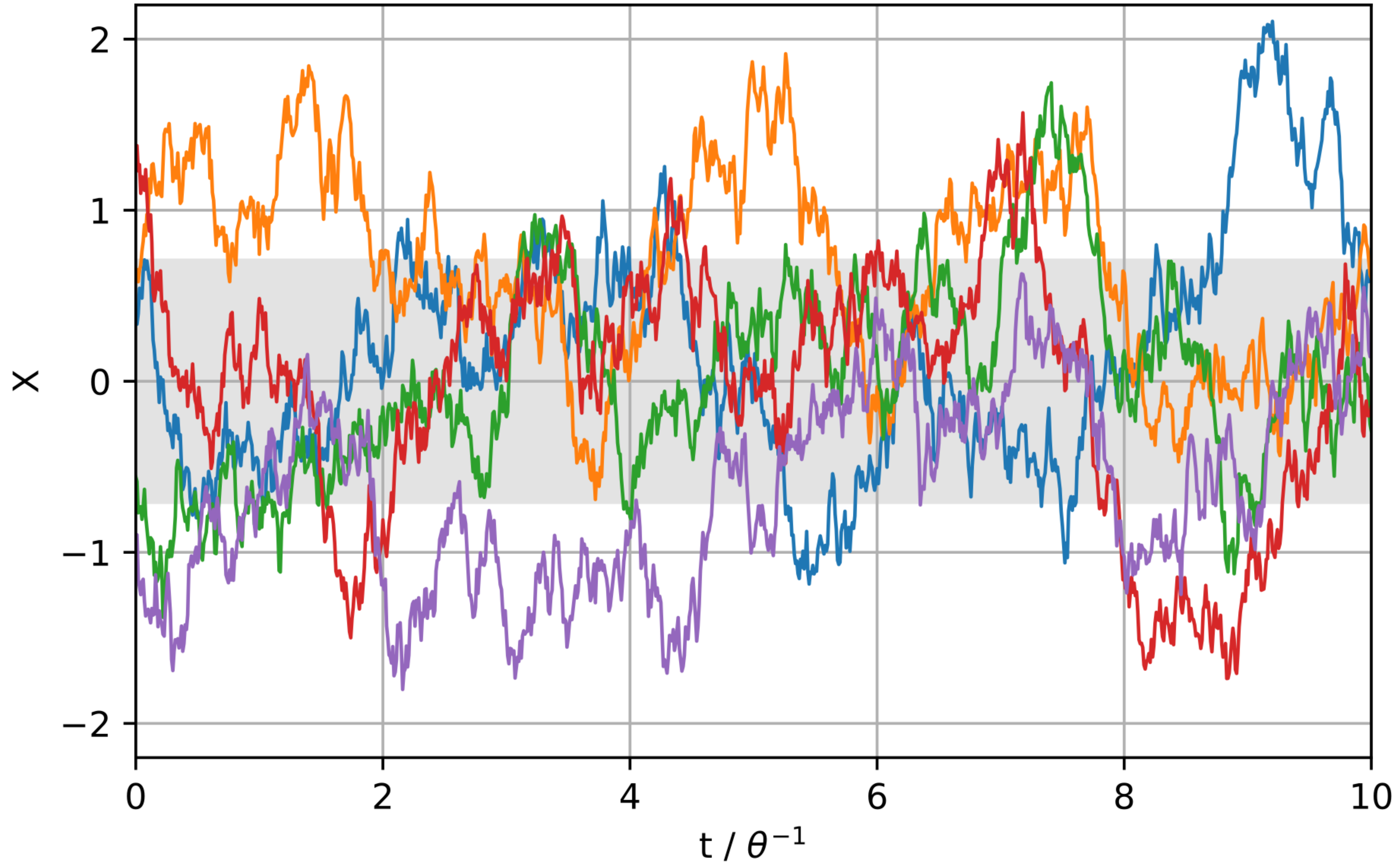
Mini-Course: Geometric Functional Inequalities for Markov Chains

University of Warwick, Department of Statistics
25 April 2025

Sam Power, University of Bristol

A Prompt for the Audience



“how do *you* like to study the long-time behaviour of a Markov process”

A Personal Perspective

- (some) principles for analysing the long-time behaviour of Markov processes
  - ‘probabilistic’: { pathwise, couplings, regenerations, local, … }
  - ‘functional-analytic’: { semigroup, divergences, entropies, global, … }
- i have a lot of appreciation for the former approach, when it applies well
- some specific applications led me to become interested in the latter approach
- i hope to share the reasons for my enthusiasm

Some Motivating Questions

- what is a ‘functional inequality’, and what are the ‘standard’ functional inequalities?
    - when ‘should’ they hold, and what are their consequences?
- to which settings is the functional-analytic approach well-suited?
    - highlight that it can be *easy to use* and *robust to (certain) details*
- how do probabilistic and functional-analytic techniques interface?
    - explain how to obtain functional-analytic consequences from probabilistic insights

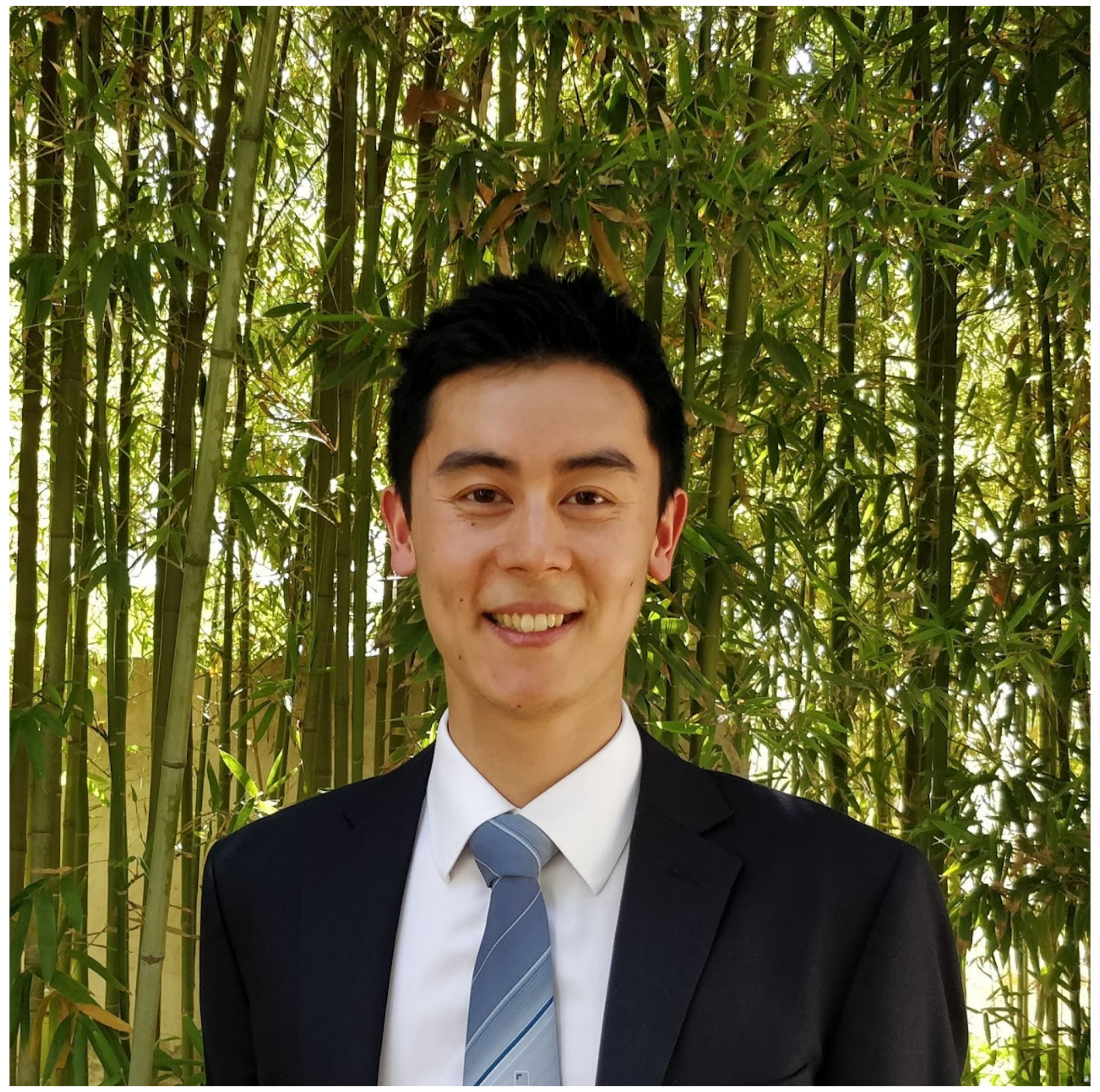
Coarse-Grained Plan

- Prelude: Why Take a Functional Approach?
- Part 1: The Story for Reversible Diffusion Processes // OLD (π)
- Part 2: The Story for Reversible Discrete-Time Markov Chains

Some Compromises

- **time-limited**: broad strokes, intuition, generous with references
- **Markov-centric**: limited discussion of geometry, concentration, etc.
- **reversibility-centric**: story is cleaner, though more is possible
- $\mathbf{L^2}$**-centric**: related to the above; will touch upon entropies somewhat
- **MCMC-centric**: my examples will inevitably reflect my own interests

this journey would not have started without:
Christophe Andrieu, Anthony Lee (Bristol), Andi Wang (Warwick)



feel free to stop me at any point

“robust *to what*?”

Before I Say Anything Interesting:

- P is a π-invariant Markov semigroup, often reversible and positive
- we initialise at $X_0 \sim \mu_0$, and propagate to μ_t
- we are looking to quantify how $\mu_t \to \pi$ as $t \to \infty$

Prelude: Why Take a Functional Approach?

or, ‘How On Earth Shall I Quantify Convergence?’

(Some) Notions of Long-Time Convergence

- $\mathsf{TV}\left(p,q\right) = \frac{1}{2} \cdot \int \left|p - q\right| \left(\mathrm{d}x\right)$
- $\mathsf{TV}_{\varphi}\left(p,q\right) = \int \varphi\left(x\right) \cdot \left|p - q\right| \left(\mathrm{d}x\right)$
- $\mathscr{T}_1\left(p,q\right) = \inf \left\{ \mathbf{E}_\gamma \left[\mathsf{d}\left(X,Y\right)\right] : \gamma \in \mathsf{Couplings}\left(p,q\right) \right\}$
- ‘information-theoretic convergence’

Information-Theoretic Divergences

- for me: ‘Density Ratio Divergences’; also ‘f-divergence’, ‘Φ-entropy’, …
- let Φ be convex, non-negative, $\Phi(1) = 0$, and define

$$\mathsf{D}_{\Phi}\left(p, q\right) = \int q\left(\mathrm{d}x\right) \cdot \Phi\left(\frac{\mathrm{d}p}{\mathrm{d}q}(x)\right)$$

- examples: { TV, KL, Chi-Squared / Pearson, Hellinger, … }
- many well-understood inter-relations; TV convergence usually follows

Information-Theoretic Contraction

- if all goes well, we should hope to write that

$$\mathsf{D}_{\Phi}\left(\mu_t, \pi\right) \leq \exp\left(-c \cdot t\right) \cdot \mathsf{D}_{\Phi}\left(\mu_0, \pi\right)$$

  - not very a useful assumption as written; need something ‘checkable’

- one idea: ‘reverse-Grönwall’

Information-Theoretic Contraction

- one idea: ‘reverse-Grönwall’
  - reverse-Grönwall: differentiate at $t = 0$, and interpret this

$$\mathtt{I}_{\Phi}\left(\mu_0, \pi\right) := \left[-\partial_t \mathsf{D}_{\Phi}\left(\mu_t, \pi\right)\right]_{t=0}$$

$$\mathtt{I}_{\Phi}\left(\mu_0, \pi\right) \geq c \cdot \mathsf{D}_{\Phi}\left(\mu_0, \pi\right)$$

  - a priori: a bit optimistic to start with the desired conclusion?
    - it will turn out here to be a very fruitful strategy
    - let us first study it in a more concrete setting

Part 1: Reversible Diffusions

or, ‘The Tale of the Overdamped Langevin Diffusion’

The Overdamped Langevin Diffusion

- let π be a probability density on $\mathbf{R}^d$
- then, OLD (π) is the Ito diffusion

$$\mathrm{d}X_t = \nabla \log \pi \left(X_t \right) \, \mathrm{d}t + \sqrt{2} \, \mathrm{d}W_t$$

- define the semigroup $\left(P_t \right)_{t \geq 0}$ by $P_t f(x) = \mathbf{E}^x \left[f\left(X_t \right) \right]$
  - very interesting, not particularly tractable

From Semigroup to Generator

- define the infinitesimal generator $\mathscr{L}$ by

$$\mathscr{L}f = \lim_{t \to 0^+} \frac{P_t f - f}{t}, \qquad \rightsquigarrow P_t = \exp\left(t \cdot \mathscr{L}\right)$$

- equally: $\partial_t P_t = \mathscr{L} P_t = P_t \mathscr{L}$
- concretely, this writes as

$$\mathscr{L}f(x) = \langle \nabla \log \pi(x), \nabla f(x) \rangle + \Delta f(x)$$

The Dirichlet Energy Form

- another useful object is the bilinear form

$$\begin{aligned} \mathscr{E}\left(f, g\right) &= \mathbf{E}\left[f \cdot \left(-\mathscr{L} g\right)\right] \\ &= \mathbf{E}\left[\langle \nabla f, \nabla g \rangle\right] \end{aligned}$$

(integration by parts)

- provides useful language for describing dissipation of various functionals
- particularly pertinent for <u>reversible</u> processes

Functional Forms

- we want to make statements of the form $\mathsf{D}_\Phi\left(\mu_t, \pi\right) \lesssim_t \mathsf{D}_\Phi\left(\mu_0, \pi\right)$
- by reversibility, can show that $\mathrm{d}\mu_t/\mathrm{d}\pi = P_t\left(\mathrm{d}\mu_0/\mathrm{d}\pi\right)$
- so, defining $\mathsf{Ent}_\Phi(F) = \mathbf{E}[\Phi \circ F] - \Phi \circ \mathbf{E}[F]$, it holds that

$$\mathsf{D}_\Phi\left(\mu_t, \pi\right) = \mathsf{Ent}_\Phi\left(P_t \rho_0\right), \qquad \rho_0 = \mathrm{d}\mu_0/\mathrm{d}\pi$$

- hence, sufficient to show that $\mathsf{Ent}_\Phi\left(P_t F\right) \lesssim_t \mathsf{Ent}_\Phi(F)$ for suitable F

Let Us Compute

- compute explicitly that

$$
\begin{aligned}
-\partial_t \mathrm{Ent}_\Phi\left(P_t F\right) &= -\,\partial_t \mathbf{E}\left[\Phi\left(P_t F\right)\right] \\
&= \mathbf{E}\left[\Phi'\left(P_t F\right)\cdot\left(-\mathscr{L} P_t F\right)\right] \\
&= \mathscr{E}\left(\Phi'\left(P_t F\right), P_t F\right) \\
&= \mathbf{E}\left[\Phi''\left(P_t F\right)\cdot\left|\,\nabla\left(P_t F\right)\right|^2\right]
\end{aligned}
$$

