LOCALIZATION FOR AFFINE $\mathcal{W}$-ALGEBRAS

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Abstract. We prove a localization theorem for affine $\mathcal{W}$-algebras in the spirit of Beilinson–Bernstein and Kashiwara–Tanisaki. More precisely, for any non-critical regular weight λ, we identify λ-monodromic Whittaker D-modules on the enhanced affine flag variety with a full subcategory of Category $\mathcal{O}$ for the $\mathcal{W}$-algebra.

To identify the essential image of our functor, we provide a new realization of Category $\mathcal{O}$ for affine $\mathcal{W}$-algebras using Iwahori–Whittaker modules for the corresponding Kac–Moody algebra. Using these methods, we also obtain a new proof of Arakawa's character formulae for simple positive energy representations of the $\mathcal{W}$-algebra.

CONTENTS

1. Introduction

1.1. **Localization.** In their celebrated resolution [BB81] of the Kazhdan–Lusztig conjecture, Beilinson and Bernstein introduced a new method for studying irreducible modules in representation theory. Their idea was to *localize* representations, realizing modules over certain rings as suitable categories of holonomic D-modules. The resulting localization theorem provided Beilinson–Bernstein access to deep semi-simplicity results from algebraic geometry derived from Hodge theory and its extensions, including Deligne's resolution [Del80] of the Weil conjectures.

Since its emergence in [BB81], localization has played central role in many parts of representation theory, including affine algebras and modular representations of reductive groups.

1.2. **$\mathcal{W}$-algebras.** In this paper, we prove a localization theorem for affine $\mathcal{W}$-algebras. Recall that for any reductive Lie algebra $\mathfrak{g}$ equipped with a *level* (i.e., Ad-invariant symmetric bilinear form) κ, there is an associated $\mathcal{W}$-algebra denoted $\mathcal{W}_\kappa = \mathcal{W}_{\mathfrak{g},\kappa}$. These algebras have drawn interest for several reasons.

1.2.1. First, these algebras exhibit Feigin–Frenkel duality: $\mathcal{W}_{\mathfrak{g},\kappa} \simeq \mathcal{W}_{\check{\mathfrak{g}},\check{\kappa}}$. Here $\check{\mathfrak{g}}$ is the Langlands dual Lie algebra, and $\check{\kappa}$ is the dual level (say, for κ non-degenerate). This identification is expected to provide the interface for quantum local geometric Langlands; see [FF91], [BD], [Gai16], and [Ras16b].

1.2.2.   For $\mathfrak{sl}_2$, the algebra $\mathcal{W}_\kappa$ is the completed enveloping algebra of a Virasoro algebra with central charge depending on κ. The Virasoro algebra is an infinite dimensional Lie algebra that has been extensively studied due to its appearance as a symmetry in conformal field theory and string theory.

There is a category $\mathcal{O}$ of lowest energy Virasoro representations. Remarkably, fairly general objects of this category appear in physical models. For example, highest weight modules in singular blocks appear already in free field theory, while continuous families of Verma modules appear in Liouville theory. For these reasons, lowest energy representations of the Virasoro algebra have received considerable study.

1.2.3.   If $\mathfrak{g}$ has a simple factor of semisimple rank greater than one, $\mathcal{W}_\kappa$ is no longer associated with an infinite dimensional Lie algebra. However, there is a vertex algebra associated with $\mathcal{W}_\kappa$, which is conformal away from critical level. Therefore, $\mathcal{W}$-algebras provide fundamental examples of nonlinear symmetry algebras in conformal field theory.

1.2.4.   Despite considerable interest in representation theory of $\mathcal{W}$-algebras, and folklore analogies between this subject and the geometry of affine flag variety, a direct connection between the two has not previously been established. Our main result, a localization theorem for affine $\mathcal{W}$-algebras, realizes this picture in a strong sense. Even for the Virasoro algebra, no geometric description of a category of its representations was previously known.

1.3. **Statement of the main results.** To state our main results, we first introduce some notation and conventions, which are developed along with other preliminary material in greater detail in Section 2.

1.3.1.   We first introduce the relevant Lie-theoretic data. Let G be a semisimple, simply connected group[1] with Lie algebra $\mathfrak{g}$, and κ a noncritical level. Write T for the abstract Cartan, $\mathfrak{t}$ for its Lie algebra, and W_{f} and W for the finite and affine Weyl groups. Let $\lambda \in \mathfrak{t}^*$ be a weight that is regular of level κ. That is, we suppose it has trivial stabilizer under the level κ dot action of the affine Weyl group, cf. Section 2.1.5.

We fix once and for all a Borel subgroup B with unipotent radical N, so that $B/N \simeq T$. In addition, we fix a nondegenerate *Whittaker* character of conductor zero for the algebraic loop group N_F of N

$$\psi : N_F \to \mathbb{G}_a.$$

1.3.2.   The geometric side of our localization theorem is the DG category of κ-twisted, Whittaker equivariant, λ-monodromic D-modules on the enhanced affine flag variety. Below, we denote this category as

$$\mathsf{Whit}_{\kappa,\lambda-mon}(\mathrm{Fl}),$$

cf. Section 6.1.2 for our precise conventions.

1.3.3.   The algebraic side of the localization theorem is the DG category of modules for $\mathcal{W}_\kappa$-algebra, cf. Section 2.3.3 for more detail. Below, we denote this category as

$$\mathcal{W}_\kappa-\mathsf{mod}.$$

Recall that the Verma modules M_χ for the $\mathcal{W}_\kappa$-algebra are parametrized by the central characters χ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. We identify the set of central characters with the set of closed points of $W_{\mathrm{f}} \backslash \mathfrak{t}^*$ via the Harish–Chandra homomorphism

$$\pi : \mathfrak{t}^* \to W_{\mathrm{f}} \backslash \mathfrak{t}^*,$$

cf. Lemma 5.1.7 for our precise normalizations.

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[1]The results we prove straightforwardly extend, *mutatis mutandis*, to the case of any reductive group.

1.3.4.   With these notations in hand, we may state our main theorem.

Theorem 1.3.5. *The functor of global sections on the affine flag variety yields a fully faithful embedding*

$$(1.1) \qquad\qquad \Gamma(\mathrm{Fl}, -) : \mathsf{Whit}_{\kappa, \lambda-mon}(\mathrm{Fl}) \to \mathcal{W}_\kappa\text{--mod}.$$

Moreover, its essential image is the full subcategory of $\mathcal{W}_\kappa$--mod generated under colimits and shifts by the Verma modules

$$M_\chi, \qquad for \quad \chi \in \pi(W \cdot \lambda).$$

That is, we obtain the subcategory generated by Verma modules with highest weights in the image of the affine Weyl group dot orbit of λ under the Harish–Chandra homomorphism.

1.3.6.   As is typical in localization theory, the situation is especially nice when λ is moreover antidominant of level κ. We refer to Section 2.1.5 for the standard definition of this notion, but emphasize that antidominant integral weights exist only for negative κ. We also recall from [Ras16b] that there is a canonical construction of t-structures on DG categories of Whittaker equivariant D-modules. The construction is of semi-infinite nature, cf. Section 1.5.1.

Theorem 1.3.7. *If λ is regular and antidominant of level κ, then the functor $\Gamma(\mathrm{Fl}, -)$ from Theorem 1.3.5 is t-exact. Moreover, this functor preserves standard, costandard, and simple objects.*

In particular, passing to hearts of the t-structures, we obtain a fully faithful embedding:

$$(1.2) \qquad\qquad \mathsf{Whit}_{\kappa, \lambda-mon}(\mathrm{Fl})^\heartsuit \to \mathcal{O}$$

with essential image an explicit direct sum of blocks, cf. Theorem 6.3.1. Here $\mathcal{O}$ is a suitable version of monodromic Category $\mathcal{O}$ for $\mathcal{W}_\kappa$, whose block decomposition was determined in [Dhi]. Indeed, a principal motivation for *loc. cit.* was to understand the essential image of our localization theorem.

Remark 1.3.8. As we will see in the course of the proof, both abelian categories in (1.2) are compactly generated by their subcategories of finite length objects. Moreover, the DG category on the left-hand side of (1.1) and its essential image in the right-hand side can be canonically reconstructed from these abelian categories. Therefore, in the antidominant case, our results amount to an equivalence of Artinian abelian categories.

Remark 1.3.9. The essential image under (1.2) of the abelian category of λ-equivariant (not merely monodromic) D-modules provides highest weight categories of $\mathcal{W}$-algebra representations at negative level. As far as we are aware, such categories have not previously been constructed.

As a consequence, we obtain the following calculation of characters of simple modules for $\mathcal{W}_\kappa$.

Corollary 1.3.10. *Let λ be a regular antidominant weight of level κ. Then characters of simple modules in the block of $\mathcal{O}$ containing M_λ are calculated via parabolic Kazhdan–Lusztig polynomials for*

$$W_{\lambda,\mathrm{f}} \backslash W_\lambda$$

from the characters of Verma modules; here W_λ denotes the integral Weyl group of λ and $W_{\lambda,\mathrm{f}}$ its intersection with the finite Weyl group.

Corollary 1.3.10 and its extension to arbitrary blocks were originally proved by Arakawa in the seminal paper [Ara07]. In the antidominant case considered here, Corollary 1.3.10 provides a conceptual proof. Following the original methods [KL80] of Kazhdan–Lusztig, various Kazhdan–Lusztig polynomials are well-known to calculate multiplicities in geometric settings. For example, in our setting, it is essentially standard that the Kazhdan–Lusztig polynomials considered here calculate multiplicities in $\mathsf{Whit}_{\kappa, \lambda-mon}(\mathrm{Fl})^\heartsuit$. Therefore, we obtain Corollary 1.3.10 as a direct consequence of Theorem 1.3.7.

In the work of Arakawa, a Drinfeld-Sokolov reduction functor from $\widehat{\mathfrak{g}}_\kappa$--mod to $\mathcal{W}_\kappa$--mod was shown to send Vermas to Vermas and simples to simples or zero, extending a conjecture of Frenkel–Kac–Wakimoto [FKW92]. His results reduced the calculation of simple characters for $\mathcal{W}_\kappa$ to those for $\widehat{\mathfrak{g}}_\kappa$. These had been computed by Kashiwara–Tanisaki using localization onto the affine flag variety, as in the resolution of the original Kazhdan–Lusztig conjecture [KT95],[KT98],[KT00]. Our work provides a direct geometric

representation theoretic explanation for the relation between parabolic Kazhdan-Lusztig polynomials and $\mathcal{W}$-characters.

1.4. **Relation to other work.** As far as we are aware, Theorems 1.3.5 and 1.3.7 were not explicitly formulated as conjectures in the literature. However, some related results may be found.

