$\mathcal{W}$-ALGEBRAS AND WHITTAKER CATEGORIES

SAM RASKIN

For Arthur

ABSTRACT. This article is concerned with Whittaker models in geometric representation theory, and gives applications to the study of affine $\mathcal{W}$-algebras.

The main new innovation connects Whittaker models to invariants for compact-open subgroups of the loop group. This method, which has a counterpart for p-adic groups, settles a conjecture of Gaitsgory in the categorical setting. This method shows that Whittaker sheaves in geometric representation theory admit t-structures, as had previously been observed in some special cases.

We then apply this method to the setting of affine $\mathcal{W}$-algebras. We study a new family of modules for affine $\mathcal{W}$-algebras, which can be thought of as analogues of certain tautological ("generalized vaccuum") modules over the Kac-Moody algebra. Using the above t-structure, we obtain an affine analogue of Skryabin's theorem that connects affine $\mathcal{W}$-algebras and Whittaker models.

This theorem allows various geometric methods to be used to study affine $\mathcal{W}$-algebras. As one such application, we offer a new proof of one of Arakawa's foundational results in the theory.

CONTENTS

1. INTRODUCTION

1.1. **Overview.** This paper is about generalizing Skryabin's theorem, a simple result about *finite* $\mathcal{W}$-algebras,[1] to the more subtle setting of *affine* $\mathcal{W}$-algebras.

The main new construction of the paper is of a more general nature. It provides a *compact approximation* to the *Whittaker model*, which corresponds under the conjectural local geometric

---

October 21, 2016. *Updated:* September 1, 2020.

[1]There are competing stories in the literature for the origins of this peculiar name.

That the "W" is taken from the analogy with Whittaker models in harmonic analysis has been suggested repeatedly, e.g. in [DSK]. As the subject was started in [Kos], this seems quite reasonable. (We remark that the word "Whittaker" in the title of this paper is explicitly meant to evoke exactly this meaning.)

Arakawa [Ara1] suggests the name comes because affine $\mathcal{W}$-algebras generalize the Virasoro algebra, and W succeeds V in the alphabet.

Langlands correspondence to the stratification of the moduli space of de Rham local systems by slope. This method, which we call the *adolescent Whittaker* construction, is a new one, though closely related to [Rod]. It appears to be fundamental in geometric Langlands, and may be of interest to specialists in the Langlands program without an interest in $\mathcal{W}$-algebras.

But in what follows, we emphasize the role of $\mathcal{W}$-algebras, which are ultimately the main players in this paper. We provide a survey of the subject below. The reader with no interest in this part of the paper can safely skip that material.

1.2. **Some notation.** We work over a ground field k of characteristic 0 throughout the paper.

Let G be a split reductive group over k, let N be the radical of a Borel subgroup B of G, and let T be the Cartan B/N. As usual, we let $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{b}$, and $\mathfrak{t}$ denote the corresponding Lie algebras. We let Λ (resp. $\check{\Lambda}$) denote the lattice of weights (resp. coweights). We fix Chevalley generators $e_i \in \mathfrak{n}$. Let $\psi : \mathfrak{n} \to k$ be the *non-degenerate* character with $\psi(e_i) = 1$ for all i.

Throughout the paper, notation is *always* assumed derived (see §1.5.6 for an explanation why). Our derived categories are considered as DG categories.[2] They all admit arbitrary (small) colimits (equivalently, arbitrary direct sums), i.e., they are *cocomplete.* They are considered as objects of DGCat_{cont}, the ∞-category of presentable[3] DG categories and continuous functors (i.e., functors preserving all colimits) .

E.g., Vect means the DG category of chain complexes of (k-)vector spaces. Similarly, $\mathfrak{g}$–mod denotes the DG category of chain complexes of $\mathfrak{g}$-modules. We use the notation $\mathsf{Vect}^\heartsuit$, $\mathfrak{g}$–$\mathsf{mod}^\heartsuit$, etc. to refer to the usual abelian categories (since these are the hearts of standard t-structures on the DG categories).

We use $\underline{\mathrm{Hom}}(-, -)$ to denote the chain complex of maps in a DG category.

1.3. **Finite $\mathcal{W}$-algebras.**

1.3.1. We begin by describing what $\mathcal{W}$-algebras are in the finite-dimensional setting, and what about them we wish to generalize.

1.3.2. We have the *finite Drinfeld-Sokolov* functor:

$$\Psi^{fin} : \mathfrak{g}\text{–}\mathsf{mod} \to \mathsf{Vect}$$

defined by:

$$M \mapsto C^\bullet(\mathfrak{n}, M \otimes -\psi)$$

De Sole and Kac also suggest in [DSK] §0.2 that because $\mathcal{W}$-algebras quantize functions on the space of invariant polynomials of the group, which can be thought of as invariant polynomials for the Weyl group considered as a Chevalley group, that the name derives from *Weyl*.

I tried to hunt the answer down in the literature, with limited success. For $\mathfrak{sl}_3$, the affine $\mathcal{W}$-algebra has two 1-parameter families of generators; one family has to do with Virasoro, so is denoted L_n by standard tradition. In Zamolodchikov's first paper [Zam] on the subject, which introduced the affine $\mathcal{W}$-algebra for $\mathfrak{sl}_3$, he denotes the other family by V_n. However, in his *second* paper [FZ] on the subject, joint with Fateev, the second family is denoted by W_n.

As far as I could tell, the name originates from this choice of notation in the second paper. I do not know what this choice was made. The connection to Virasoro was transparent at that time, but I am not sure about the connection to Whittaker models and Kostant's work. I am not even sure that the connection to $\mathfrak{sl}_3$ would have been clear yet.

[2]This means that they are enriched over chain complexes of vector spaces (and satisfy a few additional hypotheses). However, this notion should be considered in the homotopic sense, i.e., as ∞-categories in the sense of Lurie, cf. [Gai4].

[3]We remind that *presentable* means cocomplete and accessible, where the latter is a mild set-theoretic condition.

where $C^\bullet$ indicates the cohomological Chevalley complex (i.e., Lie algebra cohomology), and $-\psi$ is abusive notation for the 1-dimensional $\mathfrak{n}$-module defined by the character $-\psi$ (the reason for the sign will be apparent later).

The non-derived version of this functor was introduced in [Kos], where its basic properties were established.[4]

Define the DG algebra $\mathcal{W}^{fin}$ as the *endomorphisms* of this functor.[5] One has:

Theorem 1.3.1 ($\approx$ Kostant [Kos]). *(1) $\mathcal{W}^{fin}$ is concentrated in cohomological degree 0, i.e., $\mathcal{W}^{fin} = H^0(\mathcal{W}^{fin})$.*

(2) $\mathcal{W}^{fin}$ carries a canonical filtration whose associated graded is slightly non-canonically isomorphic to the algebra of functions on the Kostant slice $f + \mathfrak{b}^e \simeq f + \mathfrak{b}/N \simeq \mathfrak{g}//G$. Here f is a principal nilpotent element related to ψ, e fits into a principal $\mathfrak{sl}_2$ with f with $[e,f] \in \mathfrak{b}$. We use the quotient symbol $/$ to indicate the stack quotient (which happens to be an affine scheme in this case), and $//$ to indicate the GIT quotient.

This isomorphism is completely determined by a choice of Ad-invariant isomorphism $\mathfrak{g} \simeq \mathfrak{g}^\vee$. Then for $\pi : \mathfrak{g} \to \mathfrak{g}^\vee \to \mathfrak{n}^\vee$, f should be the unique nilpotent element in $\pi^{-1}(\psi)$.

The proof is quick using *Kazhdan-Kostant* filtrations, cf. [Kos] §1-2, [GG] §4, or §A.5 from the appendix of the present paper.

Remark 1.3.2. In fact, it is straightforward to show using these methods that the canonical map $Z(\mathfrak{g}) \to \mathcal{W}^{fin}$ is an isomorphism; in particular, $\mathcal{W}^{fin}$ is commutative. (This is also all but proved in [Kos].)

We encourage the reader to forget this fact as much as possible. The affine $\mathcal{W}$-algebras are not (usually) commutative. There is a more subtle point as well: this identification is not true *derivedly*, i.e., there are non-vanishing higher Hochschild cohomology groups for $U(\mathfrak{g})$. (From this perspective, the algebra $\mathcal{W}^{fin}$ can be thought of as a *construction* of the *usual* (non-derived) center of $U(\mathfrak{g})$ adapted to derived settings.)

1.3.3. *Skryabin's theorem.* One has the following description of the category of modules over $\mathcal{W}^{fin}$.

Let $\mathfrak{g}\text{–mod}^{N,\psi} \subseteq \mathfrak{g}\text{–mod}$ denote the full subcategory consisting of twisted Harish-Chandra modules, i.e,. the full subcategory consisting of complexes on whose cohomologies the operators $x - \psi(x)$ act locally nilpotently for every $x \in \mathfrak{n}$.[6]

Theorem 1.3.3 (Skryabin's theorem). *There is a canonical t-exact equivalence of DG categories $\mathfrak{g}\text{–mod}^{N,\psi} \xrightarrow{\simeq} \mathcal{W}^{fin}\text{–mod}$ fitting into a commutative diagram:*

$$
\begin{array}{ccc}
\mathfrak{g}\text{–mod}^{N,\psi} & \xrightarrow{\ \simeq\ } & \mathcal{W}^{fin}\text{–mod} \\
& \searrow{\scriptstyle \Psi^{fin}} \quad {\scriptstyle \mathrm{Oblv}}\swarrow & \\
& \mathrm{Vect.} &
\end{array}
$$

Here Oblv *denotes the forgetful functor.*

---

[4]Almost established, in any case. One often finds this source cited for results which are not proved there, but whose proofs can readily be extracted from it.

[5]Explicitly, this means we take $\underline{\mathrm{End}}_{\mathfrak{g}\text{–mod}}(\Psi^{fin}(U(\mathfrak{g})))$, where $\Psi^{fin}(U(\mathfrak{g}))$ is regarded as a $\mathfrak{g}$-module through the bimodule structure on $U(\mathfrak{g})$.

More explicitly still, note that $C_\bullet(\mathfrak{n},-) = C^\bullet(\mathfrak{n},-)[\dim \mathfrak{n}] \otimes \det(\mathfrak{n})$ (with $C_\bullet$ being Lie algebra homology), so $\Psi^{fin}(U(\mathfrak{g})) = \mathrm{ind}_{\mathfrak{n}}^{\mathfrak{g}}(-\psi)[-\dim \mathfrak{n}] \otimes \det(\mathfrak{n})^\vee$ (the sign occurs in switching between right and left actions). So we compute that $\underline{\mathrm{End}}_{\mathfrak{g}\text{–mod}}(\Psi^{fin}(U(\mathfrak{g})))$ is $C^\bullet\big(\mathfrak{n}, C_\bullet(\mathfrak{n}, U(\mathfrak{g}) \otimes -\psi) \otimes \psi\big)$.

[6]To be clear: this is not the general notion of Harish-Chandra module, but is equivalent in this case because N is unipotent.

The proof is easy: the induced module $\mathrm{ind}_{\mathfrak{n}}^{\mathfrak{g}}(\psi) \in \mathfrak{g}\text{–mod}^{N,\psi}$ is a compact generator of this DG category (it generates because N is unipotent), so by general DG category principles, we have an equivalence:[7]

$$\underline{\mathrm{Hom}}_{\mathfrak{g}\text{–mod}^{N,\psi}}(\mathrm{ind}_{\mathfrak{n}}^{\mathfrak{g}}(\psi), -) : \mathfrak{g}\text{–mod}^{N,\psi} \simeq \mathrm{End}_{\mathfrak{g}\text{–mod}^{N,\psi}}(\mathrm{ind}_{\mathfrak{n}}^{\mathfrak{g}}(\psi))^{op}\text{–mod}.$$

By definition, this functor is Ψ^{fin}, and $\mathcal{W}^{fin}$ was defined as these endomorphisms. We remark that t-exactness of the functor follows as $\mathfrak{g}\text{–mod}^{N,\psi,\leqslant 0}$ is generated under colimits by $\mathrm{ind}_{\mathfrak{n}}^{\mathfrak{g}}(\psi)$ while $\mathcal{W}^{fin}\text{–mod}$ is generated under colimits by $\mathcal{W}^{fin}$.

Remark 1.3.4. In particular, we see that the *only* role played by the non-degeneracy of the character ψ here is to make $\mathcal{W}^{fin}$ a classical (i.e., non-DG) associative algebra. The result remains true in general, as long as we systematically work in the DG setting. This will not be the case anymore once we pass to the affine setting.

1.3.4. *Some references.* There are a many good places to learn more about finite $\mathcal{W}$-algebras, and we name just a few here for the reader's convenience. However, we note that many authors are interested in a more subtle generalization taking an arbitrary (i.e., possibly non-principal) nilpotent element in $\mathfrak{g}$ as input.

The original source is [Kos], as noted above, and it contains most of the ideas indicated above. The general definition may be found in [Pre], whose appendix by Skryabin contains the original proof of Theorem 1.3.3.

There are many convenient surveys, e.g. [Los] and [Wan]. See also [Ara3] and [DSK] for treatments that emphasize the affine point of view as well.

1.4. **Affine $\mathcal{W}$-algebras.**

1.4.1.   Here $\mathfrak{g}$ is replaced by the loop algebra $\mathfrak{g}((t)) := \mathfrak{g} \otimes_k k((t))$.

More generally, recall that an Ad-invariant bilinear form κ defines the *affine Kac-Moody* algebra $\widehat{\mathfrak{g}}_\kappa$, which is a central extension of $\mathfrak{g}((t))$ by the abelian 1-dimensional Lie algebra k. The form κ is called the *level*. We recall that the Kac-Moody cocycle vanishes on $\mathfrak{g}[[t]]$ and $\mathfrak{n}((t))$. When we speak of modules for the loop/Kac-Moody algebras, we agree that they are *discrete* (or *smooth*), i.e., every vector is annihilated by $t^N \mathfrak{g}[[t]]$ for some N (depending on the vector), and that the generator of the central k in the Kac-Moody algebra acts by the identity.

Remark 1.4.1. The reader may ignore the level κ, reading $\widehat{\mathfrak{g}}_\kappa$ as $\mathfrak{g}((t))$ everywhere, and not miss out on much fun in this paper. The major downside is that the level plays a key role in Feigin-Frenkel duality, which is a major source of motivation here.

In the affine theory, the role of $\mathfrak{n}$ is then replaced by $\mathfrak{n}((t))$, and the character $\psi : \mathfrak{n} \to k$ is replaced by the (conductor 0) *Whittaker* character:

$$\mathfrak{n}((t)) \to \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]((t)) = \oplus_{i \in \mathcal{I}_G} k((t)) \xrightarrow{\Sigma} k((t)) \xrightarrow{\mathrm{Res}} k$$

where Res is the residue with respect to dt[8] and $\mathcal{I}_G$ is the set of simple roots (alias: vertices of the Dynkin diagram of G). Abusing notation, we let $\psi : \mathfrak{n}((t)) \to k$ denote the corresponding character.

---

[7]The superscript *op* here indicates that we take the opposite algebra structure, i.e., take right modules over this DG algebra. (A posteriori, this is irrelevant by commutativity.)

[8]In more general settings relevant for global geometric Langlands, it can be important to remove the choice of 1-form by incorporating twists. cf. [Ras2] or [FGKV].

Remark 1.4.2. Affine W-algebras were introduced in mathematical physics by Zamolodchikov [Zam] in the case $\mathfrak{g} = \mathfrak{sl}_3$. (The above perspective via Lie algebras and quantum Hamiltonian reduction is found in Feigin-Frenkel [FF1] and its antecedents, which include [DS] and [BO] and others.) We refer to [BS] for further discussion of the role of W-algebras in physics.

Remark 1.4.3. The relevance in local geometric Langlands is that for $K = k((t))$, the loop group[9] $G(K)$ plays the role of the p-adic reductive group in usual harmonic analysis. Then $\mathfrak{g}((t))$ is its Lie algebra, and it is not surprising that one accesses the group through its Lie algebra. In fact, in the Frenkel-Gaitsgory philosophy ([FG1], [Fre2]), the DG category of modules $\mathfrak{g}((t))$–mod (or in truth, the Kac-Moody representations of *critical* level) is regarded as an important canonical *categorified* representation of $G(K)$. The goal of the affine Skryabin theorem below is to explain that its Whittaker model is closely tied to affine W-algebras.

1.4.2.   Below, we use $\mathfrak{g}((t))$–mod and $\widehat{\mathfrak{g}}_\kappa$–mod to denote the DG categories of modules over the loop and Kac-Moody algebras. These categories are defined in [FG3], and there are subtle points that we discuss in §1.5.6.

However, for now the reader may ignore these points, and we will discuss them in what follows. It is enough to know that they have t-structures with hearts the relevant abelian categories of (discrete) modules, and that they are almost-but-not-quite the derived categories of these abelian categories.

1.4.3.   We have not actually defined the affine W-algebra above, and we will not quite do this below. It is not so easy as in the finite situation because of the use of topological algebras, which are well-adapted to abelian categories but not so well to DG categories. Because (semi-infinite) Lie algebra cohomology plays such a key role, (even in the finite case), one has to be careful balancing abelian and derived categories. (This is one of the major technical issues settled by the affine Skryabin theorem.)

But still, let us review the major features in what follows.

1.4.4. *Semi-infinite cohomology.* One has the *semi-infinite* cohomology functor:

$$H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], -) : \mathfrak{n}((t))\text{–mod} \to \mathsf{Vect}.$$

We refer to §A.1.2 in the appendix for a simple construction via Lie algebra cohomology. We simply remark that it can be thought of as mixing Lie algebra cohomology for $\mathfrak{n}[[t]]$ with Lie algebra homology for $\mathfrak{n}((t))/\mathfrak{n}[[t]]$, except that the latter does not make sense. (Properly, one takes a direct limit over cohomologies for an increasing sequence of compact open subalgebras.)

We obtain the (quantum) *Drinfeld-Sokolov* functor:

$$\Psi = H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], (-) \otimes -\psi) : \widehat{\mathfrak{g}}_\kappa\text{–mod} \to \mathsf{Vect}.$$

(Here we recall that there is a forgetful functor $\widehat{\mathfrak{g}}_\kappa$–mod $\to \mathfrak{n}((t))$–mod because the Kac-Moody cocycle vanishes on $\mathfrak{n}((t)) \subseteq \mathfrak{g}((t))$.)

1.4.5.   The W-algebra at level κ comes in two guises: as a vertex algebra $\mathcal{W}_\kappa$ and as a topological associative algebra (in the sense of [BD2]) $\mathcal{W}_\kappa^{as}$. We remind that whatever a vertex algebra structure is, it allows one to construct a topological associative algebra, and this is the relationship between the two constructions above. The algebra $\mathcal{W}_\kappa^{as}$ is analogous to the completion of the enveloping algebra of $\mathfrak{g}((t))$ defined by the left ideals generated by $t^N\mathfrak{g}[[t]]$ over all $N \geqslant 0$; note that discrete

---

[9]The author prefers to denote loop Lie algebras as $\mathfrak{g}((t))$ and loop Lie groups as $G(K)$.

modules for this algebra are the same as discrete modules for $\mathfrak{g}((t))$ as defined before, and there is a similar relationship here.

The vertex algebra $\mathcal{W}_\kappa$ is easier to define: $\mathcal{W}_\kappa = \Psi(\mathbb{V}_\kappa)(= H^0\Psi(\mathbb{V}_\kappa))$, where $\mathbb{V}_\kappa = \mathrm{ind}_{\mathfrak{g}[[t]]}^{\widehat{\mathfrak{g}}_\kappa}(k) \in \widehat{\mathfrak{g}}_\kappa\text{–mod}^\heartsuit$ is the *vacuum* representation, which we remind is a vertex algebra with the same abelian category of modules as the Kac-Moody algebra itself.

Either perspective defines the same abelian category of modules, and we denote it by $\mathcal{W}_\kappa\text{–mod}^\heartsuit$. The relationship to the above is that the cohomologies of $\Psi(M)$ for any $M \in \widehat{\mathfrak{g}}_\kappa\text{–mod}$ are acted on by the $\mathcal{W}$-algebra (at level κ).

Remark 1.4.4. We refer to [FBZ] §15 and [BD2] §3.8 for more complete discussion on the constructions. See also §4.4 below.

1.4.6.   We now review the major properties of affine $\mathcal{W}$-algebras for the reader's convenience. Note the parallel with Theorem 1.3.1.

Theorem 1.4.5.   *(1) $\mathcal{W}_\kappa$ is concentrated in cohomological degree 0, i.e., $\mathcal{W}_\kappa = H^0\Psi(\mathbb{V}_\kappa)$.*
   (2) $\mathcal{W}_\kappa$ and $\mathcal{W}_\kappa^{as}$ carry canonical filtrations whose associated graded are slightly non-canonically isomorphic to the algebra of functions on the affine Kostant slices $f + \mathfrak{b}^e[[t]] \simeq f + \mathfrak{b}[[t]]/N(O) \simeq (\mathfrak{g}//G)(O)$ and $f + \mathfrak{b}^e((t)) \simeq f + \mathfrak{b}((t))/N(K) \simeq (\mathfrak{g}//G)(K)$ respectively.
      These isomorphisms are completely determined by a choice of Ad-invariant isomorphism $\mathfrak{g} \simeq \mathfrak{g}^\vee$ and the non-vanishing 1-form dt on the formal disc.
   (3) (Feigin-Frenkel duality, [FF2].) There is a canonical vertex algebra isomorphism:

$$\mathcal{W}_{\mathfrak{g},\kappa} \simeq \mathcal{W}_{\check{\mathfrak{g}},\check{\kappa}}$$

   where the notation indicates the affine $\mathcal{W}$-algebras for $\mathfrak{g}$ and for the Langlands dual Lie algebra. The construction of the dual level $\check{\kappa}$ is reviewed in §7.3.3. (Note Warning 7.3.1.)

Remark 1.4.6. The proofs of the first two results are similar to the finite-dimensional case up to a subtlety about the convergence of a spectral sequence. See [FBZ] Chapter 15 and [Ara3] for details; the arguments are due originally to [dBT] (at least in the case of $\mathfrak{sl}_n$). (Below, §4 provides a slightly non-standard approach to dealing with the convergence of the spectral sequence.)

Remark 1.4.7. The vertex algebra $\mathcal{W}_\kappa$ is filtered *as a vertex algebra*, and the associated graded is commutative here, so it makes sense to speak of an algebra structure on it.

Remark 1.4.8. For $\kappa = \kappa_{crit}$ *critical*, i.e., $-\frac{1}{2}$ times the Killing form for $\mathfrak{g}$, $\mathcal{W}_{crit} := \mathcal{W}_{\kappa_{crit}}$ (resp. $\mathcal{W}_{crit}^{as}$) is commutative and coincides with $\mathbb{V}_{crit}^{G(O)}$ (resp. the center of the topological enveloping algebra of $\widehat{\mathfrak{g}}_{crit}$). Here the Feigin-Frenkel duality relates the center with opers for the Langlands dual group.

At other levels, the affine $\mathcal{W}$-algebra is non-commutative (e.g., it contains a copy of the Virasoro algebra).

Remark 1.4.9. The Feigin-Frenkel duality is highly suggestive in local geometric Langlands. We refer to [FG1] and [Fre2] for a discussion of its role in the traditional subject, and to [Gai2] for a formulation of *quantum* local geometric Langlands, which is a general conjectural framework that helps to explain the meaning of the Feigin-Frenkel result.

1.5. **Affine Skryabin, group actions, and Whittaker.**

1.5.1. Frenkel-Gaitsgory proposed a framework for local geometric Langlands based on the idea of *group actions on categories* for group indschemes like $G(K)$. At a certain point, it became clear that the natural class of categories to be acted on are cocomplete DG categories.

For derived categories of abelian categories, an appropriate substitute for the general theory was developed in the appendices to [FG1]. But this theory is inadequate for the general local Langlands framework, and Gaitsgory and his collaborators have spent a number of years and expended a great deal of energy developing the necessary language and methods here.

His ideas about actions of group indschemes (such as $G(K)$ or $N(K)$) on DG categories were developed in detail in [Ber], and we refer the reader there to learn this material. In particular, for $\mathcal{C} \in \mathsf{DGCat}_{cont}$ acted on (strongly) by $G(K)$ (or the twisted version of this notion incorporating the level κ), one can form the *Whittaker category* $\mathsf{Whit}(\mathcal{C}) \in \mathsf{DGCat}_{cont}$ as *invariants* (or coinvariants: see §1.5.4) for $N(K)$ with respect to the character $\psi : N(K) \to \mathbb{G}_a$.

Although the definitions in this theory are simple, the idea to take this very infinite-dimensional (and potentially quite pathological) construction seriously was a quite nontrivial one. This breakthrough was made by Gaitsgory in his notes [Gai3], where he showed several quite nontrivial results, including a comparison with an ad hoc definition with good properties used in [FGV]. These ideas were further advanced by Beraldo in [Ber], who extended Gaitsgory's results and thereby gave more evidence that the Whittaker construction is a good one. (The most psychologically difficult points about this formalism are highlighted below in §1.5.6.)

We comment more on the Whittaker construction in §1.5.4. For now, we merely state:

- Basic examples of $\mathcal{C}$ acted on by $G(K)$ are D-modules on a suitably nice space acted on by $G(K)$, or more relevantly for us, $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}$.
- One can fairly write $\mathsf{Whit}(\mathcal{C}) = \mathcal{C}^{N(K),\psi}$ in such a way that the general formalism in the finite-dimensional setting would produce $\mathfrak{g}$–$\mathsf{mod}^{N,\psi}$ from §1.3.3.
- There are canonical functors $\mathsf{Whit}(\mathcal{C}) \to \mathcal{C}$ and $\mathcal{C} \to \mathsf{Whit}(\mathcal{C})$.

    However, they do *not* satisfy any adjunction. This reflects of the infinite-dimensional nature of $N(K)$, specifically that it is an indscheme but not a scheme. The functor $\mathsf{Whit}(\mathcal{C}) \to \mathcal{C}$ should be regarded as the forgetful functor. The functor $\mathcal{C} \to \mathsf{Whit}(\mathcal{C})$ can be informally thought of as "Av_*^ψ plus an infinite cohomological shift," although neither Av_*^ψ nor the infinite cohomological shift make sense.

1.5.2. We can now formulate the main theorem of this paper:

Theorem (Affine Skryabin theorem, Thm. 5.1.1). *There is a canonical equivalence* $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}) \simeq \mathcal{W}_\kappa$–$\mathsf{mod}$ *such that:*

$$\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \to \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}) \to \mathcal{W}_\kappa\text{–}\mathsf{mod} \to \mathsf{Vect}$$

is computed by the Drinfeld-Sokolov functor Ψ.

As a consequence[10] of affine Skryabin and Feigin-Frenkel duality, we obtain:

Theorem (Categorical Feigin-Frenkel duality, Thm. 7.3.2). *There is a canonical equivalence of categories:*

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}) \simeq \mathsf{Whit}(\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}).$$

---

[10]At this point, we should note that the category $\mathcal{W}_\kappa$–mod has not been properly defined: the issues are discussed in §1.5.6 and §1.5.9. For this reason, Theorem 7.3.2 requires somewhat more input than just Theorem 5.1.1: this is the content of §6.

At critical level, this identifies $\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–mod})$ with $\mathsf{QCoh}(\mathrm{Op}_{\check{G}}(\mathring{D}))$, i.e., the DG category of quasi-coherent sheaves on the indscheme of opers for the Langlands dual group.

Remark 1.5.1. Categorical Feigin-Frenkel duality, especially at the critical level, was the initial motivation for the present paper. Although it was anticipated for a long time, it does not seem that there is a more straightforward approach than the one given in this paper.

Although it may seem quite formal given usual Feigin-Frenkel duality, it allows one to easily convert results about the Whittaker model from local geometric Langlands to statements about Kac-Moody algebras. This allows for simpler arguments (and many extensions) of the Frenkel-Gaitsgory work at critical level.

Remark 1.5.2. Each of the above theorems will come as no surprise to experts in the area. It has long been known that something like this must be true. For example, immediately after the relevant definitions were given in [FG3] and [Gai3], Gaitsgory explicitly postulated both results in [Gai2].[11],[12]

1.5.3. *Why was this result not proved sooner?* There are several reasons, discussed in more detail in what follows:

- The Whit notation is a priori ambiguous. (See §1.5.4.)
- Representation theorists have not encountered objects of $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ before. (See §1.5.6.)
- In the proof of the classical Skryabin theorem, we compared both sides by matching Exts (as A_∞-algebras) between compact generators. How can we find compact generators of $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$? How can we find compact generators of $\mathcal{W}_\kappa\text{–mod}$?[13] (See §1.5.9.)

Remark 1.5.3. The difference between the finite and affine Skryabin theorems can be appreciated by analogy with the following finite-dimensional picture.

The proof of Theorem 1.3.3 given above shows the following more general result. Suppose H is a (finite-dimensional) algebraic group with Lie algebra $\mathfrak{h}$, and A is an algebra with an action of H of Harish-Chandra type datum $i : \mathfrak{h} \to A$ (cf. [BL] §1.4). Then H acts on A–mod, and one can show that modules over the BRST reduction $\mathrm{BRST}(H, A)$ (derived H-invariants in the derived $\mathfrak{h}$-coinvariants of A) embed fully-faithfully into A–mod^H. To say that the induced functor is an equivalence means that Harish-Chandra modules for the pair (A, H) are generated by their H-invariant vectors; in particular, this is automatic for unipotent H.

It is reasonable to expect a similar picture to hold in the infinite-dimensional setup. However, by [GR1], representations of a unipotent group *indscheme* (e.g., $\mathbb{G}_a(K)$) may not be generated by their invariants in any sense. So in the above analogy, $N(K)$ may be taken to be more analogous to a non-unipotent finite-dimensional group, and we cannot expect some general such considerations to be sufficient in proving this affine Skryabin theorem.

The special feature we use in this setup is the *adolescent Whittaker* construction, which is briefly discussed in §1.6 below and in more detail in §2. Informally, this construction shows that Whittaker

---

[11]However, we note that away from the commutative cases, Theorem 5.1.1 could not have been formulated as an honest conjecture because it was not known how to define the full derived category $\mathcal{W}_\kappa$–mod, cf. §1.5.9.

[12]Perhaps each of the results above should be attributed to Gaitsgory as conjectures. I learned this circle of ideas as his graduate student, and am not completely sure where the boundary between folklore and his ideas is on this particular point. In any case, the necessary language to formulate such a result was developed by him and his collaborators explicitly so that such theorems could be formulated and proved.

[13]Here the fact that $\mathcal{W}_\kappa^{as}$ is a topological algebra and not a discrete one is key. For a usual algebra, A is a compact generator of A–mod. For a topological algebra, A does not make sense as an object of A–mod (which is not even a clearly defined DG category in all cases, as we discuss below).

invariants (in the presence of a $G(K)$-action) behave like invariants for prounipotent group schemes (not indschemes).

1.5.4. *What is the Whittaker category?* The discussion here follows [Gai3] and [Ber]. We refer to the latter for details on the definitions and constructions.

There are two a priori candidates for $\mathsf{Whit}(\mathcal{C})$: invariants and coinvariants, denoted by $\mathcal{C}^{N(K),\psi}$ and $\mathcal{C}_{N(K),\psi}$ respectively. The invariants are equipped with a tautological fully-faithful functor to $\mathcal{C}$, and the coinvariants receive a canonical functor from $\mathcal{C}$. Each satisfy natural universal properties in the language of group actions on categories.

In the formalism of group actions on categories, invariants and coinvariants coincide when taken with respect to a group *scheme*, such as $N(O)$ (or $G(O)$). But since $N(K)$ is a group indscheme, it is easy to see that there is no such equivalence in general.

So there is a serious question: do we mean Whittaker invariants or coinvariants?

1.5.5. There is a canonical functor $\mathcal{C}_{N(K),\psi} \to \mathcal{C}^{N(K),\psi}$ constructed by Gaitsgory in [Gai3] (it is denoted by Θ in [Ber]). Gaitsgory conjectured that for $\mathcal{C}$ acted on by $G(K)$ (as opposed to merely by $N(K)$), this functor is an equivalence, meaning that $\mathsf{Whit}(\mathcal{C})$ could be understood unambiguously. This conjecture was shown by Gaitsgory for $G = GL_2$ and by Beraldo for $G = GL_n$.

The first major result of this paper settles Gaitsgory's conjecture for general reductive G:

Theorem (Thm. 2.1.1). *For any reductive G and any $\mathcal{C}$ acted on by $G(K)$ (possibly with level κ), Gaitsgory's functor yields an equivalence $\mathcal{C}_{N(K),\psi} \simeq \mathcal{C}^{N(K),\psi}$.*

So we unambiguously have $\mathsf{Whit}(\mathcal{C})$, and we have the canonical functors $\mathsf{Whit}(\mathcal{C}) \simeq \mathcal{C}^{N(K),\psi} \to \mathcal{C}$ and $\mathcal{C} \to \mathcal{C}_{N(K),\psi} \simeq \mathsf{Whit}(\mathcal{C})$.

Remark 1.5.4. This result is obviously of a technical nature, and may not especially excite the reader more interested in W-algebras than in Whittaker categories. But in fact, the proof, which uses the adolescent Whittaker construction, is more interesting, and has significant implications for affine W-algebras. This is discussed further in what follows (and in §4).

1.5.6. *Where are these Whittaker modules?* Recall that the heart of the t-structure on $\mathfrak{g}$–$\mathsf{mod}^{N,\psi}$ consisted of $\mathfrak{g}$-modules $M \in \mathsf{Vect}^{\heartsuit}$ such that $x - \psi(x)$ act locally nilpotently for every $x \in \mathfrak{n}$. Moreover, $\mathfrak{g}$–$\mathsf{mod}^{N,\psi}$ is the derived category of this abelian category.

We might expect to interpret $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod})$ similarly. However, *there are no discrete modules of $\widehat{\mathfrak{g}}_\kappa$ for which $x - \psi(x)$ acts locally nilpotently for every $x \in \mathfrak{n}((t))$.*

1.5.7. How does this not contradict the affine Skryabin theorem?

The mechanism lies in subtleties of the DG category $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}$. We use the (*renormalized*) version of this DG category defined in [FG3]. We refer to the notes [Gai6] for a more detailed treatment. The point is that $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}$ has a t-structure such that the bounded below derived category $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}^+$ is the bounded below derived (DG) category of the abelian category $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}^{\heartsuit}$. However, there are "many" objects "in cohomological degree $-\infty$,"[14] and in fact, all Whittaker objects (for $\mathfrak{g}$ nonabelian) have this property.

So, somewhat remarkably, the affine Skryabin theorem finds all of the representations of the W-algebra in the "invisible" part of the DG category.

Remark 1.5.5. That Whittaker D-modules lie in cohomological degree $-\infty$ should not come as a surprise, as this happens geometrically as well. Indeed, e.g., the Whittaker D-module on $\mathrm{Gr}_N =$

---

[14]Formally, this means the object lies in $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}^{\leqslant -N}$ for all N.

$N(K)/N(O)$ is the dualizing D-module twisted by the exponential character. Since Gr_N is isomorphic to (ind-)infinite-dimensional affine space, its dualizing D-module unsurprisingly lies in cohomological degree $-\infty$.

1.5.8.   Let us offer a few more words on how to think about objects of $\widehat{\mathfrak{g}}_\kappa$–mod. We are used to thinking of A-modules as abelian groups with an action of A. How should we think about objects of $\widehat{\mathfrak{g}}_\kappa$–mod?

The answer is that for $M \in \widehat{\mathfrak{g}}_\kappa$–mod, and for every $N \geqslant 0$, we have the ability to form:

$$C^\bullet(t^N \mathfrak{g}[[t]], M) \in \mathsf{Vect}$$

where this notation indicates the Lie algebra cohomology of $t^N \mathfrak{g}[[t]]$ with coefficients in M. Understanding this properly: that these functors should be continuous in the variable M and should satisfy certain functoriality properties, one obtains the definition from [FG3].

Note that one recovers the vector space underlying M as:

$$\operatorname*{colim}_N C^\bullet(t^N \mathfrak{g}[[t]], M).$$

So the mechanism for non-zero objects to exist in cohomological degree $-\infty$ is that they have non-vanishing Lie algebra cohomologies with respect to some $t^N \mathfrak{g}[[t]]$, but in the direct limit this cohomology dies.

Remark 1.5.6. It will be helpful in what follows to have formally reviewed one construction of $\widehat{\mathfrak{g}}_\kappa$–mod. One takes $D^+(\widehat{\mathfrak{g}}_\kappa\text{–mod}^\heartsuit)$ the (DG) bounded below derived category, takes the full subcategory generated under cones by the induced modules $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{t^N \mathfrak{g}[[t]]}(k)$, and then forms the ind-category of this. According to [FG3] §23, this has a t-structure with the anticipated bounded below part.

We mention this definition to highlight the key role played by the family of modules $t^N \mathfrak{g}[[t]]$ in the definition.

1.5.9. *Compact generators.* A more serious problem is to identify compact generators on both sides. Because $N(K)$ is a group indscheme (not a group scheme), it is not at all clear that $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ is compactly generated.

As far as I know, even the compact generators in $\mathcal{W}_\kappa$–mod were not previously constructed, even though this basic problem about affine $\mathcal{W}$-algebras can be understood without any categorical formalism. The natural expectation is that there are modules $\mathcal{W}_\kappa^n \in \mathcal{W}_\kappa\text{–mod}^\heartsuit$ similar to the modules $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{t^n \mathfrak{g}[[t]]} k \in \widehat{\mathfrak{g}}_\kappa\text{–mod}^\heartsuit$. (So for $n = 0$, we should have the "vacuum" representation $\mathcal{W}_\kappa$, and for the Virasoro algebra, the construction is clear; at critical level, there is a theory of *opers with singularities* due to [BD1] that settles the issue; for higher rank $\mathfrak{g}$ and $n > 0$, the construction is not so clear.)

We refer the interested reader to the beginning of §4 where the naive expectations are formulated in detail; this material does not require having read the preceding parts of the paper except, at a certain point, needing some elementary Lie theoretic notation from §2.

Remark 1.5.7. As in Remark 1.5.6, the modules $\mathcal{W}_\kappa^n$ actually play an essential role in defining the DG category $\mathcal{W}_\kappa$–mod. That is, $\mathcal{W}_\kappa$–mod has a t-structure for which $\mathcal{W}_\kappa\text{–mod}^+$ is the bounded below derived category of $\mathcal{W}_\kappa\text{–mod}^\heartsuit$, but the actual definition of the full derived category takes the modules $\mathcal{W}_\kappa^n$ as input.

1.6. **The adolescent Whittaker construction.**

1.6.1.   Each of the problems above are naturally solved using the *adolescent Whittaker construction*, which shows that the Whittaker category is more highly structured than was previously known. This construction gives a *stratification* of the Whittaker category by simpler pieces. We refer to §2, most notably Theorem 2.3.1, where the results and constructions are formulated in detail. Here we give a more tactile description.

Example 1.6.1 (cf. Cor. 7.3.5). Recall from categorical Feigin-Frenkel duality that $\mathsf{Whit}(\hat{\mathfrak{g}}_{crit}\text{–mod})$ is equivalent to $\mathsf{QCoh}(\mathrm{Op}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$. For $n > 0$, define $\mathsf{Whit}^{\leqslant n}(\hat{\mathfrak{g}}_{crit}\text{–mod}) \subseteq \mathsf{Whit}(\hat{\mathfrak{g}}_{crit}\text{–mod})$ to be the full subcategory corresponding to the subcategory of quasi-coherent sheaves set-theoretically supported on $\mathrm{Op}_{\check{G}}^{\leqslant n} \subseteq \mathrm{Op}_{\check{G}}(\overset{\circ}{\mathcal{D}})$, the subscheme of opers *with singularity* $\leqslant n$ (in the sense of [BD1] §3.8, see also [FG1]).

The surprising fact is that the above construction makes sense for every $\mathcal{C}$ acted on by $G(K)$ (possibly with level κ). Namely, we define categories $\mathsf{Whit}^{\leqslant n}(\mathcal{C})$ for all $n \geqslant 0$. These are connected by adjoint functors:

$$\mathsf{Whit}^{\leqslant n}(\mathcal{C}) \rightleftarrows \mathsf{Whit}^{\leqslant n+1}(\mathcal{C}) \in \mathsf{DGCat}_{cont}.$$

The direct limit over n (formed in DGCat_{cont}) is $\mathsf{Whit}(\mathcal{C})$. For $n > 0$, the induced functor $\mathsf{Whit}^{\leqslant n}(\mathcal{C}) \to \mathsf{Whit}(\mathcal{C})$ is fully-faithful.

For low values of n, this construction was previously known:

- For $n = 0$, $\mathsf{Whit}^{\leqslant 0}(\mathcal{C}) = \mathcal{C}^{G(O)}$.
- For $n = 1$, $\mathsf{Whit}^{\leqslant 1}(\mathcal{C})$ is the *baby Whittaker* category of [AB] and [**?**]. (This is the reason for the terminology *adolescent*.)

Also, for $G = T$ a torus, we have:

- $\mathsf{Whit}(\mathcal{C}) = \mathcal{C}$ (tautologically) and $\mathsf{Whit}^{\leqslant n}(\mathcal{C}) = \mathcal{C}^{\mathcal{K}_n}$ for $\mathcal{K}_n \subseteq T(O)$ the nth congruence subgroup.

Remark 1.6.2. For κ integral, local geometric Langlands predicts that $\mathsf{Whit}(\mathcal{C})$ is a *category over* $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ (see [Ras4] for a discussion of this notion). Motivated by the case of critical level Kac-Moody representations and the Frenkel-Gaitsgory philosophy, we expect $\mathsf{Whit}^{\leqslant n}(\mathcal{C})$ to be the base-change of $\mathsf{Whit}(\mathcal{C})$ along $\mathrm{LocSys}_{\check{G}}^{\leqslant n-1} \to \mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$, where $\mathrm{LocSys}_{\check{G}}^{\leqslant n-1}$ is the locus of local systems with slope $\leqslant n-1$; by definition, the map $\mathrm{LocSys}_{\check{G}}^{\leqslant n-1} \to \mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ is formally étale for $n \neq 0$, and for $n = 0$ we agree $\mathrm{LocSys}_{\check{G}}^{\leqslant -1} = \mathbb{B}\check{G}$, i.e., the trivial local system.

This gives a conceptual explanation for the use of the baby Whittaker category in [AB]: they are working with regular singular (alias: slope 0) local systems, so the baby Whittaker category $\mathsf{Whit}^{\leqslant 1}$ is enough.

Remark 1.6.3. In the classical framework of p-adic groups, a similar construction was given by Rodier in [Rod] (although his normalizations meaningfully differ for $\mathrm{rank}(G) > 1$).

1.6.2.   In general, $\mathsf{Whit}^{\leqslant n}(\mathcal{C})$ is defined as invariants with respect to a compact open subgroup of $G(K)$. Since the functors $\mathsf{Whit}^{\leqslant n}(\mathcal{C}) \to \mathsf{Whit}(\mathcal{C})$ admit continuous right adjoints, this means in practice that $\mathsf{Whit}(\mathcal{C})$ *is as good as invariants with respect to a group scheme*. E.g., these functors preserve compact objects, so compact generation with respect to congruence subgroups implies it for the Whittaker category. This is true for $\mathcal{C} = D(X)$ for $G(K)$ acting on X a reasonable indscheme, or for $\mathcal{C} = \hat{\mathfrak{g}}_{\kappa}\text{–mod}$.

Since the functor $\Psi : \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) \to \mathcal{W}_\kappa\text{–mod}$ is supposed to preserve compacts (being an equivalence), we obtain candidates for the modules $\mathcal{W}_\kappa^n$. This idea turns out to be fruitful, and is pursued in §4.

As an another instance of the above idea, the comparison Theorem 2.1.1 between invariants and coinvariants falls out immediately from the adolescent formalism, as does Gaitsgory's functor between them.

Example 1.6.4. For instance, replacing G by $G \times G$, we find that the DG category of D-modules on $G(K)$ with both left and right Whittaker equivariance is compactly generated. Local geometric Langlands predicts that at integral level, this category is equivalent to $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$. The corresponding compact generation on the spectral side was obtained in [Ras4].

1.6.3. *Key construction: t-structures on Whittaker categories.* In the proof of Theorem 5.1.1, we use a general construction of t-structures on Whittaker categories, which we highlight here because it may be of interest outside of the theory of $\mathcal{W}$-algebras.

The point is that (up to shift) the canonical functors $\mathsf{Whit}^{\leqslant n}(\mathcal{C}) \to \mathsf{Whit}^{\leqslant n+1}(\mathcal{C})$ are t-exact essentially whenever $\mathcal{C}$ has a t-structure compatible with the action of $G(K)$. Roughly, this is because by Theorem 2.3.1, this functor may be realized either as the top possible cohomology of a $*$-averaging functor, or the bottom possible cohomology of a !-averaging functor, and therefore is exact.[15]

Ultimately, this is what allows us to prove Theorem 5.1.1: being topological algebras, $\mathcal{W}$-algebras are of abelian categorical nature (as remarked above), and we analyze $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ with abelian categories via this t-structure.

1.6.4. In particular, if $D(X)$ is a reasonable indscheme with X acted on by $G(K)$, then $\mathsf{Whit}(D(X))$ has a canonical t-structure.

For instance, in the setting of Example 1.6.4, one finds that the category of bi-Whittaker equivariant D-modules on $G(K)$ has a canonical t-structure.[16]

Therefore, we should anticipate there being a canonical t-structure on $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$. Can it be constructed independently of local geometric Langlands? This would generalize the perverse coherent t-structure on the nilpotent cone constructed in [Bez] by Bezrukavnikov, but to a much more complicated setting.

Remark 1.6.5. More generally, $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$ can be regarded as the $\kappa \to \infty$ limit of the categories $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{G(K),w}$ of *Harish-Chandra bimodules* for the Kac-Moody algebra. The (conjectural) perverse coherent t-structure on $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$ should deform in this setting. Indeed, local geometric Langlands (in the quantum setting) continues to predict that Harish-Chandra bimodules should be equivalent to bi-Whittaker D-modules at an appropriate level.

1.6.5. *Summary.* The above story may be summarized as follows: the adolescent Whittaker method from §2, which arises from the geometry of loop groups and "pure"[17] local geometric Langlands, can be imported to the setting of affine $\mathcal{W}$-algebras to illuminate the subject and solve some basic problems.

---

[15]Naively, the argument here is standard: see e.g. [BBM]. We refer to Appendix B for a discussion of technical points.

[16]The proof of the affine Skryabin theorem can be modified to identify the corresponding abelian category with (classical) chiral modules over the chiral algebra $(\Psi \boxtimes \Psi)(CDO_\kappa)$, where CDO_κ is the level κ chiral differential operators, as constructed e.g. in [AG].

[17]That is, that part that explicitly ties to the traditional arithmetic Langlands program, so e.g. does not mention Kac-Moody algebras.

But we may have reversed the logic in this presentation. The existence of the modules $\mathcal{W}_\kappa^n \in \mathcal{W}_\kappa\text{–mod}^\heartsuit$ was readily anticipated by anyone who considered the issue. Moreover, the affine Skryabin theorem predicts the existence of the t-structure on $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$. The theory of opers with singularities (cf. Example 1.6.1) was long known. So perhaps our knowledge of affine $\mathcal{W}$-algebras should rather have anticipated the adolescent Whittaker theory.

1.7. Structure of the paper.

We now outline the contents of the paper, trying to highlight what parts may be of interest to different types of readers.

The adolescent Whittaker theory is developed in §2. A *semi-classical* version of the theory is given in §3. This material, plus the analogue for D-module categories of the t-structure construction from §5, is the new material on local geometric Langlands from this paper that does not mention $\mathcal{W}$-algebras.

In §4, we solve the problem from §1.5.9 of finding generating modules for affine $\mathcal{W}$-algebras. The construction uses ideas from §2, but another construction using more classical ideas is given in §6. These two sections do not use any categorical machinery; rather, they formulate and solve problems purely about affine $\mathcal{W}$-algebras.

The affine Skryabin theorem is proved in §5. This combines the categorical methods with Kac-Moody algebras.

Finally, some applications, such as the categorical Feigin-Frenkel theorem, are given in §7. We discuss exactness properties of the Drinfeld-Sokolov functor here.

There are two appendices. Appendix A gives background material on homological algebra for Tate Lie algebras. It includes supporting material on semi-infinite cohomology and its calculation. All the material there is standard, but perhaps it warranted an updated exposition. Appendix B is about t-exactness of !-averaging functors, which as indicated above, plays a key role.

1.8. Notation and conventions.

1.8.1. We assume the reader is familiar with the theory of D-modules in infinite type and group actions on (cocomplete DG) categories. We refer to [Ber] and [Ras3] for these subjects. For a group H acting on $\mathcal{C}$, we let $\mathcal{C}^H$ denote the strong invariants, $\mathcal{C}^{H,w}$ denote the weak invariants, and $\mathcal{C}^{H,\psi}$ denote the invariants twisted by the exponential D-module along a character $\psi : H \to \mathbb{G}_a$ (cf. [Ber] §2.5.4).

We use homotopical algebra methods freely, although this is inessential at some points. Our default categorical language is the ∞-categorical language of Lurie. So *category* means $(\infty, 1)$-category, and the rest of the categorical notions (notably limits and colimits) are understood correspondingly.

We use $\otimes$ to denote the natural symmetric monoidal operation on DGCat_{cont}.

1.8.2. *Kac-Moody groups.* We let:

$$1 \to Z_{KM} \to \widehat{G(K)} \to G(K) \to 1$$

denote the Kac-Moody extension. Some explanation is in order.

Here Z_{KM} is a certain (finite-dimensional) torus, which is $\mathbb{G}_m$ for G a simple group. Precisely, it suffices to define its character lattice; it will be a certain sublattice of the k-vector space of *levels*, i.e., Ad-invariant symmetric bilinear forms on $\mathfrak{g}$.

Recall that $\mathfrak{g} = \mathfrak{z}_\mathfrak{g} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{z}_\mathfrak{g}$ is the center. Moreover, recall that $[\mathfrak{g}, \mathfrak{g}]$ is canonically a direct sum of simple Lie algebras $\mathfrak{g}_i$ (its minimal normal subalgebras). For any level κ, the spaces $\mathfrak{z}_\mathfrak{g}$ and $\mathfrak{g}_i$ are pairwise orthogonal. We then take the lattice of levels such that $\kappa|_{\mathfrak{z}_\mathfrak{g}}$ is *even*, i.e., the pairing of any two coweights of $G/[G, G]$ is an even integer, and $\kappa|_{\mathfrak{g}_i}$ is an integral multiple of the Killing form. Clearly such levels k-span the space of all levels.

For any level of the above type, we obtain a central extension of $G(K)$ by $\mathbb{G}_m$ whose Lie algebra is the Kac-Moody extension $\hat{\mathfrak{g}}_\kappa$ of $\mathfrak{g}((t))$. Indeed, the adjoint group G^{ad} is canonically a product of groups G_i, so we obtain $G \to G/[G,G] \times \prod_i G_i$; our extension of $G(K)$ is the Baer sum of the extensions of $G/[G,G](K)$ defined by the Contou-Carrére symbol and the extensions of $G_i(K)$ defined by the usual determinant line method. This is additive in the level, so it is equivalent to say that there is an extension $\widehat{G(K)}$ of $G(K)$ by Z_{KM} whose pushout by a character $Z_{KM} \to \mathbb{G}_m$ (equivalently, a level of the specified type) is the one we just explained.

