ON THE NOTION OF SPECTRAL DECOMPOSITION IN LOCAL GEOMETRIC LANGLANDS

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ABSTRACT. The geometric Langlands program is distinguished in assigning spectral decompositions to all representations, not only the irreducible ones. However, it is not even clear what is meant by a spectral decomposition when one works with non-abelian reductive groups and with ramification. The present work compares two notions, showing that one is a special case of the other.

More broadly, we study the moduli space of (possibly irregular) de Rham local systems from the perspective of homological algebra. We show that, in spite of its infinite-dimensional nature, this moduli space shares some of the nice features of an Artin stack.

Along the way, we give some apparently new, if unsurprising, results about the algebraic geometry of the moduli space of connections, using Babbitt-Varadarajan's reduction theory for differential equations.

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1. INTRODUCTION

1.1. Local geometric Langlands. Let k be a ground field of characteristic zero, let $K := k((t))$ for t an indeterminate. Let $\overset{\circ}{\mathcal{D}} := \mathrm{Spec}(K)$ be the ("formal") punctured disc. Let G be a split reductive group over k, and let $\check{G}$ be a split reductive group over k with root datum dual to that of G.

Recall the format of local geometric Langlands from [FG1]: there it is suggested that, roughly speaking, DG categories[1] acted on (strongly) by the loop group $G(K)$ should be equivalent to DG categories over the moduli space $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ of *de Rham local Langlands parameters*, i.e., $\check{G}$-bundles on $\overset{\circ}{\mathcal{D}}$ with connection. Note that we work allow irregular singularities in our local systems, which by an old analogy is parallel to wild ramification in the arithmetic theory.

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[1]Throughout this introduction, *DG category* means *presentable* (i.e., cocomplete plus a set theoretic condition) DG category.

1.2.   Some remarks are in order.

- For the reader who is not completely comfortable with the translation from the arithmetic theory:

  A DG category acted on by $G(K)$ should be thought of as analogous to a smooth representation. The basic example is the category of D-modules on a space acted on by $G(K)$ (e.g., D-modules on the affine Grassmannian).

  One can then think of a category over[2] $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ as something like a measure on the space of Langlands parameters. Then this measure is measuring the spectral decomposition of the corresponding smooth representation.

- This idea is quite appealing. One of the distinguishing features of geometric Langlands, in contrast to the arithmetic theory, is the existence of geometric structures on the sets of spectral parameters, which has led to the suggestion (c.f. [BD], [Gai4]) that pointwise spectral descriptions should extend in families over these moduli spaces. The use of spectral decompositions in families then allows to remove irreducibility hypotheses from the analogous arithmetic questions.

- Such an equivalence is not expected to hold literally as is, morally because of the existence of Arthur parameters in the arithmetic spectral theory.

  However, some form of this conjecture may be true as is if we take *tempered* categories acted on by $G(K)$, c.f. [AG]. This notion warrants further study: one can make a number of precise conjectures for which there are not obvious solutions.

- There are two main pieces of evidence for believing in such theory.

  First, Beilinson long ago observed[3] that Contou-Carrère's construction of Cartier self-duality of $\mathbb{G}_m(K)$ implies a precise form of the conjecture for a torus.[4]

  Second, (derived) geometric Satake (c.f. [MV], [FGV], [BF] and [AG]) and Bezrukavnikov's geometric affine Hecke theory (c.f. [AB], [Bez]) fit elegantly into this framework. They completely settle the questions over (the formal completion of) the locus of unramified connections and regular connections with unipotent monodromy respectively (and are instructive about how to understand the temperedness issues).

- Local geometric Langlands is supposed to satisfy various compatibilities as in the arithmetic theory, e.g. there should be a compatibilities with parabolic induction and with Whittaker models.

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[2]This paper is an attempt to understand the phrase "category over," so we ask the reader to suspend disbelief and accept that there is such a notion for the moment. The main feature is that one should have the ability to take fibers at points.

[3]See e.g. [Bei] Proposition 1.4. (It is necessary here to find the published version of the article and not the preprint.)

[4]More precisely, one should use Gaitsgory's notion of (and results on) 1-affineness [Gai5], as will be discussed later: see especially §1.27.

More[5] intriguing is the compatibility with Kac-Moody representations at the critical level: this idea is implicit in [BD], and is explicitly proposed in [FG1]. There is nothing[6] analogous in the arithmetic theory, so this marks one of the major points of departure of geometric Langlands from the usual theory of automorphic forms. Moreover, such a compatibility has deep implicitations in representation theory that are of independent interest.

1.3. **Local geometric Langlands?** Though there are exceptions in special cases, local geometric Langlands (for non-abelian G) essentially stalled out after Bezrukavnikov's theory.

There may be many reasons for this: Bezrukavnikov's theory invokes many brilliant constructions, and perhaps no one has figured out how to extend these. But more fundamentally, there are serious technical challenges when one reaches beyond Iwahori.

On the geometric side, for compact open subgroups $\mathcal{K} \subseteq G(K)$ smaller than Iwahori, the $\mathcal{K}$-orbits on $G(K)/\mathcal{K}$ are not "combinatorial," i.e., the orbits are not discretely parameterized. For starters, this means one must handle non-holonomic D-modules. More seriously, this transition abandons the comfort zone of classical geometric representation theory.

On the spectral side, recall that the formal completion of the locus of regular singular (de Rham) local systems with unipotent monodromy is isomorphic to the formal completion of $\check{\mathcal{N}}/\check{G} \subseteq \check{\mathfrak{g}}/\check{G}$ for $\check{\mathcal{N}} \subseteq \check{\mathfrak{g}}$ the nilpotent cone. However, beyond this locus, $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ is no longer an Artin stack, and is not even finite type. So again, we find ourselves out our comfort zone.

This is to say the skeptic would not be rash in asking if there is any true local geometric Langlands (which for starters would incorporate irregular singularities); nor to suggest that Iwahori ramification is the limit of what is apparently just a geometric shadow of a much richer arithmetic theory. If our interlocutor is right, then there is not much more to explore in geometric Langlands, and the subject is nearing exhaustion.

1.4.   But we remember Beilinson's class field theory for de Rham local systems and draw some resilience from it.

1.5.   This paper is essentially an attempt to study how bad the geometry on the spectral side of local geometric Langlands is.

The results are surprisingly positive: from the perspective of homological algebra, $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ has some the favorable features of an Artin stack.

In short summary, our main result compares two (a priori quite different, c.f. below) notions of category over $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$, showing that one is a special case of the other. We conjecture that the two do coincide.

As a happy output of our methods, we also show that $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$ is compactly generated as a DG category, which comes as a bit of a surprise.

Remark 1.5.1. As we will discuss below, these results can be understood as a geometric strengthening of the simple observation that tangent spaces at field-valued points of LocSys are finite-dimensional.

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[5]This paragraph is included for the sake of completeness, but we explicitly note that it plays no role in what follows. The reader unfamiliar with this idea does not need to follow it, and does not need to turn to [BD] and [FG1] to catch up.

[6]Perhaps this is too strong a claim: as is well-known among experts, the appearance of opers can be understood as a spectral analogue of the Whittaker model, which normally only appears on the geometric side. Nevertheless, such a thing is impossible arithmetically.

1.6. **History.** We were first asked this question by Dennis Gaitsgory in 2010. He informs us that the ambiguity in what was meant by a category over $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ was the reason that no formulation of a local geometric Langlands conjecture was given by him and Frenkel in [FG1], nor by Beilinson earlier.

1.7. **Changing notation.** At this point, we stop referring to any Langlands duality. For the remainder of the paper, let G be an affine algebraic group over k, which will play the role that $\check{G}$ played above. In particular, we do not always assume that G is reductive in what follows.

1.8. **Main results.** We now proceed to give a detailed description of the results of this paper.

We proceed in increasing technical sophistication. We begin in §1.9-1.18 with the main novel geometric result of this paper, which is less technical than the rest of the paper in that it does not involve DG categories.

In §1.19, we will comment on the compact generation of QCoh, indicating why the claim is non-trivial (i.e., why it encodes at a technical level the idea that $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ is like an Artin stack).

Finally, in §1.20, we will begin to discuss 1-affineness and what could be meant by a category over $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$. Recall that our main result here compares two different notions.

1.9. **Geometry of** $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$**.** We now provide a more detailed description of the results of this paper.

Our treatment will be thorough, since there are not great references treating the moduli of local systems on the punctured disc as an object of algebraic geometry (though it is old folklore), and hoping that the expert reader will forgive this. So we will give a precise definition of $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ followed by some examples, and then state our main geometric result.

1.10. First, we need to define $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$.

Let AffSch denote the category of affine schemes.[7] We will presently define a functor:

$$\mathsf{AffSch} \to 1{-}\mathsf{Gpd}$$

$$S \mapsto \mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})(S)$$

where $1{-}\mathsf{Gpd}$ is the category of 1-groupoids (i.e., the usual notion of groupoid, not a higher groupoid). This will be the geometric object we mean by $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$: in the language [Gai2] of derived algebraic geometry, we say that $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ is a 1-truncated classical prestack.

Warning 1.10.1. We emphasize from the onset that we will *not* sheafify this functor for any topology; though we are trained always to sheafify, not sheafifying actually avoids some useless and unnecessary confusion,[8] and by descent does not change the categories we are interested in (namely, quasi-coherent sheaves and sheaves of categories).

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[7]Here these are *classical* affine schemes in the language of derived algebraic geometry. Fortunately, derived algebraic geometry only plays a minor role in the present work (though it always lurks in the background). Moreover, manipulations with connections (which is only possible in classical algebraic geometry) is crucial in §2, so our default convention is the language of classical algebraic geometry.

[8]E.g., we never need to worry about the subtle questions of projectivity of an $A((t))$-module versus local freeness on $\mathrm{Spec}(A)$: this question is irrelevant for us. So the cost is some awkwardness, and the benefit is that we never engage with some subtleties.

1.11. Let Ω^1_K denote the K-line of k-linear continuous differentials for K. So we have a map $d : K \to \Omega^1_K$, and our coordinate t defines the basis element $dt \in \Omega^1_K$ so that $df = f' \cdot dt$. For economy, we usually use the notation Kdt for Ω^1_K, though we will not be using our coordinate in any symmetry breaking way.

More generally, for A a k-algebra and V an $A((t))$-module, we let $Vdt := \Omega^1_K \otimes_K V$.

1.12. **Description of** $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ **as a quotient.** We recall that there is a *gauge* action of the indscheme $G(K)$ on the indscheme[9] $\mathfrak{g}((t))dt$.

Morally, this is given by the formula:

$$G(K) \times \mathfrak{g}((t))dt \to \mathfrak{g}((t))$$

$$(g, \Gamma dt) \mapsto \mathrm{Gauge}_g(\Gamma dt) := \mathrm{Ad}_g(\Gamma)dt - (dg)g^{-1}$$

though we will give a precise construction in what follows. Modulo the construction of this action, we define $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ as the prestack (i.e., non-sheafified) quotient $\mathfrak{g}((t))dt/G(K)$, where we are using the gauge action.

One approach to constructing this action is to notice that for $G = GL_n$, this formula makes sense as is, and then to use the Tannakian formalism to reduce to this case. Indeed, the only difficulty in making sense of this formula is the term $dg \cdot g^{-1}$, and this case a clear meaning for matrices.

Alternatively, recall that G carries the canonical $\mathfrak{g}$-valued 1-form (the Cartan form). This is a right invariant $\mathfrak{g}$-valued 1-form on G. Note that right invariant $\mathfrak{g}$-valued 1-forms are the same as vectors in $\mathfrak{g} \otimes \mathfrak{g}^\vee$, and our 1-form then corresponds to the identity matrix in $\mathfrak{g} \otimes \mathfrak{g}^\vee = \mathrm{End}(\mathfrak{g})$. Note that for $G = GL_n$, the Cartan form is given by the formula $dg \cdot g^{-1}$.

For $S = \mathrm{Spec}(A) \in \mathsf{AffSch}$, let $\overset{\circ}{\mathcal{D}}_S := \mathrm{Spec}(A((t)))$. Given $g : \overset{\circ}{\mathcal{D}}_S \to G$ (i.e., an S-point of $G(K)$), one can pullback[10] the Cartan form to obtain an $\mathfrak{g}$-valued S-relative differential form on $\overset{\circ}{\mathcal{D}}_S$, i.e., an element of $(\mathfrak{g} \otimes A)((t))dt = (\mathfrak{g}((t))dt)(S)$.

1.13. **Description of** $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ **via local systems.** We now give a somewhat more concrete description of $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$, especially for $G = GL_n$.

Definition 1.13.1. A *differential module* on $\overset{\circ}{\mathcal{D}}_S$ is a finite rank free $A((t))$-module V equipped with an A-linear map:

$$\nabla : V \to Vdt$$

satisfying the Leibniz rule:

$$\nabla(fv) = f\nabla(v) + df \otimes v.$$

For $G = GL_n$, we define $\mathrm{LocSys}_{GL_n}(\overset{\circ}{\mathcal{D}})(S)$ as the groupoid of differential modules on $\overset{\circ}{\mathcal{D}}_S$ of rank n (as an $A((t))$-module). Indeed, for an $(n \times n)$-matrix $\Gamma dt \in \mathfrak{g}((t))$, $\nabla := d + \Gamma dt$ defines a connection on $A((t))^{\oplus n}$, and it is standard to see that this gives an equivalence of groupoids.

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[9]We begin a practice here where sometimes we view $\mathfrak{g}((t))dt$ (or $\mathfrak{g}[[t]]dt$) as a geometric object, i.e., an indscheme (or scheme), and sometimes as a linear algebra object, i.e., a K-vector space. We promise always to be careful to distinguish which of the two perspectives we are using.

[10]For precision, since $A((t))dt$ is a bit outside the usual format of differential algebraic geometry: the composition $\mathcal{O}_G \xrightarrow{g^*} A((t)) \xrightarrow{d} A((t))dt$ is a derivation, so induces a map $\Omega^1_G \to A((t))dt$. Similarly, we obtain $\mathfrak{g} \otimes \Omega^1_G \to (\mathfrak{g} \otimes A)((t))dt$, and our form is the image of the Cartan form under this map.

More generally, we obtain a Tannakian description of $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$. (But we should be careful not to sheafify in S!) So we take the groupoid of symmetric monoidal functors from finite-dimensional representations of G to differential modules on $\overset{\circ}{\mathcal{D}}_S$, such that the resulting functor from finite-dimensional representations of G to free $A((t))$-modules admits an isomorphism with $\mathrm{Oblv} \otimes_k A((t))$, for Oblv the forgetful functor for G-representations.

1.14. **Example:** $G = \mathbb{G}_m$**.** We now give some important explicit examples of $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$, which the reader should always keep in mind. We begin with the case $G = \mathbb{G}_m$.

Since G is commutative, the gauge action is given by $(g, \omega) \in \mathbb{G}_m(K) \times K dt \mapsto \omega - d\log(g)$.

We will compute the quotient by the gauge action in stages, ultimately showing that $\mathrm{LocSys}_{\mathbb{G}_m}(\overset{\circ}{\mathcal{D}})$ is a product of three terms: $\mathbb{G}_a/\mathbb{Z}$ where $\mathbb{Z}$ acts by translations, the quotient of an ind-infinite dimensional affine space by its underlying formal group (i.e., $\mathrm{colim}_n \mathbb{A}^n_{dR}$), and $\mathbb{B}\mathbb{G}_m$ the classifying stack of $\mathbb{G}_m$.

First, we quotient by the first congruence subgroup $\mathcal{K}_1 := \mathrm{Ker}(\mathbb{G}_m(O) \to \mathbb{G}_m)$ of $\mathbb{G}_m(K)$ (here and everywhere, $O := k[[t]]$).

Since $\mathcal{K}_1$ is prounipotent, $g \in \mathcal{K}_1$ can be written canonically as $\exp(\xi)$ for $\xi \in tk[[t]]$. Here we are abusing notation: by (e.g.) $g \in \mathcal{K}_1$, we implicitly are taking an S-point (for S a test affine scheme), and we are considering $tk[[t]]$ as a scheme in the obvious way. That aside, the action of $\mathcal{K}_1$ then sends $(g = \exp(\xi), \omega)$ to $\omega - d\xi$. Therefore, we see that $K dt/\mathcal{K}_1 = K dt/O dt$, since $d : tk[[t]] \xrightarrow{\simeq} O dt$. Note that $K dt/O dt$ is an ind-affine space, and is ind-finite type.

Quotienting by $\mathbb{G}_m(O)$, we see that $\mathbb{G}_m = \mathbb{G}_m(O)/\mathcal{K}_1$ acts on $K dt/O dt$ trivially, and therefore the quotient is $K dt/O dt \times \mathbb{B}\mathbb{G}_m$.

Next, we quotient by $\widehat{\mathbb{G}_m(O)}$, the formal completion of $\mathbb{G}_m(O)$ in $\mathbb{G}_m(K)$, i.e., the connected component of the identity in $\mathbb{G}_m(K)$. The exponential map defines a canonical isomorphism $(K/O)^\wedge_0 \xrightarrow{\simeq} \widehat{\mathbb{G}_m(O)}/\mathbb{G}_m(O)$, where $(K/O)^\wedge_0$ is the formal completion at the identity of the group indscheme K/O (considered as a group under addition). Moreover, the resulting action on $K dt/O dt \times \mathbb{B}\mathbb{G}_m$ is trivial on the second factor, and on the first factor (up to sign) is induced from the translation action of $K dt/O dt$ on itself and the homomorphism $K/O \xrightarrow{d} K dt/O dt$. Since the latter homomorphism identifies the source with the kernel of the residue map, the resulting quotient is:

$$\mathbb{G}_a \cdot \frac{dt}{t} \times \mathrm{Ker}(\mathrm{Res} : K dt/O dt \to \mathbb{G}_a)_{dR} \times \mathbb{B}\mathbb{G}_m.$$

Here we are using the well-known de Rham prestack construction, and the fact that de Rham of a group is the quotient of the group by its underlying formal group.

Finally, we quotient by $\mathbb{G}_m(K)/\widehat{\mathbb{G}_m(O)} = \mathbb{Z}$. It is easy to see that the generator acts as translation by $\frac{dt}{t}$ on the factor $\mathbb{G}_a \frac{dt}{t}$ and trivially on the other factors above, so the resulting quotient is:

$$(\mathbb{G}_a \cdot \frac{dt}{t})/(\mathbb{Z} \cdot \frac{dt}{t}) \times \mathrm{Ker}(\mathrm{Res} : K dt/O dt \to \mathbb{G}_a)_{dR} \times \mathbb{B}\mathbb{G}_m$$

as originally claimed.

1.15. **Example: G is unipotent.** Next, we treat the case where G is unipotent. We will show that $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ is isomorphic to the stack $\mathfrak{g}/G$, where G acts via the adjoint action. More precisely, we claim that the map:

$$\mathfrak{g}/G \to \mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$$

$$\xi \mapsto d + \xi \frac{dt}{t}$$

gives an isomorphism.

First, suppose that G is a commutative unipotent group. We have an exponential isomorphism $\exp : \mathfrak{g}((t)) \xrightarrow{\cong} G(K)$, and by commutativity, the gauge action then becomes:

$$\mathfrak{g}((t)) \times \mathfrak{g}((t))dt \xrightarrow{\simeq} G(K) \times \mathfrak{g}((t))dt \to \mathfrak{g}((t))dt$$

$$(\xi, \Gamma dt) \mapsto \Gamma dt - d\xi.$$

This clearly gives the claim in this case.

In the general case, let Z be the center of G. By commutativity, $\mathrm{LocSys}_Z(\overset{\circ}{\mathcal{D}})$ has commutative group structure as a prestack. Moreover, by centrality it acts on $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$, and the prestack quotient for this action is $\mathrm{LocSys}_{G/Z}(\overset{\circ}{\mathcal{D}})$. Therefore, we can apply induction on the degree of nilpotence to reduce to the commutative case.

1.16. **Example:** $G = \mathbb{G}_m \ltimes \mathbb{G}_a$. In this case, we will see that $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ is *not* locally of finite type, unlike the previous examples. Note that the nature of this example forces similar bad behavior for $G = GL_2$, or any other nonabelian (connected) reductive group.

In what follows, we realize G as a matrix group in GL_2 in the usual way.

Consider $O = \{f = \sum_{i=0}^{\infty} a_i t^i\}$ as a scheme, and let $\mathbb{A}^1 \times O$ map to $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ though the map:

$$(\lambda, f) \in \mathbb{A}^1 \times O \mapsto d + \begin{pmatrix} \frac{\lambda}{t} & f \\ 0 & 0 \end{pmatrix} dt.$$

If $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ were locally of finite type, then this map would factor through $\mathbb{A}^1 \times O/t^n O$ for some n, i.e., it would be isomorphic to a map depending only on the first n of the a_i in the above notation. We claim this is not the case.

To this end, we claim that the G-connection:

$$d + \begin{pmatrix} -\frac{n}{t} & t^{n-1} \\ 0 & 0 \end{pmatrix} dt$$

is not isomorphic to the trivial G-connection for $n \in \mathbb{Z}^{\geq 0}$, while the connection $d + \begin{pmatrix} -\frac{n}{t} & 0 \\ 0 & 0 \end{pmatrix} dt$ is isomorphic to a trivial connection. Note that this immediately gives a contradiction to the locally finite type claim.

To see these results, we apply a gauge transformation by $\begin{pmatrix} t^{-n} & 0 \\ 0 & 1 \end{pmatrix} \in G(K)$. The former connection becomes:

$$d + \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix} dt$$

which is easily seen to be nontrivial (one solution to the corresponding order 2 differential equation is the logarithm), while the the latter connection becomes trivial, as desired.

Remark 1.16.1. This example also shows that $t^{-1}\mathfrak{g}[[t]]dt/G(O)$ is not locally of finite type.

Remark 1.16.2. This example admits another interpretation. We will be slightly informal here, since we will not use this language later in the paper.

Recall that to a two step complex of vector spaces (or vector bundles), we can associate a stack. Then $\operatorname{LocSys}_{\mathbb{G}_m \ltimes \mathbb{G}_a}(\overset{\circ}{\mathcal{D}}) \to \operatorname{LocSys}_{\mathbb{G}_m}(\overset{\circ}{\mathcal{D}})$ is obtained by the (Tate analogue of) this construction for the two step complex on $\operatorname{LocSys}_{\mathbb{G}_m}(\overset{\circ}{\mathcal{D}})$ computing de Rham cohomology of a rank 1 local system. In these terms, the above corresponds to the fact that regular singular $\mathbb{G}_m$-connections can have de Rham cocycles with an arbitrarily high order of zero.

1.17. **Semi-infinite motivation.**

The difficulty in working with $\operatorname{LocSys}_G(\overset{\circ}{\mathcal{D}})$ as a geometric object is that it is the quotient of something very infinite-dimensional by something else very infinite-dimensional. Here "very infinite-dimensional" means "an indscheme of ind-infinite type." So it is not at all an Artin stack (except for G unipotent).

But $\operatorname{LocSys}_G(\overset{\circ}{\mathcal{D}})$ *does* share some properties of an Artin stack. E.g., the cotangent complex is computed using de Rham cohomology, and therefore has finite-dimensional cohomology groups at *field-valued* points.

So as a first approximation, one should think that the infinite-dimensional forces are canceling each other out into something almost finite-dimensional. We have seen that this does happen in a fairly precise the special cases where G is unipotent or commutative, though it is harder to say what we mean by this if G is merely, say, solvable.

In fact, everything in this paper marked as a "theorem" may be regarded as an attempt to say in what sense $\operatorname{LocSys}_G(\overset{\circ}{\mathcal{D}})$ behaves like an Artin stack (i.e., that it has better geometric properties than, say, $\mathfrak{g}((t))/G(K)$ under the adjoint action). Moreover, already for $G = GL_2$, these results are the only such statements of which I am aware.

1.18.   Motivated by the example of $\mathbb{G}_m \ltimes \mathbb{G}_a$, we will show the following results in §2. Here G can be any affine algebraic group.

We begin with the case of regular connections.

Theorem (Thm. 2.12.1 and Ex. 2.8.4)**.** *Let $\mathcal{K}_1 \subseteq G(O)$ denote the first congruence subgroup (i.e., the kernel of the evaluation map $G(O) \to G$).*

For $\Gamma_{-1} \in \mathfrak{g}$ a k-point, consider the gauge action of $\mathcal{K}_1$ on the subscheme:

$$\Gamma_{-1}\frac{dt}{t} + \mathfrak{g}[[t]]dt \subseteq \mathfrak{g}((t))dt.$$

Then the quotient of this scheme by $\mathcal{K}_1$ is an Artin stack smooth over k (in particular, it is of finite type).

In other words, when we fix the polar part of the connection, the quotient is an Artin stack.

Remark 1.18.1. To emphasize, we have are encountering a strange pathology in infinite type algebraic geometry. $\mathcal{K}_1$ acts on $t^{-1}\mathfrak{g}[[t]]dt$ preserving the fibers of the residue map $t^{-1}\mathfrak{g}[[t]]dt \to t^{-1}\mathfrak{g}[[t]]dt/\mathfrak{g}[[t]]dt = \mathfrak{g}$; then the quotient by $\mathcal{K}_1$ is an Artin stack on geometric fibers, but is *not* an Artin stack before we take fibers (since it is not of finite type by §1.16).

In more naive terms, for every fiber some congruence subgroup acts freely, but the subgroup cannot be taken independently of the fiber (again, by §1.16).

The result above is not quite true as is when we allow higher order poles, but we will see that the following variant holds:

Theorem (Thms. 2.12.1 and 2.19.1). *For every $r > 0$, there exists an integer ρ (depending also on G) such that for every $\Gamma_{-r}, \ldots, \Gamma_{-r+\rho}$ k-points of $\mathfrak{g}$, the quotient of the action of the $(\rho + 1)$-congruence group $\mathcal{K}_{\rho+1} := \mathrm{Ker}(G(O) \to G(O/t^{\rho+1}O))$ on the subscheme of connections of the form:*

$$\Gamma_{-r}t^{-r}dt + \ldots + \Gamma_{-r+\rho}t^{-r+\rho}dt + t^{-r+\rho+1}\mathfrak{g}[[t]]dt \subseteq \mathfrak{g}((t))dt$$

is an Artin stack smooth over k.

Note that the main difficulty in proving this is showing that the quotient is locally of finite type.

Remark 1.18.2. This second result is a special case of the first result: the only additional claim is that for $r = -1$, we can take $\rho = 0$. Moreover, as indicated above, this will be clear from the formulation of Theorem 2.12.1 and from Example 2.8.4. We emphasize that the only difference from the previous theorem is that for $r > 1$, we typically have $\rho \geqslant r$, i.e., we have to fix more than just the polar part of the connection.

Remark 1.18.3. Note that the latter cited theorem, Theorem 2.19.1, is strongly influenced by Babbitt-Varadarajan [BV], and closely follows their method for treating connections on the formal punctured disc.

Remark 1.18.4. The proof of Theorem 2.12.1, which says that an infinitesimal finiteness hypothesis implies quotients of the above kind are finite type, is surprisingly tricky, especially considering how coarse the hypothesis and the conclusion of this result are. I would be very glad to hear a simpler proof.

Remark 1.18.5. The regular singularities case stated above is substantially more elementary than the case of higher order singularities, and is left as an instructive exercise for the reader. We have mentioned it separately here only because the formulation is somewhat more straightforward than the general case. In particular, this case does not need to pass through the infinitesimal analysis; this is essentially because it is easy to find so-called *good lattices* for regular singular connections.

1.19. **Compact generation of** QCoh.