- this is a sort of ‘Φ-information’ functional

A Generic Energy-Entropy Inequality

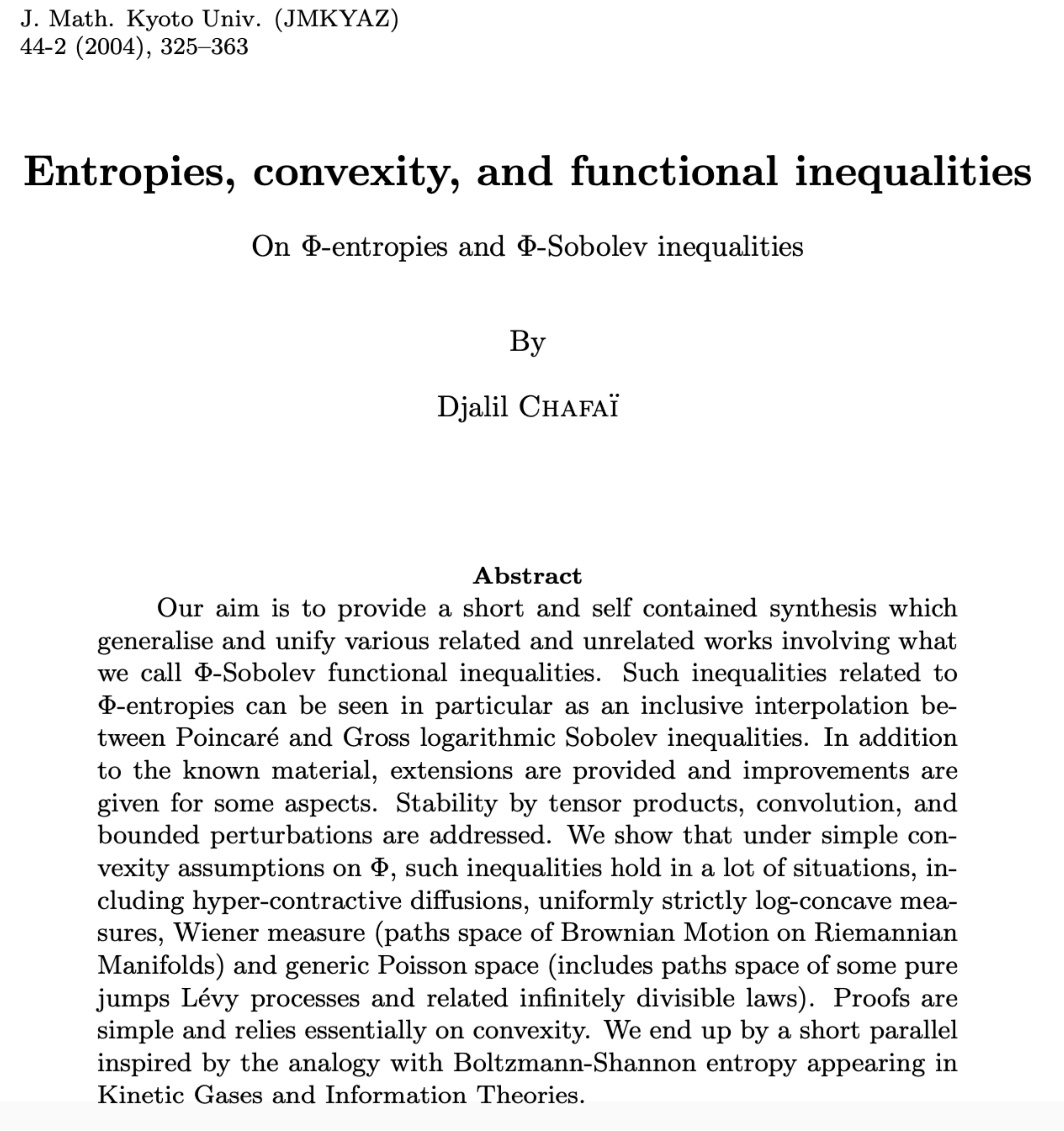
- say that π satisfies a Φ-Sobolev inequality with constant c if
  - for all suitable F, there holds the estimate

$$\mathbf{E}\left[\Phi''(F) \cdot \left| \nabla F \right|^2\right] \geq c \cdot \mathsf{Ent}_\Phi(F)$$

- if this holds, then apply Grönwall, and we are done
- my convention: big constant $\sim$ good mixing (not universal)

For more, see …

- Chafaï, “Entropies, Convexity, and Functional Inequalities”
- studies general convex Φ, for which many conclusions persist
- bonus results appear particularly for $\Phi(r) \sim \left\{ r^p : 1^+ < p \leq 2 \right\}$
  - appears crucial to have convexity of all of $\{\Phi, \Phi'', -1/\Phi''\}$



J. Math. Kyoto Univ. (JMKYAZ)
44-2 (2004), 325–363

Entropies, convexity, and functional inequalities

On Φ-entropies and Φ-Sobolev inequalities

By

Djalil Chafaï

Abstract

Our aim is to provide a short and self contained synthesis which generalise and unify various related and unrelated works involving what we call Φ-Sobolev functional inequalities. Such inequalities related to Φ-entropies can be seen in particular as an inclusive interpolation between Poincaré and Gross logarithmic Sobolev inequalities. In addition to the known material, extensions are provided and improvements are given for some aspects. Stability by tensor products, convolution, and bounded perturbations are addressed. We show that under simple convexity assumptions on Φ, such inequalities hold in a lot of situations, including hyper-contractive diffusions, uniformly strictly log-concave measures, Wiener measure (paths space of Brownian Motion on Riemannian Manifolds) and generic Poisson space (includes paths space of some pure jumps Lévy processes and related infinitely divisible laws). Proofs are simple and relies essentially on convexity. We end up by a short parallel inspired by the analogy with Boltzmann-Shannon entropy appearing in Kinetic Gases and Information Theories.

For more, see …

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- studies general convex Φ, for which many conclusions persist
- bonus results appear particularly for $\Phi(r) \sim \left\{ r^p : 1^+ < p \leq 2 \right\}$
    - appears crucial to have convexity of all of $\{\Phi, \Phi'', -1/\Phi''\}$

Two Famous Characters

- $\Phi(r) = (r-1)^2$
  - the ‘Poincaré inequality’ (PI)

$$\gamma \cdot \mathrm{var}_\pi(F) \le \mathbf{E}\left[\left|\nabla F\right|^2\right]$$

- $\Phi(r) = r \cdot \log r - r + 1$
  - the ‘Logarithmic Sobolev Inequality’ (LSI)

$$2 \cdot \lambda \cdot \mathsf{Ent}(F) \le \mathbf{E}\left[F^{-1} \cdot \left|\nabla F\right|^2\right]$$

$$\rightsquigarrow \frac{\lambda}{2} \cdot \mathsf{Ent}\left(F^2\right) \le \mathbf{E}\left[\left|\nabla F\right|^2\right] \text{ (2-hom)}$$

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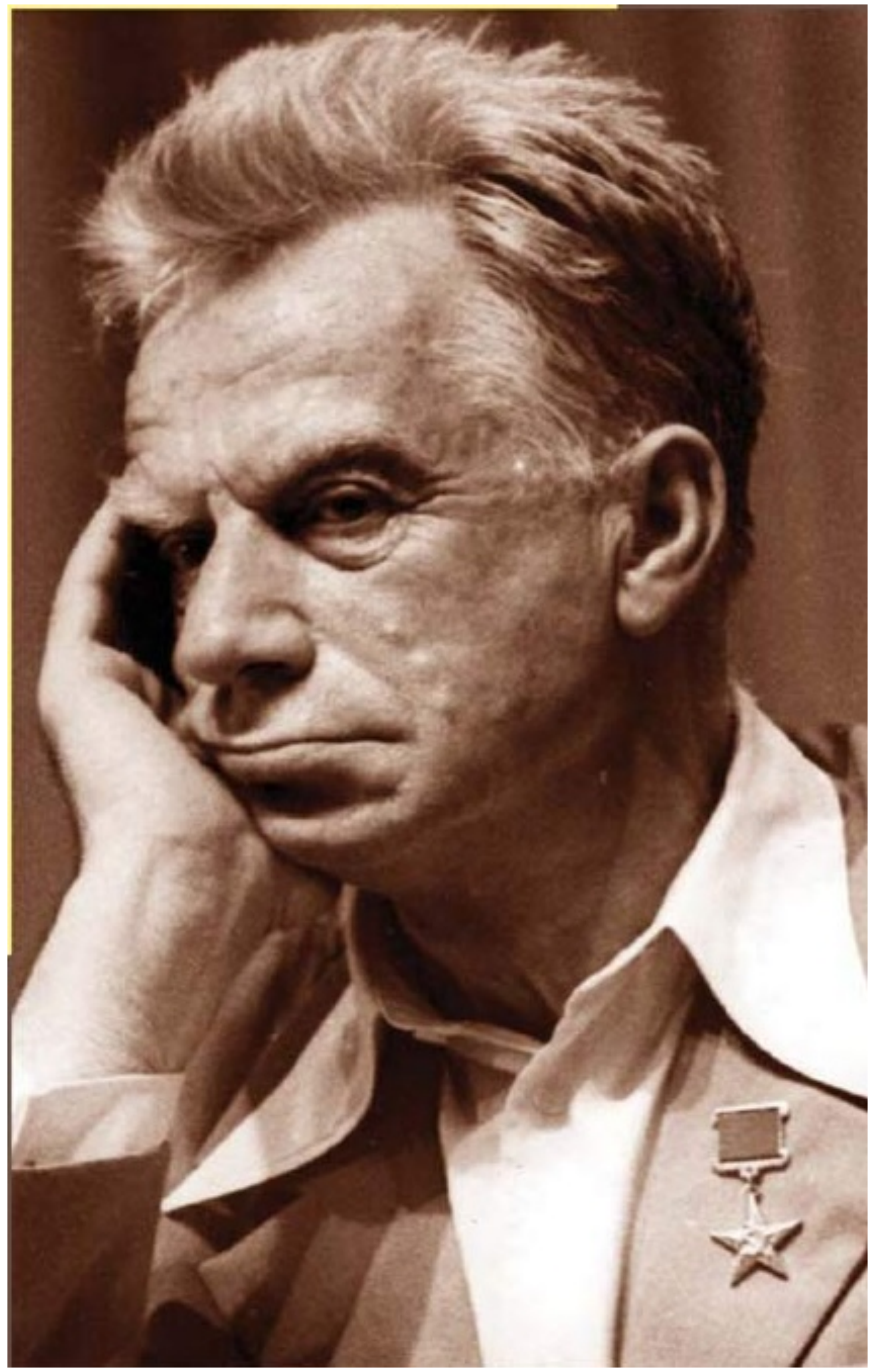


Poincaré and Log-Sobolev

- LSI implies PI with same effective rate
  - rough idea: $r \cdot \log r - r + 1 \asymp (r-1)^2/2$ as $r \to 1$
  - equally, look at $\mathsf{Ent}\,(1 + \varepsilon \cdot F)$ as $\varepsilon \to 0^+$
- LSI is strictly stronger (examples to come)
- inequalities weaker than PI map onto slower-than-exponential ergodicity
- inequalities stronger than LSI cannot tensorise; “too good to be true”

Illustrative Examples (1)

- let π be ρ-strongly log-concave, with $\rho > 0$, i.e.

$$\nabla^2 \left(-\log \pi\right)(x) \succeq \rho \cdot \mathbf{I}_d$$

- then, LSI (hence PI) holds with constant ρ, i.e.

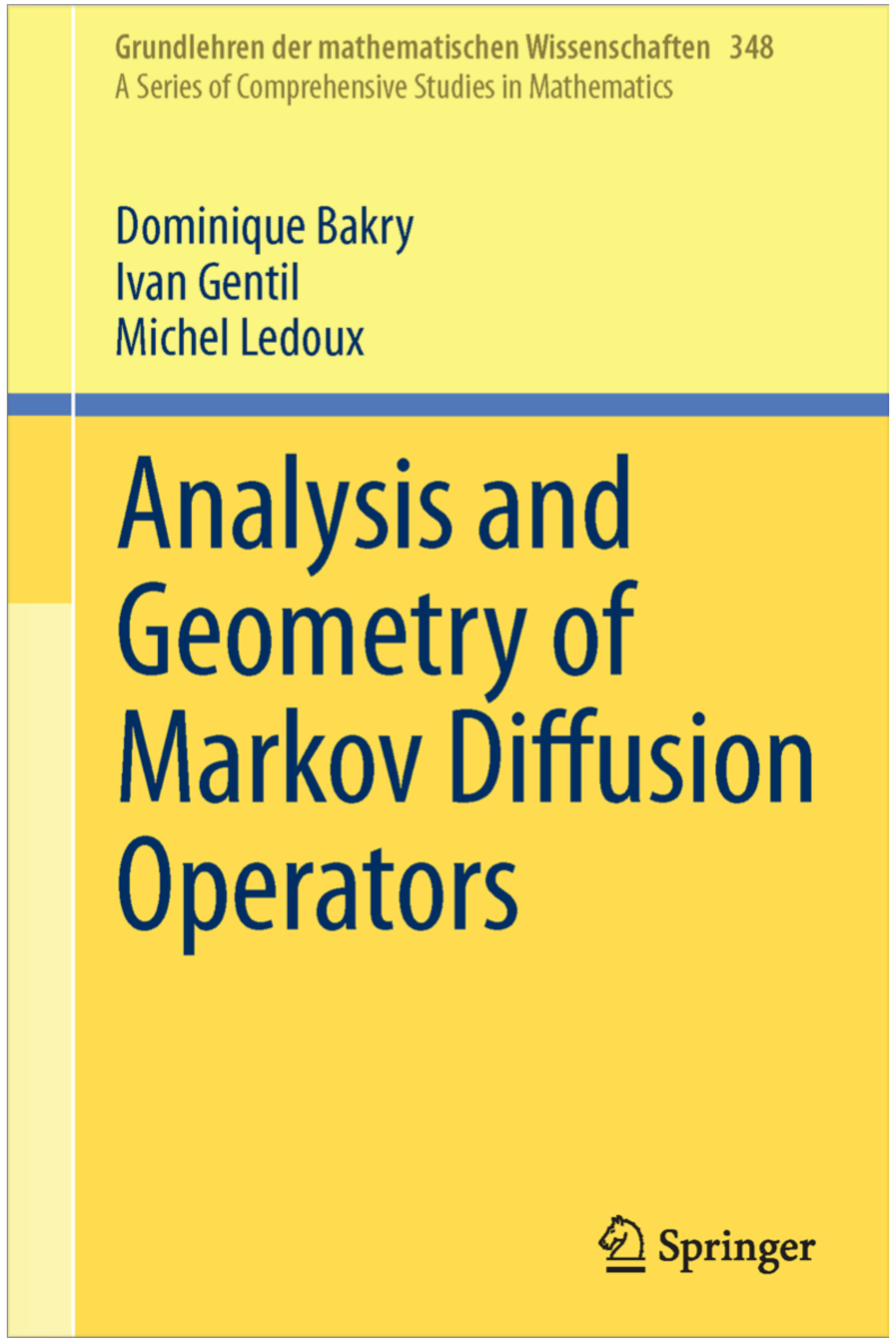
$$\frac{\rho}{2} \cdot \text{Ent}\left(F^2\right) \leq \mathbf{E}\left[\left|\nabla F\right|^2\right]$$

  - many ways to establish this, will comment shortly
  - the ‘Gamma Calculus’ of Bakry-Émery yields even more

For the curious, see …

- Bakry, Gentil, Ledoux - “Analysis and Geometry of Markov Diffusion Operators”

- pursues ‘all possible’ consequences of this assumption
    - can be somewhat brittle



Illustrative Examples (2)

- let π be the $\text{Exp}(1)$ distribution on $\mathbf{R}^+$, i.e. $\pi(x) = \exp(-x) \cdot \mathbf{1}_{x>0}$
- then, PI holds with constant $1/4$, i.e. $\text{var}_\pi(F) \leq 4 \cdot \mathbf{E}\left[\left|\nabla F\right|^2\right]$
  - can study eigenproblem more-or-less directly; try $F(x) = \exp(\lambda \cdot x)$
  - LSI **cannot** hold (will comment on why shortly)

Transfer Principles

- how can we prove that such inequalities hold for more interesting π?
- develop an ‘algebra’ (very loosely) for functional inequalities
  - { tensorisation, transport, convolution, bounded change of measure, mixtures }
- i will state (and perhaps prove) for the PI, but they will also hold for LSI
  - and often also general Φ-Sobolev inequalities

Tensorisation

- suppose that for $i \in [N]$, μ_i satisfies a PI with constant $\gamma_i > 0$
- upon defining $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_N$,
  - it holds that μ satisfies a PI with constant $\gamma \geq \min_{i \in \mathcal{I}} \gamma_i > 0$
- "you are only as bad as your worst one-dimensional component"
- enables application to high-dimensional problems

Probability in High Dimension

- many good references on this topic
- useful for concentration of measure, but also far beyond
- van Handel does a particularly good job of emphasising the significance of obtaining ‘dimension-free’ bounds

Ramon van Handel

Probability in High Dimension

APC 550 Lecture Notes
Princeton University

Transport

- suppose that μ satisfies a PI with constant $\gamma > 0$

- let $T : \mathbf{R}^d \to \mathbf{R}^{d'}$ be L-Lipschitz-continuous, and set $\pi = T_{\#}\mu$

- then, with $g = f \circ T$, see that

$$\mathrm{var}_\mu\left(g\right) = \mathrm{var}_\pi\left(f\right), \qquad \mathbf{E}_\mu\left[\left|\nabla g\right|^2\right] \leq L^2 \cdot \mathbf{E}_\pi\left[\left|\nabla f\right|^2\right]$$

  - corollary: π satisfies a PI with constant $\gamma \cdot L^{-2} > 0$

Constructive Lipschitz Transport

- very active and exciting area
- *related* to ‘optimal transport’; distinct goals
- recent developments use ‘heat flow’
  - pioneered by Kim-Milman
  - greatly advanced by Mikulincer-Shenfeld
  - surprising (?) links to diffusion models

Math. Ann.
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Mathematatische Annalen

A generalization of Caffarelli’s contraction theorem via (reverse) heat flow

Young-Heon Kim · Emanuel Milman

Received: 1 April 2010 / Revised: 14 April 2011

Abstract A theorem of L. Caffarelli implies the existence of a map, pushing forward a source Gaussian measure to a target measure which is more log-concave than the source one, which contracts Euclidean distance (in fact, Caffarelli showed that the optimal-transport Brenier map T_{opt} is a contraction in this case). We generalize this result to more general source and target measures, using a condition on the third derivative of the potential, by providing two different proofs. The first uses a map T, whose inverse is constructed as a flow along an advection field associated to an appropriate heat-diffusion process. The contraction property is then reduced to showing that log-concavity is preserved along the corresponding diffusion semi-group, by using a maximum principle for parabolic PDE. In particular, Caffarelli’s original result immediately follows by using the Ornstein–Uhlenbeck process and the Prékopa–Leindler Theorem. The second uses the map T_{opt} by generalizing Caffarelli’s argument, employing in addition further results of Caffarelli. As applications, we obtain new correlation and isoperimetric inequalities.