1.8.3. For *any* level κ, there is a canonical multiplicative D-module on Z_{KM} whose underlying $\mathcal{O}$-module is multiplicatively trivialized. E.g., if $Z_{KM} = \mathbb{G}_m$, this is the D-module "z^λ" for λ the given scalar. We let $D_\kappa(G(K))$ denote the corresponding category of twisted D-modules: by definition, this is the category of D-modules on $\widehat{G(K)}$ that are Z_{KM}-equivariant against our multiplicative character. This DG category is equipped with a convolution monoidal structure.

As in [Ber], this allows us to speak about DG categories acted on at level κ (for a general level κ).

1.9. **Acknowledgements.** I have benefited from discussions with a number of people in working on this material. I'm happy to thank Dima Arinkin, Sasha Beilinson, David Ben-Zvi, Dario Beraldo, Roman Bezrukavnikov, Sasha Braverman, Vladimir Drinfeld, Giorgia Fortuna, Edward Frenkel, Reimundo Heluani, Masoud Kamgarpour, Ivan Mirkovic, and Xinwen Zhu for their support, encouragement, and for their generosity in sharing their ideas.

Special thanks are due to Dennis Gaitsgory. This spirit of representation theory was invented by him, and his singular influence extends throughout this work.

2. Compact approximation to the Whittaker model

2.1. **Invariants and coinvariants.** In this section, we will prove the following theorem:

Theorem 2.1.1. *For every* $\mathcal{C}$ *acted on by* $G(K)$ *at level* κ*, there is an equivalence:*

$$\mathcal{C}_{N(K),\psi} \xrightarrow{\simeq} \mathcal{C}^{N(K),\psi}$$

functorial in $\mathcal{C}$*.*

Remark 2.1.2. The functor[18] realizing this equivalence was constructed by Gaitsgory in [Gai3], and conjectured to be an equivalence. This theorem was proved in *loc. cit.* for rank 1 groups, and in [Ber] for $G = GL_n$.

Remark 2.1.3. From the onset, we draw the reader's attention to Theorem 2.3.1, which is the essential tool in proving Theorem 2.1.1, and which plays a key role throughout this paper.

2.2. **The adolescent Whittaker constructions.**

---

[18]This functor admits a very simple description in the language [Ras3] of D-modules on infinite type (ind)schemes. First, note that convolution by any $(N(K), \psi)$-biequivariant D-module on $G(K)$ induces a functor $\mathcal{C}_{N(K),\psi} \to \mathcal{C}^{N(K),\psi}$. Then we should take the convolution with the *renormalized* pushforward from $N(K)$ to $G(K)$ of the character sheaf. We remind that the word *renormalized* indicates something specific to the infinite type setup, and in particular that it indicates that we have chosen trivializations of dimension torsors.

In particular, we see that this functor makes sense for any category acted on by $N(K)$, i.e., the $G(K)$ action is not necessary. However, we remind that a peculiarity of the infinite type framework is that if we took $\mathcal{C} = \mathsf{Vect} \cdot \psi$ (i.e., the $N(K)$ action on Vect corresponding to the character ψ), then this functor is zero, even though $\mathcal{C}_{N(K),\psi} = \mathcal{C}^{N(K),\psi} = \mathsf{Vect}$ (the identification being realized here by a different functor).

In any case, the explicit description of the functor will be immediate from the proof given below, and we do not particularly emphasize it.

2.2.1. The main tool for proving Theorem 2.1.1 is the following sequence of subgroups of $G(K)$.

Definition 2.2.1. For $n \geqslant 0$, we let $\mathring{I}_n$ denote the subgroup $\mathrm{Ad}_{-n\check{\rho}(t)}(G(O) \times_{G(O/t^n)} N(O/t^n))$.

In other words, we take elements of $G(O)$ that lie in N modulo t^n, and conjugate this subgroup in a way that enlarges congruence subgroups of $N(K)$, fixes $T(K)$, and shrinks congruence subgroups of $N^-(K)$.

Example 2.2.2. For $n = 0$, $\mathring{I}_0 = G(O)$.[19] For $n = 1$, $\mathring{I}_1$ is conjugated by $\check{\rho}(t)$ from the radical of Iwahori. While $\mathring{I}_n$ neither contains nor is contained in $\mathring{I}_{n+1}$, these groups limit to $N(K)$.

Notation 2.2.3. Motivated by the limiting behavior above, we add $N(K)$ to this family of subgroups by setting $\mathring{I}_n = N(K)$ for $n = \infty$.

Example 2.2.4. For $G = GL_2$ and $0 < n < \infty$, $\mathring{I}_n$ is the subgroup of matrices:

$$\begin{pmatrix} 1 + t^n a & t^{-n} b \\ t^{2n} c & 1 + t^n d \end{pmatrix}$$

for $a, b, c, d \in O$. Similarly, for $G = GL_3$, we obtain the subgroup of matrices:

$$\begin{pmatrix} 1 + t^n a & t^{-n} b & t^{-2n} c \\ t^{2n} d & 1 + t^n e & t^{-n} f \\ t^{3n} g & t^{2n} h & 1 + t^n i \end{pmatrix}.$$

Example 2.2.5. For $G = T$ a torus, $\mathring{I}_n$ is the nth congruence subgroup.

Remark 2.2.6 (Triangular decomposition). We will repeatedly use the following fact without mention. For all $n > 0$, note that $\mathring{I}_n$ admits a *triangular decomposition*:

$$\mathring{I}_n = (\mathring{I}_n \cap N^-(K)) \times (\mathring{I}_n \cap T(K)) \times (\mathring{I}_n \cap N(K))$$

with the isomorphism induced by the multiplication map. The same is true if we reverse the order of the factors.

Remark 2.2.7 (Splitting the Kac-Moody extension). Note that $\widehat{G(K)} \to G(K)$ is canonically split over $G(O)$. By transport of structure, it is canonically split over $\mathrm{Ad}_{-n\check{\rho}(t)}(G(O))$ as well, and in particular, over $\mathring{I}_n$.

Varying n, these splittings coincide on all intersections $\mathring{I}_n \cap \mathring{I}_m$. Indeed, we may safely assume one of n and m is non-zero, in which case this intersection is prounipotent. But the splittings differ by a homomorphism $\mathring{I}_n \cap \mathring{I}_m \to \mathbb{G}_m$, which must be trivial by prounipotence.

Therefore, $D_\kappa(\mathring{I}_n) \simeq D(\mathring{I}_n)$ as monoidal categories, and as n varies these equivalences are compatible with restriction to intersections between the subgroups $\mathring{I}_n$.

---

[19]This is the only case where $\mathring{I}_n$ is not prounipotent. So for many problems, a claim about all $\mathring{I}_n$ is proved by treating the $n = 0$ case separately, where the claim may be degenerate anyway. Despite this clumsiness, it seems to be most natural to include the $n = 0$ case on equal footing wherever possible.

2.2.2.   One has the following straightforward construction of characters of the $\mathring{I}_n$.

Lemma 2.2.8. *For every n, there is a unique homomorphism $\psi_{\mathring{I}_n} : \mathring{I}_n \to \mathbb{G}_a$ annihilating $B^-(O) \cap \mathring{I}_n$ and, with $\psi_{\mathring{I}_n}|_{N(K) \cap \mathring{I}_n} = \psi|_{\mathring{I}_n \cap N(K)}$. For a pair of integers n, m, the corresponding characters coincide on the intersection $\mathring{I}_n \cap \mathring{I}_m$.*

Notation 2.2.9. To encourage the reader to think of the characters $\psi_{\mathring{I}_n}$ and ψ as all being "the same," we denote them all by ψ when there is no risk of confusion.

Example 2.2.10. Since the $n = 0$ case can be the most confusing: the character is trivial in this case.

2.2.3.   In the remainder of this section, $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is equipped with a $G(K)$ action of level κ.

Definition 2.2.11. For $0 \leqslant n \leqslant \infty$, we define the nth *adolescent Whittaker category* as:[20]

$$\mathsf{Whit}^{\leqslant n}(\mathcal{C}) := \mathcal{C}^{\mathring{I}_n, \psi}.$$

Remark 2.2.12. For $n > 0$, $\mathsf{Whit}^{\leqslant n}(\mathcal{C})$ is a subcategory of $\mathcal{C}$ since $\mathring{I}_n$ is prounipotent if $n < \infty$ and ind-prounipotent if $n = \infty$.

Remark 2.2.13. In line with Example 2.2.2, $\mathsf{Whit}^{\leqslant 0}(\mathcal{C}) = \mathcal{C}^{G(O)}$ is the *spherical* category, $\mathsf{Whit}^{\leqslant 1}(\mathcal{C})$ is[21] the *baby* Whittaker category of [AB], and in the limit as $n \to \infty$, we have $\mathsf{Whit}^{\leqslant \infty}(\mathcal{C}) = \mathcal{C}^{N(K), \psi}$ the (grown-up) Whittaker category.

2.2.4. *How are the categories $\mathsf{Whit}^{\leqslant n}$ related as we vary n?* Since the $\mathring{I}_n$ groups are not contained one in another, we can naturally relate these categories via averaging. In Theorem 2.3.1, we will find particularly nice behavior of these averaging functors in this setting.

2.2.5.   For $n \leqslant m < \infty$, we have a functor:

$$\iota_{n,m,*} : \mathsf{Whit}^{\leqslant n}(\mathcal{C}) \to \mathsf{Whit}^{\leqslant m}(\mathcal{C})$$

given as the composition:

$$\mathcal{C}^{\mathring{I}_n, \psi} \xrightarrow{\mathrm{Oblv}} \mathcal{C}^{\mathring{I}_n \cap \mathring{I}_m, \psi} \xrightarrow{\mathrm{Av}_*^{\psi}} \mathcal{C}^{\mathring{I}_m, \psi}$$

and for all $n \leqslant m$ we have a functor:

$$\iota_{n,m}^! : \mathsf{Whit}^{\leqslant m}(\mathcal{C}) \to \mathsf{Whit}^{\leqslant n}(\mathcal{C})$$

given as the composition:

$$\mathcal{C}^{\mathring{I}_m, \psi} \xrightarrow{\mathrm{Oblv}} \mathcal{C}^{\mathring{I}_n \cap \mathring{I}_m, \psi} \xrightarrow{\mathrm{Av}_*^{\psi}} \mathcal{C}^{\mathring{I}_n, \psi}.$$

Notation 2.2.14. We use the following convention systematically: for $m = \infty$, we suppress m from the notation. So we use the notation $\iota_n^!$ instead of $\iota_{n,\infty}^!$.

---

[20]Note that this category makes sense because of Remark 2.2.7.

[21]At least if the center of G is connected, so that $\mathring{I}_1$ is actually conjugate to the radical of Iwahori by an element of $G(K)$ (and not merely $G^{ad}(K)$), but the reader is advised to ignore this point.

Remark 2.2.15. We remind that $*$-averaging only makes sense for group schemes, not group ind-schemes, so $\iota_{n,*} = \iota_{n,\infty,*}$ does not make sense. (Here *makes sense* means that while there is a non-continuous right adjoint for formal reasons, this functor is pathological.)

Remark 2.2.16. These functors compose well: e.g., for $\ell \leqslant n \leqslant m < \infty$, we have $\iota_{n,m,*} \circ \iota_{\ell,n,*} = \iota_{\ell,m,*}$.

Remark 2.2.17. There is no a priori adjunction between the functors $\iota_{n,m,*}$ and $\iota^!_{n,m}$, since Oblv is a left adjoint while Av^ψ_* is a right adjoint. Rather, their relationship, in the terminology of [Gai4], is that these functors are *dual* to one other (when $\mathcal{C}$ is dualizable).

Warning 2.2.18. Neither of the constructions $\iota_{n,m,*}$ or $\iota^!_{n,m}$ is compatible with forgetful functors to $\mathcal{C}$. We advise to mostly forget about the forgetful functors to $\mathcal{C}$ and to remember these functors instead.

2.3. Formulation of the main result. Let $\Delta := 2(\check\rho, \rho) \in \mathbb{Z}^{\geqslant 0}$. Note that:[22]

$$n\Delta = \dim(\mathrm{Ad}_{-n\check\rho(t)} N(O)/N(O)).$$

The main result of this section is the following.

Theorem 2.3.1.   *(1) For all $n \leqslant m \leqslant \infty$, the functor $\iota^!_{n,m}$ admits a left adjoint $\iota_{n,m,!}$.*
   (2) For all $0 < n \leqslant m \leqslant \infty$, $\iota_{n,m,!}$ is fully-faithful.
   (3) If $m \neq \infty$, there is a canonical isomorphism:

$$\iota_{n,m,!} \simeq \iota_{n,m,*}[2(m-n)\Delta].$$

   These isomorphisms are compatible with compositions, e.g. the induced isomorphisms between $\iota_{n,m,*} \circ \iota_{\ell,n,*} = \iota_{\ell,m,*}$ and $\iota_{n,m,!} \circ \iota_{\ell,n,!} = \iota_{\ell,m,!}$ canonically coincide.

Remark 2.3.2. The method below is also used in a finite-dimensional situation in [BBM], so we consider this an affine analogue of the first part of their Theorem 1.5 (1).

Remark 2.3.3. The functor $\iota_{n,m,!}$ is tautologically computed by forgetting down to $\mathcal{C}^{\mathring{I}_n \cap \mathring{I}_m, \psi}$ and then !-averaging to $\mathcal{C}^{\mathring{I}_m, \psi}$: the claim in this theorem is that the !-averaging is actually defined.

Remark 2.3.4. In the $n = 0$ and $m = \infty$ case this result says that we can !-average spherical objects to obtain Whittaker equivariant objects. This is an old observation that has been known for as long as the words have made sense. (Actually, the $\mathring{I}_n$ groups were found by reverse engineering while trying to generalize that argument along the lines of the proof of Theorem 2.3.1 (1) given below in the $m = \infty$ case.)

Remark 2.3.5. A quite similar pattern appeared long ago in the p-adic setting in [Rod], though with a (mildly) different series of subgroups in place of the $\mathring{I}_n$.

Remark 2.3.6 (Relationship to the work of Beraldo-Gaitsgory). In the case $G = GL_r$ (resp. $G = GL_2$), these results all follow from [Ber] (resp. [Gai3]). Indeed, in *loc. cit.*, the authors construct closed subgroups[23] $H_n \subseteq \mathring{I}_n$ (specific to GL_r) such that the maps:

$$H_n/H_n \cap H_m \to \mathring{I}_n/\mathring{I}_n \cap \mathring{I}_m$$

---

[22]Just for fun, we remark that Δ can also be calculated as $\sum_i \binom{d_i}{2}$, where the d_i are the exponents of the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$.

[23]For example, for GL_2, one has:

are isomorphisms for all $m \geqslant n$. Moreover, using Fourier techniques reminiscent of the mirabolic theory, Beraldo shows that $*$-averaging $\mathcal{C}^{H_{n+1},\psi} \to \mathcal{C}^{H_n,\psi}$ is an equivalence for all $\mathcal{C}$ as above, with inverse given by the appropriately shifted $*$-averaging functor. These facts are easily seen to imply Theorem 2.3.1 in this case.

The methods for a general reductive group are (by necessity) quite different.

(We also remark that for the application to $\mathcal{W}$-algebras, it is essential to work with compact *open* subgroups of $G(K)$.)

2.4. **Application to co/invariants.** Next, we we will deduce Theorem 2.1.1 from Theorem 2.3.1. We begin by making precise the sense in which the adolescent Whittaker constructions limit to the usual Whittaker construction.

Let $\mathcal{C}$ be as always. For all $0 < n \leqslant m < \infty$, observe that the diagrams:

$$
\begin{array}{ccc}
\mathcal{C}^{N(K) \cap \overset{\circ}{I}_m, \psi} & \xrightarrow{\quad \text{Oblv} \quad} & \mathcal{C}^{N(K) \cap \overset{\circ}{I}_n, \psi} \\
\downarrow{\scriptstyle \mathrm{Av}_*^\psi} & & \downarrow{\scriptstyle \mathrm{Av}_*^\psi} \\
\mathcal{C}^{\overset{\circ}{I}_m, \psi} = \mathsf{Whit}^{\leqslant m}(\mathcal{C}) & \xrightarrow{\ \iota_{n,m}^! \ } & \mathsf{Whit}^{\leqslant n}(\mathcal{C}) = \mathcal{C}^{\overset{\circ}{I}_n, \psi}
\end{array}
$$

and:

$$
\begin{array}{ccc}
\mathcal{C}^{\overset{\circ}{I}_n, \psi} = \mathsf{Whit}^{\leqslant n}(\mathcal{C}) & \xrightarrow{\ \iota_{n,m,*} \ } & \mathsf{Whit}^{\leqslant m}(\mathcal{C}) = \mathcal{C}^{\overset{\circ}{I}_m, \psi} \\
\downarrow{\scriptstyle \text{Oblv}} & & \downarrow{\scriptstyle \text{Oblv}} \\
\mathcal{C}^{N(K) \cap \overset{\circ}{I}_n, \psi} & \xrightarrow{\quad \mathrm{Av}_*^\psi \quad} & \mathcal{C}^{N(K) \cap \overset{\circ}{I}_m, \psi}
\end{array}
\tag{2.4.1}
$$

commute.

Lemma 2.4.1. *The induced functors in* DGCat_{cont}:

$$
\mathcal{C}^{N(K),\psi} := \lim_{n,\text{Oblv}} \mathcal{C}^{N(K) \cap \overset{\circ}{I}_n, \psi} \to \lim_{n, \iota_{n,m}^!} \mathsf{Whit}^{\leqslant n}(\mathcal{C})
$$

$$
\operatorname*{colim}_{n, \iota_{n,m,*}} \mathsf{Whit}^{\leqslant n}(\mathcal{C}) \to \operatorname*{colim}_{n, \mathrm{Av}_*^\psi} \mathcal{C}^{N(K) \cap \overset{\circ}{I}_n, \psi} =: \mathcal{C}_{N(K),\psi}
$$

are equivalences.

Proof. First, we treat the coinvariants statement.

Suppose $\Gamma_i \subseteq G(K)$ is a cofiltered sequence of prounipotent group subschemes and let $\Gamma_\infty := \cap_i \Gamma_i$. Note that:

$$
G(K)/\Gamma_\infty = \lim_i G(K)/\Gamma_i.
$$

Then we claim that:

$$
\operatorname*{colim}_i \mathcal{C}^{\Gamma_i} = \mathcal{C}^\Gamma
$$

$$
H_n = \left\{ \begin{pmatrix} 1 + t^n a & t^{-n} b \\ 0 & 1 \end{pmatrix} \mid a, b \in O \right\}.
$$

In the general case, H_n is the intersection of $\overset{\circ}{I}_n$ with the mirabolic subgroup.

where the structure maps in the colimit are forgetful functors. Indeed, this follows because the D-module $\delta_\Gamma \in D(G(K))$ is the colimit of the δ_{Γ_i}, and convolution with these δ D-modules give the appropriate (colocalizing) averaging functors.

Now for $\infty > m \geqslant n > 0$, define $\mathring{I}_{n,m}$ as $(\mathring{I}_m \cap B^-(K)) \cdot (\mathring{I}_n \cap N(K))$. Note that $\mathring{I}_{n,m}$ is a group scheme, $\mathring{I}_{n,m} \supseteq \mathring{I}_{n,m+1}$, and $\cap_m \mathring{I}_n = N(K) \cap \mathring{I}_n$.

Moreover, for a pair $(m', n') \in \mathbb{Z}^{>0} \times \mathbb{Z}^{>0}$ with $m' \geqslant n', m$, the functor:

$$\mathrm{Av}_*^\psi : \mathcal{C}^{N(K) \cap \mathring{I}_n, \psi} \to \mathcal{C}^{N(K) \cap \mathring{I}_{n'}, \psi}$$

takes the subcategory $\mathcal{C}^{\mathring{I}_{n,m}, \psi}$ into $\mathcal{C}^{\mathring{I}_{n',m'}, \psi}$. Therefore, we have:

$$\operatorname*{colim}_{n, \mathrm{Av}_*^\psi} \mathcal{C}^{N(K) \cap \mathring{I}_n, \psi} = \operatorname*{colim}_{m \geqslant n} \mathcal{C}^{\mathring{I}_{n,m}, \psi} = \operatorname*{colim}_{n} \mathcal{C}^{\mathring{I}_n, \psi}$$

as desired.

The version for invariants follows formally from the coinvariants version:

$$\mathcal{C}^{N(K), \psi} = \mathsf{Hom}_{D_\kappa(G(K)) - \mathsf{mod}}(D_\kappa(G(K))_{N(K), \psi}, \mathcal{C}) \xrightarrow{\simeq}$$

$$\mathsf{Hom}_{D_\kappa(G(K)) - \mathsf{mod}}(\operatorname*{colim}_n D_\kappa(G(K))^{\mathring{I}_n, \psi}, \mathcal{C}) =$$

$$\mathsf{Hom}_{D_\kappa(G(K)) - \mathsf{mod}}(\operatorname*{colim}_n D_\kappa(G(K))_{\mathring{I}_n, \psi}, \mathcal{C}) =$$

$$\operatorname*{lim}_n \mathsf{Hom}_{D_\kappa(G(K)) - \mathsf{mod}}(D_\kappa(G(K))_{\mathring{I}_n, \psi}, \mathcal{C}) = \operatorname*{lim}_n \mathcal{C}^{\mathring{I}_n, \psi}.$$

$\square$

Proof that Theorem 2.3.1 implies Theorem 2.1.1. By Lemma 2.4.1, $\mathcal{C}_{N(K), \psi}$ is the colimit in DGCat_{cont} of the $\mathsf{Whit}^{\leqslant n}(\mathcal{C})$ with the functors $\iota_{n,m,*}$ as structural functors. Intertwining via the autoequivalences:

$$\mathsf{Whit}^{\leqslant n}(\mathcal{C}) \xrightarrow{\simeq} \mathsf{Whit}^{\leqslant n}(\mathcal{C})$$

$$\mathcal{F} \mapsto \mathcal{F}[2n\Delta]$$

we see that $\mathcal{C}_{N(K), \psi}$ is also equivalent to the colimit where instead we use the structural functors:

$$\iota_{n,m,*}[2(m-n)\Delta] \stackrel{Thm.2.3.1(3)}{\simeq} \iota_{n,m,!}.$$

Now this is a colimit in DGCat_{cont} under left adjoints, so it is equivalent to the limit under the right adjoints $\iota_{n,m}^!$. Applying Lemma 2.4.1 again gives this limit as $\mathcal{C}^{N(K), \psi}$, as desired.

$\square$

We record the following consequence of the argument.

Corollary 2.4.2. *The equivalent categories $\mathcal{C}^{N(K), \psi}$ and $\mathcal{C}_{N(K), \psi}$ are obtained from the categories $\mathsf{Whit}^{\leqslant n}(\mathcal{C})$ by either taking the colimit under the functors $\iota_{n,m,!}$, or the limit under their right adjoints $\iota_{n,m}^!$.*

Remark 2.4.3. For the sake of clarity: note that we did not use Theorem 2.3.1 (2) in this deduction. But this result *will* be used in proving the other parts of the theorem.

2.5. **Notation.** In the remainder of this section, we prove Theorem 2.3.1.

To keep the notation from becoming too overburdened: for a subgroup $H \subseteq G(K)$, we use the notation H^n for $\mathrm{Ad}_{-n\check{\rho}(t)}(H)$. For example, $N(O)^n = N(K) \cap \mathring{I}_n$.

We also use the notation $\mathsf{Whit}(\mathcal{C})$ to indicate Whittaker *invariants* $\mathcal{C}^{N(K),\psi} = \mathsf{Whit}^{\leqslant \infty}(\mathcal{C})$.

2.6. **Proof of Theorem 2.3.1** (1) **for** $m = \infty$.

2.6.1. We begin by showing that objects of $\mathsf{Whit}^{\leqslant n}(\mathcal{C})$ can be !-averaged to $\mathsf{Whit}(\mathcal{C})$.

2.6.2. We begin by giving the argument in the geometric setting, where it is easier to understand, and later will explain how to adapt to the categorical setup. So suppose $\mathcal{C} = D(X)$ for X a suitably nice (e.g., ind-finite type) indscheme acted on by $G(K)$, ignoring the irrelevant level for the time being.[24]

For $\mathcal{F} \in \mathsf{Whit}^{\leqslant n}(D(X))$, we want to show that $\mathrm{act}_!(\psi \widetilde{\boxtimes} \mathcal{F})$ is defined, where:[25]

$$\psi \widetilde{\boxtimes} \mathcal{F} \in D(N(K) \overset{N(O)^n}{\times} X)$$

is the descent of $\psi \boxtimes \mathcal{F}$ using equivariance of $\mathcal{F}$ and act denotes the action map $N(K) \overset{N(O)^n}{\times} X \to X$.

So first we should compactify the map act. Note that:

$$N(K) \overset{N(O)^n}{\times} X = N(K)\mathring{I}_n \overset{\mathring{I}_n}{\times} X = N(K)G(O)^n \overset{G(O)^n}{\times} X$$

since $G(O)^n \cap N(K) = N(K) \cap \mathring{I}_n = N(O)^n$.

Then define $\overline{N(K)G(O)^n}$ as the pullback to $G(K)$ of the closure of the $N(K)$ orbit through 1 in $G(K)/G(O)^n$. Since the latter is isomorphic to the affine Grassmannian, so is ind-proper, $\overline{N(K)G(O)^n}/G(O)^n$ is ind-proper as well. Therefore, the map:

$$\overline{\mathrm{act}} : \overline{N(K)G(O)^n} \overset{G(O)^n}{\times} X \to X$$

is ind-proper as well.[26]

Let j denote the open embedding:

$$N(K) \overset{N(O)^n}{\times} X = N(K)G(O)^n \overset{G(O)^n}{\times} X \hookrightarrow \overline{N(K)G(O)^n} \overset{G(O)^n}{\times} X.$$

Then it remains to verify the *cleanness* result:

$$j_!(\psi \widetilde{\boxtimes} \mathcal{F}) \overset{\simeq}{\longrightarrow} j_{*,dR}(\psi \widetilde{\boxtimes} \mathcal{F}).$$

In particular, the left hand side is defined.

To this end, we begin with a lemma, where we reintroduce the level κ for clarity in the generalization.

Let $\mathcal{K}_n \subseteq G(O)$ denote the nth congruence subgroup, so $\mathcal{K}_n^n = \mathrm{Ad}_{-n\check{\rho}(t)}(\mathcal{K}_n)$ by our convention.

Lemma 2.6.1. *(1)* $\mathsf{Whit}(D_\kappa(\overline{N(K)G(O)^n}/\mathcal{K}_n^n)) = \mathsf{Whit}(D_\kappa(N(K)G(O)^n/\mathcal{K}_n^n))$.

---

[24]If we were not forgetting the level, we would equip X with a Z_{KM}-torsor $\mathcal{P}$ with an action of $\widehat{G(K)}$ extending the given action of the central Z_{KM}.

[25]Here $N(K) \overset{N(O)^n}{\times} X$ is the standard notation for the quotient of $N(K) \times X$ by the diagonal action of $N(O)^n$ acting on the right on $N(K)$ and "on the left" on X.

[26]The notation is potentially confusing: $\overline{\mathrm{act}}$ is just induced by the usual action map $G(K) \times X \to X$.

(2) *The functor $j_!$ is defined on objects $\mathcal{F} \in \mathsf{Whit}(D_\kappa(N(K)G(O)^n/\mathcal{K}_n^n))$, and the natural morphism $j_!(\mathcal{F}) \to j_{*,dR}(\mathcal{F})$ is an isomorphism.*

Proof. We begin with (1). Recall that $\overline{N(K)G(O)^n}$ is stratified by the locally closed strata $N(K)\check{\lambda}(t)G(O)^n$ for $\check{\lambda} \in -\check{\Lambda}^{pos}$.[27] Therefore, it suffices to show that $\mathsf{Whit}(D_\kappa(N(K)\check{\lambda}(t)G(O)^n/\mathcal{K}_n^n)) = 0$ for all $0 \neq \check{\lambda} \in -\check{\Lambda}^{pos}$.

Using:

$$N(K)\check{\lambda}(t)G(O)^n/\mathcal{K}_n^n = \check{\lambda}(t)N(K)G(O)^n/\mathcal{K}_n = \check{\lambda}(t)N(K) \overset{N(O)^n}{\times} G(O)^n/\mathcal{K}_n$$

we obtain:

$$\mathsf{Whit}(D_\kappa(N(K)\check{\lambda}(t)G(O)^n/\mathcal{K}_n^n)) = D(G(O)^n/\mathcal{K}_n^n)^{N(O)^n, \psi^{\check{\lambda}}}$$

where $\psi^{\check{\lambda}} := \psi \circ \mathrm{Ad}_{\check{\lambda}(t)}$. (We forget κ because we are dealing with a conjugate of $G(O)$ now.)

As $\mathcal{K}_n^n \subseteq G(O)$ is normal, it acts trivially on $G(O)^n/\mathcal{K}_n^n$. Because $N(O) \subseteq \mathcal{K}_n^n$, we see that the $N(O)$ action on $G(O)^n/\mathcal{K}_n^n$ is trivial. Therefore, it suffices to see that $\psi^{\check{\lambda}}|_{N(O)}$ is non-trivial for $0 \neq \check{\lambda} \in -\check{\Lambda}^{pos}$.

This is standard: $\psi^{\check{\lambda}}(\exp(t^m e_i)) = \psi(\exp(t^{m+(\check{\lambda},\alpha_i)}e_i)$, which is 1 if $m + (\check{\lambda}, \alpha_i) = -1$. So it suffices to show that $(\check{\lambda}, \alpha_i) < 0$ for some $i \in \mathcal{I}_G$ (then take $m = -1 - (\check{\lambda}, \alpha_i) \geqslant 0$). Since $(\check{\lambda}, \rho) < 0$ by assumption on $\check{\lambda}$, this is clear.

We now show (2).[28] We show that for $\mathcal{F} \in \mathsf{Whit}(D_\kappa(N(K)G(O)^n/\mathcal{K}_n^n))$, $j_{*,dR}(\mathcal{F})$ satisfies the defining property of $j_!(\mathcal{F})$.

Let $i : Z \hookrightarrow \overline{N(K)G(O)^n}/\mathcal{K}_n^n$ be the (reduced) complement to $N(K)G(O)^n/\mathcal{K}_n^n$. We need to show that for any $\mathcal{G} \in D_\kappa(Z)$, every morphism $j_{*,dR}(\mathcal{F}) \to i_{*,dR}(\mathcal{G})$ is zero. Any such morphism lifts canonically to $\lim_m \mathrm{Av}_*^{N(O)^m, \psi} i_{*,dR}(\mathcal{G})$, so it suffices to show that this limit vanishes.[29]

Let $S \subseteq \overline{N(K)G(O)^n}/\mathcal{K}_n^n$ be a closed, finite type subscheme. The above claim is equivalent to showing $\lim_m \mathrm{Av}_*^{N(O)^m, \psi} i_{*,dR}(\mathcal{G})$ restricts to S as zero. In fact, we claim that there is an integer m depending only on S such that $\mathrm{Av}_*^{N(O)^m, \psi} i_{*,dR}(\mathcal{G})$ restricts to S as zero.

For this, let $\overset{\circ}{S} = S \cap N(K)G(O)^n/\mathcal{K}_n^n$ and let $S_Z = S \cap Z$. The restriction of $\mathrm{Av}_*^{N(O)^m, \psi} i_{*,dR}(\mathcal{G})$ to $\overset{\circ}{S}$ is zero for any m by base-change. Then for m large enough, $N(O)^m$ clearly contains the stabilizers in $N(K)$ of points of S. By our analysis from (1), our character is non-trivial on each of these stabilizers, so any $(N(O)^m, \psi)$-equivariant object restricts to 0 on S for m large enough; this gives our claim. $\square$

Remark 2.6.2. More generally, this argument shows that any D-module on $\overline{N(K)G(O)^n} \times X$ satisfying Whittaker equivariance for the left action on the first factor and $\mathcal{K}_n^n$-equivariance for the right action on the first factor is cleanly extended from $N(K)G(O)^n \times X$. Note that the action of $G(K)$ on X is not used here. (In the categorical setting, one should instead note that for any $\mathcal{C} \in \mathsf{DGCat}_{cont}$ considered with a trivial $G(K)$-action, $\mathsf{Whit}(D_\kappa(\overline{N(K)G(O)^n}/\mathcal{K}_n^n) \otimes \mathcal{C}) = \mathsf{Whit}(D_\kappa(N(K)G(O)^n/\mathcal{K}_n^n) \otimes \mathcal{C}).$)

---

[27]Here $\check{\Lambda}^{pos} \subseteq \check{\Lambda}$ is the subset of positive coroots, i.e., the $\mathbb{Z}^{\geqslant 0}$-span of $\{\check{\alpha}_i\}_{i \in \mathcal{I}_G}$.

[28]We thank Dennis Gaitsgory for pointing out that there is something non-obvious still to show.

[29]This would be obvious in the finite-dimensional setting, but because of the inverse limit, it is not completely formal that the restriction of this object to the open cell is zero.

Therefore, it suffices to see that the pullback of $\psi\widetilde{\boxtimes}\mathcal{F}$ to $N(K)G(O)^n \times X$ is Whittaker equivariant for the action of $N(K)$ on the first factor, and $\mathcal{K}_n^n$-equivariant for the right action on the first factor. Note that the map $N(K)G(O)^n \times X \to N(K) \overset{N(O)^n}{\times} X$ is given by the formula $(g_1 \cdot g_2, x) \mapsto (g_1, g_2 \cdot x)$, so the first claim is obvious.

For the second claim, note that it suffices to check equivariance after further pullback to $N(K) \times G(O)^n \times X$. Indeed, this follows as the project $N(K) \times G(O)^n \to N(K)G(O)^n$ is a torsor for a prounipotent group, so pullback is fully-faithful. Our map to $N(K) \overset{N(O)^n}{\times} X$ then lifts to $N(K) \times X$. Now observe that $\psi \boxtimes \mathcal{F}$ is $\mathcal{K}_n^n$-equivariant for the action of $\mathcal{K}_n^n$ on the X-factor, since $\mathcal{K}_n^n \subseteq \overset{\circ}{I}_n$ and $\psi_{\overset{\circ}{I}_n}|_{\mathcal{K}_n^n}$ is trivial. Moreover, the map:

$$N(K) \times G(O)^n \times X \xrightarrow{(g_1,g_2,x) \mapsto (g_1,g_2\cdot x)} N(K) \times X \to N(K) \times X/\mathcal{K}_n^n$$

descends to a map from $N(K) \times G(O)^n/\mathcal{K}_n^n \times X$, by normality of $\mathcal{K}_n^n \subseteq G(O)^n$. This gives the claim, completing the argument.

2.6.3. We now indicate what changes should be made in the general categorical setting. So let $\mathcal{C}$ be acted on by $G(K)$ at level κ.

We have a coaction functor:

$$\mathcal{C} \to D(N(K)) \otimes \mathcal{C}$$

(obtained from the coaction functor $\mathcal{C} \to D_\kappa(G(K)) \otimes \mathcal{C}$ encoding the action of $G(K)$ by !-restriction along $N(K) \hookrightarrow G(K)$). This induces a functor:

$$\mathrm{act}^! : \mathsf{Whit}^{\leqslant n}(\mathcal{C}) = \mathcal{C}^{\overset{\circ}{I}_n,\psi} \to D(N(K)) \overset{N(O)^n}{\otimes} \mathcal{C}$$

where the superscript $N(O)^n$ indicates we take invariants for the diagonal action mixing the action on $\mathcal{C}$ with the right action on $D(N(K))$.

Similarly, we have $\mathrm{act}_{*,dR} : D(N(K)) \overset{N(O)^n}{\otimes} \mathcal{C} \to \mathsf{Whit}^{\leqslant n}(\mathcal{C})$.

As before, we need to show that the left adjoint $\mathrm{act}_!$ to $\mathrm{act}^!$ is defined on $\psi\widetilde{\boxtimes}\mathcal{F}$ for every $\mathcal{F} \in \mathsf{Whit}^{\leqslant n}(\mathcal{C})$. Again, it suffices to show the corresponding clean extension property for:[30]

$$\psi\widetilde{\boxtimes}\mathcal{F} \in D(N(K)) \overset{N(O)^n}{\otimes} \mathcal{C} = D(N(K)G(O)^n) \overset{G(O)^n}{\otimes} \mathcal{C} \overset{j_{*,dR}}{\subseteq} D_\kappa(\overline{N(K)G(O)^n}) \overset{G(O)^n}{\otimes} \mathcal{C}.$$

From here, the argument proceeds as explained in the geometric setting.

2.7. **An auxiliary lemma.** We will need the following lemma before proceeding.[31]

---

[30]Because the Kac-Moody cocycle is non-trivial on $\mathfrak{n}((t)) \times \mathrm{Ad}_{-n\check\rho(t)} \mathfrak{g}[[t]]$, there is risk of thinking that we should be including κ in the middle term here. But in fact, the Z_{KM}-torsor $\widetilde{G(K)} \to G(K)$ is canonically $N(K) \times G(O)^n$-equivariantly trivial over this locus, essentially because the determinant line bundle is canonically trivial over $\mathrm{Gr}_N \subseteq \mathrm{Gr}_G$. So there is no risk of making a mistake here.

[31]This result is essentially [Rod] Lemma 4, except that he works with a slightly different series of subgroups (but with similar enough properties that the same arguments should work uniformly for both). Unfortunately, the argument there is not correct: it relies on Lemma 13 from *loc. cit.*, which in particular says that every element of $f + t^N \mathfrak{g}[[t]]$ can be conjugated into a Borel; this is not true since we can approximate f by elliptic elements in the Kostant slice (e.g. $G = GL_2$ and take $\begin{pmatrix} 0 & t^{2N+1} \\ 1 & 0 \end{pmatrix}$).

We remark that the argument given here immediately adapts to the mixed characteristic setting of *loc. cit.*

Lemma 2.7.1. *Suppose $n > 0$. Then for $g \in N(K)$, we have:*

$$\psi_{\mathring{I}_n}\big|_{\mathring{I}_n \cap \mathrm{Ad}_g \mathring{I}_n} = \psi_{\mathring{I}_n} \circ \mathrm{Ad}_{g^{-1}}\big|_{\mathring{I}_n \cap \mathrm{Ad}_g \mathring{I}_n}$$

if and only if $g \in N(K) \cap \mathring{I}_n$.

In other words, if $g \in N(K)$ but $\notin \mathring{I}_n$, there exists $h \in \mathring{I}_n \cap \mathrm{Ad}_g \mathring{I}_n$ with $\psi_{\mathring{I}_n}(h) \neq \psi_{\mathring{I}_n}(g^{-1}hg)$.

We will deduce this result in turn from the following lemma. Here it is convenient to use the notation $I \subseteq G(O)$ for the Iwahori subgroup $I = G(O) \times_G B$ (which should not be confused with the groups $\mathring{I}_n$).

Lemma 2.7.2. *Suppose $\check{\lambda} \in \check{\Lambda}^+$ is a dominant coweight. Then the intersection:*

$$N(K) \cap I\check{\lambda}(t)G(O)$$

is non-empty if and only if $\check{\lambda} = 0$. In this case, the intersection is exactly $N(O)$.

Proof. Let $I_+ = N(O) = I \cap N(K)$ and let $I_- = I \cap B^-(K)$, i.e., T times the first congruence subgroup in $B^-(O)$. Recall that I has a triangular decomposition $I_+ \times I_-$. In particular, we see that our intersection is non-empty if and only if the intersection of $N(K)$ with $I_-\check{\lambda}(t)G(O)$ is.

Then dominance of $\check{\lambda}(t)$ implies $\mathrm{Ad}_{-\check{\lambda}(t)} I_- \subseteq I_-$, i.e., $I_-\check{\lambda}(t) \subseteq \check{\lambda}(t)I_-$. Therefore, we have:

$$I_-\check{\lambda}(t)G(O) \subseteq \check{\lambda}(t)I_-G(O) = \check{\lambda}(t)G(O).$$

But by the Iwasawa decomposition, $\check{\lambda}(t)G(O)$ can only intersect $N(K)$ for $\check{\lambda} = 0$. $\square$

Proof of Lemma 2.7.1. By the Cartan decomposition, g is in the[32] $(I^n, G(O)^n)$-double coset $\check{\lambda}(t)$ for some $\check{\lambda} \in \check{\Lambda}$. We write:

$$g = \gamma_1 \check{\lambda}(t)\gamma_2, \quad \gamma_1 \in I^n, \quad \gamma_2 \in G(O)^n.$$

Our assumption on g is equivalent to saying $\mathrm{Ad}_{n\check{\rho}(t)}(g) \notin N(O)$, so by Lemma 2.7.2 $\check{\lambda}$ is not dominant. Choose a simple root α_i with $(\check{\lambda}, \alpha_i) < 0$.

Now take $h := \mathrm{Ad}_{\gamma_1} \exp(\frac{e_i}{t})$ (the exponential being taken in $N(K)$, where it has the evident meaning). We claim this h satisfies the desired conclusions.

First, because $\gamma_1 \in I^n$, we have $h \in \mathring{I}_n$ with $\psi_{\mathring{I}_n}(h) \neq 0$. Therefore, it suffices to show that $\mathrm{Ad}_{g^{-1}}(h) \in \mathring{I}_n$ with $\psi_{\mathring{I}_n}(\mathrm{Ad}_{g^{-1}}(h)) = 0$.

To see this, we compute:

$$\mathrm{Ad}_{g^{-1}}(h) = \mathrm{Ad}_{\gamma_2^{-1}} \mathrm{Ad}_{-\check{\lambda}(t)} \exp(\frac{e_i}{t}) = \mathrm{Ad}_{\gamma_2^{-1}} \exp(t^{-(\check{\lambda}, \alpha_i)-1}e_i).$$

Then $\exp(t^{-(\check{\lambda}, \alpha_i)-1}e_i) \in N(O) \subseteq N(K)$, and so this element lies in the nth congruence subgroup $\mathcal{K}_n^n$ of $G(O)^n$. Therefore, the same is true of $\mathrm{Ad}_{\gamma_2^{-1}}$, giving:

$$\mathrm{Ad}_{g^{-1}}(h) \in \mathcal{K}_n^n \subseteq \mathrm{Ker}(\psi_{\mathring{I}_n}) \subseteq \mathring{I}_n$$

as desired. $\square$

---

[32] We are using the notation from §2.5 here.

2.8. **Proof of Theorem 2.3.1 (2) for $m = \infty$.** Next, we discuss the fully-faithfulness of $\iota_{n,!}$. Throughout this section, $n > 0$ (as this was an obviously necessary hypothesis).

Proof of Theorem 2.3.1 (2) for $m = \infty$.

Step 1. First, note that $\iota_{n,!}$ must be given by convolution with a kernel $\mathcal{K}_n \in D_\kappa(G(K))^{(N(K),\psi),(\overset{\circ}{I}_n,-\psi)}$, where the notation indicates that the kernel is $(N(K),\psi)$-equivariant for the left action. and $(\overset{\circ}{I}_n, -\psi)$-equivariant for the right action.

Indeed, consider the $D_\kappa(G(K))$ as a $D_{-\kappa}(G(K))$-module[33] category via the right action, so that this action commutes with the $D_\kappa(G(K))$-module structure. With invariants understood with respect to the *left* action, we have the functor:

$$\iota_{n,!} : D_\kappa(G(K))^{\overset{\circ}{I}_n,\psi} \to D_\kappa(G(K))^{N(K),\psi}$$

which is a morphism of $D_{-\kappa}(G(K))$-module categories for formal reasons[34] Since the $(\overset{\circ}{I}_n,\psi)$-invariants coincide with coinvariants, universal properties produce an object:

$$\mathcal{K}_n \in D_\kappa(G(K))^{(N(K),\psi),(\overset{\circ}{I}_n,-\psi)} \simeq \mathsf{Hom}_{D_\kappa(G(K))-\mathsf{mod}}(D_\kappa(G(K))^{\overset{\circ}{I}_n,\psi}, D_\kappa(G(K))^{N(K),\psi})$$

corresponding to $\iota_{n,!}$.

It is tautological that for $\mathcal{C} = D_\kappa(G(K))$, $\iota_{n,!}$ is given by convolution by $\mathcal{K}_n$. For general $\mathcal{C}$, this follows formally by functoriality, since:

$$\iota_{n,!} : \mathcal{C}^{\overset{\circ}{I}_n,\psi} = D_\kappa(G(K))^{\overset{\circ}{I}_n,\psi} \underset{D_\kappa(G(K))}{\otimes} \mathcal{C} \to D_\kappa(G(K))^{N(K),\psi} \underset{D_\kappa(G(K))}{\otimes} \mathcal{C} \to \mathcal{C}^{N(K),\psi}.$$

Step 2. We are trying to show that the unit map $\mathrm{id} \to \iota_n^! \iota_{n,!}$ is an equivalence. Let us rewrite this goal in terms of kernels.

Note that $\iota_n^!$ is given as the composition: $\mathsf{Whit}(\mathcal{C}) = \mathcal{C}^{N(K),\psi} \xrightarrow{\mathrm{Oblv}} \mathcal{C} \xrightarrow{\mathrm{Av}_*^\psi} \mathcal{C}^{\overset{\circ}{I}_n,\psi} = \mathsf{Whit}^{\leqslant n}(\mathcal{C})$.[35] In other words, $\iota_n^!$ is given by convolution with $\delta_{\overset{\circ}{I}_n}^\psi \in D_\kappa(G(K))$, where this notation indicates the de Rham pushforward of the character sheaf on $\overset{\circ}{I}_n$.

Therefore, it suffices to show that $\delta_{\overset{\circ}{I}_n}^\psi \xrightarrow{\simeq} \delta_{\overset{\circ}{I}_n}^\psi \star \mathcal{K}_n = \iota_n^! \iota_{n,!}(\delta_{\overset{\circ}{I}_n}^\psi)$ is an equivalence.

Step 3. Next, we observe that $\mathcal{K}_n$ can be readily calculated:

Namely, it suffices to calculate $\iota_{n,!}$ applied to $\delta_{\overset{\circ}{I}_n}^\psi$ (this is the de Rham pushforward of the character sheaf on $\overset{\circ}{I}_n$). This object is obtained by !-averaging, so is obtained by !-extending $\delta_{N(K)\overset{\circ}{I}_n}^\psi$, the (pullback of the) exponential D-module from $N(K)\overset{\circ}{I}_n$.[36] Note that by Lemma 2.6.1, this extension is clean, i.e., the !-extension coincides with the $*$-extension.

---

[33]Since we are exclusively working with D-modules in this section and not quasi-coherent sheaves, there is no need to incorporate any critical twist here.

[34]Namely, the proof of Theorem 2.3.1 (1) ($m = \infty$) shows that $\iota_{n,!}$ upgrades to a natural transformation between the functors $\mathsf{Whit}^{\leqslant n}, \mathsf{Whit} : D_\kappa(G(K))-\mathsf{mod} \to \mathsf{DGCat}_{cont}$ considered as morphisms of DGCat_{cont}-enriched categories. Then use the fact that $D_{-\kappa}(G(K))$ acts on $D_\kappa(G(K)) \in D_\kappa(G(K))-\mathsf{mod}$.

[35]We are using the hypothesis that $n > 0$ here, so that $\overset{\circ}{I}_n$ is prounipotent.

[36]Because the given splittings of the Kac-Moody extension for $N(K)$ and $\overset{\circ}{I}_n$ coincide on their intersection, and because the characters ψ coincide here as well, this twisted D-module makes sense.

Step 4. By the cleanness noted above, the convolution we are trying to compute is the renormalized D-module pushforward of $\delta^{\psi}_{\mathring{I}_n} \boxtimes \delta^{\psi}_{N(K)\mathring{I}_n}$ along the multiplication map:

$$\mathring{I}_n \times N(K)\mathring{I}_n \to G(K).$$

We claim that this D-module is supported on $\mathring{I}_n$, i.e., is obtained by de Rham pushforward of some D-module on this subscheme. In this step, we will show this for $\kappa = 0$, and in the next step, we will show it for general κ.

Indeed, since this D-module is equivariant for compact open subgroups, this is really a problem about D-modules on ind-finite type schemes. So it suffices to show that the !-fibers of this convolution at all geometric points vanish outside of $\mathring{I}_n$.

For $\gamma \in G(K)$ a geometric point, the !-fiber obviously vanishes unless γ can be written as $h_1 g h_2$ with $h_i \in \mathring{I}_n$ and $g \in N(K)$. It suffices to show that the fiber vanishes unless $g \in N(K) \cap \mathring{I}_n$.

We will do this using the following paradigm: if H is a prounipotent group acting on X, $\psi : H \to \mathbb{G}_a$ is a character, and $x \in X$ is a geometric point with stabilizer $H_x \subseteq H$, then if $\psi|_{H_x}$ is non-trivial, any (H, ψ)-equivariant D-module on X vanishes along the H-orbit through x.

So consider $G(K)$ as acted on by $\mathring{I}_n \times \mathring{I}_n$. Note that our convolution is equivariant with respect to the character $(\psi, -\psi)$ (the minus sign occurring due to the sign appearing for the right action). Note that the stabilizer of g for this action is the subgroup:

$$\mathring{I}_n \cap \mathrm{Ad}_g(\mathring{I}_n) \overset{h \mapsto (h, \mathrm{Ad}_{g^{-1}} h)}{\hookrightarrow} \mathring{I}_n \times \mathring{I}_n.$$

By Lemma 2.7.1, if $g \notin N(K) \cap \mathring{I}_n$, then the character $(\psi, -\psi)$ restricted to this stabilizer subgroup is non-trivial, so our !-fiber vanishes, as desired.

Step 5. Next, we explain how the above calculation works for κ-twisted D-modules.

Recall that κ-twisted D-modules are D-modules on $\widehat{G(K)}$ satisfying some equivariance with respect to the Kac-Moody center, which will actually not be relevant here. We want to show that any $(\mathring{I}_n, \psi)$-biequivariant[37] D-module has vanishing !-fiber at any point $\tilde{g} \in \widehat{G(K)}$ mapping to $g \in N(K) \subseteq G(K)$ with $g \notin \mathring{I}_n$. (We remind that this equivariance makes sense in the first place because the Kac-Moody extension is split over $\mathring{I}_n$.)

So we need a version of Lemma 2.7.1 for $\tilde{g}$ instead of g. Let $\sigma : \mathring{I}_n \to \widehat{G(K)}$ denote the splitting. Then we want the conclusion of Lemma 2.7.1 but for some $h \in \mathring{I}_n$ with

$$\sigma(h) \in \sigma(\mathring{I}_n) \cap \mathrm{Ad}_g \sigma(\mathring{I}_n)$$

instead. Note here that Ad_g makes sense on the left hand side, and coincides with $\mathrm{Ad}_{\tilde{g}}$, because Z_{KM} is central in $\widehat{G(K)}$. We will actually show:

$$\sigma(\mathring{I}_n) \cap \mathrm{Ad}_g \sigma(\mathring{I}_n) = \sigma(\mathring{I}_n \cap \mathrm{Ad}_g \mathring{I}_n) \tag{2.8.1}$$

which immediately gives the claim by Lemma 2.7.1.

Let π denote the projection $\widehat{G(K)} \to G(K)$. This map is $G(K)$-equivariant for the adjoint actions, which immediately implies that the left hand side of (2.8.1) is contained in the right hand side.

Now note that $\mathring{I}_n \cap \mathrm{Ad}_g \mathring{I}_n$ acts on $\pi^{-1}(g)$ via the action map:

---

[37]Here "biequivariant" should certainly be understood with the sign change on the character on the right.

$$\mathring{I}_n \cap \mathrm{Ad}_g \mathring{I}_n \times \pi^{-1}(g) \to \pi^{-1}(g)$$

$$(h, \widetilde{g}) \mapsto \sigma(h)\widetilde{g}\sigma(g^{-1}h^{-1}g).$$

The right hand side obviously lies in $\pi^{-1}(g)$, and this honestly defines an action map because σ is a homomorphism, and $\sigma \circ \mathrm{Ad}_{g^{-1}} : \mathrm{Ad}_g(\mathring{I}_n) \to \widehat{G(K)}$ is too.