1.4.1. For finite $\mathcal{W}$-algebras, parallel results were proved by Ginzburg, Losev, and Webster [Gin09], [Los12], [Web11]. The expectation that something similar should hold in affine type was mentioned in work of Backelin–Kremnizer [BK15].

1.4.2. For finite $\mathcal{W}$-algebras associated to a general nilpotent element f of $\mathfrak{g}$, Dodd–Kremnizer realized representations with fixed central character in terms of asymptotic differential operators on the standard resolution of the intersection of the nilpotent cone with the Slodowy slice defined by f [DK09]. Arakawa–Kuwabara–Malikov then gave a chiralization of their construction to realize affine $\mathcal{W}$-algebras at critical level with fixed central character [AKM15].

For a principal nilpotent f, as in our results, the intersection of the Slodowy slice and the nilpotent cone is a point. In addition, we work at non-critical levels. Therefore, our localization results are disjoint from those of [AKM15].

1.4.3. For $G = SL_n$, Fredrickson–Neitzke observed a bijection between cells in a moduli spaces of wild Higgs bundles in genus zero and minimal models for the affine $\mathcal{W}$-algebra matching certain parameters [FN17]. Our results do not speak to the phenomena they found.

1.5. **Outline of the arguments.** Below, we first highlight the key technical aspect of the current work. We then discuss the main features of the proofs of Theorems 1.3.5 and 1.3.7.

1.5.1. *Semi-infinite categorical methods.* The primary technical issue in proving Theorem 1.3.5 is its semi-infinite nature. As in [Ras19], semi-infinite phenomena present themselves when pairing abstract categorical methods and infinite-dimensional Lie groups. Below, we highlight some specific phenomena that appear in our present setting.

Our overall strategy is to prove Theorem 1.3.5 by passing to Whittaker equivariant objects in Kashiwara–Tanisaki's localization theorem, which relates D-modules on the enhanced affine flag variety and Kac–Moody representations. However, there are no Whittaker equivariant objects in the abelian category of twisted D-modules on Fl; this follows because N_F-orbits on Fl are infinite dimensional. Relatedly, there are no Whittaker equivariant objects in the abelian category of Kac–Moody representations.

However, for the corresponding *DG categories* of D-modules and Kac–Moody representations, there are robust categories of Whittaker equivariant objects. By the previous paragraph, such objects are necessarily concentrated in cohomological degree $-\infty$, i.e., these objects are in degrees $\leqslant -n$ for all integers n.[2] Thus, when we form global sections of a Whittaker D-module in Theorem 1.3.5, the underlying complex of vector spaces vanishes, but the corresponding object of $\mathcal{W}_\kappa$–mod does not.

The connection between Whittaker equivariant objects in this setting and representations of $\mathcal{W}$-algebras was established by the second author in [Ras16b]. In particular, one consequence of the main construction of [Ras16b], which is crucial for the present work, is a systematic way to realize *abelian* categories of Whittaker equivariant objects, even though such objects lie in cohomological degree $-\infty$ when forgetting the equivariance.

The importance of considering categories of representations with objects in degree $-\infty$ in infinite dimensional algebras was first highlighted by Frenkel–Gaitsgory [FG09b]. We follow them in sometimes referring to such methods as *renormalization*, and the resulting DG categories as *renormalized*.

Therefore, to handle the described issues of semi-infinite nature, and in particular to pass to Whittaker equivariant objects, we use a version of Kashiwara–Tanisaki localization which relates the renormalized categories of twisted D-modules and Kac–Moody representations and includes equivariance for categorical actions of loop groups. Such an enhancement is provided by the first author and J. Campbell in [CD20].

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[2]In particular, as we explain in more detail below, these DG categories are *not* the unbounded derived categories of the corresponding abelian categories.

Remark 1.5.2. The dream of localization theorems in semi-infinite contexts is an old one, going back to conjectures of Lusztig and Feigin–Frenkel on critical level representation theory and localization on the semi-infinite flag manifold [Lus91], [FF90]. As far as we are aware, our work is a first instance of a semi-infinite localization theorem.

1.5.3. In the remainder of the introduction, we discuss some novel ingredients used to determine its essential image. On both sides of Theorem 1.3.5, the adolescent Whittaker filtration plays an essential role. This is a functorial filtration on Whittaker invariants of categorical loop group representations that was introduced in [Ras16b].

1.5.4. On the geometric side, we show in Section 3 that the first term of the adolescent Whittaker filtration suffices to describe Whittaker D-modules on Fl, i.e., we obtain a canonical equivalence between the Iwahori–Whittaker and Whittaker categories of D-modules on Fl.

In the untwisted setting, this result was shown in [Ras16a]. However, here we deduce this result from a more general assertion. In effect,[3] we show that Whittaker invariants for a categorical representation generated by objects equivariant for the nth congruence subgroup of $\mathfrak{L}G$ are exhausted by the nth step in its adolescent Whittaker filtration, in line with the philosophy of [Ras16b] Remark 1.22.2.

1.5.5. On the representation theoretic side, the adolescent Whittaker construction realizes subcategories of $\mathcal{W}_\kappa$–mod$^\heartsuit$ as certain categories of Harish-Chandra modules for $\widehat{\mathfrak{g}}_\kappa$.

We show that Category $\mathcal{O}$ for $\mathcal{W}_\kappa$ may be realized in the first step of the adolescent Whittaker filtration, i.e., as Iwahori–Whittaker modules for the Kac–Moody algebra. In Definition A.2.1, we provide an explicit description of the corresponding subcategory of Iwahori–Whittaker Kac–Moody representations; this subcategory is mapped isomorphically onto Category $\mathcal{O}$ for $\mathcal{W}_\kappa$ via Drinfeld–Sokolov reduction.

To our knowledge, the existence of such an Iwahori–Whittaker model for Category $\mathcal{O}$ was not anticipated prior to [Ras16b]. As an application of this perspective on Category $\mathcal{O}$, we give a short new proof of the character formula for simple positive energy representations of $\mathcal{W}_\kappa$, originally obtained by Arakawa in the seminal paper [Ara07].

1.5.6. In order to obtain the above results, we give a new description of the adolescent Whittaker filtration of $\mathcal{W}_\kappa$–mod. Unlike the descriptions in [Ras16b], which involved Kac–Moody algebras, our description is intrinsic to the $\mathcal{W}$-algebra, involving local nilpotency of certain explicit elements.

One important consequence of our description is that the abelian subcategories of $\mathcal{W}_\kappa$–mod$^\heartsuit$ provided by the adolescent Whittaker construction are closed under subobjects; this was not clear to the second author when writing [Ras16b].

1.5.7. Having completed our analysis of the relevant steps in the adolescent Whittaker filtration, the determination of the essential image reduces to a problem on baby Whittaker categories. The latter question, which is now of finite-dimensional (i.e., no longer semi-infinite) nature, follows the pattern of standard arguments in geometric representation theory.

1.6. **Organization of the paper.** In Section 2, we collect notation and recall preliminary material. In Section 3, we prove a general result relating depth and adolescent Whittaker filtrations, and specialize it to the affine flag variety. In Section 4, we give a new interpretation of the adolescent Whittaker filtration on $\mathcal{W}_\kappa$–mod, intrisic to the action of $\mathcal{W}_\kappa$ on representations. Building on this, in Section 5 we give an explicit realization of Category $\mathcal{O}$ for the $\mathcal{W}_\kappa$-algebra via the baby Whittaker category, and use it to give a new proof of the formulae for simple characters. In Section 6, we obtain the localization theorem and apply the previous material to develop its basic properties. Finally, we include an appendix describing the analog of q-characters for our baby Whittaker model of Category $\mathcal{O}$. This is used to establish some technical assertions invoked in Sections 5 and 6.

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[3]We do not formulate the result in these terms. The assertion stated here follows from our Theorem 3.0.1 using the methods of [BZGO18].

2. Preliminary material

Below, we collect notation and constructions from Lie theory and infinite dimensional geometry. The reader may wish to skip to the next section, and refer back only as needed.

2.1. **Lie theoretic notation.**

2.1.1.   Fix a simply-connected semisimple algebraic group G with Cartan and Borel subgroups $T \subset B$, and let N denote the radical of B. Let B^- be the corresponding opposed Borel. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}$ and $\mathfrak{t}$ denote the corresponding Lie algebras.

2.1.2. *The finite root system.* We write Λ for the weight lattice, i.e., the characters of T, and $\check{\Lambda}$ for the coweight lattice, i.e., the cocharacters of T. Within $\check{\Lambda}$, we denote the coroots by $\check{\Phi}_{\mathrm{f}}$, and within them the simple coroots by

$$\check{\alpha}_i, \quad \text{for } i \in \mathfrak{I}.$$

We will write W_{f} for the finite Weyl group. In addition to its linear action on the dual Cartan $\mathfrak{t}^\vee$, we will use its dot action. I.e., if we write ρ for the half sum of positive roots in $\mathfrak{t}^*$, then the dot action of an element w in W_{f} on $\lambda \in \mathfrak{t}^*$ is defined by

$$w \cdot \lambda := w(\lambda + \rho) - \rho,$$

where the right-hand action is the linear one.

2.1.3. *Levels.* Recall that κ denotes a *level*, i.e., an Ad-invariant bilinear form on $\mathfrak{g}$. There are two particular levels that play distinguished roles. We will denote by κ_c the *critical* level, i.e., minus one half times the Killing form. We denote by κ_b the *basic* level. If $\mathfrak{g}$ is simple, κ_b is the unique level for which the short coroots have squared length two. I.e., for a short coroot $\check{\alpha}$, one has

$$\kappa_b(\check{\alpha}, \check{\alpha}) = 2.$$

For semisimple $\mathfrak{g}$, κ_b is the unique level restricting to the basic form on each simple factor. For G simple, we recall that κ_c is necessarily a scalar multiple of κ_b; by definition, that scalar is minus the dual Coxeter number of G.

If $\mathfrak{g}$ is a simple Lie algebra, we call a level κ noncritical if it does not equal κ_c. We call a level κ positive if it lies in

$$\kappa_c + \mathbb{Q}^{\geqslant 0} \kappa_b,$$

and κ negative if it is not positive. If $\mathfrak{g}$ is a semisimple Lie algebra, write it as a sum of simple Lie algebras

$$\mathfrak{g} \simeq \bigoplus_{j \in \mathfrak{J}} \mathfrak{g}_j$$

We say a level κ for $\mathfrak{g}$ is noncritical if its restriction to each $\mathfrak{g}_j$ is noncritical. Similarly, we say κ is positive if its restriction to each $\mathfrak{g}_j$ is positive, and κ is negative if its restriction to each $\mathfrak{g}_j$ is negative.