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On the Lipschitz Properties of Transportation Along Heat Flows

Dan Mikulincer and Yair Shenfeld

Abstract We prove new Lipschitz properties for transport maps along heat flows, constructed by Kim and Milman. For (semi)-log-concave measures and Gaussian mixtures, our bounds have several applications: eigenvalues comparisons, dimensional functional inequalities, and domination of distribution functions.

Constructive Lipschitz Transport

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Heat flow, log-concavity, and Lipschitz transport maps

Giovanni Brigati* and Francesco Pedrotti†

Institute of Science and Technology Austria

May 8, 2024

Abstract

In this paper we derive estimates for the Hessian of the logarithm (log-Hessian) for solutions to the heat equation. For initial data in the form of log-Lipschitz perturbation of strongly log-concave measures, the log-Hessian admits an explicit, uniform (in space) lower bound. This yields a new estimate for the Lipschitz constant of a transport map pushing forward the standard Gaussian to a measure in this class. Further connections are discussed with score-based diffusion models and improved Gaussian logarithmic Sobolev inequalities. Finally, we show that assuming only fast decay of the tails of the initial datum does not suffice to guarantee uniform log-Hessian upper bounds.

MSC2020: 26D10 (primary), 39B62, 35K05, 49Q22, 39P15 (secondary).

Keywords: Heat semigroup, log-Hessian estimates, Lipschitz transport maps, log-concavity, logarithmic Sobolev inequality, score-based diffusion models.

Micro-Application: Convolution

- let $X_i \sim \mu_i$ be independent, each μ_i satisfies a PI with constant $\gamma_i > 0$
- let $Y = X_1 + X_2 + \cdots \cdot X_N$
- then the law of Y satisfies a PI with constant $\left(\sum \gamma_i^{-1} \right)^{-1}$
- proof sketch: tensorisation, followed by transport / projection

Bounded Change-of-Measure

- suppose that μ satisfies a PI with constant $\gamma > 0$
- suppose that $\pi \equiv \mu$ strongly, i.e. $0 < \sup \mathrm{d}\pi/\mathrm{d}\mu, \sup \mathrm{d}\mu/\mathrm{d}\pi < \infty$
  - then π satisfies a PI with constant $\gamma \cdot \kappa^{-1}$, where

$$\kappa := \left(\sup \mathrm{d}\mu/\mathrm{d}\pi\right) \cdot \left(\sup \mathrm{d}\pi/\mathrm{d}\mu\right) \geq 1$$

- proof sketch:

$$\mathrm{var}_\pi\left(f\right) \leq \left(\sup \mathrm{d}\pi/\mathrm{d}\mu\right) \cdot \mathrm{var}_\mu\left(f\right)$$

$$\mathbf{E}_\mu\left[\left|\nabla f\right|^2\right] \leq \left(\sup \mathrm{d}\mu/\mathrm{d}\pi\right) \cdot \mathbf{E}_\pi\left[\left|\nabla f\right|^2\right]$$

Mixtures

- a bit more particular:
  - let (P_x) all satisfy PI (γ),
  - suppose that $\sup_{x,x'} \chi^2(P_x, P_{x'}) \leq \bar{\chi}$
    - (actual assumptions weaker)
- then the mixture $\pi = \mu P$ also satisfies a PI, with constant depending only on $(\gamma, \bar{\chi})$
- same holds for LSI

Dimension-free log-Sobolev inequalities for mixture distributions

Hong-Bin Chen [a], Sinho Chewi [b], Jonathan Niles-Weed [a,*]

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[b] *Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA*

A R T I C L E I N F O

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Mixture distribution

A B S T R A C T

We prove that if $(P_x)_{x \in \mathscr{X}}$ is a family of probability measures which satisfy the log-Sobolev inequality and whose pairwise chi-squared divergences are uniformly bounded, and μ is any mixing distribution on $\mathscr{X}$, then the mixture $\int P_x \, d\mu(x)$ satisfies a log-Sobolev inequality. In various settings of interest, the resulting log-Sobolev constant is dimension-free. In particular, our result implies a conjecture of Zimmermann and Bardet et al. that Gaussian convolutions of measures with bounded support enjoy dimension-free log-Sobolev inequalities.

Other Routes to { PI, LSI, … }

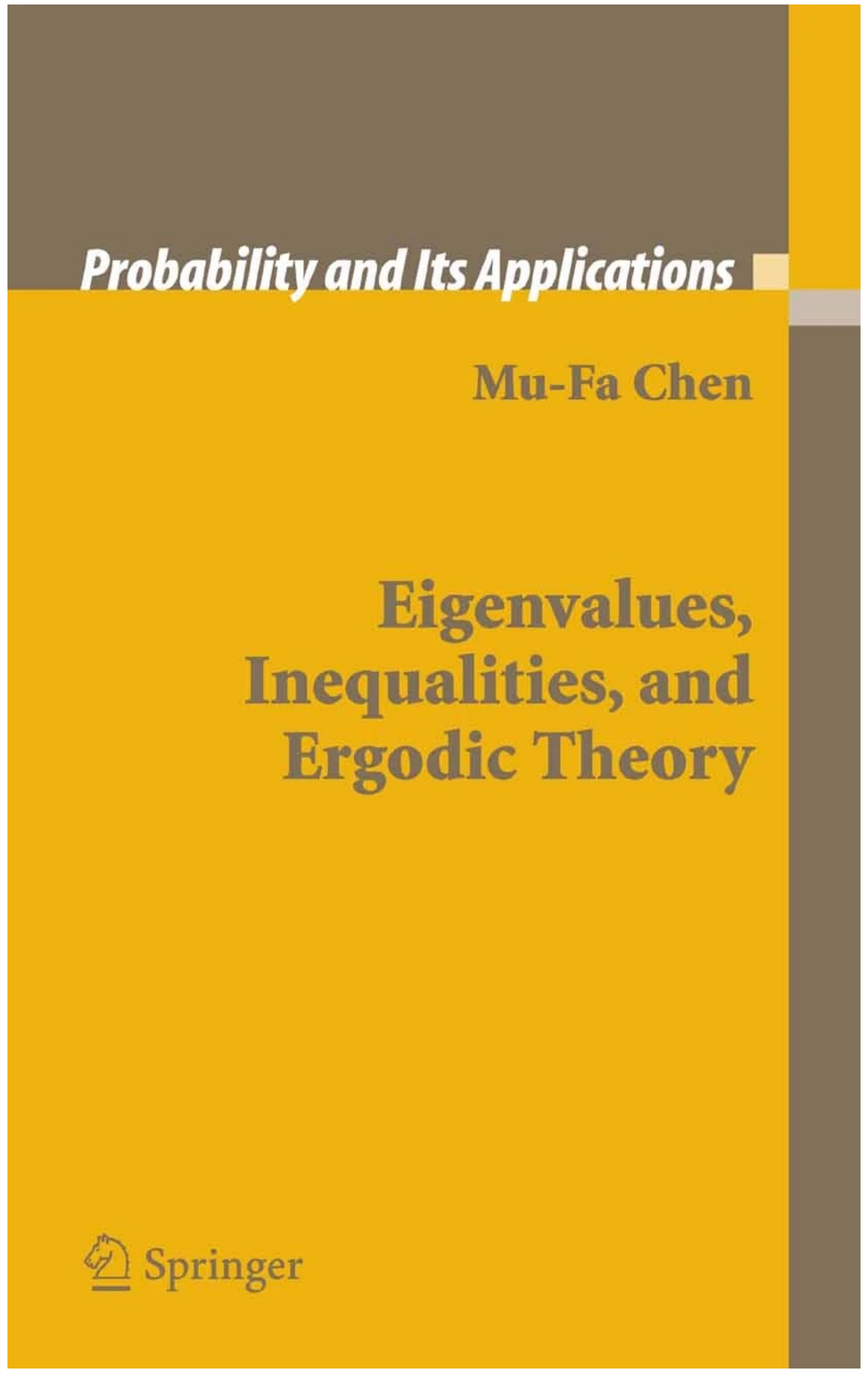
A Taster Menu

{ PI, LSI, …} from Coupling

- contraction in $\mathcal{T}_1$ yields PI with same constant
  - if dual is non-expansive in $\mathcal{T}_\infty$, then yields LSI with same constant
    - actually, can even accommodate a little bit of expansion
- can hence solve strongly log-concave setting probabilistically
  - and by being creative with the metric, can solve yet more cases
    - read { Chen-Wang, Eberle, Monmarché, … }

Transport and Functional Inequalities

- that contraction implies PI is known for some years
  - (e.g. to MF Chen, FY Wang, LM Wu, etc.)
- extension to LSI was conjectured, and recently resolved by Caputo-Salez-Münch
  - an additional ingredient is genuinely needed; counter-example of Münch
- other impacts of contractivity assumption are numerous and remarkable; well-documented by Ollivier, Djellout-Guillin-Wu



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Entropy and curvature: beyond the Peres-Tetali conjecture

Pietro Caputo, Florentin Münch and Justin Salez

February 12, 2024

Abstract

We study Markov chains with non-negative sectional curvature on finite metric spaces. Neither reversibility, nor the restriction to a particular combinatorial distance are imposed. In this level of generality, we prove that a 1-step contraction in the Wasserstein distance implies a 1-step contraction in relative entropy, by the same amount. Our result substantially strengthens a recent breakthrough of the second author, and has the advantage of being applicable to arbitrary scales. This leads to a time-varying refinement of the standard Modified Log-Sobolev Inequality (MLSI), which allows us to leverage the well-acknowledged fact that *curvature improves at large scales*. We illustrate this principle with several applications, including birth and death chains, colored exclusion processes, permutation walks, Gibbs samplers for high-temperature spin systems, and attractive zero-range dynamics. In particular, we prove a MLSI with constant equal to the minimal rate increment for the mean-field zero-range process, thereby answering a long-standing question.

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Ricci curvature of Markov chains on metric spaces

Yann Ollivier

ENS Lyon, UMPA, 46 allée d'Italie, Lyon, France

Received 20 December 2007; accepted 3 November 2008

Available online 28 November 2008

Communicated by Cédric Villani

Abstract

We define the coarse Ricci curvature of metric spaces in terms of how much small balls are closer (in Wasserstein transportation distance) than their centers are. This definition naturally extends to any Markov chain on a metric space. For a Riemannian manifold this gives back, after scaling, the value of Ricci curvature of a tangent vector. Examples of positively curved spaces for this definition include the discrete cube and discrete versions of the Ornstein–Uhlenbeck process. Moreover this generalization is consistent with the Bakry–Émery Ricci curvature for Brownian motion with a drift on a Riemannian manifold.

Positive Ricci curvature is shown to imply a spectral gap, a Lévy–Gromov–like Gaussian concentration theorem and a kind of modified logarithmic Sobolev inequality. The bounds obtained are sharp in a variety of examples.

Keywords: Ricci curvature; Markov chains; Metric geometry; Concentration of measure

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2004, Vol. 32, No. 3B, 2702–2732
DOI: 10.1214/009117904000000531

TRANSPORTATION COST-INFORMATION INEQUALITIES AND APPLICATIONS TO RANDOM DYNAMICAL SYSTEMS AND DIFFUSIONS

BY H. DJELLOUT, A. GUILLIN AND L. WU

Université Blaise Pascal, Université Paris IX Dauphine and Université Blaise Pascal

We first give a characterization of the L^1-transportation cost-information inequality on a metric space and next find some appropriate sufficient condition to transportation cost-information inequalities for dependent sequences. Applications to random dynamical systems and diffusions are studied.

Transport and Functional Inequalities

- that contraction implies PI is known for some years
  - (e.g. to MF Chen, FY Wang, LM Wu, etc.)
- extension to LSI was conjectured, and recently resolved by Caputo-Salez-Münch
  - an additional ingredient is genuinely needed; counter-example of Münch
- other impacts of contractivity assumption are numerous and remarkable; well-documented by Ollivier, Djellout-Guillin-Wu

A ‘Robust’ Curvature Approach

- suppose that we have a non-uniform curvature $\rho(x) > 0$, i.e. for all x,

$$\nabla^2\left(-\log\pi\right)(x) \succeq \rho(x)\cdot\mathbf{I}_d$$

- <u>Veysseire</u> shows that π enjoys a spectral gap of $\gamma \geq \mathbf{E}\left[\rho(X)^{-1}\right]^{-1}$
    - it is compelling to seek $\gamma \geq \mathbf{E}\left[\rho(X)\right]$; unfortunately, this *does not hold*
- <u>Cattiaux-Guillin-Fathi</u> pursue same setting, obtain new (less clean) bounds

{ PI, LSI, … } from Drift (1)

- well-known that drift and minorisation yields exponential ergodicity
  - that is, assuming
    - $\mathscr{L}V \leq -\gamma \cdot V + K$,
    - minorisation on $C_\iota := \left\{x : V(x) \leq \iota \cdot K/\gamma\right\}$ for some $\iota > 1$
- also well-known that such strategies can be quantitatively poor
  - one hint: lack of tensorisation property
  - the culprit is not drift, rather *minorisation*

{ PI, LSI, … } from Drift (2)

- circa 2008, a group of French authors took a new perspective:
  - keep the drift condition, but *replace the minorisation condition*
  - instead, require that the process satisfies a ‘restricted PI’ on C_ι, i.e.

$$\mathrm{var}_{\pi_{C_\iota}}(f) \leq \kappa_\iota \cdot \pi\left(\left| \nabla f \right|^2 \right)$$

- this strategy can yield reasonable (though not excellent) estimates
- Taghvaei-Mehta recently brought this story full circle

Lyapunov Functions and Functional Inequalities

- Bakry, Barthe, Cattiaux, Guillin - "A Simple Proof of the Poincaré Inequality for a Large Class of Probability Measures, Including the Log-Concave Case"
- actually, the strategy applies more generally, and allows to prove inequalities which are { stronger, weaker } than the basic Poincaré Inequality under { stronger, weaker } conditions; see also book chapter

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in PROBABILITY

A SIMPLE PROOF OF THE POINCARÉ INEQUALITY FOR A LARGE CLASS OF PROBABILITY MEASURES INCLUDING THE LOG-CONCAVE CASE

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AMS 2000 Subject classification: Lyapunov functions, Poincaré inequality, log-concave measure
Keywords: 26D10, 47D07, 60G10, 60J60

Lyapunov Functions and Functional Inequalities

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Lyapunov conditions for Super Poincaré inequalities

Patrick Cattiaux [a], Arnaud Guillin [b,c,*], Feng-Yu Wang [d,e], Liming Wu [f,g]

[a] *Université Paul Sabatier, Institut de Mathématiques, Laboratoire de Statistique et Probabilités, UMR C 5583, 118 route de Narbonne, F-31062 Toulouse Cedex 09, France*
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[c] *Université Blaise Pascal, 33, avenue des landais, 63177 Aubières Cedex, France*
[d] *School of Mathematical Science, Beijing Normal University, Beijing 100875, China*
[e] *Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, Swansea, UK*
[f] *Laboratoire de Mathématiques Appliquées, CNRS-UMR 6620, Université Blaise Pascal, 63177 Aubière, France*
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Communicated by C. Villani

Abstract

We show how to use Lyapunov functions to obtain functional inequalities which are stronger than Poincaré inequality (for instance logarithmic Sobolev or F-Sobolev). The case of Poincaré and weak Poincaré inequalities was studied in [D. Bakry, P. Cattiaux, A. Guillin, Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré, J. Funct. Anal. 254 (3) (2008) 727–759. Available on Mathematics arXiv:math.PR/0703355, 2007]. This approach allows us to recover and extend in a unified way some known criteria in the euclidean case (Bakry and Emery, Wang, Kusuoka and Stroock, ...).

Keywords: Ergodic processes; Lyapunov functions; Poincaré inequalities; Super Poincaré inequalities; Logarithmic Sobolev inequalities

Lyapunov Functions and Functional Inequalities

- Bakry, Barthe, Cattiaux, Guillin - "A Simple Proof of the Poincaré Inequality for a Large Class of Probability Measures, Including the Log-Concave Case"

- actually, the strategy applies more generally, and allows to prove inequalities which are { stronger, weaker } than the basic Poincaré Inequality under { stronger, weaker } conditions; see also book chapter

Functional inequalities for heavy tailed distributions and application to isoperimetry

Patrick Cattiaux* Nathael GOZLAN† Arnaud Guillin‡ Cyril Roberto§

Abstract

This paper is devoted to the study of probability measures with heavy tails. Using the Lyapunov function approach we prove that such measures satisfy different kind of functional inequalities such as weak Poincaré and weak Cheeger, weighted Poincaré and weighted Cheeger inequalities and their dual forms. Proofs are short and we cover very large situations. For product measures on $\mathbb{R}^n$ we obtain the optimal dimension dependence using the mass transportation method. Then we derive (optimal) isoperimetric inequalities. Finally we deal with spherically symmetric measures. We recover and improve many previous result.