Moreover, this action commutes with the Z_{KM}-action on the fiber, so is induced by a homomorphism $\mathring{I}_n \to Z_{KM}$. This homomorphism must be trivial because $\mathring{I}_n$ is prounipotent while Z_{KM} is a torus. Therefore, we obtain:

$$\sigma(h)\widetilde{g}\sigma(g^{-1}h^{-1}g) = \widetilde{g}$$

i.e.:

$$\mathrm{Ad}_{g^{-1}}(\sigma(h)) = \sigma(\mathrm{Ad}_{g^{-1}}(h)), \qquad h \in \mathring{I}_n \cap \mathrm{Ad}_g(\mathring{I}_n).$$

This implies that the right hand side of (2.8.1) is contained in the left hand side, since for h as above, $\sigma(h) = \mathrm{Ad}_g(\sigma(\mathrm{Ad}_{g^{-1}} h)) \in \mathrm{Ad}_g(\sigma(\mathring{I}_n))$.

Step 6. At this point, we have seen that $\delta^{\psi}_{\mathring{I}_n} \star \mathcal{K}_n$ is supported on $\mathring{I}_n$. Obviously, it is $(\mathring{I}_n, \psi)$-equivariant, and its !-fiber at the identity is k by prounipotence of $\mathring{I}_n$. Therefore, this convolution is isomorphic to $\delta^{\psi}_{\mathring{I}_n}$ and the unit map is an isomorphism, as desired.

$\square$

2.9. **Proof of Theorem 2.3.1** (1) **and** (2) **for** m **general.** Next, we claim that the work we have done so far implies the corresponding results on $\iota_{n,m,!}$ for m general.

First, we need:

Lemma 2.9.1. *Let $G_2 : \mathcal{C}_2 \to \mathcal{C}_1$ and $G_1 : \mathcal{C}_1 \to \mathcal{C}_0$ be functors such that G_2 admits a fully-faithful left adjoint F_2 and $G_1 \circ G_2$ admits a left adjoint Ξ. Then G_1 admits a left adjoint, which is computed as $G_2 \circ \Xi$.*

Proof. It suffices to show that Ψ maps $\mathcal{C}_0$ into the subcategory $F_2(\mathcal{C}_1) \subseteq \mathcal{C}_2$, which is equivalent to saying that:

$$F_2 G_2 \Xi \to \Xi$$

is an isomorphism. But note that we have a map:

$$\Xi \to F_2 G_2 \Xi$$

induced by adjunction from the unit map:

$$\mathrm{id}_{\mathcal{C}_0} \to G_1 G_2 F_2 G_2 \Xi = G_1 G_2 \Xi$$

where we are using $\mathrm{id}_{\mathcal{C}_1} = G_2 F_2$. It is straightforward to show that this map is inverse to the given one.[38]

---

[38]Here is a more conceptual argument that does not require checking anything, but which feels too abstract for such a simple claim. For notational reasons, suppose all finite colimits exist and are preserved by every functor in sight (otherwise, use opposite Yoneda categories instead of Pro-categories), and assume all categories are accessible (otherwise, play with universes). Then $\mathsf{Pro}(G_1) : \mathsf{Pro}(\mathcal{C}_1) \to \mathsf{Pro}(\mathcal{C}_0)$ admits a left adjoint $\widetilde{F}_1$, and we have $\widetilde{F}_1 = \mathsf{Pro}(G_2)\mathsf{Pro}(F_2)\widetilde{F}_1 = \mathsf{Pro}(G_2)\mathsf{Pro}(\Xi)$. The right hand side obviously maps $\mathcal{C}_0$ into $\mathcal{C}_1$, so we obtain $\widetilde{F}_1 = \mathsf{Pro}(G_2\Xi)$ as desired.

$\square$

Regarding the theorem, we may suppose $m > n$, so in particular, $m \neq 0$ and $\iota_{m,!}$ is fully-faithful. Then the lemma implies $\iota_{n,m,!} = \iota_m^! \circ \iota_{n,!}$. Moreover, if $n > 0$, then because $\iota_{m,!} \circ \iota_{n,m,!} = \iota_{n,!}$ and both $\iota_{m,!}$ and $\iota_{n,!}$ are fully-faithful, clearly $\iota_{n,m,!}$ is as well.

2.10. **Proof of Theorem 2.3.1** (3). It remains to show that for $n \leqslant m < \infty$, $\iota_{n,m,!}$ and $\iota_{n,m,*}$ differ by a cohomological shift. We may safely assume $n < m$, so that $m \neq 0$.

Let $\delta^\psi_{\mathring{I}_m \mathring{I}_n} \in D(\mathring{I}_m \mathring{I}_n)$ denote the pullback of the exponential D-module along the (well-defined) map $\psi : \mathring{I}_m \mathring{I}_n \to \mathbb{G}_a$. Note that the Kac-Moody extension canonically splits over $\mathring{I}_m \mathring{I}_n$, since the splittings over each of these subgroups coincides on their intersection; in particular, $\delta^\psi_{\mathring{I}_m \mathring{I}_n}$ can be considered as κ-twisted.

The main geometric result is:

Lemma 2.10.1. *The D-module $\delta^\psi_{\mathring{I}_m \mathring{I}_n}$ cleanly extends to $D_\kappa(G(K))$.*

Proof. Let $\overline{\mathring{I}_m \mathring{I}_n}$ denote the closure in $G(K)$, and let $j : \mathring{I}_m \mathring{I}_n \hookrightarrow \overline{\mathring{I}_m \mathring{I}_n}$ denote the open embedding. We want to show:

$$j_!(\delta^\psi_{\mathring{I}_m \mathring{I}_n}) \xrightarrow{\simeq} j_{*,dR}(\delta^\psi_{\mathring{I}_m \mathring{I}_n}). \tag{2.10.1}$$

Note that these κ-twisted D-modules are $(\mathring{I}_m, \psi)$ equivariant with respect to the left action of $G(K)$ on itself. Therefore, by Theorem 2.3.1 (2), it suffices to show that this map is an isomorphism after applying $\iota_{m,!}$, i.e., !-averaging to Whittaker with respect to the left action.

Note that $\mathring{I}_m \mathring{I}_n \subseteq N(K)\mathring{I}_n$, so the same is true of their closures. Therefore, the !-averages of both sides of (2.10.1) are supported on $\overline{N(K)\mathring{I}_n}$.

We claim that $\overline{N(K)\mathring{I}_n} \cap N(K)G(O)^n = N(K)\mathring{I}_n$. Indeed, it suffices to show that $N(K)\mathring{I}_n \subseteq N(K)G(O)^n$ is closed, and for this it suffices to show this mod $N(K)$, i.e., that:

$$\mathring{I}_n/N(O)^n \to G(O)^n/N(O)^n$$

is a closed embedding. The $n = 0$ case is obvious, and for $n > 0$, this is the embedding of an orbit for a prounipotent group on a quasi-affine scheme, so is a closed embedding.

Therefore, by Lemma 2.6.1, our !-averages are cleanly extended from $N(K)\mathring{I}_n$. Moreover, they are $(N(K), \psi)$ and $(\mathring{I}_n, -\psi)$-equivariant, so it suffices to show that our map induces an isomorphism between fibers at $1 \in G(K)$, which is clear.

$\square$

We now conclude the proof of Theorem 2.3.1. As in the proof of Theorem 2.3.1 (2), $\iota_{n,m,!}$ and $\iota_{n,m,*}$ are defined by kernels; in the notation above, the former is given by $j_!(\delta^\psi_{\mathring{I}_m \mathring{I}_n})$ and the latter by $j_{*,dR}(\delta^\psi_{\mathring{I}_m \mathring{I}_n})[-2(m-n)\Delta]$. Note that the the shift appears because in the finite-dimensional setting, $*$-averaging is given by convolution with the constant sheaf, not the dualizing sheaf.

3. Semi-classical counterpart

3.1. A semi-classical counterpart of Theorem 3.1.1.

For $V \subseteq \mathfrak{g}((t))$, let $V^\perp \subseteq \mathfrak{g}((t))^\vee = \mathfrak{g}((t))dt = \mathfrak{g}((t))$ denote the perpendicular subspace with respect to the residue pairing.[39]

The main result of this section is the following.

Theorem 3.1.1. *For every $0 \leqslant n \leqslant m \leqslant \infty$, the morphism:*

$$f + \operatorname{Lie} \mathring{I}_n^\perp \cap \operatorname{Lie} \mathring{I}_m^\perp / \mathring{I}_n \cap \mathring{I}_m \to f + \operatorname{Lie} \mathring{I}_m^\perp / \mathring{I}_m$$

is a finitely presented closed embedding.[40]

Remark 3.1.2. This result will be needed in §5.

Remark 3.1.3. We consider this as a semi-classical version of Theorem 2.3.1; let us explain why. The reader may safely skip this. It freely uses some ideas from §4 and Appendix A.

First, note that Theorem 2.3.1 (1) can be reformulated as saying that $\iota_{n,m,*}$ preserves compacts.

The category $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_n, \psi}$ has the Kazhdan-Kostant filtration with *semi-classical* category $\mathsf{QCoh}^{ren}(f + \operatorname{Lie} \mathring{I}_n^\perp / \mathring{I}_n)$ (see *loc. cit.* for the notation). The $*$-averaging functor $\iota_{n,m,*} : \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_n, \psi} \to \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_m, \psi}$ has associated semi-classical functor given — up to a *mild correction* — by pull-push along the correspondence:

$$f + \operatorname{Lie} \mathring{I}_n^\perp \cap \operatorname{Lie} \mathring{I}_m^\perp / \mathring{I}_n \cap \mathring{I}_m$$

$$f + \operatorname{Lie} \mathring{I}_n^\perp / \mathring{I}_n \qquad\qquad\qquad f + \operatorname{Lie} \mathring{I}_m^\perp / \mathring{I}_m.$$

The pullback along the left arrow obviously preserves compacts, whereas pushforward along the right arrow does because it is a (finitely-presented) regular embedding.

A word on the "mild correction:" running the calculation properly, one finds that the pullback should actually be a $*$-pullback followed by a !-pullback for the morphisms:

$$f + \operatorname{Lie} \mathring{I}_n^\perp \cap \operatorname{Lie} \mathring{I}_m^\perp / \mathring{I}_n \cap \mathring{I}_m \to f + \operatorname{Lie} \mathring{I}_n^\perp / \mathring{I}_n \cap \mathring{I}_m \to f + \operatorname{Lie} \mathring{I}_n^\perp / \mathring{I}_n.$$

But this does not affect the discussion above. (The reason this appears is that $\iota_{n,m,*}$ is the composition $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_n, \psi} \xrightarrow{\operatorname{Oblv}} \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_n \cap \mathring{I}_m, \psi} \xrightarrow{\operatorname{Av}_*^\psi} \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_n, \psi}$; the semi-classical version of a forgetful functor involves a $*$-pullback along a correspondence, while the semi-classical version of averaging involves a !-pullback along a regular embedding.)

The proof of the above result will occupy the remainder of this section.

3.2. Regular centralizers.

We need an auxiliary result of independent interest, which calculates in some explicit terms the semi-classical analogue of the category $\mathsf{Whit}^{\leqslant n} \widehat{\mathfrak{g}}_\kappa\text{–mod}$.

More precisely, we will calculate the quotient stack $f + \operatorname{Lie} \mathring{I}_n^\perp / \mathring{I}_n$ in more explicit terms.

Let $\mathcal{J}$ denote the group scheme of regular centralizers over $f + \mathfrak{b}^e \xrightarrow{\sim} f + \mathfrak{b}/N$. We recall that $\mathcal{J}$ is the fiber product of $f + \mathfrak{b}^e$ with itself over $\mathfrak{g}_{reg}/G$, that $\mathcal{J}$ is smooth, and that $\mathfrak{g}_{reg}/G$ is the classifying stack $\mathbb{B}\mathcal{J}$ of $\mathcal{J}$ (considered as a group scheme, so e.g. $\mathbb{B}\mathcal{J}$ maps smoothly to $f + \mathfrak{b}^e$).

---

[39]It would be better practice for a variety of reasons to include the symbol dt in what follows, but to simplify the notation we choose coordinates and omit this twist.

[40]For $m = \infty$ and from the DAG perspective, we should really say ind-finitely presented, as for $0 \hookrightarrow \operatorname{colim}_n \mathbb{A}^n$.

Let $\mathcal{J}(O)$ denote the corresponding pro-smooth group scheme over $f + \mathfrak{b}^e[[t]]$. Let $\mathcal{J}_n \subseteq \mathcal{J}(O)$ denote its nth congruence subgroup.

Finally, let $\mathcal{J}_n^n$ denote the group scheme over $f + t^{-n}\operatorname{Ad}_{-n\check{\rho}(t)}\mathfrak{b}^e[[t]]$ obtained from $\mathcal{J}_n$ by pullback along the isomorphism:

$$t^n\operatorname{Ad}_{n\check{\rho}(t)}(-) : f + t^{-n}\operatorname{Ad}_{-n\check{\rho}(t)}\mathfrak{b}^e[[t]] \xrightarrow{\simeq} f + \mathfrak{b}^e[[t]].$$

(Somewhat more naturally, $\mathcal{J}_n^n$ is the group scheme of centralizers lying in the nth congruence subgroup of the constant group scheme $G(O)^n$, so our notation here accords with that of §2.5.)

Lemma 3.2.1. *For $n > 0$, the natural map:*

$$\mathbb{B}\mathcal{J}_n^n \to f + \operatorname{Lie}\mathring{I}_n^{\circ\perp}/\mathring{I}_n$

is an isomorphism. (As above, $\mathbb{B}\mathcal{J}_n^n$ denotes the classifying stack of the group scheme $\mathcal{J}_n^n$ over $f + t^{-n}\operatorname{Ad}_{n\check{\rho}(t)}\mathfrak{b}^e[[t]].$)

Proof. Applying $\operatorname{Ad}_{n\check{\rho}(t)}$, it is equivalent to show that the map:

$$\mathbb{B}\mathcal{J}_n \to f + t^n\operatorname{Ad}_{n\check{\rho}(t)}(\operatorname{Lie}\mathring{I}_n^{\circ\perp})/\operatorname{Ad}_{n\check{\rho}(t)}\mathring{I}_n$$

is an isomorphism.

We have:

$$f + t^n\operatorname{Ad}_{n\check{\rho}(t)}(\operatorname{Lie}\mathring{I}_n^{\circ\perp})/\operatorname{Ad}_{n\check{\rho}(t)}\mathring{I}_n = (f + \mathfrak{b}[[t]] + t^n\mathfrak{g}[[t]])/\big(G(O) \underset{G(O/t^n)}{\times} N(O/t^n)\big).$$

Therefore, we may interpret the quotient stack appearing above as the stack of maps from the formal disc to $\mathfrak{g}_{reg}/G$ equipped with an order n lift to $f + \mathfrak{b}/N$. Then writing $\mathfrak{g}_{reg}/G = \mathbb{B}\mathcal{J}$ makes this assertion a tautology.

$\square$

3.3. **Proof of Theorem 3.1.1.** We now return to the result in question.

Proof of Theorem 3.1.1. It suffices to show that the diagram:

$$
\begin{array}{ccc}
f + \operatorname{Lie}\mathring{I}_n^{\circ\perp} \cap \operatorname{Lie}\mathring{I}_m^{\circ\perp}/\mathring{I}_n \cap \mathring{I}_m & \longrightarrow & f + \operatorname{Lie}\mathring{I}_m^{\circ\perp}/\mathring{I}_m \\
\downarrow & & \downarrow \\
f + t^{-n}\operatorname{Ad}_{-n\check{\rho}(t)}\mathfrak{b}^e[[t]] & \hookrightarrow & f + t^{-m}\operatorname{Ad}_{-m\check{\rho}(t)}\mathfrak{b}^e[[t]]
\end{array}
$$

is Cartesian. We can safely assume $m > n$, and in particular that $m \neq 0$.

We show the assertion in this form in what follows.

Step 1. Suppose $n > 0$ for the moment. Fix a point:[41]

$$\xi \in f + \operatorname{Lie}\mathring{I}_n^{\circ\perp}.$$

Let $Z(\xi) \subseteq G(K)$ denote the centralizer of ξ. We will show that any $z \in Z(\xi)$ lying in $N(K)\mathring{I}_n$ actually lies in $\mathcal{K}_n^n \subseteq \mathring{I}_n$ (reminding that $\mathcal{K}_n^n := \operatorname{Ad}_{-n\check{\rho}(t)}\mathcal{K}_n \subseteq G(O)^n$ is conjugated from the nth congruence subgroup of $G(O)$).

---

[41]For clarity: in this argument, we use the notation "$\in$" to refer to A-points of stacks for some implicit fixed commutative k-algebra A. Similarly, where we say $Z(\xi) \subseteq G(K)$ below, we really are working over $\operatorname{Spec}(A)$ and considering $Z(\xi) \subseteq G(K) \times \operatorname{Spec}(A)$.

Indeed, given an element z as above, we can write $z = gh$ for $g \in N(K)$ and $h \in \mathring{I}_n$. As $\xi \in f + \text{Lie}\,\mathring{I}_n^\perp$, the functional:

$$\text{Lie}\,\mathring{I}_n \hookrightarrow \mathfrak{g}((t)) \simeq \mathfrak{g}((t))^\vee \xrightarrow{\text{ev}_\xi} k$$

is given by the character $\psi_{\mathring{I}_n}$ (where ξ indicates evaluation on ξ). As $\text{Ad}_z(\xi) = \xi$, we have $\text{Ad}_{g^{-1}}(\xi) = \text{Ad}_h(\xi) \in f + \text{Lie}\,\mathring{I}_n^\perp$, so the composition:

$$\mathring{I}_n \cap \text{Ad}_g\,\text{Lie}\,\mathring{I}_n \hookrightarrow \mathfrak{g}((t)) \simeq \mathfrak{g}((t))^\vee \xrightarrow{\text{ev}_{\text{Ad}_{g^{-1}}\xi}} k$$

is also calculated by the restriction of $\psi_{\mathring{I}_n}$ to $\text{Ad}_g\,\text{Lie}\,\mathring{I}_n \cap \mathring{I}_n$.

Therefore, by Lemma 2.7.1, $g \in N(O)^n = N(K) \cap \mathring{I}_n$. Therefore, $z = gh \in \mathring{I}_n$.

It remains to show $z \in \mathcal{K}_n^n \subseteq \mathring{I}_n$. Applying the automorphism $t^n \text{Ad}_{n\check{\rho}(t)}(-)$, we can equivalently show that for:

$$\xi \in f + \mathfrak{b}[[t]] + t^n \mathfrak{g}[[t]]$$

any $z \in N(O/t^n) \times_{G(O/t^n)} G(O)$ centralizing ξ lies in $\mathcal{K}_n$. Reducing modulo t^n, we need to show that for any:

$$\xi \in f + \mathfrak{b}[[t]]/t^n \subseteq \mathfrak{g}[[t]]/t^n$$

any element $z \in N(O/t^n)$ centralizing ξ is the identity.

For $n = 1$, this is the well-known fact that the centralizer of $\xi \in f + \mathfrak{b}$ intersects N only at the identity.[42] For higher n, it follows from the fact that that intersection is moreover transverse (by passing to n-jets).

Step 2. Now fix:

$$\xi \in f + \text{Lie}\,\mathring{I}_n^\perp \cap \text{Lie}\,\mathring{I}_m^\perp.$$

We claim that there exists $g \in \mathring{I}_n \cap \mathring{I}_m$ conjugating ξ into an element $f + t^{-n}\text{Ad}_{-n\check{\rho}(t)}\mathfrak{b}^e[[t]]$ (necessarily corresponding to the characteristic polynomial of ξ via the Kostant section).

First, assume $n \neq 0$. By Lemma 3.2.1, there exists an element $h \in \mathring{I}_n$ (resp. $g \in \mathring{I}_m$) with $\text{Ad}_h(\xi) \in f + t^{-n}\text{Ad}_{n\check{\rho}(t)}\mathfrak{b}^e[[t]]$ (resp. $\text{Ad}_g(\xi) \in f + t^{-n}\text{Ad}_{n\check{\rho}(t)}\mathfrak{b}^e[[t]]$). Considering characteristic polynomials, we must have:

$$\text{Ad}_h(\xi) = \text{Ad}_g(\xi).$$

Therefore, $g^{-1}h \in Z(\xi)$. Clearly $g^{-1}h \in \mathring{I}_m\mathring{I}_n \subseteq N(K)\mathring{I}_n$. By Step 1, we have $g^{-1}h \in \mathcal{K}_n^n \subseteq \mathring{I}_n$, so $g \in \mathring{I}_n$. As $g \in \mathring{I}_m$ by assumption, this yields the claim.

---

[42]In more detail:

First, one shows that the Springer fiber $\text{Spr}^\xi \subseteq G/B$ is contained in the open cell $Bw_0B/B \subseteq G/B$: under the $\mathbb{G}_m$ action on G/B defined by $\check{\rho}$, any field-valued point of Spr^ξ limits to a field-valued point of Spr^f, so to w_0; this is equivalent to the assertion by the Bruhat decomposition.

Next, observe that the natural action of the centralizer $Z(\xi)$ of ξ on Spr^ξ is trivial. Indeed, by regularity of ξ, $Z(\xi)$ is generated by its identity component and $Z(G)$; as Spr^ξ is finite (by regularity again), the former action is trivial, while the action of $Z(G)$ is obviously trivial.

Therefore, any $z \in Z(\xi)$ is contained in any Borel containing ξ, and any such is transverse to B. Therefore, $Z(\xi) \cap N = \{1\}$.

The transversality asserted below amounts to the (easier) infinitesimal version of this same assertion.

The case $n = 0$ is similar. The only difference is that we should take h lying in the radical $\mathring{I} = N \times_G G(O)$ of the Iwahori; this may be done as $\xi \in f + \operatorname{Lie} \mathring{I}_m^{\circ\perp} \cap \mathfrak{g}[[t]]$ for $m > 0$ implies $\xi \in f + \operatorname{Lie} I \subseteq \mathfrak{g}[[t]]$. The $n = 1$ version of the assertion of Step 1 (suitably conjugated by $\check{\rho}(t)$) yields that $g^{-1}h \in N(K)\mathring{I} \cap Z(\xi) \subseteq \mathcal{K}_1 \subseteq \mathring{I}$. Therefore, $g \in \mathring{I}$, and therefore lies in $\mathring{I} \cap \mathring{I}_m \subseteq G(O) \cap \mathring{I}_m$ as desired.

Step 3. It follows from Step 2 and Kostant theory that:

$$f + \operatorname{Lie} \mathring{I}_n^{\circ\perp} \cap \operatorname{Lie} \mathring{I}_m^{\circ\perp}/\mathring{I}_n \cap \mathring{I}_m$$

is the classifying stack over $f + t^{-n} \operatorname{Ad}_{-n\check{\rho}(t)} \mathfrak{b}^e[[t]]$ for the group scheme of loops into $\mathcal{J}$ that lie in $\mathring{I}_n \cap \mathring{I}_m \subseteq G(K)$. By Step 1, it is the same as to take $\mathring{I}_n \cap \mathcal{K}_m^m$ in place of $\mathring{I}_n \cap \mathring{I}_m$.

Applying Lemma 3.2.1, we are reduced to showing that if $\gamma \in \mathcal{K}_m^m \subseteq \mathring{I}_m$ (for $\mathcal{K}_m \subseteq G(O)$ the mth congruence subgroup) stabilizes $\xi \in f + t^{-n} \operatorname{Ad}_{-n\check{\rho}(t)} \mathfrak{b}^e[[t]]$, then $\gamma \in \mathring{I}_n$.

As before, let us first assume $n \neq 0$. Let $\gamma = \gamma^+ \cdot \gamma^-$ for $\gamma^+ \in N(K)$ and $\gamma^- \in B^-(K)$. Then we have:

$$\xi = \operatorname{Ad}_\gamma(\xi) = \operatorname{Ad}_{\gamma^+} \operatorname{Ad}_{\gamma^-}(\xi).$$

Note that $\gamma^- \in \mathring{I}_m \cap B^-(K) \subseteq \mathring{I}_n$, so:

$$\operatorname{Ad}_{\gamma^-}(\xi) \in f + \operatorname{Lie} \mathring{I}_n^{\circ\perp}.$$

By Lemma 3.2.1, there is an element $g \in \mathring{I}_n$ with:

$$\operatorname{Ad}_g \operatorname{Ad}_{\gamma^-}(\xi) = \xi.$$

It follows that $\gamma^+ g^{-1} \in Z(\xi)$ and in $N(K)\mathring{I}_n$, so $\gamma^+ \in \mathring{I}_n$ by Step 1, so $\gamma \in \mathring{I}_n$ as desired.

The $n = 0$ case is treated using the same modification from Step 2: replace $\mathring{I}_n = G(O)$ above by $\mathring{I}$.

$\square$

4. Drinfeld-Sokolov realization of the generalized vacuum representations

4.1. **A question.**

4.1.1. We begin this section with a basic question about $\mathcal{W}$-algebras: what are its *generalized vacuum* modules? By this, at first pass, we mean that we expect a projective system of modules $\mathcal{W}_\kappa^n \in \mathcal{W}_\kappa\text{–mod}^\heartsuit$ ($n \geq 0$) playing a similar role to the modules $\operatorname{ind}_{t^n\mathfrak{g}[[t]]}^{\widehat{\mathfrak{g}}_\kappa}(k)$ for the Kac-Moody algebra.

In more detail, we want that:

- $\mathcal{W}_\kappa^0$ is the vacuum representation, i.e., $\mathcal{W}_\kappa^0 = \mathcal{W}_\kappa \in \mathcal{W}_\kappa\text{–mod}^\heartsuit$.
- The inverse limit of the $\mathcal{W}_\kappa^n$ is the topological chiral[43] algebra $\mathcal{W}_\kappa^{as}$ associated with the vertex algebra $\mathcal{W}_\kappa$ (cf. [BD2] §3.6.2).

---

[43]This funny name is taken from [Bei], who asks that it be used. It means an associative algebra with respect to the $\overset{\to}{\otimes}$-monoidal product on $\mathsf{Pro}(\mathsf{Vect}^\heartsuit)$ from *loc. cit.* Note that in [BD2], such a thing is called a *topological associative algebra.*

- For $\mathfrak{g} = \mathfrak{sl}_2$, so that $\mathcal{W}_\kappa$ is a Virasoro algebra, the modules $\mathcal{W}_\kappa^n$ should be induced from the subalgebra $t^{2n} \operatorname{Der}(\mathcal{D})$.[44,45]
- The module $\mathcal{W}_\kappa^n$ should have a canonical filtration[46] $F_\bullet \mathcal{W}_\kappa^n$ compatible with the filtration on $\mathcal{W}_\kappa$ and with $F_{-1}\mathcal{W}_\kappa^n = 0$. Recalling that the associated graded of $\mathcal{W}_\kappa^{as}$ is[47] the algebra of functions on the indscheme $f + \mathfrak{b}((t))/N(K)$, the associated graded of $\mathcal{W}_\kappa^n$ should be identified with the structure sheaf of the closed subscheme:[48]

$$t^{-n} \operatorname{Ad}_{-n\check\rho(t)} \big(f + \mathfrak{b}[[t]]/N(O) \big) \subseteq f + \mathfrak{b}((t))/N(K).$$

The morphism $\mathcal{W}_\kappa^{n+1} \to \mathcal{W}_\kappa^n$ should be strictly compatible with filtrations and should induce the restriction of functions map when we identify the associated graded as above.
- The $\mathcal{W}_\kappa^n$ should form a flat family of modules as we vary κ, and moreover, this family should extend to allow $\kappa \to \infty$ for κ non-degenerate. In this case, recall that $\mathcal{W}_\infty^{as}$ is the algebra of functions on the indscheme $\operatorname{Op}_G(\overset{\circ}{\mathcal{D}})$ of opers, i.e., $(f + \mathfrak{b}((t)))dt/N(K)$ where $N(K)$ acts by the *gauge* action (not the adjoint action). Then $\mathcal{W}_\infty^n$ should be the structure sheaf of the subscheme $\operatorname{Op}_G^{\leqslant n}$ of *opers with singularities of order $\leqslant n$* as defined in [BD1] §3.8 (see also [FG1] §2). We remind that this subscheme is:[49]

$$\operatorname{Op}_G^{\leqslant n} := \big(f + \operatorname{Ad}_{-n\check\rho(t)}(\mathfrak{b}[[t]]) \big)dt / \operatorname{Ad}_{-n\check\rho(t)} N(O)$$

with $\operatorname{Ad}_{-n\check\rho(t)} N(O)$ acting through the gauge action.[50]

The purpose of this section and the next is to construct such modules, which seem not to exist elsewhere in the literature.

4.1.2. In fact, we will give two constructions of these modules.

We will present the two constructions somewhat out of order: first, we give a construction via Drinfeld-Sokolov reduction, and then in §6 give a (perhaps) more elementary construction via the *free-field realization* of the $\mathcal{W}$-algebra. The reason is that the former is the one relevant for proving the affine Skryabin theorem. The latter is included in this paper for the sake of completeness, and because it plays a technical role in deducing the categorical Feigin-Frenkel theorem from the affine Skryabin theorem.

---

[44]The 2 here is needed for compatibility with the next expectation. At the critical level, this might be compared to the fact that the Sugawara element corresponding to the derivation $t^n \partial_t$ ($n \geqslant 0$) acts by zero on the module $\operatorname{ind}_{t^{\lfloor \frac{n}{2} \rfloor} \mathfrak{g}[[t]]}^{\hat{\mathfrak{g}}_{crit}}(k)$, cf. [BD1] Theorem 3.7.9.

[45]The other expectations in this list will be verified in this section, but not this one. For this comparison, see Example 6.1.3.

[46]We maintain the conventions of Appendix A, so filtrations are assumed exhaustive. Moreover, since we are speaking about objects of an abelian category, we are tacitly assuming here that each $F_i\mathcal{W}_\kappa^n \to \mathcal{W}_\kappa^n$ is injective.

[47]This is not quite canonical: we are using our choice of dt and some choice of non-degenerate Ad-invariant bilinear form on $\mathfrak{g}$ to make $\mathfrak{g}((t))$ self-dual. More canonical would be to take $\mu^{-1}(\psi)/N(K)$ for $\mu : \mathfrak{g}((t))^\vee \to \mathfrak{n}((t))^\vee$ the canonical map.

[48]Recall that $f + \mathfrak{b}((t))/N(K) = f + \mathfrak{b}^e((t))$ (for e fitting into the $\mathfrak{sl}_2$-triple $(e, 2\operatorname{Lie}(\check\rho)(1), f)$) with $f + \mathfrak{b}[[t]]/N(O) = f + \mathfrak{b}^e[[t]]$. So for $n = 0$, we really do get a closed subscheme, and for general n this follows because $t^{-n}\operatorname{Ad}_{-n\check\rho(t)}$ is an automorphism of our indscheme.

[49]One uses $\check\rho(t)$ to define the structure maps as we vary n.

[50]So from an *opers-centric* worldview, the modules $\mathcal{W}_\kappa^n$ are quantizations of opers with singularity $\leqslant n$. The existence of such quantizations implies that the subschemes $\operatorname{Op}_G^{\leqslant n} \subseteq \operatorname{Op}_G(\overset{\circ}{\mathcal{D}})$ are coisotropic with respect to the canonical Poisson structure on $\operatorname{Op}_G(\overset{\circ}{\mathcal{D}})$. This is straightforward to see from the given description of $\operatorname{Op}^{\leqslant n}$, cf. [FG1] Lemma 4.4.1 and various points in the discussion of [BD1] §3.6-3.8.

4.1.3. *Adolescent Whittaker construction.* The main idea of this section is to take:

$$\mathcal{W}_\kappa^n = \Psi(\mathrm{ind}_{\mathrm{Lie}\, \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} \psi_{\mathring{I}_n})$$

up to a cohomological shift, and show that this construction satisfies our expectations.

Remark 4.1.1. To make some of our treatment more elementary, this result is split across Theorem 4.2.1 and Theorem 4.5.1; the former does not explicitly mention W-algebras, while the latter does.

Remark 4.1.2. This construction is motivated by the following considerations.
    Note that:

$$\mathrm{ind}_{\mathrm{Lie}\, \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi) \in \widehat{\mathfrak{g}}_\kappa\text{--mod}^{\mathring{I}_n,\psi} = \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{--mod})$$

is compact, and generates this category whenever $n > 0$. Therefore, as we vary n, the objects $\iota_{n,!}(\mathrm{ind}_{\mathrm{Lie}\, \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) \in \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{--mod})$ give compact generators.
    Anticipating the affine Skryabin theorem $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{--mod}) \simeq \mathcal{W}_\kappa\text{--mod}$ and its compatibility with the functor Ψ, we are computing the images of our compact generators as modules over the W-algebra.

Remark 4.1.3. For $n = 0$, this theorem says that $\Psi(\mathbb{V}_\kappa) = \mathcal{W}_\kappa$, which is the definition of the right hand side. The argument given below recovers the fundamental results on $\mathcal{W}_\kappa$: that it is in cohomological degree 0, and that it is a filtered vertex algebra[51] with commutative associated graded the algebra of functions on $f + \mathfrak{b}[[t]]/N(O)$.
    The argument is somewhat more conceptual than appears elsewhere (cf. [dBT] and [FBZ] Chapter 15). It substantially overlaps with the presentation given in the recent survey [Ara3], but the issue of the convergence of the spectral sequence is dealt with by a different argument. The key advantage of the method below is that it avoids the *tensor product decomposition* appearing in other treatments (even in [FG5]), which is always justified by explicit formulae I have not been able to understand conceptually.[52]

4.2. **Vanishing of cohomology.**

4.2.1. *Drinfeld-Sokolov reduction.* Recall that there is a continuous functor:[53]

$$\Psi : \widehat{\mathfrak{g}}_\kappa\text{--mod} \to \mathsf{Vect}$$

defined as the composition:

$$\widehat{\mathfrak{g}}_\kappa\text{--mod} \overset{\mathrm{Oblv}}{\to} \mathfrak{n}((t))\text{--mod} \xrightarrow{C^{\frac{\infty}{2}}(\mathfrak{n}((t)),\mathfrak{n}[[t]],(-)\otimes-\psi)} \mathsf{Vect}$$

where $-\psi$ is the 1-dimensional $\mathfrak{n}((t))$-module corresponding to the character $-\psi$, and $C^{\frac{\infty}{2}}$ is the semi-infinite cohomology functor as defined in §A.8.4. We remind that $\mathfrak{n}[[t]]$ appears in the notation, but plays a very mild role.

---

[51]One should use factorization techniques for this, cf. [BD2] §3.8.

[52]Though it may well be that a conceptual explanation of this method exists and I just could not find it. Or perhaps the argument we give here for the convergence, which has some remarkable similarities with the tensor product argument, is that conceptual explanation.

[53]We remind at this point that e.g. Vect denotes the *DG category* of chain complexes of k-vector spaces, and that *canonical* means defined up to canonical quasi-isomorphism (in the ∞-categorical sense).

4.2.2. The first main result of this section is the following.

Theorem 4.2.1. *For every $n \geqslant 0$, the complex $\Psi(\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \mathring{I}_n} \psi_{\mathring{I}_n})$ is concentrated in cohomological degree $-n\Delta$. (Here Δ is used as in §2.3, i.e., $\Delta = 2(\rho, \check{\rho})$.)*

The argument will be given in §4.2.5 below.

Remark 4.2.2. The main issue in what follows is the convergence of a certain spectral sequence. The approach given below seems to be the most versatile one for $n = 0$. However, for $n > 0$, the method of §B.3 can also be adapted to this purpose and is much more flexible.

4.2.3. We will prove the above by an argument about passage to the associated graded, using the methods of Appendix A.

First, note that $G(K)$ acts on $\widehat{\mathfrak{g}}_\kappa$, so we obtain an action of $\mathbb{G}_m$ on the Kac-Moody algebra via:

$$-\check{\rho} : \mathbb{G}_m \to G \subseteq G(O) \subseteq G(K).$$

This action preserves $\mathring{I}_n$ for all $n \geqslant 0$. Moreover, the character $\psi : \mathrm{Lie}\, \mathring{I}_n \to k$ is $\mathbb{G}_m$-equivariant if we give the target k the degree -1 grading.

Therefore, Appendix A produces *PBW* and *Kazhdan-Kostant* (KK) filtrations on the categories $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_n,\psi}$ and $\mathfrak{n}((t))\text{–mod}^{\mathrm{Ad}_{-n\check{\rho}(t)}\, N(O),\psi}$. (We remind that $\mathrm{Ad}_{-n\check{\rho}(t)}\, N(O) = N(K) \cap \mathring{I}_n$.)

4.2.4. We recall the basic facts about these filtrations that we will need. We will need some of the language and notation from Appendix A: see especially the material about filtrations in §A.2 and renormalization in §A.7.2.

Throughout, we fix an Ad-invariant identification $\mathfrak{g} \simeq \mathfrak{g}^\vee$ and a 1-form dt to obtain $\mathfrak{g}((t))^\vee \simeq \mathfrak{g}((t))$; note that this induces:

$$\mathfrak{n}((t))^\vee \simeq \mathfrak{g}((t))/\mathfrak{b}((t)).$$

As in §3.1, for $V \subseteq \mathfrak{g}((t))$, we let $V^\perp$ denote $(\mathfrak{g}((t))/V)^\vee \subseteq \mathfrak{g}((t))^\vee = \mathfrak{g}((t))$; we recall from *loc. cit.* that:

$$\mathrm{Lie}\, \mathring{I}_n^\perp = \mathrm{Ad}_{-n\check{\rho}(t)}(\mathfrak{g}[[t]] + t^{-n}\mathfrak{b}[[t]]).$$

- We have *semi-classical* categories:

$$\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_n,\psi,PBW-cl} = \mathsf{QCoh}^{ren}(\mathrm{Lie}\, \mathring{I}_n^\perp/\mathring{I}_n)$$

$$\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_n,\psi,KK-cl} = \mathsf{QCoh}^{ren}(f + \mathrm{Lie}\, \mathring{I}_n^\perp/\mathring{I}_n)$$

for $f \in \mathfrak{n}^-((t))$ our principal nilpotent corresponding to ψ.

For the $\mathfrak{n}((t))$-categories, observe that:[54]

$$\mathfrak{n}((t))\text{–mod}^{N(K)\cap\mathring{I}_n,\psi,PBW-cl} = \mathsf{QCoh}^{ren}\left(\mathrm{Ad}_{-n\check{\rho}(t)}(\mathfrak{g}[[t]]/\mathfrak{b}[[t]])/\mathrm{Ad}_{-n\check{\rho}(t)}\, N(O)\right)$$

$$\mathfrak{n}((t))\text{–mod}^{N(K)\cap\mathring{I}_n,\psi,KK-cl} = \mathsf{QCoh}^{ren}\left(f + \mathrm{Ad}_{-n\check{\rho}(t)}(\mathfrak{g}[[t]]/\mathfrak{b}[[t]])/\mathrm{Ad}_{-n\check{\rho}(t)}\, N(O)\right)$$

where these isomorphisms arise from the identification:

---

[54]At various points in our discussion, the abundance of symbols $\mathrm{Ad}_{-n\check{\rho}(t)}$ means something is canonically isomorphic to the same expression with all such symbols removed. We still retain the notation since it is an important bookkeeping device, especially as we vary n.

$$\text{Lie } \mathring{I}_n^{\perp}/\big(\text{Lie } \mathring{I}_n^{\perp} \cap \mathfrak{b}((t))\big) = \text{Ad}_{-n\check\rho(t)}\big((\mathfrak{g}[[t]] + t^{-n}\mathfrak{b}[[t]])/t^{-n}\mathfrak{b}[[t]]\big) = \text{Ad}_{-n\check\rho(t)}(\mathfrak{g}[[t]]/\mathfrak{b}[[t]])$$

- The forgetful functor:

$$\widehat{\mathfrak{g}}_{\kappa}\text{--mod}^{\mathring{I}_n,\psi} \to \mathfrak{n}((t))\text{--mod}^{\text{Ad}_{-n\check\rho(t)} N(O),\psi}$$

is filtered for either filtration. Its underlying PBW semi-classical functor:

$$\mathsf{QCoh}^{ren}(\text{Lie } \mathring{I}_n^{\perp}/\mathring{I}_n) \to \mathsf{QCoh}^{ren}\Big(\text{Ad}_{-n\check\rho(t)}(\mathfrak{g}[[t]]/\mathfrak{b}[[t]])/\text{Ad}_{-n\check\rho(t)} N(O)\Big)$$

is given by pullback/pushforward along the obvious structure maps from:

$$\text{Lie } \mathring{I}_n^{\perp}/\text{Ad}_{-n\check\rho(t)} N(O).$$

- The functor:

$$C^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \text{Ad}_{-n\check\rho(t)} N(O), (-)\otimes-\psi\big) : \mathfrak{n}((t))\text{--mod}^{\text{Ad}_{-n\check\rho(t)} N(O),\psi} \to \mathsf{Vect} \qquad (4.2.1)$$

is filtered for each of these filtrations. Here we remind the reader of Notation A.8.5, which says:

$$C^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \text{Ad}_{-n\check\rho(t)} N(O), -\big) = C^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \text{Ad}_{-n\check\rho(t)} \mathfrak{n}[[t]], \text{Ad}_{-n\check\rho(t)} N(O), -\big) =$$
$$C^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \text{Ad}_{-n\check\rho(t)} \mathfrak{n}[[t]], -\big)$$

where the first equality is a definition and the second equality is only an equality of functors, not of filtered functors (the fact that it is an equality of functors is a consequence of the prounipotence of our group scheme $\text{Ad}_{-n\check\rho(t)} N(O)$).

Its underlying PBW (resp. KK) semi-classical functors are given by $*$-restriction to:

$$0/\text{Ad}_{-n\check\rho(t)} N(O) \subseteq \text{Ad}_{-n\check\rho(t)}(\mathfrak{g}[[t]]/\mathfrak{b}[[t]])/\text{Ad}_{-n\check\rho(t)} N(O)$$

(resp. $f/\text{Ad}_{-n\check\rho(t)} N(O) \subseteq f + \text{Ad}_{-n\check\rho(t)}(\mathfrak{g}[[t]]/\mathfrak{b}[[t]])/\text{Ad}_{-n\check\rho(t)} N(O)$) followed by global sections, i.e., group cohomology with respect to $\text{Ad}_{-n\check\rho(t)} N(O)$.

- These two filtrations fit into a *bifiltration* (cf. §A.5.4).

In the language of §A.5.4, the KK filtration on the PBW-semi-classical categories is induced via Example A.2.6 from the action:[55]

$$\lambda \cdot x = \lambda^{-1} \cdot \text{Ad}_{\check\rho(\lambda^{-1})}(x)$$

of $\mathbb{G}_m$ on $\text{Lie } \mathring{I}_n^{\perp}/\mathring{I}_n$ and $\text{Ad}_{-n\check\rho(t)}(\mathfrak{g}[[t]]/\mathfrak{b}[[t]])/\text{Ad}_{-n\check\rho(t)} N(O)$ respectively (cf. Remark A.5.12).

---

[55]Note that in Appendix A, we use the sometimes confusing convention that the action of $\mathbb{G}_m$ on a scheme is *expanding* if functions are non-negatively graded. The reason is that if any group acts on a scheme, there is an inverse sign for the induced action on functions. But e.g., the grading on $\text{Sym}(\mathfrak{g}) = \text{gr}_\bullet U(\mathfrak{g})$ corresponds to the action of $\mathbb{G}_m$ by inverse homotheties on $\mathfrak{g}^{\vee}$.

4.2.5.   With these preliminaries aside, we are prepared to prove the above result.

First, we introduce the following notation, which significantly reduces the burden in what follows.

Notation 4.2.3. For all $n \geqslant 0$, let ℓ^n denote the line $\det(\mathrm{Ad}_{-n\check{\rho}(t)} \mathfrak{n}[[t]]/\mathfrak{n}[[t]])$. For $n \leqslant m$, we let:

$$\ell^{n,m} := \det(\mathrm{Ad}_{-m\check{\rho}(t)} \mathfrak{n}[[t]]/ \mathrm{Ad}_{-n\check{\rho}(t)} \mathfrak{n}[[t]]) = \ell^m \otimes \ell^{n,\vee}.$$

Proof of Theorem 4.2.1. Here is the strategy: we will construct a Kazhdan-Kostant filtration on $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi)$ to obtain one on $\Psi(\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi))$, and it will be easy to show that the associated graded for the latter is in one cohomological degree.

We would be done, but that the Kazhdan-Kostant filtration on $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi)$ *is not bounded below* (and not even complete). We will nevertheless show that the induced filtration *on its Drinfeld-Sokolov* reduction is bounded below. This will be achieved by a comparison with its PBW filtration, which is bounded below (although its associated graded is not in a single cohomological degree).

Step 1. Observe that:

$$\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi) \in \widehat{\mathfrak{g}}_\kappa\mathrm{-mod}^{\overset{\circ}{I}_n,\psi}$$

has a canonical KK filtration, denoted $F^{KK}_\bullet \mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi)$. Indeed, this follows by functoriality from Example A.5.13. This filtration fits into a bifiltration with the PBW filtration $F^{PBW}_\bullet \mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi)$.

The induced KK filtration on the vector space:

$$\mathrm{gr}^{PBW}_\bullet \mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi) = \mathrm{Sym}^\bullet \mathfrak{g}((t))/\mathrm{Lie}\, \overset{\circ}{I}_n$$

is easy to understand concretely – the KK filtration is:

$$F^{KK}_i \mathrm{gr}^{PBW}_\bullet \mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi) = \underset{i' \leqslant i}{\oplus} \underset{j \in \mathbb{Z}}{\oplus} \left(\mathrm{Sym}^{i'-j} \mathfrak{g}((t))/\mathrm{Lie}\, \overset{\circ}{I}_n \right)^j$$

where the superscript j indicates the $-\check{\rho}$-grading (note the sign!). (For example, for a positive root α, $t^m e_\alpha$ has grading $-(\check{\rho},\alpha)$, so the corresponding element lies in $F^{KK}_{1-(\check{\rho},\alpha)}$ but, if non-zero, this element will not lie in $F^{KK}_{-(\check{\rho},\alpha)}$.)

Step 2. Note that we can compute $\Psi(\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi))$ as the *Harish-Chandra* version of semi-infinite cohomology (cf. §A.8.5):

$$\Psi(\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi)) = C^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathrm{Ad}_{-n\check{\rho}(t)} N(O), \mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi) \otimes -\psi)$$

by prounipotence of $\mathrm{Ad}_{-n\check{\rho}(t)} N(O)$.

Since the functors:

$$\widehat{\mathfrak{g}}_\kappa\mathrm{-mod}^{\overset{\circ}{I}_n,\psi} \xrightarrow{\mathrm{Oblv}} \mathfrak{n}((t))\mathrm{-mod}^{\mathrm{Ad}_{-n\check{\rho}(t)} N(O),\psi} \xrightarrow{C^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathrm{Ad}_{-n\check{\rho}(t)} N(O), -\otimes -\psi)} \mathsf{Vect}$$

are bifiltered, we obtain a bifiltration on $\Psi(\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi)) \in \mathsf{Vect}$. In particular, we obtain the filtrations $F^{PBW}_\bullet \Psi(\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi))$ and $F^{KK}_\bullet \Psi(\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\, \overset{\circ}{I}_n} (\psi))$. We will now compute the associated graded complexes.

First, it is convenient to slightly modify the functor Ψ. We define:

$$\Psi_n(-) = C^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathrm{Ad}_{-n\check{\rho}(t)}\,\mathfrak{n}[[t]], -) = \Psi(-) \otimes \ell^{n,\vee}[-n\Delta] \qquad (4.2.2)$$

(cf. (4.2.1)). Obviously it differs from Ψ only by a cohomological shift and tensoring with a 1-dimensional vector space. (At the level of filtered functors, there is also a shift in the indexing by $n\Delta$.)

Then we claim that:

$$\mathrm{gr}_\bullet^{PBW} \Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) = C^\bullet\big(\mathrm{Ad}_{-n\check{\rho}(t)} N(O), \mathrm{Fun}(\mathrm{Ad}_{-n\check{\rho}(t)}\,t^{-n}\mathfrak{b}[[t]])\big) \qquad (4.2.3)$$

i.e., (derived) global sections of the structure sheaf of the stack $\mathrm{Ad}_{-n\check{\rho}(t)}\,t^{-n}\mathfrak{b}[[t]]/\mathrm{Ad}_{-n\check{\rho}(t)} N(O)$.

This is a straightforward verification using §4.2.4. Writing $\Psi_n|_{\widehat{\mathfrak{g}}_\kappa-\mathsf{mod}^{\mathring{I}_n,\psi}}$ as the appropriate composition:

$$\widehat{\mathfrak{g}}_\kappa-\mathsf{mod}^{\mathring{I}_n,\psi} \xrightarrow{\mathrm{Oblv}} \mathfrak{n}((t))-\mathsf{mod}^{\mathrm{Ad}_{-n\check{\rho}(t)} N(O),\psi} \to \mathsf{Vect}$$

we find that its semi-classical functor:

$$\mathsf{QCoh}^{ren}(\mathrm{Lie}\,\mathring{I}_n^\perp/\mathring{I}_n) \to \mathsf{Vect}$$

is given by successive pullbacks and (renormalized) pushforwards along the diagram:

$$\mathrm{Lie}\,\mathring{I}_n^\perp/\mathrm{Ad}_{-n\check{\rho}(t)} N(O) \qquad\qquad\qquad 0/\mathrm{Ad}_{-n\check{\rho}(t)} N(O)$$

$$\mathrm{Lie}\,\mathring{I}_n^\perp/\mathring{I}_n \qquad\quad \mathrm{Ad}_{-n\check{\rho}(t)}(\mathfrak{g}[[t]]/\mathfrak{b}[[t]])/\mathrm{Ad}_{-n\check{\rho}(t)} N(O) \qquad\quad \mathrm{Spec}(k).$$

We recall that $\mathrm{Lie}\,\mathring{I}_n^\perp = \mathrm{Ad}_{-n\check{\rho}(t)}(\mathfrak{g}[[t]] + t^{-n}\mathfrak{b}[[t]])$. Therefore, since $\mathrm{gr}_\bullet^{PBW}\,\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)$ is the structure sheaf of $\mathrm{Lie}\,\mathring{I}_n^\perp/\mathrm{Ad}_{-n\check{\rho}(t)}$, we obtain the claim.

The same analysis applies in the KK setting, and we obtain:

$$\mathrm{gr}_\bullet^{KK} \Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) = C^\bullet\big(\mathrm{Ad}_{-n\check{\rho}(t)} N(O), \mathrm{Fun}(f + \mathrm{Ad}_{-n\check{\rho}(t)}\,t^{-n}\mathfrak{b}[[t]])\big)$$

The major difference is that $\mathrm{Ad}_{-n\check{\rho}(t)} N(O)$ acts *freely* on this locus and the quotient is an affine scheme. Indeed, by Kostant theory we have:

$$f + \mathrm{Ad}_{-n\check{\rho}(t)}\,t^{-n}\mathfrak{b}[[t]]/\mathrm{Ad}_{-n\check{\rho}(t)} N(O) = t^{-n}\,\mathrm{Ad}_{-n\check{\rho}(t)}(f + \mathfrak{b}[[t]])/\mathrm{Ad}_{-n\check{\rho}(t)} N(O) =$$
$$t^{-n}\,\mathrm{Ad}_{-n\check{\rho}(t)}(f + \mathfrak{b}^e[[t]])$$

for e fitting into our principal $\mathfrak{sl}_2$ as usual.