2.1.4. *The affine root system.* Write $F := \mathbb{C}((t))$ denote the field of Laurent series. Write G_F for the algebraic loop group of G, and $\mathfrak{g}_F$ for its Lie algebra. Here and below, we will mean topological Lie algebras, so in this case

$$\mathfrak{g}_F := \mathfrak{g} \underset{\mathbb{C}}{\otimes} F.$$

Associated to our level κ is the affine Lie algebra, given as a central extension

$$0 \to \bigoplus_{j \in \mathfrak{J}} \mathbb{C}\mathbf{c}_j \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}_F \to 0.$$

Explicitly, for elements X and Y of $\mathfrak{g}_i$ and $\mathfrak{g}_j$, for i and j in $\mathfrak{J}$, and Laurent series f and g in $\mathbb{C}((t))$, the bracket is given by

$$[X \otimes f, Y \otimes g] \ = \ [X, Y] \otimes fg \ + \ \delta_{i,j} \cdot \kappa(X, Y) \cdot \operatorname{Res} df g \cdot \mathbf{c}_i,$$

where Res denotes the residue, and $\delta_{i,j}$ the Kronecker delta function.

Consider the affine Cartan

$$\mathfrak{t} \oplus \bigoplus_{j \in \mathcal{J}} \mathbb{C}\mathbf{c}_j.$$

We may write its linear dual as

(2.1)
$$\mathfrak{t}^* \oplus \bigoplus_{j \in \mathcal{J}} \mathbb{C}\mathbf{c}_j^*,$$

where $\mathbf{c}_j^*$ pairs with $\mathfrak{t}$ and the $\mathbf{c}_i$, for $i \neq j$, by zero and pairs with $\mathbf{c}_j$ by one. Let us denote the real affine coroots by $\check{\Phi}$, and the simple affine coroots by

$$\check{\alpha}_i, \quad \text{for } i \in \hat{\mathcal{J}}.$$

Explicitly, $\widehat{\mathcal{J}} := \mathcal{I} \sqcup \mathcal{J}$. To write the corresponding elements of $\mathfrak{t} \oplus \mathbb{C}\mathbf{c}$, if we denote by $\check{\theta}_j$ the short dominant coroot of $\mathfrak{g}_j$, by κ_j the restriction of κ to $\mathfrak{g}_j$, and $\kappa_{b,j}$ the basic level for $\mathfrak{g}_j$, then they are given by

$$\check{\alpha}_i, \quad \text{for } i \in \mathcal{I}, \qquad \text{and} \qquad \check{\alpha}_j := -\check{\theta}_j + \frac{\kappa_j}{\kappa_{b,j}} \mathbf{c}_j, \quad \text{for } j \in \mathcal{J}.$$

We will write W for the affine Weyl group. In addition to its linear action on the dual affine Cartan, we will use its dot action. I.e., if we write $\check{\rho}$ for the unique element of (2.1) satisfying

$$\langle \check{\rho}, \check{\alpha}_i \rangle = 1, \quad \text{for } i \in \widehat{\mathcal{J}},$$

then the dot action of w in W on an element λ of (2.1) is given by

$$w \cdot \lambda = w(\lambda + \hat{\rho}) - \hat{\rho}.$$

We may identify $\mathfrak{t}^*$ with the affine subspace

$$\mathfrak{t}^* + \sum_{j \in \mathcal{J}} \mathbf{c}_j^*.$$

This affine subspace is preserved by both the linear and dot actions of W, and we will only need these induced affine linear actions on $\mathfrak{t}^*$ in what follows.

2.1.5. *Weights and integral Weyl groups.* Recall that a weight λ of $\mathfrak{t}^*$ is *antidominant* if

$$\langle \lambda + \check{\rho}, \check{\alpha}_i \rangle \notin \mathbb{Z}^{>0}, \quad \text{for } i \in \widehat{\mathcal{J}},$$

and that λ is *dominant* if

$$\langle \lambda + \check{\rho}, \check{\alpha}_i \rangle \notin \mathbb{Z}^{<0}, \quad \text{for } i \in \widehat{\mathcal{J}}.$$

If κ is positive, then the dot orbit of λ always contains a dominant weight, and if κ is negative, then the dot orbit of λ always contains an antidominant weight.

Associated to any weight λ of $\mathfrak{t}^*$ is the subset of integral real affine coroots

$$\check{\Phi}_\lambda := \{\check{\alpha} \in \check{\Phi} : \langle \check{\alpha}, \lambda \rangle \in \mathbb{Z}\}.$$

The integral Weyl group W_λ is the subgroup of W generated by the reflections indexed by the integral coroots.

We recall that a weight λ is said to be regular if its stabilizer in W under the dot action is trivial. If κ is not critical, this coincides with its stabilizer within W_λ under the dot action.

2.2. The $\mathcal{W}_\kappa$-algebra.

2.2.1. For $\widehat{\mathfrak{g}}_\kappa$ the affine algebra as above, we let $\mathcal{W}_\kappa$ denote the $\mathcal{W}$-algebra associated to $\widehat{\mathfrak{g}}_\kappa$ and a principal nilpotent element f of $\mathfrak{g}$. I.e., $\mathcal{W}_\kappa$ is the vertex algebra obtained as the quantum Drinfeld-Sokolov reduction of the vacuum representation of $\widehat{\mathfrak{g}}_\kappa$, cf. [Ara17] and Chapter 15 of [FBZ04]. For κ noncritical, $\mathcal{W}_\kappa$ contains a canonical conformal vector. Throughout this paper, unless otherwise specified, we assume κ is noncritical.

2.2.2.   We recall that the Zhu algebra of $\mathcal{W}_\kappa$ identifies with the center of the universal enveloping algebra of $\mathfrak{g}$. We refer to Lemma 5.1.7 for our precise normalization of this isomorphism. In particular, for a central character χ, we denote by M_χ the corresponding Verma module for $\mathcal{W}_\kappa$, and by L_χ its unique simple quotient.

2.3. Categories of representations.

2.3.1.   For a DG category $\mathcal{C}$ equipped with a t-structure, we write $\mathcal{C}^\heartsuit$ for the abelian category of objects lying in its heart, and $\mathcal{C}^+$ for its full subcategory consisting of bounded below, i.e., eventually coconnective, objects.

2.3.2.   We denote the abelian category of smooth modules for $\widehat{\mathfrak{g}}_\kappa$ on which each $\mathbf{c}_j$, for $j \in \mathcal{J}$, acts via the identity by

$$(2.2) \qquad\qquad\qquad\qquad \widehat{\mathfrak{g}}_\kappa\text{–mod}^\heartsuit.$$

We denote by $\widehat{\mathfrak{g}}_\kappa$–mod the DG category of smooth modules for $\widehat{\mathfrak{g}}_\kappa$, which is a renormalization of the un-bounded DG derived category of $\widehat{\mathfrak{g}}_\kappa$–mod$^\heartsuit$ introduced by Frenkel–Gaitsgory [FG09a]. Explicitly, within the bounded derived category of (2.2), consider the subcategory generated under cones and shifts from modules induced from trivial representations of compact open subalgebras of $\widehat{\mathfrak{g}}_\kappa$; by definition, $\widehat{\mathfrak{g}}_\kappa$–mod is the ind-completion of this category. The category $\widehat{\mathfrak{g}}_\kappa$–mod carries a t-structure with heart (2.2); its bounded below part identifies canonically with the bounded below derived category of (2.2). Further details may be found in Sections 22 and 23 of [FG09a] as well as the very readable Section 2 of [Gai15].

2.3.3.   We denote the abelian category of modules, in the sense of vertex algebras, over $\mathcal{W}_\kappa$ by

$$(2.3) \qquad\qquad\qquad\qquad \mathcal{W}_\kappa\text{–mod}^\heartsuit.$$

We denote by $\mathcal{W}_\kappa$–mod the DG category of modules for $\mathcal{W}_\kappa$. This is a renormalization of the unbounded DG derived category of $\mathcal{W}_\kappa$–mod$^\heartsuit$ introduced in [Ras16b]. As in the Kac–Moody case, it may be constructed as the ind-completion of the subcategory generated by an explicit collection of modules within the bounded derived category of (2.3), cf. Section 4 of *loc. cit.* and Section 4 of the present paper.

2.4. Conventions on DG categories.

2.4.1.   We denote by DGCat the $(\infty, 1)$-category of DG categories and DG functors between them. We let $\mathsf{DGCat}_{cont} \subset \mathsf{DGCat}$ denote the 1-full subcategory of of cocomplete DG categories and continuous DG functors between them. We refer to [GR17] for background material on the ∞-categorical perspective on DG categories.

2.5. D-modules.

2.5.1.   For a general treatment of D-modules on infinite dimensional varieties, we refer the reader to [Ras15] and [Ber17]. However, with the exception of (κ-twisted) D-modules on the loop group, we will only deal with the DG category of D-modules on ind-schemes X of ind-finite type.

Let us summarize the relevant aspects of the theory in this simpler case. For an indscheme X written[4] as an ascending union

$$Z_0 \to Z_1 \to Z_2 \to \cdots$$

of schemes of finite type along closed embeddings, the category $D(X)$ of D-modules on X is the colimit in DGCat_{cont} of the corresponding categories for the Z_i along $*$-pushforwards, i.e.,

$$(2.4) \qquad\qquad D(X) \simeq \operatorname{colim} D(Z_0) \to D(Z_1) \to D(Z_2) \to \cdots.$$

In particular, by the t-exactness of these pushforwards, $D(X)$ has a natural t-structure. Moreover, its bounded below part $D(X)^+$ identifies with the bounded below derived category of its heart $D(X)^\heartsuit$, cf. Lemma 5.4.3 of [Ras16b]. Finally, we should note that $D(X)$ is compactly generated, with

$$D(X)^c \simeq \operatorname{colim} D(Z_0)^c \to D(Z_1)^c \to D(Z_2)^c \to \cdots,$$

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[4]Of course, for cardinality reasons, not every ind-scheme can be written as such a union. However, this is satisfied for our examples of interest, so for concreteness, we assume it.

where the superscript c denotes compact objects and the appearing colimit is taken in DGCat. Plainly, a compact object of $D(X)$ is simply a bounded complex of D-modules with coherent cohomology $*$-extended from some Z_n, and Hom between two such objects may be computed in any Z_n containing both their supports.

2.5.2. We recall that κ defines categories of twisted D-modules the loop group G_F and its descendants. More precisely, there is a monoidal DG category $D_\kappa(G_F)$ defined in [Ras16b] Section 1.30 and [Ras19]; in the latter source, this category is denoted $D_\kappa^*(G(K))$. This twisting is canonically trivialized on the compact open subgroup G_O of regular maps from the disc to G, and below we use the restriction of this trivialization to $I \subset G_O$. Similarly, the twisting is canonically trivialized on the group ind-scheme $N_F \subset G_F$ of loops into N.