Next, let $\mathsf{QCoh}(\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}))$ denote the (cocomplete) DG category of quasi-coherent sheaves on $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$. Recall that QCoh is defined as an appropriate homotopy limit for any prestack, and we are simply applying this construction in the case of $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$.

Theorem (Thm. 4.4.1). *If G is reductive, then $\mathsf{QCoh}(\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}))$ is compactly generated.*

Let us comment on why this result is nontrivial.

In forming $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$, we take a certain quotient by $G(K)$. Since G is reductive, $G(K)/G(O) = \mathrm{Gr}_G$ is ind-proper, and the major difficulty arises in quotienting by $G(O)$.

Indeed, one can easily see that $\mathsf{QCoh}(\mathbb{B}G(O))$ has *no* non-zero compact objects. Ultimately, this is because the trivial representation has infinite cohomological amplitude, because of the infinite dimensional pro-unipotent tail of $G(O)$ (more precisely, one should combine this observation with left completeness of the canonical t-structure on this category). In other words, the global sections functor on $\mathbb{B}G(O)$ has infinite cohomological amplitude, ruling out compactness.

We will prove the compact generation for $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ by showing that global sections for $t^{-r}\mathfrak{g}[[t]]dt/G(O)$ is cohomologically bounded (for the natural t-structure on this quotient).

Here's a sketch of the proof. This result can be checked after replacing $G(O)$ by the ρth congruence subgroup $\mathcal{K}_\rho$. Then the claim follows from a Cousin spectral sequence argument by noting that the

geometric fibers of the map $t^{-r}\mathfrak{g}[[t]]dt/\mathcal{K}_\rho \to t^{-r}\mathfrak{g}[[t]]dt/t^{-r+\rho}\mathfrak{g}[[t]]dt$ are Artin stacks for ρ large enough (by the earlier geometric theorems), and the further (easy) observation that the dimensions of these fibers are uniformly bounded (in terms of ρ and G).

Remark 1.19.1. This argument illustrates the main new idea of this work: $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ is nice from the homological perspective because its worst pathologies are rooted in the poor behavior that occurs as we move between the fibers of the map $t^{-r}\mathfrak{g}[[t]]dt/\mathcal{K}_\rho \to t^{-r}\mathfrak{g}[[t]]dt/t^{-r+\rho}\mathfrak{g}[[t]]dt$ (as has long been known), and the Cousin filtration means that these pathologies disappear in the derived category.

Question 1.19.2. The above theorem relies on the properness of Gr_G, which is why I only know it for G reductive. We know it for G unipotent by separate means. Is $\mathsf{QCoh}(\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}))$ compactly generated for general G? Already for $G = \mathbb{G}_m \ltimes \mathbb{G}_a$, I do not know the answer.

1.20. **1-affineness.**
We now discuss the notion of *1-affineness* from [Gai5], which plays a major role in this text.

Remark 1.20.1. As some motivation for what follows: 1-affineness appears to play a key technical role in this flavor of geometric representation theory. Indeed, I think I am not overstepping in asserting that every non-trivial formal manipulation in the subject is an application of 1-affineness, or that the theorems on 1-affineness, all of which are contained in [Gai5], are what fundamentally undergird the "functional analysis" of the subject.

Remark 1.20.2. This is the only review of 1-affineness given in the text, and it may be slightly too detailed for an introduction. We apologize to the reader if it seems to be so, and suggest to skip anything that does not appear to be urgent.

1.21.
First, we briefly need to recall the linear algebra of DG categories.

We always work in the higher categorical framework, so our default language is that a category is an $(\infty, 1)$-category in the sense of Lurie et al.

By a cocomplete DG category, we will always mean a presentable one, i.e., a DG category admitting (small) colimits and satisfying a set-theoretic condition. The relevant set theory will lie under the surface in our applications to e.g. the adjoint functor theorem, and life is better for us all if we suppress (without forgetting) it to the largest extent possible.

Let DGCat_{cont} denote the category of cocomplete (i.e., presentable) DG categories, with morphisms being continuous (i.e., commuting with filtered colimits) DG functors. Note that these functors actually commute with all colimits, since DG functors tautologically commute with finite colimits.

We let Vect denote the DG category of (complexes of) vector spaces. For $A \in \mathsf{Alg}(\mathsf{Vect})$, we let A–mod denote the DG category of left A-modules. For A connective, we let A–$\mathsf{mod}^\heartsuit$ denote the heart of the t-structure on A–mod.

Recall that DGCat_{cont} is equipped with a standard tensor product $\otimes$ with unit object Vect. We remark that this tensor product is generally hard to computeexplicitly with, and questions of 1-affineness generally boil down to the calculation of many tensor products.

For $\mathcal{A} \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$, we will generally use $\mathcal{A}$–mod to mean $\mathcal{A}$–$\mathsf{mod}(\mathsf{DGCat}_{cont})$. We sometimes say that a functor $F : \mathcal{C} \to \mathcal{D}$ between objects of $\mathcal{A}$–mod is $\mathcal{A}$-linear if it is (equipped with a structure of) morphism in $\mathcal{A}$–mod: in particular, this means that the functor F commutes with colimits.

1.22.
Suppose that $\mathcal{Y}$ is a prestack in the sense of [Gai2]: note that this is inherently a notion of derived algebraic geometry. What should we mean by "a (DG) category over $\mathcal{Y}$?"

First, if $\mathcal{Y} = \mathrm{Spec}(A)$ is an affine DG scheme, all roads lead to Rome. I.e., the following structures on $\mathcal{C} \in \mathsf{DGCat}_{cont}$ are equivalent:

- Functorially making Homs in $\mathcal{C}$ into A-modules, i.e., giving a morphism of $\mathcal{E}_2$-algebras $A \to Z(\mathcal{C})$, where $Z(\mathcal{C})$ is the Hochschild cohomology (aka derived Bernstein center) of $\mathcal{C}$.
- Giving $\mathcal{C}$ the structure of A–mod-module in DGCat_{cont}.

1.23.   For a general prestack $\mathcal{Y}$, we have two options.

First, we could ask for a DG category *tensored* over $\mathcal{Y}$, i.e., an object of $\mathsf{QCoh}(\mathcal{Y})$–$\mathsf{mod} :=$ $\mathsf{QCoh}(\mathcal{Y})$–$\mathsf{mod}(\mathsf{DGCat}_{cont})$ (where usual tensor products of quasi-coherent sheaves makes $\mathsf{QCoh}(\mathcal{Y})$ into a commutative algebra object of DGCat_{cont}).

More abstractly, we could also ask for a (functorial) assignment for every $f : \mathrm{Spec}(A) \to \mathcal{Y}$, of an assignment of an A-linear category $f^*(\mathcal{C})$, with identifications:

$$f^*(\mathcal{C}) \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod} = (f \circ g)^*(\mathcal{C})$$

for every $\mathrm{Spec}(B) \xrightarrow{g} \mathrm{Spec}(A) \xrightarrow{f} \mathcal{Y}$, and satisfying higher (homotopical) compatibilities. I.e., we ask for an object of the homotopy limit of the diagram indexed by $\{\mathrm{Spec}(A) \to \mathcal{Y}\}$ and with value the category of A-linear categories, with induction as the structure functors. We denote the resulting category by $\mathsf{ShvCat}_{/\mathcal{Y}}$.

Roughly speaking, we should think that the former notion is more concrete, and that the latter notion has better functoriality properties.

Remark 1.23.1. A toy model: a categorical level down, for $\mathcal{Y}$ a prestack, these two ideas give two notions of "vector space over $\mathcal{Y}$," namely, a $\Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$-module, or a quasi-coherent sheaf on $\mathcal{Y}$.

As in this analogy, we have adjoint functors:

$$\mathsf{QCoh}(\mathcal{Y})\text{–}\mathsf{mod} \underset{\Gamma=\Gamma(\mathcal{Y},-)}{\overset{\mathrm{Loc}=\mathrm{Loc}_\mathcal{Y}}{\rightleftarrows}} \mathsf{ShvCat}_{/\mathcal{Y}}$$

Definition 1.23.2 (Gaitsgory, [Gai5]). $\mathcal{Y}$ is *1-affine* if these functors are mutually inverse equivalences.

Remark 1.23.3. 1-affineness is a much more flexible notion than usual affineness, as we will see in §1.25 below.

Remark 1.23.4. We let $\mathsf{QCoh}_{/\mathcal{Y}}$ denote the sheaf of categories $\mathrm{Loc}(\mathsf{QCoh}(\mathcal{Y}))$, i.e., this is the sheaf of categories that assigns A–mod to every $\mathrm{Spec}(A) \to \mathcal{Y}$. Note that $\Gamma(\mathsf{QCoh}_{/\mathcal{Y}})$ is always equal to $\mathsf{QCoh}(\mathcal{Y})$.

1.24. **Regarding sheafification.** A quick aside: by descent for sheaves of categories ([Gai5] Appendix A), $\mathsf{ShvCat}_{/-}$ is immune to fppf sheafification. Therefore, we will often *not* sheafify, since this is simpler and more convenient in many circumstances.

We have used this convention once already in defining $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$. We will further use it in forming quotients by group schemes G: for G acting on S, S/G will denote the *prestack* quotient, and $\mathbb{B}G$ will denote $\mathrm{Spec}(k)/G$.

1.25. **Examples.** We now give the basic examples and counterexamples of 1-affineness. With one exception, all of these results are proved in [Gai5]: see §2 of *loc. cit* and the local references given there.

The one exception is the failure of 1-affineness of[11] $\mathbb{A}^\infty := \operatorname{colim}_n \mathbb{A}^n$: the argument given in [Gai5] is not correct, and we refer instead to the erratum [GR1].

Theorem 1.25.1 (Gaitsgory). *The following prestacks are 1-affine:*
- *Any quasi-compact quasi-separated DG scheme.*
- *Any (classically) finite type algebraic stack, or more generally, any eventually coconnective almost finite type DG Artin stack. In particular, the classifying stack of an algebraic group is 1-affine.*
- *For any ind-finite type indscheme S, S_{dR} is 1-affine.*
- *The formal completion $T_S^\wedge$ of any quasi-compact quasi-separated DG scheme T along a closed subscheme $S \hookrightarrow T$ with $S^{cl} \subseteq T^{cl}$ defined by a locally finitely generated sheaf of ideals.*
- *For G an algebraic group, the classifying (pre)stack $\mathbb{B}G_e^\wedge$ of its formal group is 1-affine.*

The following prestacks are not *1-affine:*
- *The indscheme $\mathbb{A}^\infty := \operatorname{colim}_n \mathbb{A}^n$. Same for its formal completion at the origin.*
- *The classifying prestack $\mathbb{B}(\prod_{i=1}^\infty \mathbb{G}_a)$.*
- *The classifying prestack $\mathbb{B}\mathbb{A}^\infty$, with $\mathbb{A}^\infty$ being the ind-infinite dimensional affine space. The same holds for its formal group.*

Remark 1.25.2. In remembering some of these examples, a helpful[12] mnemonic is that infinite-dimensional tangent spaces are the primary obstruction to 1-affineness. E.g., $\mathbb{A}^\infty := \operatorname{colim}_n \mathbb{A}^n$ is not 1-affine "since" it has infinite dimensional tangent spaces, but $\mathbb{A}_{dR}^\infty$ *is* 1-affine, and its tangent spaces vanish.

Remark 1.25.3. In what follows, it is helpful to know two examples where 1-affineness fails, but partially holds.

Namely, for $\mathbb{A}^\infty$, the functor:

$$\operatorname{Loc} : \mathsf{QCoh}(\mathbb{A}^\infty)\text{–mod} \to \mathsf{ShvCat}_{/\mathbb{A}^\infty}$$

is fully-faithful. In contrast, for $\mathbb{B}(\prod_{i=1}^\infty \mathbb{G}_a)$, the functor:

$$\Gamma : \mathsf{ShvCat}_{/\mathbb{B}(\prod_{i=1}^\infty \mathbb{G}_a)} \to \mathsf{QCoh}(\mathbb{B}(\prod_{i=1}^\infty \mathbb{G}_a))\text{–mod}$$

is fully-faithful.[13]

1.26.   Our main conjecture is the following:

Conjecture 1. *For any affine algebraic group G, $\operatorname{LocSys}_G(\overset{\circ}{\mathcal{D}})$ is 1-affine.*

The following partial result is the main theorem of this paper.

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[11]In this paper, the notation $\mathbb{A}^\infty$ should be regarded as "locally defined": it may refer to either a pro-infinite dimensional affine space or to an ind-infinite dimensional affine space, and we will specify locally which we mean.

[12]But imperfect: many examples in [Gai5] contradict this principle. Still, it is helpful for the purposes of the present paper.

[13]This is not quite stated in [Gai5]: Theorem 5.1.5 of *loc. cit.* does not apply because the structure sheaf of this stack is not compact. However, the claim is given in Proposition 3.5.1 below.

Main Theorem. *For G a reductive group, the functor:*

$$\mathrm{Loc} : \mathsf{QCoh}(\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}))\text{--}\mathsf{mod} \to \mathsf{ShvCat}_{/\,\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})}$$

is fully-faithful.

Remark 1.26.1. We remind that this theorem partially answers a question of Gaitsgory.

Remark 1.26.2. Note that the space of gauge forms $\mathfrak{g}((t))dt$ is an indscheme that is isomorphic to a product of ind-infinite dimensional affine space and pro-infinite dimensional affine space, and therefore it is not 1-affine.

Moreover, we are quotienting not just by $G(O)$, which itself tends to create prestacks that are not 1-affine (like $\mathbb{B}G(O)$), but by $G(K)$. And we have seen in Remark 1.25.3 that 1-affineness fails in two different ways here.

Essentially, the above theorem shows that quotienting by $G(O)$ is no problem, and is unable to treat the obstruction to 1-affineness coming from $\mathfrak{g}((t))dt$ not being 1-affine.

1.27. **Example: $G = \mathbb{G}_m$.** It is instructive to analyze $G = \mathbb{G}_m$, where 1-affineness quickly reduces to Gaitsgory's results. We assume that the reader has retained (or at least revisited) the material of §1.14.

Note that the intermediate $Kdt/\mathbb{G}_m(O)$ is *not* 1-affine, since there is an ind-infinite dimensional affine space as a factor.

Instead, we need to quotient gauge forms Kdt by $\widehat{\mathbb{G}_m(O)} \coloneqq$ the formal completion of $\mathbb{G}_m(O)$ in $\mathbb{G}_m(K)$. In this case, we obtain:

$$\mathbb{G}_a \cdot \frac{dt}{t} \times \mathrm{Ker}(\mathrm{Res} : Kdt/Odt \to \mathbb{G}_a)_{dR} \times \mathbb{B}\mathbb{G}_m$$

as in §1.14. Since the infinite-dimensional affine space is replaced by its de Rham version, this quotient *is* 1-affine.

It remains to quotient by $\mathbb{G}_m(K)/\widehat{\mathbb{G}_m(O)} = \mathbb{Z}$. One can readily show that $\mathbb{B}\mathbb{Z}$ is 1-affine.[14]

Then the morphism:

$$\mathrm{LocSys}_{\mathbb{G}_m}(\overset{\circ}{\mathcal{D}}) = (Kdt/\widehat{\mathbb{G}_m(O)})/\mathbb{Z} \to \mathbb{B}\mathbb{Z}$$

has 1-affine fibers (since these fibers are $Kdt/\widehat{\mathbb{G}_m(O)}$). Since $\mathbb{B}\mathbb{Z}$ is 1-affine, this implies that $\mathrm{LocSys}_{\mathbb{G}_m}(\overset{\circ}{\mathcal{D}})$ is also 1-affine by [Gai5] Corollary 3.2.7.

1.28. For general reductive G, such an explicit analysis does not work. However, we will show the following results, inspired by the above.

The following is the main result in §3.

Theorem (Thm. 3.9.1). *For any affine algebraic group G, $t^{-r}\mathfrak{g}[[t]]dt/G(O)$ is 1-affine.*

As for $\mathbb{A}^\infty$, we obtain that Loc is fully-faithful for $\mathfrak{g}((t))dt/G(O)$ (and this is the best possible result for this prestack). We then use ind-properness of $G(K)/G(O)$ to deduce the main theorem.

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[14]A proof, for the interested: tautologically, categories over $\mathbb{B}\mathbb{Z}$ are equivalent to categories with an automorphism (the fiber functor corresponds to pullback $\mathrm{Spec}(k) \to \mathbb{B}\mathbb{Z}$). Moreover, $\mathsf{QCoh}(\mathbb{B}\mathbb{Z})$ is equivalent to $\mathsf{QCoh}(\mathbb{G}_m)$ with its convolution structure, so module categories for $\mathsf{QCoh}(\mathbb{B}\mathbb{Z})$ are equivalent to categories over $\mathbb{B}\mathbb{G}_m$. By 1-affineness of $\mathbb{B}\mathbb{G}_m$, the latter are equivalent to $\mathsf{Rep}(\mathbb{G}_m)$-module categories, i.e., to categories with an automorphism. Then it is a simple matter of chasing the constructions to see that $\Gamma : \mathsf{ShvCat}_{/\mathbb{B}\mathbb{Z}} \to \mathsf{QCoh}(\mathbb{B}\mathbb{Z})\text{--}\mathsf{mod}$ corresponds to the identity functor for categories with an automorphism, and therefore is an equivalence.

Or, if one likes, this follows more conceptually from the method of [Gai5] §11.

1.29.   A remark on what is not in this paper:

The problem in proving Conjecture 1, at least for reductive G, is that Theorem 3.9.1 is not so good for passing to the limit in r, since we cannot possibly obtain a 1-affine prestack.

One can show that following conjecture would formally imply Conjecture 1 for reductive G.

Conjecture 2. *For any affine algebraic group G, the formal completion of $t^{-r}\mathfrak{g}[[t]]dt$ in $\mathfrak{g}((t))dt$ modulo the gauge action of $G(K)^{\wedge}_{G(O)}$ (the formal completion of $G(O)$ in $G(K)$) is 1-affine.*

Namely, this result for finite r would also imply 1-affineness of $\mathfrak{g}((t))dt/G(K)^{\wedge}_{G(O)}$, i.e., it is well-adapted to passing to the limit.

1.30. **A heuristic.** Let us give a heuristic explanation for why our geometric results imply Theorem 3.9.1, i.e., the 1-affineness of $t^{-r}\mathfrak{g}[[t]]dt/G(O)$ and the first major step towards the main theorem. The reader may safely skip this section.

First, why is $\mathbb{B}G(O)$ *not* 1-affine? Here is a heuristic, which is less scientific than the proof given in [Gai5]. It relies on some general notions from the theory of group actions on categories that are reviewed in §3.

It is easy to see that if $\mathbb{B}G(O)$ were 1-affine, then invariants and coinvariants for $\mathsf{QCoh}(G(O))$-module categories would coincide.[15]

However, the identity functor for Vect induces a functor $\mathsf{Vect}_{G(O),w} \to \mathsf{Vect}$. If if $\mathsf{Vect}_{G(O),w} \xrightarrow{\simeq} \mathsf{Vect}^{G(O),w}$, we obtain an induced functor $\mathsf{QCoh}(\mathbb{B}G(O)) = \mathsf{Vect}^{G(O),w} \to \mathsf{Vect}$, and morally this functor computes $G(O)$-invariants of representations. Moreover, this functor tautologically is continuous, since that is built into our framework.

However, as discussed above, the trivial representation in $\mathsf{QCoh}(\mathbb{B}G(O))$ is *not* compact, so $G(O)$-invariants does not commute with colimits. What would the functor above be? And indeed, Gaitsgory's result that $\mathbb{B}G(O)$ is not 1-affine rules out the existence of this functor.

Then the heuristic explanation for the difference between $t^{-r}\mathfrak{g}[[t]]dt/G(O)$ and $\mathrm{Spec}(k)/G(O)$ is that the former has a continuous global sections functor, as was explained in §1.19.

1.31. **Structure of this paper.** We have basically given it already above.

In §2, we give our geometric results on $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$, as described above. In §3, we show that $t^{-r}\mathfrak{g}[[t]]dt/G(O)$ is 1-affine; the main ideas were summarized in §1.19. In §4, we prove the compact generation for $\mathsf{QCoh}(\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}))$ (for G reductive). Finally, in §5, we complete the proof of the main theorem.

1.32. **Some conventions.** We use higher categorical language throughout, letting *category* mean $(\infty, 1)$-*category*, (co)limit means *homotopy (co)limit*, etc.

Most of our conventions about DG categories were recalled in §1.21. One warning: following the above conventions, we use the notation $\mathrm{Coker}(\mathcal{F} \to \mathcal{G})$ where others would use Cone, and we use $\mathrm{Ker}(\mathcal{F} \to \mathcal{G})$ where others would use $\mathrm{Cone}[-1]$. If we mean to take a co/kernel in an abelian category, not in the corresponding derived category, we will be cautious to indicate this desire to the reader.

For $\mathcal{C}$ a DG category with t-structure, we let $\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0} \subseteq \mathcal{C}$ denote the corresponding subcategories, where we use *cohomological* grading throughout. We let $\mathcal{C}^{\heartsuit} := \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ denote the heart of the t-structure.

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[15]Proof: $\mathsf{QCoh}(G(O))$-module categories are easily seen to be the same as sheaves of categories on $\mathbb{B}G(O)$, and global sections matches up with invariants. So if $\mathbb{B}G(O)$ were 1-affine, global sections would commute with colimits and be DGCat_{cont}-linear. This would allow us to reduce to checking that the norm functor $\mathcal{C}_{G(O),w} \to \mathcal{C}^{G(O),w}$ is an equivalence for $\mathcal{C} = \mathsf{QCoh}(G(O))$, where it is clear.

Finally, we assume that the reader is quite comfortable with the linear algebra of DG categories. We refer to [Gai1], [Gai5], [Gai3], [GR3] (esp. §7) and other foundational papers on Gaitsgory's website for an introduction to the subject.

2. Semi-infinite geometry of de Rham local systems

2.1.   The motto of this section: in spite of all the evil in the world (e.g., $\text{LocSys}(\overset{\circ}{\mathcal{D}})$ is far from an Artin stack; not all ∇ are Fredholm), there is some current of good (c.f. Theorems 2.12.1 and 2.19.1).

Remark 2.1.1. We remind the reader to visit §1.9-1.18 for a proper introduction to this material. In particular, §1.16 is essential for understanding why we need to work with geometric fibers over the leading terms space.

Remark 2.1.2. To the reader overly steeped in derived algebraic geometry, we emphasize that the manipulations in this section are really about classical algebraic geometry, and we allow ourselves the full toolkit of classical commutative algebra throughout.

2.2. **Tate's linear algebra.** We give a quick introduction to the language of Tate objects in the derived setting. This language plays a fairly supporting role in what follows, and we include it only for convenience.

These definitions were found independently by [Hen], and use *loc. cit.* as a reference for this material.

2.3.   Let $\mathcal{C} \in \mathsf{DGCat}$ be a fixed compactly generated DG category and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the full subcategory of compact objects.

Definition 2.3.1. The Tate category $\mathsf{Tate}(\mathcal{C})$ is the full subcategory of $\mathsf{Pro}(\mathcal{C})$ Karoubi-generated (i.e., generated under finite colimits and retracts) by $\mathcal{C} \subseteq \mathsf{Pro}(\mathcal{C})$ and $\mathsf{Pro}(\mathcal{C}^0)$.

Objects of $\mathsf{Pro}(\mathcal{C}^0) \subseteq \mathsf{Tate}(\mathcal{C})$ are sometimes called *lattices*, and objects of $\mathcal{C} \subseteq \mathsf{Tate}(\mathcal{C})$ are sometimes called *colattices*.

2.4.   Theorem 2 of [Hen] gives a more symmetric characterization of $\mathsf{Tate}(\mathcal{C})$. In particular, one obtains:

Corollary 2.4.1. *The Tate construction commutes with duality:* $\mathsf{Tate}(\mathcal{C})^{op} = \mathsf{Tate}(\mathcal{C}^\vee)$.

2.5. **Fredholm operators.** Following [BBE] §2, we make the following definition.

Definition 2.5.1. A morphism $T : \mathcal{F} \to \mathcal{G} \in \mathsf{Tate}(\mathcal{C})$ is *Fredholm* if $\text{Coker}(T) \in \mathcal{C}^0 \subseteq \mathsf{Tate}(\mathcal{C})$.

2.6. **Example: Laurent series.** Suppose A is a (classical) commutative ring and V is a rank n free $A((t))$-module. Then V inherits an obvious structure of object of $\mathsf{Tate}(A\text{–}\mathsf{mod})$.

Remark 2.6.1. One advantage of using the Tate formalism (or at least pro-objects) is that we can use formulae like $A((t)) \otimes_A B = B((t))$, as long as we are understanding (as we always will) $A((t))$ as an object of $\mathsf{Pro}(A\text{–}\mathsf{mod})$ and $B((t))$ as an object of $\mathsf{Pro}(B\text{–}\mathsf{mod})$.

In this setting, we will use the following terminology.

Definition 2.6.2. A *lattice* in V is an object[16] $\Lambda \in \mathsf{Pro}(A\text{–}\mathsf{mod})^\heartsuit$ that can be written as a limit in[17] $\mathsf{Pro}(A\text{–}\mathsf{mod})$ of finite rank projective A-modules, and which has been equipped with an admissible monomorphism $\Lambda \hookrightarrow V$ with $\mathrm{Coker}(\Lambda \to V)$ lying in[18] $A\text{–}\mathsf{mod} \subseteq \mathsf{Pro}(A\text{–}\mathsf{mod})$ and flat.

Lemma 2.6.3. *For any $\Lambda_1 \hookrightarrow \Lambda_2 \hookrightarrow V$ a pair of lattices, Λ_2/Λ_1 is a finite rank projective A-module.*

We will deduce this using the following general result.

Lemma 2.6.4. *For $\mathcal{C}$ a compactly generated DG category, the intersection $\mathsf{Pro}(\mathcal{C}^0) \cap \mathcal{C}$ in $\mathsf{Tate}(\mathcal{C})$ equals $\mathcal{C}^0$.*

Proof. Tautologically, this intersection is formed in $\mathsf{Pro}(\mathcal{C})$. If $\mathcal{F} \in \mathsf{Pro}(\mathcal{C}^0) \cap \mathcal{C}$, write $\mathcal{F} = \lim_i \mathcal{F}_i$ with $\mathcal{F}_i \in \mathcal{C}^0$. Since $\mathcal{F} \in \mathcal{C}$, the identity map for $\mathcal{F}$ must factor through some $\mathcal{F}_i$, meaning that $\mathcal{F}$ is a retract of $\mathcal{F}_i$. But $\mathcal{C}^0$ is closed under retracts, so we obtain the claim.

$\square$

Proof of Lemma 2.6.3. First, observe that $\Lambda_2/\Lambda_1 \in \mathsf{Pro}(\mathsf{Perf}(A\text{–}\mathsf{mod})) \cap A\text{–}\mathsf{mod}$. Indeed, it obviously lies in $\mathsf{Pro}(\mathsf{Perf}(A\text{–}\mathsf{mod}))$, and it lies in $A\text{–}\mathsf{mod}$ since it sits in an exact triangle with V/Λ_1 and V/Λ_2. Therefore, Lemma 2.6.4 implies that it lies in $\mathsf{Perf}(A\text{–}\mathsf{mod})$.

Since this quotient lies in cohomological degree 0, we see that it is finitely presented. Therefore, it suffices to show that this quotient is flat. This follows again from the resolution:

$$\Lambda_2/\Lambda_1 = \mathrm{Ker}(V/\Lambda_1 \to V/\Lambda_2).$$

$\square$

Definition 2.6.5. An $A[[t]]$-*lattice* in V is a lattice $\Lambda \subseteq V$ which is an $A[[t]]$-submodule, (equivalently: for which $t\Lambda \subseteq \Lambda$).