In particular, we find that $\mathrm{gr}_\bullet^{KK} \Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) \in \mathsf{Vect}^\heartsuit$. Since Ψ_n differs from Ψ by the cohomological shift by $n\Delta$, we find that $\mathrm{gr}_\bullet^{KK} \Psi(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi))$ is concentrated in cohomological degree $-n\Delta$, as expected.

Step 3. As indicated in the preamble, it remains to show that $F_i^{KK}\Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) = 0$ for all $i < 0$.[56]

---

[56]Note that due to the filtering conventions, this means that F_i^{KK} of Ψ of this module vanishes for $i < n\Delta$.

As a first step toward this, we first observe that $F_i^{PBW}\Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) = 0$ for $i < 0$. Indeed, as in Remark A.8.4, we have:

$$F_i^{PBW}\Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) =$$

$$\operatorname*{colim}_{m \geqslant n} F_{i-(m-n)\Delta}^{PBW} C^\bullet(\mathrm{Ad}_{-m\check{\rho}(t)}\,\mathfrak{n}[[t]], \mathrm{Ad}_{-n\check{\rho}(t)}\,N(O), \mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) \otimes \ell^{n,m}[(m-n)\Delta].$$

Then since the PBW filtration on $\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)$ vanishes in negative degrees, the PBW filtration on:

$$C^\bullet(\mathrm{Ad}_{-m\check{\rho}(t)}\,\mathfrak{n}[[t]], \mathrm{Ad}_{-n\check{\rho}(t)}\,N(O), \mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi))$$

vanishes in degrees $< -(m-n)\Delta$ (cf. Remark A.3.16, i.e., this vanishing follows from using the *standard* filtration on Harish-Chandra cohomology). This obviously gives the claim because of the shift of filtration that occurs in the colimit.

Step 4. Now recall that $\Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi))$ is *bi*filtered. In particular:

$$\mathrm{gr}_\bullet^{PBW}\Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) \overset{(4.2.3)}{=} \Gamma(\mathrm{Ad}_{-n\check{\rho}(t)}\,t^{-n}\mathfrak{b}[[t]]/\,\mathrm{Ad}_{-n\check{\rho}(t)}\,N(O), \mathcal{O}_{\mathrm{Ad}_{-n\check{\rho}(t)}\,t^{-n}\mathfrak{b}[[t]]/\,\mathrm{Ad}_{-n\check{\rho}(t)}\,N(O)})$$

inherits a KK filtration. We claim that:

$$F_i^{KK}\,\mathrm{gr}_\bullet^{PBW}\Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi))$$

vanishes in negative degrees.

We will verify this by giving an explicit description of the KK filtration in this case.

Note that $\mathbb{G}_m \times \mathbb{G}_m$ acts on:

$$\mathrm{Ad}_{-n\check{\rho}(t)}\,t^{-n}\mathfrak{b}[[t]]/\,\mathrm{Ad}_{-n\check{\rho}(t)}\,N(O)$$

where the action of one factor is induced by the inverse homothety action on the vector space $\mathrm{Ad}_{-n\check{\rho}(t)}\,t^{-n}\mathfrak{b}[[t]]$, and the action of the other factor is given by the adjoint action induced by the cocharacter $\check{\rho}$ (of the adjoint group). In what follows, we consider this stack as equipped with the induced *diagonal* action of $\mathbb{G}_m$; in particular, the global sections of its structure sheaf inherit a grading.

Then by Step 1, the KK filtration is induced from the grading as:

$$F_i^{KK}\,\mathrm{gr}_\bullet^{PBW}\Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) = \tag{4.2.4}$$

$$\oplus_{i' \leqslant i}\Gamma(\mathrm{Ad}_{-n\check{\rho}(t)}\,t^{-n}\mathfrak{b}[[t]]/\,\mathrm{Ad}_{-n\check{\rho}(t)}\,N(O), \mathcal{O}_{\mathrm{Ad}_{-n\check{\rho}(t)}\,t^{-n}\mathfrak{b}[[t]]/\,\mathrm{Ad}_{-n\check{\rho}(t)}\,N(O)})^{i'}$$

where the outer superscript in the global sections indicates the grading defined above.

Now we claim that our $\mathbb{G}_m$-action is *expanding*. Obviously the inverse homothety action is expanding. Moreover, the $-\check{\rho}$ adjoint action on both $\mathfrak{b}$ (so on $\mathfrak{b}((t))$) and on $N(K)$ are, so the same is true for our quotient stack above. Therefore, the diagonal action is expanding as well.

Therefore, our grading is in non-negative degrees only. By (4.2.4), we obtain the desired vanishing.

Step 5. We now make the (simpler) observation that the PBW filtration on $\mathrm{gr}_\bullet^{KK}\Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi))$ vanishes in negative degrees.

Indeed, this PBW filtration on functions on the scheme $f + \mathrm{Ad}_{-n\check{\rho}(t)} t^{-n}\mathfrak{b}[[t]]/\mathrm{Ad}_{-n\check{\rho}(t)} N(O)$ comes from degenerating f. Precisely, we have a prestack over $\mathbb{A}^1_\hbar/\mathbb{G}_m$ defined by the family:

$$\hbar f + \mathrm{Ad}_{-n\check{\rho}(t)} t^{-n}\mathfrak{b}[[t]]/\mathrm{Ad}_{-n\check{\rho}(t)} N(O)$$

and the[57] pushforward of the structure sheaf defines the PBW filtration on our complex.

The monoid $\mathbb{A}^1$ acts on the total space of this fibration through the homothety action of this monoid on $\mathfrak{g}((t))$: this implies the claim.

Step 6. From here, the claim is formal:

Suppose $V \in \mathsf{BiFil\,Vect}$ is any bifiltered vector space. We denote its underlying filtrations as $F_\bullet^{PBW}$ and $F_\bullet^{KK}$. We suppose that $F_i^{PBW}V$, $F_i^{KK}\mathrm{gr}_\bullet^{PBW}V$, and $F_i^{PBW}\mathrm{gr}_\bullet^{KK}$ vanish for $i < 0$. Then we claim that $F_i^{KK}V$ vanishes for $i < 0$ as well.

We denote:

$$F_{i,j}V = F_i^{KK}F_j^{PBW}V = F_j^{PBW}F_i^{KK}V.$$

Then observe that:

$$F_i^{KK}V = \operatorname*{colim}_j F_{i,j}V.$$

So it suffices to show that if $i < 0$, then $F_{i,j}V = 0$.

Since $\mathrm{Coker}(F_{i,j-1}V \to F_{i,j}V) = F_i^{KK}\mathrm{gr}_j^{PBW}V = 0$, $F_{i,j}V$ is independent of j. Therefore, it suffices to show the above claim when $i, j < 0$.

But the same argument shows that $F_{i,j}V$ is independent of i when j is negative. Therefore, for $i, j < 0$, we have:

$$F_{i,j}V \xrightarrow{\simeq} \operatorname*{colim}_{i'} F_{i',j}V = F_j^{PBW}V = 0$$

as desired.

$\square$

Remark 4.2.4. The method used here bears a striking resemblance to the method of *tensor product decomposition* used traditionally (e.g. in [dBT] and [FBZ]) in the $n = 0$ case above, i.e., to compute $\Psi(\mathbb{V}_\kappa)$.

There, one finds a quasi-isomorphic subcomplex[58] of the usual complex of Drinfeld-Sokolov semi-infinite chains with "size" $\mathrm{Sym}(\mathfrak{b}((t))/\mathfrak{b}[[t]]) \otimes \Lambda^\bullet\mathfrak{n}[[t]]^\vee$; this complex is closed under the KK filtration and bounded from below with respect to it, which solves the boundedness problem.

The method used above settled the convergence by a comparison with the PBW filtration, whose associated graded also has this "size," since it is Lie algebra cohomology for $\mathfrak{n}[[t]]$ with coefficients in $\mathrm{Fun}(\mathfrak{b}[[t]]) = \mathrm{Sym}(\mathfrak{b}((t))/\mathfrak{b}[[t]])$.

4.3. $\mathcal{W}_\kappa$-**module structures.**

---

[57]Note that we should work with QCoh^{ren}, so the pushforward is continuous. But this is a place where one can only feel anxiety, but cannot make a mistake: since our complexes are bounded from below, renormalized pushforward coincides with any other notion.

[58]In [FBZ], it is introduced in §15.2 and denoted by $C_k^\bullet(\mathfrak{g})_0$.

4.3.1.   Define:

$$\mathcal{W}_\kappa^n := H^0 \Psi_n(\mathrm{ind}_{\mathrm{Lie}\, \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} \psi_{\mathring{I}_n}) = H^{-n\Delta}\Psi(\mathrm{ind}_{\mathrm{Lie}\, \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} \psi_{\mathring{I}_n}) \otimes \ell^{n,\vee}$$

By Theorem 4.2.1, this is the only non-vanishing cohomology group. We recall that Ψ_n was defined in (4.2.2). Note that tensoring with this line is extremely mild.

   We now make some observations about the $\mathcal{W}_\kappa$-*module* structure on $\mathcal{W}_\kappa^n$. (For further observations, see §6.)

4.4. **Recollections on $\mathcal{W}$-algebras.** Before proceeding, we summarize what facts we will need about $\mathcal{W}_\kappa$.

4.4.1.   Recall that $\mathcal{W}_\kappa := \Psi(\mathbb{V}_\kappa) = \mathcal{W}_\kappa^0$ is a *vertex algebra*, which for our purposes means the vacuum representation of a chiral (or factorization) algebra in the sense of [BD2].

   Here are two conceptual explanations of this fact.

   One may use [BD2] §3.8 to give a model for semi-infinite cochains that factorizes.

   Alternatively, one can work as in [Gai6] and show that the categories $\widehat{\mathfrak{g}}_\kappa$–mod is (the fiber of) a unital chiral category in the sense of [Ras1] with unit the factorization algebra $\mathbb{V}_\kappa$ (or rather, the Kac-Moody factorization algebra). Moreover, one can show that our definition of $\Psi : \widehat{\mathfrak{g}}_\kappa$–mod $\to$ Vect factorizes. This formally implies that the image of the unit is a vertex algebra in the above sense.

   (There is also a traditional vertex algebra description: see [FBZ] §15.)

4.4.2.   In either of the above pictures, we find that Ψ upgrades to a functor:

$$\Psi : \widehat{\mathfrak{g}}_\kappa\text{–mod} \to \mathcal{W}_\kappa\text{–mod}^{\mathrm{fact}}$$

where the right hand side denotes the DG category of *factorization modules* for (the factorization algebra with underlying vacuum representation) $\mathcal{W}_\kappa$; this notion is defined in [Ras1].

   Some remarks about this notion are in order.

   $\mathcal{W}_\kappa$–mod$^{\mathrm{fact}}$ has a canonical t-structure, and by [BD2] §3.6, the heart is the abelian category of discrete $\mathcal{W}_\kappa^{as}$-modules, which coincides with vertex modules in the usual sense. We denote this category by $\mathcal{W}_\kappa$–mod$^\heartsuit$.

   It is possible to compute $\mathcal{W}_\kappa$–mod$^{\mathrm{fact}}$ explicitly: it turns out to be the left completion of the derived category of this abelian category. However, such arguments have not been given in the literature previously (e.g., the corresponding fact for Kac-Moody algebras is not published), and are somewhat involved.

   Moreover, *we will not need to compute this DG category* so explicitly: we will only need to know the heart of its t-structure.

4.4.3.   By the above, for $M \in \widehat{\mathfrak{g}}_\kappa$–mod, $H^k\Psi(M) \in \mathcal{W}_\kappa$–mod$^\heartsuit$, i.e., this object of Vect$^\heartsuit$ has a canonical action of the $\mathcal{W}_\kappa$-algebra.

   Moreover, we find that the functors $(H^k\Psi, k \in \mathbb{Z})$ are *cohomological*, i.e., for a morphism $f : M \to N \in \widehat{\mathfrak{g}}_\kappa$–mod, the boundary morphism:

$$H^k(\mathrm{Coker}(f)) \to H^{k+1}(M)$$

is a morphism of $\mathcal{W}_\kappa$-modules.

4.4.4.   We also need some compatibilities with filtrations.

First, the KK filtration on $\mathbb{V}_\kappa$ makes sense factorizably, so defines on $\mathbb{V}_\kappa$ a structure of *filtered* vertex algebra (cf. [BD2] §3.3.11 or [FBZ] §15).

Then one can show by mixing [Gai6] and Appendix A that the KK filtration on $\widehat{\mathfrak{g}}_\kappa$–mod defines a filtration on this category *as a unital factorization category* and that Ψ is a factorizable functor.

Since the induced KK filtration on $\mathcal{W}_\kappa = \Psi(\mathbb{V}_\kappa) = H^0\Psi(\mathbb{V}_\kappa)$ is a filtration in the abelian category of vector spaces, we deduce that $\mathcal{W}_\kappa$ is a filtered vertex algebra. Concretely, this means that $\mathcal{W}_\kappa^{as}$ is filtered as a topological chiral algebra. We remind that $\mathrm{gr}_\bullet^{KK} \mathcal{W}_\kappa$ is the commutative vertex algebra of functions on $f + \mathfrak{b}[[t]]/N(O) = f + \mathfrak{b}^e[[t]]$, and that the associated graded of $\mathcal{W}_\kappa^{as}$ is the corresponding commutative topological algebra of functions on the affine Kostant slice $f + \mathfrak{b}((t))/N(K) = f + \mathfrak{b}^e((t))$.

4.4.5.   Suppose $M \in \mathsf{Fil}^{KK}\widehat{\mathfrak{g}}_\kappa$–mod is a KK filtered Kac-Moody representation. Recall from Appendix A that $\Psi(M)$ inherits a KK filtration.

From the above discussion, for any integer k, $\mathrm{gr}_\bullet^{KK} \Psi(M) \in \mathrm{gr}_\bullet^{KK} \mathcal{W}_\kappa$–mod$^{\mathrm{fact}}$, which as before is a DG category with a t-structure whose heart is $\mathsf{IndCoh}(f + \mathfrak{b}((t))/N(K))^\heartsuit$.[59] Therefore, for any integer k, $H^k(\mathrm{gr}_\bullet^{KK} \Psi(M)) \in \mathsf{IndCoh}(f + \mathfrak{b}((t))/N(K))^\heartsuit$.

4.5. **Description of the modules $\mathcal{W}_\kappa^n$.** Now observe that the modules $\mathcal{W}_\kappa^n$ are naturally filtered. Indeed, as above, each module $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\underset{\mathrm{Lie}\,\mathring{I}_n}{\circ}}(\psi)$ carries a canonical KK filtration, so it Drinfeld-Sokolov reduction does as well. However, we will renumber the filtration by shifting the indices by $n\Delta$: this amounts to considering the natural filtration on $H^0\Psi_n(\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\underset{\mathrm{Lie}\,\mathring{I}_n}{\circ}}(\psi))$ instead of on $H^{n\Delta}\Psi$ of this module. We denote this filtration $F_\bullet^{KK}\mathcal{W}_\kappa^n$.

We now have the following outcome of the proof of Theorem 4.2.1:

Theorem 4.5.1. *The filtration on $\mathcal{W}_\kappa^n$ satisfies:*
- $F_i^{KK}\mathcal{W}_\kappa^n = 0$ *for $i < 0$.*
- $H^k(\mathrm{gr}_\bullet^{KK}\mathcal{W}_\kappa^n)$ *vanishes for $k \neq 0$.*
- $H^0(\mathrm{gr}_\bullet^{KK}\mathcal{W}_\kappa^n) \in \mathsf{IndCoh}(f + \mathfrak{b}((t))/N(K))^\heartsuit$ *is the structure sheaf of the closed subscheme $t^{-n}\mathrm{Ad}_{-n\check{\rho}(t)}(f + \mathfrak{b}[[t]]/N(O))$ with its natural grading as a $\mathbb{G}_m$-invariant subscheme.*

Remark 4.5.2 (Cyclicity of $\mathcal{W}_\kappa^n$). Note that the algebra of functions on $f + \mathfrak{b}/N = f + \mathfrak{b}^e$ is a polynomial algebra, and with respect to the KK grading, is generated by elements of degree $\geqslant 1$ (and $\geqslant 2$ if $\mathfrak{g}$ is semisimple). The same holds for the affine version, or for the structure sheaf of a $\mathbb{G}_m$-invariant subscheme of $f + \mathfrak{b}((t))/N(K)$.

Because $F_{-1}^{KK}\mathcal{W}_\kappa^n = 0$, we deduce that $F_0^{KK}\mathcal{W}_\kappa^n = \mathrm{Gr}_0^{KK}\mathcal{W}_\kappa^n = k$, where this copy of k corresponds to the constant functions on $t^{-n}\mathrm{Ad}_{-n\check{\rho}(t)}(f + \mathfrak{b}[[t]]/N(O))$. In particular, we obtain a canonical *vacuum vector* $1 \in \mathcal{W}_\kappa^n$.

Moreover, the map $\mathcal{W}_\kappa^{as} \to \mathcal{W}_\kappa^n$ given by acting on this vector is surjective; indeed, this follows because it is true at the associated graded level. Therefore, we obtain that $\mathcal{W}_\kappa^n$ is a *cyclic* module for the $\mathcal{W}_\kappa$-algebra.

4.6. **Varying n.** We conclude this section with the following result.

Theorem 4.6.1.   *(1) For $m \geqslant n$, there is a unique map $\mathcal{W}_\kappa^m \to \mathcal{W}_\kappa^n$ preserving the vacuum vectors introduced in Remark 4.5.2.*

---

[59]We recall that the heart of the natural t-structure here is tautologically the same as the abelian category of discrete modules over the algebra of functions on this indscheme.

(2) *This map is surjective and strictly compatible with filtrations. Upon passage to the associated graded, it yields the restriction map for functions along the canonical embedding:*

$$t^{-n} \operatorname{Ad}_{-n\check{\rho}(t)}(f + \mathfrak{b}[[t]]/N(O)) \hookrightarrow t^{-m} \operatorname{Ad}_{-m\check{\rho}(t)}(f + \mathfrak{b}[[t]]/N(O)).$$

(3) *The compatible system of maps $\mathcal{W}_\kappa^{as} \to \mathcal{W}_\kappa^n$ defined by the vacuum vectors defines an isomorphism:*

$$\mathcal{W}_\kappa^{as} \to \lim_n \mathcal{W}_\kappa^n \in \mathsf{Pro}(\mathsf{Vect}^\heartsuit)$$

of pro-vector spaces.

Proof.

Step 1. First, we observe that these maps are unique if they exist. Indeed, this follows immediately from the cyclicity of the modules $\mathcal{W}_\kappa^m$.

Step 2. We now construct a map $\alpha : \mathcal{W}_\kappa^m \to \mathcal{W}_\kappa^n$.
    First, note that we have:

$$\operatorname{ind}_{\operatorname{Lie}(\mathring{I}_n \cap \mathring{I}_m)}^{\widehat{\mathfrak{g}}_\kappa}(\psi) \twoheadrightarrow \operatorname{ind}_{\operatorname{Lie}\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi).$$

Then we claim that there is a canonical map:

$$\operatorname{ind}_{\operatorname{Lie}\mathring{I}_m}^{\widehat{\mathfrak{g}}_\kappa}(\psi) \to \operatorname{ind}_{\operatorname{Lie}(\mathring{I}_n \cap \mathring{I}_m)}^{\widehat{\mathfrak{g}}_\kappa}(\psi) \otimes \ell^{n,m}[(m-n)\Delta] \in \widehat{\mathfrak{g}}_\kappa\text{--mod}.$$

It suffices to construct:

$$\psi \to \operatorname{ind}_{\operatorname{Lie}(\mathring{I}_n \cap \mathring{I}_m)}^{\operatorname{Lie}\mathring{I}_m}(\psi) \otimes \ell^{n,m}[(m-n)\Delta] \in \operatorname{Lie}\mathring{I}_m\text{--mod}.$$

More generally, suppose $\mathfrak{h}_1 \subseteq \mathfrak{h}_2$ is an open Lie subalgebra in a profinite dimensional Lie algebra. Then for any module $M \in \mathfrak{h}_2\text{--mod}$, we claim that there is a canonical map:

$$M \to \operatorname{ind}_{\mathfrak{h}_1}^{\mathfrak{h}_2}\big(\operatorname{Oblv}(M) \otimes \det(\mathfrak{h}_2/\mathfrak{h}_1)[\dim \mathfrak{h}_2/\mathfrak{h}_1]\big).$$

Indeed, by the projection formula it suffices to construct this for $M = k$ the trivial module, and then it is given by Lemma A.7.10.
    By composition, we obtain:

$$\operatorname{ind}_{\operatorname{Lie}\mathring{I}_m}^{\widehat{\mathfrak{g}}_\kappa}(\psi) \to \operatorname{ind}_{\operatorname{Lie}\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi) \otimes \ell^{n,m}[(m-n)\Delta]. \tag{4.6.1}$$

Applying Ψ now gives the desired map.
    Note that these morphisms compose well as we vary n and m.

Step 3. By our generalities on filtrations from Appendix A, $\alpha : \mathcal{W}_\kappa^m \to \mathcal{W}_\kappa^n$ is filtered, and on passage to the associated graded, it yields the restriction map along our closed embeddings.
    This observation actually implies the rest of the results. The map preserves vacuum vectors by their construction. It is surjective because it is surjective at the associated graded level. For a surjective morphism of filtered abelian groups with filtrations bounded from below, surjectivity at the associated graded level is equivalent to strictness. Finally, the projective system $\{\mathcal{W}_\kappa^n\}$ gives $\mathcal{W}_\kappa^{as}$ because this is true at the associated graded level.

$\square$

5. Affine Skryabin theorem

5.1. Formulation of the result.

Let $D^+(\mathcal{W}_\kappa\text{--mod}^\heartsuit)$ denote the bounded below derived category considered as a DG category. Then define:

$$\mathcal{W}_\kappa\text{--mod}^c \subseteq D^+(\mathcal{W}_\kappa\text{--mod}^\heartsuit)$$

as the full subcategory generated by the objects $\mathcal{W}_\kappa^n \in \mathcal{W}_\kappa\text{--mod}^\heartsuit$ under cones and shifts; recall that these objects were defined in §4. Finally, following [FG3] §22-3, we define:

$$\mathcal{W}_\kappa\text{--mod} := \mathsf{Ind}(\mathcal{W}_\kappa\text{--mod}^c).$$

The purpose of this section is to prove:

Theorem 5.1.1 (Affine Skryabin theorem)**.** *There is a canonical equivalence:*

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{--mod}) \simeq \mathcal{W}_\kappa\text{--mod}.$$

This equivalence has the property that the composition:

$$\widehat{\mathfrak{g}}_\kappa\text{--mod} \to \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{--mod}) \simeq \mathcal{W}_\kappa\text{--mod} \xrightarrow{\mathrm{Oblv}} \mathsf{Vect}$$

is the Drinfeld-Sokolov functor Ψ; here the first morphism is the canonical morphism coming from identifying Whit with coinvariants.

5.2. Construction of the t-structure.

5.2.1. The core of the proof is a construction of a canonical t-structure on $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{--mod})$ and an analysis of some nice properties that it has.

5.2.2. Recall from §2 that we have the categories $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{--mod})$, and adjoint functors $(\iota_{n,m,!}, \iota_{n,m}^!)$ for $0 \leqslant n \leqslant m \leqslant \infty$.

For $n < \infty$, note that $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{--mod}) = \widehat{\mathfrak{g}}_\kappa\text{--mod}^{\mathring{I}_n,\psi}$ has a canonical t-structure (the forgetful functor to $\widehat{\mathfrak{g}}_\kappa\text{--mod}$ is t-exact).

The key observation is:

Lemma 5.2.1. *For all $n \leqslant m < \infty$, the functor:*

$$\iota_{n,m,!}[-(m-n)\Delta] \overset{Thm.2.3.1(3)}{=} \iota_{n,m,*}[(m-n)\Delta]$$

is t-exact. (Again, $\Delta := 2(\rho, \check{\rho})$ as in §2.3.)

Remark 5.2.2. We prove Lemma 5.2.1 in Appendix B after giving a sketch below.

Proof sketch for Lemma 5.2.1. The idea is that for general reasons $*$-averaging $\mathring{I}_n \cap \mathring{I}_m$ to $\mathring{I}_m$ has cohomological amplitude:

$$[0, \dim \mathring{I}_m/\mathring{I}_n \cap \mathring{I}_n] = [0, (m-n)\Delta]$$

while !-averaging has amplitude:

$$[-\dim \mathring{I}_m/\mathring{I}_n \cap \mathring{I}_n, 0] = [-(m-n)\Delta, 0].$$

Indeed, this follows since they are essentially given by $*$ and !-versions of de Rham cohomology along the *affine* scheme $\mathring{I}_m/\mathring{I}_n \cap \mathring{I}_n$.

Then recall that $\iota_{n,m,*}$ and $\iota_{n,m,!}$ are a composition of a forgetful functor, which is t-exact, with such an averaging functor. So we find that $\iota_{n,m,*}[(m-n)\Delta]$ is right t-exact, while $\iota_{n,m,!}[-(m-n)\Delta]$ is left t-exact. These functors coincide by Theorem 2.3.1 (3), so they are t-exact.

$\square$

5.2.3.   We now have the following result:

Proposition 5.2.3.   *(1) There exists a unique t-structure on $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ compatible with filtered colimits such that the functors:*

$$\iota_{n,!}[n\Delta]$$

are t-exact.

(2) With respect to this t-structure, $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^+$ is the (DG) bounded below derived category $D^+(\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod}^\heartsuit))$ of its heart.

Convention 5.2.4. Here we make the decision to always equip $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ with the canonical t-structure on Harish-Chandra modules, i.e., realizing the category as $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_{n},\psi}$. This can be confusing, since we need the shifts above for the functors to be t-exact; but shifting all our t-structures would probably be more confusing.

Using the automorphisms $[n\Delta] : \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) \to \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$, we have:

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) = \operatorname*{colim}_{n,\iota_{n,m,!}[-(m-n)\Delta]} \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod}).$$

Therefore, the above result is an immediate consequence of the following plus Lemma A.8.1 (the Bernstein-Lunts theorem).

Lemma 5.2.5. *Suppose $i \mapsto \mathcal{C}_i \in \mathsf{DGCat}_{cont}$ is a filtered diagram of cocomplete DG categories, each equipped with an (accessible) t-structure compatible with filtered colimits. Suppose every structure functor $\psi_{i,j} : \mathcal{C}_i \to \mathcal{C}_j$ is t-exact and admits a continuous right adjoint $\varphi_{i,j}$.*

(1) Then $\mathcal{C} := \operatorname{colim}_i \mathcal{C}_i$ admits a unique t-structure such that each $\psi_i : \mathcal{C}_i \to \mathcal{C}$ is t-exact.

(2) If $\mathcal{C}_i^+ = D^+(\mathcal{C}_i^\heartsuit)$ is the bounded below derived category of its heart for each i, and if the filtered category indexing our colimit is countable, then $\mathcal{C}^+ = D^+(\mathcal{C}^\heartsuit)$.

Proof.

Step 1. Define a t-structure on $\mathcal{C}$ by declaring $\mathcal{C}^{\leqslant 0}$ to be generated under colimits by the subcategories $\psi_i(\mathcal{C}_i^{\leqslant 0})$. It is equivalent to say that $\mathcal{F} \in \mathcal{C}^{\geqslant 0}$ if and only if $\varphi_i(\mathcal{F}) \in \mathcal{C}_i^{\geqslant 0}$ for all i; here $\varphi_i : \mathcal{C} \to \mathcal{C}_i$ is the (continuous) right adjoint to ψ_i.

We want to show that the functors ψ_i are t-exact. Clearly they are right t-exact, so it remains to show left t-exactness. Suppose $\mathcal{F} \in \mathcal{C}_i^{\geqslant 0}$; we need to show $\psi_i(\mathcal{F}) \in \mathcal{C}^{\geqslant 0}$. It suffices to show $\varphi_j \psi_i(\mathcal{F}) \in \mathcal{C}_j^{\geqslant 0}$ for all indices j. By standard generalities about filtered co/limits (see [Gai4]), we have:

$$\varphi_j \psi_i(\mathcal{F}) = \operatorname*{colim}_{i \to k \leftarrow j} \varphi_{j,k} \psi_{i,k}(\mathcal{F}_i) \in \mathcal{C}_j.$$

But $\psi_{i,k}(\mathcal{F}_i) \in \mathcal{C}_k^{\geqslant 0}$ by t-exactness, and $\varphi_{j,k}(\mathcal{C}_k^{\geqslant 0}) \subseteq \mathcal{C}_j^{\geqslant 0}$ because it is right adjoint to a t-exact functor, so each term in our colimit is in degrees $\geqslant 0$. Since our t-structures are compatible with filtered colimits, we obtain the claim.

Step 2. Now suppose that $\mathcal{C}_i^+ = D^+(\mathcal{C}_i^\heartsuit)$ is the bounded below derived category of its heart. We want to see the same property for $\mathcal{C}$.

Some technical comments first: note that the t-structure on $\mathcal{C}_i$ is right separated by assumption. By the compatibility of the t-structures with filtered colimits, it is therefore also right complete. It follows formally that the t-structure on $\mathcal{C}$ is right separated, and again, that it is right complete.

Therefore, it suffices to show that an injective object I in the abelian category $\mathcal{C}^\heartsuit$ is "actually injective" in $\mathcal{C}$, i.e., that the following equivalent conditions are satisfied:

- $\underline{\mathrm{Hom}}_\mathcal{C}(-, I) : \mathcal{C} \to \mathsf{Vect}^{op}$ is t-exact.
- For $\mathcal{F} \in \mathcal{C}^{\geqslant 0}$, $\underline{\mathrm{Hom}}_\mathcal{C}(\mathcal{F}, I) \in \mathsf{Vect}^{\leqslant 0}$.
- For $\mathcal{F} \in \mathcal{C}^\heartsuit$, $\underline{\mathrm{Hom}}_\mathcal{C}(\mathcal{F}, I) \in \mathsf{Vect}^\heartsuit$.

We will do this in what follows.

Step 3. We need some general properties about homological algebra for abelian categories.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are Grothendieck abelian categories and $F : D^+(\mathcal{A}) \rightleftarrows D^+(\mathcal{B}) : G$ are adjoint with F t-exact. Then recall that G is the derived functor of $H^0(G) : \mathcal{B} \to \mathcal{A}$, that is, for any injective $I \in \mathcal{B}$, $G(I) = H^0(G(I))$.

Indeed, for $\mathcal{F} \in \mathcal{A}$, we have:

$$\mathrm{Ext}^i_\mathcal{A}(\mathcal{F}, G(I)) = \mathrm{Ext}^i_\mathcal{B}(F(\mathcal{F}), I) = 0 \text{ for } i > 0.$$

A standard argument shows as well that $H^0 G(I)$ is injective. Therefore, by the long exact sequence, $\tau^{>0} G(I) = \mathrm{Coker}(H^0(G(I)) \to G(I)) \in D^+(\mathcal{A})$ satisfies the above vanishing as well. But since $\tau^{>0} G(I)$ is in degrees > 0, the vanishing of positive Exts is enough to guarantee that it is zero.

Therefore, in our setting, the functors $H^0 \varphi_{i,j} : \mathcal{C}_j^\heartsuit \to \mathcal{C}_i^\heartsuit$ preserve injective objects, and $\varphi_{i,j} = H^0 \varphi_{i,j}$ when evaluated on such an object.

Step 4. Now we show that $\varphi_i(I) = H^0(\varphi_i(I))$ for every i. (The argument is unfortunately a little indirect.)

Suppose $i \to j$ is given. We form the cone:

$$\mathrm{Coker}(\varphi_{i,j} H^0(\varphi_j(I)) \to \varphi_i(I)) = \mathrm{Coker}\left(\varphi_{i,j} H^0(\varphi_j(I)) \to \varphi_{i,j}\varphi_j(I)\right) = \varphi_{i,j}\tau^{>0}\varphi_j(I).$$

Obviously the last term is in cohomological degrees > 0. Moreover, because $H^0(\varphi_j(I))$ is injective in $\mathcal{C}_j^\heartsuit$, the first term $\varphi_{i,j} H^0(\varphi_j(I))$ in our distinguished triangle is in cohomological degree 0 (by the previous step). Therefore, this is a truncation sequence, and in particular we have:

$$H^0(\varphi_{i,j} H^0(\varphi_j(I))) \xrightarrow{\simeq} H^0(\varphi_i(I)).$$

We remark again that $H^0(\varphi_{i,j} H^0(\varphi_j(I))) = \varphi_{i,j} H^0(\varphi_j(I))$. Therefore, the objects $j \mapsto H^0(\varphi_j(I))$ defines an object of $\lim_j \mathcal{C}_j = \mathcal{C}$ (the structure maps in this limit being the φ_j).

Recall that by filteredness, any object $\mathcal{F}$ of $\mathcal{C}$ can be written as:

$$\operatorname*{colim}_i \psi_i \varphi_i(\mathcal{F}).$$

Therefore, the object of $\mathcal{C}$ constructed above is:

$$\operatorname*{colim}_i \psi_i H^0(\varphi_j(I)) = H^0(\operatorname*{colim}_i \psi_i \varphi_j(I)) = H^0(I) = I.$$

This proves the claim.

Step 5. We now conclude the argument. For $\mathcal{F} \in \mathcal{C}$, we have $\mathcal{F} = \mathrm{colim}_i \, \psi_i \varphi_i(\mathcal{F}) \in \mathcal{C}$, so:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, I) = \lim_i \underline{\mathrm{Hom}}_{\mathcal{C}_i}(\psi_i \varphi_i(\mathcal{F}), I) = \lim_i \underline{\mathrm{Hom}}_{\mathcal{C}_i}(\varphi_i(\mathcal{F}), \varphi_i(I)).$$

Now suppose $\mathcal{F} \in \mathcal{C}^{\heartsuit}$. Then $\varphi_i(\mathcal{F}_i) \in \mathcal{C}_i^{\geq 0}$ and $\varphi_i(I)$ is injective by the above, so each term in this limit is in $\mathsf{Vect}^{\leq 0}$. Because this is a countable limit by assumption, the limit is in $\mathsf{Vect}^{\leq 1}$. But now recall that:

$$H^1(\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, I)) = \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{F}, I) = \mathrm{Ext}^1_{\mathcal{C}^{\heartsuit}}(\mathcal{F}, I) = 0$$

by injectivity of I, completing the argument.

$\square$

Notation 5.2.6. There is risk for notational confusion in using this t-structure: e.g., $\mathsf{Whit}^{\leq n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ continues to denote the adolescent Whittaker category, while $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^{\leq n}$ denotes the subcategory of $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ consisting of objects cohomologically bounded from above by n.

5.3. Fiber functor.

5.3.1.   Observe that we have:

where we regard Whit as coinvariants. Indeed, this follows because each functor $C^\bullet(\mathrm{Ad}_{-n\check{\rho}(t)} \mathfrak{n}[[t]], (-) \otimes -\psi)$ obviously[60] factors as:

We will prove Theorem 5.1.1 by an analysis of this functor.

Warning 5.3.1. The cohomological shifts can cause a great deal of confusion here. By the above, for $m \geq n$ the following diagram commutes:

---

[60]Indeed, it suffices to show that the natural transformation $\mathrm{Oblv} \, \mathrm{Av}_*^\psi \to \mathrm{id}$ induces an isomorphism on this Lie algebra cohomology; (here everything is with respect to $\mathrm{Ad}_{-n\check{\rho}(t)} N(O)$ and its Lie algebra). It obviously suffices to prove this statement for $\mathrm{Ad}_{-n\check{\rho}(t)} \mathfrak{n}[[t]]\text{–mod}$ in place of $\widehat{\mathfrak{g}}_\kappa\text{–mod}$. Then the Lie algebra cohomology functor is corepresented by the trivial representation, so with the twist by the character it is corepresented by the 1-dimensional module defined by ψ. Since this object lies in $\mathrm{Ad}_{-n\check{\rho}(t)} \mathfrak{n}[[t]]\text{–mod}^{\mathrm{Ad}_{-n\check{\rho}(t)} N(O), \psi}$, and since the forgetful functor from this category to $\mathrm{Ad}_{-n\check{\rho}(t)} \mathfrak{n}[[t]]\text{–mod}$ is fully-faithful, we obtain the result.

Then recall that in the proof of Theorem 2.1.1, we used cohomological shifts by $2n\Delta$ on $\mathsf{Whit}^{\leqslant n}$ to identify $\mathsf{Whit}(\mathcal{C}) := \mathrm{colim}_{n,\iota_{n,m,*}} \mathsf{Whit}^{\leqslant n}(\mathcal{C})$ with the colimit under the functors $\iota_{n,m,!}$.

Therefore, we obtain the commutativity of the following diagram:

$$
\begin{array}{ccccc}
\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{--mod}) & \xrightarrow{\ \mathrm{Oblv}\ } & \widehat{\mathfrak{g}}_\kappa\text{--mod} & \xrightarrow{\ \Psi[-2n\Delta]\ } & \mathsf{Vect} \\
{\scriptstyle \iota_{n,!}}\big\downarrow & & & & \big\| \\
\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{--mod}) & & \xrightarrow{\hspace{4.5cm}\Psi^{\mathsf{Whit}}\hspace{4.5cm}} & & \mathsf{Vect}.
\end{array}
$$

This is the reason we use the notation Ψ^{Whit} (rather than Ψ): we continue to use the notation Ψ for the functor $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{--mod}) \to \mathsf{Vect}$ obtained by thinking of this category as Harish-Chandra modules, and then $\Psi^{\mathsf{Whit}} \circ \iota_{n,!} = \Psi[-2n\Delta]$.

(Note that the confusion arises here because we are not adhering to the principle espoused in Warning 2.2.18.)

5.3.2. The main observation is that $\Psi^{\mathsf{Whit}} : \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{--mod}) \to \mathsf{Vect}$ is canonically corepresented in a nice way. Heuristically, if we think of the source category as the category of modules over the $\mathcal{W}_\kappa$-algebra, we want to show that it is corepresented by the projective system $\{\mathcal{W}_\kappa^n\}_{n\geqslant 0}$ from §4.

5.3.3. To make this idea precise, define $'\mathcal{W}_\kappa^n \in \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{--mod})$ as:

$$
\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} (\psi) \otimes \ell^{n,\vee}[n\Delta] \in \widehat{\mathfrak{g}}_\kappa\text{--mod}^{\mathring{I}_n,\psi} = \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{--mod}).
$$

Here (and throughout this section) we are freely using Notation 4.2.3, so ℓ is shorthand for a determinant line. We use the same notation for the induced object $\iota_{n,!}('\mathcal{W}_\kappa^n)$ of $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{--mod})$.

We have the following basic properties.

- $'\mathcal{W}_\kappa^n \in \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{--mod})^\heartsuit$. Indeed, recall that $\iota_{n,!}[n\Delta]$ is t-exact, giving the claim.
- The notation is compatible with §4 in the sense that:

$$
\Psi^{\mathsf{Whit}}('\mathcal{W}_\kappa^n) = \mathcal{W}_\kappa^n \in \mathsf{Vect}^\heartsuit \subseteq \mathsf{Vect}.
$$

  Indeed, this was the definition of the right hand side (we remark that Warning 5.3.1 is important here).

5.3.4. Now observe that in the proof of Theorem 4.6.1, we in effect constructed a canonical map $\alpha_{n,m} : '\mathcal{W}_\kappa^m \to '\mathcal{W}_\kappa^n$ for $m \geqslant n$.

Indeed, in (4.6.1), we produced a map:

$$
\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_m}^{\widehat{\mathfrak{g}}_\kappa} (\psi) \to \mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} (\psi) \otimes \ell^{n,m}[(m-n)\Delta] \in \widehat{\mathfrak{g}}_\kappa\text{--mod}^{\mathring{I}_n \cap \mathring{I}_m, \psi}
$$

This induces a morphism:

$$
\alpha'_{n,m} : \mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_m}^{\widehat{\mathfrak{g}}_\kappa} (\psi) \to \mathrm{Av}_*^{\mathring{I}_m, \psi}(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} (\psi) \otimes \ell^{n,m}[(m-n)\Delta]) \in \widehat{\mathfrak{g}}_\kappa\text{--mod}^{\mathring{I}_m, \psi}. \tag{5.3.1}
$$

Note that $\mathrm{Av}_*^{\mathring{I}_m, \psi}$ is the same as $\iota_{n,m,*}$ here. Therefore, incorporating the determinant twists and cohomological shifts, and switching to adolescent Whittaker notation, we obtain:[61]

---

[61]Regarding the arithmetic of cohomological shifts: in the last equality, up to the factor of Δ we have a contribution $(2m-n)$ from the previous line; switching from $\iota_{n,m,*}$ to $\iota_{n,m,!}$ means we need to add a shift by $-2(m-n)$, producing the $-n$ that is displayed.

$$\alpha_{n,m} : {}'\mathcal{W}_\kappa^m = \operatorname{ind}_{\operatorname{Lie} \mathring{I}_m}^{\widehat{\mathfrak{g}}_\kappa} (\psi) \otimes \ell^{m,\vee}[m\Delta] \to \iota_{n,m,*}\big(\operatorname{ind}_{\operatorname{Lie} \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} (\psi)\big) \otimes \ell^{n,m}[(m-n)\Delta] \otimes \ell^{m,\vee}[m\Delta] =$$

$$\iota_{n,m,!}\big(\operatorname{ind}_{\operatorname{Lie} \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} (\psi) \otimes \ell^{n,\vee}[-n\Delta]\big) = \iota_{n,m,!}({}'\mathcal{W}_\kappa^n) \in \mathsf{Whit}^{\leqslant m}(\widehat{\mathfrak{g}}_\kappa\text{–mod}).$$

We have the following important fact.

Lemma 5.3.2. *The morphism* $\alpha_{n,m} : {}'\mathcal{W}_\kappa^m \to {}'\mathcal{W}_\kappa^n \in \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit$ *is an epimorphism in this abelian category.*

Proof.

Step 1. By t-exactness of $\iota_{m,!}[m\Delta]$, it suffices to show the corresponding fact in $\mathsf{Whit}^{\leqslant m}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$. Tautologically, this reduces to a fact about the morphism $\alpha'_{n,m}$ from (5.3.1).

Observe that both the source and the target have canonical KK filtrations in $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_m,\psi}$. We will first verify that the associated graded morphism is an epimorphism. Afterwards, we will explain why this suffices to deduce the result (the issue being that the KK filtration is not bounded from below).

Step 2. We have:

$$\operatorname{gr}_\bullet^{KK} \operatorname{ind}_{\operatorname{Lie} \mathring{I}_m}^{\widehat{\mathfrak{g}}_\kappa} (\psi) = \mathcal{O}_{f + \operatorname{Lie} \mathring{I}_m^\perp / \mathring{I}_m} \in \mathsf{QCoh}^{ren}(f + \operatorname{Lie} \mathring{I}_m^\perp / \mathring{I}_m).$$

As in Remark 3.1.3, we have:

$$\operatorname{gr}_\bullet^{KK} \operatorname{Av}_*^{\mathring{I}_m,\psi}(\operatorname{ind}_{\operatorname{Lie} \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} (\psi) \otimes \ell^{n,m}[(m-n)\Delta]) = \pi_*(\mathcal{O}_{f + \operatorname{Lie} \mathring{I}_n^\perp \cap \operatorname{Lie} \mathring{I}_m^\perp / \mathring{I}_n \cap \mathring{I}_m})$$

for $\pi : f + \operatorname{Lie} \mathring{I}_n^\perp \cap \operatorname{Lie} \mathring{I}_m^\perp / \mathring{I}_n \cap \mathring{I}_m \to f + \operatorname{Lie} \mathring{I}_n^\perp / \mathring{I}_n$ the projection; note that the determinant twist and cohomological shift are absorbed due to the "mild correction" from *loc. cit.*

By construction, our map is the canonical adjunction morphism. Because π is a closed embedding by Theorem 3.1.1, this map is an epimorphism as desired.

Step 3. Now we explain why the associated graded map being an isomorphism suffices.

Let $L_0 = t\partial_t$. We use the term $(L_0 + \check{\rho})$-*grading* to refer to the corresponding grading on $\mathfrak{g}((t))$ induced by taking the diagonal action of $\mathbb{G}_m$ with respect to loop rotation and $\operatorname{Ad}_{\check{\rho}(-)}$. Note that $\psi : \mathring{I}_n \to k$ is graded for k being given degree 0: the point is that $\frac{e_i}{t}$ has $(L_0 + \check{\rho})$-degree 0.

Therefore, $\operatorname{ind}_{\operatorname{Lie} \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}$ carries a canonical $(L_0 + \check{\rho})$-grading. The same formally holds for:

$$\operatorname{Av}_*^{\mathring{I}_m,\psi}(\operatorname{ind}_{\operatorname{Lie} \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} (\psi) \otimes \ell^{n,m}[(m-n)\Delta])$$

and our map $\alpha'_{n,m}$ is compatible with these gradings.

Obviously it suffices to show $\alpha'_{n,m}$ is an epimorphism in the case $m > 0$, since otherwise $m = n = 0$. We claim that in this case, for every integer i, the KK filtration on the ith $(L_0 + \check{\rho})$-graded component of our modules is bounded from below. This combined with the corresponding semi-classical statement obviously suffices to show the surjectivity of the morphism $\alpha'_{n,m}$, since it implies it on each graded component.

We show this below.

Step 4. To verify the claim about the $(L_0 + \check{\rho})$-grading on $\operatorname{ind}_{\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa} (\psi)$, note that because the KK filtration is compatible with the grading and separated (non-derivedly), it suffices to show that the

$(L_0 + \check\rho)$-eigenvalues on $\mathrm{gr}_i^{KK} \mathrm{ind}_{\mathrm{Lie}\, \mathring{I}_m}^{\widehat{\mathfrak{g}}_\kappa}(\psi)$ are bounded above by some function of i going to $-\infty$ as i does.

Recall that:

$$\mathrm{gr}_i^{KK} \mathrm{ind}_{\mathrm{Lie}\, \mathring{I}_m}^{\widehat{\mathfrak{g}}_\kappa}(\psi) = \bigoplus_j \mathrm{Sym}^j(\mathfrak{g}((t))/\mathrm{Lie}\, \mathring{I}_n)^{i-j}$$

where the superscript indicates the $(i-j)$th graded degree with respect to the $-\check\rho$-grading.

Below, for α in the root lattice of G, we use the notation $|\alpha|$ for $(\check\rho, \alpha)$.

The jth summand above is spanned by elements of the form:

$$\frac{e_{\alpha_1}}{t^{r_1}} \frac{e_{\alpha_2}}{t^{r_2}} \cdots \frac{e_{\alpha_k}}{t^{r_k}} \frac{f_{\beta_{k+1}}}{t^{r_{k+1}}} \cdots \frac{f_{\beta_j}}{t^{r_j}}$$

where $e_{\alpha_\ell} \in \mathfrak{n}$ is a non-zero vector of weight α_ℓ, $f_{\beta_\ell} \in \mathfrak{b}^-$ is a non-zero vector of weight $-\beta_\ell$. (Note that β_ℓ can be a positive root or zero, and in the latter case f_0 can be any non-zero vector in $\mathfrak{t}$.) That this vector has degree $i - j$ means that:

$$-\sum_{\ell=1}^{k} |\alpha_\ell| + \sum_{\ell=k+1}^{j} |\beta_\ell| = i - j. \tag{5.3.2}$$

Finally, note that:

$$\begin{cases} r_\ell \geqslant 1 + m|\alpha_\ell| & \text{if } 1 \leqslant \ell \leqslant k \\ r_\ell \geqslant 1 - m(|\beta_\ell| + 1) & \text{if } k < \ell \leqslant j. \end{cases}$$

by definition of $\mathring{I}_m$, and the fact that $m > 0$.

Then the $(L_0 + \check\rho)$-degree of an element as above is:

$$\sum_{\ell=1}^{k} -r_\ell + |\alpha_\ell| + \sum_{\ell=k+1}^{j} -r_\ell - |\beta_\ell| \leqslant$$

$$\sum_{\ell=1}^{k} -1 - (m-1)|\alpha_\ell| + \sum_{\ell=k+1}^{j} -1 + (m-1)|\beta_\ell| + m \overset{(5.3.2)}{=}$$

$$-j + (m-1)(i-j) + m(j-k) =$$

$$(m-1)i - mk.$$

If $m - 1 > 0$, then clearly this goes to $-\infty$ as i does.

To treat the general case (so additionally allowing $m = 1$), we need to bound k in terms of i.

For this, we let α_{max} denote the longest root of G. We then apply (5.3.2) and the fact that $j \geqslant 0$ (by its definition) to obtain:

$$i \geqslant i - j = -\sum_{\ell=1}^{k} |\alpha_\ell| + \sum_{\ell=k+1}^{j} |\beta_\ell| \geqslant -k|\alpha_{\max}|.$$

Therefore:

$$-k \leqslant \frac{i}{|\alpha_{max}|}$$

(safely assuming G is not a torus, so this fraction makes sense).

Applying this above, we find that the $(L_0 + \check{\rho})$ degrees are at most:

$$(m-1)i - mk \leqslant (m-1+\frac{m}{|\alpha_{max}|})i$$

which does indeed go to $-\infty$ as i does.[62]

A similar calculation treats $\mathrm{Av}_*^{\mathring{I}_m,\psi}(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi) \otimes \ell^{n,m}[(m-n)\Delta])$. We note that this averaging can be explicitly described as $\Gamma(G(K), \delta^\psi_{\mathring{I}_m \mathring{I}_n})^{\mathring{I}_n}$ where $\delta^\psi_{\mathring{I}_m \mathring{I}_n}$ was defined in §2.10, and where the notation indicates that we take global sections on the loop group with coefficients in this D-module, and take right $\mathring{I}_n$-invariant sections.[63]

$\square$

5.3.5. We have the following key observation.

Proposition 5.3.3. *The pro-object* $\{'\mathcal{W}_\kappa^n\}_{n\geqslant 0} \in \mathsf{Pro}(\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}))$ *canonically corepresents the functor* Ψ^{Whit}.

Proof. By definition, $'\mathcal{W}_\kappa^n \in \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod})$ corepresents the functor:

$$\mathcal{F} \mapsto C^\bullet(\mathrm{Lie}\,\mathring{I}_n, \iota_n^!(\mathcal{F}) \otimes -\psi) \otimes \ell^n[-n\Delta].$$

Note that this complex maps by restriction of invariants to:

$$C^\bullet(\mathrm{Ad}_{-n\check{\rho}(t)}\,\mathfrak{n}[[t]], \iota_n^!(\mathcal{F}) \otimes -\psi) \otimes \ell^n[-n\Delta].$$

If we had a shift by *positive* $n\Delta$ instead, this in turn would canonically map to $\Psi(\iota_n^!(\mathcal{F}))$ by definition of Ψ. As it is, it maps instead to:

$$\Psi(\iota_n^!(\mathcal{F}))[-2n\Delta] = \Psi^{\mathsf{Whit}}(\iota_n^!(\mathcal{F})).$$

Passing to the colimit in n, we get a canonical morphism:

$$\mathrm{colim}_n C^\bullet(\mathrm{Lie}\,\mathring{I}_n, \iota_n^!(\mathcal{F}) \otimes -\psi) \otimes \ell^n[-n\Delta] \to \mathrm{colim}_n \Psi^{\mathsf{Whit}}(\iota_{n,!}\iota_n^!(\mathcal{F})) = \Psi^{\mathsf{Whit}}(\mathcal{F}).$$

The left hand side is the functor corepresented by our pro-object, so we need to see that this morphism is an isomorphism.