2.6. Group actions on categories.

2.6.1. Throughout this paper, we make essential use of techniques from categorical representation theory. We review below some basic features we employ; the reader may wish to consult the user-friendly [Ber17] or the foundational paper [Ras19] as further references. We also refer to [ABC+18] for some useful lecture notes on the subject.

2.6.2. Given a group ind-scheme H, by functoriality the multiplication on H endows the category $D(H)$ with a monoidal structure given by convolution. We denote the associated $(\infty-)$category of DG categories equipped with an action of $D(H)$ by

$$D(H)\text{–}\mathsf{mod} := D(H)\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont}).$$

2.6.3. Given a $D(H)$-module $\mathcal{C}$, one may form its categories of invariants and coinvariants, respectively given by

$$\mathcal{C}^H := \mathsf{Hom}_{D(H)\text{–}\mathsf{mod}}(\mathsf{Vect}, \mathcal{C}) \quad \text{and} \quad \mathcal{C}_H := \mathsf{Vect} \underset{D(H)}{\otimes} \mathcal{C}.$$

Example 2.6.4. Suppose for simplicity that H is of finite type, and acts on a scheme X. This endows $D(X)$ with an action of $D(H)$ by convolution. Using the bar resolution, one may identify $D(X)^H$ with the category of H-equivariant D-modules on X, i.e. the category of D-modules on the stack X/H. For a more general discussion including infinite type see [Ras15], particularly Section 6.7.

There are tautological 'forgetting and averaging' adjunctions

$$\mathrm{Oblv} : \mathcal{C}^H \rightleftarrows \mathcal{C} : \mathrm{Av}_{H,*}$$

$$\mathrm{ins}^L : \mathcal{C}_H \rightleftarrows \mathcal{C} : \mathrm{ins}.$$

2.6.5. If H is a group scheme whose pro-unipotent radical is of finite codimension, then there is a canonical identification of invariants and coinvariants. Namely, the map $\mathcal{C}_H \to \mathcal{C}^H$ induced by $\mathrm{Av}_{H,*}$ is an equivalence. We will use these equivalences implicitly throughout.

2.6.6. We will also need the twists of the above by a character. The data of a multiplicative D-module χ on H,[5] is equivalent to an action of $D(H)$ on Vect which we denote by Vect_χ. One has associated categories of twisted invariants and coinvariants

$$\mathcal{C}^{H,\chi} := \mathsf{Hom}_{D(H)\text{–}\mathsf{mod}}(\mathsf{Vect}_\chi, \mathcal{C}) \quad \text{and} \quad \mathcal{C}_{H,\chi} := \mathsf{Vect}_\chi \underset{D(H)}{\otimes} \mathcal{C}.$$

These fit into adjunctions as above, and under the same hypotheses on H one may canonically identify twisted invariants and coinvariants.

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[5]Strictly speaking, in this generality this should be regarded as an object of the *dual* category $D^!(H)$.

2.6.7.   We will need two examples of nontrivial multiplicative D-modules. First, given an element

$$\lambda \in \mathrm{Hom}(H, \mathbb{G}_m) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$$

one has an associated multiplicative D-module "t^λ" on H. We denote the corresponding twisted invariants or coinvariants by a superscript or subscript (H, λ), respectively. In what follows, this will principally be applied to Iwahori subgroups of G_F.

Second, given an element $\psi \in \mathrm{Hom}(H, \mathbb{G}_a)$ one has an associated multiplicative D-module "e^ψ" on H. We denote the corresponding twisted invariants and coinvariants by a superscript or subscript of (H, ψ), respectively. This will principally be applied to N_F and related prounipotent subgroups of G_F.

2.7. Whittaker models and the adolescent Whittaker filtration.

2.7.1.   Applying the general constructions of the previous section, to a $D_\kappa(G_F)$-module $\mathcal{C}$ one may attach its Whittaker invariants and coinvariants

$$\mathcal{C}^{N_F, \psi} \quad \text{and} \quad \mathcal{C}_{N_F, \psi}.$$

A principal result of [Ras16b] is that these may be canonically identified. This is non-trivial because N_F is not a group scheme, but rather a group ind-scheme. We employ this identification throughout. Unless an argument is biased toward one of these perspectives, we refer to them both as the *Whittaker model* of $\mathcal{C}$, and we denote this category by $\mathsf{Whit}(\mathcal{C})$.

2.7.2.   Refining the equivalence between invariants and coinvariants above, [Ras16b] constructed a canonical filtration of any Whittaker model by full subcategories

$$\mathsf{Whit}^{\leqslant 1}(\mathcal{C}) \subset \mathsf{Whit}^{\leqslant 2}(\mathcal{C}) \subset \cdots \qquad \underset{n}{\mathrm{colim}}\, \mathsf{Whit}^{\leqslant n}(\mathcal{C}) \simeq \mathsf{Whit}(\mathcal{C}).$$

As we presently review, these may be identified with the twisted invariants for certain compact open subgroups $\mathring{I}_n$ of G_F.

2.7.3.   Fix a positive integer n. Consider the nth congruence subgroup K_n of G_O, and write G_{n-1} for the quotient, and similarly consider N_{n-1}. One may form the subgroup of G_O consisting of arcs which until nth order lie in N, i.e.

$$\mathring{J}_n := G_O \underset{G_{n-1}}{\times} N_{n-1}.$$

One then obtains $\mathring{I}_n$ by conjugation, namely

$$\mathring{I}_n := \mathrm{Ad}_{t^{-n\check{\rho}}} \mathring{J}_n.$$

2.7.4.   Each $\mathring{I}_n$ admits a unique additive character ψ which is (i) trivial on $\mathring{I}_n \cap B_F^-$ and (ii) agrees with the Whittaker character on $\mathring{I}_n \cap N_F$. There are canonical identifications

$$\mathsf{Whit}^{\leqslant n}(\mathcal{C}) \simeq \mathcal{C}^{\mathring{I}_n, \psi},$$

and the transition functors in the above filtration are given by averaging, cf. Section 2 of [Ras16b] for more details.

2.7.5.   The first case step in the filtration will be of particular importance to us. Note that $\mathring{I}_1$ is the prounipotent radical of an Iwahori subgroup. For ease of notation, we often denote them by $\mathring{I}$ and I, respectively. In what follows, we sometimes refer to the corresponding invariants as the *baby Whittaker model*.

3. DEPTH AND ADOLESCENT WHITTAKER

The goal of this section is to show that the Whittaker category on the enhanced affine flag variety is exhausted by the first step in its adolescent Whittaker filtration, i.e. canonically identifies with the baby Whittaker category.

We will deduce the above statement from the following more general assertion. For a nonnegative integer n, denote by K_n the corresponding congruence subgroup of G_O.

Theorem 3.0.1. *The adolescent Whittaker filtration for $D_\kappa(G_F/K_n)$ is exhausted by its nth step, i.e.*

$$\iota_n : \mathsf{Whit}^{\leqslant n}(D_\kappa(G_F/K_n)) \simeq \mathsf{Whit}(D_\kappa(G_F/K_n)).$$

Proof. As ι_n is fully faithful, it suffices to show its essential surjectivity. Recalling that $\check{\Lambda}$ denotes the cocharacter lattice of T, the Iwasawa decomposition yields a stratification of G_F/K_n with strata

$$\mathcal{S}^{\check{\lambda}} := N_F\, t^{\check{\lambda}}\, G_O/K_n, \qquad \text{for} \quad \lambda \in \check{\Lambda}.$$

Using the normality of K_n in G_O, a standard argument shows that the Whittaker category on the stratum vanishes, i.e. that

$$\mathsf{Whit}(D_\kappa(\mathcal{S}^{\check{\lambda}})) \simeq 0,$$

unless $\check{\lambda} + n\check{\rho}$ is a dominant cocharacter. It follows that the closure of each stratum contains only finitely many other strata which support Whittaker sheaves. In particular, to see the essential surjectivity of ι_n it suffices to consider objects $*$-extended from a single stratum.

To prove the latter claim, fix a cocharacter $\check{\lambda}$ such that $\check{\lambda} + n\check{\rho}$ is dominant. Under this assumption, using the triangular decomposition of $\mathring{I}_n$ it is straightforward to see that

$$(3.1) \qquad \mathring{I}_n\, t^{\check{\lambda}}\, G_O/K_n = (\mathring{I}_n \cap N_F)\, t^{\check{\lambda}}\, G_O/K_n,$$

i.e. that these coincide as locally closed sub-ind-varieties of G/K_n. Similarly, our assumption on $\check{\lambda}$ implies that

$$(3.2) \qquad (N_F \cap t^{\check{\lambda}} G_O t^{-\check{\lambda}}) \subset (\mathring{I}_n \cap N_F).$$

It follows from (3.2) that we have

$$(3.3) \qquad \mathcal{S}^{\check{\lambda}} \simeq N_F \overset{N_F \cap \mathring{I}_n}{\times} \mathring{I}_n\, t^{\check{\lambda}} G_O/K_n.$$

Recalling that $\mathring{I}_n$ is prounipotent, we deduce equivalences

$$(3.4) \qquad D_\kappa(\mathring{I}_n\, t^{\check{\lambda}}\, G_O/K_n)_{\mathring{I}_n,\psi} \overset{(3.1)}{\simeq} D_\kappa(\mathring{I}_n\, t^{\check{\lambda}}\, G_O/K_n)_{\mathring{I}_n \cap N_F,\psi} \overset{(3.3)}{\simeq} D_\kappa(\mathcal{S}^{\check{\lambda}})_{N_F,\psi}.$$

It is straightforward to see that $*$-pushforward to G_F/K_N intertwines (3.4) and ι_n, which yields the claimed essential surjectivity. $\qquad\square$

We now deduce the desired consequence for the affine flag variety.

