in \mathrm{L}^2(\pi)$,

$$\mathcal{E}(H, f) \leq \mathcal{E}(U, f), \tag{9}$$

and conversely, for all $s > 0$, $f \in \mathrm{L}^2(\pi)$,

$$\mathcal{E}(U, f) \leq s \cdot \mathcal{E}(H, f) + \beta(s) \cdot \|f\|_{\mathrm{osc}}^2, \tag{10}$$

where $\beta : (0,\infty) \to [0,\infty)$ is given by

$$\beta(s) := c^{-1} \cdot \int_{\mathsf{T}} \beta(s,t) \cdot m\left(t\right) \,\mathrm{d}t.$$

Furthermore, β satisfies the conditions for a WPI in Definition 1.

Metropolis Chains as HSS

4.1 Metropolis chains

Definition 22. *Let π a probability measure admitting a density $\varpi = \frac{\mathrm{d}\pi}{\mathrm{d}\nu}$ with respect to some σ-finite measure ν on G, and let Q be a ν-reversible Markov kernel; we say that such triples (π, ν, Q) are* compatible. *Define* $\mathsf{Metropolis}\,(\pi, \nu, Q)$ *to be the Markov kernel P given by*

$$P(x, A) = \int_A Q(x, \mathrm{d}y) \cdot \alpha(x, y) + \mathbf{1}_A(x) \cdot \bar{\alpha}(x), \qquad x \in \mathsf{G}, A \in \mathcal{G},$$

where for $x, y \in \mathsf{G}$,

$$\alpha(x, y) := 1 \wedge \frac{\varpi(y)}{\varpi(x)}, \quad \bar{\alpha}(x) := 1 - \alpha(x), \quad \alpha(x) := \int_{\mathsf{G}} Q(x, \mathrm{d}y) \cdot \alpha(x, y).$$

It is known that all such kernels are π-reversible. In what follows, we will use (π, ν, Q) informally to refer to generic compatible triples in the sense described above.

Lemma 27. *For any compatible triple (π, ν, Q), it holds that* $\mathsf{Metropolis}\,(\pi, \nu, Q) = \mathsf{HybridSlice}\left(\pi, \nu, (H_t)\right)$, *where*

$$H_t(x, A) = Q\left(x, A \cap \mathsf{G}(t)\right) + \mathbf{1}_A(x) \cdot Q\left(x, \mathsf{G}(t)^{\complement}\right).$$

Some Metropolis Chains

Example 23. *When $\nu = \mathrm{Leb}$ and Q is a symmetric, ν-reversible kernel, then we can define the* Random Walk Metropolis *(RWM) kernel,*

$$\mathsf{RWM}(\pi, Q) := \mathsf{Metropolis}\left(\pi, \mathrm{Leb}, Q\right).$$

It is conventional to work with $Q_\sigma\left(x, \mathrm{d}y\right) = \mathcal{N}\left(\mathrm{d}y \mid x, \sigma^2 \cdot I_d\right)$ for some step-size $\sigma > 0$; we will work under this assumption going forward. See also Section 6.3.2 of [29].

Example 24. *When ν is a sufficiently-tractable probability measure, we may take $Q\left(x, \cdot\right) = \nu$ directly, independently of x. We can thus define the* Independent Metropolis–Hastings *(IMH) kernel with 'proposal' ν; see [29, Section 6.3.1]:*

$$\mathsf{IMH}(\pi, \nu) := \mathsf{Metropolis}\left(\pi, \nu, \nu\right).$$

Example 25. *When $\nu = \gamma_{\mathsf{m},\mathsf{C}}$ is a Gaussian measure with mean m and covariance operator C, then one may take $\rho, \eta \in (0,1)$ such that $\rho^2 + \eta^2 = 1$ and define the autoregressive proposal $Q_\eta\left(x, \mathrm{d}y\right) = \mathcal{N}\left(\mathrm{d}y \mid \mathsf{m} + \rho \cdot (x - \mathsf{m}), \eta \cdot \mathsf{C}\right)$. The resulting Metropolis chain is known as the* Preconditioned Crank-Nicolson *(pCN) kernel with Gaussian reference $\gamma_{\mathsf{m},\mathsf{C}}$ and step-size η; see e.g. [9]:*

$$\mathsf{pCN}(\pi, \mathsf{m}, \mathsf{C}, \eta) := \mathsf{Metropolis}\left(\pi, \gamma_{\mathsf{m},\mathsf{C}}, Q_\eta\right).$$

Beyond Hit-and-Run

There are various routes left open by this work; we list here a few.

For one, we have largely focused on Simple Slice Sampling as the ideal algorithm, and using Hit-and-Run as on-slice kernels, due to their relative genericity and ease of implementation. In specific settings, other on-slice kernels are of substantial interest. For super-level sets with some coordinate-related structure, Gibbs sampling (also 'Coordinate Hit-and-Run') is a natural option, for which results have recently been obtained in the convex setting; see [17]. For super-level sets which take the form of polytopes, yet more on-Slice Samplers are available, including the Dikin walk [16], the Vaidya and John walks [7], and various gradient-based samplers which introduce additional geometric structure to the problem (e.g. [20]); some of these methods come with theoretical guarantees in the form of estimates on the conductance or spectral gap of the kernel, which can readily be used in our framework.