This is a straightforward verification. By definition of Ψ^{Whit}, it suffices to show that for $M \in \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}$, the morphism:

$$\mathrm{colim}_n C^\bullet(\mathrm{Lie}\,\mathring{I}_n, M \otimes -\psi) \otimes \ell^n[n\Delta] \to \Psi(M) = \mathrm{colim}_n C^\bullet(\mathrm{Ad}_{-n\check{\rho}(t)}\,\mathfrak{n}[[t]], M \otimes -\psi) \otimes \ell^n[n\Delta]$$

is an isomorphism.

In the notation from the proof of Lemma 2.4.1 and using the same method, this follows from the identity:

---

[62]Of course, this only makes sense if G is not a torus. In that case, $\mathrm{gr}_i^{KK} = 0$ for $i < 0$, so the game is over before it even started.

[63]In other words, the κ-twisted D-module $\delta^\psi_{\mathring{I}_m \mathring{I}_n}$ does not descend to $G(K)/\mathring{I}_n$ because of the presence of the character ψ. But its underlying quasi-coherent sheaf does descend, and we are taking its global sections (which will still be acted on by $\widehat{\mathfrak{g}}_\kappa$); this is because the exponential D-module has trivial underlying multiplicative quasi-coherent sheaf.

$$\operatorname*{colim}_{m} C^{\bullet}(\operatorname{Lie} \mathring{I}_{n,m}, M) \xrightarrow{\simeq} C^{\bullet}(\operatorname{Ad}_{-n\check{\rho}(t)} \mathfrak{n}[[t]], M)$$

for $n > 0$, which is a straightforward verification using the fact that $\mathfrak{h}$–mod is a co/limit in the standard way for $\mathfrak{h}$ a profinite-dimensional Lie algebra. (We have omitted ψ because this holds for any $M \in \operatorname{Lie} \mathring{I}_n$–mod.)

$\square$

5.3.6.   We obtain the following important consequence of the above results.

Theorem 5.3.4. *The functor* Ψ^{Whit} *is t-exact. Its restriction to* $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^+$ *is conservative.*

Proof.

Step 1. By Proposition 5.3.3, Ψ^{Whit} is corepresented by the pro-object $\{'\mathcal{W}_\kappa^n\}_{n\geqslant 0}$. Because each of these objects lies in the heart of the t-structure, we obtain that Ψ^{Whit} is left t-exact.

Now note that $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ is compactly generated by the objects $'\mathcal{W}_\kappa^n$. Indeed, this follows from the co/limit formalism and the observation that for $n > 0$, $\operatorname{ind}_{\operatorname{Lie} \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)$ compactly generates $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_n, \psi}$ (by prounipotence of $\mathring{I}_n$).

Since these compact generators lie in $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^{\leqslant 0}$, to verify the fact that Ψ is right t-exact, it suffices to show that $\Psi^{\mathsf{Whit}}('\mathcal{W}_\kappa^n) \in \mathsf{Vect}^{\leqslant 0}$. But as we noted before, this object is $\mathcal{W}_\kappa^n$, which lies in $\mathsf{Vect}^\heartsuit$.

Step 2. Suppose $\mathcal{F} \in \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^{\geqslant 0}$ is an object with $\Psi^{\mathsf{Whit}}(\mathcal{F}) \in \mathsf{Vect}^{>0}$. By right completeness of the t-structure on $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$, the conservativeness will follow if we can show $\mathcal{F} \in \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^{>0}$.

Because the objects $'\mathcal{W}_\kappa^n \in \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit$ generate $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^{\leqslant 0}$ under colimits, it suffices so show that:

$$\underline{\operatorname{Hom}}_{\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})}('\mathcal{W}_\kappa^n, \mathcal{F}) \in \mathsf{Vect}^{>0}$$

for all n. Clearly this complex is in $\mathsf{Vect}^{\geqslant 0}$, so we need to show that its H^0 vanishes.

Observe that:

$$H^0 \underline{\operatorname{Hom}}_{\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})}('\mathcal{W}_\kappa^n, \mathcal{F}) = \operatorname{Hom}_{\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit}('\mathcal{W}_\kappa^n, H^0(\mathcal{F})).$$

Therefore, by Lemma 5.3.2, the map:

$$H^0 \underline{\operatorname{Hom}}_{\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})}('\mathcal{W}_\kappa^n, \mathcal{F}) \to H^0 \underline{\operatorname{Hom}}_{\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})}('\mathcal{W}_\kappa^{n+1}, \mathcal{F})$$

of restriction along $\alpha_{n,n+1}$ is *injective*. Therefore, it suffices to show that the colimit under n vanishes. But we have:

$$\operatorname*{colim}_{n} H^0 \underline{\operatorname{Hom}}_{\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})}('\mathcal{W}_\kappa^n, \mathcal{F}) = H^0\big(\operatorname*{colim}_{n} \underline{\operatorname{Hom}}_{\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})}('\mathcal{W}_\kappa^n, \mathcal{F})\big) = H^0 \Psi^{\mathsf{Whit}}(\mathcal{F}) = 0$$

by assumption, giving the result.

$\square$

5.4. **Affine Skryabin.** We now prove the result with which we began this section.

Proof of Theorem 5.1.1.

Step 1. The main step is to compute the heart of our t-structure on $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$.

We will do this using the following paradigm. Suppose $\mathcal{A}$ is a Grothendieck abelian category and $F : \mathcal{A} \to \mathsf{Vect}^\heartsuit$ is a conservative exact functor that commutes with colimits.

Recall that endomorphisms of the functor F can naturally be considered as a pro-vector space $\mathrm{End}(F) \in \mathsf{Pro}(\mathsf{Vect}^\heartsuit)$. To compute it explicitly, take $\{\mathcal{F}_i\} \in \mathcal{A}$ pro-representing the functor F and then evaluate $F(\mathcal{F}_i)$ as a pro-vector space. It is standard that $\mathrm{End}(F)$ is a topological chiral algebra and that the canonical functor:

$$\mathcal{A} \to \mathrm{End}(F)\text{–mod}(\mathsf{Vect}^\heartsuit)$$

is an equivalence. (cf. [BD2] §3.6.)

We apply this with $\mathcal{A} = \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit$ and $F = \Psi^{\mathsf{Whit},\heartsuit} : \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit \to \mathsf{Vect}^\heartsuit$. Note that $\Psi^{\mathsf{Whit},\heartsuit}$ is exact and conservative by Theorem 5.3.4. We want to show the topological chiral algebra defined by this data is $\mathcal{W}_\kappa^{as}$, i.e., the one associated with the vertex algebra $\mathcal{W}_\kappa$.

We have a canonical morphism of topological chiral algebras

$$\mathcal{W}_\kappa^{as} \to \mathrm{End}(\Psi^{\mathsf{Whit},\heartsuit})$$

because $\mathcal{W}_\kappa^{as}$ acts on the cohomologies of Ψ of any object of $\widehat{\mathfrak{g}}_\kappa\text{–mod}$ (cf. §4.4.3).

Therefore, to show that this map is an isomorphism, we just need to show it at the level of pro-vector spaces. Because $\{'\mathcal{W}_\kappa^n\}_{n\geqslant 0}$ corepresents $\Psi^{\mathsf{Whit},\heartsuit}$, $\mathrm{End}(\Psi^{\mathsf{Whit},\heartsuit})$ is the pro-vector space given by:

$$\{\Psi^{\mathsf{Whit}}('\mathcal{W}_\kappa^n)\}_{n\geqslant 0} = \{\mathcal{W}_\kappa^n\}_{n\geqslant 0} = \mathcal{W}_\kappa^{as}$$

where the last equality is Theorem 4.6.1 (3). Clearly this identification is compatible with the map above, so we obtain the claim.

Step 2. From here, the theorem is straightforward. By Proposition 5.2.3 (2) and the above, we have:

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^+ \simeq D^+(\mathcal{W}_\kappa\text{–mod}^\heartsuit).$$

Because $\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ is compactly generated by the objects:

$$'\mathcal{W}_\kappa^n \in \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^+$$

which correspond under this equivalence to $\mathcal{W}_\kappa^n \in D^+(\mathcal{W}_\kappa\text{–mod}^\heartsuit)$, we obtain the theorem by definition of $\mathcal{W}_\kappa\text{–mod}$.

$\square$

6. Free-field realization of the generalized vacuum representations

6.1. **Overview.**

6.1.1.   In this section, we give another construction of the modules $\mathcal{W}_\kappa^n$. Though the construction is interesting in its own right, it also plays a technical role in the proof of Theorem 7.3.2.

6.1.2.   Let $\kappa' = -\kappa + \kappa_{crit}$.[64] Let $\widehat{\mathfrak{t}}_{\kappa'}$ denote the Heisenberg extension:

---

[64]The correction by κ_{crit} plays an essentially negligible role in what follows; see [FG1] §10 and [Gai6] for some explanations why it is needed.

$$0 \to k \to \widehat{\mathfrak{t}}_{\kappa'} \to \mathfrak{t}((t)) \to 0.$$

(This is another name for the Kac-Moody extension associated to κ' considered as a symmetric bilinear form for $\mathfrak{t}$.) Let $\mathbb{V}_{\mathfrak{t},\kappa'}$ denote the vacuum representation $\mathrm{ind}_{\mathfrak{t}[[t]]}^{\widehat{\mathfrak{t}}_{\kappa'}}(k)$, considered as a vertex algebra.

Recall that there is an injective *free-field* homomorphism:

$$\varphi : \mathcal{W}_\kappa \to \mathbb{V}_{\mathfrak{t},\kappa'}$$

that is a map of vertex algebras; its construction is recalled in §6.2.3. In particular, this means that any module over $\widehat{\mathfrak{t}}_{\kappa'}$ can be considered as a module over $\mathcal{W}_\kappa$ by restriction.

6.1.3. Now let us revisit the problem from §4.1. We have the generalized vacuum representations:

$$\mathbb{V}_{\mathfrak{t},\kappa'}^n := \mathrm{ind}_{t^n\mathfrak{t}[[t]]}^{\widehat{\mathfrak{t}}_{\kappa'}}(k)$$

of the Heisenberg algebra. We can use these to construct cyclic modules over $\mathcal{W}_\kappa$: $\mathbb{V}_{\mathfrak{t},\kappa'}^n$ is a $\mathcal{W}_\kappa$-module by restriction along φ, and we can take the sub-$\mathcal{W}_\kappa$-module generated by the canonical vacuum vector in $\mathbb{V}_{\mathfrak{t},\kappa'}^n$.

The main result of this section is:

Theorem 6.1.1. *This construction produces $\mathcal{W}_\kappa^n$ equipped with its canonical vacuum vector.*

Remark 6.1.2. Note that because we are comparing cyclic modules with preferred generators, this theorem uniquely characterizes the isomorphism it describes.

The above result is quite useful in practice for computing the modules $\mathcal{W}_\kappa^n$ in cases where $\mathcal{W}_\kappa$ has an explicit description.

Example 6.1.3. For $\mathfrak{g} = \mathfrak{sl}_2$, this theorem and the explicit formulae[65] for φ from [FBZ] §15.4.14 together with the above result allow to recover the explicit description of the modules $\mathcal{W}_\kappa^n$ proposed in this case in §4.1.

Example 6.1.4. At the critical level $\kappa = \kappa_{crit}$, this theorem implies that $\mathcal{W}_\kappa^n$ is the structure sheaf of $\mathrm{Op}_{\check{G}}^{\leqslant n}$ under Feigin-Frenkel. See Lemma 7.3.3 for more on this.

6.1.4. *Proof sketch.* The proof of the theorem is based on a straightforward generalization of the map φ. Namely, we will construct maps:

$$\varphi_n : \mathcal{W}_\kappa^n \to \mathbb{V}_{\mathfrak{t},\kappa'}^n$$

of $\mathcal{W}_\kappa^n$-modules. We will show that it is injective and preserves vacuum vectors by constructing filtrations and computing this map at the associated graded level (recall that this is how the vacuum vector in $\mathcal{W}_\kappa^n$ was constructed).

Obviously this would suffice to prove the theorem: a generator-preserving injective map between cyclic modules is an isomorphism.

---

[65]For the reader's convenience, if $L_n = -t^{n+1}\partial_t$ in the Virasoro algebra and $h_i \in \widehat{\mathfrak{t}}_\kappa'$ is defined by the element $t^i \in k((t)) = \mathfrak{t}((t)) \subseteq \widehat{\mathfrak{t}}_\kappa'$ (recalling that the Heisenberg algebra has a canonical vector space splitting, i.e., it is defined by a 2-cocycle), these formulae say:

$$\varphi(L_n) = \sum_{i+j=n} : h_i h_j : -(n+1)\lambda h_n$$

for an appropriate scalar λ depending on the level. Here $: h_i h_j :$ is the normally-ordered product, so $h_i h_j$ if $j \geqslant i$ and $h_j h_i$ otherwise.

Remark 6.1.5 (Screening operators?). Dennis Gaitsgory has suggested that φ_n might be the first map in a resolution of $\mathcal{W}_\kappa^n$, as in the $n = 0$ case (at least in the irrational and critical level cases, see [FF3] and [FG2]). We record his idea here as a sort-of-conjecture.

6.2. The Wakimoto vertex algebra.

6.2.1.   The construction of φ passes through the theory of Wakimoto modules. Since we are trying to generalize this construction, we must review these. We also refer the reader to [FG1] §10-11 and [Fre1] for some other introductions.

We also use the theory of global sections of D-modules on the loop group, but only in a minor way. The reader familiar with [AG] (cf. also [FG1] §21 and [Gai6]) will have more than enough information at hand for these constructions.

Let I and I^- denote the Iwahori subgroups defined by B and B^-. We then form:

$$\Gamma(G(K), j_{*,dR}(\omega_{I \cdot I^-})$$

where this pushforward is as a κ-twisted D-module. Note that because $I \cdot I^- \subseteq G(O)$ is open (it is jets on the open cell BB^-), the theory of D-modules on the loop group normalizes this object to lie in cohomological degree 0. There is a left action on this vector space by $\widehat{\mathfrak{g}}_\kappa$ (with the central element acting by the identity), and a commuting right action by[66] $\widehat{\mathfrak{g}}_{-\kappa + 2\kappa_{crit}}$.

We define $\mathbb{W}_\kappa$ as the semi-infinite cohomology:

$$C^{\frac{\infty}{2}}\left(\mathfrak{n}^-((t)) + \mathfrak{b}^-[[t]], B^-(O), \Gamma\big(G(K), j_{*,dR}(\omega_{I \cdot I^-})\big)\right)$$

formed with respect to the right action. Because we have the commuting left action, $\mathbb{W}_\kappa \in \widehat{\mathfrak{g}}_\kappa$–mod.

Remark 6.2.1. In a suitable sense, this is global sections of the semi-infinite flag variety $\mathrm{Fl}^{\frac{\infty}{2}} = G(K)/N^-(K)T(O)$ with coefficients in the D-module on it induced by $j_{*,dR}(\omega_{I \cdot I^-}) \in D_\kappa(G(K))$. See [Ras6], where some of these ideas are introduced. (But I do not mean to suggest that this perspective is especially enlightening.)

There is a canonical morphism:

$$\mathbb{V}_\kappa \to \mathbb{W}_\kappa \tag{6.2.1}$$

induced by the composition:

$$\mathbb{V}_\kappa = \Gamma(G(K), \delta_{G(O)})^{G(O)} \to \Gamma(G(K), \delta_{G(O)})^{B^-(O)} \to$$

$$\Gamma(G(K), j_{*,dR}(\omega_{I \cdot I^-}))^{B^-(O)} \to C^{\frac{\infty}{2}}\left(\mathfrak{n}^-((t)) + \mathfrak{b}^-[[t]], B^-(O), \Gamma\big(G(K), j_{*,dR}(\omega_{I \cdot I^-})\big)\right). \tag{6.2.2}$$

Here $\delta_{G(O)}$ is the pushforward of $\omega_{G(O)}$, considered as a κ-twisted D-module, and invariants are for the right actions.

Remark 6.2.2. Factorization shows that $\mathbb{W}_\kappa$ is a vertex algebra, and $\mathbb{V}_\kappa \to \mathbb{W}_\kappa$ is a morphism of vertex algebras.

---

[66]We will only need the action of $\widehat{\mathfrak{b}}^-_{-\kappa + 2\kappa_{crit}}$, the induced central extension of $\mathfrak{b}^-((t))$. Moreover, the action of $\mathfrak{n}^-((t)) + \mathfrak{b}^-[[t]]$, which is substantially easier to construct, will play the main role.

6.2.2.   We now compute $\mathbb{W}_\kappa$ more explicitly. Note that:

$$\Gamma(G(K), j_{*,dR}(\omega_{I.I^-}))) = \Gamma(N(K), \delta_{N(O)}) \otimes \Gamma(B^-(K), \delta_{B^-(O)}).$$

This is compatible with the left action of $\mathfrak{n}((t))$ and the right action of $\widehat{\mathfrak{b}}_{-\kappa+2\kappa_{crit}}$. (Each of these global sections is usually called a *CDO* for the respective groups.)

Note that:

$$C^\bullet(\mathfrak{n}^-((t)) + \mathfrak{b}^-[[t]], B^-(O), \Gamma(B^-(K), \delta_{B^-(O)})) = \mathbb{V}_{t,\kappa'}.$$

Indeed, the invariants $B^-(O)$ leave us with a vacuum representation for a central extension of $\mathfrak{b}((t))$, and the rest of the semi-infinite cohomology reduces us to a central extension for T (cf. [AG] Theorem 5.5).

The calculation of the exact level has to do with finer points about Tate extensions: we refer to the sources above.[67]

So we find $\mathbb{W}_\kappa$ is isomorphic to a tensor product of the CDO for N and a Heisenberg algebra; in particular, it lies in cohomological degree zero.

Remark 6.2.3. The morphism $\mathbb{V}_\kappa \to \mathbb{W}_\kappa$ is injective: see [Fre1] Theorem 5.1. The argument is proved by constructing filtrations and passing to the associated graded; we will essentially reconstruct it (and generalize it) in what follows.

6.2.3. *Construction of the free-field homomorphism.* Note that:

$$\Psi(\Gamma(N(K), \delta_{N(O)})) = k$$

by a similar calculation as above. Therefore, $\Psi(\mathbb{W}_\kappa) = \mathbb{V}_{t,\kappa'}$.

By functoriality, we obtain the morphism $\varphi : \mathbb{W}_\kappa \to \mathbb{V}_{t,\kappa'}$ from (6.2.1). Factorization makes clear that φ is a morphism of vertex algebras. As in Remark 6.2.3, one can show that it is injective using filtrations; this argument will be generalized in what follows.

6.3. **Generalization to higher n.**

6.3.1.   We now wish to construct the maps φ_n. We do this through a straightforward generalization of the above, but adapted to the modules $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\,\mathring{I}_n}(\psi)$ in place of $\mathbb{V}_\kappa$.

For $n > 0$, we let $\delta^\psi_{\mathring{I}_n}$ denote the D-module on $G(K)$ given by pushforward from the character sheaf on $\mathring{I}_n$ defined by ψ; we normalize it to lie in cohomological degree 0. For $n = 0$, we let $\delta^\psi_{\mathring{I}_n}$ be the D-module considered before: the pushforward from $I \cdot I^-$.

Let $\mathring{I}^-_n$ denote $B^-(O) \cap \mathring{I}^-_n$.

Define $\mathbb{W}^n_\kappa$ as the semi-infinite cohomology:

$$C^{\frac{\infty}{2}}(\mathfrak{n}^-((t)) + \mathfrak{b}^-[[t]], \mathring{I}^-_n, \Gamma(G(K), \delta^\psi_{\mathring{I}_n}))$$

where the semi-infinite cohomology is again taken with respect to the right action. Note that $\mathbb{W}^n_\kappa \in \widehat{\mathfrak{g}}_\kappa$–mod again.

Writing $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\,\mathring{I}_n}(\psi)$ as $\Gamma(G(K), \delta^\psi_{\mathring{I}_n})^{\mathring{I}_n}$ (the invariants being for the right action), we obtain a map $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\,\mathring{I}_n}(\psi) \to \mathbb{W}^n_\kappa$, as in (6.2.2).

---

[67]In [FG1], there is a potentially frustrating typo in the beginning of §10.2 that might thwart the reader who turns there: what is denoted κ' there should be $-\kappa + 2\kappa_{crit}$ (the sign is wrong there in the second summand).

6.3.2. Let us compute $\mathbb{W}_\kappa^n$ more explicitly.

Let $\delta_{\mathring{I}_n^-} \in D_\kappa(B^-(K))$ denote the pushforward of $\omega_{\mathring{I}_n^-}$ with a cohomological shift to put it in the heart of the t-structure.

We use the notation $\mathring{I}_n^+$ for $\mathrm{Ad}_{-n\check{\rho}(t)} N(O) = \mathring{I}_n \cap N(K)$. Let $\delta_{\mathring{I}_n^+}^\psi \in D_\kappa(N(K))$ be the pushforward of the character sheaf on $\mathring{I}_n^+$ defined by ψ, again normalized to be in cohomological degree 0.

Then the triangular decomposition $\mathring{I}_n = \mathring{I}_n^+ \cdot \mathring{I}_n^-$ readily implies:

$$\Gamma(G(K), \delta_{\mathring{I}_n}^\psi) = \Gamma(N(K), \delta_{\mathring{I}_n^+}^\psi) \otimes \Gamma(B^-(K), \delta_{\mathring{I}_n^-})$$

compatible with the left $\mathfrak{n}((t))$ and right $\widehat{\mathfrak{b}}^-_{-\kappa+2\kappa_{crit}}$ actions. Note that:

$$C^{\frac{\infty}{2}}\left(\mathfrak{n}^-((t)) + \mathfrak{b}^-[[t]], \mathring{I}_n^-, \Gamma(B^-(K), \delta_{\mathring{I}_n^-})\right)$$

is $\mathbb{V}_{\mathfrak{t},\kappa'}^n$: this follows because $\mathrm{Lie}\,\mathring{I}_n \cap \mathfrak{t}[[t]] = t^n \mathfrak{t}[[t]]$.

6.3.3. *Generalized free-field morphisms.* Note that:

$$\Psi_n(\Gamma(N(K), \delta_{\mathring{I}_n^+}^\psi)) = k$$

for Ψ_n the Drinfeld-Sokolov functor defined relative to the lattice $\mathrm{Lie}\,\mathring{I}_n^+ \subseteq \mathfrak{n}((t))$, as in (4.2.2).

Combining this with the above, we obtain $\Psi_n(\mathbb{W}_\kappa^n) = \mathbb{V}_{\mathfrak{t},\kappa'}^n$. Factorization geometry makes $\Psi_n(\mathbb{W}_\kappa^n)$ a vertex module for $\Psi(\mathbb{W}_\kappa) = \mathbb{V}_{\mathfrak{t},\kappa'}^n$, and this isomorphism is compatible.

6.3.4. Now by functoriality, we obtain the desired map:

$$\varphi_n : \mathcal{W}_\kappa^n := \Psi_n(\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)) \to \Psi_n(\mathbb{W}_\kappa^n) = \mathbb{V}_{\mathfrak{t},\kappa'}^n.$$

This is obviously a morphism of $\mathcal{W}_\kappa$-modules by functoriality.

As in §6.1.4, it remains to show the following.

Lemma 6.3.1. *The morphism φ_n is injective and preserves vacuum vectors.*

Proof. Note that $\mathbb{W}_\kappa^n$ carries a canonical KK (i.e., Kazhdan-Kostant) filtration such that the morphism from $\mathrm{ind}_{\mathrm{Lie}\,\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)$ is filtered. Indeed, both are derived from the KK filtration on $\Gamma(G(K), \delta_{\mathring{I}_n}^\psi)$. The resulting filtration on $\Psi(\mathbb{W}_\kappa^n)$ is the PBW filtration on this Heisenberg module: this follows because $\mathfrak{t}((t))$ has zero $-\check{\rho}$-grading.

In particular, the KK filtrations on the source and target are both bounded from below.

At the associated level, we obtain a morphism:

$$\mathrm{Fun}(\mathcal{O}_{f+t^{-n}\,\mathrm{Ad}_{n\check{\rho}(t)}\,\mathfrak{b}[[t]]}/\mathrm{Ad}_{-n\check{\rho}(t)}\,N(O)) \to \mathrm{Fun}(\mathcal{O}_{t^{-n}\mathfrak{t}[[t]]}).$$

It is routine (and similar to the methods from §4) to see that this map is obtained by pullback from the *Miura transform*:

$$t^{-n}\mathfrak{t}[[t]] \xrightarrow{\xi \mapsto f+\xi} f + t^{-n}\,\mathrm{Ad}_{n\check{\rho}(t)}\,\mathfrak{b}[[t]]/\mathrm{Ad}_{-n\check{\rho}(t)}\,N(O).$$

So obviously this map preserves vacuum vectors: they correspond to the constant function with value 1. Since the filtrations on $\mathcal{W}_\kappa^n$ and $\mathbb{V}_{\mathfrak{t},\kappa'}^n$ begin in degree 0 with 1-dimensional gr_0, this implies that φ_n preserves vacuum vectors as well.

Then it remains to show that the Miura transform is dominant. Applying the isomorphism $t^n \operatorname{Ad}_{n\check{\rho}(t)}$, we are reduced to the $n = 0$ case, where it is well-known: over the open in $f + \mathfrak{b}[[t]]/N(O)$ corresponding to regular semisimple elements, the map is finite étale.

$\square$

7. Applications

7.1. **Overview.** In this section, we give some applications of the above results. First, we discuss how Theorem 5.1.1 provides a systematic framework for understanding exactness properties of the Drinfeld-Sokolov functor Ψ. Then we give a categorical form of the Feigin-Frenkel theorem, which is Theorem 7.3.2.

7.2. **Exactness results for Ψ.**

7.2.1. Our main general result is the following.

Theorem 7.2.1. *For every $n \geqslant 0$, the functor:*

$$\Psi[-n\Delta] : \widehat{\mathfrak{g}}_\kappa\text{--mod}^{\mathring{I}_n, \psi} \to \mathsf{Vect}$$

is t-exact. Moreover, for every $M \in \widehat{\mathfrak{g}}_\kappa\text{--mod}^{\mathring{I}_n, \psi, \heartsuit}$, the canonical morphism:

$$H^0((M \otimes \ell^n)^{\mathring{I}_n, \psi}) \to H^{-n\Delta}\Psi(M)$$

is injective; here ℓ^n is a determinant line as before, the superscript $\mathring{I}_n, \psi$ indicates invariants, and we included H^0 to emphasize that these are non-derived invariants.

Proof. The exactness follows Theorem 5.3.4 because $\Psi^{\mathsf{Whit}} \iota_{n,!} = \Psi[-2n\Delta]$, and $\iota_{n,!}[n\Delta]$ is t-exact. The injectivity follows immediately from Lemma 5.3.2.

$\square$

In the particular case $n = 0$, we obtain:

Corollary 7.2.2. *The functor $\Psi : \widehat{\mathfrak{g}}_\kappa\text{--mod}^{G(O)} \to \mathsf{Vect}$ is t-exact.*

Remark 7.2.3. At critical level, this is a part of [FG4] Theorem 3.2. At non-critical level (say for $\mathfrak{g}$ simple), this follows from Arakawa exactness, cf., below. (The relevant deduction is the proof of Proposition 2 in [FG5], although this reference is ostensibly at critical level.)

7.2.2. *Arakawa exactness.* We now show how the $n = 1$ case of the above recovers *Arakawa exactness.*[68]

Let I^- be the negative Iwahori group $G(O) \times_G B^-$, and let $\mathring{I}^-$ denote its prounipotent radical $G(O) \times_G N^-$.

Corollary 7.2.4 (Arakawa exactness). *The functor:*

$$\Psi[-\Delta + \dim(N)] : \widehat{\mathfrak{g}}_\kappa\text{--mod}^{\operatorname{Ad}_{-\check{\rho}(t)} \mathring{I}^-} \to \mathsf{Vect}$$

is t-exact.

---

[68]The ideas used here can be extended to reprove the main results from [Ara2] and other results from the representation theory of affine W-algebras. We plan to pursue this in future work.

Remark 7.2.5. This result generalizes [Ara2] Main Theorem 1 (1). First, *loc. cit.* actually uses $\mathrm{Ad}_{-\check{\rho}(t)} I^-$ instead of its prounipotent radical. Moreover, it assumes that there is $\mathbb{Z}$-grading given on our modules compatible with the L_0-grading on the Kac-Moody algebra – the above result removes this restriction (which is only substantial at critical level).

Proof of Corollary 7.2.4. Recall the main theorem of [BBM]: for any $\mathcal{C}$ acted on by G, the functor $\mathrm{Av}_!^{N,\psi}$ is defined on $\mathcal{C}^{N^-}$, and $\mathrm{Av}_!^{N,\psi} = \mathrm{Av}_*^{N,\psi}[2\dim N]$. (This is not how the authors formulate the result, but the proof goes through using the methods from the proof of Theorem 2.3.1, which was modeled on [BBM].)

We can write $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathrm{Ad}_{-\check{\rho}(t)} \overset{\circ}{I}{}^-}$ in two steps, by first taking invariants with respect to the conjugated first congruence subgroup $\mathrm{Ad}_{-\check{\rho}(t)} \mathcal{K}_1$, and then invariants with respect to $N^- = \mathrm{Ad}_{-\check{\rho}(t)} \overset{\circ}{I}{}^- / \mathrm{Ad}_{-\check{\rho}(t)} \mathcal{K}_1$.

Therefore, as in Lemma 5.2.1 by Lemmas B.2.2 and B.2.3, this implies that the functor:

$$\mathrm{Av}_*^{N,\psi}[\dim N] : \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathrm{Ad}_{-\check{\rho}(t)} \overset{\circ}{I}{}^-} \to \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\overset{\circ}{I}_1,\psi}$$

is t-exact.

Clearly $\Psi(M) = \Psi(\mathrm{Av}_*^{N,\psi}(M))$ for such $M \in \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathrm{Ad}_{-\check{\rho}(t)} \overset{\circ}{I}{}^-}$. Since $\Psi[-\Delta]$ is t-exact on $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\overset{\circ}{I}_1,\psi}$, we obtain the claim.

$\square$

7.3. **Feigin-Frenkel redux.**

7.3.1.   We now show a (long[69] anticipated) version of Feigin-Frenkel duality.

7.3.2.   Let $\check{G}$ denote the Langlands dual group to G, defined by some choice of Borel B, maximal torus T, and Chevalley generators $e_i \in \mathfrak{n}$. We obtain Langlands dual data for $\check{G}$, which we denote similarly.

Note that $\check{\mathfrak{t}} = \mathfrak{t}^\vee$, since T and $\check{T}$ are dual tori; this identification is compatible with the natural actions of the Weyl group W. Recall that a level κ is the same as a W-invariant bilinear form on $\mathfrak{t}$. Therefore, if κ is *non-degenerate*, then $\frac{1}{\kappa}$ makes sense as a level for $\check{\mathfrak{g}}$: κ is a W-invariant isomorphism $\mathfrak{t} \simeq \mathfrak{t}^\vee$, and $\frac{1}{\kappa}$ is its inverse.

Recall that for κ a fixed non-degenerate bilinear form, we can take limits of "many" constructions as $\kappa \to \infty$, i.e., the definitions of the topological enveloping algebra, and its category of representations, and the $\mathcal{W}_\kappa$-algebra, etc., extend naturally over $\mathbb{P}^1_\kappa$.

7.3.3.   For κ as above, we let $\check{\kappa}$ denote the level of $\check{\mathfrak{g}}$ given by:

$$\check{\kappa} := \frac{1}{\kappa - \kappa_{crit}} + \kappa_{crit}$$

where in the denominator we are using the critical level for $\mathfrak{g}$ and in the second term it is the critical level for $\check{\mathfrak{g}}$. Note that the map $\kappa \mapsto \check{\kappa}$ is involutive and sends κ_{crit} to ∞.

The Feigin-Frenkel duality theorem from [FF2] says:

$$\mathcal{W}_{\mathfrak{g},\kappa} \simeq \mathcal{W}_{\check{\mathfrak{g}},\check{\kappa}}$$

as vertex algebras. Here e.g. $\mathcal{W}_{\mathfrak{g},\kappa}$ is what we were denoting $\mathcal{W}_\kappa$ before, and the right hand side is the $\mathcal{W}$-algebra on the Langlands dual side with respect to the Kac-Moody extension defined by the dual level. This duality theorem makes sense and is defined in the limit $\kappa \to \infty$.

---

[69]Certainly it was written in [Gai2] from 2007, though it must have been anticipated earlier still.

Warning 7.3.1. In truth, I don't know where in the literature to find this statement: Feigin and Frenkel only prove it for irrational levels and critical/level ∞. It seems to be folklore[70] that it is true in this full generality: see e.g. [Ara3] Remark 5.24. We assume duality holds at all levels in what follows.

7.3.4. We now have the following form of Feigin-Frenkel.

Theorem 7.3.2 (Categorical Feigin-Frenkel duality). *There is a canonical equivalence of categories:*

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) \simeq \mathsf{Whit}(\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–mod}).$$

Here Whit on one side is with respect to $N(K)$, and on the other side is with respect to $\check{N}(K)$.

To prove this, observe that we have an equivalence of abelian categories $\mathcal{W}_{\mathfrak{g},\kappa}\text{–mod}^\heartsuit \simeq \mathcal{W}_{\check{\mathfrak{g}},\check{\kappa}}\text{–mod}^\heartsuit$ coming from usual Feigin-Frenkel. This fact combined with affine Skryabin (Theorem 5.1.1) gives Theorem 7.3.2 on bounded below derived categories. To deduce all of Theorem 7.3.2, we need to match the compact generators under this equivalence. This follows from:

Lemma 7.3.3. *Under Feigin-Frenkel duality, there is a unique isomorphism:*

$$\mathcal{W}^n_{\mathfrak{g},\kappa} \simeq \mathcal{W}^n_{\check{\mathfrak{g}},\check{\kappa}}$$

preserving vacuum vectors and compatible with Feigin-Frenkel duality.

Proof. The basic property satisfied by Feigin-Frenkel is that the diagram:

commutes, where the bottom arrow is the obvious isomorphism.

Therefore, this compatibility follows from Theorem 6.1.1.

$\square$

Remark 7.3.4. We are constantly neglecting twists involving forms on the disc by having chosen our dt. Of course, it is better to incorporate these twists systematically as in [Ras6] §2. They come out in the wash: Whittaker is properly defined with twists incorporated, as is this category of Kac-Moody representations, as is Feigin-Frenkel duality.

7.3.5. *Critical level.* For $\kappa = \kappa_{crit}$, the above gives:

Corollary 7.3.5. *There is an equivalence:*[71]

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–mod}) \simeq \mathsf{QCoh}(\mathrm{Op}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$$

for $\mathrm{Op}_{\check{G}}(\overset{\circ}{\mathcal{D}}) = (f + \mathfrak{b}((t)))dt/N(K)$ *the indscheme of opers on the punctured disc.*

For $n > 0$, *the subcategory* $\mathsf{Whit}^{\leq n}(\widehat{\mathfrak{g}}_{crit}\text{–mod}) \subseteq \mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–mod})$ *is the subcategory of quasi-coherent sheaves set-theoretically supported on* $\mathrm{Op}_{\check{G}}^{\leq n}$, *i.e., the subscheme of opers with singularities of order* $\leq n$.

---

[70]We learned an argument deducing the general duality theorem from the generic version from Edward Frenkel.

[71]Since $\mathrm{Op}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ is an ind-pro affine space, IndCoh defined in any sense coincides with QCoh.

The only thing still to explain is the calculation of $\mathsf{Whit}^{\leqslant n}$. This follows because we know $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_{crit}\text{–mod})$ is compactly generated by $\iota_{n,!}(\mathrm{ind}^{\widehat{\mathfrak{g}}_{crit}}_{\mathrm{Lie}\,I_n}(\psi))$, which goes under this equivalence to $\Psi^{\mathsf{Whit}}(\mathrm{ind}^{\widehat{\mathfrak{g}}_{crit}}_{\mathrm{Lie}\,\mathring{I}_n}(\psi))$. Up to shift and determinant twist, this is the structure sheaf of $\mathrm{Op}^{\leqslant n}_{\check{G}}$ by Lemma 7.3.3.

APPENDIX A. FILTRATIONS, HARISH-CHANDRA MODULES, AND SEMI-INFINITE COHOMOLOGY

A.1. **Overview.**

A.1.1.   In this appendix, we give a slightly non-standard construction of the (quantum) Drinfeld-Sokolov reduction $\Psi : \widehat{\mathfrak{g}}_{\kappa}\text{–mod} \to \mathsf{Vect}$, and discuss its compatibility with various filtrations. This material supports the calculations of §4.

A.1.2.   The treatment we give here is quite lengthy, but this does not reflect the seriousness of the contents. There is a small number of ideas, and we summarize them here as themes to keep an eye towards:

- Recall that the Drinfeld-Sokolov functor is defined as $M \mapsto C^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], M \otimes (-\psi))$, where $C^{\frac{\infty}{2}}$ is the *semi-infinite cohomology* functor. This functor should be thought of as "cohomology along $\mathfrak{n}[[t]]$ and homology along $\mathfrak{n}((t))/\mathfrak{n}[[t]]$," although this does not quite make sense. Here we recall that cohomology is well-behaved[72] for pro-finite dimensional Lie algebras, while homology is well-behaved for "discrete" Lie algebras (i.e., non-topologized ones).

  We give a slightly non-standard treatment of this semi-infinite cohomology functor, avoiding irrelevant Clifford algebras. Rather, we define:

$$C^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], -) := \underset{n \geqslant 0}{\mathrm{colim}}\, C^{\bullet}(\mathrm{Ad}_{-n\check{\rho}(t)}\,\mathfrak{n}[[t]], -)[\dim(\mathrm{Ad}_{-n\check{\rho}(t)}\,\mathfrak{n}[[t]]/\mathfrak{n}[[t]])].$$

  Here we recall that for finite-dimensional Lie algebras, Lie algebra cohomology and homology differ by a determinant twist and cohomological shift, so satisfy both covariant and contravariant functoriality (up to these twists and shifts) with respect to the Lie algebra. There is a relic of this for Lie algebra cohomology for profinite-dimensional Lie algebras, giving the structure morphisms in the above colimit. (So in fact, we should have included a twist by the line $\det(\mathrm{Ad}_{-n\check{\rho}(t)}\,\mathfrak{n}[[t]]/\mathfrak{n}[[t]])$ to make the above structure maps canonical.)

  So much attention is paid to the construction of such morphisms.

- The other major theme is *filtrations*.

  The first question is given a (PBW) filtered module over a Lie algebra, how is its co/homology filtered? What is its associated graded?

  But there is a more subtle point in working with (finite or affine) $\mathcal{W}$-algebras: one would like the associated graded of a 1-dimensional module corresponding to a character ψ of $\mathfrak{n}$ (or similarly for $\mathfrak{n}((t))$) to be the skyscraper sheaf at $\psi \in \mathfrak{n}^{\vee}$ in $\mathsf{QCoh}(\mathfrak{n}^{\vee}) = \mathrm{Sym}(\mathfrak{n})\text{–mod} = \mathrm{gr}_{\bullet}\,U(\mathfrak{n})\text{–mod}$. However, this is impossible: it is a graded module, so must be at the origin in $\mathfrak{n}^{\vee}$.

  The (standard) solution to this problem is the *Kazhdan-Kostant* method, which *twists* the filtration using $\check{\rho} : \mathbb{G}_m \to T$. This is the *only* canonical way of obtaining (non-derived) filtrations on $\mathcal{W}$-algebras.

---

[72]E.g., continuous, at least for an appropriate definition of the source category.

Finally, we emphasize the relationship between the PBW and Kazhdan-Kostant filtrations using *bifiltrations*; this material is used in §4 to settle a subtle homological algebra point regarding some Kazhdan-Kostant filtrations.

Each of these amount to completely elementary constructions and statements about Lie algebra co/homology, and its relation to Harish-Chandra conditions.

The reason this section is so long is rather out of a commitment to develop the theory with an emphasis on the categorical aspects. A primary reason for this is because, following [FG3], the derived category of $\mathfrak{g}[[t]]$ or $\mathfrak{g}((t))$-modules is subtle, and is better to understand through categories than (topological) algebras. One thing this requires, however, is some general formalism for working with filtrations on *categories* rather than on algebras and their modules.

This section also renders the theory in the IndCoh formalism of [GR2]. It has the advantage that it provides a robust framework for Lie algebra cohomology that does not rely on explicit formulae. But this accounts for some portion of the length: we have explained some elementary points in detail, in the hopes that this is instructive for understanding the formalism. We also hope that an IndCoh treatment gives the feeling why things are the way they are, and that they could never have been another way.

Finally, we advise the reader to look to §4.2.4, where we give another summary of what is actually needed from this section, which should also help identify what is most important here.

A.2. **Filtrations.**

A.2.1.   We begin with some abstract language about filtrations.

Definition A.2.1. Let $\mathcal{C} \in \mathsf{DGCat}_{cont}$ be given. A *filtration* on $\mathcal{C}$ is a datum of $\widetilde{\mathcal{C}} \in \mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)$–mod plus an isomorphism:

$$\mathcal{C} \simeq \mathsf{Fil}\,\mathcal{C} \underset{\mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)}{\otimes} \mathsf{QCoh}((\mathbb{A}^1_\hbar\backslash 0)/\mathbb{G}_m) = \underset{\mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)}{\otimes} \mathsf{Vect}.$$

Here $\mathbb{A}^1_\hbar$ is $\mathbb{A}^1$ with coordinate $\hbar$, and $\mathbb{G}_m$ acts by inverse[73] homotheties.

An object of $\mathsf{Fil}\,\mathcal{C}$ is called a *filtered* object of $\mathcal{C}$. Note that there is a canonical restriction functor $\mathsf{Fil}\,\mathcal{C} \to \mathcal{C}$. For $\mathcal{F} \in \mathcal{C}$, we refer to an extension $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to $\mathsf{Fil}\,\mathcal{C}$ as a *filtration* on $\mathcal{F}$.

We say a morphism $F : \mathcal{C} \to \mathcal{D}$ between filtered categories is *filtered* if we are given the data of $\mathsf{Fil}\,F : \mathsf{Fil}\,\mathcal{C} \to \mathsf{Fil}\,\mathcal{D}$ a $\mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)$-linear functor.

In the language of [Gai7], we would say $\mathsf{Fil}\,\mathcal{C}$ is a *sheaf of categories* over $\mathbb{A}^1_\hbar/\mathbb{G}_m$ with fiber $\mathcal{C}$ at the open point of this stack.

Notation A.2.2. For a filtered category[74] $\mathcal{C}$ as above, we let $\mathcal{C}^{cl}$ denote the fiber of $\mathsf{Fil}\,\mathcal{C}$ at 0; it is called the associated *semi-classical category*. Note that $\mathbb{G}_m$ acts weakly on $\mathcal{C}^{cl}$. For $\mathcal{F} \in \mathcal{C}$ filtered, we let $\mathrm{gr}_\bullet\,\mathcal{F}$ denote the induced object of $\mathcal{C}^{cl}$, obtained by taking the fiber at $\hbar = 0$. Note that $\mathrm{gr}_\bullet(\mathcal{F})$ comes from an object of $\mathcal{C}^{cl,\mathbb{G}_m,w}$, which we also denote by $\mathrm{gr}_\bullet(\mathcal{F})$. For a filtered functor $F : \mathcal{C} \to \mathcal{D}$, we obtain a functor $F^{cl} : \mathcal{C}^{cl} \to \mathcal{D}^{cl}$, which we refer to as the corresponding *semi-classical* functor.

Example A.2.3. Vect is canonically filtered, with $\mathsf{Fil}\,\mathsf{Vect} := \mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)$.[75] In this case, a filtered object is a $\mathbb{G}_m$-representation with a degree 1 endomorphism. A $\mathbb{G}_m$-representation is the same as

---

[73]This is so that $\hbar \in \Gamma(\mathbb{A}^1_\hbar, \mathcal{O}_{\mathbb{A}^1_\hbar})$ has weight 1 with respect to the $\mathbb{G}_m$-action: we remind that when a group acts on a scheme, there is an inverse sign in the formula for the induced action on the algebra of functions.

[74]We use this terminology, even though our categories are always of a special type: DG and cocomplete.

[75]So we can rewrite the original definition more evocatively by saying that a filtered category is a $\mathsf{Fil}\,\mathsf{Vect}$-module category (in DGCat_{cont}.

a $\mathbb{Z}$-graded vector space, and we suggestively denote the nth term of this graded vector space as $F_n(V)$. Then our degree 1 endomorphism is a sequence of maps $F_n(V) \to F_{n+1}(V)$.

The induced object of Vect is obtained by inverting this degree 1 endomorphism $\hbar$ and taking the degree 0 component: this is computed as the colimit $V := \operatorname{colim}_n F_n(V)$. (So in this formalism, filtrations are by definition exhaustive.) We compute $\operatorname{gr} V$ by taking the cokernel of $\hbar$ acting on $\oplus_n F_n V$, which is $\oplus_n \operatorname{Coker}(F_{n-1} V \to F_n V)$. In this example, we let $\operatorname{gr}_n V$ denote the nth summand.

Example A.2.4. The above example generalizes to general $\mathcal{C}$ in place of Vect: it has a canonical filtration defined by the category $\mathcal{C} \otimes \mathsf{QCoh}(\mathbb{A}^1_{\hbar}/\mathbb{G}_m)$, and the above calculations render as is. We refer to this as the *constant* filtration on $\mathcal{C}$.

Example A.2.5. For A a filtered associative DG algebra, the Rees construction provides an algebra object $A_{\hbar} \in \mathsf{QCoh}(\mathbb{A}^1_{\hbar}/\mathbb{G}_m)$ with generic fiber A. Then the category $\mathsf{Fil}\, A\text{–mod} := A_{\hbar}\text{–mod}(\mathsf{QCoh}(\mathbb{A}^1_{\hbar}/\mathbb{G}_m))$ provides a filtration on A–mod. Note that A–mod^{cl} is the DG category of $\operatorname{gr}_{\bullet} A$-modules. The induced functor A–mod $\to$ Vect is canonically filtered.

Example A.2.6. Suppose $\mathcal{C}$ carries a weak action of $\mathbb{G}_m$. Then we define:

$$\mathsf{Fil}\, \mathcal{C} := \mathcal{C}^{\mathbb{G}_m, w} \underset{\mathsf{Rep}(\mathbb{G}_m)}{\otimes} \mathsf{QCoh}(\mathbb{A}^1_{\hbar}/\mathbb{G}_m).$$

Here $\mathsf{Rep}(\mathbb{G}_m)$ acts on each of these categories as on any weak $\mathbb{G}_m$-invariants. This tensor product is considered as acted on by $\mathsf{QCoh}(\mathbb{A}^1_{\hbar}/\mathbb{G}_m)$ in the obvious way. Note that this really does define a filtration on $\mathcal{C}$, and that there is a canonical functor $\mathcal{C}^{\mathbb{G}_m, w} \to \mathsf{Fil}\, \mathcal{C}$, given by exterior product with the structure sheaf of $\mathbb{A}^1_{\hbar}/\mathbb{G}_m$.

Concretely, suppose that $\mathcal{C} = A$–mod for $A = \oplus_{i \in \mathbb{Z}} A^i$ a $\mathbb{Z}$-graded algebra. Note that A inherits a filtration from its grading: set $F_i A = \oplus_{j \leqslant i} A^j$.[76] Then $\mathsf{Fil}\, A$–mod is the DG category of filtered modules over this filtered algebra. In this case, functor A–mod$^{\mathbb{G}_m, w} \to \mathsf{Fil}\, A$–mod takes a graded module and creates a filtered one using the same construction as above.

A.2.2.   We will use the following terminology in what follows: a filtered vector space $F_{\bullet} V$ is *bounded from below* if for all $i \ll 0$, $F_i V = 0$. Note that the functor $\operatorname{gr}_{\bullet} : \mathsf{Fil}\, \mathsf{Vect} \to \mathsf{Vect}$ is conservative when restricted to the subcategory of filtered vector spaces with bounded below filtrations.[77]

A.3. **Finite-dimensional setting.**

A.3.1.   Our formalism for semi-infinite cohomology will be built from the finite-dimensional setting, so we spend a while discussing this case.

We fix $\mathfrak{h} \in \mathsf{Vect}^{\heartsuit}$ a finite-dimensional Lie algebra.

A.3.2.   The[78] PBW filtration on $U(\mathfrak{h})$ defines a filtration on $\mathfrak{h}$–mod with $\mathfrak{h}$–mod$^{cl} = \mathsf{QCoh}(\mathfrak{h}^{\vee})$.

Here is a more geometric perspective. Let $\exp(\mathfrak{h})$ denote the formal group associated with $\mathfrak{h}$. Recall that $\mathfrak{h}$–mod $= \mathsf{IndCoh}(\mathbb{B}\exp(\mathfrak{h}))$ so that the forgetful functor corresponds to !-pullback to a point.

We have an induced family $\mathfrak{h}_{\hbar}$ of Lie algebras over $\mathbb{A}^1_{\hbar}$ given by $\mathfrak{h} \otimes \mathcal{O}_{\mathbb{A}^1_{\hbar}}$ with the bracket given by $\hbar$ times the bracket coming from $\mathfrak{h}$. This family is obviously $\mathbb{G}_m$-equivariant, so defines $\mathsf{Fil}\, \mathfrak{h}$ a Lie

---

[76]Geometrically, this construction amounts to pullback along the map $\mathbb{A}^1_{\hbar}/\mathbb{G}_m \to \mathbb{B}\mathbb{G}_m$.

[77]Although boundedness from below does not make sense for an arbitrary filtered category, *(derived) completeness* of a filtration does. However, we will not use this notion in any significant level of generality, remarking only that it is not so straightforward to verify in many of our examples.

[78]Personally, I find the discussion that follows to be not so interesting, but its Harish-Chandra generalization (which we will discuss next) to at least be somewhat clarifying.

algebra over $\mathbb{A}^1_\hbar/\mathbb{G}_m$. The generic fiber of $\mathsf{Fil}\,\mathfrak{h}$ is $\mathfrak{h}$, and special fiber is the vector space $\mathfrak{h}$ *equipped with the abelian Lie algebra structure*. We remark that the latter abelian Lie algebra has underlying formal group $\mathfrak{h}^\wedge_0$, that is, the formal completion at 0 *of the vector space* $\mathfrak{h}$; we use this notation at various points to emphasize the abelian nature.

Therefore, we obtain $\mathsf{Fil}\,\mathfrak{h}\text{–}\mathsf{mod} := \mathsf{IndCoh}(\mathbb{B}\exp(\mathsf{Fil}\,\mathfrak{h}))$, which is a filtration on $\mathfrak{h}\text{–}\mathsf{mod}$. Note that the "special fiber" $\mathfrak{h}\text{–}\mathsf{mod}^{cl}$ is $\mathsf{IndCoh}(\mathbb{B}\mathfrak{h}^\wedge_0) = \mathrm{Sym}(\mathfrak{h})\text{–}\mathsf{mod} = \mathsf{QCoh}(\mathfrak{h}^\vee)$.

Warning A.3.1. For obvious reasons, we prefer to think of $\mathsf{IndCoh}(\mathbb{B}\mathfrak{h}^\wedge_0)$ as $\mathsf{QCoh}(\mathfrak{h}^\vee)$, remembering that this category actually arises from the classifying space of a commutative formal group. Part of remembering this means internalizing that a duality occurs in this transition, so various pullbacks and pushforwards get swapped. For example, the fiber functor $\mathrm{Sym}(\mathfrak{h})\text{–}\mathsf{mod} \to \mathsf{Vect}$ (which corresponds to !-pullback $\mathrm{Spec}(k) \to \mathbb{B}\mathfrak{h}^\wedge_0$) goes to global sections $\mathsf{QCoh}(\mathfrak{h}^\vee) \to \mathsf{Vect}$.

Example A.3.2. $U(\mathfrak{h})$ is filtered, and its associated graded is the structure sheaf of $\mathfrak{h}^\vee$.