Corollary 3.0.2. *The adolescent Whittaker filtration on $\mathsf{Whit}_{\kappa,\lambda-mon}(\mathrm{Fl})$ is exhausted by its first step, i.e.*

$$\iota_1 : \mathsf{Whit}^{\leqslant 1}_{\kappa,\lambda-mon}(\mathrm{Fl}) \simeq \mathsf{Whit}_{\kappa,\lambda-mon}(\mathrm{Fl}).$$

Proof. We first note that a fully faithful embedding of $D_\kappa(G_F)$-modules $\mathring{\mathcal{C}} \to \mathcal{C}$ induces equivalences

$$(3.5) \qquad \mathsf{Whit}^{\leqslant n}(\mathring{\mathcal{C}}) \simeq (\mathring{\mathcal{C}} \cap \mathsf{Whit}^{\leqslant n}(\mathcal{C})),$$

i.e., they coincide as full subcategories of $\mathcal{C}$. Indeed, this claim follows tautologically from the identification, for any $D_\kappa(G_F)$-module $\mathcal{S}$,

$$\mathsf{Whit}^{\leqslant n}(\mathcal{S}) \simeq \mathsf{Hom}_{D_\kappa(G_F)-\mathsf{mod}}(D_\kappa(G_F)^{\mathring{I}_n,\psi}, \mathcal{S}).$$

To use this, note that pullback yields a fully faithful embedding

$$(3.6) \qquad D_\kappa(G_F)^{I,\lambda-mon} \to D_\kappa(G_F)^{K_1}.$$

By Theorem 3.0.1, the adolescent Whittaker filtration for the right-hand side of (3.6) is exhausted by its first step, hence by (3.5) the same holds for the left-hand side, as desired. $\qquad\square$

4. The adolescent Whittaker filtration on representations of the $\mathcal{W}_\kappa$-algebra revisited

The goal for this section is to show the following result.

Theorem 4.0.1. *The subcategory* $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{--}\mathsf{mod})^\heartsuit \subset \mathcal{W}_\kappa\text{--}\mathsf{mod}^\heartsuit$ *is closed under subobjects.*

This is *a priori* nonobvious because the definition of adolescent Whittaker filtration is in terms of Harish–Chandra modules for varying compact open subgroups of G_F which are related by averaging functors. We will prove this result by realizing the subcategory $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{--}\mathsf{mod})^\heartsuit$ in more explicit terms, intrinsic to the action of the $\mathcal{W}_\kappa$-algebra on a module.

4.1. **Background.** Recall that [Ras16b] introduced the generalized vacuum modules, i.e. a sequence of representations $\mathcal{W}_\kappa^n \in \mathcal{W}_\kappa\text{--}\mathsf{mod}^\heartsuit$. We presently review their basic properties.

We remind that these modules have distinguished *vacuum* vectors $\mathrm{vac}_n \in \mathcal{W}_\kappa^n$, and there are natural surjections
$$\alpha_n : \mathcal{W}_\kappa^{n+1} \twoheadrightarrow \mathcal{W}_\kappa^n$$
which send vac_{n+1} to vac_n. More generally, for $m \geqslant n$, we have a composition which we denote by
$$(4.1) \qquad\qquad\qquad\qquad \alpha_{n,m} : \mathcal{W}_\kappa^m \twoheadrightarrow \mathcal{W}_\kappa^n.$$
As constructed in [Ras16b], each module $\mathcal{W}_\kappa^n$ carries a canonical *Kazhdan-Kostant* filtration. We denote these filtrations by $F_\bullet^{KK}\mathcal{W}_\kappa^n$. These satisfy
$$F_{-1}^{KK}\mathcal{W}_\kappa^n = 0 \quad \text{and} \quad F_0^{KK}\mathcal{W}_\kappa^n = k \cdot \mathrm{vac}_n.$$
The morphisms (4.1) are strictly compatible with the Kazhdan–Kostant filtrations.

Finally, we recall that the underlying vector space of each $\mathcal{W}_\kappa^n$ has a grading defined by loop rotation. We denote the jth graded piece of $\mathcal{W}_\kappa^n$ by $\mathcal{W}_\kappa^n(j)$. More importantly, for any j, define the subspace
$$\mathcal{W}_\kappa^n(\geqslant j) := \bigoplus_{k \geqslant j} \mathcal{W}_\kappa^n(k).$$

4.2. **Main construction.** Suppose $m \geqslant n$ are nonnegative integers. We will construct generators for $\ker(\alpha_{n,m})$ in terms of the above data. To do so, define
$$V_{n,m} := \sum_{i > 0} \left(F_i^{KK}\mathcal{W}_\kappa^m \cap \mathcal{W}_\kappa^m(\geqslant n \cdot i) \right) \subset \mathcal{W}_\kappa^m.$$

The following result plays a key role.

Lemma 4.2.1. $V_{n,m}$ *is contained in* $\ker(\alpha_{n,m})$. *Moreover,* $V_{n,m}$ *generates this kernel, i.e.,* $\ker(\alpha_{n,m})$ *is the minimal subobject of* $\mathcal{W}_\kappa^m$ *in* $\mathcal{W}_\kappa\text{--}\mathsf{mod}^\heartsuit$ *containing* $V_{n,m}$.

Proof. We break the proof into several steps.

Step 1. First, we explain that the analogous result for Kac–Moody algebras is straightforward. Let $\mathfrak{h}$ be a reductive Lie algebra, and let $\kappa_\mathfrak{h}$ be a level for $\mathfrak{h}$.

The analogues of the generalized vacuum modules are given by
$$\mathbb{V}_{\mathfrak{h},\kappa_\mathfrak{h}}^n := \mathrm{ind}_{t^n\mathfrak{h}_O}^{\widehat{\mathfrak{h}}_{\kappa_\mathfrak{h}}}(k).$$
These module are equipped with PBW filtrations and gradings defined by loop rotation. We have evident structure morphisms
$$\beta_{n,m} : \mathbb{V}_{\mathfrak{h},\kappa_\mathfrak{h}}^m \to \mathbb{V}_{\mathfrak{h},\kappa_\mathfrak{h}}^n$$
for $m \geqslant n$ that are strictly compatible with the PBW filtration and compatible with the grading.

Define $W_{n,m} \subset \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}}$ by analogy with $V_{n,m}$. We claim that it is contained in $\ker(\beta_{n,m})$ and generates it as a Kac–Moody representation. Indeed, to show the containment it suffices to show for every positive integer i that

$$(F_i^{PBW} \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}} \cap \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}}(\geqslant n \cdot i)) \subset \ker(\beta_{n,m}).$$

We will proceed by induction on i. For $i = 1$, we have a natural identification

$$F_1^{PBW} \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}} \simeq \widehat{\mathfrak{h}}_{\kappa_{\mathfrak{h}}} / t^m \mathfrak{h}_O.$$

Clearly, the subspace of vectors of degree $\geqslant n$ in the left-hand side identifies with $t^n \mathfrak{h}_O / t^m \mathfrak{h}_O$, and hence lies in $\ker(\beta_{n,m})$ by definition.

Now inductively, fix $i > 1$ and consider an element

$$\xi \in F_i^{PBW} \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}} \cap \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}}(\geqslant n \cdot i).$$

By definition, its symbol

$$\sigma(\xi) \in \mathrm{gr}_i^{PBW} \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}} = \mathrm{Sym}^i(\mathfrak{h}_F / t^m \mathfrak{h}_O)$$

has degree $\geqslant n \cdot i$, which readily implies that it lies in $\mathrm{Sym}^{i-1}\left(\mathfrak{h}_F / t^m \mathfrak{h}_O\right) \cdot t^n \mathfrak{h}_F / t^m \mathfrak{h}_O$. Therefore, there exists $\widetilde{\xi} \in F_{i-1}^{PBW} U(\widehat{\mathfrak{h}}_{\kappa_{\mathfrak{h}}}) \cdot t^n \mathfrak{h}_O$ with $\xi - \widetilde{\xi} \cdot \mathrm{vac}_n \in F_{i-1}^{PBW} \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}}$. Moreover, we may assume $\widetilde{\xi} \cdot \mathrm{vac}_n$ also lies in degrees $\geqslant n \cdot i$. With this, the difference $\xi - \widetilde{\xi} \cdot \mathrm{vac}_n$ lies in

$$F_{i-1}^{PBW} \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}} \cap \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}}(\geqslant n \cdot i) \subset F_{i-1}^{PBW} \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}} \cap \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}}(\geqslant n \cdot (i-1)).$$

By induction, $\xi - \widetilde{\xi} \in \ker(\beta_{n,m})$, and by construction, $\widetilde{\xi} \in \ker(\beta_{n,m})$, so we obtain the claim.

Generation is clear as the above calculation for $i = 1$ showed that the first step, i.e.

$$F_1^{PBW} \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}} \cap \mathbb{V}^m_{\mathfrak{h},\kappa_{\mathfrak{h}}}(\geqslant n) = t^n \mathfrak{h}_O / t^m \mathfrak{h}_O,$$

already generates $\ker(\beta_{n,m})$.

Step 2. We now deduce the containment in Lemma 4.2.1 from the previous step. We remind that $\mathfrak{t}$ denotes the Lie algebra of the Cartan T of G.

Recall from [Ras16b] that there is a certain level $\kappa_{\mathfrak{t}}$ of $\mathfrak{t}$ such that there exist *free-field* homomorphisms

$$\varphi_n : \mathcal{W}^n_\kappa \to \mathbb{V}^n_{\mathfrak{t},\kappa_{\mathfrak{t}}}.$$

By construction, these are injective morphisms that are compatible with loop rotation, and by [Ras16b] are strictly compatible with the Kazhdan-Kostant and PBW filtrations. Moreover, the free-field morphisms intertwine the transition maps, i.e. one has a commutative diagram

$$\begin{array}{ccc} \mathcal{W}^m_\kappa & \xrightarrow{\ \varphi_m\ } & \mathbb{V}^m_{\mathfrak{t},\kappa_{\mathfrak{t}}} \\ {\scriptstyle \alpha_{n,m}} \downarrow & & \downarrow {\scriptstyle \beta_{n,m}} \\ \mathcal{W}^n_\kappa & \xrightarrow{\ \varphi_n\ } & \mathbb{V}^n_{\mathfrak{t},\kappa_{\mathfrak{t}}}. \end{array}$$

Therefore, $\varphi_m(V_{n,m}) \subset W_{n,m} \subset \ker(\beta_{n,m})$, which implies that $V_{n,m} \subset \ker(\varphi_n \alpha_{n,m}) = \ker(\alpha_{n,m})$, as desired.

Step 3. It remains to show generation of $\ker(\alpha_{n,m})$ by $V_{n,m}$.

Let $f \in \mathfrak{b}^-$ denote a principal nilpotent element of degree -1 with respect to $\check{\rho}$ and let $\mathfrak{c}$ denote the scheme $(f + \mathfrak{b})/N$. We consider $\mathbb{G}_m$ acting on $\mathfrak{c}$ in the standard way: the action of $\mathbb{G}_m$ on $\mathfrak{g}$ defined by

$$\xi \mapsto \lambda \mathrm{Ad}_{\check{\rho}(\lambda)}(\xi), \qquad \text{for} \quad \lambda \in \mathbb{G}_m, \xi \in \mathfrak{g}$$

induces an action on $\mathfrak{c} = (f + \mathfrak{b})/N$. It is well-known from work of Kostant that one has a $\mathbb{G}_m$-equivariant isomorphism

$$\mathfrak{c} \cong \prod_{i=1}^{\mathrm{rank}\, \mathfrak{g}} \mathbb{A}^1,$$

where, if we write d_i for the ith exponent of $\mathfrak{g}$, $\mathbb{G}_m$ acts on the ith factor of $\mathbb{A}^1$ by the d_ith power of the standard homothety action.