We have the following structural result.

Lemma 2.6.6. *Any $A[[t]]$-lattice $\Lambda \subseteq V$ is a projective $A[[t]]$-module with $\Lambda \otimes_{A[[t]]} A((t)) \xrightarrow{\simeq} V$. (In particular, Λ has rank n over $A[[t]]$.)*

Proof. Note that no Tors are formed when we form $\Lambda/t^r\Lambda = \Lambda \otimes_{A[[t]]} A[[t]]/t^r$, and by Lemma 2.6.3, $\Lambda/t\Lambda$ is a finite rank projective A-module. It follows that each $\Lambda/t^r\Lambda$ is projective $A[t]/t^r$-module with rank independent of r. By a well-known argument, this implies that Λ is projective over $A[[t]]$.

For a choice of isomorphism $V \xrightarrow{\simeq} A((t))^{\oplus n}$, Λ is wedged between two $A[[t]]$-lattices of the form $t^s A[[t]]^{\oplus n}$ for some choices of $s \in \mathbb{Z}$. It then immediately follows that $\Lambda \otimes_{A[[t]]} A((t)) \xrightarrow{\simeq} V$.

$\square$

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[16]We are using the standard t-structure on $\mathsf{Pro}(A\text{–}\mathsf{mod})^\heartsuit$, characterized by the fact that it is compatible with filtered limits and restricts to the usual t-structure on $A\text{–}\mathsf{mod}$.

[17]I.e., we are asking $R^i \lim$ to vanish for $i > 0$.

[18]Of course, it lies in $A\text{–}\mathsf{mod}^\heartsuit$ then.

2.7. **Example: local de Rham cohomology.** Let $S := \operatorname{Spec}(A)$, and suppose that we are given a differential module $\chi = (V, \nabla)$ over $\overset{\circ}{\mathcal{D}}_S$, i.e., V is an $A((t))$-module free of some rank n and $\nabla : V \to V\,dt$ is A-linear and satisfies the Leibniz rule.

Below, we will construct the local de Rham cohomology[19] $H^*_{dR}(\overset{\circ}{\mathcal{D}}_S, \chi)$ as a Tate A-module, i.e., as an object of $\mathsf{Tate}(A\text{–mod})$.

Note that $V \simeq A((t))^{\oplus n}$ obviously defines an object of $\mathsf{Tate}(A\text{–mod})$.

Let $\Lambda \subseteq V$ be an $A[[t]]$-lattice, i.e., a finite rank free $A[[t]]$-submodule spanning under the action of $A((t))$. By definition of differential module, there exists an integer r such that $\nabla(\Lambda) \subseteq t^{-r}\Lambda\,dt$ for every choice of $A[[t]]$-lattice Λ.[20]

We see that for each integer s, the map:

$$\Lambda \to t^{-r}\Lambda\,dt \to t^{-r}\Lambda\,dt/t^s\Lambda\,dt$$

factors through $\Lambda/t^{r+s}\Lambda$.

Therefore, we obtain a morphism $\Lambda \to t^{-r}\Lambda\,dt$ in $\mathsf{Pro}(\mathsf{Perf}(A\text{–mod}))$, where we consider Λ and $t^{-r}\Lambda\,dt$ as objects of this category in the obvious way. Taking the kernel of this morphism, we obtain an object of $\mathsf{Pro}(\mathsf{Perf}(A\text{–mod}))$ encoding the complex $\Lambda \to t^{-r}\Lambda$ (considered as a complex in degrees 0 and 1).

Passing to the colimit in $\mathsf{Pro}(A\text{–mod})$ under all such choices of $A[[t]]$-lattice, we obviously obtain an object of $\mathsf{Tate}(A\text{–mod}) \subseteq \mathsf{Pro}(A\text{–mod})$, since V/Λ and $V\,dt/t^{-r}\Lambda\,dt$ are both objects of $A\text{–mod} \subseteq \mathsf{Pro}(A\text{–mod})$.

Note that $H^*_{dR}(\overset{\circ}{\mathcal{D}}_S, \chi)$ is the kernel of the morphism:

$$\nabla : V \to V\,dt \in \mathsf{Tate}(A\text{–mod}).$$

Example 2.7.1. In this formalism, formation of the de Rham complex commutes with base-change in the A-variable as is.

2.8. The following notion will play a key role in what follows.

Definition 2.8.1. A differential module $\chi = (V, \nabla)$ on $\overset{\circ}{\mathcal{D}}_S$ is *Fredholm*[21] if $H^*_{dR}(\overset{\circ}{\mathcal{D}}_S, \chi) \in \mathsf{Perf}(A\text{–mod}) \subseteq \mathsf{Tate}(A\text{–mod})$, i.e., if $\nabla : V \to V\,dt$ is Fredholm.

Example 2.8.2. If $A = F$ is a field, then it is well-known that every differential module (V, ∇) is Fredholm: indeed, this follows at once from the finite-dimensionality of the de Rham cohomology in this setting (see [BBE] §5.9 for related discussion).

Example 2.8.3. Suppose that we are given a connection on $A((t))^{\oplus n}$ written as:

$$d + \Gamma_{-r}t^{-r}\,dt + \text{lower order terms}$$

where Γ_{-r} is an $n \times n$-matrix with entries in A. Suppose that $r > 1$ and Γ_{-r} is invertible. Then the corresponding differential module is Fredholm.

---

[19]The notation is misleading: the reader should think de Rham cochains, not merely de Rham cohomology. We will be careful to use an i instead of $*$ when we mean to refer to a specific de Rham cohomology group, and promise the reader never to refer to the graded vector space usually denoted in this way.

[20]To see this: choose a basis, so $V \xrightarrow{\simeq} A((t))^{\oplus n}$ and $\nabla = d + \Gamma\,dt$ for some matrix Γ. Then combine the fact that Γ has a pole of bounded order with the fact that every lattice is wedged between two lattices of the form $t^s A[[t]]$, $s \in \mathbb{Z}$.

[21]A closely related notion was introduced in [BBE], where it was called ε-*nice* due to the nice behavior of ε-factors under this hypothesis.

Indeed, for $\Lambda = A[[t]]^{\oplus n}$ and any integer $s > 0$, the map:

$$t^{-s}\Lambda \xrightarrow{\nabla} t^{-r-s}\Lambda dt$$

is a quasi-isomorphism, as is readily seen using the t-adic filtrations on both sides. Passing to the limit, we see that $H_{dR}^*(\overset{\circ}{\mathcal{D}}_S, \chi)$ vanishes in this case.

In particular, in the rank 1 case, if the pole order is at least 2 and does not jump (i.e., if the leading term is invertible), then the corresponding connection is Fredholm.

Example 2.8.4. Suppose that we are given a *regular singular* connection on $A((t))^{\oplus n}$, so:

$$\nabla = d + \Gamma_{-1}t^{-1}dt + \text{lower order terms}$$

where Γ_{-1} is an $n \times n$-matrix with entries in A. Suppose that $N \operatorname{Id} + \Gamma_{-1}$ is invertible for almost every integer N.

Then using t-adic filtrations as above, we find that χ is Fredholm. Indeed, let $\Lambda = A[[t]]^{\oplus n}$ (for A our ring of coefficients, as usual), and note that ∇ maps $t^N\Lambda$ to $t^{N-1}\Lambda dt$ by assumption. Moreover, at the associated graded level, ∇ induces the map:

$$t^N\Lambda/t^{N+1}\Lambda = A^{\oplus n} \xrightarrow{\Gamma_{-1}+N\cdot\mathrm{id}} A^{\oplus n} = t^{N-1}\Lambda dt/t^N\Lambda dt.$$

Therefore, for $s \gg 0$, the map:

$$t^s A[[t]]^{\oplus n} \xrightarrow{\nabla} t^{s-1}A[[t]]^{\oplus n}dt$$

is an isomorphism, while for $r \gg 0$ the map:

$$A((t))^{\oplus n}/t^{-r}A[[t]]^{\oplus n} \xrightarrow{\nabla} A((t))^{\oplus n}dt/t^{-r-1}A[[t]]^{\oplus n}dt$$

is an isomorphism.

Counterexample 2.8.5. For $A = k[\lambda]$ with λ an indeterminate, the connection:

$$V = A((t)), \nabla = d + \lambda\frac{dt}{t}$$

is *not* Fredholm. Indeed, for this connection, H_{dR}^1 is the sum of skyscraper sheaves supported on $\mathbb{Z} \subseteq \mathbb{A}^1 = \operatorname{Spec}(k[\lambda])$.

2.9.   The following basic results will be of use to us.

Lemma 2.9.1. *Let $\chi = (V, \nabla)$ be a Fredholm differential module over $S = \operatorname{Spec}(A)$, and let $\Lambda \subseteq V$ and $\Lambda' \subseteq V'$ be $A[[t]]$-lattices with the property that $\nabla(\Lambda) \subseteq \Lambda'$. Then:*

(1) The map:

$$\nabla : \Lambda \to \Lambda' \in \mathsf{Pro}(\mathsf{Perf}(A\text{–}\mathsf{mod})) \subseteq \mathsf{Tate}(A\text{–}\mathsf{mod})$$

is Fredholm.

(2) For $N \gg 0$, the complex:

$$\operatorname{Coker}(t^N\Lambda \to \Lambda') \in \mathsf{Perf}(A\text{–}\mathsf{mod})$$

is of the form (i.e., quasi-isomorphic to a complex) $P[-1]$ for P a finite rank projective A-module.

Proof. For (1):

The 2-step complex $\Lambda \xrightarrow{\nabla} \Lambda'$ lies in $\mathsf{Pro}(A\text{–mod})$ (by construction). Moreover, it sits in an exact triangle with $H_{dR}^*(\overset{\circ}{\mathcal{D}}_S, \chi)$, which by assumption lies in $\mathsf{Perf}(A\text{–mod})$, and the complex:

$$\mathrm{Coker}(V/\Lambda \xrightarrow{\nabla} V\,dt/\Lambda')$$

which obviously lies in $A\text{–mod} \subseteq \mathsf{Tate}(A\text{–mod})$. Therefore, we obtain the claim from Lemma 2.6.4.

We now deduce (2).

First, note that ∇ maps $t^N \Lambda$ to $t^{N-1}\Lambda'$ for every $N \geqslant 0$. Indeed, for $s \in \Lambda$, $\nabla(ts) = s\,dt + t\nabla(s)$, so we see that $s\,dt \in \Lambda'$ (since $\nabla(ts)$ and $t\nabla(s)$ are). Therefore, $\nabla(t^N s) = N t^{N-1} s\,dt + t^N \nabla(s)$ lies in $t^{N-1}\Lambda'$ as desired.

We now claim that for $N \gg 0$, the map of 2-step complexes:

$$
\begin{array}{ccc}
t^N \Lambda & \xrightarrow{\ \nabla\ } & t^{N-1}\Lambda' \\
\downarrow & & \downarrow \\
V & \xrightarrow{\ \nabla\ } & V\,dt
\end{array}
$$

is (isomorphic to) the zero map in $\mathsf{Tate}(A\text{–mod})$.

Indeed, since $\Lambda \to \Lambda' = \lim_N \Lambda/t^N\Lambda \to \Lambda'/t^{N-1}\Lambda'$ as a pro-object, and since $H_{dR}^*(\overset{\circ}{\mathcal{D}}_S, \chi)$ lies in $\mathsf{Perf}(A\text{–mod})$, the map $(\Lambda \to \Lambda') \to H_{dR}^*(\overset{\circ}{\mathcal{D}}_S, \chi)$ must factor through $(\Lambda/t^N\Lambda \to \Lambda'/t^{N-1}\Lambda')$ for some N, giving the claim.

We now claim that taking N with this property suffices for the conclusion. Indeed, we have seen in (1) that $\mathrm{Coker}(\nabla : t^N\Lambda \to \Lambda')$ is perfect as an A-module, so it suffices to see that it has Tor-amplitude 0. It obviously suffices to show this for $\mathrm{Coker}(\nabla : t^N\Lambda \to t^{N-1}\Lambda')$ instead. By construction, this object has Tor-amplitude in $[-1, 0]$, so it suffices to show that it has Tor-amplitude $\geqslant 0$.

To this end, note that since the above map is zero, we obtain an isomorphism:

$$\mathrm{Ker}(V/t^N\Lambda \xrightarrow{\nabla} V\,dt/t^{N-1}\Lambda') \simeq H_{dR}^*(\overset{\circ}{\mathcal{D}}_S, \chi) \oplus \mathrm{Coker}(t^N\Lambda \xrightarrow{\nabla} t^{N-1}\Lambda')$$

upon taking its cone. Therefore, our cokernel is a direct summand of a complex visibly of Tor-amplitude $[0, 1]$, and therefore itself has Tor-amplitude $[0, 1]$ as desired. $\qquad\square$

2.10. With a bit more work, we have the following more precise version of Lemma 2.9.1 (2).

Lemma 2.10.1. *In the notation of Lemma 2.9.1:*

Suppose that A has finitely many minimal prime ideals and its nilradical is nilpotent.[22]

Then there exist integers ℓ and r_0 such that for all $r \geqslant r_0$, $t^{r+\ell}\Lambda' \subseteq \nabla(t^r\Lambda)$ with finite rank projective quotient.

Proof. We proceed by steps.

Step 1. First, observe that projectivity of the quotient follows at once if we know the inclusion:

$$t^{r+\ell}\Lambda' \subseteq \nabla(t^r\Lambda). \tag{2.10.1}$$

---

[22]E.g., A is Noetherian, or an integral domain, or a (possibly infinite) polynomial algebra over a Noetherian ring.

Indeed, first note that we may safely assume r_0 is large enough such that the conclusion of Lemma 2.9.1 holds (i.e., so that $\nabla(t^r \Lambda)$ is a lattice). Then we are taking the quotient of one lattice by another, so the claim follows from Lemma 2.6.3.

Step 2. Next, we reduce to the case where A is reduced. More precisely, suppose $I \subseteq A$ is a nilpotent ideal. We claim that if the lemma holds for our lattices modulo I, then it holds for our lattices. We obviously can reduce to the case where $I^2 = 0$ (just to make the numerics simpler).

Suppose (r_0, ℓ) satisfy the conclusion of the lemma for our lattices modulo I. We will show that $(r_0, 2\ell + r_0)$ satisfies the conclusion of the lemma for A.

Using Lemma 2.9.1, we can make sure to choose r_0 so that $t^{r_0-1}\Lambda dt \subseteq \Lambda'$ and $\nabla(t^{r_0}\Lambda) \subseteq \Lambda'$, in which case:

$$\nabla(t^r \Lambda) \subseteq t^{r-r_0}\Lambda' \tag{2.10.2}$$

for all $r \geqslant r_0$. Indeed, for $s \in \Lambda$, we then have:

$$\nabla(t^r s) = \nabla(t^{r-r_0} \cdot (t^{r_0}s)) = (r - r_0)t^{r-1}sdt + t^{r-r_0}\nabla(t^{r_0}s)$$

We now claim that:

$$t^{r+\ell}\Lambda' \subseteq \nabla(t^r \Lambda) + It^{r-r_0}\Lambda'. \tag{2.10.3}$$

Indeed, for $\omega \in t^{r+\ell}\Lambda'$, note that $\omega = \nabla(s) + \varepsilon$ where $s \in t^r \Lambda$ and $\varepsilon \in I \cdot V$, since we are assuming (r_0, ℓ) satisfies our hypotheses modulo I. Moreover, by (2.10.2), $\varepsilon \in IV \cap t^{r-r_0}\Lambda'$, so, it suffices to note that $IV \cap t^{r-r_0}\Lambda' = I \cdot t^{r-r_0}\Lambda'$. In turn, this equality holds because $IV = I \otimes_A V$ and $I\Lambda' = I \otimes_A \Lambda'$ by pro-projectivity, and then we see that (with e.g. Ker denoting homotopy kernels everywhere):

$$\big(IV \cap t^{r-r_0}\Lambda'\big)/I \cdot t^{r-r_0}\Lambda' = H^0\big(\mathrm{Ker}(t^{r-r_0}\Lambda'/It^{r-r_0}\Lambda' \to V/IV)\big) =$$
$$H^0\big(A/I \underset{A}{\otimes} \mathrm{Ker}(t^{r-r_0}\Lambda' \to V)\big) = H^0\big(A/I \underset{A}{\otimes} V/t^{r-r_0}\Lambda'[-1]\big) = 0 \in \mathsf{Tate}(A\text{--}\mathsf{mod})$$

as desired.

Finally, we show that $(r_0, 2\ell + r_0)$ satisfies for the lemma. Indeed, iterating (2.10.3), we have:

$$t^{r+r_0+2\ell}\Lambda' \subseteq \nabla(t^{r+r_0+\ell}\Lambda) + It^{r+\ell}\Lambda' \subseteq \nabla(t^{r+r_0+\ell}\Lambda) + I(\nabla(t^r\Lambda) + It^{r-r_0}\Lambda') =$$
$$\nabla(t^{r+r_0+\ell}\Lambda) + I\nabla(t^r\Lambda) \subseteq \nabla(t^r\Lambda)$$

giving the claim.

Step 3. We now reduce to the case where $A = F$ is an algebraically closed field.

First, suppose $A \hookrightarrow B$ is any embedding of commutative rings. We claim that if (r_0, ℓ) suffice for our lattices tensored with B, then they suffice before tensoring with B as well.

By Step 1, it suffices to show that the inclusion (2.10.1) holds if and only if it holds after tensoring with B. Indeed, such an inclusion is equivalent to the fact that the map $t^{r+\ell}\Lambda' \to \Lambda'/\nabla(t^r\Lambda)$ is zero, and since the quotient on the right is projective, it embeds into its tensor product with B.

Now the fact that A is reduced with finitely many minimal primes means[23] that we have:

$$A \hookrightarrow \prod_{\mathfrak{p} \subseteq A \text{ minimal}} A/\mathfrak{p} \hookrightarrow \prod_{\mathfrak{p} \subseteq A \text{ minimal}} F_\mathfrak{p}$$

---

[23]Recall that the intersection of minimal prime ideals in any commutative ring is the nilradical.

where $F_{\mathfrak{p}}$ is a choice of algebraic closure of the fraction field of $A/\mathfrak{p}$. Since this product is finite, we can reduce to the case where A coincides with one of the factors, as desired.

Step 4. In the next step, we will show that for $A = F$ algebraically closed, there is an $F[[t]]$-lattice $\Lambda_0 \subseteq V$ with the property that:

$$t^{r-1}\Lambda_0 dt \subseteq \nabla(t^r \Lambda_0)$$

for all $r \geqslant 0$. Assume this for now, and we will deduce the lemma.

We may then find $r_0 \geqslant 0$ such that $t^{r_0+1}\Lambda' \subseteq \Lambda_0 dt$ and $\ell \geqslant 0$ such that $t^\ell \Lambda_0 \subseteq t^{r_0}\Lambda$. We claim that this choice of integers suffices.

Indeed, for any $r \geqslant r_0$, we have $t^{\ell+r-r_0}\Lambda_0 \subseteq t^r \Lambda$, and applying our hypothesis on Λ_0, we obtain:

$$t^{\ell+r-r_0-1}\Lambda_0 dt \subseteq \nabla(t^{\ell+r-r_0}\Lambda_0) \subseteq \nabla(t^{\ell+r}\Lambda).$$

We then have:

$$t^{\ell+r}\Lambda' \subseteq t^{\ell+r-r_0-1}t^{r_0+1}\Lambda' \subseteq t^{\ell+r-r_0-1}\Lambda_0 dt$$

as desired.

Step 5. We now construct Λ_0 as above using the Levelt-Turrittin decomposition[24] c.f. [Lev].

First, suppose that for some $e \in \mathbb{Z}^{>0}$ we have constructed:

$$\Lambda_1 \subseteq V \underset{F((t))}{\otimes} F((t^{\frac{1}{e}}))$$

satisfying the corresponding property, i.e., such that:

$$(t^{\frac{1}{e}})^{r-1}\Lambda_1 d(t^{\frac{1}{e}}) = t^{\frac{r}{e}-1}\Lambda_1 dt \subseteq \nabla(t^{\frac{r}{e}}\Lambda_1)$$

for every $r \geqslant 0$. Define Λ_0 as $\Lambda_1 \cap V \subseteq V \otimes_{F((t))} F((t^{\frac{1}{e}}))$. Clearly Λ_0 is an $A[[t]]$-lattice, and we claim it satisfies the desired property:

Indeed, if $r \in \mathbb{Z}^{\geqslant 0}$ and $s \in t^{r-1}\Lambda_0 dt$, then we can find $\sigma \in t^r \Lambda_1$ with $\nabla(\sigma) = s$. Note that $V \otimes_{F((t))} F((t^{\frac{1}{e}}))$ decomposes as an $F((t))$-module as $\oplus_{i=0}^{e-1} V t^{\frac{i}{e}}$, and similarly for $V \otimes_{F((t))} F((t^{\frac{1}{e}}))dt$, and these decompositions are compatible with ∇. Since $\nabla(\sigma) = s \in V dt$, the component $\sigma_0 \in V \subseteq \oplus_{i=0}^{e-1} V t^{\frac{i}{e}}$ of σ also maps to s under ∇. Since $\sigma_0 \in t^r \Lambda_0$, this gives the desired claim.

Therefore, we may replace $F((t))$ by $F((t^{\frac{1}{e}}))$ for any $e > 0$. Then, because F is algebraically closed, the Levelt-Turrittin theorem says[25] that (after such an extension) V is a direct sum of differential modules each of which is either regular or else (up to gauge transformation) has an invertible leading term, reducing us to treating each of these cases.

But these cases were treated already in Examples 2.8.3 and 2.8.4, completing the proof of the existence of the lattice claimed in Step 4.

$\square$

---

[24] It would be great to have a more direct argument here.

[25] More specially, it says that after such an extension a differential module decomposes as a direct sum of a regular module and modules that are tensor products of a rank 1 irregular connection with a regular connection. The latter kind of connection obviously has an invertible (diagonal) leading term.

2.11.   We also record the following simple result for later use.

Lemma 2.11.1. *Let (V, ∇) be a differential module over $S = \mathrm{Spec}(A)$. Suppose $d \in \mathbb{Z}^{>0}$ is given.*
Then if $V[t^{\frac{1}{d}}] := V \otimes_{A((t))} A((t^{\frac{1}{d}}))$ equipped with its natural connection is Fredholm, then (V, ∇) is Fredholm.

Proof. As a differential module over $A((t))$, $V[t^{\frac{1}{d}}]$ is isomorphic to:

$$\oplus_{i=0}^{d-1} V \otimes \chi_{\frac{i}{d}}$$

where $\chi_{\frac{i}{d}}$ is the rank 1 connection:

$$d + \frac{i}{d} \frac{dt}{t}.$$

This obviously gives the claim, since V is a direct summand of this differential module.

$\square$

2.12. **Application to algebraicity of some stacks.** Our principal use of the above notions is the following technical theorem, which should be understood as dreaming that every connection is Fredholm, and deducing that $t^{-r}\mathfrak{g}[[t]]dt/G(O)$ is an algebraic stack of finite type.

   The reader who is interested to first see the construction of a large supply of Fredholm local systems may safely skip ahead to §2.19.

Theorem 2.12.1. *Let G be an affine algebraic group with $\mathfrak{g} := \mathrm{Lie}(G)$, and integers $r > 0$ and $s \geqslant 0$.*

   Let T be a Noetherian affine scheme with a morphism $T \to t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt$, and let S denote the fiber product:

$$S := T \underset{t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt}{\times} t^{-r}\mathfrak{g}[[t]]dt.$$

   Note that S is equipped with an action of the congruence subgroup $\mathcal{K}_{r+s} := \mathrm{Ker}(G(O) \to G(O/t^{r+s}O))$: indeed, the gauge action of this group on forms with poles of order $\leqslant r$ leaves the first $r + s$ coefficients fixed.[26]

   Note that the structure morphism $S \to t^{-r}\mathfrak{g}[[t]]$ defines a G-local system on $\overset{\circ}{\mathcal{D}}_S$. Suppose that the local system associated to this G-local system by the adjoint representation is Fredholm.

   Then the prestack quotient $S/\mathcal{K}_{r+s}$ is an Artin stack, and the morphism $S/\mathcal{K}_{r+s} \to T$ is smooth (in particular, finitely presented).

Example 2.12.2. Suppose G is commutative. Then the adjoint representation is trivial, and therefore the above connection on S is trivial, and in particular Fredholm. It follows that:

$$t^{-r}\mathfrak{g}[[t]]dt/G(O)$$

is an Artin stack in this case. More generally, if $\mathrm{Lie}(G)$ is a successive extension of trivial representations (e.g., if G is unipotent), then the above connection is a successive extension of trivial local systems and therefore Fredholm, so the same conclusion holds.

   We will prove Theorem 2.12.1 in §2.15-2.18 below, after some preliminary remarks in §2.13-2.14.

---

[26]The numerics here is the reason we assume $r > 0$; for $r = 0$, we would need to take $\mathcal{K}_{r+s+1}$ instead of $\mathcal{K}_{r+s}$ everywhere.

2.13. The most serious difficulty in proving Theorem 2.12.1 is that we need to "integrate" from infinitesimal information about the gauge action (through the Fredholm condition on the adjoint representation) to global information. We will use two results, Lemmas 2.13.2 and 2.14.1, to do this. Together, they give a way to check that a morphism of affine spaces is finitely presented: the former gives a reasonably soft[27] criterion for such a map to be a closed embedding, while the latter gives an infinitesimal criterion for a closed embedding to be finitely presented.

Let S be a base scheme, say affine with $S = \mathrm{Spec}(A)$ for expositional ease only.

We define an $(\aleph_0\text{-})$*affine space* V over S to be an S-scheme arising by the following construction:

Suppose $P = P_V \in A\text{–}\mathsf{mod}^\heartsuit$ is a colimit $P = \mathrm{colim}_{i \geqslant 0} P_i$ with P_i a finite rank projective A-module and each structure map $P_i \to P_j$ being injective with projective quotient. Then we set $V := \mathrm{Spec}(\mathrm{Sym}_A(P))$. Note that $V(S) = \lim P_i^\vee$, so informally we should think of V as the pro-projective A-module $\lim_i P_i^\vee \in \mathsf{Pro}(A\text{–}\mathsf{mod}^\heartsuit)$.

Construction 2.13.1. Let $f : V \to W$ be a morphism of affine spaces preserving zero sections (i.e., commuting with the canonical sections from S). We define the *linearization* $\mathfrak{L}(f) : V \to W$ as the induced morphism by considering V and W as the tangent spaces of V and W at 0.

More precisely, if P (resp. Q) is the ind-projective module defining V (resp. W) and $I_V \subseteq \mathrm{Sym}_A(P)$ is the augmentation ideal, then $I_V/I_V^2 = P$ (resp. $I_W/I_W^2 = Q$). Moreover, the map $f^* : \mathrm{Sym}_A(Q) \to \mathrm{Sym}_A(P)$ preserves augmentation ideal by assumption, so we obtain a morphism:

$$Q = I_W/I_W^2 \to I_V/I_V^2 = P$$

and by covariant functoriality of the assignment $P \mapsto V$, we obtain the desired map $\mathfrak{L}(f) : V \to W$.

Suppose P and Q are as above, and suppose $P = \mathrm{colim}_{i \geqslant 0} P_i$ and $Q = \mathrm{colim}_{i \geqslant 0} Q_i$ as in the definition of affine space. Let V_i (resp. W_i) denote $\mathrm{Spec}(\mathrm{Sym}_A(P_i))$ (resp. $\mathrm{Spec}(\mathrm{Sym}_A(Q_i))$), so that e.g. $V = \lim V_i$ with each morphism $V \to V_i$ being dominant (and a fibration with affine space fibers).