Key words: weighted Poincaré inequalities, weighted Cheeger inequalities, Lyapunov function, weak inequalities, isoperimetric profile.

AMS 2000 Subject Classification: Primary 60E15 ; 26D10.

Submitted to EJP on January 15, 2009, final version accepted March 29, 2010.

Lyapunov Functions and Functional Inequalities

- Bakry, Barthe, Cattiaux, Guillin - “A Simple Proof of the Poincaré Inequality for a Large Class of Probability Measures, Including the Log-Concave Case"

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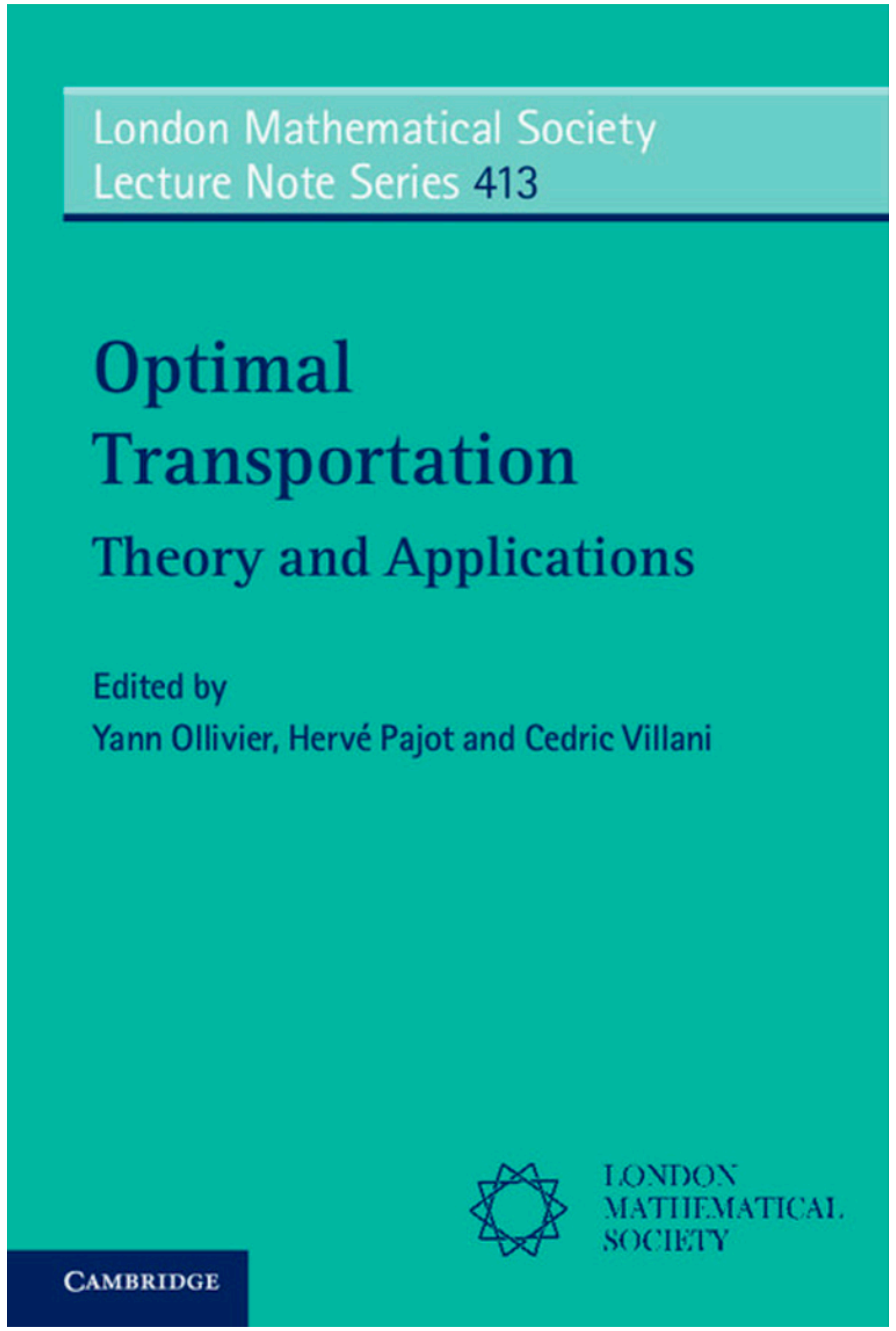

Lyapunov Functions and Functional Inequalities

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11

Functional inequalities via Lyapunov conditions

PATRICK CATTIAUX[1] AND ARNAUD GUILLIN[2]

[1] *Université de Toulouse*
[2] *Université Blaise Pascal*

Abstract

We review here some recent results by the authors, and various coauthors, on (weak, super) Poincaré inequalities, transportation-information inequalities or logarithmic Sobolev inequality via a quite simple and efficient technique: Lyapunov conditions.

Lyapunov Functions and Functional Inequalities

- Bakry, Barthe, Cattiaux, Guillin - "A Simple Proof of the Poincaré Inequality for a Large Class of Probability Measures, Including the Log-Concave Case"

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Lyapunov Functions and Functional Inequalities

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Some Miscellanea

- Roberts-Tweedie: "Geometric L2 and L1 convergence are equivalent for reversible Markov chains"
- "localisation" methods (e.g. Kannan-Lovasz-Simonovits)
- symmetry arguments
  - reflection symmetry by Barthe-Klartag
  - spherical symmetry by Bobkov, Huet, Cattiaux-Guillin-Wu

From Checking to Using

Implications of PI (1)

- suppose that $\gamma \cdot \mathrm{var}_\pi[F] \leq \mathbf{E}\left[\left|\nabla F\right|^2\right]$; write $\mathsf{C} = \mathrm{cov}_\pi\left[\mathrm{Id}\right]$
- taking $F(x) = v^\top x$, see that $\gamma \cdot \mathsf{C} \preceq \mathbf{I}_d$, and so $\gamma \leq \|\mathsf{C}\|_{\mathrm{op}}^{-1}$
- KLS Conjecture: under log-concavity, get $\gamma \geq c_{\mathrm{KLS}} \cdot \|\mathsf{C}\|_{\mathrm{op}}^{-1}$
    - interpretation: only impediment to mixing is 'long' directions
    - proven up to $\sqrt{\log d}$ by Klartag, see also Y. Chen, Klartag-Lehec

Implications of PI (2)

- let G be 1-Lipschitz-continuous, and take $F = \exp(\theta \cdot G)$
  - elementary inductive argument shows that

$$\mathbf{E}\left[\exp\left(\theta \cdot \left(G - \mathbf{E}\left[G\right]\right)\right)\right] \leq \left(1 - \frac{\theta^2}{4 \cdot \gamma}\right)^{-2}$$

  - consequence: π has lighter-than-exponential tails
    - exponential example: can't expect better in general

Implications of LSI (1)

- suppose that $\mathbf{E}\left[F^{-1} \cdot \left|\nabla F\right|^2\right] \geq 2 \cdot \lambda \cdot \mathrm{Ent}\left(F\right)$
- taking $F\left(x\right) = \exp\left(v^\top x\right)$, see that $\lambda \cdot \log \mathbf{E}\left[\exp\left(v^\top x\right)\right] \leq \frac{\left|v\right|^2}{2}$
- <u>Bizeul KLS-Type Conjecture</u>: under log-concavity,
    - if $\log \mathbf{E}\left[\exp\left(v^\top x\right)\right] \leq \frac{v^\top \mathsf{C} v}{2}$, then $\lambda \geq c_{\mathrm{Biz}} \cdot \|\mathsf{C}\|_{\mathrm{op}}^{-1}$

Implications of LSI (2)

- let G be 1-Lipschitz-continuous, and take $F = \exp\left(\theta \cdot G\right)$
  - similar elementary ('Herbst') argument shows that

$$\log\left(\mathbf{E}\left[\exp\left(\theta\cdot\left(G - \mathbf{E}\left[G\right]\right)\right)\right]\right) \leq \frac{\theta^2}{2\cdot\lambda}$$

  - consequence: π has lighter-than-Gaussian tails
    - Gaussian example: can't expect better in general

Implications of LSI (3)

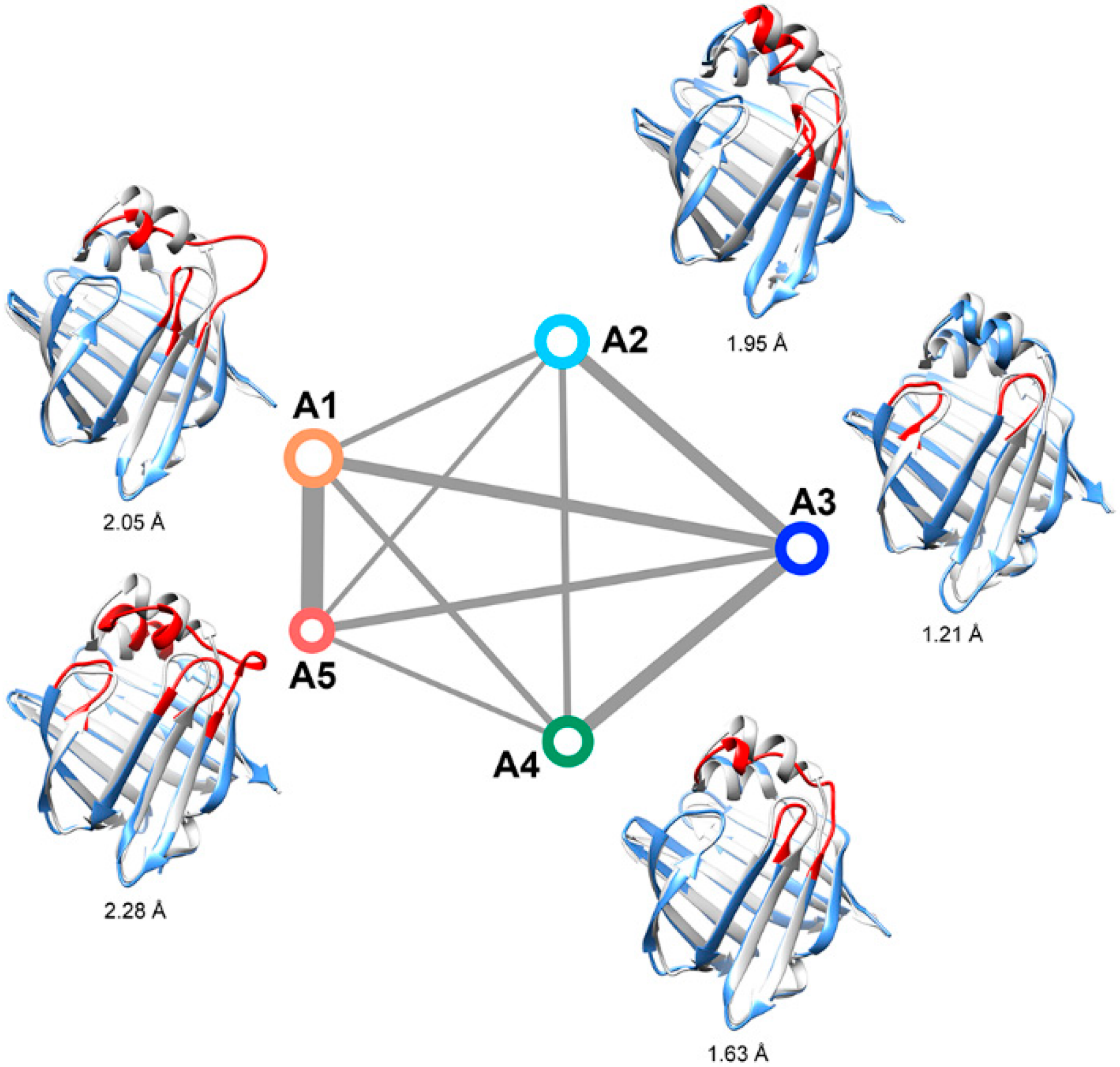
- the basic inequality shows us that we contract from one entropy to itself
- in fact, under LSI, we can contract into better norms:
  - let $p > 1, t > 0, p(t) = 1 + \left(p - 1\right) \cdot \exp(2 \cdot \lambda \cdot t)$. then

$$\left|P_t f\right|_{p(t)} \leq \left|f\right|_p$$

  - this is Nelson's "hypercontractivity"
  - actually, this also implies back the LSI

Some Related Inequalities of Interest

- { Cheeger-type / “L^1 Poincaré” } Inequalities
- { Weak, Super } Poincaré Inequalities
- { Weak, Super } Log-Sobolev Inequalities
- Weighted { Poincaré, Log-Sobolev } Inequalities
- Polyak-Łojasiewicz Inequalities in Optimisation
- A Glimpse Beyond “Energy-Entropy” Inequalities



Rapid-Fire

some directions, some references

Cheeger-Type Inequalities

- Poincaré asks that if $\pi(F) = 0$, it holds

$$\mathbf{E}\left[|F|^2\right] \lesssim \mathbf{E}\left[|\nabla F|^2\right]$$

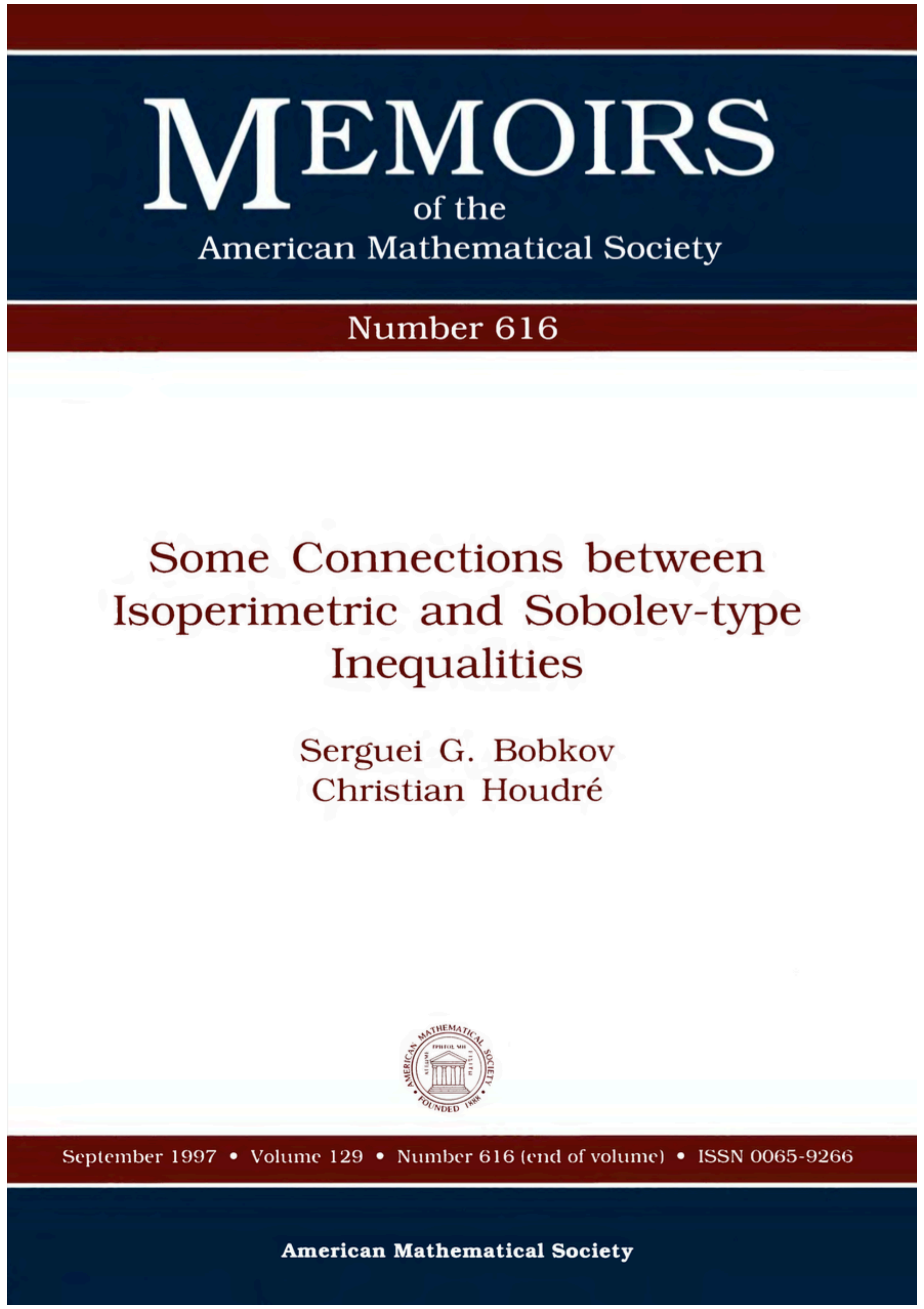
- Cheeger asks that if $\mathrm{med}_\pi(F) = 0$, it holds

$$\mathbf{E}\left[|F|\right] \lesssim \mathbf{E}\left[|\nabla F|\right]$$

- Weak Cheeger refines this to

$$\mathbf{E}\left[|F|\right] \lesssim \mathbf{E}\left[|\nabla F|\right] + \mathrm{Osc}(F)$$