Warning A.3.3. The construction $\mathfrak{h} \mapsto \mathsf{Fil}\,\mathfrak{h}$ has some subtleties. We can rewrite its output: $\mathsf{Fil}\,\mathfrak{h} \in \mathsf{LieAlg}(\mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m))$ is the same as a filtration on $\mathfrak{h}$ plus a Lie bracket on that filtered vector space extending the one on $\mathfrak{h}$. In our case, the filtration is $F_i\mathfrak{h} = 0$ for $i \leqslant 0$ and $F_i\mathfrak{h} = \mathfrak{h}$ for $i > 0$, so that the bracket $[-,-] : F_i\mathfrak{h} \otimes F_j\mathfrak{h} \to F_{i+j}\mathfrak{h}$ happens to factor through $F_{i+j-1}\mathfrak{h}$ (so that $\mathrm{gr}_\bullet\,\mathfrak{h}$ is abelian, as we expect). In particular, $\mathfrak{h} = \mathrm{gr}_1\,\mathfrak{h}$.

This is slightly confusing for $\mathfrak{h}$ being abelian: since the generic and special fibers are the same, it is easy to wrongly confuse this filtration with the constant one.

A.3.3. Now recall that the functor:

$$C_\bullet(\mathfrak{h}, -) : \mathfrak{h}\text{–}\mathsf{mod} \to \mathsf{Vect}$$

corresponds to the IndCoh-pushforward functor for $\mathbb{B}\exp(\mathfrak{h})$. Therefore, we see that this functor is naturally filtered, i.e., it extends to give $\mathsf{Fil}\,\mathfrak{h}\text{–}\mathsf{mod} \to \mathsf{Fil}\,\mathsf{Vect}$. Note that the corresponding semi-classical functor $\mathsf{QCoh}(\mathfrak{h}^\vee) \to \mathsf{Vect}$ is $*$-restriction to 0.

Remark A.3.4. Say $F_\bullet M \in \mathsf{Fil}\,\mathfrak{h}\text{–}\mathsf{mod}$, and for simplicity let us assume M is in the heart of the t-structure and that $F_iM \hookrightarrow F_{i+1}M$. Recall that $C_\bullet(\mathfrak{h}, M)$ is computed by the complex:

$$\ldots \to \Lambda^2\mathfrak{h} \otimes M \to \mathfrak{h} \otimes M \to M \to 0 \to 0 \to \ldots$$

with appropriate differentials. Then the resulting filtration on $C_\bullet(\mathfrak{h}, M)$ has F_i-term:

$$\ldots \to \Lambda^2\mathfrak{h} \otimes F_{i-2}M \to \mathfrak{h} \otimes F_{i-1}M \to F_iM \to 0 \to 0 \to \ldots$$

A.3.4. Moreover, recall that $\omega_{\mathbb{B}\exp(\mathfrak{h})}$ is *compact*, as is its extension $\omega_{\mathbb{B}\exp(\mathsf{Fil}\,\mathfrak{h})}$.[79]

Note that $\omega_{\mathbb{B}\exp(\mathfrak{h})}$ corresponds to the trivial representation in $\mathfrak{h}\text{–}\mathsf{mod}$. Therefore, the functor:

$$C^\bullet(\mathfrak{h}, -) : \mathfrak{h}\text{–}\mathsf{mod} \to \mathsf{Vect}$$

of Lie algebra cohomology has a canonical filtered structure with graded $\mathsf{QCoh}(\mathfrak{h}^\vee) \to \mathsf{Vect}$ given by the $(\mathsf{QCoh}, !)$-pullback to $0 \in \mathfrak{h}^\vee$, i.e., the (continuous) right adjoint to the pushforward functor from 0.

---

[79]This uses our assumptions on $\mathfrak{h}$ in a serious way. We recall that the standard filtration on ω of any formal prestack ("with deformation theory" in the [GR2] sense) gives a bounded free resolution in this case. This corresponds to the standard resolution of the trivial module for $U(\mathfrak{h})$ (or its Rees algebra).

Remark A.3.5. We retain the notation of Remark A.3.4. Recall that $C^\bullet(\mathfrak{h}, M)$ is computed by the complex:

$$\ldots \to 0 \to M \to \mathfrak{h}^\vee \otimes M \to \Lambda^2 \mathfrak{h}^\vee \otimes M \to \ldots$$

with appropriate differentials. Then the resulting filtration on $C^\bullet(\mathfrak{h}, M)$ has F_i-term:

$$\ldots \to 0 \to F_i M \to \mathfrak{h}^\vee \otimes F_{i+1} M \to \Lambda^2 \mathfrak{h}^\vee \otimes F_{i+2} M \to \ldots$$

Note that (unlike the case of Lie algebra homology) even if the filtration on M has $F_{-1} M = 0$, we may not have $F_{-1} C^\bullet(\mathfrak{h}, M) = 0$ (though $F_{-\dim \mathfrak{h} - 1} C^\bullet(\mathfrak{h}, M)$ will be zero).

A.3.5. We now recall the following fact.

Lemma A.3.6. *There is a canonical isomorphism of functors:*

$$C^\bullet(\mathfrak{h}, (-) \otimes \det(\mathfrak{h})[\dim \mathfrak{h}]) \simeq C_\bullet(\mathfrak{h}, -).$$

Moreover, suppose that we regard $\det(\mathfrak{h})[\dim \mathfrak{h}]$ as a filtered $\mathfrak{h}$-module with $F_{\dim \mathfrak{h} - i - 1} \det(\mathfrak{h})[\dim \mathfrak{h}] = 0$ and $F_{\dim \mathfrak{h} + i} \det(\mathfrak{h})[\dim \mathfrak{h}] = \det(\mathfrak{h})[\dim \mathfrak{h}]$ for all $i \geqslant 0$. Then this isomorphism extends to an isomorphism of filtered functors, g inducing the usual isomorphism-up-to-twist-and-shift between $*$ and $!$-restriction to the point $0 \in \mathfrak{h}^\vee$.

Proof. Recall that for any $M \in \mathfrak{h}$–mod (possibly non-filtered), $C^\bullet(\mathfrak{h}, M)$ has a canonical filtration $F_\bullet^{Chev} C^\bullet(\mathfrak{h}, M)$ (indexed by non-positive integers but bounded below) with $\mathrm{gr}_{-i}^{Chev} C^\bullet(\mathfrak{h}, M) = M \otimes \Lambda^i \mathfrak{h}^\vee[-i]$. Similarly, the functor of !-restriction along $0 \hookrightarrow \mathfrak{h}^\vee$ has a filtration with the same associated graded. One immediately sees that these "glue": $C^\bullet(\mathfrak{h}, -)$ has a canonical filtration *considered as a filtered functor.*

In particular, we obtain a natural transformation of filtered functors $\mathfrak{h}$–mod $\to$ Vect:

$$(-) \otimes \det(\mathfrak{h})^\vee[-\dim \mathfrak{h}] \to C^\bullet(\mathfrak{h}, -)$$

where on the left hand side $\det(\mathfrak{h})^\vee[-\dim \mathfrak{h}]$ is equipped with the filtration with one jump in degree $-\dim \mathfrak{h}$. Evaluating this on $U(\mathfrak{h})$, we claim that the composition:

$$\det(\mathfrak{h})^\vee[-\dim \mathfrak{h}] \to U(\mathfrak{h}) \otimes \det(\mathfrak{h})^\vee[-\dim \mathfrak{h}] \to C^\bullet(\mathfrak{h}, U(\mathfrak{h}))$$

is an isomorphism of filtered complexes. Indeed, it suffices to check this at the associated graded level, where the claim is standard.

Then observe that using the (filtered) bimodule structure on $U(\mathfrak{h})$, $C^\bullet(\mathfrak{h}, U(\mathfrak{h}))$ can actually be considered as a filtered $\mathfrak{h}$-module. We claim that the above computation is true with this extra structure, considering $\det(\mathfrak{h})^\vee$ as an $\mathfrak{h}$-module in the obvious way.

Indeed, considering $U(\mathfrak{h})$ as a filtered $\mathfrak{h}$-module via the *right* action, the morphism:

$$U(\mathfrak{h}) \otimes \det(\mathfrak{h})^\vee[-\dim \mathfrak{h}] \to C^\bullet(\mathfrak{h}, U(\mathfrak{h}))$$

is a tautologically a morphism of filtered $\mathfrak{h}$-modules. Moreover, by a standard argument, this is also true if we consider $U(\mathfrak{h})$ as a filtered $\mathfrak{h}$-module via the *adjoint* action. This shows the claim.

Now observe that by usual Morita theory, the datum of $C^\bullet(\mathfrak{h}, U(\mathfrak{h}))$ as a filtered $\mathfrak{h}$-module completely is equivalent to the datum of the filtered functor $C^\bullet$. So the result follows from the observation that:

$$C^\bullet(\mathfrak{h}, U(\mathfrak{h}) \otimes \det(\mathfrak{h})[\dim \mathfrak{h}]) = k = C_\bullet(\mathfrak{h}, U(\mathfrak{h}))$$

as filtered $\mathfrak{h}$-modules.

$\square$

Remark A.3.7. From the perspective of usual complexes: for M a complex of $\mathfrak{h}$-modules, Lie algebra homology is $M \otimes \Lambda^\bullet \mathfrak{h}$ with the appropriate grading and differential, while cohomology is $M \otimes \Lambda^\bullet \mathfrak{h}^\vee$. Noting that $\Lambda^i \mathfrak{h} = \det(\mathfrak{h}) \otimes \Lambda^{\dim \mathfrak{h} - i} \mathfrak{h}^\vee$ and matching up the differentials and gradings then gives the claim.

Remark A.3.8. This isomorphism amounts to the calculation of $C^\bullet(U(\mathfrak{h}))$ as a filtered complex acted on by $\mathfrak{h}$ (using the bimodule structure on $U(\mathfrak{h})$). Since the claim is that the result is 1-dimensional in some cohomological degree with filtration jumping in one degree, the isomorphism above is easy to pin down uniquely.

A.3.6.   We also observe that the above generalizes in the natural way to the setting of a morphism $\mathfrak{h}_1 \to \mathfrak{h}_2$ between (finite-dimensional, non-derived) Lie algebras. Here generalizing means that we obtain the previous discussion by considering the structure map $\mathfrak{h} \to 0$.

We draw attention to the following consequence of Lemma A.3.6 in this setting, which will play an important role in what follows. Let $\mathrm{ind}_{\mathfrak{h}_1}^{\mathfrak{h}_2} : \mathfrak{h}_1\text{–}\mathsf{mod} \to \mathfrak{h}_2\text{–}\mathsf{mod}$ denote the left adjoint to the forgetful functor. Note that by realizing $\mathrm{ind}_{\mathfrak{h}_1}^{\mathfrak{h}_2}$ as a IndCoh-pushforward, we find that it is naturally filtered with semi-classical functor the quasi-coherent pullback along $\mathfrak{h}_2^\vee \to \mathfrak{h}_1^\vee$.

Corollary A.3.9. *There is a canonical isomorphism of filtered functors:*

$$C^\bullet\left(\mathfrak{h}_2, \mathrm{ind}_{\mathfrak{h}_1}^{\mathfrak{h}_2}\left((-) \otimes \det(\mathfrak{h}_2/\mathfrak{h}_1)[\dim \mathfrak{h}_2/\mathfrak{h}_1]\right)\right) = C^\bullet(\mathfrak{h}_1, -).$$

Here $\det(\mathfrak{h}_2/\mathfrak{h}_1)[\dim \mathfrak{h}_2/\mathfrak{h}_1]$ *is considered as trivially filtered with a single jump in degree* $\mathfrak{h}_2/\mathfrak{h}_1$.

Remark A.3.10. If $\mathfrak{h}_1 \to \mathfrak{h}_2$ is not injective, the quotient above should be replaced by the cone.

A.3.7. *Harish-Chandra setting.* Now suppose that $(\mathfrak{h}, K)$ is a finite-dimensional Harish-Chandra pair, so $\mathfrak{h}$ is as above, K is an affine algebraic group acting on $\mathfrak{h}$, and we are given a K-equivariant morphism $\mathrm{Lie}(K) =: \mathfrak{k} \to \mathfrak{h}$.

Recall that a group always acts trivially on its classifying stack. Therefore, the induced action of K on $\mathbb{B}\exp(\mathfrak{h})$ factors through an action of K_{dR} on $\mathbb{B}\exp(\mathfrak{h})$. Using somewhat strange notation, we let[80] $\mathbb{B}(\mathfrak{h}, K)$ denote the quotient $\mathbb{B}\exp(\mathfrak{h})/K_{dR}$. We let $\mathfrak{h}\text{–}\mathsf{mod}^K$ denote $\mathsf{IndCoh}(\mathbb{B}\exp(\mathfrak{h})/K_{dR})$, noting that this category is tautologically the K-equivariant category for the induced strong action of K on $\mathfrak{h}\text{–}\mathsf{mod}$.

Example A.3.11. For the Harish-Chandra pair $(\mathfrak{k}, K)$, we obtain $\mathbb{B}(\mathfrak{k}, K) = \mathbb{B}K$.

Example A.3.12. Note that the projection map $\mathrm{Spec}(k) \to \mathbb{B}(\mathfrak{h}, K)$ defines a group $(\mathfrak{h}, K)$ whose classifying stack is as notated. If $\mathfrak{h} = \mathrm{Lie}(H)$ for $K \subseteq H$, then $(\mathfrak{h}, K)$ is the formal completion of K in H.

A.3.8.   Now we recall that for any ind-affine nil-isomorphism $f : X \to Y$ of prestacks, [GR2] §IV.5.2 associates a *deformation* of this map, i.e., a prestack $Y_\hbar$ over $\mathbb{A}^1_\hbar/\mathbb{G}_m$ and an $\mathbb{A}^1_\hbar/\mathbb{G}_m$-morphism $X \times \mathbb{A}^1_\hbar/\mathbb{G}_m \to Y_\hbar$ giving f over the generic point.

For example, for $\mathrm{Spec}(k) \to \mathbb{B}\exp(\mathfrak{h})$, we obtain the deformation $\mathbb{B}\mathsf{Fil}\,\mathfrak{h}$ corresponding to the PBW filtration.

---

[80]We do not include exp in the notation because we believe that with the comma, there is no risk for confusing the vector space underlying $\mathfrak{h}$ and its associated formal group, the way there could be in the notation $\mathbb{B}\mathfrak{h}$.

For a Harish-Chandra datum as above, we obtain a deformation $\mathbb{B}\mathsf{Fil}\,(\mathfrak{h}, K)$ associated with the map $\mathbb{B}K \to \mathbb{B}(\mathfrak{h}, K)$. The special fiber of this deformation is $(\mathbb{B}(\mathfrak{h}/\mathfrak{k})_0^\wedge)/K$, where we note that $(\mathfrak{h}/\mathfrak{k})_0^\wedge$ is[81] $\exp(\mathfrak{h}/\mathfrak{k})$ with $\mathfrak{h}/\mathfrak{k}$ *considered as an abelian Lie algebra.*

Therefore, we obtain a filtration on $\mathfrak{h}\text{–}\mathsf{mod}^K$ with $\mathfrak{h}\text{–}\mathsf{mod}^{K,cl} = \mathsf{QCoh}((\mathfrak{h}/\mathfrak{k})^\vee/K)$.

Example A.3.13. When $\mathfrak{k} \xrightarrow{\simeq} \mathfrak{h}$, this is the constant filtration on $\mathsf{Rep}(K)$ (in the sense of Example A.2.4).

A.3.9.   From now on, we assume $\mathfrak{k} \to \mathfrak{h}$ is injective.

A.3.10. IndCoh pushforward and pullback along the ind-affine nil-isomorphism $\mathbb{B}K \to \mathbb{B}(\mathfrak{h}, K)$ induces a pair of adjoint functors:

$$\mathrm{ind} = \mathrm{ind}_{\mathfrak{k}}^{\mathfrak{h}} : \mathsf{Rep}(K) \rightleftarrows \mathfrak{h}\text{–}\mathsf{mod}^K : \mathrm{Oblv}\,.$$

These functors are evidently compatible with filtrations. At the semi-classical level, they induce the adjunction:

$$\mathsf{Rep}(K) \rightleftarrows \mathsf{QCoh}((\mathfrak{h}/\mathfrak{k})^\vee/K)$$

given by pullback and pushforward along the structure map $(\mathfrak{h}/\mathfrak{k})^\vee/K \to \mathbb{B}K$.

In particular, we find that $\mathfrak{h}\text{–}\mathsf{mod}^K$ admits a canonical t-structure for which $\mathrm{Oblv} : \mathfrak{h}\text{–}\mathsf{mod}^K \to \mathsf{Rep}(K)$ is t-exact, and that the functor ind is t-exact for this t-structure as well.

A.3.11.   We focus on the case of *Harish-Chandra cohomology.*

Using the standard resolution of $\omega_{\mathbb{B}(\mathfrak{h},K)}$ induced by the ind-affine nil-isomorphism $\mathbb{B}K \to \mathbb{B}(\mathfrak{h}, K)$, we find that $\omega_{\mathbb{B}(\mathfrak{h},K)}$ is compact. Moreover, this compactness remains true (for the same reason) for $\omega_{\mathbb{B}\mathsf{Fil}\,(\mathfrak{h},K)}$.

Therefore, mapping out of it defines a filtered functor:

$$C^\bullet(\mathfrak{h}, K, -) : \mathfrak{h}\text{–}\mathsf{mod}^K \to \mathsf{Vect}.$$

On associated, graded, this functor is given by taking !-restriction along $0/K \hookrightarrow (\mathfrak{h}/\mathfrak{k})^\vee/K$, and then global sections (i.e., group cohomology for K).

Example A.3.14. If $\mathfrak{k} \xrightarrow{\simeq} \mathfrak{h}$, then $C^\bullet(\mathfrak{k}, K, -)$ computes group cohomology $\mathfrak{k}\text{–}\mathsf{mod}^K = \mathsf{Rep}(K) \to \mathsf{Vect}$. The filtration on this functor is the constant one.

Example A.3.15. If K is unipotent, then $C^\bullet(\mathfrak{h}, K, -)$ coincides with $C^\bullet(\mathfrak{h}, -)$ (composed with the forgetful functor $\mathfrak{h}\text{–}\mathsf{mod}^K \to \mathfrak{h}\text{–}\mathsf{mod}$). However, the filtered structures are substantially affected by the presence of K, as can be seen e.g. by looking at the semi-classical level.

Remark A.3.16. We explain the above constructions using usual complexes. Suppose for simplicity that K is unipotent, so $C^\bullet(\mathfrak{h}, K, -) = C^\bullet(\mathfrak{h}, -)$ as a non-filtered functor. In the notation of Remark A.3.4, the filtration $F_\bullet M$ is a filtration in $\mathfrak{h}\text{–}\mathsf{mod}^K$ if the $F_i M$ are K-submodules of M, i.e., if it induces a filtration on the image of $M \in \mathfrak{k}\text{–}\mathsf{mod}^K = \mathsf{Rep}(K)$.

In this case, we obtain a filtration on the cohomological Chevalley complex with ith term:

$$\ldots \to 0 \to F_i M \to (\mathfrak{h}/\mathfrak{k})^\vee \otimes F_{i+1} M + \mathfrak{h}^\vee \otimes F_i M \to \Lambda^2(\mathfrak{h}/\mathfrak{k})^\vee \otimes F_{i+2} M + (\mathfrak{h}/\mathfrak{k})^\vee \wedge \mathfrak{h}^\vee \otimes F_{i+1} M + \Lambda^2 \mathfrak{h}^\vee \otimes F_i M \to \ldots$$

---

[81]We will not need this in practice, but if $\mathfrak{k} \to \mathfrak{h}$ is not injective, then $\mathfrak{h}/\mathfrak{k}$ should be considered as a complex and understood as the cone; the corresponding scheme should be understood in the usual sense.

A.3.12. We now give versions of Corollary A.3.9 in this setting.

Lemma A.3.17. *Let* ind *denote the induction functor* $\mathsf{Rep}(K) \to \mathfrak{h}\text{--}\mathsf{mod}^K$. *There is a canonical isomorphism:*

$$C^\bullet\Big(\mathfrak{h}, K, \mathrm{ind}\left((-) \otimes \det(\mathfrak{h}/K)[\dim \mathfrak{h}/\mathfrak{k}]\right)\Big) = C^\bullet(K, -) : \mathsf{Rep}(K) \to \mathsf{Vect}$$

of filtered functors. As before, $\det(\mathfrak{h}/\mathfrak{k})[\dim \mathfrak{h}/\mathfrak{k}])$ *is filtered with a single jump in degree* $\dim \mathfrak{h}/\mathfrak{k}$.

Proof. The proof is similar to Lemma A.3.6: we need to evaluate both sides on the regular representation[82] $\mathcal{O}_K$ and consider the result as a filtered K-module. For the right hand side, we obtain the trivial representation (in cohomological degree 0, with the filtration jumping only in degree 0).

Then one calculates $C^\bullet(\mathfrak{h}, K, \mathrm{ind}\, \mathcal{O}_K)$ as $\det(\mathfrak{h}/\mathfrak{k})^\vee[-\dim \mathfrak{h}/\mathfrak{k}]$ using standard filtrations as in Lemma A.3.6, which gives the claim. Here we note that for $C^\bullet(\mathfrak{h}, K, -)$ (considered merely as a filtered functor) the standard filtration has associated graded:

$$\mathrm{gr}_{-i} C^\bullet(\mathfrak{h}, K, -) = C^\bullet(K, (-) \otimes \Lambda^i(\mathfrak{h}/\mathfrak{k})^\vee[-i]).$$

$\square$

Suppose now that $(\mathfrak{h}_1, K) \to (\mathfrak{h}_2, K)$ is a morphism of Harish-Chandra pairs. Note that IndCoh pushforward defines a functor $\mathrm{ind}_{\mathfrak{h}_1}^{\mathfrak{h}_2} : \mathfrak{h}_1\text{--}\mathsf{mod}^K \to \mathfrak{h}_2\text{--}\mathsf{mod}^K$ compatible with forgetful functors and satisfying similar properties as before.

We have the following generalization of Lemma A.3.17.

Lemma A.3.18. *There is a canonical isomorphism of filtered functors:*

$$C^\bullet\Big(\mathfrak{h}_2, K, \mathrm{ind}_{\mathfrak{h}_1}^{\mathfrak{h}_2}\left((-) \otimes \det(\mathfrak{h}_2/\mathfrak{h}_1)[\dim \mathfrak{h}_2/\mathfrak{h}_1]\right)\Big) = C^\bullet(\mathfrak{h}_1, K, -) : \mathfrak{h}_1\text{--}\mathsf{mod}^K \to \mathsf{Vect}.$$

As before, $\det(\mathfrak{h}_2/\mathfrak{h}_1)[\dim \mathfrak{h}_2/\mathfrak{h}_1]$ *is filtered with a single jump in degree* $\dim \mathfrak{h}_2/\mathfrak{h}_1$.

The argument is similar to those that preceded it, so we omit it.

A.3.13. *Group actions on filtered categories.* We now put the above into a more conceptual framework whose perspective will be convenient at some points. We essentially rewrite the above using a version of the theory of group actions on categories.

Associated with the ind-affine nil-isomorphism, $K \to K_{dR}$ one has the deformation $\mathsf{Fil}\, K_{dR}$, which is a *relative group*[83] *prestack* over $\mathbb{A}_\hbar^1/\mathbb{G}_m$ with generic fiber K_{dR} and special fiber $K \ltimes \mathbb{B}\mathfrak{k}_0^\wedge$. Note that because $\mathfrak{k}_0^\wedge$ is a commutative formal group, $\mathbb{B}\mathfrak{k}_0^\wedge$ actually is a commutative group prestack (acted on by K).

Observe that we have a fiber sequence of relative groups:

$$1 \to \exp(\mathsf{Fil}\, \mathfrak{k}) \to \mathsf{Fil}\, K \to \mathsf{Fil}\, K_{dR} \to 1 \tag{A.3.1}$$

where $\mathsf{Fil}\, K$ just means $K \times \mathbb{A}_\hbar^1/\mathbb{G}_m$.

Note that the usual convolution monoidal structure makes $\mathsf{IndCoh}(\mathsf{Fil}\, K_{dR}) \in \mathsf{Alg}(\mathsf{QCoh}(\mathbb{A}_\hbar^1/\mathbb{G}_m)\text{--}\mathsf{mod})$.

Definition A.3.19. Let $\mathcal{C}$ be a filtered category. A *(strong) action of* K *on* $\mathsf{Fil}\, \mathcal{C}$ is a $\mathsf{IndCoh}(\mathsf{Fil}\, K_{dR})$-module structure on $\mathsf{Fil}\, \mathcal{C} \in \mathsf{QCoh}(\mathbb{A}_\hbar^1/\mathbb{G}_m)\text{--}\mathsf{mod}$.[84]

---

[82]We are lazily not distinguishing between $\mathcal{O}_K$ considered as a sheaf and its global sections.

[83]One way to see that the deformation formalism from [GR2] forces a (relative) group structure is to instead work with the classifying prestacks.

[84]One may work equivalently with co-actions by duality. This has the advantage that one may equate IndCoh with QCoh using formal smoothness, and then at least try to forget IndCoh.

Warning A.3.20. There is a redundancy in the terminology: K acts on $\mathsf{Fil}\,\mathcal{C}$ could wrongly be understood to mean that $\mathsf{Fil}\,\mathcal{C} \in \mathsf{DGCat}_{cont}$ is a $D(K)$-module category. However, we are confident that there is no risk for confusion in our use of the above terminology: the meaning is completely determined by whether or not there is a Fil in front of what is acted upon.

We also note that there is no such risk for *weak* actions, i.e., if K acts on $\mathsf{Fil}\,\mathcal{C}$, then K acts weakly on $\mathsf{Fil}\,\mathcal{C}$ in the sense that $\mathsf{Fil}\,\mathcal{C}$ is a $\mathsf{QCoh}(K)$-module category.

Note that if K acts on $\mathsf{Fil}\,\mathcal{C}$, then K in particular acts on $\mathcal{C}$ (in the usual sense).

Moreover, $\mathcal{C}^{cl}$ receives an action of $\mathsf{IndCoh}(K \ltimes \mathbb{B}\mathfrak{k}_0^\wedge)$. This is equivalent to saying that $\mathsf{QCoh}(\mathfrak{k}^\vee)$ (with the usual symmetric monoidal structure) acts on $\mathcal{C}^{cl}$, and K acts weakly on $\mathcal{C}^{cl}$, and these two actions are compatible under the adjoint action of K on $\mathfrak{k}^\vee$. In particular, $\mathcal{C}^{cl,K,w}$ may be thought of as a sheaf of categories on $\mathfrak{k}^\vee/K$.

Using the "trivial" action of K on $\mathsf{Fil}\,\mathsf{Vect}$ as a filtered category, we obtain an invariants and coinvariants formalism. By the same arguments as in [Ber] §2, the two are canonically identified.

In particular, we obtain a filtration on $\mathcal{C}^K$ with $\mathcal{C}^{K,cl} = (\mathcal{C}^{cl} \underset{\mathsf{QCoh}(\mathfrak{k}^\vee)}{\otimes} \mathsf{Vect})^{K,w}$, where $\mathsf{QCoh}(\mathfrak{k}^\vee)$ acts on Vect through the restriction to 0 functor. Geometrically, this means we take our sheaf of categories on $\mathfrak{k}^\vee/K$, restrict to $0/K$, and then take global sections.

The adjoint functors $\mathrm{Oblv} : \mathcal{C}^K \rightleftarrows \mathcal{C} : \mathrm{Av}_*$ carry natural filtrations. semi-classically, Oblv^{cl} is the functor:

$$(\mathcal{C}^{cl} \underset{\mathsf{QCoh}(\mathfrak{k}^\vee)}{\otimes} \mathsf{Vect})^{K,w} \to \mathcal{C}^{cl} \underset{\mathsf{QCoh}(\mathfrak{k}^\vee)}{\otimes} \mathsf{Vect} \to \mathcal{C}^{cl}$$

where the first functor is forgetting and the second is $*$-pushforward along $0 \hookrightarrow \mathfrak{k}^\vee$. Of course, the semi-clclassical Av_*^{cl} is the right adjoint to the functor we just described: $!$-pullback $0 \hookrightarrow \mathfrak{k}^\vee$ and then $*$-pushforward to $\mathbb{B}K$.

Example A.3.21. We can reformulate some of our earlier constructions in saying that K acts on $\mathsf{Fil}\,\mathfrak{h}\mathrm{-mod}$ for a Harish-Chandra pair $(\mathfrak{h}, K)$. The associated filtration on $\mathfrak{h}\mathrm{-mod}^K$ is our earlier one.

A.4. **Derived categories.** The following result will play an important role for us, but may be skipped at first pass.

Lemma A.4.1 (Bernstein-Lunts, [BL])**.** $\mathfrak{h}\mathrm{-mod}^K$ *is the derived category of its heart. (Here we continue to assume $\mathfrak{k} \to \mathfrak{h}$ is injective.)*

Proof. If $\mathfrak{k} \to \mathfrak{h}$ is an isomorphism, then $\mathfrak{h}\mathrm{-mod}^K = \mathsf{Rep}(K)$, and this is a general property about algebraic stacks (see e.g. [Gai1] Proposition 5.4.3).

In general, we have a t-exact forgetful functor $\mathrm{Oblv} : \mathfrak{h}\mathrm{-mod}^K \to \mathsf{Rep}(K)$. Moreover, this functor admits a left adjoint ind, geometrically given by IndCoh-pushforward.

Now observe that our hypothesis implies that the tangent complex of the morphism $\mathbb{B}K \to \mathbb{B}(\mathfrak{h}, K)$ is concentrated in cohomological degree 0: it is $\mathfrak{h}/\mathfrak{k}$ considered as a K-representation. Therefore, the monad $\mathrm{Oblv} \circ \mathrm{ind}$ has a standard filtration with associated graded given by tensoring with $\mathrm{Sym}(\mathfrak{h}/\mathfrak{k})$, so in particular, this monad is t-exact. Since Oblv is t-exact, we find that ind is as well.

Therefore, we obtain a similar pair of adjoint functors between $D(\mathsf{Rep}(K)^\heartsuit) = \mathsf{Rep}(K)$ and $D(\mathfrak{h}\mathrm{-mod}^{K,\heartsuit})$. Both forgetful functors $\mathfrak{h}\mathrm{-mod}^K \to \mathsf{Rep}(K)$ and $D(\mathfrak{h}\mathrm{-mod}^{K,\heartsuit}) \to \mathsf{Rep}(K)$ are conservative and commute with colimits, so are monadic. Then we observe that the monads on $\mathsf{Rep}(K)$ are naturally identified, giving the claim.

$\square$

A.5. **Kazhdan-Kostant twists.**

A.5.1.   We now discuss how to render the standard solution to a standard problem in our framework.

We begin by describing the issue. Suppose $\mathfrak{g}$ and $\mathfrak{n}$ are as usual, and $\psi : \mathfrak{n} \to k$ is a non-degenerate character. We choose G-equivariant symmetric $\mathfrak{g} \simeq \mathfrak{g}^\vee$ and take $f \in \mathfrak{g} = \mathfrak{g}^\vee$ the principal nilpotent mapping to $\psi \in \mathfrak{n}^\vee$.

Recall that $\mathcal{W}^{fin}\text{–mod} \simeq \mathfrak{g}\text{–mod}^{N,\psi}$, and that $\mathcal{W}^{fin}$ is filtered with associated graded being the algebra of functions on $f + \mathfrak{b}/N$. In particular, $C^\bullet(\mathfrak{n}, (-\psi) \otimes \text{ind}_{\mathfrak{n}}^{\mathfrak{g}}(\psi))$ is filtered with associated graded being this algebra of functions.

However, this filtration is *not* induced by the obvious PBW filtration on $\text{ind}_{\mathfrak{n}}^{\mathfrak{g}}(\psi)$. Indeed, suppose more generally that M is any (PBW) filtered $\mathfrak{n}$-module. Then the induced filtration on $M \otimes -\psi$ *has the same associated graded* as M. (So in the case above, we will see the DG algebra $\Gamma(\mathfrak{b}/N, \mathcal{O}_{\mathfrak{b}/N})$ instead.)

The issue is with the filtration on the 1-dimensional representation ψ: jumping in a single degree, its associated graded is going to be the augmentation module of $\text{Sym}(\mathfrak{n})$. In other words, its associated graded will be a $\mathbb{G}_m$-equivariant quasi-coherent sheaf on $\mathfrak{n}^\vee$, so it must be the skyscraper at the origin, though we would rather see the skyscraper at $\psi \in \mathfrak{n}^\vee$.

The solution[85] to this problem is to use $\check{\rho}$ to modify the filtration on $U(\mathfrak{n})$, so that its associated graded is again $\text{Sym}(\mathfrak{n})$ (but with a modified grading!), and the module ψ is filtered with associated graded being the skyscraper at $\psi \in \mathfrak{n}^\vee$ instead. Namely, we set:

$$F_i^{KK}U(\mathfrak{n}) = \bigoplus_j F_{i-j}^{PBW}U(\mathfrak{n}) \cap U(\mathfrak{n})^j$$

$$U(\mathfrak{n})^j := \{x \in U(\mathfrak{n}) \mid \text{Ad}_{-\check{\rho}(\lambda)}(x) = \lambda^j \cdot x\}.$$

We emphasize that the grading used here is induced by $-\check{\rho} : \mathbb{G}_m \to G^{ad}$, not $\check{\rho}$ itself (so $\mathfrak{n}$ is negatively graded). For example, for $i \in \mathcal{I}_G$, $e_i \in F_0^{KK}U(\mathfrak{n})$, and for α a general root, $e_\alpha \in F_{1-(\check{\rho},\alpha)}^{KK}U(\mathfrak{n})$.

One has a similar filtration on $U(\mathfrak{g})$. Moreover, these filtrations naturally induce the "correct" filtration on $\mathcal{W}^{fin}$ (e.g., it is a filtration in the abelian category, not just the derived category).

Our present goal is to render the above ideas in the categorical framework. We will begin by discussing some generalities, and then apply these to the example of Harish-Chandra modules.

Warning A.5.1. We immediately see that Kazhdan-Kostant filtrations are typically unbounded from below and are not complete filtrations. This causes some technical problems (e.g., in §4), and requires care.

A.5.2.   We begin by discussing how to twist a filtration by a grading to obtain a new filtration. Motivated by our particular concerns, we use the notation PBW to indicate an "old" filtration and KK to indicate a "new" filtration.

So suppose that $F_\bullet^{PBW}V \in \text{Fil Rep}(\mathbb{G}_m)$, i.e., V is equipped with a grading $V = \oplus_j V^j$ and a filtration $F_\bullet^{PBW}V$ *as a graded vector space*, so we have a grading $F_i^{PBW}V = \oplus_j F_i^{PBW}V^j$ compatible with varying i in the natural sense.

Then we can *twist* the filtration $F_\bullet^{PBW}$ by the grading to obtain the filtration:

$$F_i^{KK}V = \bigoplus_j F_{i-j}^{PBW}V^j$$

as before.

---

[85]Essentially introduced, I believe, in [Kos] §1, where it is attributed to Kazhdan.

Here is a geometric interpretation. Recall that a filtered vector space is the same as a quasi-coherent sheaf on $\mathbb{A}^1_\hbar/\mathbb{G}_m$. The data of a compatible filtration and grading as above is equivalent to a quasi-coherent sheaf on $\mathbb{A}^1_\hbar/\mathbb{G}_m \times \mathbb{B}\mathbb{G}_m$, where we recover the underlying filtered vector space by pulling back along the first projection.

Formation of the KK filtration corresponds to pulling back along the graph of the projection $\mathbb{A}^1_\hbar/\mathbb{G}_m \to \mathbb{B}\mathbb{G}_m$ instead.

In either perspective, we immediately find that:

$$\mathrm{gr}_i^{KK} V = \bigoplus_j \mathrm{gr}_{i-j}^{PBW} V^j.$$

That is, $\mathrm{gr}_\bullet^{KK} V = \mathrm{gr}_\bullet^{PBW} V$ as vector spaces, although the gradings are different.

A.5.3.   We now give a categorical version of the above.

Suppose $\mathcal{C}$ is a filtered category. We use the notation $\mathsf{Fil}^{PBW}\mathcal{C}$ for the underlying $\mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)$-module category, since we wish to construct another filtration on $\mathcal{C}$.

The extra data we need is an action of $\mathsf{QCoh}(\mathbb{G}_m)$ (with its convolution monoidal structure) on $\mathsf{Fil}^{PBW}\mathcal{C}$ commuting with the $\mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)$. In this case, we can again *twist* our filtration by this action in forming:

$$\mathsf{Fil}^{KK}\mathcal{C} := \mathsf{Fil}^{PBW}\mathcal{C}^{\mathbb{G}_m,w} \underset{\mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)\otimes\mathsf{Rep}(\mathbb{G}_m)}{\otimes} \mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)$$

where the action on the right term is induced by the symmetric monoidal functor:

$$\mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m) \otimes \mathsf{Rep}(\mathbb{G}_m) \to \mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)$$

of pullback along the graph of the structure map $\mathbb{A}^1_\hbar/\mathbb{G}_m \to \mathbb{B}\mathbb{G}_m$.

More geometrically: we are given the datum of a sheaf of categories on $\mathbb{A}^1_\hbar/\mathbb{G}_m \times \mathbb{B}\mathbb{G}_m$, and we observe that we can form *two* filtered categories from it, via pullback along the maps:

$$\mathrm{id} \times p, \Gamma : \mathbb{A}^1_\hbar/\mathbb{G}_m \to \mathbb{A}^1_\hbar/\mathbb{G}_m \times \mathbb{B}\mathbb{G}_m$$

where $p : \mathrm{Spec}(k) \to \mathbb{B}\mathbb{G}_m$ is the tautological projection, and Γ is the graph of the structure map as above. These two maps coincide over the open point $\mathbb{A}^1_\hbar\backslash 0/\mathbb{G}_m$, so are filtrations on the same category $\mathcal{C}$. By definition, the pullback along $\mathrm{id} \times p$ defines the "PBW" filtration, and pullback along Γ defines the KK filtration.

Remark A.5.2. There is a tautological functor $\mathsf{Fil}^{PBW}\mathcal{C}^{\mathbb{G}_m,w} \to \mathsf{Fil}^{KK}\mathcal{C}$.

Remark A.5.3. Note that $\mathcal{C}^{cl}$ is also the fiber at 0 of $\mathsf{Fil}^{KK}\mathcal{C}$, but the weak $\mathbb{G}_m$-action is different: it is the diagonal action mixing the standard action of $\mathbb{G}_m$ on $\mathcal{C}^{cl}$ with the action coming from the weak $\mathbb{G}_m$-action on $\mathsf{Fil}^{PBW}\mathcal{C}$.

Example A.5.4. In §A.5.2, it is tautological that the KK twisting construction is symmetric monoidal. So if A is an algebra with compatible filtration $F_\bullet^{PBW}$ and grading, we obtain a filtration $F_\bullet^{KK}$ on A (as an algebra). Therefore, we obtain two filtrations $\mathsf{Fil}^{PBW}, \mathsf{Fil}^{KK}$ on A–mod. Of course, the grading on A induces a weak $\mathbb{G}_m$-action on $\mathsf{Fil}^{PBW}A$, and the general categorical construction above produces $\mathsf{Fil}^{KK}A$. The functor $\mathsf{Fil}^{PBW}A$–$\mathsf{mod}^{\mathbb{G}_m,w} \to \mathsf{Fil}^{KK}A$–$\mathsf{mod}$ corresponds to taking a graded and PBW filtered A-module and then applying the corresponding KK twist to obtain a KK filtered A-module.

A.5.4.   The reader may skip this material for the time being, and return to it as necessary. Its purpose is closely tied to Warning A.5.1: Kazhdan-Kostant filtrations are typically incomplete and unbounded from below, even when a corresponding PBW filtration is bounded from below. To deal with this issue, we wish to yoke the two filtrations on our category.

Definition A.5.5. A *bifiltration* on $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is a $\mathsf{QCoh}(\mathbb{A}^1_{\hbar_1}/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m)$-module category $\mathsf{BiFil}\,\mathcal{C}$ plus an isomorphism:

$$\mathcal{C} \simeq \mathsf{BiFil}\,\mathcal{C} \underset{\mathsf{QCoh}(\mathbb{A}^1_{\hbar_1}/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m)}{\otimes} \mathsf{QCoh}((\mathbb{A}^1_{\hbar_1}\backslash 0)/\mathbb{G}_m \times (\mathbb{A}^1_{\hbar_2}\backslash 0)/\mathbb{G}_m) = \mathsf{BiFil}\,\mathcal{C} \underset{\mathsf{QCoh}(\mathbb{A}^1_{\hbar_1}/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m)}{\otimes} \mathsf{Vect}.$$

A *bifiltration* on $\mathcal{F}$ in $\mathcal{C}$ is an object of $\mathsf{BiFil}\,\mathcal{C}$ restricting to $\mathcal{F}$.

Much of our earlier discussion generalies. E.g., we have an obvious notion of bifiltered vector spaces, and so on.

Note that a bifiltration on $\mathcal{C}$ indeed gives rise to two filtrations on $\mathcal{C}$, denoted $\mathsf{Fil}^{PBW}\mathcal{C}$ and $\mathsf{Fil}^{KK}\mathcal{C}$. These are respectively obtained by restricting $\mathsf{BiFil}\,\mathcal{C}$ to the loci:

$$\mathbb{A}^1_{\hbar_1}/\mathbb{G}_m \times (\mathbb{A}^1_{\hbar_2}\backslash 0)/\mathbb{G}_m = \mathbb{A}^1_{\hbar}/\mathbb{G}_m$$
$$(\mathbb{A}^1_{\hbar_1}\backslash 0)/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m = \mathbb{A}^1_{\hbar}/\mathbb{G}_m.$$

Note that a bifiltration on $\mathcal{F} \in \mathcal{C}$ gives rise to PBW and KK filtrations on $\mathcal{F}$.

Remark A.5.6. Let $\mathcal{C}^{PBW-cl}$ and $\mathcal{C}^{KK-cl}$ denote the semi-classical categories associated with each of these filtrations. We claim that e.g. $\mathcal{C}^{PBW-cl}$ carries a natural KK filtration $\mathsf{Fil}^{KK}\mathcal{C}^{PBW-cl}$; moreover, the weak $\mathbb{G}_m$-action on $\mathcal{C}^{PBW-cl}$ extends to one on $\mathsf{Fil}^{KK}\mathcal{C}^{PBW-cl}$, and this action commutes with the action of $\mathsf{QCoh}(\mathbb{A}^1_{\hbar}/\mathbb{G}_m)$.

Indeed, note that $\mathcal{C}^{PBW-cl,\mathbb{G}_m,w}$ is the restriction to:

$$\mathbb{B}\mathbb{G}_m \times (\mathbb{A}^1_{\hbar_2}\backslash 0)/\mathbb{G}_m \subseteq \mathbb{A}^1_{\hbar_1}/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m$$

so taking $\mathsf{Fil}^{KK}\mathcal{C}^{PBW-cl,\mathbb{G}_m,w}$ as the restriction to:

$$\mathbb{B}\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m$$

(and applying de-equivariantization) gives the desired construction.

Of course, this works symmetrically in PBW and KK.

Example A.5.7. Suppose that we are in the setting of §A.5.3, so $\mathcal{C}$ carries a single (PBW) filtration and a compatible weak $\mathbb{G}_m$-action. We claim that this data induces a bifiltration on $\mathcal{C}$ inducing the PBW and KK filtrations in the sense of §A.5.3. For this, we note that we have the morphism:[86]

$$\mathbb{A}^1_{\hbar_1}/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m \to \mathbb{A}^1_{\hbar}/\mathbb{G}_m \times \mathbb{B}\mathbb{G}_m$$
$$(s_1 \in \mathcal{L}_1, s_2 \in \mathcal{L}_2) \mapsto (s_1 \otimes s_2 \in \mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{L}_2)$$

whose restriction to $\mathbb{A}^1_{\hbar_1}/\mathbb{G}_m \times (\mathbb{A}^1_{\hbar_2}\backslash 0)/\mathbb{G}_m$ is the $\mathrm{id} \times p$ and whose restriction to $(\mathbb{A}^1_{\hbar_1}\backslash 0)/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m = \mathbb{A}^1_{\hbar}/\mathbb{G}_m$ is Γ; so the operation of pullback (in the sheaf of categories language) along this morphism gives the desired structure.

Recall that $\mathcal{C}^{PBW-cl}$ is canonically isomorphic to $\mathcal{C}^{KK-cl}$ in this case, so we denote them each by $\mathcal{C}^{cl}$.

---

[86]The notation $s \in \mathcal{L}$ for points of $\mathbb{A}^1_{\hbar}/\mathbb{G}_m$ is used because this stack is the moduli of a line bundle plus a section.

Recall that $\mathbb{G}_m \times \mathbb{G}_m$ weakly acts on $\mathcal{C}^{cl}$: one factor acts because this is always true for the semi-classical category, and the other factor acts because of the weak $\mathbb{G}_m$-action on $\mathsf{Fil}^{PBW}\mathcal{C}$. Then it is straightforward to verify that the KK filtration on $\mathcal{C}^{PBW-cl} = \mathcal{C}^{cl}$ is induced by taking the diagonal weak action of $\mathbb{G}_m$ on $\mathcal{C}^{cl}$ and applying Example A.2.6.

The situation with $\mathcal{C}^{KK-cl} = \mathcal{C}^{cl}$ is similar: its filtration is induced by the "canonical" weak $\mathbb{G}_m$-action on $\mathcal{C}^{cl}$, i.e., the one from the first $\mathbb{G}_m$-factor above (so is unrelated to the weak $\mathbb{G}_m$-action on $\mathsf{Fil}^{PBW}\mathcal{C}$).

A.5.5. We now begin to apply the above in the setting of Lie algebras and Harish-Chandra modules.

Before discussing Kazhdan-Kostant directly, let us discuss what we can obtain without the additional "grading" (i.e., weak $\mathbb{G}_m$-action). We use the language of §A.3.13.

Suppose $\mathbb{G}_a$ acts on $\mathsf{Fil}\,\mathcal{C}$. Let ψ denote the exponential (alias: Artin-Schreier) character sheaf on $\mathbb{G}_a$. Our problem is to construct a filtration on $\mathcal{C}^{\mathbb{G}_a,\psi}$.

First, note that (forgetting the filtrations) we can write $\mathcal{C} \mapsto \mathcal{C}^{\mathbb{G}_a,\psi}$ in two steps: for $\widehat{\mathbb{G}}_a$ the formal completion of $\mathbb{G}_a$ at the origin, the group prestack $\mathbb{B}\widehat{\mathbb{G}}_a = \mathbb{G}_{dR}/\mathbb{G}_a$ acts on $\mathcal{C}^{\mathbb{G}_a,w}$. Note that $\mathsf{QCoh}(\mathbb{B}\widehat{\mathbb{G}}_a) = \mathsf{QCoh}(\mathbb{A}^1_{\mathrm{Lie}})$ with the convolution structure on the LHS corresponding to the tensor product structure on the RHS; here the subscript Lie is used so we later remember the Lie-theoretic origins of this copy of the affine line. Therefore, $\mathcal{C}^{\mathbb{G}_a,w}$ fibers over $\mathbb{A}^1_{\mathrm{Lie}}$, and we can take its fiber at $1 \in \mathbb{A}^1_{\mathrm{Lie}}$, i.e., we can form:

$$\mathcal{C}^{\mathbb{G}_a,w} \underset{\mathsf{QCoh}(\mathbb{A}^1_{\mathrm{Lie}})}{\otimes} \mathsf{Vect}$$

using the restriction functor along $1 \hookrightarrow \mathbb{A}^1_{\mathrm{Lie}}$. It is immediate to see that this tensor product is $\mathcal{C}^{\mathbb{G}_a,\psi}$.

In the filtered setting, let $\mathsf{Fil}\,\widehat{\mathbb{G}}_a$ denote the (commutative) relative formal group over $\mathbb{A}^1_\hbar/\mathbb{G}_m$ defined by $\mathsf{Fil}\,\mathrm{Lie}(\mathbb{G}_a)$. As above, $\mathbb{B}\mathsf{Fil}\,\widehat{\mathbb{G}}_a$ acts on $\mathsf{Fil}\,\mathcal{C}^{\mathbb{G}_a,w}$. Note that:

$$\mathsf{QCoh}(\mathbb{B}\mathsf{Fil}\,\widehat{\mathbb{G}}_a) \simeq \mathsf{QCoh}((\mathbb{A}^1_{\mathrm{Lie}} \times \mathbb{A}^1_\hbar)/\mathbb{G}_m) \in \mathsf{Alg}(\mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)\text{–}\mathsf{mod})$$

where the left hand side is equipped with the convolution monoidal structure, and the right hand side is equipped with the tensor product monoidal structure. Moreover, since $\mathbb{A}^1_{\mathrm{Lie}}$ occurs here as the *co*adjoint space $\mathrm{Lie}(\mathbb{G}_a)^\vee$, it is naturally equipped with the action of $\mathbb{G}_m$ by inverse homotheties; so both $\mathbb{A}^1$-factors are acted on in this way, and our graded algebra of functions is a polynomial algebra on the two degree 1 generators $\hbar$ and $x\hbar$ for $x \in \mathrm{Lie}(\mathbb{G}_a)$ the generator.

Warning A.5.8. The reader confused why we see $(\mathbb{A}^1_{\mathrm{Lie}} \times \mathbb{A}^1_\hbar)/\mathbb{G}_m$ and not $\mathbb{A}^1_{\mathrm{Lie}} \times (\mathbb{A}^1_\hbar/\mathbb{G}_m)$ should return to Warning A.3.3.

The upshot is that we obtain a filtration on $\mathcal{C}^{\mathbb{G}_a,\psi}$ by taking $\mathsf{Fil}\,\mathcal{C}^{\mathbb{G}_a,\psi}$ as $\mathsf{Fil}\,\mathcal{C}^{\mathbb{G}_a,w}$ and restricting along the diagonal map:

$$\mathbb{A}^1_\hbar/\mathbb{G}_m \xrightarrow{x \mapsto (x,x)} (\mathbb{A}^1_{\mathrm{Lie}} \times \mathbb{A}^1_\hbar)/\mathbb{G}_m.$$

When there is risk for confusion, we refer to this as the *PBW* filtration on $\mathcal{C}^{\mathbb{G}_a,\psi}$ and denote it by $\mathsf{Fil}^{PBW}\mathcal{C}^{\mathbb{G}_a,\psi}$.

Remark A.5.9. But in our hearts, we know that we would rather restrict along the *non-existing map*:

$$\mathbb{A}^1_\hbar/\mathbb{G}_m \xrightarrow{x \mapsto (1,x)} (\mathbb{A}^1_{\mathrm{Lie}} \times \mathbb{A}^1_\hbar)/\mathbb{G}_m.$$

Remark A.5.10. Note that $\mathcal{C}^{\mathbb{G}_a,\psi,cl} = \mathcal{C}^{\mathbb{G}_a,cl}$ under this construction.

A.5.6. Let $\mathbb{G}_m$ act on the group $\mathbb{G}_a$ through inverse[87] homotheties. Since the morphism $\mathbb{G}_a \to \mathbb{G}_{a,dR}$ is $\mathbb{G}_m$-equivariant, $\mathbb{G}_m$ acts on $\mathsf{Fil}\,\mathbb{G}_{a,dR}$. For $\mathcal{C}$ "PBW" filtered, it makes sense to ask that a weak $\mathbb{G}_m$-action and a (strong) $\mathbb{G}_a$-action on $\mathsf{Fil}^{PBW}\mathcal{C}$ be compatible (with the action of $\mathbb{G}_m$ on $\mathsf{Fil}\,\mathbb{G}_a$). Note that this in particular gives a weak $\mathbb{G}_m$-action on $\mathsf{Fil}^{PBW}\mathcal{C}^{\mathbb{G}_a,w}$, so we may form $\mathsf{Fil}^{KK}\mathcal{C}^{\mathbb{G}_a,w}$.