Lemma 2.13.2. *Suppose $f : V \to W$ is a morphism of S-schemes preserving 0. Suppose that:*

- *For every i, we have a (necessarily unique, by dominance) map:*

$$
\begin{array}{ccc}
V & \xrightarrow{\ f\ } & W \\
\downarrow & & \downarrow \\
V_i & \xdashrightarrow{\ f_i\ } & W_i.
\end{array}
$$

- *For every i, there is a (necessarily unique) factorization:*

$$
\begin{array}{ccc}
V_i & \xrightarrow{\ f_i - \mathfrak{L}(f_i)\ } & W_i \\
\downarrow & \nearrow & \\
V_{i-1}. & {\scriptstyle \alpha_i} &
\end{array}
$$

  where for $i = 0$ we use the normalization $V_{i-1} = \mathrm{Spec}(A)$.
- *Each morphism $\mathfrak{L}(f_i) : V_i \to W_i$ is a closed embedding.*

Then f is a closed embedding.

---

[27]Note that there is no hope of obtaining a purely infinitesimal criterion, because the Jacobian conjecture is unknown. The criterion below is instead modeled on the simplest construction of non-linear automorphisms of affine space, c.f. Example 2.13.3.

Example 2.13.3. In the above, the lemma remains true if "closed embedding" is replaced by "isomorphism" everywhere. Then a toy model for the above lemma is the fact that any morphism of the form:

$$\mathbb{A}^2 \xrightarrow{(x,y)\mapsto(x,y+p(x))} \mathbb{A}^2$$

(with p any polynomial) is an isomorphism.

Proof of Lemma 2.13.2. Passing to the limit, it suffices to show that each f_i is a closed embedding. We will prove this by induction on i.

Note that $\alpha_0 : \mathrm{Spec}(A) \to W_i$ must be the zero section, since $f_0 - \mathfrak{L}(f_0)$ preserves zero sections. Therefore, $f_0 = \mathfrak{L}(f_0)$, i.e., f_0 is linear. Since $\mathfrak{L}(f_0)$ is assumed to be a closed embedding, this of course implies that f_0 is as well.

We now show f_i is a closed embedding, assuming that f_{i-1} is. Since V_i and W_i are affine, it suffices to show that pullback of functions along f_i is surjective.

Suppose φ is a function on V_i. Since $\mathfrak{L}(f_i)$ is an isomorphism, there is a function ψ on W_i with $\mathfrak{L}(f_i)^*(\psi) = \varphi$. We obtain:

$$\varphi = f_i^*(\psi) - (f_i - \mathfrak{L}(f_i))^*(\psi).$$

Using α_i, we see that $(f_i - \mathfrak{L}(f_i))^*(\psi)$ descends to a function on V_{i-1}, and therefore is obtained by restriction (along f_{i-1}) from a function on W_{i-1} by induction. In particular, $(f_i - \mathfrak{L}(f_i))^*(\psi)$ is obtained by restriction (along f_i) from a function on W_i, as desired.

$\square$

2.14.   Above, we gave a way to check that a morphism of affine spaces is a closed embedding. We also have the following criterion for checking when a closed embedding as above is actually finitely presented.

Lemma 2.14.1. *Suppose I_1 and I_2 are sets and we are given a closed embedding:*

$$i : \mathbb{A}_T^{I_1} \hookrightarrow \mathbb{A}_T^{I_2}$$

of affine spaces over a Noetherian base T, and suppose that i is compatible with zero sections, i.e., the diagram:

$$
\begin{array}{ccc}
\mathbb{A}_T^{I_1} & \xrightarrow{\quad i \quad} & \mathbb{A}_T^{I_2} \\
& \searrow{\scriptstyle 0} \quad {\scriptstyle 0}\nearrow & \\
& T &
\end{array}
$$

commutes.

Then i is finitely presented if and only if its conormal sheaf $N^*_{\mathbb{A}^{I_1}/\mathbb{A}^{I_2}} \in \mathsf{QCoh}(\mathbb{A}^{I_1})^\heartsuit$ is coherent.

Proof. We can assume T affine, so $T = \mathrm{Spec}(B)$. Moreover, straightforward reductions allow us to assume T is integral (and in all our applications, B will actually be a field).

The key fact we will need from commutative algebra is that a prime ideal in any (possibly infinitely generated) polynomial algebra over a Noetherian ring (e.g., k) is finitely generated if and only if it has finite height (in the usual sense of commutative algebra). Indeed, this result is essentially given by [GH] Theorem 4 (see also [Ras] Proposition 4.3, which completes the argument in some simple respects).[28]

---

[28]For the reader's convenience, we quickly sketch the proof down here in the footnotes.

Let $J \subseteq B[\{x_i\}_{i \in I_2}]$ be the ideal of the closed embedding. Note that $\mathbb{A}_T^{I_1}$ is integral (since T is), and therefore J is a prime ideal.

By assumption, J/J^2 is finitely generated. Choose $f_1, \ldots, f_n \in J$ generating modulo J^2.

Suppose that $f_1, \ldots, f_n$ lie in $B[\{x_i\}_{i \in I_2'}]$ for $I_2' \subseteq I_2$ a finite subset (as we may safely do). We claim that J is contained in the ideal generated by $\{x_i\}_{i \in I_2'}$. Note that this implies the claim: clearly $\{x_i\}_{i \in I_2'}$ is contained in a finitely generated prime ideal (since B is Noetherian), and as was noted above, prime subideals of finitely generated prime ideals in $B[\{x_i\}_{i \in I_2}]$ are themselves finitely generated.

Let $\overline{J}$ denote the reduction of J modulo $(\{x_i\}_{i \in I_2'})$ (the ideal generated by our finite subset of x_i's). It suffices to see that $\overline{J} = 0$.

First, note that $\overline{J} = \overline{J}^2$ by construction of our generators. Moreover:

$$\overline{J} \subseteq B[\{x_i\}_{i \in I_2}]/(\{x_i\}_{i \in I_2'}) = B[\{x_i\}_{i \in I_2 \setminus I_2'}]$$

is contained in the ideal generated by all the x_i: indeed, this is true for J in the original ring $B[\{x_i\}_{i \in I_2}]$ by the compatibility with zero sections, implying the corresponding statement for $\overline{J}$.

But now we have:

$$\overline{J} = \bigcap_{r=1}^{\infty} \overline{J}^r \subseteq \bigcap_{r=1}^{\infty} (\{x_i\}_{i \in I_2 \setminus I_2'})^r = 0$$

giving the claim.

$\square$

2.15. **Proof of Theorem 2.12.1.** We now prove Theorem 2.12.1. The proof will occupy §2.15-2.18.

2.16. Note that, since $r > 0$, $\mathcal{K}_{r+s}$ is pro-unipotent. Because torsors for unipotent groups are trivial on affine schemes, $\mathcal{K}_{r+s}$-torsors on affine schemes are as well. Therefore, the prestack quotient $S/\mathcal{K}_{r+s}$ is already a sheaf.

2.17. **Formulation of the key lemma.** The following result will be the key input to the proof of Theorem 2.12.1.

Lemma 2.17.1. *For all sufficiently large integers $N \geqslant r + s$, the map:*

$$\begin{aligned}
\mathcal{K}_N \times S &\to t^{-r}\mathfrak{g}[[t]]dt \times S \\
(\gamma, s) &\mapsto (\mathrm{Gauge}_\gamma(\Gamma(s)dt), s)
\end{aligned}$$

(2.17.1)

is a finitely presented closed embedding, where here $s \mapsto \Gamma(s)dt$ is the structure map $S \to t^{-r}\mathfrak{g}[[t]]dt$.

The main point is the following simple observation: if $Z_{n+1} \subseteq \mathbb{A}^{n+1}$ is a closed and integral subscheme, then the scheme-theoretic image $Z_n := \overline{p(Z_{n+1})} \subseteq \mathbb{A}^n$ of Z_{n+1} along the projection $p : \mathbb{A}^{n+1} \to \mathbb{A}^n$ either has smaller codimension, or else $Z_{n+1} = p^{-1}(Z_n)$. Indeed, $Z_{n+1} \subseteq p^{-1}(Z_n)$, and either the latter has codimension one less than Z_{n+1}, or else they have equal codimension, and so are equal by irreducibility.

Now for $Z \subseteq \mathbb{A}^\infty := \mathrm{Spec}(k[x_1, x_2, \ldots])$ (note that k can be any Noetherian ring, and we could be less lazy and allow uncountable indexing sets for our indeterminates) a closed and integral subscheme corresponding to a finite height prime ideal, the scheme-theoretic image $Z_n \subseteq \mathbb{A}^n$ of Z along the projection $p_n : \mathbb{A}^\infty \to \mathbb{A}^n$ corresponds to a prime ideal of the same height for n large enough, and then the above analysis shows that $Z = p_n^{-1}(Z_n)$. In particular, Z is defined by a finitely presented ideal.

Similarly, if Z is defined by a finitely presented ideal, then $Z = p_n^{-1}(Z_n)$ for n large enough, and our earlier analysis shows that the height does not increase as we pullback to finite affine spaces; since finite height prime ideals in $k[x_1, x_2, \ldots]$ are finitely presented, one readily deduces that the height of a prime ideal in this ring is bounded by the number of generators (since this holds in each of the Noetherian rings $k[x_1, \ldots, x_m]$ by Krull's theorem).

Proof. We proceed by steps. Let ∇ be the connection on the $\mathfrak{g}_A[[t]]$ defined by $\nabla := d - [\Gamma, -]dt$ throughout.

Step 1. First, we construct N:

Note that because T is Noetherian and $S \to T$ is a fibration with affine space fibers, S satisfies the hypotheses of Lemma 2.10.1. Applying the lemma (with $\Lambda = \mathfrak{g}_A[[t]]$ and $\Lambda' = t^{-r}\mathfrak{g}_A[[t]]dt$), we find $\ell \geqslant 0$ and an integer $N \geqslant r + s$ such that $\nabla|_{t^N \mathfrak{g}_A[[t]]}$ is injective, and for all $i \geqslant 0$ we have:

$$t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt \subseteq \nabla(t^{N+i}\mathfrak{g}_A[[t]]) \subseteq t^{N+i-r}\mathfrak{g}_A[[t]]dt$$

with finite rank projective quotients. Note that $\nabla(t^{N+i}\mathfrak{g}_A[[t]])$ is a lattice in this case.

Finally, note that we can safely replace N by $N+i$ for any i as above and the conclusion remains, so we may assume $N \geqslant \ell + 1$.

Step 2. In Steps 2-6, we will apply Lemma 2.13.2 to see that (2.17.1) is a closed embedding. In the present step, we just set up notation for this.

We abuse notation in letting e.g. $\mathfrak{g}_A[[t]]$ denote the affine space over $S = \operatorname{Spec}(A)$ associated with $\mathfrak{g}_A[[t]]$.

To be in the setting of Lemma 2.13.2, we need to have a map between affine spaces over a Noetherian base preserving zero. First, we use the exponential isomorphism $t^N\mathfrak{g}_A[[t]]dt \xrightarrow{\simeq} K_N \times S$ so that our map goes between affine spaces. Moreover, since $S \to T$ is an affine space over T, our map is between affine spaces over a Noetherian base (namely, T). Our map does not quite preserve zero: $0 \mapsto \Gamma dt \in t^{-r}\mathfrak{g}_A[[t]]dt$ the given gauge form. But of course, we can just correct this by subtracting off Γdt.

Define:

$$\Lambda_i := \nabla^{-1}(t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt) \cap t^{N+i}\mathfrak{g}_A[[t]].$$

Note that Λ_i is a lattice, since it maps isomorphically onto the lattice $t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt$. In particular, it has an associated affine space over S, which we again denote by Λ_i.

Step 3. In the next step, we will show that the composite map:

$$t^N\mathfrak{g}_A[[t]] \xrightarrow{\xi \mapsto \operatorname{Gauge}_{\exp(\xi)}(\Gamma dt) - \Gamma dt} t^{-r}\mathfrak{g}_A[[t]]dt \xrightarrow{\operatorname{proj.}} t^{-r}\mathfrak{g}_A[[t]]dt/t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt \qquad (2.17.2)$$

factors through $t^N\mathfrak{g}_A[[t]]/\Lambda_i$ (therefore satisfying the first condition from Lemma 2.13.2).

In other words, for[29] $\xi \in t^N\mathfrak{g}_A[[t]]$ and $\eta \in \Lambda_i$, we need to see that:

$$\operatorname{Gauge}_{\exp(\xi)}(\Gamma dt) - \Gamma dt) \text{ and}$$
$$\operatorname{Gauge}_{\exp(\xi+\eta)}(\Gamma dt) - \Gamma dt)$$

differ by an element of $t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt$. We will do this by explicitly computing $\operatorname{Gauge}_{\exp(\xi+\eta)}(\Gamma dt) - \Gamma dt)$.

In a first motion towards this, we analyze the relationship between the "matrix" exponential appearing above and the term "$dg \cdot g^{-1}$" appearing in the definition of the gauge action. More precisely, this step will show the following identity of points of $\mathfrak{g}_A[[t]]dt$:

$$\big(d\exp(\xi+\eta)\big) \cdot \exp(-\xi-\eta) \in \big(d\exp(\xi)\big) \cdot \exp(-\xi) + d\eta + t^{2N+i-1}\mathfrak{g}_A[[t]]dt. \qquad (2.17.3)$$

---

[29]Here ξ and η are points of these schemes with values in some test affine scheme. We suppress the affine scheme from the notation to keep the notation simple, and we maintain such abuses throughout.

Note that e.g. $\exp(\xi)^{-1} = \exp(-\xi)$, hence the appearance of these terms.

By the Tannakian formalism, it suffices to prove (2.17.3) for the general linear group. Then note that:

$$\exp(\xi + \eta) \in \exp(\xi)\exp(\eta) + t^{2N+i}\mathfrak{g}_A[[t]].$$

Indeed, the Campbell-Baker-Hausdorff formula says:

$$\exp(\xi)\exp(\eta) = \exp(\xi + \eta + \underbrace{\tfrac{1}{2}[\xi, \eta] + \text{etc.}}_{\in\, t^{2N+i}\mathfrak{g}_A[[t]]}).$$

(Here we recall that $\xi \in t^N\mathfrak{g}_A[[t]]$ and $\eta \in t^{N+i}\mathfrak{g}_A[[t]]dt$.) Then for $a := \xi + \eta$ and $b := \frac{1}{2}[\xi, \eta] + \text{etc.} \in t^{2N+i}\mathfrak{g}_A[[t]]$, we have:

$$\exp(a + b) = \mathrm{id} + (a + b) + \frac{1}{2}(a + b)^2 + \ldots =$$
$$\mathrm{id} + a + \frac{1}{2}a^2 + \ldots + b + \frac{1}{2}(ab + ba + b^2) + \ldots \in \exp(a) + t^{2N+i}\mathfrak{g}_A[[t]]$$

as desired.

We now show (2.17.3). Let $g = \exp(\xi)$ and $h = \exp(\eta)$. By the above, we have:

$$\big(d\exp(\xi + \eta)\big) \cdot \exp(-\xi - \eta) \in$$
$$(d(gh) + t^{2N+i-1}\mathfrak{g}_A[[t]]dt) \cdot (h^{-1}g^{-1} + t^{2N+i}\mathfrak{g}_A[[t]]) =$$
$$d(gh)h^{-1}g^{-1} + d(gh) \cdot t^{2N+i}\mathfrak{g}_A[[t]]dt + t^{2N+i-1}\mathfrak{g}_A[[t]]dt \cdot h^{-1}g^{-1} + t^{4N+2i-1}\mathfrak{g}_A[[t]]dt \subseteq$$
$$d(gh)h^{-1}g^{-1} + t^{2N+i-1}\mathfrak{g}_A[[t]]dt$$

where the last line follows from the observations that $d(gh) \in \mathfrak{g}_A[[t]]dt$ and $h^{-1}g^{-1} \in G(O)$.

Note that $d(gh)h^{-1}g^{-1} = dg \cdot g^{-1} + g(dh \cdot h^{-1})g^{-1}$. We compute:

$$dh \cdot h^{-1} = (d\exp(\eta)) \cdot \exp(-\eta) = (d\eta + \tfrac{1}{2}d(\eta^2) + \ldots)(\mathrm{id} - \eta + \ldots) \in d\eta + t^{2(N+i)-1}\mathfrak{g}_A[[t]]dt. \quad (2.17.4)$$

Since $g \in \mathcal{K}_N$ and since $d\eta \in t^{N+i-1}\mathfrak{g}_A[[t]]dt$, we have $g(d\eta)g^{-1} \in d\eta + t^{2N+i-1}\mathfrak{g}_A[[t]]dt$. Since Ad_g clearly preserves $t^{2(N+i)-1}\mathfrak{g}_A[[t]]$, this combines with the above to give:

$$dg \cdot g^{-1} + g(dh \cdot h^{-1})g^{-1} \in dg \cdot g^{-1} + d\eta + t^{2N+i-1}\mathfrak{g}_A[[t]]$$

as was claimed in (2.17.3).

Step 4. We now complete the factorization claim from the beginning of Step 3.

Note that the map (2.17.2) sends $\xi + \eta$ to:[30]

---

[30]For clarity, the notation $\mathrm{Ad}^n_{\xi+\eta}(\Gamma dt)$ means $[\xi + \eta, [\xi + \eta, [\ldots, [\xi + \eta, \Gamma dt]\ldots]]]$.

$$\mathrm{Gauge}_{\exp(\xi+\eta)}(\Gamma dt) - \Gamma dt = \mathrm{Ad}_{\exp(\xi+\eta)}(\Gamma dt) - d\big(\exp(\xi+\eta)\big) \cdot \exp(-\xi-\eta) - \Gamma dt =$$

$$[\xi+\eta, \Gamma dt] + \sum_{n=2}^{\infty} \frac{1}{n!} \mathrm{Ad}_{\xi+\eta}^n(\Gamma dt) - d\big(\exp(\xi+\eta)\big) \cdot \exp(-\xi-\eta) \in$$

$$[\xi+\eta, \Gamma dt] + \sum_{n=2}^{\infty} \frac{1}{n!} \mathrm{Ad}_{\xi+\eta}^n(\Gamma dt) - (d\exp(\xi)) \cdot \exp(-\xi) - d\eta + t^{2N+i-1}\mathfrak{g}_A[[t]]dt = \qquad (2.17.5)$$

$$-\nabla(\eta) + [\xi, \Gamma dt] + \sum_{n=2}^{\infty} \frac{1}{n!} \mathrm{Ad}_{\xi+\eta}^n(\Gamma dt) - (d\exp(\xi)) \cdot \exp(-\xi) + t^{2N+i-1}\mathfrak{g}_A[[t]]dt.$$

Here we have applied the calculation (2.17.3). Note that, since $N \geqslant \ell + 1$, we have $2N + i - 1 \geqslant N + i + \ell \geqslant N + i + \ell - r$, and therefore in the last equation we can replace $t^{2N+i-1}\mathfrak{g}_A[[t]]dt$ by the larger lattice $t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt$.

We will show the following points all lie in $t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt$:

Next, observe that $\nabla(\eta) \in t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt$ by definition of Λ_i. Moreover, below we will show that that for any $n \geqslant 2$:

$$\mathrm{Ad}_{\xi+\eta}^n(\Gamma dt) \in \mathrm{Ad}_\xi^n(\Gamma dt) + t^{N+i+1+\ell-r}\mathfrak{g}_A[[t]]dt. \qquad (2.17.6)$$

Assume this for the moment, and we will conclude the argument. We then see that the last term in (2.17.5) lies in:

$$[\xi, \Gamma dt] + \sum_{n=2}^{\infty} \frac{1}{n!} \mathrm{Ad}_\xi^n(\Gamma dt) - (d\exp(\xi)) \cdot \exp(-\xi) + t^{N+\ell+i-r}\mathfrak{g}_A[[t]]dt =$$

$$\mathrm{Ad}_{\exp(\xi)}(\Gamma dt) - (d\exp(\xi)) \cdot \exp(-\xi) + t^{N+\ell+i-r}\mathfrak{g}_A[[t]]dt =$$

$$\mathrm{Gauge}_{\exp(\xi)}(\Gamma dt) + t^{N+\ell+i-r}\mathfrak{g}_A[[t]]dt$$

as desired.

It remains to show (2.17.6). More precisely, we claim that for any $n \geqslant 0$, we have:

$$\mathrm{Ad}_{\xi+\eta}^n(\Gamma dt) \in \mathrm{Ad}_\xi^n(\Gamma dt) + t^{Nn+i-r}\mathfrak{g}_A[[t]]dt \qquad (2.17.7)$$

which would imply the claim of (2.17.6), since if $n \geqslant 2$, we have $Nn + i - r \geqslant 2N + i - r \geqslant N + i + 1 + \ell - r$ (since $N \geqslant \ell + 1$ by definition).

We show (2.17.7) by induction, the base case $n = 0$ being obvious. Assume (2.17.7) holds for n, and we will show it holds for $n + 1$. We apply $\mathrm{Ad}_{\xi+\eta} = \mathrm{Ad}_\xi + \mathrm{Ad}_\eta$ to both sides. Obviously the left hand side of (2.17.7) transforms as desired. For the right hand side, note that $\mathrm{Ad}_{\xi+\eta}$ maps $t^{Nn+i-r}\mathfrak{g}_A[[t]]dt$ into $t^{N(n+1)+i-r}\mathfrak{g}_A[[t]]dt$, since $\xi + \eta \in t^N\mathfrak{g}_A[[t]]dt$; moreover, since $\mathrm{Ad}_\xi^n(\Gamma dt) \in t^{Nn-r}\mathfrak{g}_A[[t]]dt$, Ad_η maps it into $t^{Nn-r+N+i}\mathfrak{g}_A[[t]]dt = t^{N(n+1)+i-r}\mathfrak{g}_A[[t]]dt$.

Step 5. We now verify the remaining conditions of Lemma 2.13.2.

The linearization of (2.17.2) is:

$$-\nabla : t^N\mathfrak{g}_A[[t]]/\Lambda_i \to t^{-r}\mathfrak{g}_A[[t]]dt/t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt$$

which is an embedding: indeed, if $\xi \in t^N\mathfrak{g}_A[[t]]$ with $\nabla(\xi) \in t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt$, then we have $\xi \in t^{N+i}\mathfrak{g}_A[[t]]$ by construction, and Λ_i is exactly $\nabla^{-1}(t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt) \cap t^{N+i}\mathfrak{g}_A[[t]]$.

Step 6. Finally, we show that the maps α_i from Lemma 2.13.2 exist. I.e., we should show that if $\eta \in \Lambda_{i-1}$, then:

$$\text{Gauge}_{\exp(\xi+\eta)}(\Gamma dt) - \Gamma dt + \nabla(\eta) \bmod t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt$$

is independent of η.

This follows readily from our earlier work. We compute the left hand side using (2.17.5), and we find that it equals:

$$[\xi, \Gamma dt] + \sum_{n=2}^{\infty} \frac{1}{n!} \text{Ad}_{\xi+\eta}^n(\Gamma dt) - (d\exp(\xi)) \cdot \exp(-\xi) + t^{2N+i-2}\mathfrak{g}_A[[t]]dt.$$

Now the only terms involving η occur inside that infinite sum, and by (2.17.6) (substituting $i - 1$ for i in *loc. cit.*), we have (for $n \geqslant 2$):

$$\text{Ad}_{\xi+\eta}^n(\Gamma dt) \in \text{Ad}_{\xi}^n(\Gamma dt) + t^{N+i+\ell-r}\mathfrak{g}_A[[t]]dt$$

It remains to observe that $2N + i - 2 \geqslant N + i + \ell - r$, since $2N + i - 2 \geqslant N + \ell + i - 1$ and since we have $r \geqslant 1$ by a running assumption.

Step 7. Finally, it remains to see that our closed embedding is finitely presented. Of course, we do so via Lemma 2.14.1; *loc. cit.* means that we should show that the conormal sheaf is finitely generated.

This is quite easy, in fact. Our map:

$$K_N \times S \to t^{-r}\mathfrak{g}_A[[t]]dt$$

is given by acting on an S-point of the right hand side, so is obviously K_N-equivariant. Therefore, its cornormal sheaf is also K_N-equivariant, so is determined by its restriction to S. Now the cotangent complex of the above map restricted to S is dual (is the sense of pro-vector bundles) to the complex:

$$-\nabla : t^N\mathfrak{g}_A[[t]] \to t^{-r}\mathfrak{g}_A[[t]]dt.$$

Therefore, the restriction of the conormal bundle is dual to the cokernel of this map and therefore a finite rank vector bundle.

$\square$

2.18.   We now deduce the theorem.

Proof of Theorem 2.12.1. Let N be as in Lemma 2.17.1. It obviously suffices to show that $S/\mathcal{K}_N$ is an algebraic space smooth over T, as we will show below.

Step 1. For each integer $m \geqslant s$, let S_m denote the fiber product:

$$T \underset{t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt}{\times} t^{-r}\mathfrak{g}[[t]]dt/t^m\mathfrak{g}[[t]]dt.$$

Note that as we vary m, the structure maps are smooth, and $\lim_m S_m = S \in \mathsf{AffSch}$.

We claim that the structure map:

$$S \to S/\mathcal{K}_N$$

factors through S_m for $m \gg 0$ (depending on N).

We will show this in Steps 2-4.

Step 2. In this step, we say in explicit terms what it takes to give a factorization $S \to S_m \to S/\mathcal{K}_N$, and in the next step we provide such a construction.

We claim that the data of a section $\sigma_m : S_m \to S$ to the structure map $\pi_m : S \to S_m$ plus a map $\upsilon : S \to \mathcal{K}_N$ such that the composite:

$$S \xrightarrow{\pi_m \times \upsilon} S_m \times \mathcal{K}_N \xrightarrow{\sigma_m \times \mathrm{id}} S \times \mathcal{K}_m \xrightarrow{\mathrm{act}} S$$

is the identity gives rise to a section as above.[31,32]

Indeed, the map $S_m \times \mathcal{K}_N \to S$ is equivariant for the is $\mathcal{K}_N$-equivariant for the usual action of $\mathcal{K}_N$ on S and for the action of $\mathcal{K}_N$ on $S_m \times \mathcal{K}_N$ defined by the trivial action on S_m and the left action on $\mathcal{K}_N$. Passing to the quotient, we obtain a map $S_m \to S/\mathcal{K}_N$.

The composite map $S \to S_m \to S/\mathcal{K}_N$ is the composite of $\sigma_m \pi_m$ with the tautological projection of S to $S/\mathcal{K}_N$. But the map υ provides an obvious way to identify these two maps.

Now observe that any choice of uniformizer t gives an obvious section σ_m. Below, we will construct υ with the desired properties (for m sufficiently large).

Step 3. Next, we claim that the morphism:

$$\mathcal{K}_N \times S \xrightarrow{\mathrm{act} \times p_2} S \underset{T}{\times} S$$

is a finitely presented closed embedding.

Observe that:

$$S \underset{T}{\times} S = t^{-r}\mathfrak{g}[[t]]dt \underset{t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt}{\times} S$$

and that when we compose the above map with the structure map to $t^{-r}\mathfrak{g}[[t]]dt \times S$, it sends $(\upsilon, s) \in \mathcal{K}_N \times S$ to $(\mathrm{Gauge}_\upsilon(\omega_s), s)$, where $s \mapsto \omega_s$ is the structure map $S \to t^{-r}\mathfrak{g}[[t]]dt$. It suffices to show that this composite map is a finitely presented closed embedding, but this is the content of Lemma 2.17.1.