  - equivalent to isoperimetric profile



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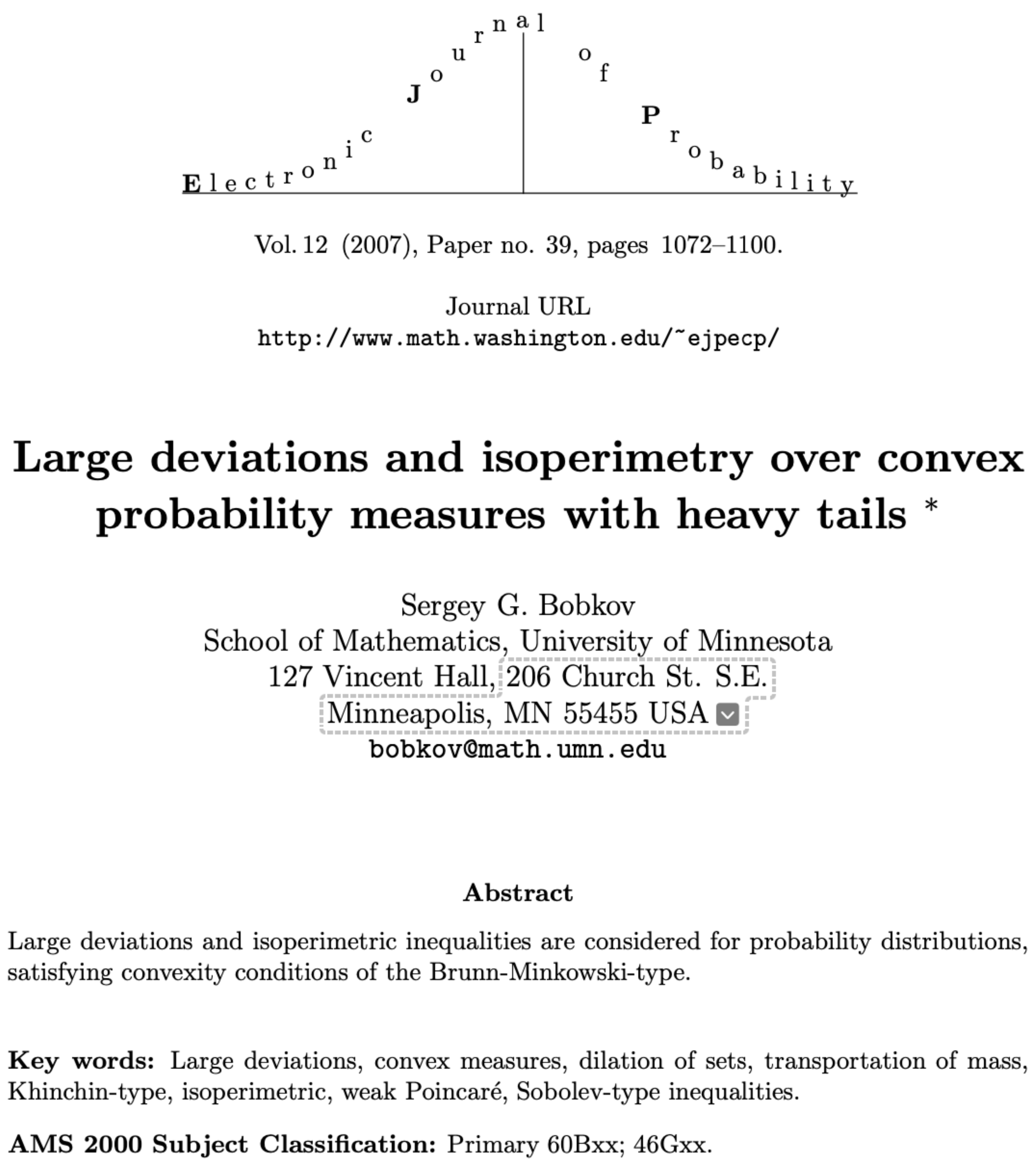

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Journal URL
http://www.math.washington.edu/~ejpecp/

Large deviations and isoperimetry over convex probability measures with heavy tails *

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Abstract

Large deviations and isoperimetric inequalities are considered for probability distributions, satisfying convexity conditions of the Brunn-Minkowski-type.

Key words: Large deviations, convex measures, dilation of sets, transportation of mass, Khinchin-type, isoperimetric, weak Poincaré, Sobolev-type inequalities.

AMS 2000 Subject Classification: Primary 60Bxx; 46Gxx.

Cheeger-Type Inequalities

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  - equivalent to isoperimetric profile

Refined Poincaré Inequalities

- Poincaré asks that if $\pi(F) = 0$, it holds

$$\mathbf{E}\left[|F|^2\right] \lesssim \mathbf{E}\left[|\nabla F|^2\right]$$

- Weak Poincaré refines this to

$$\mathbf{E}\left[|F|^2\right] \lesssim \mathbf{E}\left[|\nabla F|^2\right] + \mathrm{Osc}(F)^2$$

- Super Poincaré refines this to

$$\mathbf{E}\left[|F|^2\right] \lesssim \mathbf{E}\left[|\nabla F|^2\right] + |F|_1^2$$

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Weak Poincaré Inequalities and L^2-Convergence Rates of Markov Semigroups[1]

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In order to describe L^2-convergence rates slower than exponential, the weak Poincaré inequality is introduced. It is shown that the convergence rate of a Markov semigroup and the corresponding weak Poincaré inequality can be determined by each other. Conditions for the weak Poincaré inequality to hold are presented, which are easy to check and which hold in many applications. The weak Poincaré inequality is also studied by using isoperimetric inequalities for diffusion and jump processes. Some typical examples are given to illustrate the general results. In particular, our results are applied to the stochastic quantization of field theory in finite volume. Moreover, a sharp criterion of weak Poincaré inequalities is presented for Poisson measures on configuration spaces.

Key Words: weak Poincaré inequality; isoperimetric inequality; L^2-convergence rate; Markov semigroup.

Refined Poincaré Inequalities

- Poincaré asks that if $\pi(F) = 0$, it holds

$$\mathbf{E}\left[\left|F\right|^2\right] \lesssim \mathbf{E}\left[\left|\nabla F\right|^2\right]$$

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Journal of Functional Analysis **170**, 219–245 (2000)
Article ID jfan.1999.3516, available online at http://www.idealibrary.com on IDEAL®

Functional Inequalities for Empty Essential Spectrum

Feng-Yu Wang[1, 2]

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Received April 19, 1999; revised July 4, 1999; accepted August 21, 1999

In terms of the equivalence of Poincaré inequality and the existence of spectral gap, the super-Poincaré inequality is suggested in the paper for the study of essential spectrum. It is proved for symmetric diffusions that, such an inequality is equivalent to empty essential spectrum of the corresponding diffusion operator. This inequality recovers known Sobolev and Nash type ones. It is also equivalent to an isoperimetric inequality provided the curvature of the operator is bounded from below. Some results are also proved for a more general setting including symmetric jump processes. Moreover, estimates of inequality constants are also presented, which lead to a proof of a result on ultracontractivity suggested recently by D. Stroock. Finally, concentration of reference measures for super-Poincaré inequalities is studied, the resulting estimates extend previous ones for Poincaré and log-Sobolev inequalities.

Key Words: Super-Poincaré inequality; *F*-Sobolev inequality; Nash type inequality; isoperimetric inequality; essential spectrum; ultracontractivity; concentration.

Refined Poincaré Inequalities

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Refined Entropic Inequalities

- Log-Sobolev asks that

$$\mathrm{Ent}\left(F^2\right) \lesssim \mathbf{E}\left[\left|\nabla F\right|^2\right]$$

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Probab. Theory Relat. Fields (2007) 139:563–603
DOI 10.1007/s00440-007-0054-5

Weak logarithmic Sobolev inequalities and entropic convergence

P. Cattiaux · I. Gentil · A. Guillin

Received: 17 November 2005 / Revised: 4 December 2006 / Published online: 24 March 2007

Abstract In this paper we introduce and study a weakened form of logarithmic Sobolev inequalities in connection with various others functional inequalities (weak Poincaré inequalities, general Beckner inequalities, etc.). We also discuss the quantitative behaviour of relative entropy along a symmetric diffusion semi-group. In particular, we exhibit an example where Poincaré inequality can not be used for deriving entropic convergence whence weak logarithmic Sobolev inequality ensures the result.

Keywords Logarithmic Sobolev inequalities · Concentration inequalities · Entropy

Mathematics Subject Classification (2000) 26D10 · 60E15

Refined Entropic Inequalities

- Log-Sobolev asks that

$$\text{Ent}\left(F^2\right) \lesssim \mathbf{E}\left[\left|\nabla F\right|^2\right]$$

- Weak Log-Sobolev refines this to

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J. Math. Kyoto Univ. (JMKYAZ)
38-2 (1998), 295-318

Herbst inequalities for supercontractive semigroups

By

Leonard GROSS* and Oscar ROTHAUS

March 31, 1997

Refined Entropic Inequalities

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Weighted Inequalities

- Poincaré asks that if $\pi(F) = 0$, it holds

$$\mathbf{E}\left[\left|F\right|^2\right] \lesssim \mathbf{E}\left[\left|\nabla F\right|^2\right]$$

- Weighted Poincaré instead asks for

$$\mathbf{E}\left[\left|F\right|^2\right] \lesssim \mathbf{E}\left[\omega \cdot \left|\nabla F\right|^2\right]$$

- ω relates to Langevin w/ multiplicative noise
    - can imagine 'preconditioned' Poincaré
- can seek the same for { LSI, etc. }

The Annals of Probability
2009, Vol. 37, No. 2, 403–427
DOI: 10.1214/08-AOP407

WEIGHTED POINCARÉ-TYPE INEQUALITIES FOR CAUCHY AND OTHER CONVEX MEASURES[1]

BY SERGEY G. BOBKOV AND MICHEL LEDOUX

University of Minnesota and Université Paul-Sabatier

Brascamp–Lieb-type, weighted Poincaré-type and related analytic inequalities are studied for multidimensional Cauchy distributions and more general κ-concave probability measures (in the hierarchy of convex measures). In analogy with the limiting (infinite-dimensional log-concave) Gaussian model, the weighted inequalities fully describe the measure concentration and large deviation properties of this family of measures. Cheeger-type isoperimetric inequalities are investigated similarly, giving rise to a common weight in the class of concave probability measures under consideration.

Weighted Inequalities

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$$\mathbf{E}\left[\left|F\right|^2\right] \lesssim \mathbf{E}\left[\left|\nabla F\right|^2\right]$$

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Weighted Poincaré inequalities, concentration inequalities and tail bounds related to Stein kernels in dimension one

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We investigate links between the so-called Stein's density approach in dimension one and some functional and concentration inequalities. We show that measures having a finite first moment and a density with connected support satisfy a weighted Poincaré inequality with the weight being the Stein kernel, that indeed exists and is unique in this case. Furthermore, we prove weighted log-Sobolev and asymmetric Brascamp–Lieb type inequalities related to Stein kernels. We also show that existence of a uniformly bounded Stein kernel is sufficient to ensure a positive Cheeger isoperimetric constant. Then we derive new concentration inequalities. In particular, we prove generalized Mills' type inequalities when a Stein kernel is uniformly bounded and sub-gamma concentration for Lipschitz functions of a variable with a sub-linear Stein kernel. More generally, when some exponential moments are finite, the Laplace transform of the random variable of interest is shown to bounded from above by the Laplace transform of the Stein kernel. Along the way, we prove a general lemma for bounding the Laplace transform of a random variable, that may be of independent interest. We also provide density and tail formulas as well as tail bounds, generalizing previous results that where obtained in the context of Malliavin calculus.

Keywords: concentration inequality; covariance identity; isoperimetric constant; Stein kernel; tail bound; weighted log-Sobolev inequality; weighted Poincaré inequality

Weighted Inequalities

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- ω relates to Langevin w/ multiplicative noise
  - can imagine ‘preconditioned’ Poincaré
- can seek the same for { LSI, etc. }

Functional Inequalities in Optimisation

- our inequalities look like

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}(x(t)) \gtrsim \mathscr{F}(x(t)) - \mathscr{F}_\star$$

- for deterministic gradient flows, this reads as

$$\left|\nabla F\right|^2 \gtrsim F - F_\star$$

- 'Polyak-Łojasiewicz Inequality'
- can hold for non-convex F
  - e.g. nonlinear least-squares, deep learning

Linear Convergence of Gradient and Proximal-Gradient Methods Under the Polyak-Łojasiewicz Condition

Hamed Karimi, Julie Nutini, and Mark Schmidt(✉)

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Abstract. In 1963, Polyak proposed a simple condition that is sufficient to show a global linear convergence rate for gradient descent. This condition is a special case of the Łojasiewicz inequality proposed in the same year, and it does not require strong convexity (or even convexity). In this work, we show that this much-older Polyak-Łojasiewicz (PL) inequality is actually weaker than the main conditions that have been explored to show linear convergence rates without strong convexity over the last 25 years. We also use the PL inequality to give new analyses of coordinate descent and stochastic gradient for many non-strongly-convex (and some non-convex) functions. We further propose a generalization that applies to proximal-gradient methods for non-smooth optimization, leading to simple proofs of linear convergence for support vector machines and L1-regularized least squares without additional assumptions.

Keywords: Gradient descent · Coordinate descent · Stochastic gradient · Variance-reduction · Boosting · Support vector machines · L1-regularization

Beyond “Energy-Entropy” Inequalities

- i focused on “energy-entropy” inequalities
- there is an extended hierarchy of related inequalities, each of great independent interest in concentration and geometry
- in general, the implications ‘go downwards’
- under a curvature lower bound, these implications can also be reversed to ‘go upwards’

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Uspekhi Mat. Nauk **67**:5 3–110
DOI 10.1070/RM2012v067n05ABEH004808

Dedicated to the centenary of the birth of Leonid Vital’evich Kantorovich

The Monge–Kantorovich problem: achievements, connections, and perspectives

V. I. Bogachev and A. V. Kolesnikov

Abstract. This article gives a survey of recent research related to the Monge–Kantorovich problem. Principle results are presented on the existence of solutions and their properties both in the Monge optimal transportation problem and the Kantorovich optimal plan problem, along with results on the connections between both problems and the cases when they are equivalent. Diverse applications of these problems in non-linear analysis, probability theory, and differential geometry are discussed.

Bibliography: 196 titles.

Keywords: Monge problem, Kantorovich problem, optimal transportation, transport inequality, Kantorovich–Rubinshtein metric.

Beyond "Energy-Entropy" Inequalities

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3.5. The hierarchy of inequalities

The examples considered above reveal a certain regularity. In the class of probability measures there is the following hierarchy of functional inequalities:

1) isoperimetric inequalities $\mathscr{I}_\mu(t) \geqslant t\varphi(t),\ t \leqslant \frac{1}{2}$

$\Downarrow$

2) Sobolev-type inequalities $\mathrm{Ent}_\mu f^2 \leqslant \lambda_S \int c^*\left(\left\|\frac{\nabla f}{f}\right\|\right) f^2\, d\mu$

$\Downarrow$

3) transport inequalities $W_c(\mu, f\cdot\mu) \leqslant \lambda_T\ \mathrm{Ent}_\mu f$

$\Downarrow$

4) concentration inequalities $\mu(A^r) \geqslant 1 - \mathrm{e}^{-\lambda_C c(r)}$

$\Downarrow$

5) exponential integrability $\exists \varepsilon > 0\colon\ \int \mathrm{e}^{\varepsilon c(|x|)}\, \mu(dx) < \infty.$

Beyond “Energy-Entropy” Inequalities

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On the role of convexity in isoperimetry, spectral gap and concentration

Emanuel Milman

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Abstract We show that for convex domains in Euclidean space, Cheeger’s isoperimetric inequality, spectral gap of the Neumann Laplacian, exponential concentration of Lipschitz functions, and the a-priori weakest requirement that Lipschitz functions have *arbitrarily slow* uniform tail-decay, are all quantitatively equivalent (to within universal constants, independent of the dimension). This substantially extends previous results of Maz’ya, Cheeger, Gromov–Milman, Buser and Ledoux. As an application, we conclude a sharp quantitative stability result for the spectral gap of convex domains under convex perturbations which preserve volume (up to constants) and under maps which are “on-average” Lipschitz. We also provide a new characterization (up to constants) of the spectral gap of a convex domain, as one over the square of the average distance from the “worst” subset having half the measure of the domain. In addition, we easily recover and extend many previously known lower bounds on the spectral gap of convex domains, due to Payne–Weinberger, Li–Yau, Kannan–Lovász–Simonovits, Bobkov and Sodin. The proof involves estimates on the diffusion semi-group following Bakry–Ledoux and a result from Riemannian Geometry on the concavity of the isoperimetric profile. Our results extend to the more general setting of Riemannian manifolds with density which satisfy the $CD(0, \infty)$ curvature-dimension condition of Bakry-Émery.