Observe that if we regard $\mathrm{Lie}\,\mathbb{G}_a$ as a filtered Lie algebra through the PBW method of §A.3.2 (remembering Warning A.3.3) and equip it with the above (degree -1) grading, then Kazhdan-Kostant twisting gives $\mathrm{Lie}\,\mathbb{G}_a$ equipped with the *constant* filtration (jumping only in degree 0).

It follows formally that $\mathbb{B}\widehat{\mathbb{G}}_a$ (with no Fil!) acts on $\mathsf{Fil}^{KK}\mathcal{C}^{\mathbb{G}_a,w}$. Therefore, $\mathsf{Fil}^{KK}\mathcal{C}^{\mathbb{G}_a,w}$ has an action of $\mathsf{QCoh}(\mathbb{A}^1_{\mathrm{Lie}})$, and we may take its fiber at $1 \in \mathbb{A}^1_{\mathrm{Lie}}$ (by approriately tensoring with Vect). By definition, this is $\mathsf{Fil}^{KK}\mathcal{C}^{\mathbb{G}_a,\psi}$, the *Kazhdan-Kostant* filtration on $\mathcal{C}^{\mathbb{G}_a,\psi}$.

Recall that $\mathcal{C}^{cl}$ has commuting (because $\mathbb{G}_a$ is commutative) actions of $\mathsf{QCoh}(\mathbb{G}_a)$ (under convolution) and $\mathsf{QCoh}(\mathrm{Lie}(\mathbb{G}_a)^\vee) = \mathsf{QCoh}(\mathbb{A}^1_{\mathrm{Lie}})$. One immediately finds that $\mathcal{C}^{\mathbb{G}_a,\psi,KK-cl}$, the "semi-classical" category for the special fiber, is $\mathcal{C}^{cl,\mathbb{G}_a,w}|_{1\in\mathbb{A}^1_{\mathrm{Lie}}}$, where the restriction notation means we form the appropriate tensor product.

Warning A.5.11. This Kazhdan-Kostant filtration on $\mathcal{C}^{\mathbb{G}_a,\psi}$ is *not* obtained by applying the method of §A.5.3 to the PBW filtration. Indeed, the semi-classical categories are different.

A.5.7. We now repeat the above to produce a *bi*filtration on $\mathcal{C}^{\mathbb{G}_a,\psi}$ inducing the PBW and KK filtrations.

Note that $\mathcal{C}^{\mathbb{G}_a,w}$ carries a canonical bifiltration from §A.5.4: it is induced by the weak $\mathbb{G}_m$-action on $\mathsf{Fil}^{PBW}\mathcal{C}^{\mathbb{G}_a,w}$. Moreover, because $\mathrm{Lie}\,\mathbb{G}_a$ is bifiltered by its PBW and KK filtrations by §A.5.4, it follows that the corresponding bifiltered formal group acts on $\mathsf{BiFil}\,\mathcal{C}^{\mathbb{G}_a,w}$. Combining our analysis in the PBW and KK cases, we find that the action of $\mathsf{QCoh}(\mathbb{A}^1_{\hbar_1}/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m)$ extends to an action of:

$$\mathsf{QCoh}\Big((\mathbb{A}^1_{\mathrm{Lie}} \times \mathbb{A}^1_{\hbar_1})/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m\Big)$$

where the action on the first two factors is diagonal. So we obtain our desired bifiltration by setting $\mathsf{BiFil}\,\mathcal{C}^{\mathbb{G}_a,\psi}$ to be the restriction of $\mathsf{BiFil}\,\mathcal{C}^{\mathbb{G}_a,w}$ along the map:

$$\mathbb{A}^1_{\hbar_1}/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m \xrightarrow{\Delta/\mathbb{G}_m\times\mathrm{id}} (\mathbb{A}^1_{\mathrm{Lie}} \times \mathbb{A}^1_{\hbar_1})/\mathbb{G}_m \times \mathbb{A}^1_{\hbar_2}/\mathbb{G}_m$$

where Δ is the diagonal map $\mathbb{A}^1_{\hbar_1} = \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^1_{\mathrm{Lie}} \times \mathbb{A}^1_{\hbar_1}$. By construction, it induces the PBW and KK filtrations on $\mathcal{C}^{\mathbb{G}_a,\psi}$, with terminological conventions consistent with §A.5.4.

Remark A.5.12. Recall from Remark A.5.6 that our bifiltration induces filtrations on $\mathcal{C}^{\mathbb{G}_a,\psi,PBW-cl}$ and $\mathcal{C}^{\mathbb{G}_a,\psi,KK-cl}$ (in the notation of *loc. cit.*). It is straightforward to verify that the (KK) filtration on $\mathcal{C}^{\mathbb{G}_a,\psi,PBW-cl} = \mathcal{C}^{cl,\mathbb{G}_a,w}|_{0\in\mathbb{A}^1_{\mathrm{Lie}}}$ is as in Example A.5.7, i.e., induced by the diagonal $\mathbb{G}_m$-action via Example A.2.6. The (PBW) filtration on $\mathcal{C}^{\mathbb{G}_a,\psi,KK-cl} = \mathcal{C}^{cl,\mathbb{G}_a,w}|_{1\in\mathbb{A}^1_{\mathrm{Lie}}}$ is obtained from degenerating the character, i.e., it is the $\mathbb{G}_m$-equivariant sheaf of categories over $\mathbb{A}^1$ with fiber $\mathcal{C}^{cl,\mathbb{G}_a,w}|_{\lambda\in\mathbb{A}^1_{\mathrm{Lie}}}$ at $\lambda \in \mathbb{A}^1$; of course, the $\mathbb{G}_m$-equivariance here comes from the $\mathbb{G}_m$-action on $\mathsf{Fil}^{PBW}\mathcal{C}$.

---

[87]The reader is reminded that we used $-\check{\rho}$ in §A.5, so $\mathfrak{n}$ had negative degree with respect to the $\mathbb{G}_m$-action. So the sign here is the expected one.

A.5.8.   We now apply the above in the Harish-Chandra setting.

Suppose as before that $(\mathfrak{h}, K)$ is a Harish-Chandra pair with $\mathfrak{h}$ finite-dimensional and K an affine algebraic group. We suppose $\mathrm{Lie}(K) \hookrightarrow \mathfrak{h}$ for simplicity. Suppose moreover that we are given a non-trivial character $\psi : K \to \mathbb{G}_a$; let K' denote the kernel. We also let ψ denote the induced character $\mathfrak{k} \to k$, or the corresponding 1-dimensional $\mathfrak{k}$-module; similarly for $-\psi$.

Then since K acts on $\mathsf{Fil}\,\mathfrak{h}\text{–mod}$, $\mathbb{G}_a$ acts on $\mathsf{Fil}\,\mathfrak{h}\text{–mod}^{K'}$, so by the above, we obtain a PBW filtration $\mathsf{Fil}^{PBW}\mathfrak{h}\text{–mod}^{K,\psi}$ from §A.5.5. We have $\mathfrak{h}\text{–mod}^{K,\psi,PBW-cl} = \mathsf{QCoh}((\mathfrak{h}/\mathfrak{k})^{\vee}/K)$.

Suppose now that $\mathbb{G}_m$ acts on K; that the K-action on $\mathfrak{h}$ has been extended to $\mathbb{G}_m \ltimes K$; and that the character $\psi : K \to \mathbb{G}_a$ is $\mathbb{G}_m$-equivariant for the inverse homothety action of $\mathbb{G}_m$ on $\mathbb{G}_a$. Then $\mathsf{Fil}\,\mathfrak{h}\text{–mod}^{K'}$ carries an action of $\mathbb{G}_a = K/K'$ and a weak action of $\mathbb{G}_m$, giving a datum as in §A.5.6. Therefore, we obtain a KK filtration $\mathsf{Fil}^{KK}\mathfrak{h}\text{–mod}^{K,\psi}$. We have:

$$\mathfrak{h}\text{–mod}^{K,\psi,KK-cl} = \mathsf{QCoh}(\psi + (\mathfrak{h}/\mathfrak{k})^{\vee}/K)$$

where $\psi + (\mathfrak{h}/\mathfrak{k})^{\vee} \subseteq \mathfrak{h}^{\vee}$ is the inverse image of ψ under the map $\mathfrak{h}^{\vee} \to \mathfrak{k}^{\vee}$; this locus is closed under the K-action because ψ is a character.

These two filtrations fit into a bifiltration by the general formalism.

Example A.5.13.   For $\mathfrak{k} = \mathfrak{h}$, the PBW filtration on $\mathsf{Rep}(K) \overset{-\otimes\psi}{\simeq} \mathfrak{k}\text{–mod}^{K,\psi}$ is the constant one (as we discussed before), and the KK filtration is induced from Example A.2.6 via the weak $\mathbb{G}_m$-action on $\mathsf{Rep}(K)$. In other words, we regard $\mathcal{O}_K$ as a graded coalgebra, so by *loc. cit.* it inherits a natural coalgebra filtration; then the KK filtration is obtained by considering filtered comodules. Note that the group cohomology functor $\mathsf{Rep}(K) \to \mathsf{Vect}$ is canonically bifiltered, e.g. because the trivial representation has a canonical[88] bifiltration.

It follows formally that the induction functor $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{h}} : \mathsf{Rep}(K) = \mathfrak{k}\text{–mod}^{K,\psi} \to \mathfrak{h}\text{–mod}^{K,\psi}$ is bifiltered. Applying this to the trivial representation with its canonical bifiltration, we see that the functor $C^{\bullet}(\mathfrak{k}, K, (-) \otimes -\psi) : \mathfrak{h}\text{–mod}^{K,\psi} \to \mathsf{Vect}$ is also naturally bifiltered. Its underlying semi-classical functors for the PBW and KK filtrations are the appropriate global sections functors:

$$\Gamma((\mathfrak{h}/\mathfrak{k})^{\vee}/K, -) : \mathsf{QCoh}((\mathfrak{h}/\mathfrak{k})^{\vee}/K) \to \mathsf{Vect}$$
$$\Gamma(\psi + (\mathfrak{h}/\mathfrak{k})^{\vee}/K, -) : \mathsf{QCoh}(\psi + (\mathfrak{h}/\mathfrak{k})^{\vee}/K) \to \mathsf{Vect}.$$

Somewhat more generally,[89] suppose that the character ψ is extended to $\mathfrak{h}$ and continues to satisfy the appropriate $\mathbb{G}_m$-equivariance.

Then for $M \in \mathfrak{h}\text{–mod}^{K,\psi}$, $M \otimes -\psi$ can be considered as an object of $\mathfrak{h}\text{–mod}^{K}$, so it makes sense to take the Harish-Chandra cohomology:

$$C^{\bullet}(\mathfrak{h}, K, M \otimes -\psi).$$

This functor is bifiltered, with PBW and KK semi-classical versions given by:

$$\mathsf{QCoh}((\mathfrak{h}/\mathfrak{k})^{\vee}/K) \to \mathsf{Vect}$$
$$\mathsf{QCoh}(\psi + (\mathfrak{h}/\mathfrak{k})^{\vee}/K) \to \mathsf{Vect}$$

given by !-restriction to $0/K$ or ψ/K followed by global sections on this stack (i.e., group cohomology for K).

---

[88]It is the constant bifiltration on the underlying vector space of the representation.

[89]This setup appears strange if one has the finite W-algebra example $(\mathfrak{h}, K) = (\mathfrak{g}, N)$, but its infinite-dimensional version appears in the affine W-algebra setup.

Remark A.5.14. Let us describe what a KK filtration on an object of $\mathfrak{h}\text{–mod}^{K,\psi}$ looks like concretely. Suppose $M \in \mathfrak{h}\text{–mod}^{K,\psi,\heartsuit}$ observe that $M \otimes -\psi \in \mathfrak{k}\text{–mod}^K = \mathsf{Rep}(K)$, i.e., the natural $\mathfrak{k}$-action integrates to the group. A sequence:

$$\ldots \subseteq F_i^{KK} M \subseteq F_{i+1}^{KK} M \subseteq \ldots$$

is a KK filtration if:

- It is a filtration of M considered as a module over the KK-filtered algebra $U(\mathfrak{h})$. In other words, if $\mathfrak{h}^j$ indicates the jth graded component of $\mathfrak{h}$, $\mathfrak{h}^j$ maps $F_i^{KK} M$ to $F_{i+j+1}^{KK} M$.
- Consider $\mathcal{O}_K$ as a filtered coalgebra using the $\mathbb{G}_m$-action on it and Example A.2.6. Then the coaction map $(M \otimes -\psi) \to (M \otimes -\psi) \otimes \mathcal{O}_K$ should be filtered.

  If K is connected, this is equivalent to asking that $\mathfrak{k}^j$ acting on $M \otimes -\psi$ takes $F_i^{KK} M \otimes -\psi$ to $F_{i+j}^{KK} M \otimes -\psi$.

Suppose now that ψ is $\mathbb{G}_m$-equivariantly extended to $\mathfrak{h}$. We also suppose that K is unipotent, so $C^\bullet(\mathfrak{h}, K, -)$ coincides with $C^\bullet(\mathfrak{h}, -)$ as a non-filtered functor. Then the KK filtration on $C^\bullet(\mathfrak{h}, K, M \otimes -\psi)$ is similar to the filtration from Remark A.3.16; its ith term is:

$$0 \to F_i^{KK} M \otimes -\psi \to \sum_j (\mathfrak{h}^j/\mathfrak{k}^j)^\vee \otimes F_{i+j+1}^{KK} M \otimes -\psi + (\mathfrak{h}^j)^\vee \otimes F_{i+j}^{KK} M \otimes -\psi \to \ldots$$

A.6. **Compact Lie algebras.**

A.6.1. We now begin to move to an infinite dimensional setting. Let $\mathfrak{h}$ be a *profinite-dimensional Lie algebra*, so $\mathfrak{h} = \lim_i \mathfrak{h}/\mathfrak{h}_i$ is a filtered limit of finite-dimensional Lie algebras $\mathfrak{h}/\mathfrak{h}_i \in \mathsf{LieAlg}(\mathsf{Vect}^\heartsuit)$, and with all structure maps being surjective. Of course, $\mathfrak{h}_i \subseteq \mathfrak{h}$ indicates the corresponding normal open Lie subalgebra, which is of the requisite type.

Following [FG3] §22-23, we define:

$$\mathfrak{h}\text{–mod} := \operatorname*{colim}_i \mathfrak{h}/\mathfrak{h}_i\text{–mod} \in \mathsf{DGCat}_{cont}$$

where our structure functors are forgetful functors. By our assumptions, for each structure map $\mathfrak{h}/\mathfrak{h}_i \twoheadrightarrow \mathfrak{h}/\mathfrak{h}_j$, the induced functor:

$$\mathsf{Oblv} : \mathfrak{h}/\mathfrak{h}_j\text{–mod} \to \mathfrak{h}/\mathfrak{h}_i\text{–mod}$$

has a continuous right adjoint: it is Lie algebra cohomology with respect to the $\mathfrak{h}_j/\mathfrak{h}_i$. Therefore, the above colimit is also the limit under these right adjoint functors.

By Lemma 5.2.5, $\mathfrak{h}\text{–mod}$ has a canonical t-structure compatible with filtered colimits with heart the abelian category of discrete[90] $\mathfrak{h}$-modules. Moreover, if our indexing category is countable, $\mathfrak{h}\text{–mod}^+$ is the (bounded below) derived category of this abelian category.

We see that $\mathfrak{h}\text{–mod}$ is compactly generated, and that it has a canonical trivial representation $k \in \mathfrak{h}\text{–mod}$ that is compact. Therefore, we have a continuous functor $C^\bullet(\mathfrak{h}, -) : \mathfrak{h}\text{–mod} \to \mathsf{Vect}$, which is defined as the complex of maps from the trivial representation.

Remark A.6.1. Note that the t-structure on $\mathfrak{h}\text{–mod}$ is not necessarily left complete. Indeed, suppose $\mathfrak{h}$ is abelian and infinite-dimensional. Then[91] $\mathsf{Ext}^\bullet_{\mathfrak{h}\text{–mod}}(k, k) = \Lambda^\bullet \mathfrak{h}^\vee$, so there are non-zero maps $k \to k[n]$ for each $n \geqslant 0$. If the t-structure were left complete, we would have $\oplus_{n \geqslant 0} k[n] \xrightarrow{\simeq} \prod_{n \geqslant 0} k[n]$ (proof: consider the Postnikov tower for the LHS). But this is impossible: we would have

---

[90]Recall that these are $\mathfrak{h}$-modules V such that the stabilizer of any vector in V is open in $\mathfrak{h}$.

[91]In what follows, $\mathfrak{h}^\vee$ should always be understood as the *continuous* dual to $\mathfrak{h}$.

constructed a map $k \to \oplus_{n \geqslant 0} k[n]$ that would not factor through any finite direct sum, contradicting the compactness of k.

In fact, the t-structure is not even left separated. Here is one explicit way to see this. Then $\mathfrak{h}$–mod is canonically *self-dual* in the sense of [Gai4]: indeed, each $\mathfrak{h}/\mathfrak{h}_i$–$\mathsf{mod}$ has a canonical Serre self-duality, and we tautologically have:[92]

$$\mathfrak{h}\text{--}\mathsf{mod}^\vee = \lim_{\mathrm{Oblv}^\vee} \mathfrak{h}/\mathfrak{h}_i\text{--}\mathsf{mod}$$

where the notation indicates the limit under the functors *dual* to the forgetful functors; these are given by *co*invariants with respect to the kernels, which differ from the invariants by a shift and tensoring with a 1-dimensional representation, according to Lemma A.3.6. This readily implies the claim: we should replace Serre self-duality on each $\mathfrak{h}/\mathfrak{h}_i$–$\mathsf{mod}$ by its composition with the functor of shifting by $\dim \mathfrak{h}/\mathfrak{h}_i$ and tensoring with the determinant of the adjoint representation.

It follows that we have an equivalence $\mathbb{D} : \mathsf{Pro}(\mathfrak{h}\text{--}\mathsf{mod}^c)^{op} \simeq \mathfrak{h}\text{--}\mathsf{mod}$, where $\mathfrak{h}$–mod^c indicates the subcategory of compact objects. The objects $U(\mathfrak{h}/\mathfrak{h}_i)$ are compact in $\mathfrak{h}$–mod (and even generate), and form a filtered projective system in the obvious way; we denote this pro-object by $U(\mathfrak{h})$. The object $\mathbb{D}U(\mathfrak{h}) \in \mathfrak{h}$–$\mathsf{mod}$ is then obviously non-zero, but lives in $\cap_n \mathfrak{h}\text{--}\mathsf{mod}^{\leqslant -n}$ because $\mathbb{D}U(\mathfrak{h}) = \mathrm{colim}_i \mathbb{D}U(\mathfrak{h}/\mathfrak{h}_i)$, and $\mathbb{D}U(\mathfrak{h}/\mathfrak{h}_i)$ lies in cohomological degree $-\dim \mathfrak{h}/\mathfrak{h}_i$.

Remark A.6.2. Note that $\mathfrak{h}^\vee$ is a Lie coalgebra, so there is a general formalism of taking comodules over it. It is straightforward to show that $\mathfrak{h}^\vee$–comod is the left completion of $\mathfrak{h}$–mod.

Warning A.6.3. For $M \in \mathfrak{h}$–mod^+, $C^\bullet(\mathfrak{h}, M)$ is computed by a standard complex, but this is *not* true for general $M \in \mathfrak{h}$–mod. To make this precise, note that for any $M \in \mathfrak{h}$–mod, one can form a semi-cosimplicial diagram $M \overset{\mathrm{coact}}{\underset{0}{\rightrightarrows}} M \otimes \mathfrak{h}^\vee \rightrightarrows \dots \in \mathsf{Vect}$ and the canonical morphism:

$$C^\bullet(\mathfrak{h}, M) \to \lim_{\boldsymbol{\Delta}_{inj}} \left(M \rightrightarrows M \otimes \mathfrak{h}^\vee \rightrightarrows \dots \right)$$

This map is an equivalence for $M \in \mathfrak{h}$–mod^+, but not for general M. Indeed, considering the right hand side as a functor in the variable M, it is easy to see that it will not commute with colimits.

A.7.   We have a canonical filtration on $\mathfrak{h}$–mod. Indeed, this follows immediately from the fact that the functors $\mathrm{Oblv} : \mathfrak{h}/\mathfrak{h}_j\text{--}\mathsf{mod} \to \mathfrak{h}/\mathfrak{h}_i\text{--}\mathsf{mod}$ are filtered.

We have:

$$\mathfrak{h}\text{--}\mathsf{mod}^{cl} \simeq \mathrm{colim}_i \mathsf{QCoh}((\mathfrak{h}/\mathfrak{h}_i)^\vee)$$

where the colimit is under $*$-pushforward functors along the closed embeddings $\alpha_{i,j} : (\mathfrak{h}/\mathfrak{h}_j)^\vee \hookrightarrow (\mathfrak{h}/\mathfrak{h}_i)^\vee$.

We claim that $\mathfrak{h}$–mod^{cl} is canonically isomorphic to[93] $\mathsf{IndCoh}(\mathfrak{h}^\vee)$, where $\mathfrak{h}^\vee$ is considered as an indscheme. Indeed, in the standard IndCoh notation from [Gai5], we have commutative diagrams:

---

[92]Dualizability is no issue because we are in a co/limit situation.

[93]Recall that tensoring with the dualizing sheaf induces an equivalence $\mathsf{QCoh}(\mathfrak{h}^\vee) \xrightarrow{\simeq} \mathsf{IndCoh}(\mathfrak{h}^\vee)$. We prefer to write the category of IndCoh rather than QCoh though because the notation is somewhat simpler.

(Note that the place where this equivalence is shown, [GR3] Theorem 10.0.7, has a countability hypothesis. This assumption is verified for us in our applications, so the reader may safely assume it in this section. But in fact, one readily verifies that this assumption is only used in finding nice presentations for a fairly general class of indschemes; this is no problem for our indscheme $\mathfrak{h}^\vee$, so one finds that the countability is not needed in applying their method in the present case.

$$\begin{array}{ccc} \mathsf{IndCoh}((\mathfrak{h}/\mathfrak{h}_j)^\vee) & \xrightarrow{\alpha_{i,j,*}^{\mathsf{IndCoh}}} & \mathsf{IndCoh}((\mathfrak{h}/\mathfrak{h}_i)^\vee) \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\ \mathsf{QCoh}((\mathfrak{h}/\mathfrak{h}_j)^\vee) & \xrightarrow{\alpha_{i,j,*}} & \mathsf{QCoh}((\mathfrak{h}/\mathfrak{h}_i)^\vee) \end{array}$$

with vertical arrows equivalences; of course, these commutative diagrams have the requisite compatibilities of higher category theory. Therefore, $\mathfrak{h}$–mod^{cl} is equivalent to this colimit; since the functors $\alpha_{i,j,*}^{\mathsf{IndCoh}}$ admit the continuous right adjoints $\alpha_{i,j}^!$, we obtain the claim.

As before, the functor $C^\bullet(\mathfrak{h}, -) : \mathfrak{h}$–$\mathsf{mod} \to \mathsf{Vect}$ is filtered, and with semi-classical functor $\mathsf{IndCoh}(\mathfrak{h}^\vee) \to \mathsf{Vect}$ given as !-restriction to $0 \in \mathfrak{h}^\vee$.

A.7.1. *Actions of group schemes on filtered categories.* It is straightforward to generalize the above definitions to the Harish-Chandra setting and to compute the outputs. But it is quite clarifying in this setting to generalize the language of §A.3.13, so we do so.

A.7.2. Suppose K is an affine group scheme; we write K as a filtered limit $\lim_i K/K_i$ for $K_i \subseteq K$ a normal subgroup scheme with K/K_i an affine algebraic group. Recall that an action of an algebraic group on a filtered category induced a weak action the semi-classical category. As a warm-up, we begin our discussion there.

Definition A.7.1. A *weak action* of K on a category $\mathcal{C} \in \mathsf{DGCat}_{cont}$ a $\mathsf{QCoh}(K)$-module structure on $\mathcal{C}$, where $\mathsf{QCoh}(K)$ is given the convolution monoidal structure.

A *renormalized (weak)*[94] *action of K* is an object of $\lim_i \mathsf{QCoh}(K/K_i)$–$\mathsf{mod}$, where the structure functors $\mathsf{QCoh}(K/K_i)$–$\mathsf{mod} \to \mathsf{QCoh}(K/K_j)$–$\mathsf{mod}$ are given by weak invariants with respect to K_j/K_i.

Notation A.7.2. We say a renormalized action of K is *on $\mathcal{C} \in \mathsf{DGCat}_{cont}$* if the compatible system of $\mathsf{QCoh}(K/K_i)$-module categories is denoted by $\mathcal{C}^{K_i-ren}$ and $\mathcal{C} = \mathrm{colim}\, \mathcal{C}^{K_i-ren}$. Note that this is necessarily a co/limit situation, i.e., all structure functors admit right adjoints.

Example A.7.3. Define $\mathsf{Rep}(K)$ as $\mathrm{colim}\, \mathsf{Rep}(K/K_i) \in \mathsf{DGCat}_{cont}$. (This category should not be confused with $\mathsf{QCoh}(\mathbb{B}K)$: they have t-structures and coincide on bounded below derived categories, but $\mathsf{Rep}(K)$ is always compactly generated while $\mathsf{QCoh}(\mathbb{B}K)$ may not be; rather, the latter category is the left completion of the former.)

Clearly this definition makes sense for any affine group scheme. Moreover, $\mathsf{Rep}(K_i)$ is weakly acted upon by K/K_i. So setting $\mathsf{Vect}^{K_i-ren} := \mathsf{Rep}(K_i)$, we obtain a renormalized action of K on Vect.

Remark A.7.4. Note that the trivial representation in $\mathsf{Rep}(K)$ is compact, so we have a *continuous* group cohomology functor $C^\bullet(K, -) : \mathsf{Rep}(K) \to \mathsf{Vect}$.

Remark A.7.5. Note that for any $\mathcal{C}$ with a renormalized action of K, $\mathsf{Rep}(K)$ acts on $\mathcal{C}^{K-ren}$: indeed, writing $\mathcal{C}^{K-ren} = (\mathcal{C}^{K_i-ren})^{K/K_i,w}$, we find $\mathsf{Rep}(K_i)$ acts, so taking the colimit over i gives the claim.

In fact, since the colimit defining $\mathsf{Rep}(K)$ is under (symmetric) monoidal functors, and since:

---

[94]If K is an extension of an affine algebraic group by a prounipotent one, as is always the case for us, this renormalization has no effect in the setting of *strong* group actions on categories: this a consequence of the coincidence of invariants and coinvariants for such categories (see [Ber]). So we set the convention that the word *renormalization* indicates that we are working with weak group actions.

$$\mathsf{QCoh}(K/K_i)\text{–mod} \xrightarrow{(-)^{K/K_i,w}} \mathsf{Rep}(K/K_i)\text{–mod}$$

is an equivalence (by 1-affineness of $\mathbb{B}K/K_i$), we find that the functor $\mathcal{C} \mapsto \mathcal{C}^{K-ren}$ is actually an equivalence between $\mathsf{Rep}(K)$–mod and the (2-)category of categories with a renormalized K-action.

Remark A.7.6. Suppose that X is a quasi-compact quasi-separated *classical*[95] scheme with a K-action. By Noetherian approximation, we can write $X = \lim_i X^i$ under affine morphisms and $K = \lim_i K/K_i$ as above such that X^i is finite type and K/K_i acts on X^i, with these actions being compatible in the natural sense as we vary i. Finally, we assume that all structure morphisms among the X^i are *flat*.

In this case, we obtain a renormalized action of K on $\mathsf{QCoh}(X)$ by setting:

$$\mathsf{QCoh}(X)^{K-ren} := \operatorname{colim}_i \mathsf{QCoh}(X^i/(K/K_i)).$$

To emphasize a stacky perspective, we sometimes use the notation:

$$\mathsf{QCoh}^{ren}(X/K) = \mathsf{QCoh}(X)^{K-ren}.$$

All structure maps here are affine, so this is a co/limit situation. We similarly have $\mathsf{IndCoh}(X)^{K-ren}$; the flatness of our structure maps implies that this is also a co/limit. (All of this is invariant under choices of presentations as limits.)

Note that for $X = \operatorname{Spec}(k)$, we recover $\mathsf{Rep}(K)$ by this construction.

More generally, let $X = \operatorname{colim}_j X_j$ be an indscheme, with the X_j schemes satisfying the above hypotheses. Moreover, we assume the closed embeddings among the X_j are finitely presented and *eventually coconnective* (e.g. regular). Note that $\mathsf{QCoh}(X) := \lim \mathsf{QCoh}(X_j)$ is a co/limit, and similarly for $\mathsf{IndCoh}(X)$.[96] Clearly each of these categories has a canonical renormalized action of K.

A.7.3.   We now discuss the filtered setting.

Definition A.7.7. A *(strong) action of K on a filtered category* is an object of:

$$\lim_i \mathsf{IndCoh}(\mathsf{Fil}\,(K/K_i)_{dR})\text{–mod}(\mathsf{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)\text{–mod})$$

i.e., a compatible system of filtered categories acted on by the K/K_i.

As before, we say this action is on $\mathsf{Fil}\,\mathcal{C}$ if our compatible system is denoted $\mathsf{Fil}\,\mathcal{C}^{K_i}$ and $\mathsf{Fil}\,\mathcal{C} = \operatorname{colim}_i \mathsf{Fil}\,\mathcal{C}^{K_i}$. Again, this is a co/limit.

If K has a prounipotent tail, then $\mathcal{C}$ inherits a (strong) action of K with all notation compatible, i.e., the generic fiber $\mathcal{C}^{K_i}$ of $\mathsf{Fil}\,\mathcal{C}^{K_i}$ is the K_i-invariants for this action.

Remark A.7.8. At the semi-classical level, we obtain an object of:

$$\lim_i \mathsf{QCoh}((\mathfrak{k}/\mathfrak{k}_i)^\vee/(K/K_i))\text{–mod} =: \mathsf{ShvCat}^{ren}(\mathfrak{k}^\vee/K).$$

Here we use the sheaf of categories notation because, by [GR1],

$$\mathsf{ShvCat}(\mathfrak{k}^\vee) := \lim_i \mathsf{QCoh}((\mathfrak{k}/\mathfrak{k}_i)^\vee)\text{–mod} \neq \mathsf{QCoh}(\mathfrak{k}^\vee)\text{–mod}.$$

---

[95]We have used the word *scheme* throughout to mean classical scheme, but are emphasizing it here because although it may seem unnecessary, it is important for the Noetherian approximation we are applying.

[96]Note that taking $K = \{1\}$, our earlier discussion gave a makeshift definition of $\mathsf{IndCoh}(X_i)$.

(Although the RHS is a full subcategory of the LHS.) We use the superscript *ren* because of the relationship to our notion of a renormalized action of K on a category.

In the above setup, we use the notation $\mathcal{C}^{cl}|_{(\mathfrak{k}/\mathfrak{k}_i)^\vee}^{K_i-ren}$ to indicate the corresponding object of $\mathsf{QCoh}((\mathfrak{k}/\mathfrak{k}_i)^\vee/(K/K_i))$–mod. With this notation, we are encouraging the reader to imagine $\mathcal{C}^{cl}$ as sitting over $\mathfrak{k}^\vee$ and equipped with a compatible weak (or better: renormalized) action of K.

Then observe that the filtration on $\mathcal{C}^K$ has $\mathcal{C}^{K,cl} = \mathcal{C}^{cl}|_0^{K-ren}$, and more generally, $\mathcal{C}^{K_i,cl} = \mathcal{C}^{cl}|_{(\mathfrak{k}/\mathfrak{k}_i)^\vee}^{K_i-ren}$.

Remark A.7.9. Note that in the above setting, we may reformulate our semi-classical data in saying that we have a compatible system of categories $\mathcal{C}^{cl}|_{(\mathfrak{k}/\mathfrak{k}_i)^\vee}$ equipped with renormalized K-actions and $\mathsf{QCoh}((\mathfrak{k}/\mathfrak{k}_i)^\vee)$-module category structures and satisfying the natural compatibilities.

We let $\mathcal{C}^{cl}$ denote the limit of this diagram. Note that this is actually a co/limit situation. Then $\mathcal{C}^{cl}$ can be thought of as the global sections of the sheaf of categories on $\mathfrak{k}^\vee$ that our datum induced. Note that $\mathcal{C}$ itself actually is a filtered category, with $\mathcal{C}^{cl}$ as its semi-classical version.

Note that the place to be careful about making mistakes in distinguishing sheaves of categories from module categories is that we may have:

$$\mathcal{C}^{cl}|_{(\mathfrak{k}/\mathfrak{k}_i)^\vee} \neq \mathcal{C}^{cl} \underset{\mathsf{QCoh}(\mathfrak{k}^\vee)}{\otimes} \mathsf{QCoh}((\mathfrak{k}/\mathfrak{k}_i)^\vee).$$

A.7.4. Now suppose that we are in a Harish-Chandra setting: we assume we are given a projective system of Harish-Chandra data $(\mathfrak{h}/\mathfrak{h}_i, K/K_i)$ with $\mathfrak{k}/\mathfrak{k}_i \to \mathfrak{h}/\mathfrak{h}_i$ injective. Our two projective systems $\mathfrak{h}/\mathfrak{h}_i$ and K/K_i are assumed to satisfy our earlier (e.g., finiteness) hypotheses.

We then set:

$$\mathsf{Fil}\,\mathfrak{h}\text{–mod}^K := \underset{i}{\mathrm{colim}}\,\mathsf{Fil}\,\mathfrak{h}/\mathfrak{h}_i\text{–mod}^{K/K_i} \in \mathsf{DGCat}_{cont}.$$

Note that this is a co/limit situation; the right adjoint to the forgetful functor:

$$\mathsf{Fil}\,\mathfrak{h}/\mathfrak{h}_j\text{–mod}^{K/K_j} \to \mathsf{Fil}\,\mathfrak{h}/\mathfrak{h}_i\text{–mod}^{K/K_i}$$

is given by (the filtered version of) Harish-Chandra cohomology with respect to $(\mathfrak{h}_j/\mathfrak{h}_i, K_j/K_i)$.

Note that this construction makes sense for each K_i in place of K. Moreover, K/K_i acts on $\mathsf{Fil}\,\mathfrak{h}\text{–mod}^{K_i}$, and we have natural compatibilities as we take invariants. Therefore, the above data defines an action of K on $\mathsf{Fil}\,\mathfrak{h}\text{–mod}$.

Note that $\mathfrak{h}\text{–mod}^{K,cl} = \mathsf{IndCoh}^{ren}((\mathfrak{h}/\mathfrak{k})^\vee/K)$: the calculation is the same as the one we gave for $\mathfrak{h}\text{–mod}^{cl}$.

We have a natural filtered Harish-Chandra cohomology functor $\mathfrak{h}\text{–mod}^K \to \mathsf{Vect}$ with expected semi-classical version given by !-restriction to $0/K$ followed by group cohomology.

A.7.5. Now suppose in the above setting that we have compatible $\mathbb{G}_m$-actions on each K/K_i and $\mathfrak{h}/\mathfrak{h}_i$. Suppose moreover that we are given a $\mathbb{G}_m$-equivariant character $\psi : K \to \mathbb{G}_a$ for the action of $\mathbb{G}_m$ on $\mathbb{G}_a$ by inverse homotheties.

The construction of §A.5.8 applies as is, giving a Kazhdan-Kostant filtration on $\mathfrak{h}\text{–mod}^{K,\psi}$ fitting into a bifiltration with the PBW filtration. It has similar properties to the finite-dimensional version, up to the various differences we saw above between the finite and profinite-dimensional settings.

A.7.6. Now suppose that we are given $\mathfrak{h}^0 \subseteq \mathfrak{h}$ an open subalgebra, so $\mathfrak{h}/\mathfrak{h}^0$ is finite-dimensional. We assume the pair $(\mathfrak{h}^0, K)$ satisfies the profinite-dimensional Harish-Chandra conditions as above, so $\mathfrak{k} \subseteq \mathfrak{h}^0 \subseteq \mathfrak{h}$ and $\mathfrak{h}^0$ is a K-submodule of $\mathfrak{h}$.

We have the following version of Corollary A.3.9 and Lemma A.3.18.

Lemma A.7.10. *(1) The forgetful functor $\mathfrak{h}$–mod $\to \mathfrak{h}^0$–mod admits a left adjoint $\mathrm{ind}_{\mathfrak{h}^0}^{\mathfrak{h}}$ as a filtered functor. The induced semi-classical functor $(\mathrm{ind}_{\mathfrak{h}^0}^{\mathfrak{h}})^{cl}$ is the $(\mathsf{IndCoh}, *)$ pullback functor:*

$$\mathsf{IndCoh}(\mathfrak{h}^{0,\vee}) \to \mathsf{IndCoh}(\mathfrak{h}^{\vee})$$

i.e., the left adjoint to the IndCoh-pushforward along the projection $\mathfrak{h}^{\vee} \to \mathfrak{h}^{0,\vee}$.

There is a canonical isomorphism of filtered functors $\mathfrak{h}$–mod $\to \mathsf{Vect}$:

$$C^{\bullet}\left(\mathfrak{h}, \mathrm{ind}_{\mathfrak{h}^0}^{\mathfrak{h}}\left((-) \otimes \det(\mathfrak{h}/\mathfrak{h}^0)[\dim \mathfrak{h}/\mathfrak{h}^0]\right)\right) = C^{\bullet}(\mathfrak{h}^0, -)$$

where $\det(\mathfrak{h}/\mathfrak{h}^0)[\dim \mathfrak{h}/\mathfrak{h}^0])$ is filtered with a single jump in degree $\dim \mathfrak{h}/\mathfrak{h}^0$.

(2) The functor $\mathrm{ind}_{\mathfrak{h}^0}^{\mathfrak{h}}$ is a morphism of filtered categories acted on by K. The induced functor $\mathrm{ind}_{\mathfrak{h}^0}^{\mathfrak{h}} : \mathfrak{h}^0$–mod$^K \to \mathfrak{h}$–mod^K has semi-classical version:

$$\mathsf{IndCoh}^{ren}((\mathfrak{h}^0/\mathfrak{k})^{\vee}/K) \to \mathsf{IndCoh}^{ren}((\mathfrak{h}/\mathfrak{k})^{\vee}/K)$$

again given by $(\mathsf{IndCoh}, *)$-pullback.

There is a canonical isomorphism of filtered functors $\mathfrak{h}^0$–mod$^K \to \mathsf{Vect}$:

$$C^{\bullet}\left(\mathfrak{h}, K, \mathrm{ind}_{\mathfrak{h}^0}^{\mathfrak{h}}\left((-) \otimes \det(\mathfrak{h}/\mathfrak{h}^0)[\dim \mathfrak{h}/\mathfrak{h}^0]\right)\right) = C^{\bullet}(\mathfrak{h}^0, K, -).$$

(3) Suppose now that we are given the extra data of §A.7.5, and suppose that we are given a character $\psi : \mathfrak{h} \to k$ extending the same-named character on $\mathfrak{k}$. Then there is a canonical isomorphism of bifiltered functors $\mathfrak{h}^0$–mod$^{K,\psi} \to \mathsf{Vect}$:

$$C^{\bullet}\left(\mathfrak{h}, K, \mathrm{ind}_{\mathfrak{h}^0}^{\mathfrak{h}}\left((-) \otimes (-\psi) \otimes \det(\mathfrak{h}/\mathfrak{h}^0)[\dim \mathfrak{h}/\mathfrak{h}^0]\right)\right) = C^{\bullet}(\mathfrak{h}^0, K, - \otimes (-\psi)).$$

Each of these functors has semi-classical version:

$$\mathsf{IndCoh}^{ren}(\psi + (\mathfrak{h}^0/\mathfrak{k})^{\vee}/K) \to \mathsf{Vect}$$

given by !-restriction to ψ/K followed by Γ^{IndCoh} (i.e., group cohomology with respect to K).

Proof. These results follow immediately from their finite-dimensional counterparts by passing to the limit.
$\square$

A.8. **Tate setting.**

A.8.1. Our treatment here follows [Gai6] at some points.

Suppose $\mathfrak{h} \in \mathsf{Pro}(\mathsf{Vect}^{\heartsuit})$ is a Tate Lie algebra, i.e., $\mathfrak{h}$ is a limit under surjective maps of (possibly infinite dimensional) vector spaces, has a continuous Lie bracket, and an open profinite dimensional Lie subalgebra.[97]

Suppose moreover that we are given a Harish-Chandra datum $(\mathfrak{h}, K)$ with K a group scheme and $\mathfrak{k} \hookrightarrow \mathfrak{h}$ an *open*[98] subalgebra. We are going to define a filtered category $\mathfrak{h}$–mod acted on by K.

---

[97]As in [Bei] §1.4, a topological Lie algebra structure on a Tate vector space automatically has a basis by open Lie subalgebras.

[98]This is a serious condition: for example, K can not be trivial if the topology on $\mathfrak{h}$ is non-trivial.

First, observe that the *group prestack* $(\mathfrak{h}, K)$ from §A.3.7 still makes sense, receiving a canonical ind-affine nil-isomorphism $\mathbb{B}K \to \mathbb{B}(\mathfrak{h}, K)$. The definition from *loc. cit.* does not make sense as is: de Rham spaces and formal completions are best avoided in infinite type. Instead, we assume $\mathfrak{h} = \mathrm{Lie}(H)$ for a group indscheme H with $K \subseteq H$ a compact open subgroup;[99] then $(\mathfrak{h}, K)$ is the formal completion of K in H. In general, one can appeal e.g. to [BD1] 7.11.2 (v) for the construction.

We form the simplicial diagram:

$$\ldots K\backslash(\mathfrak{h}, K) \overset{K}{\times} (\mathfrak{h}, K)/K \rightrightarrows^{\textstyle\rightarrow} K\backslash(\mathfrak{h}, K)/K \rightrightarrows \mathbb{B}K$$

given by applying the Cech construction to the morphism $\mathbb{B}K \to \mathbb{B}(\mathfrak{h}, K)$; note that the geometric realization of this diagram is $\mathbb{B}(\mathfrak{h}, K)$. Moreover, note that each term in the simplicial diagram is of the form "an ind-finite type indscheme modulo an action of K." Therefore, IndCoh^{ren} makes sense for each term of this diagram. We define $\mathfrak{h}\text{--}\mathsf{mod}^K$ as the totalization:

$$\mathfrak{h}\text{--}\mathsf{mod}^K := \mathrm{Tot}\left(\mathsf{Rep}(K) = \mathsf{IndCoh}^{ren}(\mathbb{B}K) \rightrightarrows \mathsf{IndCoh}^{ren}(K\backslash(\mathfrak{h}, K)/K) \rightrightarrows^{\textstyle\rightarrow} \ldots \right)$$

Since all the morphisms in our simplicial diagram are ind-affine nil-isomorphisms, this is a co/limit. The Beck-Chevalley formalism easily implies that the forgetful functor $\mathrm{Oblv} : \mathfrak{h}\text{--}\mathsf{mod}^K \to \mathsf{Rep}(K)$ is monadic, and in particular, admits a left adjoint $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{h}}$.

It is easy to see that $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{h}} \circ \mathrm{Oblv}$ is conservative and t-exact, so Oblv is monadic and $\mathfrak{h}\text{--}\mathsf{mod}^K$ has a canonical t-structure with Oblv and $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{h}}$ both t-exact. We have:

Lemma A.8.1 (Bernstein-Lunts, [BL]). *If $K = \lim_i K/K_i$ is a countable inverse limit, $\mathfrak{h}\text{--}\mathsf{mod}^{K,+}$ is the bounded below derived category of the heart of its t-structure.*

Indeed, $\mathsf{Rep}(K)^+ = D^+(\mathsf{Rep}(K)^\heartsuit)$ by Lemma 5.2.5 (which is where the countability hypothesis enters), and then the same argument as in the finite-dimensional Lemma A.4.1 applies.

A.8.2. Now note that the deformations defined in [GR2] §IV.5.2 makes sense in the infinite type setup and are well-behaved in our setup. Applying this to $\mathbb{B}K \to \mathbb{B}(\mathfrak{h}, K)$, we obtain a prestack $\mathsf{Fil}\,\mathbb{B}(\mathfrak{h}, K)$ over $\mathbb{A}^1_\hbar/\mathbb{G}_m$ with special fiber $(\mathbb{B}(\mathfrak{h}/\mathfrak{k})_0^\wedge)/K$.

We can form the above Cech construction along this deformation and imitate the above construction to obtain a filtration on $\mathfrak{h}\text{--}\mathsf{mod}^K$. We claim that $\mathfrak{h}\text{--}\mathsf{mod}^{K,cl}$ is canonically isomorphic to $\mathsf{QCoh}^{ren}((\mathfrak{h}/\mathfrak{k})^\vee/K)$, with $(\mathfrak{h}/\mathfrak{k})^\vee$ the continuous dual considered as an affine scheme. Indeed, we need to compute:

$$\mathrm{Tot}\left(\mathsf{IndCoh}^{ren}(\mathbb{B}K) \rightrightarrows \mathsf{IndCoh}^{ren}((\mathfrak{h}/\mathfrak{k})_0^\wedge/K) \rightrightarrows^{\textstyle\rightarrow} \ldots \right).$$

Here it makes sense to replace $\mathfrak{h}/\mathfrak{k}$ by any K-representation V. Since any K-representation is the union of its finite-dimensional representations, we find that the above is $\mathsf{IndCoh}^{ren}(\mathbb{B}(\mathfrak{h}/\mathfrak{k})_0/K)$ (where the renormalization makes sense because $\mathbb{B}(\mathfrak{h}/\mathfrak{k})_0$ is the appropriate colimit of the classifying stacks acted on by K corresponding to finite-dimensional subrepresentations of $\mathfrak{h}/\mathfrak{k}$). Clearly $\mathsf{IndCoh}^{ren}(\mathbb{B}(\mathfrak{h}/\mathfrak{k})_0^\wedge/K) \simeq \mathsf{QCoh}^{ren}((\mathfrak{h}/\mathfrak{k})^\vee/K)$ as desired.

Finally, note that if $K = \lim_i K/K_i$ as before, then the above construction makes sense for each of the compact open normal subgroup schemes $K_i \subseteq K$. Moreover, K_i/K_j-invariants for $\mathsf{Fil}\,\mathfrak{h}\text{--}\mathsf{mod}^{K_j}$ are easily seen to give $\mathsf{Fil}\,\mathfrak{h}\text{--}\mathsf{mod}^{K_i}$.

This is exactly the data to define the filtered category $\mathfrak{h}\text{--}\mathsf{mod}$ acted on by K. Note that:

---

[99]In particular, we assume H/K is ind-finite type: this forces H to be reasonable in the sense of [BD1].

$$\mathfrak{h}\text{--mod}^{cl} = \mathsf{IndCoh}(\mathfrak{h}^{\vee}) := \operatorname*{colim}_i \mathsf{QCoh}((\mathfrak{h}/\mathfrak{k}_i)^{\vee}) \in \mathsf{DGCat}_{cont}$$

with the colimit being under $*$-pushforward functors; we label this colimit as IndCoh for the same reason as in §A.7.

More precisely, recall that our semi-classical data is a renormalized sheaf of categories on $\mathfrak{k}^{\vee}/K$. This sheaf of categories is described in the same way as in §A.7.4.

A.8.3. Next, we note that the above makes sense even if $\mathfrak{k}$ is not an open subalgebra.

More precisely, and with apologies for the notation change, choose H_0 a group scheme and a Harish-Chandra datum $(H_0, \mathfrak{h})$ with $\mathfrak{h}_0 \subseteq \mathfrak{h}$ open. Then suppose that K is a group subscheme of H_0, with no hypothesis that it be compact open (e.g., K could be trivial).

Then as above, we have an action of H_0 on $\mathsf{Fil}\,\mathfrak{h}\text{--mod}$. We claim that given any action of H_0 on $\mathsf{Fil}\,\mathcal{C}$, we can restrict to obtain an action of K on $\mathsf{Fil}\,\mathcal{C}$.

Indeed, if $H_0 = \lim_i H_0/H_i$ for H_i a normal subgroup scheme, note that $H_i K$ is a compact open subgroup scheme of H; we then set:

$$\mathsf{Fil}\,\mathcal{C}^K := \operatorname*{colim}_i \mathsf{Fil}\,\mathcal{C}^{H_i K} \in \mathsf{DGCat}_{cont}.$$

We remark that this is a co/limit. Replacing K by a compact open subgroup (of K), we obtain the requisite data.

A.8.4. *Semi-infinite cohomology.* We now make a more stringent assumption on $\mathfrak{h}$: suppose that it is a *union*[100] of open pro-finite dimensional subalgebras $\mathfrak{h} = \operatorname{colim}_i \mathfrak{h}_i$. We fix an initial index "0" and let $\mathfrak{h}_0$ denote the corresponding open subalgebra.

We assume that for every $\mathfrak{h}_i \subseteq \mathfrak{h}_j$, the action of $\mathfrak{h}_i$ on $\det(\mathfrak{h}_j/\mathfrak{h}_i)$ is trivial; e.g., this is automatically the case if $\mathfrak{h}$ is ind-pronilpotent. For later use, we observe that in this case:

$$\operatorname{ind}_{\mathfrak{h}_1}^{\mathfrak{h}_2}(M \otimes \det(\mathfrak{h}_j/\mathfrak{h}_i)) = \operatorname{ind}_{\mathfrak{h}_1}^{\mathfrak{h}_2}(M) \otimes \det(\mathfrak{h}_j/\mathfrak{h}_i) \tag{A.8.1}$$

since we are just tensoring by a line. (This line is essentially just a placeholder, ensuring the canonicity of various isomorphisms.)

In this case, we obtain a *semi-infinite cohomology* functor:

$$C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, -) : \mathfrak{h}\text{--mod} \to \mathsf{Vect}$$

defined as follows.

Note that any of the compact open subalgebras $\mathfrak{h}_i$, we have a *forgetful* functor $\mathfrak{h}\text{--mod} \to \mathfrak{h}_i\text{--mod}$, which is conservative and admits the left adjoint $\operatorname{ind}_{\mathfrak{h}_i}^{\mathfrak{h}}$. Then we claim that the induced map:

$$\mathfrak{h}\text{--mod} \to \lim_i \mathfrak{h}_i\text{--mod}$$

is an equivalence. Indeed, both sides are clearly monadic over $\mathfrak{h}_0\text{--mod}$. This is a co/limit, so we also obtain:

$$\operatorname*{colim}_i \mathfrak{h}_i\text{--mod} \xrightarrow{\simeq} \mathfrak{h}\text{--mod} \in \mathsf{DGCat}_{cont}$$

where we are using the induction functors on the left hand side.

We then define a functor:

---

[100]Note that this assumption is not satisfied for the Kac-Moody Lie algebra. There is a semi-infinite cohomology theory for such algebras, but it is a more subtle and will not be needed in this paper.

$$C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, \operatorname{ind}_{\mathfrak{h}_i}^{\mathfrak{h}}(-)) : \mathfrak{h}_i\text{–mod} \to \mathsf{Vect}$$

as:

$$C^\bullet(\mathfrak{h}_i, (-) \otimes \det(\mathfrak{h}_i/\mathfrak{h}_0)[\dim \mathfrak{h}_i/\mathfrak{h}_0]).$$

By Lemma A.7.10 (1) and (A.8.1), for $\mathfrak{h}_i \subseteq \mathfrak{h}_j \subseteq \mathfrak{h}$, we have canonical isomorphisms:

$$C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, \operatorname{ind}_{\mathfrak{h}_i}^{\mathfrak{h}}(-)) \simeq C^{\frac{\infty}{2}}\big(\mathfrak{h}, \mathfrak{h}_0, \operatorname{ind}_{\mathfrak{h}_j}^{\mathfrak{h}_i} \operatorname{ind}_{\mathfrak{h}_i}^{\mathfrak{h}_j}(-)\big).$$

These are compatible as we vary indices, so we obtain the desired functor $C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, -)$.