Step 4. We now complete the proof of the claim of Step 1 by constructing a map $\upsilon : S \to \mathcal{K}_N$ with the desired properties.

Observe that $S = \lim_m S \times_{S_m} S$ as an $S \times_T S$-scheme, where everything in sight is affine. Since we have a morphism:

$$\lim_m S \underset{S_m}{\times} S = S \to S \times \mathcal{K}_N \to S \underset{T}{\times} S$$

of $S \times_T S$-schemes (sending $s \in S$ to $(s, 1) \in S \times \mathcal{K}_N$) and since the latter map is finitely presented, there must exist an integer m and a map:

$$S \underset{S_m}{\times} S \to S \times \mathcal{K}_N \to S \underset{T}{\times} S \tag{2.18.1}$$

of $S \times_T S$-schemes, and which induces the diagonal map on the diagonally embedded copy of $S \subseteq S \times_{S_m} S$.

---

[31]Since every $\mathcal{K}_N$-torsor on S_m is trivial, it is easy to see that such a datum is essentially equivalent to giving such a factorization.

[32]In words: we want to conjugate $s \in S$ to $\sigma_m \pi_m(s)$ to s by means of $\upsilon(s)$. Our Fredholm assumption and the construction of $\mathcal{K}_N$ gives an infinitesimal version of this, and our problem is to integrate to get a global version.

Composing the first map of (2.18.1) with the map $S \xrightarrow{\text{id} \times \sigma_m \pi_m} S \times_{S_m} S$ (for any choice of splitting σ_m, say the one defined by a coordinate t), we obtain a map $S \to S \times \mathcal{K}_N$ of S-schemes, i.e., a map $\upgamma : S \to \mathcal{K}_N$.

It is tautological from the construction that $\upgamma$ has the desired property.

Step 5. It follows immediately from the claim in Step 1 that $S/\mathcal{K}_N$ is a stack locally of finite presentation over T. Therefore, it suffices to show that $S_m \to S/\mathcal{K}_N$ is a smooth covering and provides an atlas.

Indeed, $S/\mathcal{K}_N$ obviously has an affine diagonal, and therefore $S_m \to S/\mathcal{K}_N$ is affine. Moreover, since S_m and $S/\mathcal{K}_N$ are locally finitely presented over T, this implies that the morphism is finitely presented (since it is affine and therefore quasi-compact). We therefore deduce smoothness by looking at cotangent complexes. Finally, we easily see that this map is a covering by base-change along $S \to S/\mathcal{K}_N$.

$\square$

2.19. **Turning points.** We have seen in Counterexample 2.8.5 that the a connection can fail to be Fredholm quite severely: for the $\mathbb{A}^1$-family of connections there, there is no stratification of $\mathbb{A}^1$ such that the restrictions of the connection to strata are Fredholm.

However, our next result says that this behavior cannot occur locally.

Theorem 2.19.1. *For every pair of positive integers $n, r > 0$, there exists a constant[33] $N_{n,r} \in \mathbb{Z}^{\geq 0}$ with the following property:*

Let T be the spectrum of a field and be equipped with a map:

$$T \to t^{-r}\mathfrak{gl}_n[[t]]dt/t^s\mathfrak{gl}_n[[t]]dt$$

with $s \geqslant N_{n,r}$. Define:

$$S := T \underset{t^{-r}\mathfrak{gl}_n[[t]]dt/t^s\mathfrak{gl}_n[[t]]dt}{\times} t^{-r}\mathfrak{gl}_n[[t]]dt$$

and note that there is a canonical rank n family of local systems on $\overset{\circ}{\mathcal{D}}$ parametrized by S.

Then the corresponding family of connections on S is Fredholm.

Remark 2.19.2. This result says that any family of connections with all leading terms the same (to some large enough order) is Fredholm.

Combining Theorem 2.12.1 with Theorem 2.19.1, we obtain:

Corollary 2.19.3. *For G an affine algebraic group, $s \gg r$ and η any (possibly non-closed) point in $t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt$, the quotient of the fiber of $t^{-r}\mathfrak{g}[[t]]dt$ at η by $\mathcal{K}_{r+s}$ is a Noetherian and regular Artin stack that is smooth over η.*

Remark 2.19.4. The proof is essentially an application of the Babbitt-Varadarajan algorithm [BV], which they introduced for finding canonical forms for linear systems of Laurent series differential equations (which is a somewhat different concern from ours here). We make no claims to originality in our methods here, and indeed, the reader who is familiar with [BV] will find the argument redundant; we rather include the argument for the reader who is not so familiar with their reduction theory for linear differential equations.

The proof of the theorem will be given in §2.21-2.35 below.

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[33]The proof of Theorem 2.19.1 below gives a very explicit recursive procedure for computing $N_{n,r}$.

2.20. **Some counterexamples.** We begin by clarifying why Theorem 2.19.1 is formulated in quite the way it is. This material may safely be skipped by the reader, though we do not particularly advise this.

First, why can we not always take $N_{n,r} = 0$?

Counterexample 2.20.1. Suppose that we could always take $N_{2,2} = 0$. It would follow that any connection of the form:

$$d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{dt}{t^2} + 0 \cdot \frac{dt}{t} + \text{lower order terms}$$

is Fredholm (this follows from the case $T = \mathrm{Spec}(k)$ in the theorem). However, we claim that this is not the case.

Indeed, let $S = \mathrm{Spec}(k[\lambda])$, and consider the connection:

$$d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{dt}{t^2} + \begin{pmatrix} 0 & 0 \\ \lambda(\lambda+1) & 0 \end{pmatrix} dt.$$

We claim that this connection is not Fredholm, in contradiction with the above.

When we specialize to $\lambda = \ell \in \mathbb{Z} \subseteq k$, we find that the de Rham complex for this connection has a 0-cocycle $\begin{pmatrix} t^\ell \\ -\ell \cdot t^{\ell+1} \end{pmatrix} \in k((t))^{\oplus 2}$.

Now observe that over the generic point of S, the de Rham complex is acyclic. Indeed, by (the proof of) Lemma 2.11.1, it suffices to see this after adjoining a square root $t^{\frac{1}{2}}$ of t. Then we can apply a gauge transformation by:

$$\begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & t^{-\frac{1}{2}} \end{pmatrix} \in GL_2(K)$$

so that our connection becomes:

$$d + \begin{pmatrix} -\frac{1}{2} & 1 \\ \lambda(\lambda+1) & \frac{1}{2} \end{pmatrix} \frac{dt}{t}.$$

The leading term of this regular singular connection has determinant $-\frac{1}{4} - \lambda(\lambda+1)$ and trace zero, and therefore is diagonalizable with eigenvalues $\pm\sqrt{\frac{1}{4} + \lambda(\lambda+1)} = \pm(\lambda + \frac{1}{2})$. But the connection $d + \frac{\eta}{t} dt$ is acyclic for $\eta \notin \mathbb{Z}$, giving the acyclicity here.[34]

As in Counterexample 2.8.5, this implies that the connection is not Fredholm: indeed, the de Rham cohomology vanishes at the generic point, but is not supported on any Zariski closed subvariety of $\mathrm{Spec}(k[\lambda])$, and therefore cannot be coherent.

We include the next counterexample just to indicate how jumps can occur as we vary our point in $t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt$.

Counterexample 2.20.2. Let $T = \mathrm{Spec}(F)$ be the spectrum of the localization of $k[\lambda]$ at 0. Then we claim that the rank one connection:

$$d + \frac{\lambda}{t^2} dt$$

---

[34]Note that $\frac{dt}{t} = 2\frac{d\sqrt{t}}{\sqrt{t}}$, so one should "really" view the eigenvalues as being $2\lambda + 1$, which explains why the complex is not acyclic for λ an integer.

is not Fredholm.

Indeed, if it were, then Lemma 2.9.1 would imply that the complex:

$$B[[t]] \xrightarrow{\nabla} t^{-2}B[[t]]dt$$

is perfect. However, after extending scalars to the fields $B[\lambda^{-1}]$ and $B/\lambda = k$, we obtain different Euler characteristics, a contradiction. Indeed, for λ invertible, we have seen in Example 2.8.3 that the complex is acyclic, while for $\lambda = 0$, the complex is obviously quasi-isomorphic[35] to $k \oplus k[-1] \oplus k[-1]$.

2.21. **Proof of Theorem 2.19.1.** We now give the proof of Theorem 2.19.1.

2.22. We let $T = \mathrm{Spec}(F)$ and let $S = \mathrm{Spec}(A)$ in what follows.

2.23. **Regular singular case.** Suppose $r = 1$. Then we claim that $N_{n,1} = 0$ suffices for any rank n for the local system.

Indeed, suppose that the map $T = \mathrm{Spec}(F) \to t^{-1}\mathfrak{gl}_n[[t]]dt/\mathfrak{gl}_n[[t]]dt = \mathfrak{gl}_n$ is defined by the matrix $\Gamma_{-1} \in \mathfrak{gl}_n(F)$.

Since the characteristic polynomial of Γ_{-1} has finitely many roots in F, there are only finitely many integers $\ell \in \mathbb{Z}$ such that $\Gamma_{-1} + \ell \cdot \mathrm{id}_n$ is not invertible. Therefore, by Example 2.8.3, the corresponding connection is Fredholm.

2.24. **The rank 1 case.** At this point, we proceed by induction on n, the rank of the local system.

We begin with the case $n = 1$. We claim that we can take $N_{1,r} = 0$ for all r.

Indeed, suppose our map:

$$T = \mathrm{Spec}(F) \to t^{-r}k[[t]]dt/k[[t]]dt$$

is defined by $f_{-r}t^{-r}dt + \ldots + f_{-1}t^{-1}dt + k[[t]]dt$ with $f_i \in F$.

If $f_i = 0$ for $i < -1$, then we are in the paradigm of §2.23. Otherwise, $f_i \neq 0$ for some $i < -1$, and by Example 2.8.3, the corresponding connection is Fredholm.

Therefore, we can assume the result true for all $n' < n$ in what follows.

2.25. We will use the following notation in what follows. Given a map:

$$T \to t^{-r}\mathfrak{gl}_n[[t]]dt/t^s\mathfrak{gl}_n[[t]]dt$$

we let $\Gamma_i \in \mathfrak{gl}_n(F), -r \leqslant i < s$ denote the coefficient of t^i above.

For all $-r \leqslant i$, we let $\Gamma_i \in \mathfrak{gl}_n(A)$ denote the coefficient of t^i in the tautological connection on S. Note that the abuse of notation is justified by the fact that for $-r \leqslant i < s$, the two matrices we have called Γ_i are the same under the embedding $\mathfrak{gl}_n(F) \hookrightarrow \mathfrak{gl}_n(A)$.

Finally, we let $\Gamma \in t^{-r}(\mathfrak{gl}_n \otimes A)[[t]]$ denote the connection matrix $\sum \Gamma_i t^i$.

(Note that Γ_{-r} is independent of choice of coordinate up to scaling, justifying the prominent role that it plays below.)

---

[35]The kernel is generated by $1 \in k[[t]]$, while the cokernel is generated by the classes of $t^{-1}dt$ and $t^{-2}dt$

2.26. **Reduction to the case of nilpotent leading term.** Suppose $N_{n,r} \geqslant N_{n',r'}$ for all $n' < n$ and $r' \leqslant r$ (where we can suppose the latter numbers are all known by induction). We then claim that the conclusion of the theorem is true for a given map $T \to t^{-r}\mathfrak{gl}_n[[t]]dt/t^s\mathfrak{gl}_n[[t]]dt$ as long as Γ_{-r} is not nilpotent.[36]

Note that we can safely assume $r > 1$ below.

Step 1. Let $\Gamma_{-r} = f + s$ with f nilpotent and s semisimple commuting elements of $\mathfrak{gl}_n(F)$ be the Jordan decomposition of Γ_{-r}.

Note that $F^{\oplus n} \simeq \mathrm{Ker}(s) \oplus \mathrm{Image}(s)$. Similarly, $\mathfrak{gl}_n(F) = \mathrm{Ker}(\mathrm{Ad}_s) \oplus \mathrm{Image}(\mathrm{Ad}_s)$, and every matrix in $\mathrm{Ker}(\mathrm{Ad}_s)$ preserves this decomposition of $F^{\oplus}$ as a direct sum.

Next, note that $\mathrm{Image}(\mathrm{Ad}_s) \subseteq \mathrm{Image}(\mathrm{Ad}_{\Gamma_{-r}})$. Indeed, since $\mathrm{Ad}_s : \mathrm{Image}(\mathrm{Ad}_s) \xrightarrow{\simeq} \mathrm{Image}(\mathrm{Ad}_s)$ and since Ad_f is a nilpotent endomorphism of $\mathrm{Image}(\mathrm{Ad}_s)$ commuting with Ad_s, $\mathrm{Ad}_{\Gamma_{-r}} = \mathrm{Ad}_s + \mathrm{Ad}_f : \mathrm{Image}(\mathrm{Ad}_s) \to \mathrm{Image}(\mathrm{Ad}_s)$ is an isomorphism.

Note that we obtain a similar decomposition:

$$A^{\oplus n} = \mathrm{Ker}(\mathrm{Ad}_s) \oplus \mathrm{Image}(\mathrm{Ad}_s)$$

where we consider s acting on $A^{\oplus n}$ by extension of scalars.

Step 2. Next, we claim that we can apply a gauge transformation by an element of $\mathcal{K}_1(S)$ so that each matrix $\Gamma_j \in \mathfrak{gl}_n(A)$ ($j \geqslant -r$) preserves each the A-submodules $\mathrm{Ker}(s)$ and $\mathrm{Image}(s)$ of $A^{\oplus n}$. (This method is very standard, and goes back at least to [Sib].)

More precisely, we will construct by induction $g_i \in \mathcal{K}_i(S)$ with the property that the first $i + 1$ matrices in $\mathrm{Gauge}_{g_i g_{i-1} \ldots g_1}(\Gamma dt)$ have the property above; note that the infinite product of the g_i makes sense in $\mathcal{K}_1$, and therefore gives a gauge transformation with the desired property.

These elements g_i will have the property, which will be important later, that g_i depends only on $\Gamma_{-r}, \ldots, \Gamma_{-r+i}$. In particular, for $1 \leqslant i < r + s$, we will have $g_i \in \mathcal{K}_1(T) \subseteq \mathcal{K}_1(S)$.

To construct the g_i: applying the gauge transformation by $g_{i-1} \ldots g_1$, we can assume that $\Gamma_{-r}, \ldots, \Gamma_{-r+i-1}$ preserve our submodules.

Then, we can (uniquely) write $\Gamma_{-r+i} = M_1 + M_2$ with $M_1 \in \mathrm{Ker}(\mathrm{Ad}_s)$ and $M_2 \in \mathrm{Image}(\mathrm{Ad}_s)$. Note that $M_2 = [\Gamma_{-r}, C]$ for some $C \in \mathfrak{gl}_n(A)$ (or $C \in \mathfrak{gl}_n(F)$ if $i < r + s$), since $\mathrm{Image}(\mathrm{Ad}_s) \subseteq \mathrm{Image}(\mathrm{Ad}_{\Gamma_{-r}})$.

We then take $g_i = \exp(-t^i C) \in \mathcal{K}_i(A)$. Then:

$$\mathrm{Gauge}_{g_i}(\Gamma dt) = \mathrm{Ad}_{\exp(-t^i C)}(\Gamma dt) - (d\exp(-t^i C)) \cdot \exp(t^i C) =$$

$$\left(\Gamma dt - [t^i C, \Gamma dt] + \text{terms divisible by } t^{2i-r}\right) + \left(t^i C + \text{terms divisible by } t^{2i-1}\right).$$

Note that $r \geqslant 1$ by running assumption, so divisibility by $2i - 1$ implies divisibility by $2i - r$, and since $i \geqslant 1$, this implies divisibility by $i - r + 1$. Then reducing modulo $t^{i-r+1}\mathfrak{gl}_n[[t]]dt$, we obtain:

$$\Gamma dt - [t^i C, \Gamma_{-r} dt] + t^{i-r+1}\mathfrak{gl}_n[[t]]dt.$$

Then we have not changed any Γ_j for $j \leqslant i - r$ except Γ_{i-r}, and we have changed it to M_1 in the above notation. Since M_1 commutes with the semisimple matrix Ad_s, it preserves the decomposition of $F^{\oplus n}$ as $\mathrm{Image}(\mathrm{Ad}_s) \oplus \mathrm{Ker}(\mathrm{Ad}_s)$.

Step 3. Finally, suppose that Γ_{-r} is not a nilpotent matrix. Then $s \neq 0$, i.e., $\mathrm{Ker}(\mathrm{Ad}_s) \neq A^{\oplus n}$.

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[36]Since A is an integral domain, there is no ambiguity in the meaning of *nilpotent* here: it just means that Γ_{-r} is nilpotent as a matrix.

Therefore, we have shown above that we can gauge transform Γ to be a direct sum of connection matrices Γ^1 and Γ^2 of smaller rank (so $\Gamma^i \in \mathfrak{gl}_{n_i}((t))$ for $n_1 + n_2 = n$, $n_i \neq 0$).

Since the g_i constructed above lie in $\mathcal{K}_1$, Γ^1 and Γ^2 have order of pole at most r.

Moreover, since the construction of g_i above depended only on the leading $i + 1$ terms of Γ, the coefficients of t^j for $-r \leqslant j < s$ depends only on the leading terms of Γ. That is, the connection $d + \Gamma^1 dt + \Gamma^2 dt$ on S has leading $r + s$-terms defined by some map:

$$T \to t^{-r}\mathfrak{gl}_n[[t]]dt/t^s\mathfrak{gl}_n[[t]]dt.$$

Therefore, in this case we obtain the claim as long as $N_{n,r}$ satisfies the inequalities we said.

2.27. Below, we will show how to handle the case the Γ_{-r} is a nilpotent matrix. We use the key idea of [BV] here: use the geometry of Slodowy slices to proceed by induction on $\dim(\mathrm{Ad}_{\Gamma_{-r}})$.

2.28. **Slodowy review.** We briefly review some facts about nilpotent elements in reductive Lie algebra. Let G be a split reductive group ($G = GL_n$ for us) with Lie algebra $\mathfrak{g}$ over a ground field of characteristic zero, which we suppress from the notation (it will be F for us later).

Let f be a nilpotent element in $\mathfrak{g}$, and extend f to an $\mathfrak{sl}_2$-triple $\{e, f, h\} \subseteq [\mathfrak{g}, \mathfrak{g}]$ via Jacobson-Morozov. Let $H : \mathbb{G}_m \to G$ be the cocharacter with derivative h.

Recall that the *Slodowy slice* $\mathcal{S}_f$ is the scheme $f + \mathrm{Ker}(\mathrm{Ad}_e)$ (considered as a scheme by thinking of $\mathrm{Ker}(\mathrm{Ad}_e)$ as an affine space).

Equip $\mathfrak{g}$ with the $\mathbb{G}_m$-action:

$$\lambda \star \xi := \lambda^2 \mathrm{Ad}_{H(\lambda)}(\xi), \quad \lambda \in \mathbb{G}_m, \xi \in \mathfrak{g}.$$

Here the λ^{-2} in front is the normal action by homotheties of $\mathbb{G}_m$ on $\mathfrak{g}$.

Note that this $\star$-action preserves $\mathcal{S}_f$ because $\mathrm{Ad}_{H(\lambda)}(f) = \lambda^{-2}f$. Moreover, this action contracts $\mathcal{S}_f$ onto the point f, since Ad_h has non-negative eigenvalues on $\mathrm{Ker}(\mathrm{Ad}_e)$ (by the representation theory of $\mathfrak{sl}_2$).

Next, observe that the $\star$-action preserves the G-orbit through f: indeed, for $g \in G$, we have:

$$\lambda \star \mathrm{Ad}_g(f) = \lambda^2 \mathrm{Ad}_{H(\lambda)} \mathrm{Ad}_g(f) = \mathrm{Ad}_{H(\lambda)} \mathrm{Ad}_g(\lambda^2 f) =$$
$$\mathrm{Ad}_{H(\lambda)} \mathrm{Ad}_g \mathrm{Ad}_{H(\lambda)^{-1}}(f) = \mathrm{Ad}_{H(\lambda)gH(\lambda)^{-1}}(f).$$

More generally, the $\star$-action preserves the G-orbit through *any* nilpotent element f'. Indeed, first notice that $\lambda f \in G \cdot f$ for any λ, since $\mathrm{Ad}_{H(\sqrt{\lambda})}(f) = \lambda f$. Here nothing about f is special, so $\lambda f' \in G \cdot f'$ for any λ. Then:

$$\lambda \star f' = \lambda^2 \mathrm{Ad}_{H(\lambda)}(f') \in G \cdot \mathrm{Ad}_{H(\lambda)}(f') = G \cdot f'.$$

With these preliminary geometric observations, we now deduce:

Lemma 2.28.1 (c.f. [BV] §2). *For any field-valued point $f' \in \mathcal{S}_f$ nilpotent, $\dim(G \cdot f')(= \dim \mathrm{Image}(\mathrm{Ad}_{f'}))$ is greater than or equal to $\dim(G \cdot f)(= \dim \mathrm{Image}(\mathrm{Ad}_f))$, with equality if and only if $f' = f$.*

Proof. If f' is as above, we must have $G \cdot f \subseteq \overline{G \cdot f'}$ (the orbit closure), since the $\star$-action preserves $G \cdot f'$ and contracts $\mathcal{S}_f$ onto f. This implies the claim on dimensions.

Since $G \cdot f'$ is open in its closure, if $\dim(G \cdot f) = \dim(G \cdot f')$ then we must have $G \cdot f = G \cdot f'$. Therefore, we should see that $G \cdot f \cap \mathcal{S}_f = f$.

Observe that the G-orbit $G \cdot f$ through f and $\mathcal{S}_f$ meet transversally at f — this follows from the identity:

$$\mathfrak{g} = \mathrm{Ker}(\mathrm{Ad}_e) \oplus \mathrm{Image}(\mathrm{Ad}_f)$$

(which is again a consequence of the representation theory of $\mathfrak{sl}_2$). Because the $\star$-action on $\mathfrak{g}$ preserves $G \cdot f$ and $\mathcal{S}_f$ and contracts onto f, it follows that $G \cdot f$ and $\mathcal{S}_f$ meet only at the point f.
$\square$

2.29. Nilpotent leading term. We now treat the case of nilpotent leading term. At this point, the reader may wish to skip ahead to §2.36, where we indicate how the reduction theory works in the simplest non-trivial case.

2.30. Let $\Gamma_{-r} = f \in \mathfrak{gl}_n(A)$ be nilpotent, and let $\delta := \dim G \cdot f$, where $G \cdot f \subseteq \mathfrak{g} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(F)$ as a scheme. We will proceed by descending induction on δ.

More precisely, below we will construct $N_{n,r,\delta}$ with the property that for $s \geqslant N_{n,r,\delta}$ and Γ_{-r} of the above type, the conclusion of the theorem holds. By induction, we can assume that knowledge of $N_{n',r'}$ for all $n' < n$, and can assume the knowledge of $N_{n,r',\delta'}$ for all $\delta' > \delta$.

Note that $\delta > \dim(G)$, the hypotheses are vacuous, giving the base case in the induction. Moreover, this makes clear that if we accomplish the above construction of $N_{n,r,\delta}$, we have completed the proof of the theorem: combining this with §2.26, we see that if we take:

$$N_{n,r} := \max \left\{ \{N_{n',r}\}_{n'<n}, \{N_{n,r,\delta}\}_{\delta \leqslant \dim(G)} \right\}$$

we have obtained a constant satisfying the conclusion of the theorem.

2.31. We take an $\mathfrak{sl}_2$-triple $\{e, f = \Gamma_{-r}, h\} \in \mathfrak{sl}_n(F)$ as before. We obtain an identification:

$$\mathfrak{gl}_n(F) = \mathrm{Ker}(\mathrm{Ad}_e) \oplus \mathrm{Image}(\mathrm{Ad}_f)$$

and similarly for $\mathfrak{gl}_n(A)$. Our $\mathfrak{sl}_2$-triple integrates to a map at the level of group schemes over F, and in particular we let $H : \mathbb{G}_{m,F} \to SL_{m,F}$ integrate h.

By the same method as in Step 2 of §2.26, we may perform a gauge transformation by an element of $\mathcal{K}_1(F)$ to obtain a new connection matrix $\Gamma' = \sum_{i \geqslant -r} \Gamma'_i t^i \in t^{-r} \mathfrak{gl}_n[[t]]$ with $\Gamma'_{-r} = \Gamma_{-r}$ and $\Gamma'_i \in \mathrm{Ker}(\mathrm{Ad}_e)$ for all $i > -r$. Moreover, by the construction of *loc. cit.*, $\Gamma'_i \in \mathfrak{gl}_n(F) \subseteq \mathfrak{gl}_n(A)$ (i.e., it "has constant coefficients") for $-r \leqslant i < s$.

Since the first $r + s$ terms of our matrix are constant, we might as well replace Γ by Γ' and thereby assume that the $\Gamma_i \in \mathrm{Ker}(\mathrm{Ad}_e)$ for all $i > -r$ (just to simplify the notation with Γ').

2.32. For $j \in \mathbb{Z}$, let $\Gamma_i^{(j)}$ be the degree j component of Γ_i with respect to the grading defined by H, so:

$$[h, \Gamma_i^{(j)}] = j \Gamma_i^{(j)}, \text{ or equivalently, } \mathrm{Ad}_{H(\lambda)}(\Gamma_i^{(j)}) = \lambda^j \Gamma_i^{(j)} \ (\lambda \in \mathbb{G}_m).$$

For example, $\Gamma_{-r} = \Gamma_{-r}^{(-2)}$. Note that for $i > -r$, since $\Gamma_i \in \mathrm{Ker}(\mathrm{Ad}_e)$, we have $\Gamma_i^{(j)} \neq 0$ only for $j \geqslant 0$.

Let $\alpha \in \mathbb{Q}^{>0}$ be defined as:[37]

$$\alpha := \min_{\Gamma_i^{(j)} \neq 0, \, -r < i < s} \frac{i + r}{j + 2}.$$

We treat the cases $\alpha \geqslant \frac{r-1}{2}$ and $\alpha < \frac{r-1}{2}$ separately in §2.34 and §2.35 respectively.

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[37]We use the standard convention that $\alpha = \infty$ if the set indexing this minimum is empty.

2.33. At this point, we impose our conditions on $N_{n,r,\delta}$. The reader may safely skip these formulae right now: we are only including them now for the sake of concreteness.

Let j_δ the maximal degree in the grading of $\mathfrak{gl}_n$ defined by H for *some* nilpotent f with orbit having dimension δ.[38] This constant is well-defined e.g. because there are only finitely many nilpotent orbits.

Below, we will show that as long as:

$$N_{n,r,\delta} \geqslant -1 + j_\delta \cdot \frac{r-1}{2},$$

$$N_{n,r,\delta} \geqslant (j_\delta + 2) \cdot j_\delta \cdot \frac{r-1}{2} + N_{n',(j_\delta+2)\cdot(r-1)+1} \text{ for } n' < n, \text{ and} \qquad (2.33.1)$$

$$N_{n,r,\delta} \geqslant (j_\delta + 2) \cdot j_\delta \cdot \frac{r-1}{2} + N_{n,(j_\delta+2)\cdot(r-1)+1,\delta'} \text{ for } \delta < \delta' \leqslant \dim GL_n.$$

the connection is Fredholm. Note that there are only finitely many conditions listed here, and they are of the desired inductive nature, and so if we can show this, then we have completed the proof of the theorem.

2.34. First, suppose that $\alpha \geqslant \frac{r-1}{2}$.

By Lemma 2.11.1, it suffices to show our connection is Fredholm after adjoining $t^{\frac{1}{2}}$.