As $\mathfrak{c}$ is an affine scheme, $\mathfrak{c}_O \subset \mathfrak{c}_F$ is a closed subscheme. In addition, $\mathfrak{c}_F$ has an action of $(\mathbb{G}_m)_F$, which arises by looping the above action. We consider the twist

$$(4.2) \qquad \mathfrak{c}_F dt := (\mathfrak{c}/\mathbb{G}_m)_F \times_{(\mathbb{B}\mathbb{G}_m)_F} \operatorname{Spec} k$$

where $\operatorname{Spec} k \to (\mathbb{B}\mathbb{G}_m)_F$ corresponds to the line bundle of 1-forms on $\operatorname{Spec} F$. So after a choice of non-vanishing 1-form, we may identify $\mathfrak{c}_F dt \simeq \mathfrak{c}_F$, but the action of loop rotations is slightly modified. We use the similar notation $\mathfrak{c}_O dt$.

Now recall from [Ras16b] that there is a canonical isomorphism

$$(4.3) \qquad \eta_n : \operatorname{gr}^{KK}_\bullet \mathcal{W}^n_\kappa \xrightarrow{\sim} \operatorname{Fun}(t^{-n} \cdot \mathfrak{c}_O dt).$$

Here Fun indicates the algebra of functions, and $t^{-n} \cdot \mathfrak{c}_O dt \subset \mathfrak{c}_F dt$ is obtained by acting via $t^{-n} \in (\mathbb{G}_m)_F(k)$. By construction, this isomorphism η_n identifies the grading on

$$\operatorname{gr}^{KK}_\bullet \mathcal{W}^n_\kappa = \bigoplus_i \operatorname{gr}^{KK}_i \mathcal{W}^n_\kappa$$

with the grading on $\operatorname{Fun}(t^{-n} \cdot \mathfrak{c}_O dt)$ coming from the action of $\mathbb{G}_m \subset (\mathbb{G}_m)_O$ on $t^{-n}\mathfrak{c}_O dt$. Moreover, η_n is equivariant for coordinate changes on the disc, and in particular, for loop rotation. Finally, for $m \geqslant n$, $\operatorname{gr}^{KK}_\bullet(\alpha_{n,m})$ corresponds to the natural restriction map

$$\operatorname{Fun}(t^{-m} \cdot \mathfrak{c}_O dt) \twoheadrightarrow \operatorname{Fun}(t^{-n} \cdot \mathfrak{c}_O dt).$$

From Kostant's description of $\mathfrak{c}$, we deduce that $\operatorname{Ker}(\operatorname{gr}^{KK}_\bullet(\alpha_{n,m}))$ is generated by elements in $\operatorname{gr}^{KK}_i(\alpha_{n,m})$ for some $1 \leqslant i \leqslant d_{\operatorname{rank}\mathfrak{g}}$ that are homogeneous for loop rotation, where we can take the loop rotation degrees of the generators to lie in the interval $[in, im)$.

We now argue that the above generators may be obtained as symbols of elements of $V_{n,m}$. To see this, consider more generally any $i \geqslant 1$ and an element

$$f \in \ker(\operatorname{gr}^{KK}_i(\alpha_{n,m}))$$

which is homogeneous for loop rotation of degree $d \geqslant in$. Lift f to $F^{KK}_i \mathcal{W}^m_\kappa$, and then let

$$\widetilde{f} \in F^{KK}_i \mathcal{W}^m_\kappa$$

be the degree d component of this lift. Clearly $\widetilde{f}$ lifts f as well, and $\widetilde{f} \in V_{n,m}$. In particular, $\widetilde{f} \in \operatorname{Ker}(\alpha_{n,m})$ by Step 2 above.

In other words, symbols of elements of $V_{n,m}$ generate $\operatorname{Ker}(\operatorname{gr}^{KK}_\bullet(\alpha_{n,m}))$. The generation immediately follows. $\qquad\square$

4.3. **Passage to the topological algebra.** Let $\mathcal{W}^{as}_\kappa$ denote the topological vector space

$$\mathcal{W}^{as}_\kappa := \lim_n \mathcal{W}^n_\kappa,$$

where the topology is the inverse limit topology. The corresponding pro-object

$$\text{``}\lim_n\text{''}\, \mathcal{W}^n_\kappa \in \mathsf{Pro}(\mathcal{W}_\kappa\text{--}\mathsf{mod}^\heartsuit)$$

corepresents the forgetful functor $\mathcal{W}_\kappa\text{--}\mathsf{mod}^\heartsuit \to \mathsf{Vect}^\heartsuit$ by [Ras16b]. This implies that $\mathcal{W}^{as}_\kappa$ identifies with endomorphisms of this forgetful functor, so is canonically a topological algebra.[6] The left ideals

$$\operatorname{Ker}(\mathcal{W}^{as}_\kappa \to \mathcal{W}^n_\kappa)$$

are open and provide a neighborhood basis of zero. The evident functor identifies $\mathcal{W}_\kappa\text{--}\mathsf{mod}^\heartsuit$ with the category of discrete $\mathcal{W}^{as}_\kappa$-modules.[7]

The Kazhdan–Kostant filtrations and loop rotation gradings extend to the topological algebra as follows. For any i, define

$$F^{KK}_i \mathcal{W}^{as}_\kappa := \lim_n \big(F^{KK}_i \mathcal{W}^n_\kappa\big).$$

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[6]More precisely, an $\overset{\to}{\otimes}$-algebra in the language of [Ras19].

[7]Although we do not need it, a version of this statement for bounded below derived categories follows from Proposition 3.7.1 of [Ras19].

These are closed subspaces of $\mathcal{W}_\kappa^{as}$ whose union is dense. Similarly, for any j define

$$\mathcal{W}_\kappa^{as}(\geqslant j) := \lim_n \left(\mathcal{W}_\kappa^n(\geqslant j)\right).$$

4.4. **Definition of certain ideals.** Let $n \in \mathbb{Q}^{\geqslant 0}$ be a positive rational number. Define

$$V_n := \sum_{i>0} \left(F_i^{KK}\mathcal{W}_\kappa^{as} \cap \mathcal{W}_\kappa^{as}(\geqslant n \cdot i)\right) \subset \mathcal{W}_\kappa^{as}.$$

Here we understand $\mathcal{W}_\kappa^{as}(\geqslant n \cdot i)$ in the evident way for $n \cdot i$ a rational number; it coincides with $\mathcal{W}_\kappa^{as}(\geqslant \lceil n \cdot i \rceil)$. Define $I_n \subset \mathcal{W}_\kappa^{as}$ as the closure of the left ideal generated by V_n.

Proposition 4.4.1. *For $n \in \mathbb{Z}^{\geqslant 0}$, I_n is the kernel of the canonical map $\alpha_n : \mathcal{W}_\kappa^{as} \to \mathcal{W}_\kappa^n$.*

Proof. Clearly $\ker(\alpha_n) = \lim_{m \geqslant n} \ker(\alpha_{n,m})$. An element of V_n maps to $\lim_{m \geqslant n} V_{n,m}$, so lies in $\ker(\alpha_n)$. Conversely, fix $\xi \in \ker(\alpha_n)$. By Lemma 4.2.1, for any $m \geqslant n$ we have

$$\xi \in \mathcal{W}_\kappa^{as} \cdot V_n + \ker(\alpha_m).$$

As the $\ker(\alpha_m)$ form a neighborhood basis of zero, this implies that ξ is in the closure of the ideal generated by V_n. $\square$

Corollary 4.4.2. *For any $n \in \mathbb{Q}^{\geqslant 0}$, I_n is open.*

Proof. Clearly $I_m \subset I_n$ for any integer $m \geqslant n$, and I_m is open by Proposition 4.4.1. $\square$

Remark 4.4.3. We obtain modules $\mathcal{W}_\kappa^n := \mathcal{W}_\kappa^{as}/I_n$ for all $n \in \mathbb{Q}^{\geqslant 0}$, generalizing the modules from [Ras16b]. Each $\mathcal{W}_\kappa^n$ carries an evident Kazhdan-Kostant filtrations, and its associated graded is the space of functions on classical opers with slopes $\leqslant n$.

4.5. We will need the following basic results describing when the ideals I_n "jump." Below, we write h for the Coxeter number of $\mathfrak{g}$, i.e. the maximal exponent $d_{\mathrm{rank}\,\mathfrak{g}}$.

Lemma 4.5.1. *For $n \in \mathbb{Q}^{\geqslant 0}$, define*

$$V_n^{\leqslant h} := \sum_{i=1}^{h} \left(F_i^{KK}\mathcal{W}_\kappa^{as} \cap \mathcal{W}_\kappa^{as}(\geqslant n \cdot i)\right) \subset V_n \subset \mathcal{W}_\kappa^{as}.$$

Then I_n is topologically generated by $V_n^{\leqslant h}$, i.e., the closure of the ideal generated by $V_n^{\leqslant h}$ is I_n.

Proof. For $n \in \mathbb{Z}^{\geqslant 0}$, this is immediate from the proof of Lemma 4.2.1 (see Step 4.2). The same argument applies for general n, using the general description of $\mathrm{gr}_\bullet \mathcal{W}_\kappa^n$ from Remark 4.4.3. $\square$

Corollary 4.5.2. *Let $h \in \mathbb{Z}^{>0}$ denote the Coexter number of G. Then for any $n \in \mathbb{Q}^{\geqslant 0}$, $I_n = I_{\frac{1}{h}\lceil nh \rceil}$. In particular, the ideals I_n are distinct only for $n \in \frac{1}{h}\mathbb{Z}^{\geqslant 0}$.*

Proof. Clearly $V_n^{\leqslant h} = V_{\frac{1}{h}\lceil nh \rceil}^{\leqslant h}$, so we obtain the claim from Lemma 4.5.1. $\square$

4.6. **Local nilpotence.** In the remainder of this section, we will show the following result.

Theorem 4.6.1. *Let n be a positive integer. Then $M \in \mathcal{W}_\kappa\text{–mod}^\heartsuit$ lies in $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit$ if and only if V_n acts locally nilpotently on M, i.e., for every $m \in M$, there exists an integer $N \geqslant 0$ such that for every $\xi_1, \ldots, \xi_N \in V_n$, $\xi_1 \ldots \xi_N m = 0$.*

Note that Theorem 4.0.1 is an immediate consequence of this result. The remainder of this section is devoted to a proof of Theorem 4.6.1.