Beyond “Energy-Entropy” Inequalities

- i focused on “energy-entropy” inequalities
- there is an extended hierarchy of related inequalities, each of great independent interest in concentration and geometry
- in general, the implications ‘go downwards’
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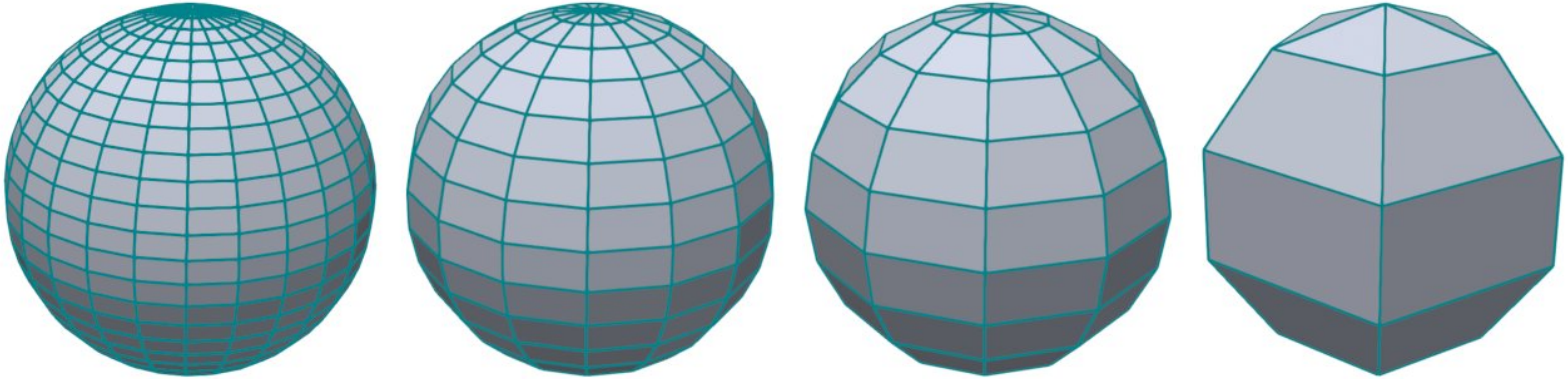
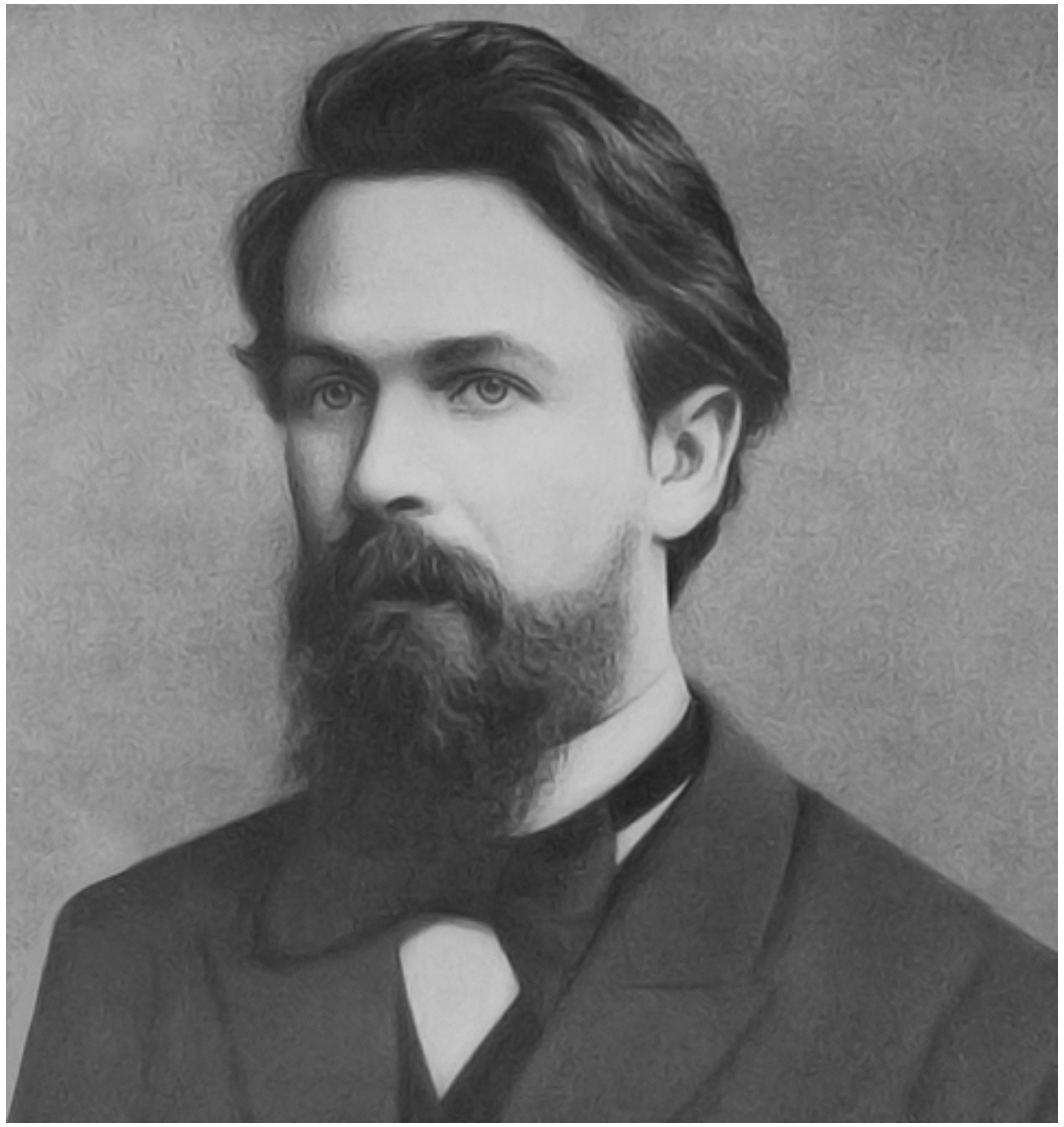


A Moment to Breathe



Part 2: Discrete-Time Markov Chains

or, ‘Functional Analysis of Practical MCMC’

Setting the Scene

- some of the Langevin picture is quite particular
  - … but luckily, much of what we have discussed is quite general!
- let us work with a positive, π-reversible Markov kernel P
- let us also focus on the L^2 or ‘Poincaré’ picture

“… somehow, Grönwall returned”

Contraction in $\mathrm{L}^2(\pi)$

- as before, we might aim to show that whenever $\pi(f) = 0$,

$$\|Pf\|_2 \leq (1-\gamma) \cdot \|f\|_2$$

- squaring and expanding, this is tantamount to the claim that

$$\mathscr{E}_{P\star P}(f,f) \gtrsim \mathrm{var}_\pi(f)$$

for some 'interesting' object $\mathscr{E}_{P\star P}$

Dirichlet Energy Forms in Discrete Time

- in our general setting, define

$$\begin{aligned} \mathscr{E}_T\left(f, g\right) &= \left\langle f, \left(\mathrm{Id} - T\right) g \right\rangle \\ &= \frac{1}{2} \int \pi\left(\mathrm{d}x\right) \cdot T\left(x, \mathrm{d}y\right) \cdot \Delta_{f,g}\left(x, y\right) \end{aligned}$$

where $$\Delta_{f,g}\left(x, y\right) = \left(f(x) - f\left(y\right)\right) \cdot \left(g(x) - g\left(y\right)\right)$$

- for reversible, positive chains, $\mathscr{E}_P$ and $\mathscr{E}_{P\star P}$ are directly comparable

- $\mathscr{E}_P$ will be easier for us to work with

Connecting to the Diffusion Picture

- let $t > 0$, let P_t be the transition kernel for OLD (π)
- as $t \to 0^+$, one checks that $t^{-1} \cdot \mathscr{E}_{P_t}\left(f, g\right) \to \mathbf{E}\left[\langle \nabla f, \nabla g\rangle\right]$
  - ‘proof’: $\mathrm{Id} - P_t \approx - t \cdot \mathscr{L}$
- so, we have ‘genuinely’ generalised the diffusion picture
- our ‘Poincaré inequality’ is now

$$\mathscr{E}_P\left(f,f\right) \geq \gamma \cdot \mathrm{var}_\pi\left(f\right)$$

Back to Square One?

- instructive to revisit earlier principles, and assess what remains
  - tensorisation, bounded change of measure, mixtures
  - metric contractivity, ‘typical’ curvature, Lyapunov arguments
- when do we lose?
  - usually: when the *chain rule* was important

An Example: What *is* the LSI?

- we might reasonably propose either of

$$\frac{\lambda}{2} \cdot \mathsf{Ent}\left(F^2\right) \leq \mathscr{E}\left(F, F\right)$$

$$2 \cdot \lambda \cdot \mathsf{Ent}\left(F\right) \leq \mathscr{E}\left(F, \log F\right)$$

- in general, the former is *strictly stronger*; see e.g. Bobkov-Tetali
- Salez-Tikhomirov-Youssef clarify when the latter can ~imply the former

Application to MCMC: Case Studies (1)

- some warm-ups in the propose-accept-reject paradigm
  - changing the acceptance probability from Metropolis to Barker (to …)
  - bounded change-of-measure for target
  - bounded change-of-measure for proposal kernel

Application to MCMC: Case Studies (2)

- exact-approximate Metropolis-Hastings methods
  - safely approximating the accept-reject decision
- exact-approximate Gibbs sampling
  - safely approximating the conditional simulation step

Set-Up: Metropolis-Hastings kernels

- protocol: at x,
  - propose move to $y \sim Q\left(x, \mathrm{d}y\right)$
  - compute $r\left(x, y\right) = \pi\left(y\right) \cdot Q\left(y, x\right) / \pi(x) \cdot Q\left(x, y\right)$
  - with probability $\alpha\left(x, y\right) = \min\left\{1, r\left(x, y\right)\right\}$, move to y
    - otherwise, stay at x

The Dirichlet Energy Form

- in this paradigm, the Dirichlet form writes as

$$\mathscr{E}_P\left(f, g\right) = \frac{1}{2}\int \pi\left(\mathrm{d}x\right) \cdot Q\left(x, \mathrm{d}y\right) \cdot \alpha\left(x, y\right) \cdot \Delta_{f,g}\left(x, y\right)$$

- note that rejections are ‘ignored’, rather than ‘separated’

Some Warm-Up Applications

- these are *a little bit* interesting, but mostly illustrate the ‘tricks of the trade’
- qualitatively, they resemble the earlier “bounded change-of-measure”
- the emphasis is that this trick can now be applied in a few different places

Warm-Up: Acceptance Functions (1)

- fix π, Q, but consider accepting moves with either of

$$\alpha^{\mathrm{MH}} = \min\{1,r\} \qquad \alpha^{\mathrm{B}} = \frac{r}{1+r}.$$

- both generate π-reversible chains, so fit for purpose
  - more generally, if $\beta : \mathbf{R}^+ \to [0,1]$ satisfies $\beta(r) = r \cdot \beta\left(r^{-1}\right)$, then we call it a 'balancing function', and can use $\alpha^{\beta} = \beta(r)$
- how shall we choose between the two options?

Warm-Up: Acceptance Functions (2)

- some painless verification:

$$\frac{1}{2} \cdot \min\{1, r\} \leq \frac{r}{1+r} \leq \min\{1, r\}$$

- consequence:

$$\frac{1}{2} \cdot \mathscr{E}_P^{\mathrm{MH}}\left(f, f\right) \leq \mathscr{E}_P^{\mathrm{B}}\left(f, f\right) \leq \mathscr{E}_P^{\mathrm{MH}}\left(f, f\right)$$

- conclusion: other things being equal, no worse off by using MH
  - … but if you do use the Barker acceptance, you won't lose too much

Warm-Up: Change of Proposal (1)

- fix π, and consider choosing between different proposal kernels Q_1, Q_2
  - suppose that they are each symmetric, so that $\alpha\left(x,y\right) = \min\left\{1, \pi\left(y\right)/\pi\left(x\right)\right\}$
- suppose that uniformly in x, y, we can bound

$$0 < \delta^- \leq \frac{Q_2\left(x,y\right)}{Q_1\left(x,y\right)} \leq \delta^+ < \infty$$

  - LHS: tails of Q_2 are at least as heavy as those of Q_1 (similar on RHS)

Warm-Up: Change of Proposal (2)

- suppose that uniformly in x, y, we can bound

$$\delta^{-} \leq \frac{Q_2\left(x, y\right)}{Q_1\left(x, y\right)} \leq \delta^{+}$$

- consequence:

$$\delta^{-} \cdot \mathscr{E}_{P_1}\left(f, f\right) \leq \mathscr{E}_{P_2}\left(f, f\right) \leq \delta^{+} \cdot \mathscr{E}_{P_1}\left(f, f\right)$$

- one interpretation: slightly heavier-tailed proposals can't hurt too much

Warm-Up: Change of Target (1)

- fix some symmetric Q, and consider changing target from π to π'
  - as earlier, assume that $0 < \sup \mathrm{d}\pi/\mathrm{d}\pi', \sup \mathrm{d}\pi'/\mathrm{d}\pi < \infty$
- want to compare

$$\mathscr{E}_P\left(f,f\right) = \frac{1}{2}\int \pi\left(\mathrm{d}x\right) \cdot Q\left(x, \mathrm{d}y\right) \cdot \alpha\left(x,y\right) \cdot \Delta_{f,f}\left(x,y\right)$$

$$\mathscr{E}_{P'}\left(f,f\right) = \frac{1}{2}\int \pi'\left(\mathrm{d}x\right) \cdot Q\left(x, \mathrm{d}y\right) \cdot \alpha'\left(x,y\right) \cdot \Delta_{f,f}\left(x,y\right)$$

Warm-Up: Change of Target (2)

- first, study acceptance probabilities

$$\begin{aligned}\alpha'\left(x,y\right) &= \min\left\{1,\pi'\left(y\right)/\pi'(x)\right\}\\ &\geq \min\left\{1,\kappa^{-1}\cdot\pi\left(y\right)/\pi(x)\right\}\\ &\geq \kappa^{-1}\cdot\min\left\{1,\pi\left(y\right)/\pi(x)\right\}\\ &= \kappa^{-1}\cdot\alpha\left(x,y\right)\end{aligned}$$

where $\kappa = \left(\sup \mathrm{d}\pi/\mathrm{d}\pi'\right)\left(\sup \mathrm{d}\pi'/\mathrm{d}\pi\right)$

Warm-Up: Change of Target (3)

- follows that

$$
\begin{aligned}
\mathscr{E}_P\left(f,f\right) &= \frac{1}{2}\int \pi\left(\mathrm{d}x\right)\cdot Q\left(x,\mathrm{d}y\right)\cdot\alpha\left(x,y\right)\cdot\Delta_{f,f}\left(x,y\right) \\
&\leq \left(\sup\mathrm{d}\pi/\mathrm{d}\pi'\right)\cdot\kappa\cdot\frac{1}{2}\int\pi'\left(\mathrm{d}x\right)\cdot Q\left(x,\mathrm{d}y\right)\cdot\alpha'\left(x,y\right)\cdot\Delta_{f,f}\left(x,y\right) \\
&= \left(\sup\mathrm{d}\pi/\mathrm{d}\pi'\right)\cdot\kappa\cdot\mathscr{E}_{P'}\left(f,f\right)
\end{aligned}
$$

- old argument still works to show that $\mathrm{var}_{\pi'}\left(f\right)\leq\left(\sup\mathrm{d}\pi/\mathrm{d}\pi'\right)\cdot\mathrm{var}_{\pi}\left(f\right)$

Warm-Up: Change of Target (4)

- so, assuming that for all suitable f,

$$\gamma \cdot \mathrm{var}_{\pi}\left(f\right) \leq \mathscr{E}_{P}\left(f,f\right),$$

we can argue that

$$\begin{aligned}
\mathrm{var}_{\pi'}\left(f\right) &\leq \left(\sup \mathrm{d}\pi/\mathrm{d}\pi'\right) \cdot \mathrm{var}_{\pi}\left(f\right) \\
&\leq \left(\sup \mathrm{d}\pi/\mathrm{d}\pi'\right) \cdot \gamma^{-1} \cdot \mathscr{E}_{P}\left(f,f\right) \\
&\leq \left(\sup \mathrm{d}\pi/\mathrm{d}\pi'\right) \cdot \gamma^{-1} \cdot \left(\sup \mathrm{d}\pi'/\mathrm{d}\pi\right) \cdot \kappa \cdot \mathscr{E}_{P'}\left(f,f\right) \\
&= \gamma^{-1} \cdot \kappa^{2} \cdot \mathscr{E}_{P'}\left(f,f\right)
\end{aligned}$$

i.e. we preserve the spectral gap, up to a factor of κ^2

Improved Argument (1)

