
Problem Set 7

Textbook questions are from “Galois Theory” (2nd edition) by Joseph Rotman.

1. Textbook exercise 84.
2. Textbook exercise 85.

Remark : Once you have solved these two exercises, I will recommend you to go through the proof of Lemma 73 one more time to better understand the result.

3. For the following polynomials in $\mathbb{Q}[x]$, determine the Galois group of their splitting fields over \mathbb{Q} . Justify your answers.

(a) $f(x) = (x^2 - 3)(x^2 - 5)(x^2 - 7)$

(b) $g(x) = \prod_{i=1}^n (x^2 - p_i)$, where p_1, p_2, \dots, p_n are distinct primes.

(c) $h(x) = (x^2 + 3)(x^2 - 5)$

4. Let $f(x), g(x) \in \mathbb{Q}[x]$ be solvable by radicals. Show that fg is solvable by radicals.

Hint : Let E, F, F' be the splitting fields of fg, f, g , over \mathbb{Q} , respectively. So F, F' are subfields of E . Show that $\text{Gal}(E/\mathbb{Q})$ is isomorphic to a subgroup of $\text{Gal}(F/\mathbb{Q}) \times \text{Gal}(F'/\mathbb{Q})$, i.e. there is an injective homomorphism $\phi : \text{Gal}(E/\mathbb{Q}) \rightarrow \text{Gal}(F/\mathbb{Q}) \times \text{Gal}(F'/\mathbb{Q})$.

5. Let $\alpha \in \mathbb{C}$ be constructible .

(a) Show that α is algebraic over \mathbb{Q} .

(b) If $p_\alpha(x)$ is the minimal polynomial of α over \mathbb{Q} , what can you say about $\partial(p_\alpha)$? Use this to construct an algebraic number that is not constructible.

(c) Show that the angle 72° can be constructed using only a straightedge and a compass.

Hint : Show that θ can be constructed as an angle if $\cos \theta$ is constructible. You may use $\cos 72^\circ = \frac{\sqrt{5}-1}{4}$ to then show that it is constructible.

2% Bonus Problem. This problem is based on the following interesting question asked by a student during the lectures : Are subfields of a radical extension, radical ? The answer is no, and this problem is about producing an example.

- (a) Let $\zeta_7 = \cos \frac{2\pi i}{7} + i \sin \frac{2\pi i}{7}$ be a 7th root of unity. So $\mathbb{Q}(\zeta_7)$ is a radical extension. Consider the subfield $L = \mathbb{Q}(\zeta_7 + \zeta_7^{-1}) \subseteq \mathbb{Q}(\zeta_7)$. Show that $[L : \mathbb{Q}] = 3$.
- (b) If L was radical, it must be a pure extension, i.e. $L = \mathbb{Q}(\alpha)$ with $\alpha^n \in \mathbb{Q}$ for some n . Show that $n = 3$. *Hint : Minimal polynomial of α is of degree 3. Figure out what its roots look like.*
- (c) Derive a contradiction by showing that the minimal polynomial of α must have repeated roots then.