4.6.2. We have the following basic observation.

Lemma 4.6.3. *For an integer $n > 0$, $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit$ is the minimal subcategory of $\mathcal{W}_\kappa\text{–mod}^\heartsuit$ containing $\mathcal{W}_\kappa^n$ and closed under colimits and extensions.*

Proof. We have $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit = \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}_n,\psi,\heartsuit}$ in the notation of [Ras16b]. By construction, the module $\mathcal{W}_\kappa^n$ in the left hand side corresponds to $\mathrm{ind}_{\mathrm{Lie}(\mathring{I}_n)}^{\widehat{\mathfrak{g}}_\kappa}(\psi)$. Now the claim follows as $\mathring{I}_n$ is pro-unipotent. $\square$

Remark 4.6.4. We can use Remark 4.4.3 to give a definition of $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ and $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit$ for $n \in \mathbb{Q}^{\geqslant 0}$: for $n = 0$ it is defined as in [Ras16b], and for $n > 0$, we generate under colimits using $\mathcal{W}_\kappa^n$. For this definition, our argument shows that Theorem 4.6.1 is true for $n \in \mathbb{Q}^{>0}$.

4.7. We begin with the following result.

Proposition 4.7.1. *For any $n \in \mathbb{Q}^{>0}$, V_n acts locally nilpotently on $\mathcal{W}_\kappa^n$.*

Proof. Fix $v \in \mathcal{W}_\kappa^n$. Choose an integer i such that $v \in F_i^{KK}\mathcal{W}_\kappa^n$ and an integer r such that $v \in \mathcal{W}_\kappa(\geqslant r)$. We first analyze the action of a single element of V_n on v. So, fix $j > 0$ and an element

$$\xi \in F_j^{KK}\mathcal{W}_\kappa^{as} \cap \mathcal{W}_\kappa^{as}(\geqslant n \cdot j) \subset V_n.$$

Clearly $\xi v \in F_{i+j}^{KK}\mathcal{W}_\kappa^n$. In fact, we claim that it is in $F_{i+j-1}^{KK}\mathcal{W}_\kappa^n$. To see this, note the symbol of ξ in $\mathrm{gr}_i \mathcal{W}_\kappa^{as}$ annihilates the vacuum vector in $\mathrm{gr}_\bullet \mathcal{W}_\kappa^n$, so as $\mathrm{gr}_\bullet \mathcal{W}_\kappa^{as}$ is commutative and $\mathrm{gr}_\bullet \mathcal{W}_\kappa^n$ is generated by its vacuum vector, the symbol of ξ annihilates all of $\mathrm{gr}_\bullet \mathcal{W}_\kappa^n$.

Now take an integer N with $N > i - \frac{r}{n}$ and suppose $\xi_1, \ldots, \xi_N \in V_n$. We claim that $\xi_1 \ldots \xi_N v = 0$. To see this, note from the definition of V_n we may assume that there are positive integers $j_k > 0$ with each

$$\xi_k \in F_{j_k}^{KK}\mathcal{W}_\kappa^{as} \cap \mathcal{W}_\kappa^{as}(\geqslant n \cdot j_k).$$

Then the above shows that $\xi_1 \ldots \xi_N v \in F_{i+j_1+\ldots+j_N-N}^{KK}\mathcal{W}_\kappa^n$. Clearly this vector is in $\mathcal{W}_\kappa(\geqslant r + n(j_1 + \ldots + j_N))$. Now observe that

$$r + n(j_1 + \ldots + j_N) > n(i + j_1 + \ldots + j_N - N)$$

by definition of N. The desired vanishing is therefore a consequence of the following Lemma 4.7.2. $\square$

Lemma 4.7.2. *For $n \in \mathbb{Q}^{>0}$, $F_i^{KK}\mathcal{W}_\kappa^n \cap \mathcal{W}_\kappa^n(\geqslant ni + 1) = 0$.*

Proof. From the description (4.3) of $\mathrm{gr}_\bullet \mathcal{W}_\kappa^n$, we find that $\mathrm{gr}_j \mathcal{W}_\kappa^n(m)$ vanishes for $m > nj$. Therefore, for $j \leqslant i$, $\mathrm{gr}_j \mathcal{W}_\kappa^n(\geqslant ni + 1) = 0$, which immediately gives the claim. $\square$

4.8. We will need one last preparatory lemma. In its statement below, we keep the notation of Lemma 4.5.1.

Lemma 4.8.1. *For $n \in \frac{1}{h}\mathbb{Z}^{>0}$ and $\xi, \varphi \in V_n$, we have $[\xi, \varphi] \in I_{n+\frac{1}{h}}$.*

Proof. We may assume that there are integers $0 < i, j \leqslant h$ such that

$$\xi \in F_i^{KK}\mathcal{W}_\kappa^{as} \cap \mathcal{W}_\kappa^{as}(\geqslant ni) \qquad \text{and} \qquad \varphi \in F_j^{KK}\mathcal{W}_\kappa^{as} \cap \mathcal{W}_\kappa^{as}(\geqslant nj).$$

With this, it follows that $[\xi, \varphi] \in F_{i+j-1}^{KK}\mathcal{W}_\kappa^{as}$, as $\mathrm{gr}_\bullet \mathcal{W}_\kappa^{as}$ is commutative. Since moreover this element lies in $\mathcal{W}_\kappa^{as}(\geqslant n(i+j))$, we have that

$$[\xi, \varphi] \in V_{\frac{i+j}{i+j-1}\cdot n}.$$

As $\frac{i+j}{i+j-1} \cdot n > n$ and $n \in \frac{1}{h}\mathbb{Z}^{>0}$, Corollary 4.5.2 implies $V_{\frac{i+j}{i+j-1}\cdot n} \subset I_{n+\frac{1}{h}}$, giving the claim. $\square$

Corollary 4.8.2. *For $n \in \mathbb{Q}^{>0}$, let M be a non-zero module on which V_n acts locally nilpotently (in the sense of Theorem 4.6.1). Then there exists a non-zero morphism $\mathcal{W}_\kappa^n \to M$.*

Proof. By Corollary 4.5.2, we may suppose $n \in \frac{1}{h}\mathbb{Z}^{>0}$. For any $m \in \frac{1}{h}\mathbb{Z}^{>0}$, let $M_m \subset M$ be the subspace of vectors annihilated by I_m. By Lemma 4.5.1, this space coincides with the subspace of vectors annihilated by $V_m^{\leqslant h}$. Vectors in M_m correspond to maps $\mathcal{W}_\kappa^m \to M$, so we equivalently must show that the nonvanishing of M_n.

We will show by descending induction that for any $m \in \frac{1}{h}\mathbb{Z}$ with $m \geqslant n$, the subspace $M_m \neq 0$. Clearly

$$M = \bigcup_m M_m,$$

and so by the nontriviality of M we have the nonvanishing of M_m for $m \gg 0$. Therefore, it suffices to show that if $m \geqslant n$ and $M_{m+\frac{1}{h}} \neq 0$, then moreover $M_m \neq 0$.

We first claim that if $\xi \in V_m$ and and $v \in M_{m+\frac{1}{h}}$, then $\xi \cdot v \in M_{m+\frac{1}{h}}$. Indeed, if $\varphi \in V_{m+\frac{1}{h}} \subset V_m$, then $[\varphi, \xi] \in I_{m+\frac{1}{h}}$ by Lemma 4.8.1, and so

$$\varphi\xi \cdot v = [\varphi, \xi] \cdot v = 0,$$

as desired.

It remains to show the nontriviality of M_m. Applying Lemma 4.8.1 again, we see that for $\xi_1, \xi_2 \in V_m$, the operators $\xi_i \cdot - : M_{m+\frac{1}{h}} \to M_{m+\frac{1}{h}}$ commute. Moreover, as $m \geqslant n$ by assumption, these operators are locally nilpotent. Therefore, for any finite collection $\xi_1, \ldots, \xi_m \in V_n$, there exists a non-zero vector in $M_{m+\frac{1}{h}}$ annihilated by each ξ_i. Now the nontriviality of M_m follows from Lemma 4.5.1 and the observation that the inclusion

$$(V_m^{\leqslant h} \cap I_{m+\frac{1}{h}}) \subset V_m^{\leqslant h}$$

is of finite codimension, which is clear e.g. from Lemma 4.7.2. □

4.9. **Conclusion.** In the remainder of the section, we finish the proof of Theorem 4.6.1.

4.10. Let $\mathcal{A}_n \subset \mathcal{W}_\kappa\text{–mod}^\heartsuit$ be the subcategory of modules on which V_n acts locally nilpotently. By Proposition 4.7.1, $\mathcal{W}_\kappa^n \in \mathcal{A}_n$. Clearly $\mathcal{A}_n$ is closed under quotients, extensions, and filtered colimits. Therefore, by Lemma 4.6.3, $\mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit \subset \mathcal{A}_n$.

4.11. Now before proceeding, recall that the canonical functor $\iota_{n,*} : \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) \to \mathcal{W}_\kappa\text{–mod}$ is fully faithful (at the derived level), t-exact, and admits a right adjoint $\iota_n^!$.

We claim that for any $M \in \mathcal{W}_\kappa\text{–mod}^\heartsuit$, the adjunction map

$$\varepsilon : H^0(\iota_{n,*}\iota_n^!(M)) \to M$$

is a monomorphism in $\mathcal{W}_\kappa\text{–mod}^\heartsuit$. As $H^0(\iota_{n,*}\iota_n^!(M)) \in \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit$, it lies in $\mathcal{A}_n$. Therefore, its subobject $\mathrm{Ker}(\varepsilon)$ lies in $\mathcal{A}_n$. We next observe that the map

$$(4.4) \qquad \mathrm{Hom}_{\mathcal{W}_\kappa\text{–mod}^\heartsuit}(\mathcal{W}_\kappa^n, H^0(\iota_{n,*}\iota_n^!(M))) \to \mathrm{Hom}_{\mathcal{W}_\kappa\text{–mod}^\heartsuit}(\mathcal{W}_\kappa^n, M)$$

is an isomorphism since $\mathcal{W}_\kappa^n \in \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})^\heartsuit$. Therefore, $\mathrm{Hom}_{\mathcal{W}_\kappa\text{–mod}^\heartsuit}(\mathcal{W}_\kappa^n, \mathrm{Ker}(\varepsilon)) = 0$. By Corollary 4.8.2, we find that $\mathrm{Ker}(\varepsilon) = 0$, as desired.