- studying target and acceptance separately is actually lossy
- useful to jointly study

$$
\begin{aligned}
J\left(\mathrm{d}\left(x, y\right)\right) &:= \pi\left(\mathrm{d}x\right) \cdot Q\left(x, \mathrm{d}y\right) \cdot \alpha\left(x, y\right) \\
&= \min\left\{\pi\left(\mathrm{d}x\right) \cdot Q\left(x, \mathrm{d}y\right), \pi\left(\mathrm{d}y\right) \cdot Q\left(y, \mathrm{d}x\right)\right\}
\end{aligned}
$$

  - changing π to π' incurs a factor of $\left(\sup \mathrm{d}\pi / \mathrm{d}\pi'\right)$
- sharpens to $\mathscr{E}_P\left(f, f\right) \leq \left(\sup \mathrm{d}\pi / \mathrm{d}\pi'\right) \cdot \mathscr{E}_{P'}\left(f, f\right)$
  - sharpens to $\mathrm{var}_{\pi'}\left(f\right) \leq \gamma^{-1} \cdot \kappa \cdot \mathscr{E}_{P'}\left(f, f\right)$

“Exact-Approximate” Methods

- the upcoming applications are instances of “exact-approximate” Monte Carlo methods
- these arise in problems where some ‘standard’ method is not possible to implement directly, due to some computational intractability
- the general idea is to replace the intractable part with a suitable Monte Carlo approximation, defining a new ‘approximate’ process
  - … which nevertheless *has the right limiting behaviour*
- these can be of substantial interest for various statistical models

Application: Exact-Approximate MH (1)

- in some settings, we cannot evaluate $\pi(x)$, even up to a constant,
    - … but we can estimate it unbiasedly
- formalisation:
    - draw $W \sim \omega_x$ s.t. $\mathbf{E}^x[W] = 1$, and observe $\widehat{\pi(x;w)} = \pi(x) \cdot w$
- using this estimator in an MH accept-reject decision *remains valid*

'the pseudo-marginal approach'

Set-Up: PMMH kernels

- protocol: at (x, w),
  - propose move to $\left(y, u\right) \sim Q\left(x, \mathrm{d}y\right) \cdot \omega_y\left(\mathrm{d}u\right)$
  - compute $r\left((x, w), \left(y, u\right)\right) = \widehat{\pi\left(y; u\right)} \cdot Q\left(y, x\right) / \widehat{\pi\left(x; w\right)} \cdot Q\left(x, y\right)$
  - with probability $\alpha\left((x, w), \left(y, u\right)\right) = \min\left\{1, r\right\}$, move to $\left(y, u\right)$
    - otherwise, stay at (x, w)

Application: Exact-Approximate MH (2)

- for PMMH, the Dirichlet Energy Form reads as

$$\mathscr{E}_{\tilde{P}}\left(f,g\right)=\frac{1}{2}\int\pi\left(\mathrm{d}x\right)\cdot\omega_{x}\left(\mathrm{d}w\right)\cdot w\cdot Q\left(x,\mathrm{d}y\right)\cdot\omega_{y}\left(\mathrm{d}u\right)\cdot\alpha\left(x,w,y,u\right)\cdot\Delta_{f,g}$$

- the ‘ideal’ algorithm would instead have Dirichlet Energy Form

$$\mathscr{E}_{P}\left(f,g\right)=\frac{1}{2}\int\pi\left(\mathrm{d}x\right)\cdot\omega_{x}\left(\mathrm{d}w\right)\cdot w\cdot Q\left(x,\mathrm{d}y\right)\cdot\omega_{y}\left(\mathrm{d}u\right)\cdot u\cdot\alpha\left(x,y\right)\cdot\Delta_{f,g}$$

Negative Result

- suppose that f depends only on x, not on w. then,

$$
\begin{aligned}
\mathscr{E}_{\tilde{P}}\left(f,f\right) &= \frac{1}{2}\int \pi\left(\mathrm{d}x\right)\cdot\omega_x\left(\mathrm{d}w\right)\cdot w\cdot Q\left(x,\mathrm{d}y\right)\cdot\omega_y\left(\mathrm{d}u\right)\cdot\alpha\left(x,w,y,u\right)\cdot\Delta_{f,f} \\
&= \frac{1}{2}\int \pi\left(\mathrm{d}x\right)\cdot\omega_x\left(\mathrm{d}w\right)\cdot w\cdot Q\left(x,\mathrm{d}y\right)\cdot\omega_y\left(\mathrm{d}u\right)\cdot\min\left\{1,r\left(x,y\right)\cdot\frac{u}{w}\right\}\cdot\Delta_{f,f} \\
&\overset{\text{Jen}}{\leq} \frac{1}{2}\int \pi\left(\mathrm{d}x\right)\cdot\omega_x\left(\mathrm{d}w\right)\cdot w\cdot Q\left(x,\mathrm{d}y\right)\cdot\min\left\{1,r\left(x,y\right)\cdot\frac{1}{w}\right\}\cdot\Delta_{f,f} \\
&= \frac{1}{2}\int \pi\left(\mathrm{d}x\right)\cdot\omega_x\left(\mathrm{d}w\right)\cdot Q\left(x,\mathrm{d}y\right)\cdot\min\left\{w,r\left(x,y\right)\right\}\cdot\Delta_{f,f} \\
&\overset{\text{Jen}}{\leq} \frac{1}{2}\int \pi\left(\mathrm{d}x\right)\cdot Q\left(x,\mathrm{d}y\right)\cdot\min\left\{1,r\left(x,y\right)\right\}\cdot\Delta_{f,f} \\
&= \mathscr{E}_P\left(f,f\right)
\end{aligned}
$$

Positive Result

- suppose that $w \leq \bar{w}$ almost surely. then,

$$
\begin{aligned}
\mathscr{E}_{\tilde{P}}\left(f,f\right) &= \frac{1}{2}\int \pi\left(\mathrm{d}x\right)\cdot\omega_x\left(\mathrm{d}w\right)\cdot w\cdot Q\left(x,\mathrm{d}y\right)\cdot\omega_y\left(\mathrm{d}u\right)\cdot\alpha\left(x,w,y,u\right)\cdot\Delta_{f,f} \\
&= \frac{1}{2}\int \pi\left(\mathrm{d}x\right)\cdot\omega_x\left(\mathrm{d}w\right)\cdot w\cdot Q\left(x,\mathrm{d}y\right)\cdot\omega_y\left(\mathrm{d}u\right)\cdot\min\left\{1,r\left(x,y\right)\cdot\frac{u}{w}\right\}\cdot\Delta_{f,f} \\
&= \frac{1}{2}\int \pi\left(\mathrm{d}x\right)\cdot\omega_x\left(\mathrm{d}w\right)\cdot w\cdot Q\left(x,\mathrm{d}y\right)\cdot\omega_y\left(\mathrm{d}u\right)\cdot u\cdot\min\left\{\frac{1}{u},r\left(x,y\right)\cdot\frac{1}{w}\right\}\cdot\Delta_{f,f} \\
&\geq \bar{w}^{-1}\cdot\left(\frac{1}{2}\int \pi\left(\mathrm{d}x\right)\cdot\omega_x\left(\mathrm{d}w\right)\cdot w\cdot Q\left(x,\mathrm{d}y\right)\cdot\omega_y\left(\mathrm{d}u\right)\cdot u\cdot\min\left\{1,r\left(x,y\right)\right\}\cdot\Delta_{f,f}\right) \\
&= \bar{w}^{-1}\cdot\mathscr{E}_{P}\left(f,f\right)
\end{aligned}
$$

Comments

- negative result doesn't easily hold for all functions, but is still useful
- positive result is illustrative of how to handle 'strong' comparisons
  - start by understanding 'uniformly-bounded perturbation' setting
- for more advanced versions, need to use 'Metropolis lemma'

$$\min\{1, a \cdot b\} \geq \min\{1, a\} \cdot \min\{1, b\}$$

  - often well-suited to MCMC applications
- in each case, important to identify good target of comparison

Application: Exact-Approximate Gibbs (1)

- exact-approximate Gibbs sampling
  - safely approximating the conditional simulation step
  - think of Metropolis-within-Gibbs, Hybrid Slice Sampling, etc.

Set-Up: (EA-)RSGS kernels

- protocol: at x,
  - sample $i \sim \text{Categorical}\left(\underline{\lambda}\right)$
  - sample $x_i' \sim \pi_i\left(\mathrm{d}x_i' \mid x_{-i}\right)$; overwrite x_i
- exact-approximate setting:
  - for each $\left(i, x_{-i}\right)$, kernel $K_i\left(\,\cdot\,,\,\cdot\,\cdot \mid x_{-i}\right)$ is { reversible, positive } for $\pi_i\left(\,\cdot \mid x_{-i}\right)$
  - replace "sample $x_i' \sim \pi_i\left(\mathrm{d}x_i' \mid x_{-i}\right)$" with "sample $x_i' \sim K_i\left(x_i, \mathrm{d}x_i' \mid x_{-i}\right)$"

Application: Exact-Approximate Gibbs (2)

- for RSGS, the Dirichlet Energy Form reads as

$$\mathscr{E}_P\left(f, g\right) = \sum_{i \in \mathscr{I}} \lambda_i \cdot \int \pi_{-i}\left(\mathrm{d}x_{-i}\right) \cdot \mathrm{cov}_{\pi_i\left(\,\cdot\,\mid x_{-i}\right)}\left(f, g\right)$$

- for EA-RSGS, it instead reads as

$$\mathscr{E}_{\tilde{P}}\left(f, g\right) = \sum_{i \in \mathscr{I}} \lambda_i \cdot \int \pi_{-i}\left(\mathrm{d}x_{-i}\right) \cdot \mathscr{E}_{K_i\left(\,\cdot\,\mid x_{-i}\right)}\left(f, g\right)$$

Application: Exact-Approximate Gibbs (3)

- for 'easy' comparison, assume that
  - for each (i, x_{-i}), $K_i(\cdot \mid x_{-i})$ has a spectral gap $\geq \gamma$
- then, write

$$\begin{aligned}
\mathcal{E}_{\tilde{P}}(f,f) &= \sum_{i \in \mathcal{I}} \lambda_i \cdot \int \pi_{-i}(\mathrm{d}x_{-i}) \cdot \mathcal{E}_{K_i(\cdot \mid x_{-i})}(f,f) \\
&\geq \sum_{i \in \mathcal{I}} \lambda_i \cdot \int \pi_{-i}(\mathrm{d}x_{-i}) \cdot \gamma \cdot \mathrm{var}_{\pi_i(\cdot \mid x_{-i})}(f) \\
&= \gamma \cdot \mathcal{E}_P(f,f)
\end{aligned}$$

Comments

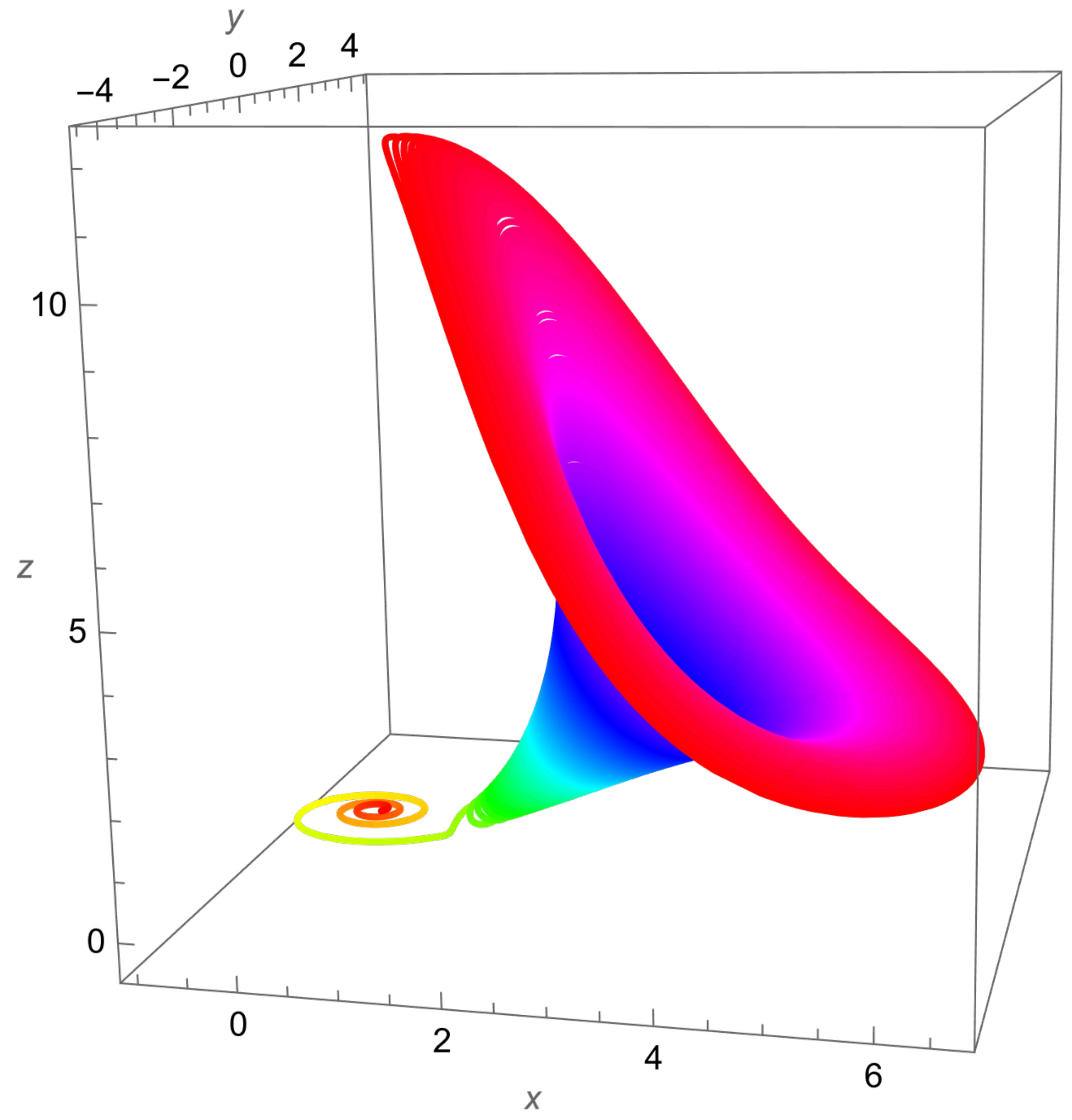
- can get negative result fairly easily through same arguments
  - (if exact conditional is available, then you should probably use it)
- immediately opens up numerous applications
  - Hit-and-Run, Slice Sampling, Particle Gibbs, …
- contrast { small-block, large-block } Gibbs

Taking Stock

- when is the functional approach applicable and easy-to-use?
  - ‘exact approximation’: still π-invariant, reversible, but replacing

    “*do [XXX] exactly*” $\leftarrow$ “*do [the Monte Carlo approximation to XXX]*”

    - difference between ‘exact’ and ‘approximate’ is a probability density
- note that the fine details of the approximation rarely matter
  - instead: { uniform bounds, moment bounds, etc. } tell quite a clear story