We then claim that after performing a gauge transformation by $H(t^{\frac{1-r}{2}})$, we obtain a connection with regular singularities.

Indeed, first note that $d\log(H(t^{\frac{1-r}{2}}))$ has a regular singularity, so it suffices to show that $\text{Ad}_{H(t^{\frac{1-r}{2}})}(\Gamma)$ has a pole of order $\leqslant 1$.

We clearly have:

$$\text{Ad}_{H(t^{\frac{1-r}{2}})}(\Gamma_{-r}t^{-r}) = t^{r-1} \cdot \Gamma_{-r}t^{-r} = \Gamma_{-r}t^{-1}.$$

Then for $-r < i < s$, we have:

$$\text{Ad}_{H(t^{\frac{1-r}{2}})}(\Gamma_i t^i) = \sum_j \Gamma_i^{(j)} t^{i+\frac{j(1-r)}{2}}.$$

We then claim that $\Gamma_i^{(j)} \neq 0$ and the definition of α implies that:

$$i + \frac{j(1-r)}{2} \geqslant -1$$

as desired. Indeed, for any such pair (i,j), we have:

$$\frac{i+r}{j+2} \geqslant \alpha \geqslant \frac{r-1}{2}$$

and rearranging terms we get:

$$i + r \geqslant j\frac{r-1}{2} + (r-1)$$

which is obviously equivalent to the desired inequality.

Finally, for $s \leqslant i$, recall from (2.33.1) that $s > -1 + j\frac{r-1}{2}$ for any degree j appearing in the grading of $\mathfrak{gl}_n$ defined by H. Therefore, for i in this range, we have:

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[38]Of course, j_δ is bounded by the maximal degree in the principal grading of $\mathfrak{gl}_n$. We could replace j_δ everywhere by this constant, but we are just trying to be somewhat more economical by retaining the dependence on δ.

$$i + \frac{j(1-r)}{2} \geqslant s + \frac{j(1-r)}{2} \geqslant -1$$

as desired.

Therefore, we see that the resulting connection is regular singular. Moreover, the above shows that the leading term of this regular singular connection is determined entirely by Γ_i for $i < s$ and therefore has coefficients in $F \subseteq A$ (noting that $d\log(H(t^{\frac{r-1}{2}}))$ also has coefficients in $F \subseteq A$). This completes the argument by §2.23.

2.35.   Finally, we treat the case where $\alpha < \frac{r-1}{2}$.

Applying Lemma 2.11.1 again, it suffices to see that our connection is Fredholm after adjoining t^α.

Performing a gauge transformation by $H(t^{-\alpha})$, we claim that we obtain a connection with a pole of order $\leqslant -r + 2\alpha$. Indeed, this follows exactly as in §2.34:

- The $d\log$ term only affects the coefficient of t^{-1}.
- $\mathrm{Ad}_{H(t^{-\alpha})}\Gamma_{-r}t^{-r} = \Gamma_{-r}t^{-r+2\alpha}$.
- For $-r < i < s$, we have
  $\mathrm{Ad}_{H(t^{-\alpha})}(\Gamma_i t^i) = \sum_j \Gamma_i^{(j)} t^{i-j\alpha}$, and for $\Gamma_i^{(j)} \neq 0$, we have:

  $$\frac{i+r}{j+2} \geqslant \alpha \Rightarrow i + r \geqslant j\alpha + 2\alpha \Rightarrow i - j\alpha \geqslant -r + 2\alpha.$$

- For $s \leqslant i$, the same argument as in §2.34 shows that the corresponding terms can only change the coefficients of t^j for $j \geqslant -1$.

Note that by assumption, we presently have $-r + 2\alpha < -1$.

For any pair (i,j) with $-r < i < s$, $\Gamma_i^{(j)} \neq 0$, and:

$$\frac{i+r}{j+2} = \alpha$$

we have:

$$i - j\alpha = i - (j+2)\alpha + 2\alpha = -r + 2\alpha.$$

Therefore, the leading term (i.e., $t^{-r+2\alpha}$-coefficient) of our resulting connection is:

$$\Gamma_{-r} + \sum_{\frac{i+r}{j+2}=\alpha, \, -r<i<s} \Gamma_i^{(j)}. \tag{2.35.1}$$

Note that this sum cannot be Γ_{-r}: indeed, there is at least one summand on the right (since $\alpha < \frac{r-1}{2} < \infty$); moreover, a summand $\Gamma_i^{(j)}$ contributes purely in degree j (with respect to the grading defined by H), and since for given j there is at most one pair (i,j) with $\frac{i+r}{j+2} = \alpha$, there can be no cancellations within the given degree j.

Therefore, this sum lies in the Slodowy slice for Γ_{-r}. By Lemma 2.28.1, it is either nilpotent with a larger dimensional orbit that Γ_{-r}, or it is not nilpotent, in which case we should bring §2.26 to bear.

Remark 2.35.1. The reader should be better off arguing for themselves that we are done at this point, since this is essentially clear. But we include the final bit of the argument (difficult though it may be to read) for completeness.

Let $\alpha = \frac{a}{b}$ with a and b coprime. Note that b is bounded in terms of δ alone: b divides $j + 2$ for some $0 \leqslant j \leqslant j_\delta$, and therefore $b \leqslant j_\delta + 2$.

Therefore, the order of pole in the extension $A((t^{\frac{1}{b}}))$ of $A((t))$ of our resulting connection is bounded in terms of δ. More precisely, we have:

$$t^{-r+2\alpha}dt = bt^{-r+2\alpha+1-\frac{1}{b}}dt^{\frac{1}{b}}$$

so that our pole has order (in the $t^{\frac{1}{b}}$-valuation) at most:

$$br - 2a - b + 1 \leqslant b(r - 1) + 1 \leqslant (j_\delta + 2) \cdot (r - 1) + 1.$$

Moreover, the coefficients before (i.e., with lower order of zero/higher order of pole):

$$t^{s-j_\delta\alpha}dt = (t^{\frac{1}{b}})^{bs-bj_\delta\alpha+b-1}dt^{\frac{1}{b}}$$

lie in $\mathfrak{gl}_n(F) \subseteq \mathfrak{gl}_n(A)$, as is clear from using our same Ad_h-eigenspace decomposition of $\mathfrak{gl}_n$. Note that we can bound the order of zero here from below independently of α:

$$bs - bj_\delta\alpha + b - 1 > bs - bj_\delta \cdot \frac{r-1}{2} + b - 1 = b(s+1) - 1 - bj_\delta \cdot \frac{r-1}{2} \geqslant$$
$$s + 1 - 1 - bj_\delta \cdot \frac{r-1}{2} \geqslant s - (j_\delta + 2) \cdot j_\delta \cdot \frac{r-1}{2}.$$

Therefore, if:

$$N_{n,r,\delta} \geqslant (j_\delta + 2) \cdot j_\delta \cdot \frac{r-1}{2} + N_{n',(j_\delta+2)\cdot(r-1)+1} \text{ for } n' < n, \text{ and}$$
$$N_{n,r,\delta} \geqslant (j_\delta + 2) \cdot j_\delta \cdot \frac{r-1}{2} + N_{n,(j_\delta+2)\cdot(r-1)+1,\delta'} \text{ for } \delta < \delta' \leqslant \dim GL_n.$$

then we obtain the claim. Indeed, the former condition takes care of the case where the leading term of $\mathrm{Gauge}_{H(t^\alpha)}(\Gamma dt)$ has a non-nilpotent leading term by §2.26; and the latter condition takes care of the case where it has nilpotent leading term, by the above analysis.

2.36. Example: n = 2, r = 2. It is instructive to see what is going on above in a simpler setup. Suppose that we have a rank 2 connection with a pole of order 2 and with nilpotent leading term. We claim that $N_{2,2} = 1$ suffices.[39]

Up to (constant) change of basis, we can write:

$$\Gamma dt = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{dt}{t^2} + \Gamma_{-1} \frac{dt}{t} + \Gamma_0 dt + \text{lower order terms.}$$

Suppose that $\Gamma_{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Gamma_0 = \begin{pmatrix} \circledast & e \\ \circledast & \circledast \end{pmatrix}$ where $\circledast$ is indicates that the term is irrelevant for our needs below.

There are two cases: when $b = 0$ and when $b \neq 0$. Note that these correspond to §2.34 and 2.35 respectively.

In the former case, we apply a gauge transformation by:

$$\begin{pmatrix} t^{-\frac{1}{2}} & 0 \\ 0 & t^{\frac{1}{2}} \end{pmatrix}$$

to obtain:

---

[39]This is a more precise estimate than we obtained from the crude estimates of §2.35.

$$\begin{pmatrix} a + \frac{1}{2} & e \\ 1 & d - \frac{1}{2} \end{pmatrix} \frac{dt}{t} + \text{lower order terms.}$$

So we see that we get a regular singular connection whose leading terms depends only on the fixed matrices Γ_{-2}, Γ_{-1}, and Γ_0, and therefore it is Fredholm. (Note that if b were $\neq 0$, it would contribute to a non-zero coefficient of $\frac{dt}{t^2}$ here.)

If $b \neq 0$, then we instead apply a gauge transformation by the matrix:

$$\begin{pmatrix} t^{-\frac{1}{4}} & 0 \\ 0 & t^{\frac{1}{4}} \end{pmatrix}$$

to obtain a connection of the form:

$$\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} \frac{dt}{t^{\frac{3}{2}}} + \begin{pmatrix} a + \frac{1}{4} & 0 \\ 0 & d - \frac{1}{4} \end{pmatrix} \frac{dt}{t} + \text{lower order terms.}$$

Since $b \neq 0$, the leading term of this connection is semisimple. Therefore, our connection can be written as a sum of two rank 1 connections whose polar parts are constant. Therefore, the connection is Fredholm by reduction to the rank 1 case.

2.37. Coda. We need some refinements to the above results, making them more effective. These refinements will be used later in the paper, and may be safely skipped for the time being.

2.38. Let G be an affine algebraic group. For $r > 0$ and $s \geqslant 0$, Theorems 2.12.1 and 2.19.1 imply that the geometric fibers of the map:

$$t^{-r}\mathfrak{g}[[t]]dt/\mathcal{K}_{r+s} \to t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt$$

are Artin stacks for $s \gg 0$.

Proposition 2.38.1. The dimensions of these fibers are bounded uniformly in terms of G and r.

Note that these Artin stacks are smooth with tangent complex:

$$t^{r+s}\mathfrak{g}[[t]] \xrightarrow{-\nabla} t^s\mathfrak{g}[[t]]dt$$

for $\Gamma dt \in t^{-r}\mathfrak{g}[[t]]dt$ a point and $\nabla := d - \Gamma dt$. The dimension of the Artin stacks above is the Euler characteristic of this complex (for Γdt any connection in the relevant fiber). Therefore, Proposition 2.38.1 follows from the next result:

Lemma 2.38.2. There is a constant $C_{n,r}$ with the following property:

Let (V, ∇) be any rank n differential module over[40] $k((t))$, and let $L \subseteq V$ be a $k[[t]]$-lattice such that ∇ maps into $t^{-r}Ldt$.

Then the absolute value of the Euler characteristic of the complex:

$$\nabla : \Lambda \to t^{-r}\Lambda dt$$

is at most $C_{n,r}$.

---

[40]Of course, k can be any field of characteristic zero, not just our ground field. The result also immediately extends to any Fredholm connection over any commutative ring.

Proof. First, note that $\dim(\mathrm{Ker}(\nabla))$ is at most n, and therefore we need to bound $\dim(t^{-r}\Lambda dt/\nabla(\Lambda))$ in terms n and r alone.

By [FZ] Lemma 8, we have:

$$\mathrm{codim}(\nabla^{-1}(\Lambda dt) \cap \Lambda \subseteq \nabla^{-1}(\Lambda dt)) \leqslant \mathrm{codim}(\nabla^{-1}(\Lambda dt) \cap \Lambda \subseteq \Lambda).$$

Moreover, the right hand side is bounded in terms of r and n: it is less than or equal to rn. Indeed, this follows from the embedding:

$$\Lambda/(\nabla^{-1}(\Lambda dt) \cap \Lambda) \xrightarrow{\nabla} t^{-r}\Lambda dt/\Lambda dt$$

so it is bounded by rn.

Now recall (see e.g. [BBE] §5.9) that $H^1_{dR}(V, \nabla)$ is at most n-dimensional. Therefore, $\nabla(\nabla^{-1}(\Lambda dt)) \subseteq \Lambda dt$ as codimension at most n. Combining this with the above, we see that:

$$\nabla(\nabla^{-1}(\Lambda dt) \cap \Lambda) \subseteq \Lambda dt$$

has codimension $\leqslant (r+1)n$, and therefore has codimension $\leqslant (2r+1)n$ in $t^{-r}\Lambda dt$.

$\square$

3. Tameness

3.1. In this section, we study the action of $G(O)$ on gauge forms, and show that through the lens of homological algebra, this action has many favorable properties.

In §3.2-3.8, we study some general aspects of (weak) actions of infinite type algebraic groups on categories. In general, this subject exhibits pathological behavior that is not present in the finite type situation.

However, we will introduce the notion of a *tame*[41] action of such a group, where such behavior is not present. The basic dichotomy the reader should keep in mind is that the regular representation is tame, while the trivial representation is not.

Then the main result of this section, Theorem 3.9.1, is that the action of $G(O)$ on gauge forms is tame.

3.2. Let $\mathcal{G}$ be an affine group scheme (possibly of infinite type) over k.

Then $\mathsf{QCoh}(\mathcal{G})$ inherits a usual *convolution* monoidal structure by $\mathcal{F} \otimes \mathcal{G} := m_*(\mathcal{F} \boxtimes \mathcal{G})$ for $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ the multiplication. This convolution structure commutes with colimits in each variable separately, and therefore defines a structure of algebra on $\mathsf{QCoh}(\mathcal{G}) \in \mathsf{DGCat}_{cont}$.

For $\mathcal{C} \in \mathsf{QCoh}(\mathcal{G})\text{–mod} := \mathsf{QCoh}(\mathcal{G})\text{–mod}(\mathsf{DGCat}_{cont})$, we define the *(weak) invariant* and *(weak) coinvariant* categories as:

$$\mathcal{C}^{\mathcal{G},w} := \varprojlim_{\boldsymbol{\Delta}} \left(\mathcal{C} \rightrightarrows \mathsf{QCoh}(\mathcal{G}) \otimes \mathcal{C} \dots \right) = \mathsf{Hom}_{\mathsf{QCoh}(\mathcal{G})\text{–mod}}(\mathsf{Vect}, \mathcal{C})$$

$$\mathcal{C}_{\mathcal{G},w} := \varinjlim_{\boldsymbol{\Delta}^{op}} \left(\dots \mathsf{QCoh}(\mathcal{G}) \otimes \mathcal{C} \rightrightarrows \mathcal{C} = \mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G})}{\otimes} \mathsf{Vect} \right).$$

Here Vect is induced with the *trivial* $\mathsf{QCoh}(\mathcal{G})$-action, i.e., it is induced by the monoidal functor $\Gamma : \mathsf{QCoh}(\mathcal{G}) \to \mathsf{Vect}$.

---

[41]Given the interests of this paper, we explicitly state that this word has nothing to do with the notion of tame ramification in Galois/D-module theory.

Example 3.2.1. If $\mathcal{Y}$ is a prestack with an action of $\mathcal{G}$, then $\mathsf{QCoh}(\mathcal{G})$ acts on $\mathsf{QCoh}(\mathcal{Y})$. Moreover, $\mathsf{QCoh}(\mathcal{Y})^{\mathcal{G},w}$ identifies with $\mathsf{QCoh}(\mathcal{Y}/\mathcal{G})$.[42]

Example 3.2.2. We have canonical identifications:

$$\mathsf{QCoh}(\mathcal{G})^{\mathcal{G},w} \simeq \mathsf{Vect}$$
$$\mathsf{QCoh}(\mathcal{G})_{\mathcal{G},w} \simeq \mathsf{Vect}.$$

Indeed, the former is a special case of Example 3.2.1, and the latter equivalence is tautological: we are computing $\mathsf{QCoh}(\mathcal{G}) \otimes_{\mathsf{QCoh}(\mathcal{G})} \mathsf{Vect}$.

3.3. **The non-renormalized category of representations.** We use the notation $\mathsf{QCoh}(\mathbb{B}\mathcal{G})$ for $\mathsf{Vect}^{\mathcal{G},w}$. Of course, this DG category is (tautologically) QCoh of the prestack $\mathbb{B}\mathcal{G}$.

Note that $\mathsf{QCoh}(\mathbb{B}\mathcal{G}) = \mathsf{Hom}_{\mathsf{QCoh}(\mathcal{G})}(\mathsf{Vect}, \mathsf{Vect})$ as monoidal categories (where the former has the usual tensor product of quasi-coherent sheaves as its monoidal structure). Therefore, $\mathcal{C}^{\mathcal{G},w}$ and $\mathcal{C}_{\mathcal{G},w}$ have canonical $\mathsf{QCoh}(\mathbb{B}G)$-module structures for any $\mathcal{C} \in \mathsf{QCoh}(\mathcal{G})$–mod.

Remark 3.3.1. For any affine scheme $\mathcal{Y}$ with an action of $\mathcal{G}$, $\mathsf{QCoh}(\mathcal{Y}/\mathcal{G})$ inherits a canonical t-structure characterized by the fact that $\mathsf{QCoh}(\mathcal{Y}/\mathcal{G}) \to \mathsf{QCoh}(\mathcal{Y})$ is t-exact; moreover, this t-structure is left (and right) complete. Indeed, these facts follow by the usual argument for Artin stacks (c.f. [Gai2]). In particular, $\mathsf{QCoh}(\mathbb{B}\mathcal{G})$ has a canonical left complete t-structure.

Remark 3.3.2. For $\mathcal{G}$ of infinite type, we emphatically do not use the notation $\mathsf{Rep}(\mathcal{G})$ for $\mathsf{QCoh}(\mathbb{B}\mathcal{G})$ out of deference to Gaitsgory, who suggests (c.f. [FG2]) to use this notation instead for a "renormalized" form of this category in which bounded complexes of finite-dimensional representations are declared to be compact. However, this renormalized form will not play any role in this paper.

3.4. **Averaging.** By the Beck-Chevalley theory, the functor $\mathsf{Oblv} : \mathcal{C}^{\mathcal{G},w} \to \mathcal{C}$ admits a continuous right adjoint $\mathsf{Av}_*^{\mathcal{G},w} = \mathsf{Av}_*^w$, and is comonadic. Moreover, the comonad $\mathsf{Oblv}\,\mathsf{Av}_*^w$ on $\mathcal{C}$ is given by convolution with $\mathcal{O}_{\mathcal{G}} \in \mathsf{QCoh}(\mathcal{G})$.

3.5.   We have the following basic result.

Proposition 3.5.1.   (1) *For every* $\mathcal{C} \in \mathsf{QCoh}(\mathcal{G})$–mod*, the natural functor:*

$$\mathcal{C}^{\mathcal{G},w} \underset{\mathsf{QCoh}(\mathbb{B}\mathcal{G})}{\otimes} \mathsf{Vect} \to \mathcal{C}$$

> *is an equivalence.*
> (2) *The invariants functor:*

$$\mathsf{QCoh}(\mathcal{G})\text{–mod} \to \mathsf{QCoh}(\mathbb{B}\mathcal{G})\text{–mod}$$

> *is fully-faithful.*

Proof. For the first part:

We claim that the averaging functor $\mathcal{C} \to \mathcal{C}^{\mathcal{G},w}$ is monadic. Indeed, its composite with the forgetful functor is tensoring with $\mathcal{O}_{\mathcal{G}}$, and therefore it is obviously conservative. Moreover, it commutes with all colimits, so this gives the claim.

Now let π denote the map $\mathrm{Spec}(k) \to \mathbb{B}\mathcal{G}$ and consider $\pi_*(k)$ as an algebra object in $\mathsf{QCoh}(\mathbb{B}\mathcal{G})$. Note that this algebra object induces the monad above. Therefore, we have:

---

[42]The main point here is that $\mathsf{QCoh}(\mathcal{Y} \times \mathcal{G}) = \mathsf{QCoh}(\mathcal{Y}) \otimes \mathsf{QCoh}(\mathcal{G})$, which is a general consequence of the fact that $\mathsf{QCoh}(\mathcal{G})$ is dualizable (being compactly generated).

$$\mathcal{C}^{\mathcal{G},w} \underset{\mathsf{QCoh}(\mathbb{B}\mathcal{G})}{\otimes} \mathsf{Vect} = \mathcal{C}^{\mathcal{G},w} \underset{\mathsf{QCoh}(\mathbb{B}\mathcal{G})}{\otimes} \pi_*(k)\text{–mod}(\mathsf{QCoh}(\mathbb{B}\mathcal{G})) = \pi_*(k)\text{–mod}(\mathcal{C}^{\mathcal{G},w}) = \mathcal{C}$$

as desired.[43]

Now the second part follows easily from the first. Indeed, this functor admits the left adjoint $- \otimes_{\mathsf{QCoh}(\mathbb{B}\mathcal{G})} \mathsf{Vect}$, and we have just checked that the counit for the adjunction is an equivalence.
□

Remark 3.5.2. This is a different argument from the one given in [Gai5], which relied on rigidity.

3.6. **Tameness.** The functor $\mathrm{Av}_*^w : \mathcal{C} \to \mathcal{C}^{\mathcal{G},w}$ induces a *norm* functor:

$$\mathrm{Nm} : \mathcal{C}_{\mathcal{G},w} \to \mathcal{C}^{\mathcal{G},w}.$$

Definition 3.6.1. $\mathcal{C}$ is *tame* (with respect to the $\mathcal{G}$ action) if the above morphism Nm is an equivalence.

Example 3.6.2. $\mathsf{QCoh}(\mathcal{G}) \in \mathsf{QCoh}(\mathcal{G})\text{–mod}$ is tame: indeed, both invariants and coinvariants are Vect, the norm functor is obviously compatible with respect to this.

Example 3.6.3. In [Gai5] §7.3, it is shown that for $\mathcal{G} = \prod_{i=1}^{\infty} \mathbb{G}_a$, Vect is *not* tame.

Example 3.6.4. If $\mathcal{G}$ is finite type, then every $\mathcal{C}$ is tame. Indeed, this follows from [Gai5] Theorem 2.2.2 and Proposition 6.2.7.

Generalizing Example 3.6.4, we obtain the following result:

Lemma 3.6.5. *For $\mathcal{G}$ acting on $\mathcal{C}$ and $\varepsilon : \mathcal{G} \to \mathcal{G}'$ a flat morphism with $\mathcal{G}'$ finite type and $\mathcal{K} := \mathrm{Ker}(\varepsilon)$, $\mathcal{C}$ is tame with respect to $\mathcal{G}$ if and only if $\mathcal{C}$ is tame with respect to $\mathcal{K}$ (with respect to the induced action).*

3.7. We have the following result, which relates the failure of tameness to the failure of the invariants functor to be a Morita equivalence.

Proposition 3.7.1. *For every $\mathcal{C} \in \mathsf{QCoh}(\mathcal{G})\text{–mod}$, the norm functor induces an equivalence:*

$$\mathcal{C}_{\mathcal{G},w} \underset{\mathsf{QCoh}(\mathbb{B}\mathcal{G})}{\otimes} \mathsf{Vect} \xrightarrow{\simeq} \mathcal{C}^{\mathcal{G},w} \underset{\mathsf{QCoh}(\mathbb{B}\mathcal{G})}{\otimes} \mathsf{Vect}.$$

Proof. By Proposition 3.5.1, the right hand side identifies with $\mathcal{C}$. In particular, both sides commutes with colimits in $\mathcal{C}$ (since this is obvious for the left hand side), and therefore we reduce to the case $\mathsf{QCoh}(\mathcal{G})$, where the result is clear.
□

3.8. **Tameness and 1-affineness.** Suppose $\mathcal{Y}$ is a prestack with an action of $\mathcal{G}$.

Definition 3.8.1. $\mathcal{Y}$ is *tame* (with respect to the action of $\mathcal{G}$) if $\mathsf{QCoh}(\mathcal{Y}) \in \mathsf{QCoh}(\mathcal{G})\text{–mod}$ is tame.

Proposition 3.8.2. *In the above notation, suppose that $\mathcal{Y}$ is 1-affine and tame. Then $\mathcal{Y}/\mathcal{G}$ is 1-affine.*

Proof. Because $\pi : \mathcal{Y} \to \mathcal{Y}/\mathcal{G}$ is a 1-affine morphism, we obtain:

$$\mathsf{ShvCat}_{/(\mathcal{Y}/\mathcal{G})} \simeq \mathcal{G} \ltimes \mathsf{QCoh}(\mathcal{Y})\text{–mod}$$

---

[43]The second to last equality follows from [Gai1] Proposition 4.8.1.

where $\mathcal{G} \ltimes \mathsf{QCoh}(\mathcal{Y})$ is the semidirect (or crossed) product of $\mathsf{QCoh}(\mathcal{G})$ with $\mathsf{QCoh}(\mathcal{Y})$ (as a mere object of DGCat_{cont}, this category is $\mathsf{QCoh}(\mathcal{G}) \otimes \mathsf{QCoh}(\mathcal{Y})$). That is, for $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}/\mathcal{G}}$, $\Gamma(\mathcal{Y}, \pi^*(\mathsf{C}))$ is acted on by $\mathsf{QCoh}(\mathcal{Y})$ and $\mathsf{QCoh}(\mathcal{G})$ satisfying the compatibility that the semi-direct product of these two algebras acts; moreover, this gives an equivalence with categories acted on by the semi-direct product.

We should show that the functor:

$$\mathcal{G} \ltimes \mathsf{QCoh}(\mathcal{Y})\text{--mod} \to \mathsf{QCoh}(\mathcal{Y}/\mathcal{G})\text{--mod} = \mathsf{QCoh}(\mathcal{Y})^{\mathcal{G},w}\text{--mod}$$
$$\mathcal{C} \mapsto \mathcal{C}^{\mathcal{G},w} = \mathsf{Hom}_{\mathcal{G} \ltimes \mathsf{QCoh}(\mathcal{Y})}(\mathsf{QCoh}(\mathcal{Y}), \mathcal{C}) \tag{3.8.1}$$

is an equivalence. This functor admits the left adjoint:

$$\big(\mathcal{D} \in \mathsf{QCoh}(\mathcal{Y}/\mathcal{G})\text{--mod}\big) \mapsto \mathcal{D} \underset{\mathsf{QCoh}(\mathcal{Y}/\mathcal{G})}{\otimes} \mathsf{QCoh}(\mathcal{Y}).$$

First, we claim (as in Proposition 3.5.1) that (3.8.1) is fully-faithful. Indeed, first note that for $\mathcal{D} \in \mathsf{QCoh}(\mathcal{Y}/\mathcal{G})\text{--mod}$, we have:

$$\mathcal{D} \underset{\mathsf{QCoh}(\mathbb{B}\mathcal{G})}{\otimes} \mathsf{Vect} \xrightarrow{\simeq} \mathcal{D} \underset{\mathsf{QCoh}(\mathcal{Y}/\mathcal{G})}{\otimes} \mathsf{QCoh}(\mathcal{Y})$$

since it suffices to check that this map is an isomorphism for $\mathsf{QCoh}(\mathcal{Y}/\mathcal{G})$ (since both sides commute with colimits), and there it follows from Proposition 3.5.1. Then the claim of fully-faithfulness follows directly from Proposition 3.5.1.