Notation A.8.2. This functor depends only in a mild way on $\mathfrak{h}_0$, but it is convenient for our purposes to keep it in the notation.

Remark A.8.3. Here is another perspective. Note that by the general co/lim formalism for a filtered diagram, any $M \in \mathfrak{h}$–mod can be written as $\operatorname{colim}_i \operatorname{ind}_{\mathfrak{h}_i}^{\mathfrak{h}} M \in \mathfrak{h}$–mod, i.e., we forget down to $\mathfrak{h}_i$ and then induce. So we find:

$$C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, M) = \operatorname*{colim}_i C^\bullet(\mathfrak{h}_i, M \otimes \det(\mathfrak{h}_i/\mathfrak{h}_0)[\dim \mathfrak{h}_i/\mathfrak{h}_0]).$$

Applying this formula to $M \in \mathfrak{h}$–mod$^\heartsuit$ (or a bounded below chain complex of such objects) and computing $C^\bullet(\mathfrak{h}_i, -)$ by the standard resolution, one recovers the usual complex computing semi-infinite cohomology in this case; i.e., this perspective recovers the classical one.

The above analysis applies just as well in the filtered setting, so we obtain a canonical filtration on $C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, -)$. The semi-classical functor:

$$\mathsf{IndCoh}(\mathfrak{h}^\vee) \to \mathsf{Vect}$$

is given by !-restricting[101] to obtain an object of $\mathsf{QCoh}((\mathfrak{h}/\mathfrak{h}_0)^\vee)$; and noting that this is QCoh of an affine scheme, we then take $*$-restriction to $0 \in (\mathfrak{h}/\mathfrak{h}_0)^\vee$. Indeed, by Lemma A.7.10, the functor $C^\bullet(\mathfrak{h}_i, (-) \otimes \det(\mathfrak{h}_i/\mathfrak{h}_0)[\dim \mathfrak{h}_i/\mathfrak{h}_0])$ semi-classically gives the functor:

$$\mathsf{IndCoh}(\mathfrak{h}_i^\vee) \to \mathsf{Vect}$$

that is the composition of !-restricting to $(\mathfrak{h}_i/\mathfrak{h}_0)^\vee$ and then $*$-restricting to 0; passing to the colimit in i gives the claim.

Remark A.8.4. From the perspective of Remark A.8.3, we have:

$$F_i C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, M) = \operatorname*{colim}_i F_{i-\dim \mathfrak{h}_i/\mathfrak{h}_0} C^\bullet(\mathfrak{h}_i, M \otimes \det(\mathfrak{h}_i/\mathfrak{h}_0)[\dim \mathfrak{h}_i/\mathfrak{h}_0]).$$

(The shift in indexing reflects the repeatedly emphasized fact that our determinant lines are considered as filtered with a jump in degree $\dim \mathfrak{h}_i/\mathfrak{h}_0$.)

---

[101]Precisely, recall that $\mathsf{IndCoh}(\mathfrak{h}^\vee)$ was defined as the colimit under pushforwards of QCoh of its reasonable subschemes; then our !-restriction here is the right adjoint to the structural functor $\mathsf{QCoh}((\mathfrak{h}/\mathfrak{h}_0)^\vee) \to \mathsf{IndCoh}(\mathfrak{h}^\vee)$.

A.8.5.   Now observe that the above all makes sense in the Harish-Chandra setting as well. Indeed, if we have the Harish-Chandra datum $(\mathfrak{h}, K)$ with K a group scheme, then note that $\mathfrak{k} \subseteq \mathfrak{h}_i$ for $i \gg 0$, so the formula:

$$C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, K, -) := \operatorname*{colim}_i C^\bullet(\mathfrak{h}_i, K, M \otimes \det(\mathfrak{h}_i/\mathfrak{h}_0)[\dim \mathfrak{h}_i/\mathfrak{h}_0])$$

makes sense by cofinality and defines a filtered functor:

$$C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, K, -) : \mathfrak{h}\text{–mod}^K \to \mathsf{Vect}.$$

If $\mathfrak{h}_0 = \mathfrak{k}$, then the corresponding semi-classical functor:

$$\mathsf{IndCoh}^{ren}((\mathfrak{h}/\mathfrak{k})^\vee/K) \to \mathsf{Vect}$$

is given by !-restriction to $0/K$ followed by group cohomology with respect to K. If $\mathfrak{h}_0 \subseteq \mathfrak{k}$ with $\det(\mathfrak{k}/\mathfrak{h}_0)$ a trivial $\mathfrak{h}_0$-representation, then we can twist to reduce to this case, i.e., the functor:

$$C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, K, (-) \otimes \det(\mathfrak{k}/\mathfrak{h}_0)^\vee[-\dim \mathfrak{k}/\mathfrak{h}_0]) : \mathfrak{h}\text{–mod}^K \to \mathsf{Vect}.$$

will have the semi-classical functor described above.

Notation A.8.5.   When $\mathfrak{h}_0 = \mathfrak{k}$, we use the notation $C^{\frac{\infty}{2}}(\mathfrak{h}, K, -)$ in place of $C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}, K, -)$.

A.8.6.   *KK version.* Suppose now that $(\mathfrak{h}, K)$ is equipped with a $\mathbb{G}_m$-action as before. Suppose moreover that K is equipped with a character $\psi : K \to \mathbb{G}_a$ that is $\mathbb{G}_m$-equivariant for the inverse homothety action on the target.

   Then $\mathfrak{h}\text{–mod}^{K,\psi}$ makes sense, and inherits PBW and KK filtrations fitting into a bifiltration as before; this is immediate from our earlier constructions and the general KK formalism.

   We have:

$$\mathfrak{h}\text{–mod}^{K,\psi,PBW-cl} = \mathsf{IndCoh}^{ren}((\mathfrak{h}/\mathfrak{k})^\vee/K)$$

$$\mathfrak{h}\text{–mod}^{K,\psi,KK-cl} = \mathsf{IndCoh}^{ren}(\psi + (\mathfrak{h}/\mathfrak{k})^\vee/K)$$

as before. The KK filtration on the former and the PBW filtration on the latter are as in Remark A.5.12.

Remark A.8.6.   Suppose now that $\mathfrak{h}$ satisfies the hypotheses of §A.8.4, and let $\mathfrak{h}_0$ be as in *loc. cit.* Suppose that $\psi : \mathfrak{k} \to k$ is extended to a character $\psi : \mathfrak{h} \to k$, so the functor:

$$C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, K, (-) \otimes -\psi) : \mathfrak{h}\text{–mod}^{K,\psi} \to \mathsf{Vect}$$

makes sense. Note that if the subalgebras $\mathfrak{h}_i \subseteq \mathfrak{h}$ are all graded subalgebras (with respect to the grading on $\mathfrak{h}$ induced by the $\mathbb{G}_m$-action), and $\psi : \mathfrak{h} \to k$ is graded for the degree -1 grading on the target, then $C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{h}_0, K, (-) \otimes -\psi)$ is naturally bifiltered. If $\mathfrak{k} = \mathfrak{h}_0$, then the induced semi-classical functors:

$$\mathsf{QCoh}^{ren}((\mathfrak{h}/\mathfrak{k})^\vee/K) \to \mathsf{Vect}$$

$$\mathsf{QCoh}^{ren}(\psi + (\mathfrak{h}/\mathfrak{k})^\vee/K) \to \mathsf{Vect}$$

are given by $*$-restriction to $0/K$ and ψ/K followed by global sections, i.e., group cohomology with respect to K. Here we note that our hypothesis means $\mathfrak{k}$ is open in $\mathfrak{h}$, so $(\mathfrak{h}/\mathfrak{k})^\vee$ is a *scheme*, not an indscheme; so our definition of IndCoh^{ren} in this infinite type setting means that it tautologically coincides with QCoh^{ren}.

A.8.7. *Central extensions.* Finally, we explain the straightforward extension of the above to *central extensions* of $\mathfrak{h}$.

A.8.8. Suppose that in the above notation, we are given $\widehat{\mathfrak{h}}$ a Tate Lie algebra and a central extension:

$$0 \to k \to \widehat{\mathfrak{h}} \to \mathfrak{h} \to 0$$

of $\mathfrak{h}$ by the abelian Lie algebra k.

We suppose that K is as in §A.8, so $\mathfrak{k} \subseteq \mathfrak{h}$ is an open subalgebra. We suppose moreover that we are given a Harish-Chandra datum $(\widehat{\mathfrak{h}}, K \times \mathbb{G}_m)$ compatible in the sense that the projections:

$$\widehat{\mathfrak{h}} \to \mathfrak{h}$$

$$K \times \mathbb{G}_m \xrightarrow{p_1} K$$

induce a morphism of Harish-Chandra data. Moreover, we assume that the structural morphism $k \xrightarrow{x \mapsto (0,x)} \mathfrak{k} \times k = \mathrm{Lie}(K \times \mathbb{G}_m) \to \widehat{\mathfrak{h}}$ is the given embedding of k into $\widehat{\mathfrak{h}}$; and that the $\mathbb{G}_m \subseteq K \times \mathbb{G}_m$-action on $\widehat{\mathfrak{h}}$ is trivial.

Remark A.8.7. Note that the extension $\widehat{\mathfrak{h}}$ is canonically split over $\mathfrak{k}$.

Remark A.8.8. Of course, everything that follows generalizes to the case where the central $\mathbb{G}_m$ is replaced by a torus. We mention this because it is necessary for the setting of affine Kac-Moody algebras for non-simple $\mathfrak{g}$, as in §1.8.2.

A.8.9.   We want to form the DG category $\widehat{\mathfrak{h}}_1$–mod, which morally is the DG category of $\widehat{\mathfrak{h}}$-modules on which $1 \in k \subseteq \widehat{\mathfrak{h}}$ acts by the identity. For this, we will construct an action of $\mathsf{QCoh}(\mathbb{A}^1)$ on $\widehat{\mathfrak{h}}$–mod; it then makes sense to take the fiber at $1 \in \mathbb{A}^1$ (by tensoring with the restriction to 1 functor $\mathsf{QCoh}(\mathbb{A}^1) \to \mathsf{Vect}$).

Indeed, note that our Harish-Chandra datum induces an action of $\mathbb{G}_{m,dR}$ on $\mathbb{B}(\widehat{\mathfrak{h}}, K)$: $\mathbb{G}_m$ acts because it acts on $\mathfrak{h}$ commuting with K, and the action of the formal group is trivial because our Harish-Chandra data was extended to $(\widehat{\mathfrak{h}}, K \times \mathbb{G}_m)$. Moreover, the underlying $\mathbb{G}_m$-action is canonically trivial, because $\mathbb{G}_m$ acts trivially on $\widehat{\mathfrak{h}}$. Therefore, we obtain an action of $\mathbb{B}\widehat{\mathbb{G}}_m = \mathbb{G}_{m,dR}/\mathbb{G}_m$ on $\mathbb{B}(\widehat{\mathfrak{h}}, K)$.[102]

Then observe that $\mathsf{QCoh}(\mathbb{B}\widehat{\mathbb{G}}_m) \simeq \mathsf{QCoh}(\mathbb{A}^1)$, with the convolution monoidal structure on the left hand side corresponding to the tensor product on the right hand side, so we obtain the desired action on $\widehat{\mathfrak{h}}$–mod by definition of this category, and therefore the definition of the category $\widehat{\mathfrak{h}}_1$–mod.

Example A.8.9. A splitting of the Lie algebra morphism $\widehat{\mathfrak{h}} \to \mathfrak{h}$ gives an identification $\widehat{\mathfrak{h}}_1$–mod $\simeq \mathfrak{h}$–mod compatible with all extra structures.

A.8.10.   There is a filtered version of the above, quite similar to §A.5.5. We use the $\mathbb{G}_m$ action on $\mathrm{Fil}\,\widehat{\mathfrak{h}}$–mod as above, finding that $\widehat{\mathfrak{h}}_1$–mod is filtered with semi-classical category $\mathsf{IndCoh}(\mathfrak{h}^\vee)$. So the situation is not sensitive to the central extension.

The rest of the usual package generalizes as is to this setting. We have an action of K on $\mathrm{Fil}\,\widehat{\mathfrak{h}}_1$–mod, so obtain a filtration on $\widehat{\mathfrak{h}}_1$–mod^K. If we have compatible $\mathbb{G}_m$-actions on $\widehat{\mathfrak{h}}$ and K with $k \subseteq \widehat{\mathfrak{h}}$ acted on trivially, we obtain a bifiltration on $\widehat{\mathfrak{h}}_1$–mod^K; if K is equipped with an appropriately $\mathbb{G}_m$-equivariant additive character, we obtain a bifiltration on $\widehat{\mathfrak{h}}_1$–mod$^{K,\psi}$ as well. The semi-classical

---

[102]Here $\widehat{\mathbb{G}}_m$ is the formal group of $\mathbb{G}_m$, i.e., the hat notation is being used in a different way from $\widehat{\mathfrak{h}}$.

categories are as expected, i.e., the same as if we worked with $\mathfrak{h}$ instead of its central extension, and the various restriction and induction functors satisfy the standard functoriality properties at the semi-classical level.

APPENDIX B. PROOF OF LEMMA 5.2.1

B.1. **Overview.** We give two proofs of this result: a geometric one based in the theory of D-modules, and a representation theoretic one.

The former approach, which was sketched after the statement of Lemma 5.2.1, is more versatile and conceptual. But for technical reasons, we only know how to apply this method for n sufficiently large.[103]

The second one is more ad hoc. The idea is that we can compute the associated graded of this functor using (the proof of) Theorem 3.1.1 and verify exactness here. However, the problems with unboundedness of Kazhdan-Kostant filtrations come in here, and we use some tricks to circumvent this.

Remark B.1.1. There is a homology between the two approaches: $(\check{\rho}, \alpha_{max})$ is involved in the technical issues on both sides. Perhaps this hints at a more systematic solution.

B.2. **Geometric approach.**

B.2.1.   We begin with the D-module approach. Since $\mathcal{C} = \widehat{\mathfrak{g}}_\kappa$–mod and its Harish-Chandra variants are fairly general examples of categories acted on by a group, we introduce some axiomatics about the relationship between such group actions and t-structures. We then establish general results about Av_* and $\mathrm{Av}_!$, and deduce Lemma 5.2.1 from here.

B.2.2. *Axiomatics.* Fix H an affine algebraic group and $\mathcal{C}$ a DG category acted on weakly by H.

Suppose $\mathcal{C}$ is equipped with a t-structure compatible with filtered colimits. Note that $\mathsf{QCoh}(H) \otimes \mathcal{C}$ inherits a t-structure: $(\mathsf{QCoh}(H) \otimes \mathcal{C})^{\leq 0}$ is generated under colimits by objects $\mathcal{O}_H \boxtimes \mathcal{F}$ for $\mathcal{F} \in \mathcal{C}^{\leq 0}$.

Lemma B.2.1. *The following conditions are equivalent:*

(1) The functor $\mathrm{Oblv} \circ \mathrm{Av}_*^w : \mathcal{C} \to \mathcal{C}$ *is t-exact.*
(2) The functor $\mathrm{act} : \mathsf{QCoh}(H) \otimes \mathcal{C} \to \mathcal{C}$ *is t-exact.*
(3) The functor $\mathrm{coact} : \mathcal{C} \to \mathsf{QCoh}(H) \otimes \mathcal{C}$ *is t-exact.*
(4) $\mathcal{C}^{H,w}$ *admits a t-structure such that* Oblv *and* Av_*^w *are t-exact.*
(5) The $\mathsf{QCoh}(H)$*-linear equivalence:*

$$\mathsf{QCoh}(H) \otimes \mathcal{C} \to \mathsf{QCoh}(H) \otimes \mathcal{C} \qquad\qquad (\mathrm{B}.2.1)$$

induced by $\mathrm{coact} : \mathcal{C} \to \mathsf{QCoh}(H) \otimes \mathcal{C}$ *is t-exact.*

Proof. Note that we have a functor $p_2^* : \mathcal{C} \xrightarrow{\mathcal{F} \mapsto \mathcal{O}_H \boxtimes \mathcal{F}} \mathsf{QCoh}(H) \otimes \mathcal{C}$, which admits the conservative right adjoint $p_{2,*}$. We claim p_2^* and $p_{2,*}$ are t-exact. Indeed, p_2^* is tautologically right t-exact, so $p_{2,*}$ is left t-exact. But from the definition of the t-structure, we see $p_{2,*}$ is right t-exact as well, so t-exact. Then since $p_{2,*}p_2^* = \mathcal{O}_H \otimes -$ is t-exact, we obtain the t-exactness of p_2^* as well.

We will deduce the other conditions from (1). Since e.g. (4) obviously implies it, this suffices.

Recall from the Beck-Chevalley formalism that we have:

---

[103]We remark that this is enough to establish Theorem 5.1.1. In turn, this is enough to show Corollary 7.3.5, which implies Lemma 5.2.1 in general. Also, for $G = GL_n$, Beraldo's refinement [Ber] of Theorem 2.3.1 can be applied to obtain Lemma 5.2.1 at general level (using D-module methods).

$$p_{2,*}\operatorname{coact} = \operatorname{act} p_2^* = \operatorname{Oblv}\operatorname{Av}_*^w.$$

Since $p_{2,*}$ is t-exact and conservative, we see that (1) implies (3).

We now deduce (2) from (1); by the above, we assume (3) as well. Note that t-exactness of coact implies that its right adjoint act is left t-exact. Since $p_2^*(\mathcal{C}^{\leq 0})$ generates $(\mathsf{QCoh}(H)\otimes\mathcal{C})^{\leq 0}$, it suffices to show $\operatorname{act} p_2^*$ is right t-exact, but this is clear since $\operatorname{Oblv}\operatorname{Av}_*^w$ is t-exact by assumption.

For (4), observe that $\mathcal{C}^{H,w}$ is the limit of a cosimplicial diagram with t-exact structure maps in the underlying semi-cosimplicial diagram (by (3)). This implies the existence of a t-structure with $\operatorname{Oblv}:\mathcal{C}^{H,w}\to\mathcal{C}$ t-exact. To see Av_*^w is t-exact, it suffices to see that $\operatorname{Oblv}\operatorname{Av}_*^w$ is, but this is given.

Finally, note that the equivalence (B.2.1) intertwines the functors p_2^* and coact. Therefore, it suffices to see that $\operatorname{coact}(\mathcal{C}^{\leq 0})$ generates $(\mathsf{QCoh}(H)\otimes\mathcal{C})^{\leq 0}$ under colimits. But this follows because act is t-exact and conservative.

$\square$

If these equivalent conditions are satisfied, we say the t-structure is *compatible* with the weak action of H.

B.2.3. Now suppose that H acts strongly on $\mathcal{C}$.

We say that the action is compatible with the t-structure if it is compatible for the weak action. It is equivalent to say that:

$$\operatorname{coact}[-\dim H]:\mathcal{C}\to D(H)\otimes\mathcal{C}$$

is t-exact. As in the weak setting, $\mathcal{C}^H$ inherits a t-structure with $\operatorname{Oblv}:\mathcal{C}^H\to\mathcal{C}$ being t-exact.

Lemma B.2.2. *In the above setting, the functor* $\operatorname{Av}_*:\mathcal{C}\to\mathcal{C}^H$ *has cohomological amplitude* $[0,\dim H]$.

More generally, for $K\subseteq H$ with H/K affine, the functor $\operatorname{Av}_*:\mathcal{C}^K\to\mathcal{C}^H$ *has cohomological amplitude* $[0,\dim H/K]$.

Proof. Av_* is left t-exact because it is right adjoint to a t-exact functor.

For the upper bound on the amplitude, note that weak averaging from $\mathcal{C}^{K,w}\to\mathcal{C}^{H,w}$ is t-exact because H/K is affine. Observe that weak averaging is given by convolution with $D_{H/K}$, $*$-averaging is given by convolution with the constant D-module $k_{H/K}$, and then use the de Rham resolution of $k_{H/K}$, which consists of free D-modules in degrees $[0,\dim H/K]$, to complete the argument.

$\square$

B.2.4. !-*averaging.* We now want a version of the above for !-averaging. It is essentially the same, but slightly more subtle because !-averaging may not be defined.

Moreover, the proof of Lemma B.2.2 in the case where H/K was affine used the fact that de Rham cohomology on an affine scheme is right t-exact. The corresponding fact for compactly supported de Rham cohomology is harder to show (for non-holonomic D-modules), and is the main theorem of [Ras5].

B.2.5. Suppose in the above setting that are given K_1, K_2 two subgroups of H, and characters $\psi_i:K_i\to\mathbb{G}_a$ that coincide on $K_1\cap K_2$. Suppose that for every $\mathcal{C}\in\mathsf{DGCat}_{cont}$ acted on by H, the functor $\operatorname{Av}_!^{\psi_2}:\mathcal{C}^{K_1,\psi_1}\to\mathcal{C}^{K_2,\psi_2}$ given by restricting to the intersection $K_1\cap K_2$ and then !-averaging is defined functorially in $\mathcal{C}$.

Lemma B.2.3. *Suppose $\mathcal{C}$ is acted on by H, and equipped with a t-structure compatible with the action. Suppose moreover that the t-structure on $\mathcal{C}$ is compactly generated, i.e., $\mathcal{C}^{\leq 0}$ is compactly generated (in the sense of general category theory).*

Then under the above hypotheses, if $K_2/K_1 \cap K_2$ is affine, then $\mathrm{Av}_!^{\psi_2} : \mathcal{C}^{K_1, \psi_1} \to \mathcal{C}^{K_2, \psi_2}$ has cohomological amplitude $[-\dim K_2/K_1 \cap K_2, 0]$.

We need the following result, which appeared already as [Gai8] Lemma 4.1.3. We include the proof for the reader's convenience.

Lemma B.2.4. *Let $\mathcal{C} \in \mathsf{DGCat}_{cont}$ be equipped with a compactly generated t-structure. Let $F : \mathcal{D}_1 \to \mathcal{D}_2 \in \mathsf{DGCat}_{cont}$ be given, and suppose that the categories $\mathcal{D}_i$ are equipped with t-structures, and that F is left t-exact. Then:*

$$\mathrm{id}_{\mathcal{C}} \otimes F : \mathcal{C} \otimes \mathcal{D}_1 \to \mathcal{C} \otimes \mathcal{D}_2$$

is left t-exact.

Proof. Let $\mathcal{F} \in \mathcal{C}^{\leq 0}$ be compact. Let $\mathbb{D}\mathcal{F} : \mathcal{C} \to \mathsf{Vect}$ denote the corresponding continuous functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -)$. Note that $\mathbb{D}\mathcal{F}$ is left t-exact because $\mathcal{F} \in \mathcal{C}^{\leq 0}$.

We have induced functors:

$$\mathbb{D}\mathcal{F} \otimes \mathrm{id}_{\mathcal{D}_i} : \mathcal{C} \otimes \mathcal{D}_i \to \mathsf{Vect} \otimes \mathcal{D}_i = \mathcal{D}_i.$$

The main observation is that $\mathcal{G} \in \mathcal{C} \otimes \mathcal{D}_i$ lies in $(\mathcal{C} \otimes \mathcal{D}_i)^{\geq 0}$ if and only if:

$$\mathbb{D}\mathcal{F} \otimes \mathrm{id}_{\mathcal{D}_i}(\mathcal{G}) \in \mathcal{D}_i^{\geq 0}$$

for all $\mathcal{F}$ as above. Indeed, for $\mathcal{H} \in \mathcal{D}_i$, the (possibly non-continuous) composite functor:

$$\mathcal{C} \otimes \mathcal{D}_i \xrightarrow{\mathbb{D}\mathcal{F} \otimes \mathrm{id}_{\mathcal{D}_i}} \mathcal{D}_i \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{D}_i}(\mathcal{H}, -)} \mathsf{Vect}$$

coincides with $\underline{\mathrm{Hom}}_{\mathcal{C} \otimes \mathcal{D}_i}(\mathcal{F} \boxtimes \mathcal{H}, -)$, as follows by observing that it is the right adjoint to the functor $k \mapsto \mathcal{F} \boxtimes \mathcal{H}$. Taking $\mathcal{H} \in \mathcal{D}_i^{\leq 0}$, this immediately implies the observation.

Therefore, we need to show that for $\mathcal{G} \in (\mathcal{C} \otimes \mathcal{D}_1)^{\geq 0}$, we have:

$$(\mathbb{D}\mathcal{F} \otimes \mathrm{id}_{\mathcal{D}_2}) \circ (\mathrm{id}_{\mathcal{C}} \otimes F)(\mathcal{G}) \in \mathcal{D}_2^{\geq 0}$$

for all compact $\mathcal{F} \in \mathcal{C}^{\leq 0}$. By functoriality, we have:

$$(\mathbb{D}\mathcal{F} \otimes \mathrm{id}_{\mathcal{D}_2}) \circ (\mathrm{id}_{\mathcal{C}} \otimes F)(\mathcal{G}) = F \circ (\mathbb{D}\mathcal{F} \otimes \mathrm{id}_{\mathcal{D}_1})(\mathcal{G}).$$

Because $(\mathbb{D}\mathcal{F} \otimes \mathrm{id}_{\mathcal{D}_1})(\mathcal{G}) \in \mathcal{D}_1^{\geq 0}$ by the above, we obtain the claim from left t-exactness of F.

$\square$

Proof of Lemma B.2.3. The functor $\mathrm{Av}_!^{\psi_2}$ is right t-exact because it is a left adjoint to a t-exact functor. So it remains to show the other bound.

First, suppose that $\mathcal{C} = D(H)$. Then $D(H)^{K_1, \psi_1}$ is compactly generated by coherent D-modules. Therefore, for the t-structure on $D(H)^{K_1, \psi_1}$, compact objects are closed under truncations. So it suffices to show that every compact object of $D(H)^{K_1, \psi_1, \geq 0}$ maps to $D(H)^{K_2, \psi_2, \geq -\dim K_2/K_1 \cap K_2}$.

This follows immediately from the fact that !-pushforward is left t-exact on coherent objects, which is Theorem 3.3.1 of [Ras5]. (The cohomological shift by $\dim K_2/K_1 \cap K_2$ arises because !-averaging is !-convolution with a dualizing D-module.)

For general $\mathcal{C}$, we use the commutative diagram:

$$
\begin{array}{ccc}
\mathcal{C}^{K_1,\psi_1} & \xrightarrow{\text{coact}} & D(H)^{K_1,\psi_1} \otimes \mathcal{C} \\
\downarrow{\scriptstyle \mathrm{Av}_!^{\psi_2}} & & \downarrow{\scriptstyle \mathrm{Av}_!^{\psi_2} \otimes \mathrm{id}_\mathcal{C}} \\
\mathcal{C}^{K_2,\psi_2} & \xrightarrow{\text{coact}} & D(H)^{K_2,\psi_2} \otimes \mathcal{C}.
\end{array}
$$

The horizontal arrows are obviously conservative and t-exact up to shift (by assumption on the action on $\mathcal{C}$), while the right vertical arrow has the correct amplitude by the above and Lemma B.2.4. This immediately implies the same for the left vertical arrow.

$\square$

Remark B.2.5. In the case $\mathcal{C} = \mathfrak{h}\text{–mod}$, the argument given amounts to using the Beilinson-Bernstein localization functor to pass from the Lie algebra to D-modules.

Remark B.2.6. The above works just as well when H is a group scheme and the K_i are compact open subgroup schemes: indeed, there is a normal compact open subgroup scheme of H contained in the K_i, reducing the problem to the finite-dimensional version. But it is not clear how to show the lemma for H being the loop group, since coact is no longer t-exact up to shift (it maps into infinitely connective objects).

Remark B.2.7. The above works just as well in the setting of twisted D-modules.

B.2.6. *Geometric proof of Lemma 5.2.1 for n large enough.* We will show Lemma 5.2.1 for $n \geqslant (\check{\rho}, \alpha_{max})$ (alias: the Coxeter number of G minus 1).

The right exactness is immediately given by Lemma B.2.2. The issue in applying Lemma B.2.3 is that we need $\mathrm{Av}_!$ to be defined and functorial for subgroups of a group scheme, not a group indscheme such as $G(K)$.

But in the given range of n, $\mathring{I}_n$ and $\mathring{I}_{n+1}$ are both contained in $\mathrm{Ad}_{-(n+1)\check{\rho}(t)} G(O)$. Since the existence and functoriality of $\mathrm{Av}_!$ is really about convolution identities, this means that for any category strongly acted on by $\mathrm{Ad}_{-(n+1)\check{\rho}(t)} G(O)$, we can !-average from $(\mathring{I}_n, \psi)$ to $(\mathring{I}_{n+1}, \psi)$, and this !-averaging coincides with the $*$-averaging up to the shift by 2Δ from Theorem 2.3.1. Now Lemma B.2.3 applies and gives the desired left t-exactness for $m = n + 1$, which evidently suffices.

Remark B.2.8. If for $\mathcal{C}$ we had D-modules on a reasonable indscheme X acted on by $G(K)$ (or the κ-twisted version of this notion), then we could apply [Ras5] directly, without needing the general Lemma B.2.3. That is, we would not need any restrictions on n.

B.3. **Representation theoretic approach.** We now indicate a representation theoretic approach to treat Lemma 5.2.1 for all n.

Proof of Lemma 5.2.1.

Step 1. Note that by the general formalism from Appendix A, $\iota_{n,m,*} : \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) \to \mathsf{Whit}^{\leqslant m}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ is filtered for the KK filtration with associated semi-classical functor:

$$
\mathsf{QCoh}^{ren}(f + \mathrm{Lie}\,\mathring{I}_n^\perp/\mathring{I}_n) \to \mathsf{QCoh}^{ren}(f + \mathrm{Lie}\,\mathring{I}_m^\perp/\mathring{I}_m)
$$

given by push/pull along the correspondence:

$$
f + \mathrm{Lie}\,\mathring{I}_n^\perp \cap \mathrm{Lie}\,\mathring{I}_m^\perp/\mathring{I}_n \cap \mathring{I}_m
$$

(B.3.1)

$$
f + \mathrm{Lie}\,\mathring{I}_n^\perp/\mathring{I}_n \qquad\qquad f + \mathrm{Lie}\,\mathring{I}_m^\perp/\mathring{I}_m
$$

up to cohomological shift and a determinant twist. The main observation is that this functor is t-exact. (The "up to cohomological shift" is compatible with the shift by $(m-n)\Delta$ in Lemma 5.2.1.)

Indeed, the pushforward in this correspondence is obviously t-exact because the map is affine. It remains to see that the left leg of the correspondence is flat (and in fact, smooth).

This follows from the explicit description of both sides from the proof of Theorem 3.1.1. Indeed, first say $n > 0$ for simplicity. Then both sides are classifying stacks over $f + t^{-n} \operatorname{Ad}_{-n\check{\rho}(t)} \mathfrak{b}^e[[t]]$ by *loc. cit.* Moreover, the relevant group schemes are congruence subgroups of jets into the group scheme of regular centralizers. We then obtain the claim from the smoothness of that group scheme.

If $n = 0$ and $m > n$, then the relevant map $f + \mathfrak{g}[[t]] \cap \operatorname{Lie} \mathring{I}_m^\perp / G(O) \cap \mathring{I}_m \to \mathfrak{g}[[t]]/G(O)$ factors through $\mathfrak{g}^{reg}(O)/G(O)$, which is the classifying stack over $f + \mathfrak{b}^e[[t]]$ of jets into regular centralizers. So the same analysis applies.

Step 2. To show $\iota_{n,m,*}[(m-n)\Delta]$ is t-exact, it suffices to show that it is left t-exact, since Lemma B.2.2 implies the right t-exactness. For this, it suffices to show that it suffices to show that for $\mathcal{F} \in \widehat{\mathfrak{g}}_\kappa\text{--mod}^{\mathring{I}_n,\psi,\heartsuit}$, $\iota_{n,m,*}(\mathcal{F})[(m-n)\Delta]$ is also in cohomological degree 0.

For $n > 0$, it suffices to take $\mathcal{F}$ to be a quotient of $\operatorname{ind}_{\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)$. Indeed, such quotients generate the abelian category under extensions and filtered colimits. Similarly, for $n = 0$, it suffices to take $\mathcal{F}$ to be a quotient of a Weyl module (i.e., a quotient of $\mathbb{V}_\kappa^\lambda := \operatorname{ind}_{\mathfrak{g}[[t]]}^{\widehat{\mathfrak{g}}_\kappa}(V^\lambda)$ for $\lambda \in \Lambda^+$ a dominant coweight, where V^λ is the highest weight representation of G, and is acted on by $\mathfrak{g}[[t]]$ through the quotient $\mathfrak{g}$).

Here is a wrong conclusion to the argument, which we correct in what follows. The modules $\operatorname{ind}_{\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)$ (resp. $\mathbb{V}_\kappa^\lambda$) have KK filtrations, so the quotient $\mathcal{F}$ inherits one as well. Therefore, $\iota_{n,m,*}(\mathcal{F})$ has a canonical filtration. By Step 1, $\operatorname{gr}_\bullet \iota_{n,m,*}(\mathcal{F})[(m-n)\Delta]$ is concentrated in cohomological degree 0.

However, because the KK filtration on $\operatorname{ind}_{\mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)$ is not bounded below, *it is not clear that the filtration on* $\Psi(\mathcal{F})$ *is bounded below in this case* (and probably it is not). That is, the argument from the proof of Theorem 4.2.1 does not adapt well to this setting. So we give a different method below, which essentially uses different bookkeeping to avoid this issue.

Step 3. Of course, it suffices to treat the case where G is not a torus, so we assume this in one follows. We first additionally suppose that $n > 0$.

Let $h \in \mathbb{Q}^{>1}$ be a rational number (greater than 1) to be specified later. This choice defines a grading on the Kac-Moody algebra with degrees lying in $h\mathbb{Z} \subseteq \mathbb{Q}$ as follows. Note that the Kac-Moody algebra has canonical $L_0 := t\partial_t$ and $\check{\rho}$-gradings. Consider it as equipped with the grading $-h\check{\rho} - (h-1)L_0$ (so e.g., $t^i e_\alpha$ has degree $-h(\check{\rho}, \alpha) - (h-1)i$).

The subalgebras $\operatorname{Lie} \mathring{I}_n, \operatorname{Lie} \mathring{I}_m$ are obviously graded. Moreover, the character $\psi : \operatorname{Lie} \mathring{I}_n \to k$ vanishes on homogeneous components apart from degree -1, so we can use the KK formalism from Appendix A. Note that there is no problem in using fractional indices, though our filtration will be graded similarly. (Clearing denominators, it is the same as renormalizing the PBW filtration to have the same associated graded, but with jumps only at multiples of the denominator of h.) Let us refer to this as the KK' filtration on $\widehat{\mathfrak{g}}_\kappa\text{--mod}$, etc. Note that if $h = 1$, this is recovering the usual KK filtration.

A straightforward calculation, which is performed in the next step, shows that we can take h so that the induced KK' filtration on $\operatorname{ind}_{\operatorname{Lie} \mathring{I}_n}^{\widehat{\mathfrak{g}}_\kappa}(\psi)$ to be *bounded from below* (it is essential that $n > 0$ here).

Of course, the same boundedness occurs for the induced KK' filtration on $\mathcal{F}$, any quotient of our induced module.

It is straightforward to see that the induced KK' filtration on:

$$C^\bullet(\mathrm{Lie}\,\mathring{I}_m, \mathring{I}_n \cap \mathring{I}_m, \mathcal{F} \otimes -\psi_{\mathring{I}_m})$$

is then bounded from below as well. First, observe that (for any $h > 1$) there is a compact open subalgebra of $\mathrm{Lie}\,\mathring{I}_n$ on which the $-h\check{\rho} - (h-1)L_0$-degrees are negative. It follows that the degrees on $\Lambda^i\big(\mathrm{Lie}\,\mathring{I}_n\big)^\vee$ are bounded from below *independently of i*, since a compact open subalgebra has finite codimension. This shows that the induced filtration on:

$$C^\bullet(\mathring{I}_n, \mathcal{F} \otimes -\psi)$$

is bounded from below, or similarly for $\mathring{I}_n \cap \mathring{I}_m$. The Harish-Chandra cohomology appearing above differs from the latter group cohomology by tensoring with the exterior algebra of $\mathrm{Ad}_{-m\check{\rho}(t)\mathfrak{n}[[t]]}/\mathrm{Ad}_{-n\check{\rho}(t)}\,\mathfrak{n}[[t]]$, so the result follows.

This Chevalley complex computes:

$$\underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_\kappa-\mathsf{mod}^{\mathring{I}_m}}\big(\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\,\mathring{I}_m}\,\psi, \iota_{n,m,*}(\mathcal{F})\big)$$

by definition of $\iota_{n,m,*}$ as $*$-averaging. To compute the associated graded, one takes $\mathrm{gr}^{KK'}_\bullet(\mathcal{F}) \in \mathsf{QCoh}(f + \mathrm{Lie}\,\mathring{I}_n^\perp/\mathring{I}_n)^\heartsuit$, applies pull-push along the correspondence (B.3.1), applies the cohomological shift by $(m-n)\Delta$ and the determinant twist, and then applies global sections on the stack $f + \mathrm{Lie}\,\mathring{I}_m^\perp/\mathring{I}_m$.

The upshot is that the resulting object of Vect is in cohomological degrees $\geqslant (m-n)\Delta$ by the exactness of our pull-push operation and because of the cohomological shift. This means the same is true for the Chevalley complex above. Because $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\,\mathring{I}_m}\,\psi$ generates $\mathsf{Whit}^{\leqslant m}(\widehat{\mathfrak{g}}_\kappa-\mathsf{mod})^{\leqslant 0}$ under colimits, we finally obtain that $\iota_{n,m,*}(\mathcal{F})$ is in cohomological degrees $\geqslant (m-n)\Delta$, hence is in degree $(m-n)\Delta$, as was desired.

Step 4. It remains to define h and check the desired boundedness. For this, let α_{max} denote the highest root, and take:

$$h := \frac{n(\check{\rho}, \alpha_{max})}{1 + (n-1)(\check{\rho}, \alpha_{max})}.$$

(E.g., for $n = 1$, h is one less than the Coxeter number of G.)

We want to see that KK' filtration on $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\,\mathring{I}_n}(\psi)$ is bounded below: in fact, we will see $F^{KK'}_{-1}\,\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\,\mathring{I}_n}(\psi) = 0$. It suffices to show that the non-zero graded degrees on:

$$\mathrm{gr}^{KK'}_\bullet\,\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}\,\mathring{I}_n}(\psi) = \mathrm{Sym}^\bullet(\mathfrak{g}((t))/\mathrm{Lie}\,\mathring{I}_n)$$

are $\geqslant 0$. Note that in the notation from the proof of Lemma 5.3.2, this associated graded is an algebra generated by elements $\frac{e_\alpha}{t^r}$ ($r \geqslant n(\check{\rho}, \alpha) + 1$) and $\frac{f_\beta}{t^r}$ ($r \geqslant -n(\check{\rho}, \beta) - n + 1$), which have gradings:

$$-h(\check{\rho}, \alpha) + (h-1)r + 1$$
$$h(\check{\rho}, \beta) + (h-1)r + 1.$$

We need to show that these numbers are each $\geqslant 0$ for α (resp. β) a positive root (resp. or zero) and r in the appropriate range.

Regarding the "α inequality," note that:

$$h \geqslant \frac{n(\check{\rho}, \alpha)}{1 + (n-1)(\check{\rho}, \alpha)}. \tag{B.3.2}$$

Then the bound on r means the KK' degree of $\frac{e_\alpha}{t^r}$ is:

$$-h(\check{\rho}, \alpha) + (h-1)r + 1 \geqslant -h(\check{\rho}, \alpha) + (h-1)(n(\check{\rho}, \alpha) + 1) + 1 = h\big(1 + (n-1)(\check{\rho}, \alpha)\big) - n(\check{\rho}, \alpha)$$

which is non-negative by (B.3.2).

For the second inequality, first note that:[104]

$$(n-1)h = \frac{n(n-1)}{\frac{1}{(\check{\rho}, \alpha_{max})} + n - 1} < \frac{n(n-1)}{n-1} = n. \tag{B.3.3}$$

Then the bound on r gives the degree of $\frac{f_\beta}{t^r}$ as:

$$h(\check{\rho}, \beta) + (h-1)r + 1 \geqslant h(\check{\rho}, \beta) + (h-1)\big(-n(\check{\rho}, \beta) - n + 1\big) + 1 = (\check{\rho}, \beta)\big(n - (n-1)h\big)$$

which is non-negative by (B.3.3) (recall our normalization that β is 0 or a *positive* root).

Step 5. Finally, we treat the case $n = 0$. Here are three arguments.

Observe that (e.g. by Theorems 5.1.1 and 5.3.4), it suffices to show that $\Psi : \widehat{\mathfrak{g}}_\kappa\text{–mod}^{G(O)} \to \mathsf{Vect}$ is t-exact.

First, this result can be found in the literature: at non-critical level, this is [FG5] Proposition 2 plus the Sugawara construction, and at critical level this is [FG4] Theorem 3.2.

Second, one can organize the above differently: [FG5] uses Arakawa exactness in an essential way, and our generalization Corollary 7.2.4 of it, which removes the use of the *extended* affine Kac-Moody algebra, allows one to use the Frenkel-Gaitsgory method directly.

Finally, note that any object of $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{G(O)}$ has a $\check{\rho}$-grading, and morphisms preserve these gradings. Therefore, $\mathcal{F}$ (:= a quotient of $\mathbb{V}_\kappa^\lambda$) has canonical $\check{\rho}$-gradings, and also inherits PBW and KK filtrations from $\mathbb{V}_\kappa^\lambda$. These satisfy the usual compatibility in the KK formalism. Therefore, we can apply the method from Theorem 4.2.1 to obtain the desired result.

$\square$

REFERENCES

[AB] Sergey Arkhipov and Roman Bezrukavnikov. Perverse sheaves on affine flags and Langlands dual group. *Israel J. Math.*, 170:135–183, 2009. With an appendix by Bezrukavrikov and Ivan Mirković.

[AG] Sergey Arkhipov and Dennis Gaitsgory. Differential operators on the loop group via chiral algebras. *International Mathematics Research Notices*, 2002(4):165–210, 2002.

[Ara1] Tomoyuki Arakawa. Infinite dimensional Lie algebras, vertex algebras and $\mathcal{W}$-algebras. , 1502:96–106, 2006.

[Ara2] Tomoyuki Arakawa. Representation theory of $\mathcal{W}$-algebras. *Invent. Math.*, 169(2):219–320, 2007.

[Ara3] Tomoyuki Arakawa. Introduction to $\mathcal{W}$-algebras and their representation theory. *arXiv preprint arXiv:1605.00138*, 2016.

[BBM] Roman Bezrukavnikov, Alexander Braverman, and Ivan Mirkovic. Some results about geometric Whittaker model. *Adv. Math.*, 186(1):143–152, 2004.

[BD1] Sasha Beilinson and Vladimir Drinfeld. Quantization of Hitchin's integrable system and Hecke eigensheaves. Available at http://www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf.

---

[104]Here the manipulation for the potentially dangerous value $n = 1$ is obviously justified.

[BD2]   Sasha Beilinson and Vladimir Drinfeld. *Chiral algebras*, volume 51 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[Bei]   Sasha Beilinson. Remarks on topological algebras. *Moscow Math. Journ*, 8(1):1–20, 2008.

[Ber]   Dario Beraldo. Loop group actions on categories and Whittaker invariants. *arXiv preprint arXiv:1310.5127*, 2013.

[Bez]   Roman Bezrukavnikov. Perverse coherent sheaves (after Deligne). *arXiv preprint math/0005152*, 2000.

[BL]   Joseph Bernstein and Valery Lunts. Localization for derived categories of $(\mathfrak{g}, K)$-modules. *Journal of the American Mathematical Society*, 8(4):819–856, 1995.

[BO]   Michael Bershadsky and Hirosi Ooguri. Heiddensl (n) symmetry in conformal field theories. *Communications in Mathematical Physics*, 126(1):49–83, 1989.

[BS]   Peter Bouwknegt and Kareljan Schoutens. W-symmetry in conformal field theory. *Physics Reports*, 223(4):183–276, 1993.

[dBT]   Jan de Boer and Tjark Tjin. The relation between quantum W-algebras and Lie algebras. *Communications in Mathematical Physics*, 160(2):317–332, 1994.

[DS]   V. G. Drinfeld and V. V. Sokolov. Lie algebras and equations of Korteweg-de Vries type. In *Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, pages 81–180. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.

[DSK]   Alberto De Sole and Victor G Kac. Finite vs affine W-algebras. *Japanese Journal of Mathematics*, 1(1):137–261, 2006.

[FBZ]   Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*. Number 88. American Mathematical Soc., 2004.

[FF1]   Boris Feigin and Edward Frenkel. Quantization of the Drinfeld-Sokolov reduction. *Physics Letters B*, 246(1):75–81, 1990.

[FF2]   Boris Feigin and Edward Frenkel. Duality in W-algebras. *International Mathematics Research Notices*, 1991(6):75–82, 1991.

[FF3]   Boris Feigin and Edward Frenkel. Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras. *International Journal of Modern Physics A*, 7:197–215, 1992.

[FG1]   Edward Frenkel and Dennis Gaitsgory. Local geometric Langlands correspondence and affine Kac-Moody algebras. In *Algebraic geometry and number theory*, pages 69–260. Springer, 2006.

[FG2]   Edward Frenkel and Dennis Gaitsgory. Geometric realizations of Wakimoto modules at the critical level. *Duke Mathematical Journal*, 143(1):117–203, 2008.

[FG3]   Edward Frenkel and Dennis Gaitsgory. D-modules on the affine flag variety and representations of affine Kac-Moody algebras. *Represent. Theory*, 13:470–608, 2009.

[FG4]   Edward Frenkel and Dennis Gaitsgory. Local geometric Langlands correspondence: the spherical case. In *Algebraic analysis and around*, volume 54 of *Adv. Stud. Pure Math.*, pages 167–186. Math. Soc. Japan, Tokyo, 2009.

[FG5]   Edward Frenkel and Dennis Gaitsgory. Weyl modules and opers without monodromy. In *Arithmetic and Geometry Around Quantization*, pages 101–121. Springer, 2010.

[FGKV]   Edward Frenkel, Dennis Gaitsgory, David Kazhdan, and Kari Vilonen. Geometric realization of Whittaker functions and the Langlands conjecture. *J. Amer. Math. Soc.*, 11(2):451–484, 1998.

[FGV]   Edward Frenkel, Dennis Gaitsgory, and Kari Vilonen. Whittaker patterns in the geometry of moduli spaces of bundles on curves. *The Annals of Mathematics*, 153(3):699–748, 2001.

[Fre1]   Edward Frenkel. Wakimoto modules, opers and the center at the critical level. *Advances in Mathematics*, 195(2):297–404, 2005.

[Fre2]   Edward Frenkel. *Langlands correspondence for loop groups*, volume 103. Cambridge University Press, 2007.

[FZ]   V. A. Fateev and A. B. Zamolodchikov. Conformal quantum field theory models in two dimensions having Z_3 symmetry. *Nuclear Phys. B*, 280(4):644–660, 1987.

[Gai1]   Dennis Gaitsgory. Quasi-coherent sheaves on stacks. Available at `http://math.harvard.edu/~gaitsgde/GL/QCohtext.pdf`.

[Gai2]   Dennis Gaitsgory. Quantum Langlands correspondence. *arXiv preprint arXiv:1601.05279*, 2007.

[Gai3]   Dennis Gaitsgory. Whittaker categories. Available at `http://math.harvard.edu/~gaitsgde/GL/LocalWhit.pdf`. Formerly titled *Les jours et les travaux*, 2008.

[Gai4]   Dennis Gaitsgory. Generalities on DG categories. Available at `http://math.harvard.edu/~gaitsgde/GL/textDG.pdf`, 2012.

[Gai5]   Dennis Gaitsgory. Ind-coherent sheaves. *Moscow Mathematical Journal*, 13(3):399–528, 2013.

[Gai6]   Dennis Gaitsgory. Kac-Moody representations (Day IV, Talk 3). Available at `https://sites.google.com/site/geometriclanglands2014/notes`. Unpublished conference notes, 2014.

[Gai7]   Dennis Gaitsgory. Sheaves of categories and the notion of 1-affineness. In T. Pantev, C. Simpson, B. Toën, M. Vaquié, and G. Vezzosi, editors, *Stacks and Categories in Geometry, Topology, and Algebra:*, Contemporary Mathematics. American Mathematical Society, 2015.

[Gai8]   Dennis Gaitsgory. Functors given by kernels, adjunctions and duality. *Journal of Algebraic Geometry*, 25(3):461–548, 2016.

[GG]   Wee Liang Gan and Victor Ginzburg. Quantization of Slodowy slices. *International Mathematics Research Notices*, 2002(5):243–255, 2002.

[GR1]   Dennis Gaitsgory and Sam Raskin. Acyclic complexes and 1-affineness. Available at `web.ma.utexas.edu/users/sraskin/Ainfty.pdf`, 2015.

[GR2]   Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry*. Available at `http://math.harvard.edu/~gaitsgde`.

[GR3]   Dennis Gaitsgory and Nick Rozenblyum. DG indschemes. *Contemp. Math*, 610:139–251, 2014.

[Kos]   Bertram Kostant. On Whittaker vectors and representation theory. *Inventiones mathematicae*, 48(2):101–184, 1978.

[Los]   Ivan Losev. Finite W-algebras. In *Proceedings of the International Congress of Mathematicians*, volume 901. World Scientific, 2010.

[Pre]   Alexander Premet. Special transverse slices and their enveloping algebras. *Advances in Mathematics*, 170(1):1–55, 2002.

[Ras1]   Sam Raskin. Chiral categories. Available at `web.ma.utexas.edu/users/sraskin/chiralcats.pdf`, 2015.

[Ras2]   Sam Raskin. Chiral principal series categories I: finite dimensional calculations. Available at `web.ma.utexas.edu/users/sraskin/cpsi.pdf`, 2015.

[Ras3]   Sam Raskin. *D*-modules on infinite dimensional varieties. Available at `web.ma.utexas.edu/users/sraskin/dmod.pdf`, 2015.

[Ras4]   Sam Raskin. On the notion of spectral decomposition in local geometric Langlands. arXiv preprint arXiv:1511.01378, 2015.

[Ras5]   Sam Raskin. A generalization of the *b*-function lemma. Available at `web.ma.utexas.edu/users/sraskin/b-function.pdf`, 2016.

[Ras6]   Sam Raskin. Chiral principal series categories II: the factorizable Whittaker category. Available at `web.ma.utexas.edu/users/sraskin/cpsii.pdf`, 2016.

[Rod]   François Rodier. Modèle de Whittaker et caractères de représentations. In *Non-commutative harmonic analysis*, pages 151–171. Springer, 1975.

[Wan]   Weiqiang Wang. Nilpotent orbits and finite W-algebras. *arXiv preprint arXiv:0912.0689*, 2009.

[Zam]   A. B. Zamolodchikov. Infinite extra symmetries in two-dimensional conformal quantum field theory. *Teoret. Mat. Fiz.*, 65(3):347–359, 1985.

University of Texas at Austin, Department of Mathematics, RLM 8.100, 2515 Speedway Stop C1200, Austin, TX 78712

Email address: `sraskin@math.utexas.edu`