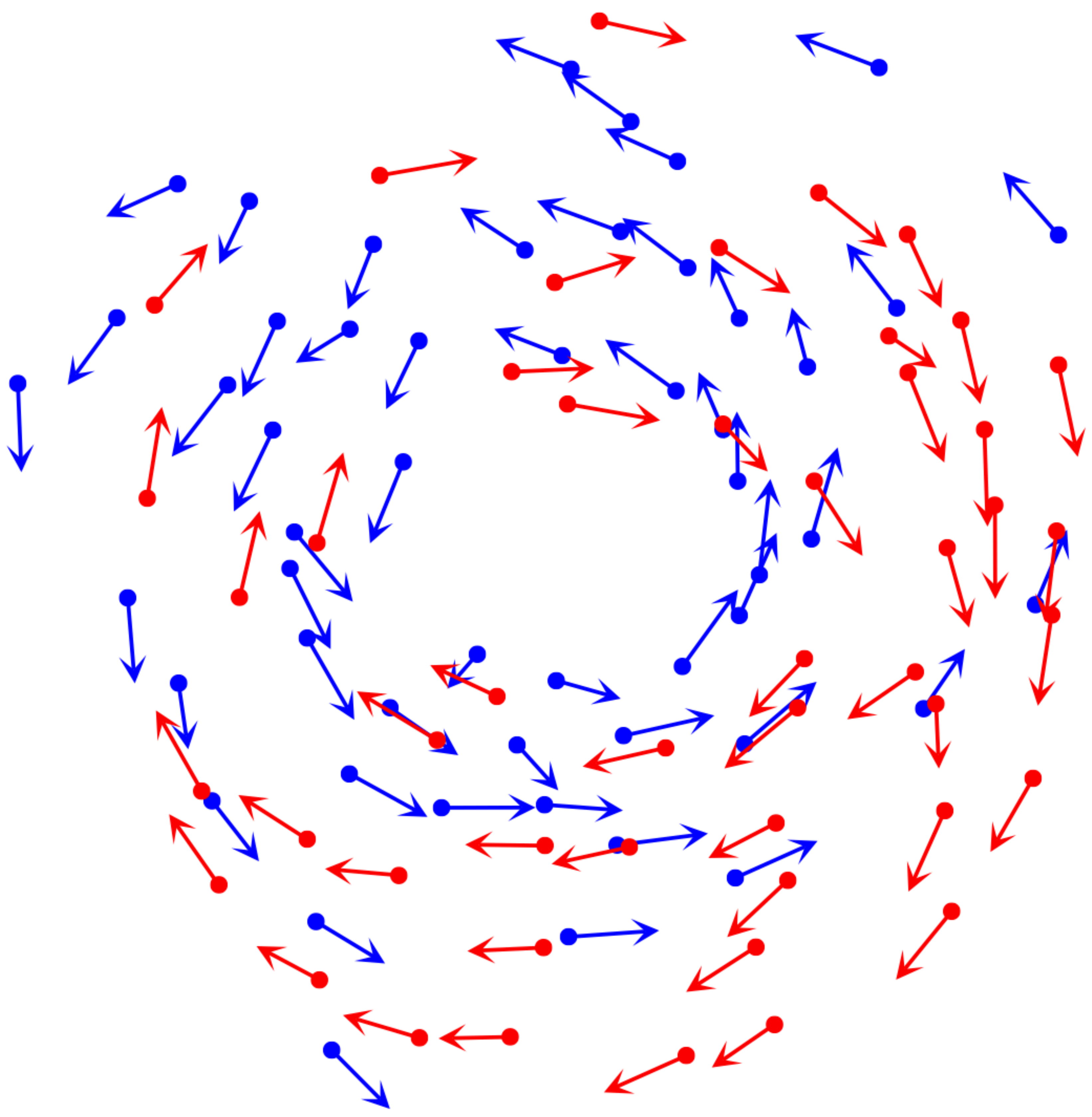
Taking Stock Briefly

Revisiting our Goals

- ‘functional inequalities’ describe the convergence behaviour of Markov processes ‘at large scales’

- such tools are appealing in high dimension, when uniform curvature is not easily available

- probabilistic techniques are often a good route to a first functional inequality for ‘nice’ problems



Some Good and Some Bad

- a functional inequality is a sensible, robust way to say that

  'this { measure / process } is nice'

- the 'global' nature of functional inequalities is instructive

- functional inequalities compose very nicely

  - particularly useful for analysis of 'meta-algorithms'

- proving a functional inequality from scratch can be some work

- working with processes which are { inhomogeneous, non-reversible processes, of unknown invariant measure, … } may not be obvious

- building intuition for functional results can require some guidance

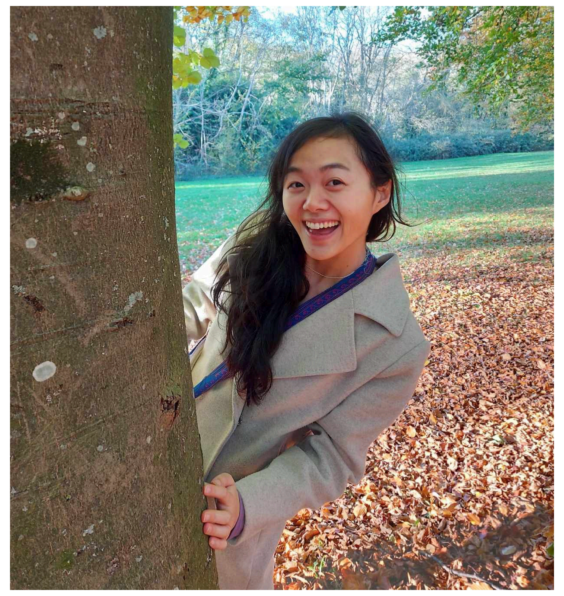
- 'conservation of (mathematical) difficulty' holds at some level

Some Recap and Outlook

- the toolbox of Markov processes and their long-time behaviour is vast
  - many nice frameworks exist, and many are somewhat compatible
- the functional-analytic framework seems rather accommodating
  - … and much is known about it already
- when studying your next Markov process, concrete or abstract,
  - … perhaps the functional-analytic perspective can offer you something!

Fin

Fin

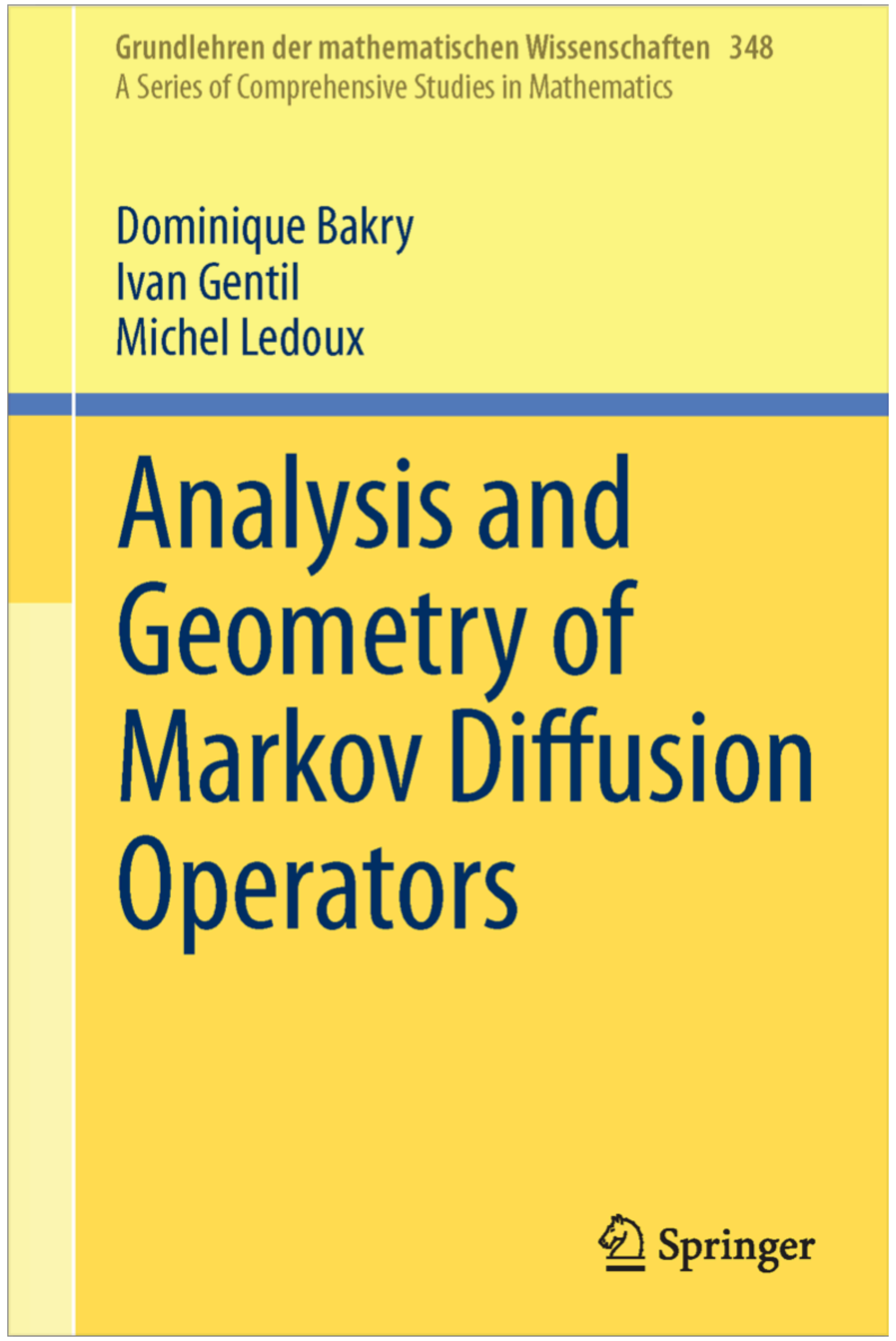


Some Key References (1)

- Bakry, Gentil, Ledoux - "Analysis and Geometry of Markov Diffusion Operators"

- Bonnefont - "Poincaré Inequality with Explicit Constant in Dimension $d \geq 1$", Lecture Notes

- Chafaï, Lehec - "Logarithmic Sobolev Inequalities Essentials", Lecture Notes



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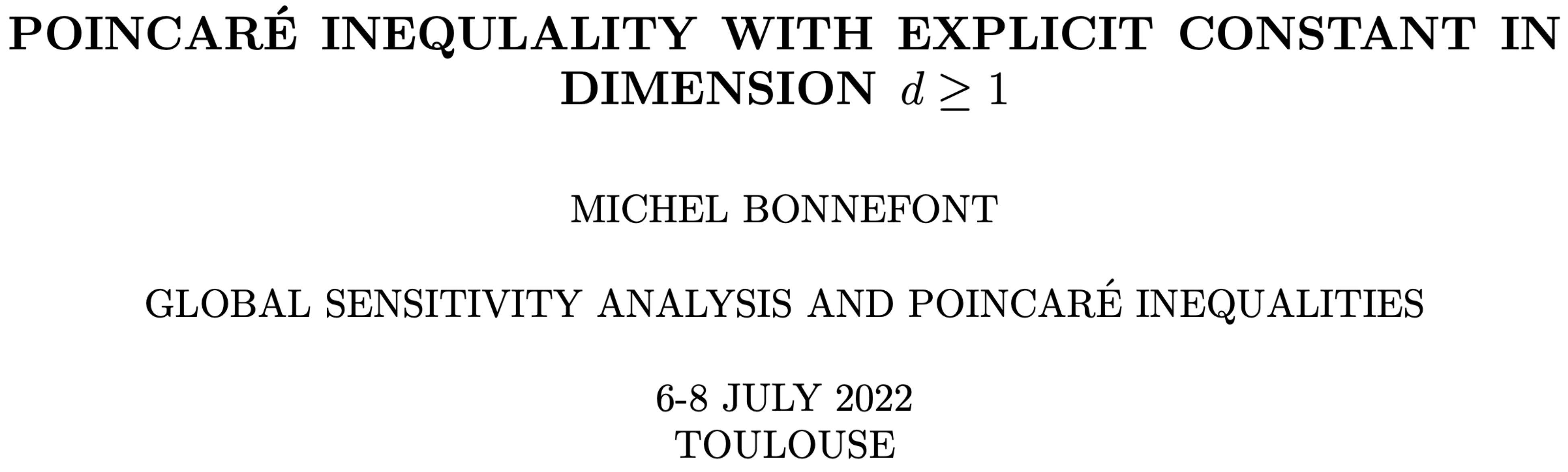

POINCARÉ INEQULALITY WITH EXPLICIT CONSTANT IN DIMENSION $d \geq 1$

MICHEL BONNEFONT

GLOBAL SENSITIVITY ANALYSIS AND POINCARÉ INEQUALITIES

6-8 JULY 2022
TOULOUSE

(back)

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Université PSL (Paris Sciences & Lettres)
Master *Mathématiques théoriques et appliquées*

Logarithmic Sobolev Inequalities Essentials
Rough lecture notes
Djalil Chafaï & Joseph Lehec
Winter 2017, Université Paris-Dauphine, Paris
Revised Spring 2017, Universidad de Chile, Santiago de Chile
Revised Winter 2023, École normale supérieure, Paris

This course is a modern overview on logarithmic Sobolev inequalities, from the probabilistic side. These inequalities have been the subject of intense activity in the recent decades in relation with the analysis and geometry of Markov processes and diffusion evolution equations. This course is designed to be accessible to a wide audience with a first year of master level in mathematics. It is divided into seven lectures of roughly three hours.

(back)

Some Key References (2)

- Chen - “Geometric Flows for Applied Mathematicians”, Lecture Notes
- Chewi - “Log-Concave Sampling”, Book-in-Progress
- van Handel - “Probability in High Dimension”, Lecture Notes

Geometric Flows for Applied Mathematicians*

Xiaohui Chen

This version: December 24, 2020

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- Chen - "Geometric Flows for Applied Mathematicians", Lecture Notes
- Chewi - "Log-Concave Sampling", Book-in-Progress
- van Handel - "Probability in High Dimension", Lecture Notes

Log-Concave Sampling

(unfinished draft)

Sinho Chewi

April 18, 2024

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- Chewi - "Log-Concave Sampling", Book-in-Progress
- van Handel - "Probability in High Dimension", Lecture Notes

Ramon van Handel

Probability in High Dimension

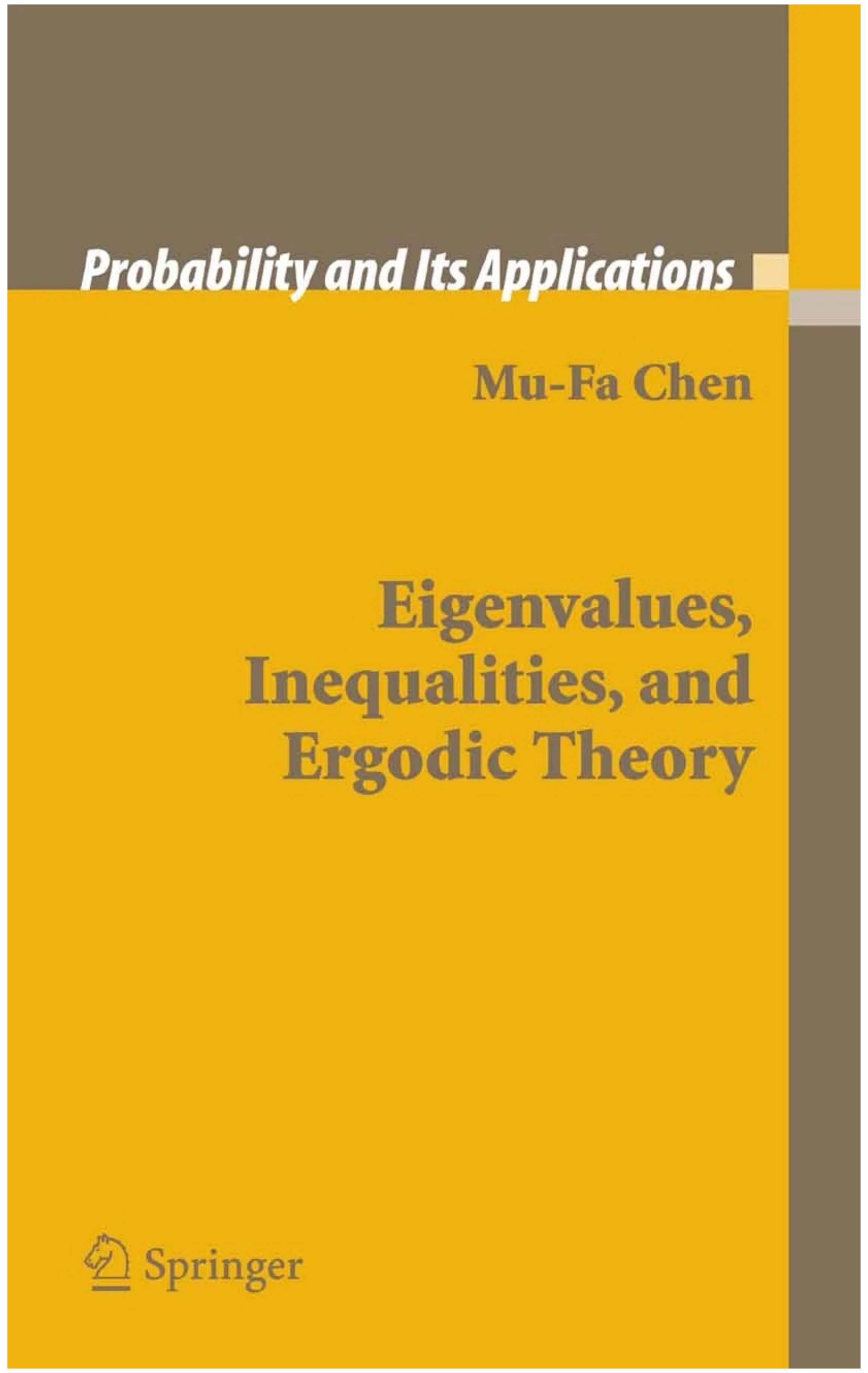
APC 550 Lecture Notes
Princeton University

Some Additional References

- Chen - “Eigenvalues, Inequalities, and Ergodic Theory”

- Montenegro, Tetali - “Mathematical Aspects of Mixing Times in Markov Chains”

- Wang - “Functional Inequalities, Markov Semigroups and Spectral Theory”



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Mathematical Aspects of Mixing Times in Markov Chains

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