It therefore suffices now to show that our left adjoint is conservative. This follows from the calculation:

$$\big(\mathcal{D} \underset{\mathsf{QCoh}(\mathcal{Y}/\mathcal{G})}{\otimes} \mathsf{QCoh}(\mathcal{Y})\big) \underset{\mathsf{QCoh}(\mathcal{G})}{\otimes} \mathsf{Vect} = \mathcal{D} \underset{\mathsf{QCoh}(\mathcal{Y}/\mathcal{G})}{\otimes} \mathsf{QCoh}(\mathcal{Y})_{\mathcal{G},w} =$$
$$\mathcal{D} \underset{\mathsf{QCoh}(\mathcal{Y}/\mathcal{G})}{\otimes} \mathsf{QCoh}(\mathcal{Y})^{\mathcal{G},w} = \mathcal{D} \underset{\mathsf{QCoh}(\mathcal{Y}/\mathcal{G})}{\otimes} \mathsf{QCoh}(\mathcal{Y}/\mathcal{G}) = \mathcal{D}$$

for any $\mathcal{D} \in \mathsf{QCoh}(\mathcal{Y}/\mathcal{G})\text{--mod}$.

$\square$

3.9.   Let G be an affine algebraic group for the remainder of this section. We now formulate the main result of this section.

Theorem 3.9.1. *For any $r \geqslant 0$, $t^{-r}\mathfrak{g}[[t]]dt$ is tame with respect to the $G(O)$-action on it.*

Remark 3.9.2. Since $\mathsf{QCoh}(\mathfrak{g}((t))dt)$ is a colimit/limit of the categories $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt)$, we immediately obtain that $\mathfrak{g}((t))dt$ is tame with respect to the action of $G(O)$ as well.

From Proposition 3.8.2, we obtain:

Corollary 3.9.3. *The quotient $t^{-r}\mathfrak{g}[[t]]dt/G(O)$ is 1-affine.*

The proof of Theorem 3.9.1 will occupy the remainder of this section.

3.10.   The main point in proving Theorem 3.9.1 is the following.

Proposition 3.10.1. *The global sections functor $\Gamma : \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O)) \to \mathsf{Vect}$ has finite cohomological amplitude.*

The proof of Proposition 3.10.1 will be given in §3.11-3.15.

3.11.   First, we observe that Γ commutes with colimits bounded uniformly from below.[44]

Indeed, this follows from t-exactness and comonadicity of the functor $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O)) \to \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt)$ by a well-known argument. We include this argument below for completeness.

Let π denote the structure map $t^{-r}\mathfrak{g}[[t]]dt \to t^{-r}\mathfrak{g}[[t]]dt/G(O)$. Suppose $\mathcal{F} \in \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O))^{\geqslant 0}$. By comonadicity, we have:

$$\mathcal{F} = \lim_{[m]\in\mathbf{\Delta}} (\pi_*\pi^*)^{m+1}(\mathcal{F}).$$

Since Γ commutes with limits (being a right adjoint), we have:

$$\Gamma(\mathcal{F}) = \lim_{[m]\in\mathbf{\Delta}} \Gamma(t^{-r}\mathfrak{g}[[t]]dt/G(O), (\pi_*\pi^*)^{m+1}(\mathcal{F})) = \lim_{[m]\in\mathbf{\Delta}} \Gamma(t^{-r}\mathfrak{g}[[t]]dt, \pi^*(\pi_*\pi^*)^{m}(\mathcal{F})).$$

Therefore, we have:

$$\tau^{\leqslant n}\Gamma(\mathcal{F}) = \tau^{\leqslant n}\lim_{[m]\in\mathbf{\Delta}} \Gamma(t^{-r}\mathfrak{g}[[t]]dt, \pi^*(\pi_*\pi^*)^{m}(\mathcal{F})) = \tau^{\leqslant n}\lim_{[m]\in\mathbf{\Delta}_{\leqslant n+1}} \Gamma(t^{-r}\mathfrak{g}[[t]]dt, \pi^*(\pi_*\pi^*)^{m}(\mathcal{F})).$$

where $\mathbf{\Delta}_{\leqslant n+1} \subseteq \mathbf{\Delta}$ is the subcategory consisting of totally ordered finite sets of size at most $n + 1$. Here the crucial second equality follows from the fact that $\tau^{\leqslant n}\lim_{\mathbf{\Delta}}$ computes the limit in the $(n + 1)$-category $\mathsf{Vect}^{[0,n]}$.

But now this expression visibly commutes with filtered colimits, since:

- The t-structure on Vect is compatible with filtered colimits.
- Γ on $t^{-r}\mathfrak{g}[[t]]dt$ commutes with all colimits (by affineness).
- π_* commutes with colimits, since π is affine.
- Our limit is now finite.

Finally, we obtain the claim from right completeness of the t-structure on Vect.

3.12.   Next, we reduce to showing that Γ has finite cohomological amplitude[45] when restricted to $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O))^+$, i.e., when restricted to objects bounded from below.

This follows from the following general lemma, noting that Γ commutes with all limits so satisfies hypothesis (1b).[46]

Lemma 3.12.1. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are DG categories equipped with left complete t-structures.*

Suppose $F : \mathcal{C} \to \mathcal{D}$ is a (possibly non-continuous) DG functor. Suppose that $F|_{\mathcal{C}^+}$ has finite cohomological amplitude.

(1) The following conditions are equivalent:
   (a) F has finite cohomological amplitude.
   (b) F commutes with left Postnikov towers, i.e.: for any $\mathcal{F} \in \mathcal{C}$, the morphism:

$$F(\mathcal{F}) = F(\lim_n \tau^{\geqslant -n}\mathcal{F}) \to \lim_n F(\tau^{\geqslant -n}\mathcal{F})$$

   is an isomorphism.

---

[44]A slightly different version of this argument follows by noting that the same fact occurs on $\mathbb{B}G(O)$, and reducing to that case using affineness of the morphism $t^{-r}\mathfrak{g}[[t]]dt/G(O) \to \mathbb{B}G(O)$; however the corresponding fact for $\mathbb{B}G(O)$ is proved by exactly the same argument as that presented here.

[45]To be totally clear: a DG functor $\mathcal{C} \to \mathcal{D}$ between DG categories with t-structures has *finite cohomological amplitude* if there is an integer δ for which $F(\mathcal{C}^{\leqslant 0}) \subseteq \mathcal{D}^{\leqslant \delta}$ and $F(\mathcal{C}^{\geqslant 0}) \subseteq \mathcal{D}^{\geqslant -\delta}$.

[46]Gaitsgory informs us that this lemma appeared already in a simpler form as [DG] Lemma 2.1.4.

(2) Suppose moreover that $\mathcal{C}$ and $\mathcal{D}$ are cocomplete, the t-structures on them are compatible with filtered colimits, and that $F|_{\mathcal{C}^+}$ commutes with colimits bounded uniformly from below. Then the equivalent conditions above imply that F is continuous.

Proof. To see that (1a) implies (1b):

We compute:

$$\lim_n F(\tau^{\geqslant -n}\mathcal{F}) = \lim_n \lim_m \tau^{\geqslant -m} F(\tau^{\geqslant -n}\mathcal{F}) = \lim_m \lim_n \tau^{\geqslant -m} F(\tau^{\geqslant -n}\mathcal{F}). \qquad (3.12.1)$$

For fixed m, we have:

$$\lim_n \tau^{\geqslant -m} F(\tau^{\geqslant -n}\mathcal{F}) = \tau^{\geqslant -m} F(\mathcal{F})$$

since the limit stabilizes to $F(\mathcal{F})$ for m large enough by the boundedness of the cohomological amplitude of F.

Therefore, we compute the right hand side of (3.12.1) as:

$$\lim_m \tau^{\geqslant -m} F(\mathcal{F}) = F(\mathcal{F})$$

by left completeness of the t-structure on $\mathcal{D}$.

Now suppose that (1b) holds.

We begin by reversing the above logic to show that the hypothesis (1b) implies that for every $\mathcal{F} \in \mathcal{C}$ and every m, we have:

$$\tau^{\geqslant -m} F(\mathcal{F}) \xrightarrow{\simeq} \lim_n \tau^{\geqslant -m} F(\tau^{\geqslant -n}\mathcal{F}). \qquad (3.12.2)$$

Because $F|_{\mathcal{C}^+}$ assumed to be cohomologically bounded, the limit on the right stabilizes. Let $\mathcal{G}_m \in \mathcal{D}$ denote this limit. Note that $\mathcal{G}_m \in \mathcal{D}^{\geqslant -m}$ since this subcategory is closed under limits. Moreover, $\tau^{\geqslant -m}(\mathcal{G}_{m+1}) = \mathcal{G}_m$ since each equals $\tau^{\geqslant -m} F(\tau^{\geqslant -n}\mathcal{F})$ for n large enough (i.e., because the limits defining $\mathcal{G}_m$ and $\mathcal{G}_{m+1}$ stabilize).

Since $m \mapsto \mathcal{G}_m$ is compatible under truncations, this system is equivalent to the datum of the object $\mathcal{G} = \lim_m \mathcal{G}_m \in \mathcal{D}$, by left completeness of the t-structure of $\mathcal{D}$. Therefore, it suffices to show that:

$$F(\mathcal{F}) \to \lim_m \mathcal{G}_m \xrightarrow{\simeq} \lim_m \lim_n \tau^{\geqslant -m} F(\tau^{\geqslant -n}\mathcal{F})$$

is an isomorphism. But this is clear, since the right hand side equals:

$$\lim_n \lim_m \tau^{\geqslant -m} F(\tau^{\geqslant -n}\mathcal{F}) = \lim_n F(\tau^{\geqslant -n}\mathcal{F}) \overset{(1b)}{=} F(\mathcal{F}).$$

Now, since we noted that the limit in the right hand side of (3.12.2) stabilizes, this makes it clear that F is cohomologically bounded, so we see that (1b) implies (1a).

Finally, suppose that $F|_{\mathcal{C}^+}$ commutes with colimits uniformly bounded from below and that the t-structures are compatible with filtered colimits. We will show that (1b) implies that F is continuous.

We need to show that for $i \mapsto \mathcal{F}_i$ a *filtered* diagram, we have:

$$\operatorname*{colim}_i F(\mathcal{F}_i) \xrightarrow{\simeq} F(\operatorname*{colim}_i \mathcal{F}_i).$$

It suffices to show this after applying $\tau^{\geqslant -m}$ for any integer m. Since the t-structures are compatible with filtered colimits, the left hand side of the above becomes $\operatorname{colim}_i \tau^{\geqslant -m} F(\mathcal{F}_i)$ upon truncation.

We now compute the right hand side as:

$$\tau^{\geqslant -m} F(\operatorname*{colim}_i \mathcal{F}_i) \overset{(3.12.2)}{=} \lim_n \tau^{\geqslant -m} F(\tau^{\geqslant -n} \operatorname*{colim}_i \mathcal{F}_i).$$

Using the compatibility between filtered colimits and t-structures, and the commutation of F with uniformly bounded colimits, this term now becomes:

$$\lim_n \operatorname*{colim}_i \tau^{\geqslant -m} F(\tau^{\geqslant -n} \mathcal{F}_i).$$

Observe that this limit is eventually constant, since $F|_{\mathcal{C}^+}$ has bounded amplitude. This justifies interchanging the limit and the colimit, so that we can now compute further:

$$\operatorname*{colim}_i \lim_n \tau^{\geqslant -m} F(\tau^{\geqslant -n} \mathcal{F}_i) \overset{(3.12.2)}{=} \operatorname*{colim}_i \tau^{\geqslant -m} F(\mathcal{F}_i)$$

as desired.

$\square$

As an auxiliary remark, Lemma 3.12.1 allows us to deduce the following result from Proposition 3.10.1:

Corollary 3.12.2. Γ *on* $t^{-r}\mathfrak{g}[[t]]dt/G(O)$ *commutes with colimits, i.e., the structure sheaf of* $t^{-r}\mathfrak{g}[[t]]dt/G(O)$ *is compact.*

We emphasize that Corollary 3.12.2 is a consequence of the (yet unproved) Proposition 3.10.1, so of course we will not appeal to it in the course of the argument below.

We will return to Corollary 3.12.2 in §4.

Remark 3.12.3. If Corollary 3.12.2 appears innocuous given §3.11, do note that it fails (as does Proposition 3.10.1) for $\mathbb{B}G(O)$ in place of $t^{-r}\mathfrak{g}[[t]]dt/G(O)$, unlike §3.11. The reader might also glance at [FG2] to see these homological subtleties play out in the context of Kac-Moody representations.

3.13. By Theorems 2.12.1 and 2.19.1, we can therefore can choose s so that the geometric fibers of the map:[47]

$$t^{-r}\mathfrak{g}[[t]]dt/\mathcal{K}_{r+s} \to t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt$$

are smooth Artin stacks.

Now observe that it suffices to prove Proposition 3.10.1 after replacing $G(O)$ by any congruence subgroup; we will prove the homological boundedness for global sections on $t^{-r}\mathfrak{g}[[t]]dt/\mathcal{K}_{r+s}$ instead.

3.14. Next, we reduce to showing that there is a *uniform* bound (see below) on the possible cohomological amplitude of $\Gamma(t^{-r}\mathfrak{g}[[t]]dt/\mathcal{K}_{r+s}, \mathcal{F})$ whenever $\mathcal{F}$ is set-theoretically supported[48] on the fiber of a schematic[49] point.

---

[47]Implicitly, we are assuming $r > 0$ here that the polar part map makes sense; of course this is fine for our purposes.

[48]For a (possibly non-closed) point $\eta \in S$ for S a smooth (finite type) scheme, we will say that $\mathcal{F}$ is supported on this point if it can be written as a filtered colimit of sheaves $i_{\eta,*}(\mathcal{G})$ for $\mathcal{G} \in \mathsf{QCoh}(\eta)$ (i.e., the DG category of vector spaces with coefficients in the residue field of η) for $i_\eta : \eta \hookrightarrow S$.

We warn that this is a bad definition for general S and is being introduced in an ad hoc way to make the language easier in our current setting.

[49]We use the term *schematic point* to mean the generic point of an integral subscheme, i.e., a point in the topological space underlying a scheme in the locally ringed spaces perspective on the subject.

Remark 3.14.1. Here by *uniform bound*, we mean that the bound should be independent of which fiber we choose. It is an immediate consequence of Theorem 2.12.1 that there is a bound fiber-by-fiber, but what will remain to show after this subsection is that we can bound the relevant cohomological dimensions uniformly.

For notational simplicity, let $\mathcal{P}$ denote the affine space $t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt$. The Cousin complex then provides a bounded resolution of $\mathcal{O}_\mathcal{P}$ by direct sums of sheaves of the form $\mathscr{C}_\eta[-\operatorname{ht}(\eta)]$ for $\eta \in \mathcal{P}$ a schematic point and $\mathscr{C}_\eta \in \mathsf{QCoh}(\mathcal{P})^\heartsuit$ set-theoretically supported on η.

Tensoring with the Cousin resolution then implies that any $\mathcal{F}$ bounded from below admits a finite filtration where the associated graded terms are direct sums of sheaves of the form $\mathcal{F}_\eta$ where $\mathcal{F}_\eta$ is supported on a point $\eta \in \mathcal{P}$, and where the $\mathcal{F}_\eta$ are bounded uniformly (in η) from below.

Because the filtration is finite, and because $\Gamma(t^{-r}\mathfrak{g}[[t]]dt/\mathcal{K}_{r+s}, -)$ commutes with direct sums bounded uniformly, we obtain the desired reduction.

3.15. Completion of the proof of Proposition 3.10.1.
Again using the fact that Γ commutes with colimits bounded uniformly from below, we see that it suffices to treat those sheaves $\mathcal{F}$ that are pushed forward from the fiber of $t^{-r}\mathfrak{g}[[t]]dt/\mathcal{K}_{r+s}$ at some schematic point in $t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt$.

Therefore, it suffices to show that the cohomological dimension of the global sections functor is bounded uniformly on the fibers the map $t^{-r}\mathfrak{g}[[t]]dt/\mathcal{K}_{r+s} \to t^s\mathfrak{g}[[t]]dt$.

Recall that the dimensions of these fibers are bounded *uniformly*: this is Proposition 2.38.1. Moreover, the dimensions of the automorphism group at a point is uniformly bounded by $\dim(\mathfrak{g})$, since the Lie algebra of the stabilizer subgroup at a point embeds into H^0_{dR} of the corresponding connection.

Therefore, the claim follows from the next result.[50]

Proposition 3.15.1. *Suppose that X is an affine scheme of finite type and K is a unipotent group acting on X.*[51]

Define the stack $\mathcal{X} := X/K$. Then the cohomological dimension of the global sections functor $\Gamma(\mathcal{X}, -)$ is bounded by:

$$\dim(\mathcal{X}) + \max_{x \in \mathcal{X} \ a \ geometric \ point} 2\dim(\operatorname{Aut}_\mathcal{X}(x)).$$

Proof. First, note that by replacing X by X^{red}, we can assume that X is reduced. The action of K also preserves irreducible components, so we can assume X is integral.

In the case that K acts transitively on X, the result is clear, since (perhaps after innocuously extending our base-field k) $\mathcal{X} = \mathbb{B}(\operatorname{Stab}_K(x))$ for a k-point $x \in X$, and then the cohomological dimension is bounded by the dimension of this stabilizer.

Let $U \subseteq X$ be a non-zero K-stable open subscheme. We claim that U contains a non-zero K-stable *affine* open subscheme.

Indeed, let $Z \subseteq X$ be the reduced complement to U. By unipotence of K, we can find a non-zero K-invariant function f vanishing on Z, since the ideal of functions vanishing on Z is a non-zero K-representation.

---

[50]There is a slight finiteness issue to clarify here, since our stacks occur a priori as quotients of infinite type affine schemes by prounipotent groups. However, since the quotient is an Artin stack, when we quotient by a small enough congruence subgroup, we obtain a scheme, and then by Noetherian descent [TT] Proposition C.6, we obtain that the quotient by a small enough congruence subgroup $\mathcal{K}_N$ is finite type affine. Then our stack of interest is the quotient of this finite type affine by the residual $\mathcal{K}_{r+s}/\mathcal{K}_N$-action, as desired.

[51]We really need to extend scalars here and work over base-fields other than the arbitrary (characteristic zero) ground field k. Of course, this is irrelevant, so for simplicity we just let k denote our ground field in this proposition.

Therefore, we can find an affine K-stable open $\varnothing \neq V \subseteq X$ such that the group scheme of stabilizers is smooth over V (since we can find some K-stable open for which this is true, namely, the open of *regular* points, for which stabilizer has the minimal dimension).

We can then compute $\Gamma(V/K, -)$ by pulling a quasi-coherent sheaf back to V and then taking Lie algebra cohomology with respect to the Lie algebra of the stabilizer group scheme. Since V is affine, this means that the cohomological dimension of V/K is at most the dimension of this stabilizer, which we bound as:

$$\max_{x \in V/K} \dim(\mathrm{Aut}_{\mathcal{X}}(x)) \leqslant \max_{x \in \mathcal{X}} \dim(\mathrm{Aut}_{\mathcal{X}}(x)) \leqslant \dim(\mathcal{X}) + \max_{x \in \mathcal{X}} 2 \dim(\mathrm{Aut}_{\mathcal{X}}(x)).$$

(Here we begin a convention throughout this argument that e.g. $\max_{x \in \mathcal{X}}$ assumes x a geometric point.)

Now let $Y \subseteq X$ be the reduced complement to V, and note that $\dim(Y) < \dim(X)$. By Noetherian descent, we can assume the result holds for $\mathcal{Y} := Y/K$. Then by an easy argument (c.f. [DG] Lemma 2.3.2), the cohomological dimension of X/K is then bounded by:[52]

$$\max \left\{ \dim(\mathcal{X}) + \max_{x \in \mathcal{X}} 2 \dim(\mathrm{Aut}_{\mathcal{X}}(x)), \ 1 + \dim(\mathcal{Y}) + \max_{y \in \mathcal{Y}} 2 \dim(\mathrm{Aut}_{\mathcal{Y}}(x)) \right\} \leqslant$$
$$\dim(\mathcal{X}) + \max_{x \in \mathcal{X}} \dim(2 \, \mathrm{Aut}_{\mathcal{X}}(x))$$

as desired.

$\square$

3.16. **Digression: t-structures and monadicity.** We digress for a moment to discuss the relationship between t-structures and monadicity. This is well-known, and appears in some form in both [Lur] and [Gai5], but we include it here because Corollary 3.16.2 is not quite stated in a conveniently referenced way in either source.

We will use the following general (and well-known) results to check monadicity.

Lemma 3.16.1. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are (possibly non-cocomplete) DG categories equipped with left complete t-structures. Suppose that $G : \mathcal{C} \to \mathcal{D}$ is right t-exact up to cohomological shift.

Then $\mathcal{C}$ admits geometric realizations of simplicial diagrams bounded uniformly from above, and G preserves these geometric realizations. I.e., the functor $\mathcal{C}^{\leqslant 0} \to \mathcal{D}$ commutes with geometric realizations.

Proof. We can assume G is right t-exact, and then this result is [Lur] Lemma 1.3.3.11 (2).

$\square$

Corollary 3.16.2. Suppose that $G : \mathcal{C} \to \mathcal{D}$ is a conservative DG functor between DG categories with left and right complete t-structures. Suppose moreover that G admits a left adjoint F such that:

- The functors F and G are of bounded cohomological amplitude.
- The upper amplitude of the functors $(FG)^n$ is bounded independently of n, i.e., there is an integer N such that each functor $(FG)^n$ $(n \geqslant 0)$ maps $\mathcal{C}^{\leqslant 0}$ into $\mathcal{C}^{\leqslant N}$.

Then G is monadic.

Remark 3.16.3. For example, the last two hypotheses hold if F and G are both t-exact. However, we will see below that one can sometimes arrange this situation *without* t-exactness of both functors.

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[52]For the comparison with [DG], note that $V/K \hookrightarrow X/K$ is affine, so the pushforward is t-exact.

Proof of Corollary 3.16.2. By the Barr-Beck theorem, it suffices to show that for every $\mathcal{F} \in \mathcal{C}$, the map (coming from the bar resolution):

$$\operatorname*{colim}_{[n] \in \mathbf{\Delta}^{op}} (FG)^{n+1}(\mathcal{F}) \to \mathcal{F}$$

is an isomorphism.

We have that $G|_{\mathcal{C}^{\leq 0}}$ commutes with geometric realizations by Lemma 3.16.1.

For $\mathcal{F} \in \mathcal{C}$ as above and for any $m \geq 0$, we claim that the map:

$$\operatorname*{colim}_{[n] \in \mathbf{\Delta}^{op}} (FG)^{n+1} \tau^{\leq m} \mathcal{F} \to \tau^{\leq m} \mathcal{F}$$

is an isomorphism. Indeed, by conservativeness, it suffices to see this after applying G. This geometric realization is of terms bounded uniformly from above by hypothesis on the endofunctors $(FG)^{n+1}$, so the colimit commutes with G; then since this simplicial diagram is G-split, we obtain the claim.

Because $(FG)^{n+1}$ is cohomologically bounded for each n (since F and G are), for every $\mathcal{F} \in \mathcal{C}$ we have:

$$(FG)^{n+1}(\mathcal{F}) = \operatorname*{colim}_m (FG)^{n+1} \tau^{\leq m} \mathcal{F}$$

by (the dual to) Lemma 3.12.1. Combining this with the above, we compute:

$$\operatorname*{colim}_{[n] \in \mathbf{\Delta}^{op}} (FG)^{n+1}(\mathcal{F}) = \operatorname*{colim}_{[n] \in \mathbf{\Delta}^{op}} \operatorname*{colim}_m (FG)^{n+1} \tau^{\leq m} \mathcal{F} = \operatorname*{colim}_m \operatorname*{colim}_{[n] \in \mathbf{\Delta}^{op}} (FG)^{n+1} \tau^{\leq m} \mathcal{F} =$$

$$\operatorname*{colim}_m \tau^{\leq m} \mathcal{F} = \mathcal{F}$$

as desired.

$\square$

3.17. We now prove Theorem 3.9.1.

Proof of Theorem 3.9.1. Let π denote the morphism $t^{-r}\mathfrak{g}[[t]]dt \to t^{-r}\mathfrak{g}[[t]]dt/G(O)$.

Step 1. By the Beck-Chevalley method (c.f. [Gai5] Appendix C), the *non-continuous* right adjoint to the canonical functor $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt) \to \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt)_{G(O),w}$ is monadic.

Let $\pi^?$ denote the (non-continuous) right adjoint to π_*. One easily checks (c.f. [Gai5] §6.3, especially Proposition 6.3.7) that the monads on $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt)$ corresponding to $\pi^?\pi_*$ and to the monad coming from coinvariants coincide, compatibly with the norm functor comparing invariants and coinvariants.

Therefore, it suffices to show that $\pi^?$ is monadic.

Step 2. We claim that $\pi^?$ is conservative. It is equivalent to say that π_* generates the target under colimits, which is the form in which we will check this result.

First, note that we can replace $G(O)$ by any congruence subgroup. Indeed, by [Gai5] Proposition 6.2.7 and 1-affineness of the classifying stack of a finite type algebraic group Γ, $\mathrm{Av}_*^{\Gamma,w} : \mathcal{C} \to \mathcal{C}^{\Gamma,w}$ generates the target under colimits for any $\mathsf{QCoh}(\Gamma)$-module category $\mathcal{C}$.

Therefore, by Theorems 2.12.1 and 2.19.1 it suffices to show this result for $t^{-r}\mathfrak{g}[[t]]dt/\mathcal{K}_{r+s}$ where we have a polar part map to $t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt$ with geometric fibers smooth Artin stacks.

Note that π_* is a morphism of $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt)$-module categories. Moreover, as for any smooth scheme, $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/t^s\mathfrak{g}[[t]]dt)$ is generated under colimits by skyscraper sheaves

at geometric points. Therefore, it suffices to show that the morphism π_* generates under colimits when restricted to any geometric fiber.

But on these geometric fibers, the map π base-changes to $\widetilde{\pi} : S \to S/K$ where S is an affine scheme (the geometric fiber of $t^{-r}\mathfrak{g}[[t]]dt$), K is a prounipotent group scheme and S/K is a smooth (finite-dimensional) Artin stack.

By Noetherian descent ([TT] Proposition C.6), there is a normal compact open subgroup $K_0 \subseteq K$ such that S/K_0 is an affine scheme of finite type. Then $S \to S/K_0$ is a K_0-torsor. Moreover, this torsor is necessarily trivial because K_0 is prounipotent and S/K_0 is affine. Therefore, the pushforward obviously generates under colimits in this case.

Then $S/K_0 \to S/K$ generates under colimits using the fact that K/K_0 is finite type, and using [Gai5] Proposition 6.2.7 once again.

Step 3. Next, we claim that $\pi^?$ has bounded cohomological amplitude.

Note that:

$$\underline{\mathrm{Hom}}_{\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt)}(\mathcal{O}_{t^{-r}\mathfrak{g}[[t]]dt}, \pi^?(\mathcal{F})) = \underline{\mathrm{Hom}}_{\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O))}(\pi_*(\mathcal{O}_{t^{-r}\mathfrak{g}[[t]]dt}), \mathcal{F}) \in \mathsf{Vect}$$

so it suffices to show that the latter complex is bounded.

But observe that $\pi_*(\mathcal{O}_{t^{-r}\mathfrak{g}[[t]]dt})$ is an ind-finite dimensional vector bundle under a countable direct limit: indeed, it is obtained by pullback from the regular representation in $\mathsf{QCoh}(\mathbb{B}G(O))$ (i.e., the pushforward of $k \in \mathsf{Vect}$ under $\mathrm{Spec}(k) \to \mathbb{B}G(O)$).

By Proposition 3.10.1 (and a dualizability argument), $\underline{\mathrm{Hom}}_{\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O))}$ out of any finite rank vector bundle has cohomological amplitude bounded independently of the vector bundle.

Note $\underline{\mathrm{Hom}}_{\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O))}(\pi_*(\mathcal{O}_{t^{-r}\mathfrak{g}[[t]]dt}), -)$ is a countable limit of such functors. Since the limit is countable, the "$R\lim$" aspect can increase the cohomological amplitude by at most 1, so this functor also has bounded cohomological amplitude.

Step 4. Finally, we claim that $(\pi_*\pi^?)^n$ has cohomological amplitude bounded independently of n. Note that this suffices by Corollary 3.16.2.

By the previous step, and since π_* is t-exact (by affineness), it suffices to show that $\pi^?\pi_*$ is t-exact. Observe that:

$$\pi^?\pi_* = \mathrm{act}_* \, p_2^? \tag{3.17.1}$$

where the maps here are $G(O) \times t^{-r}\mathfrak{g}[[t]]dt \overset{p_2}{\underset{\mathrm{act}}{\rightrightarrows}} t^{-r}\mathfrak{g}[[t]]dt$. Indeed, the identity (3.17.1) follows immediately from the base-change between upper-* and lower-* functors.

Since act_* is t-exact (by affineness of $G(O)$), we need to see that $p_2^?$ is t-exact. This follows by explicit calculation: it is equivalent to see that:

$$\underline{\mathrm{Hom}}_{\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt)}(p_{2,*}(\mathcal{O}_{G(O)\times t^{-r}\mathfrak{g}[[t]]dt}), -) = \underline{\mathrm{Hom}}_{\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt)}(\Gamma(G(O), \mathcal{O}_{G(O)})\otimes\mathcal{O}_{t^{-r}\mathfrak{g}[[t]]dt}, -)$$

is t-exact. Clearly $\Gamma(G(O), \mathcal{O}_{G(O)}) \otimes \mathcal{O}_{t^{-r}\mathfrak{g}[[t]]dt}$ is a free quasi-coherent sheaf on an affine scheme, giving the claim.

$\square$

4. Compact generation

4.1.   In this section, we discuss applications of the results of §3 to questions of compact generation.

This material logically digresses from the overall goal of proving our main theorem (c.f. §1.26). However, it is a simple application of the results of the previous section, so we include it here.

4.2. First, we give the following result, valid for any affine algebraic group G.

Proposition 4.2.1. *For every $r \geqslant 0$, $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O))$ is compactly generated. Moreover, an object in this category is compact if and only if it is perfect.*

In particular, $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O))$ is rigid symmetric monoidal.

Proof. By dualizability, perfect objects are compact if and only if the structure sheaf is compact, and this holds in our case by Corollary 3.12.2.

Now let ρ denote the structure map $t^{-r}\mathfrak{g}[[t]]dt/G(O) \to \mathbb{B}G(O) \to \mathbb{B}G$. We claim that the objects $\rho^*(V)$ for $V \in \mathsf{Rep}(G)$ a bounded complex of finite-dimensional representations form a set of compact generators. Obviously this would imply that $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O))$ is compactly generated by perfect objects, and therefore the two notions would coincide as desired.

We now claim that these objects generate under colimits. Indeed, as in the proof of Theorem 3.9.1, $\mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O))$ is generated under colimits and shifts by $\pi_*(\mathcal{O}_{t^{-r}\mathfrak{g}[[t]]dt/G(O)}) = \rho^*(\mathcal{O}_{G(O)})$. Remarks on the notation: as in §3, π is the projection map $t^{-r}\mathfrak{g}[[t]]dt \to t^{-r}\mathfrak{g}[[t]]dt/G(O)$, and we are letting $\mathcal{O}_{G(O)}$ denote the regular representation of $G(O)$.

Then observe that $\mathcal{O}_{G(O)}$ is the colimit of $\mathcal{O}_{G(O)/\mathcal{K}}$ for $\mathcal{K}$ a congruence subgroup. Since $\mathrm{Ker}(G(O)/\mathcal{K} \to G)$ is unipotent, we then see that $\mathcal{O}_{G(O)/\mathcal{K}}$ lies in the category generated under colimits by representations of G. $\qquad\square$

4.3. We readily deduce the following.

Corollary 4.3.1. $\mathsf{QCoh}(\mathfrak{g}((t))dt/G(O))$ *is compactly generated.*

Proof. The structure maps $t^{-r}\mathfrak{g}[[t]]dt/G(O) \to t^{-r-1}\mathfrak{g}[[t]]dt/G(O)$ are regular embeddings, and therefore the restriction for quasi-coherent sheaves admits a *left* adjoint. Therefore, we have:

$$\mathsf{QCoh}(\mathfrak{g}((t))dt/G(O)) = \lim_r \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O)) = \operatorname*{colim}_r \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt/G(O))$$

where the colimit is under the left adjoints. Since each of the structure functors in this colimit obviously preserves compacts (being left adjoints to continuous functors), this implies that the colimit is compactly generated as well (since $\mathsf{Ind} : \mathsf{DGCat} \to \mathsf{DGCat}_{cont}$ is a left adjoint, so commutes with colimits). $\qquad\square$

4.4. Finally, we deduce the following.

Theorem 4.4.1. *For G reductive, $\mathsf{QCoh}(\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}))$ is compactly generated.*

Proof. The map $p : \mathfrak{g}((t))dt/G(O) \to \mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}) = \mathfrak{g}((t))dt/G(K)$ is a $G(K)/G(O)$-fibration, i.e., a Gr_G-fibration up to sheafification. Therefore, this map is ind-proper (up to sheafification).

Identifying QCoh and IndCoh for Gr_G and in [GR3], we see that the pullback p^* admits a *left* adjoint. Clearly p^* is continuous and conservative, so this gives the result from Corollary 4.3.1. $\qquad\square$

5. Conclusion of the proof of the main theorem

5.1. In this section, we prove that for G *reductive*, $\mathsf{QCoh}(\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}))$–$\mathsf{mod}$ embeds fully-faithfully into $\mathsf{ShvCat}_{/\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})}$. The argument is fairly straightforward, given the work we have done already at this point.

5.2. **Tameness redux.** In what follows, let $\mathcal{G}_1$ be an affine group scheme, and let $\mathcal{G}_1 \subseteq \mathcal{G}_2$ with $\mathcal{G}_2$ a group indscheme with $\mathcal{G}_2/\mathcal{G}_1$ a formally smooth ind-proper $\aleph_0$-indscheme. We assume $\mathcal{G}_1$ has a prounipotent tail, i.e., there exists a prounipotent compact open subgroup of it.

Example 5.2.1. The example we will use is $G(O) \subseteq G(K)$ for G reductive. But this generality also would apply e.g. to $G(O)$ embedded into its formal completion in $G(K)$ (for any affine algebraic G).

5.3. **Self-duality of** $\mathsf{QCoh}(\mathcal{G}_2)$**.** First, we construct a self-duality for $\mathsf{QCoh}(\mathcal{G}_2)$. This construction will be of "semi-infinite" nature, so e.g. depends on the choice of embedding $\mathcal{G}_1 \subseteq \mathcal{G}_2$.

The construction is given in §5.4-5.5.

5.4. In what follows, $\mathcal{K}$ always denotes a (possibly variable) prounipotent compact open subgroup of $\mathcal{G}_1$.

First, note that:

$$\mathsf{QCoh}(\mathcal{G}_2) \xrightarrow{\simeq} \lim_{\mathcal{K}} \mathsf{QCoh}(\mathcal{G}_2/\mathcal{K}) = \operatorname*{colim}_{\mathcal{K}} \mathsf{QCoh}(\mathcal{G}_2/\mathcal{K}).$$

Here the limit is under pushforward functors, and the colimit is under pullback functors. Note that these pushforward functors are well-behaved because they are along affine morphisms. Adjointness gives rise to the equality of the limit and colimit here. Finally, the fact that this co/limit actually gives $\mathsf{QCoh}(\mathcal{G}_2)$ is a general fact[53] about inverse limits of prestacks along affine morphisms.

Since each of the categories $\mathsf{QCoh}(\mathcal{G}_2/\mathcal{K})$ are dualizable (e.g., being compactly generated), the fact that this is a co/limit means that $\mathsf{QCoh}(\mathcal{G}_2)$ is dualizable as well. In order to make it self-dual, we should make each category $\mathsf{QCoh}(\mathcal{G}_2/\mathcal{K})$ self-dual in a way that the pullback structure functors are dual to the pushforward structure functors. We do this in §5.5

5.5. We will accomplish the above task using IndCoh, which is an effective way of dealing with the indscheme issues that occur here.

Note that $\mathcal{G}_2/\mathcal{K}$ is formally smooth, so by [GR2], $\mathsf{QCoh}(\mathcal{G}_2/\mathcal{K}) \xrightarrow{\simeq} \mathsf{IndCoh}(\mathcal{G}_2/\mathcal{K})$, where the functor is tensoring with the dualizing sheaf. The categories $\mathsf{IndCoh}(\mathcal{G}_2/\mathcal{K})$ are self-dual via Serre duality, i.e., we have an equivalence:

$$\mathbb{D}_{\mathcal{K}}^{Serre} : \mathsf{IndCoh}(\mathcal{G}_2/\mathcal{K}) \xrightarrow{\simeq} \mathsf{IndCoh}(\mathcal{G}_2/\mathcal{K})^{\vee}.$$

Recall that under Serre duality, upper-! and lower-* functors are dual.

Using formal smoothness, we get an induced duality:

$$'\mathbb{D}_{\mathcal{K}}^{Serre} : \mathsf{QCoh}(\mathcal{G}_2/\mathcal{K}) \xrightarrow{\simeq} \mathsf{QCoh}(\mathcal{G}_2/\mathcal{K})^{\vee}.$$

We warn from the onset that we will need to modify this duality in what follows. Indeed, let $\alpha = \alpha_{\mathcal{K},\mathcal{K}'}$ denote the structure map $\mathcal{G}_2/\mathcal{K} \to \mathcal{G}_2/\mathcal{K}'$, and let us compute the dual $(\alpha^*)^{\vee}$ to the functor α^* (with respect to $'\mathbb{D}^{Serre}$). For $\mathcal{F} \in \mathsf{QCoh}(\mathcal{G}_2/\mathcal{K})$, we have:

$$(\alpha^*)^{\vee}(\mathcal{F}) \otimes \omega_{\mathcal{G}_2/\mathcal{K}'} \simeq \alpha_*^{\mathsf{IndCoh}}(\mathcal{F} \otimes \omega_{\mathcal{G}_2/\mathcal{K}}).$$

Indeed, this follows from the corresponding calculation for $\mathbb{D}^{Serre}$. Now note that in contrast, we have:

$$\alpha_*(\mathcal{F}) \otimes \omega_{\mathcal{G}_2/\mathcal{K}'} = \alpha_*^{\mathsf{IndCoh}}(\mathcal{F} \otimes \alpha^{*,\mathsf{IndCoh}}(\omega_{\mathcal{G}_2/\mathcal{K}'})) \tag{5.5.1}$$

---

[53]It reduces to the description of modules over a colimit of algebras as the limit of the corresponding categories of modules.

(c.f. [Gai3] §3.6), showing the discrepancy between $(\alpha^*)^\vee(\mathcal{F})$ and $\alpha_*(\mathcal{F})$.

Therefore, we use the following modified duality functors. For any pair $\mathcal{K} \subseteq \mathcal{K}' \subseteq \mathcal{G}_1$ as above, let $\det(\mathcal{K}', \mathcal{K})$ denote the cohomologically graded line bundle on $G_2/\mathcal{K}$ coming from the representation $\det(\mathrm{Lie}(\mathcal{K}')/\mathrm{Lie}(\mathcal{K}))[-\dim \mathcal{K}'/\mathcal{K}]$ of $\mathcal{K}$. Note that we have canonical isomorphisms:

$$\alpha_{\mathcal{K},\mathcal{K}'}^{\mathsf{IndCoh},*}(-) \simeq \det(\mathcal{K}', \mathcal{K}) \otimes \alpha_{\mathcal{K},\mathcal{K}'}^!(-).$$

In particular, we can rewrite the right hand side of (5.5.1) as:

$$\alpha_*^{\mathsf{IndCoh}}(\mathcal{F} \otimes \omega_{\mathcal{G}_2/\mathcal{K}} \otimes \det(\mathcal{K}', \mathcal{K})).$$

We then define:

$$\mathbb{D}_{\mathcal{K}}^{\frac{\infty}{2}}(\mathcal{F}) := {}'\mathbb{D}_{\mathcal{K}}^{Serre}(\det(G_1, \mathcal{K}) \otimes \mathcal{F}).$$

It then easily follows that with respect to *these* duality functors for $\mathsf{QCoh}(\mathcal{G}_2/\mathcal{K})$ and $\mathsf{QCoh}(\mathcal{G}_2/\mathcal{K}')$, α^* and α_* are dual.

As we vary our compact open subgroups, these constructions go through in a homotopy compatible way easily using the theory of [GR2], giving the desired claim.

Example 5.5.1. The functor dual to the $*$-restriction $\mathsf{QCoh}(\mathcal{G}_2) \to \mathsf{QCoh}(\mathcal{G}_1)$ is the *left* adjoint to this restriction. The functor dual to the pullback $\mathsf{Vect} \to \mathsf{QCoh}(\mathcal{G}_2)$ is given by $*$-pushing forward to $\mathcal{G}_2/\mathcal{G}_1$ and then taking IndCoh global sections (which is a left adjoint here, by assumption).

5.6. Note that the group structure on $\mathcal{G}_2$ canonically makes $\mathsf{QCoh}(\mathcal{G}_2)$ into a coalgebra object of DGCat_{cont}. If e.g. $\mathcal{G}_2$ acts a prestack $\mathcal{Y}$, the $\mathsf{QCoh}(\mathcal{Y})$ is a comodule category for $\mathsf{QCoh}(\mathcal{G}_2)$.[54] The restriction functor $\mathsf{QCoh}(\mathcal{G}_2) \to \mathsf{QCoh}(\mathcal{G}_1)$ is a morphism of coalgebras.

By self-duality, $\mathsf{QCoh}(\mathcal{G}_2)$ inherits a monoidal structure as well, so that $\mathsf{QCoh}(\mathcal{G}_2)$-module categories (in DGCat_{cont}) are the same as comodule categories for the above structure. Moreover, $\mathsf{QCoh}(\mathcal{G}_2)$ receives a monoidal functor from $\mathsf{QCoh}(\mathcal{G}_1)$.

Note that Vect has a tautological $\mathsf{QCoh}(\mathcal{G}_2)$-module structure corresponding to the trivial action of $\mathcal{G}_2$ on a point. As usual, this allows us to speak about invariants and coinvariants for $\mathsf{QCoh}(\mathcal{G}_2)$-module categories.

5.7. **Semi-infinite norm functor.** Suppose now that $\mathcal{C}$ is acted on by $\mathsf{QCoh}(\mathcal{G}_2)$. We will construct a norm functor:

$$\mathrm{Nm}^{\frac{\infty}{2}} : \mathcal{C}_{\mathcal{G}_2} \to \mathcal{C}^{\mathcal{G}_2}$$

that is functorial in $\mathcal{C}$ and an equivalence for $\mathsf{QCoh}(\mathcal{G}_2)$.

5.8. Here is an abstract description of $\mathrm{Nm}^{\frac{\infty}{2}}$, though we will give a more concrete description in what follows.

Regard $\mathsf{QCoh}(\mathcal{G}_2)$ as a bi-comodule category over itself. Then we have $\mathsf{Vect} \xrightarrow{\simeq} \mathsf{QCoh}(\mathcal{G}_2)^{\mathcal{G}_2,w}$ as $\mathsf{QCoh}(\mathcal{G}_2)$-comodule categories, where we are using the residual action on the right hand side. Indeed, since we have carefully used the comodule language everywhere, there is nothing non-standard in this claim, i.e., we are not using the self-duality of $\mathsf{QCoh}(\mathcal{G}_2)$ anywhere (which is non-standard in the sense that it has $\mathcal{G}_1$ built into it).

But by equating comodule structures and module structures by self-duality, the situation is more interesting. In particular, for $\mathcal{C} \in \mathsf{QCoh}(\mathcal{G}_2)$, we can tensor the above map to obtain:

---

[54]We need the formula $\mathsf{QCoh}(\mathcal{G}_2) \otimes \mathsf{QCoh}(\mathcal{Y}) \xrightarrow{\simeq} \mathsf{QCoh}(\mathcal{G}_2 \times \mathcal{Y})$ for this, but this formula holds because $\mathsf{QCoh}(\mathcal{G}_2)$ is dualizable.

$$\mathcal{C}_{\mathcal{G}_2,w} := \mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{Vect} = \mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{QCoh}(\mathcal{G}_2)^{\mathcal{G}_2,w} \to \Big(\mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{QCoh}(\mathcal{G}_2)\Big)^{\mathcal{G}_2,w} = \mathcal{C}^{\mathcal{G}_2,w}.$$

This is our norm map. It obviously satisfies the desired functoriality, and is obviously an equivalence for $\mathcal{C} = \mathsf{QCoh}(\mathcal{G}_2)$.

5.9.   We now give a slightly more concrete description of the norm functor above.

Suppose that $\mathcal{C}$ is a $\mathsf{QCoh}(\mathcal{G}_2)$-module category. The restriction functor $\mathsf{Oblv} : \mathcal{C}^{\mathcal{G}_2} \to \mathcal{C}^{\mathcal{G}_1}$ is conservative and admits a left adjoint $\mathrm{Av}_!^w$ by ind-properness of $\mathcal{G}_2/\mathcal{G}_1$. This functor is functorial in $\mathcal{C}$.

We claim that the composite functor:

$$\mathcal{C} \to \mathcal{C}_{\mathcal{G}_2,w} \xrightarrow{\mathrm{Nm}^{\frac{\infty}{2}}} \mathcal{C}^{\mathcal{G}_2,w}$$

is computed by:

$$\mathcal{C} \xrightarrow{\mathrm{Av}_*^w} \mathcal{C}^{\mathcal{G}_1,w} \xrightarrow{\mathrm{Av}_!^w} \mathcal{C}^{\mathcal{G}_2,w}.$$

Indeed, this follows from the commutative diagram:

$$\mathcal{C} = \mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{QCoh}(\mathcal{G}_2) \xrightarrow{\mathrm{id} \otimes \mathrm{Av}_*^w} \mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{QCoh}(\mathcal{G}_2)^{\mathcal{G}_1,w} \longrightarrow \Big(\mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{QCoh}(\mathcal{G}_2)\Big)^{\mathcal{G}_1,w} = \mathcal{C}^{\mathcal{G}_1,w}$$

with downward arrows $\mathrm{Av}_!^w$ to:

$$\mathcal{C}_{\mathcal{G}_2,w} = \mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{QCoh}(\mathcal{G}_2)^{\mathcal{G}_2,w} \longrightarrow \Big(\mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{QCoh}(\mathcal{G}_2)\Big)^{\mathcal{G}_2,w} = \mathcal{C}^{\mathcal{G}_2,w}$$

and the calculation that $\mathsf{QCoh}(\mathcal{G}_2) \xrightarrow{\mathrm{Av}_!^w \, \mathrm{Av}_*^w} \mathsf{QCoh}(\mathcal{G}_2)^{\mathcal{G}_2,w} = \mathsf{Vect}$ is the functor dual to the pullback (as follows from Example 5.5.1).

5.10.   We can now proceed as before with tameness.

Definition 5.10.1. $\mathcal{C}$ is *tame* with respect to $\mathcal{G}_2$ if the above norm map is an equivalence.

In fact, tameness for $\mathcal{G}_2$ is the same as tameness for $\mathcal{G}_1$:

Proposition 5.10.2. $\mathcal{C}$ *is tame with respect to* $\mathcal{G}_2$ *if it is tame as a* $\mathsf{QCoh}(\mathcal{G}_1)$-*module category.*

Example 5.10.3. If $\mathcal{G}_1$ is *normal* in $\mathcal{G}_2$, this result is immediate: then it is well-known (c.f. [Gai5] §11) that invariants and coinvariants coincide for $\mathcal{G}_2/\mathcal{G}_1$ via the $\mathrm{Av}_!^w$ functor.

Proof of Proposition 5.10.2. Suppose $\mathcal{C}$ is a general $\mathsf{QCoh}(\mathcal{G}_2)$-module category, i.e., forget about tameness for a moment.

We have a tautologically commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{C} & \longrightarrow & \mathcal{C}_{\mathcal{G}_1,w} & \longrightarrow & \mathcal{C}_{\mathcal{G}_2,w} \\
& {\scriptstyle \mathrm{Nm}_{\mathcal{G}_1}}\downarrow & & \downarrow {\scriptstyle \mathrm{Nm}^{\frac{\infty}{2}}_{\mathcal{G}_2}} \\
& \mathcal{C}^{\mathcal{G}_1,w} & \xrightarrow{\mathrm{Av}_!^w} & \mathcal{C}^{\mathcal{G}_2,w}.
\end{array}
$$

Note that the top arrow in this square is given by:

$$\mathcal{C}_{\mathcal{G}_1,w} = \mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{QCoh}(\mathcal{G}_2)_{\mathcal{G}_1,w} = \mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{QCoh}(\mathcal{G}_2/\mathcal{G}_1) \to \mathcal{C} \underset{\mathsf{QCoh}(\mathcal{G}_2)}{\otimes} \mathsf{Vect} = \mathcal{C}_{\mathcal{G}_2,w}.$$

corresponding to the IndCoh-global sections functor on $\mathcal{G}_2/\mathcal{G}_1$. In particular, we see that the functor $\mathcal{C}_{\mathcal{G}_1,w} \to \mathcal{C}_{\mathcal{G}_2,w}$ admits a continuous right adjoint, since $\mathcal{G}_2/\mathcal{G}_1$ is ind-proper. Moreover, this right adjoint is obviously monadic, since it is conservative and continuous.

This description of the right adjoint immediately gives the commutation of the diagram:

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{G}_1,w} & \longleftarrow & \mathcal{C}_{\mathcal{G}_2,w} \\ {\scriptstyle \mathrm{Nm}_{\mathcal{G}_1}} \downarrow & & \downarrow {\scriptstyle \mathrm{Nm}_{\mathcal{G}_2}^{\frac{\infty}{2}}} \\ \mathcal{C}^{\mathcal{G}_1,w} & \underset{\mathrm{Oblv}}{\longleftarrow} & \mathcal{C}^{\mathcal{G}_2,w} \end{array}$$

Now suppose that $\mathcal{C}$ is tame with respect to the $\mathcal{G}_1$-action. The we have a morphism of monads corresponding to the commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{G}_2,w} & \xrightarrow{\quad \mathrm{Nm}^{\frac{\infty}{2}} \quad} & \mathcal{C}^{\mathcal{G}_2,w} \\ & \searrow \qquad \swarrow & \\ & \mathcal{C}_{\mathcal{G}_1,w} = \mathcal{C}^{\mathcal{G}_1,w} & \end{array}$$

that is an isomorphism of endofunctors, by the explicit descriptions of these functors. Moreover, each of these diagonal functors is monadic (being continuous and conservative), so we obtain the claim.

$\square$

5.11. We now have the following application to the actions on gauge forms.

Proposition 5.11.1. For G a reductive group, $\mathsf{QCoh}(\mathfrak{g}((t))dt)$ is tame with respect to the $G(K)$-action by gauge transformations.

More generally, any $\mathcal{C} \in \mathsf{QCoh}(\mathfrak{g}((t))dt)$–mod equipped with a compatible action of $\mathsf{QCoh}(G(K))$ (i.e., with an action of the appropriate semidirect product, c.f. the proof of Proposition 3.8.2) is tame with respect to $G(K)$.

Proof. By Proposition 5.10.2, it suffices to prove these results with $G(O)$ replacing $G(K)$ everywhere.

Then note that in forming the limit $\mathsf{QCoh}(\mathfrak{g}((t))dt) = \lim \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt)$, each of the structural functors admits a left adjoint. Therefore, this limit is also a colimit, and formation of this limit commutes with all tensor products. Finally, note that each of the structural functors (whether left or right adjoint) is a morphism of $\mathsf{QCoh}(G(O))$-module categories.

Now for $\mathcal{C} \in \mathsf{QCoh}(\mathfrak{g}((t))dt)$–mod, we obtain:

$$\mathcal{C} = \lim_r \mathcal{C} \underset{\mathsf{QCoh}(\mathfrak{g}((t))dt)}{\otimes} \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt) = \mathrm{colim}_r \mathcal{C} \underset{\mathsf{QCoh}(\mathfrak{g}((t))dt)}{\otimes} \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt) \in \mathsf{QCoh}(G(O))\text{–mod}.$$

Note that formation of invariants commutes with formation of the limit, and formation of coinvariants commutes with formation of the colimit. Therefore, by functoriality and by Theorem 3.9.1, we obtain:

$$\mathcal{C}_{G(O),w} = \operatorname*{colim}_r \left(\mathcal{C} \underset{\mathsf{QCoh}(\mathcal{Z}_r)}{\otimes} \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt) \right)_{G(O),w} \xrightarrow{\simeq} \operatorname*{colim}_r \left(\mathcal{C} \underset{\mathsf{QCoh}(\mathcal{Z}_r)}{\otimes} \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt) \right)^{G(O),w} =$$

$$\lim_r \left(\mathcal{C} \underset{\mathsf{QCoh}(\mathcal{Z}_r)}{\otimes} \mathsf{QCoh}(t^{-r}\mathfrak{g}[[t]]dt) \right)^{G(O),w} = \mathcal{C}^{G(O),w}$$

as desired.

$\square$

5.12.   We can now show for G reductive that the functor:

$$\mathrm{Loc} : \mathsf{QCoh}(\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}))\text{–mod} \to \mathsf{ShvCat}_{/\operatorname{LocSys}_G(\overset{\circ}{\mathcal{D}})}$$

is fully-faithful.

First, we claim that the right adjoint functor:

$$\Gamma(\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}), -) : \mathsf{ShvCat}_{/\operatorname{LocSys}_G(\overset{\circ}{\mathcal{D}})} \to \mathsf{QCoh}(\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}}))\text{–mod}$$

commutes with all colimits and is a morphism of DGCat_{cont}-module categories. It suffices to check this after further composing with the forgetful functor to DGCat_{cont}.

Then note that this functor is tautologically computed by pulling back a sheaf of categories on $\mathrm{LocSys}_G(\overset{\circ}{\mathcal{D}})$ to $\mathfrak{g}((t))dt$, taking global sections there, and then forming weak $G(K)$-invariants for the resulting category. By Proposition 5.11.1, these invariants coincide with coinvariants, which makes the structural properties clear.

Therefore, the main theorem follows from the next claim.

Lemma 5.12.1. *For* $\mathcal{Y}$ *any prestack with:*

$$\Gamma(\mathcal{Y}, -) : \mathsf{ShvCat}_{/\mathcal{Y}} \to \mathsf{QCoh}(\mathcal{Y})\text{–mod}$$

commuting with colimits and a morphism of DGCat_{cont}-*module categories,* Loc *is fully-faithful.*

Proof. We need to see that $\mathrm{id} \xrightarrow{\simeq} \Gamma \circ \mathrm{Loc}$. Note that Loc is tautologically a morphism of $\mathsf{QCoh}(\mathcal{Y})$–mod-module categories. For Γ, this follows from our hypotheses, plus the usual calculation of a tensor product as a geometric realization.

Therefore, it suffices to see that $\mathrm{id} \to \Gamma \circ \mathrm{Loc}$ is an isomorphism when evaluated on the unit object $\mathsf{QCoh}(\mathcal{Y})$. But this is tautological for any prestack $\mathcal{Y}$.

$\square$

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