4.12. We now conclude the argument. Suppose $M \in \mathcal{A}_n$. As above, we need to show that the map

$$\varepsilon : H^0(\iota_{n,*}\iota_n^!(M)) \to M$$

is an isomorphism. We have shown ε is a monomorphism, so it remains to see that it is an epimorphism. Again, $\mathrm{Coker}(\varepsilon) \in \mathcal{A}_n$ as it is a quotient of M. The natural map

$$\mathrm{Ext}^1_{\mathcal{W}_\kappa\text{–mod}^\heartsuit}(\mathcal{W}_\kappa^n, H^0(\iota_{n,*}\iota_n^!(M))) \to \mathrm{Ext}^1_{\mathcal{W}_\kappa\text{–mod}}(\mathcal{W}_\kappa^n, \iota_{n,*}\iota_n^!(M))$$

is a monomorphism for cohomological degree reasons. Also, the map

$$\mathrm{Ext}^1_{\mathcal{W}_\kappa\text{–mod}}(\mathcal{W}_\kappa^n, \iota_{n,*}\iota_n^!(M)) \to \mathrm{Ext}^1_{\mathcal{W}_\kappa\text{–mod}}(\mathcal{W}_\kappa^n, M)$$

is an isomorphism as $\mathcal{W}_\kappa^n \in \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$. Considering the exact sequence

$$\mathrm{Hom}_{\mathcal{W}_\kappa\text{–mod}^\heartsuit}(\mathcal{W}_\kappa^n, \mathrm{Coker}(\varepsilon)) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{W}_\kappa\text{–mod}^\heartsuit}(\mathcal{W}_\kappa^n, H^0(\iota_{n,*}\iota_n^!(M))) \to \mathrm{Ext}^1_{\mathcal{W}_\kappa\text{–mod}^\heartsuit}(\mathcal{W}_\kappa^n, M)$$

we find that the boundary map ∂ must be injective.

Therefore, any map $\mathcal{W}_\kappa^n \to \operatorname{Coker}(\varepsilon)$ lifts to M. Applying (4.4), we see that any such map is zero. Applying Corollary 4.8.2 again, we find that $\operatorname{Coker}(\varepsilon) = 0$ as desired.

5. The baby Whittaker model of monodromic Category $\mathcal{O}$

As before, we continue to assume that κ is noncritical. In the previous section, we studied the adolescent Whittaker filtration on $\mathcal{W}_\kappa\text{–mod}^\heartsuit$. Using this, particularly the description of the steps in terms of nilpotency of Fourier modes of the generators of $\mathcal{W}_\kappa$ obtained in Theorem 4.6.1, it follows that the highest weight representations of $\mathcal{W}_\kappa$ belong to the first step in the filtration, i.e. the baby Whittaker category.

In this section, we exactly determine the corresponding objects of the baby Whittaker category. That is, we prove an equivalence between a version of Category $\mathcal{O}$ for $\mathcal{W}_\kappa$, whose definition we recall below, and a certain category of Iwahori–Whittaker modules for $\widehat{\mathfrak{g}}_\kappa$, which may not have been considered before.

5.1. **Identification of Category** $\mathcal{O}$**.** We begin by reminding the definition of the monodromic category $\mathcal{O}$ for $\mathcal{W}_\kappa$. This was introduced in [Dhi], following a definition introduced by Arakawa in [Ara07] in which the grading was an auxiliary structure, and a definition by Frenkel–Kac–Wakimoto of postive energy representations [FKW92]. Recall that $\mathcal{W}_\kappa$ contains a canonical conformal vector ω, and that L_0 denotes its corresponding energy operator, i.e., the coefficient of z^{-2} in the field $\omega(z)$.

Definition 5.1.1. The category $\mathcal{O}$ is the full subcategory of $\mathcal{W}_\kappa\text{–mod}^\heartsuit$ consisting of objects M satisfying

(1) Under the action of L_0, M decomposes into a sum of generalized eigenspaces

$$M = \bigoplus_{d \in \mathbb{C}} M_d, \quad M_d := \{m \in M : (L_0 - d)^N m = 0, N \gg 0\}.$$

(2) For each $d \in \mathbb{C}$, M_d is finite dimensional and M_{d-n} is nonzero for only finitely many $n \in \mathbb{Z}^{\geqslant 0}$.

5.1.2. *Identifying a cocomplete variant.* Define $\mathcal{O}^{loc}$ to be the full subcategory of $\mathcal{W}\text{–mod}^\heartsuit$ with objects give by direct limits of objects of $\mathcal{O}$. As $\mathcal{O}$ is closed under subobjects, a module M belongs to $\mathcal{O}^{loc}$ if and only if every finitely generated submodule of M belongs to $\mathcal{O}$. Therefore, $\mathcal{O}^{loc}$ is an abelian category that is closed under subquotients in $\mathcal{W}\text{–mod}^\heartsuit$.

Remark 5.1.3. If κ is negative, then $\mathcal{O}^{loc}$ may be identified with the ind-completion of the full subcategory of finite length objects in $\mathcal{O}$.

We now define its counterpart in $\widehat{\mathfrak{g}}_\kappa\text{–mod}^\heartsuit$. Consider the abelian baby Whittaker category

$$\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I},\psi,\heartsuit},$$

where $\mathring{I} := \mathring{I}_1$. Via restriction, any object of this carries an action of Z, the center of the universal enveloping algebra of $\operatorname{Ad}_{t-\check{\rho}}\mathfrak{g}$. Let us consider

$$\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I},\psi,Z_f,\heartsuit},$$

the full subcategory of $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I},\psi,\heartsuit}$ consisting of objects on which Z acts locally finitely. I.e., for any vector v in the representation, the subspace Zv is finite dimensional.

With this, we can state the main result of this section. Below, we denote by

$$\Psi : \widehat{\mathfrak{g}}_\kappa\text{–mod} \to \mathcal{W}_\kappa\text{–mod}$$

the functor of Drinfeld–Sokolov reduction.

Theorem 5.1.4. *The composition* $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I},\psi,Z_f,\heartsuit} \to \widehat{\mathfrak{g}}_\kappa\text{–mod} \xrightarrow{\Psi[\langle -2\check{\rho},\rho\rangle]} \mathcal{W}_\kappa\text{–mod}$ *induces an equivalence*

$$\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I},\psi,Z_f,\heartsuit} \simeq \mathcal{O}^{loc}.$$

To prove Theorem 5.1.4 we introduce the analogues of Verma modules. For a character χ of Z, consider the $\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{g}$-module

$$V_\chi := \mathbb{C}_\chi \underset{Z}{\otimes} \mathrm{ind}_{\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{n}}^{\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{g}} \mathbb{C}_\psi.$$

We will be interested in its parabolic induction to $\widehat{\mathfrak{g}}_\kappa$, namely

$$\Delta_\chi := \mathrm{pind}_{\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{g}}^{\widehat{\mathfrak{g}}_\kappa} V_\chi,$$

where as usual 'pind' denotes the induction from $\mathrm{Ad}_{t-\check{\rho}}\,L^+\mathfrak{g}$ to $\widehat{\mathfrak{g}}_\kappa$ of the inflation from $\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{g}$ to $\mathrm{Ad}_{t-\check{\rho}}\,L^+\mathfrak{g}$ of V_χ.

Lemma 5.1.5. *For any χ, the module Δ_χ belongs to $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}^{\mathring{I},\psi,Z_f,\heartsuit}$.*

Proof. The PBW filtration is a filtration by $\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{g}$ submodules with associated graded

$$\mathrm{Sym}(\widehat{\mathfrak{g}}_\kappa / \mathrm{Ad}_{t-\check{\rho}}\,L^+\mathfrak{g}) \otimes V_\chi.$$

As the tensor product of an integrable representation of $\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{g}$ and a Z locally finite representation is again Z locally finite, it follows that Z acts locally finitely on the associated graded and hence on M_χ. $\qquad\square$

We also need the following generation result.

Lemma 5.1.6. *The category $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}^{\mathring{I},\psi,Z_f,\heartsuit}$ is the smallest subcategory of $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}^{\mathring{I},\psi,\heartsuit}$ which contains the Verma modules Δ_χ and is closed under colimits and extensions.*

Proof. First, note that it suffices to show that any object of $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}^{\mathring{I},\psi,Z_f,\heartsuit}$ may be written as a quotient of a direct sum of finite successive extensions of Verma modules. To see this, it is enough to show that a finitely generated object N of $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}^{\mathring{I},\psi,Z_f,\heartsuit}$ may be written as a quotient of a finite successive extension of Verma modules. Indeed, Write $\mathring{\mathfrak{i}}$ for the Lie algebra of $\mathring{I}$. One has a fiber sequence of Lie algebras

$$(5.1) \qquad\qquad 0 \to \mathfrak{k} \to \mathring{\mathfrak{i}} \to \mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{n} \to 0,$$

where $\mathfrak{k}$ is $\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{k}_1$, and $\mathfrak{k}_1$ is the Lie algebra of first congruence subgroup of G_O. Note that $\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{g}$ normalizes $\mathfrak{k}$, and Z commutes with $\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{n}$. Using this, it follows that for any finite dimensional subspace N_0 of N, the image of the action map

$$U(\mathring{\mathfrak{i}}) \otimes Z \otimes N_0 \to N$$

is finite dimensional and stable under the action of Z and $\mathfrak{i}$. Moreover, it admits a full flag by subspaces which are stable under the action of Z and $\mathring{\mathfrak{i}}$. Taking N_0 to contain a set of generators for N and using the parabolic induction of the generated $\mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{g}$-module yields the claimed surjection. $\qquad\square$

We are now ready to prove Theorem 5.1.4, but first isolate the following claim for future reference.

Lemma 5.1.7. *The functor of Theorem 5.1.4 sends Verma modules to Verma modules, i.e. for any central character χ we have*

$$\Psi[\langle -2\check{\rho}, \rho\rangle]\,\Delta_\chi \simeq M_\chi.$$

Proof. This follows from a straightforward adaptation from the arguments of Section 4 of [Ras16b]. Briefly, the canonical generator of Δ_χ defines a vector of highest weight χ in its Drinfeld-Sokolv reduction. This yields a map from M_χ. To see this is an isomorphism, note that both carry compatible filtrations whose associated gradeds identify with the structure sheaf of

$$(5.2) \qquad\qquad\qquad \mathrm{Ad}_{t-\check{\rho}}\,\mathfrak{c}_O dt \subset \mathfrak{c}_F dt,$$

cf. Equation (4.2) for the definition of the latter. I.e., if we use the differential of the uniformizer t to identify $\mathfrak{c}_F dt$ and $\mathfrak{c}_F$, the subscheme in (5.2) consists of loops into $\mathfrak{c}$ that on the factor of $\mathbb{A}^1$ corresponding to the exponent d_i have a pole of order at most $d_i - 1$. $\qquad\square$

Proof of Theorem 5.1.4. To see that the composition lands in $\mathcal{W}_\kappa$–mod$^\heartsuit$, i.e., in cohomological degree zero, it suffices to know the analogous claim for all of $\widehat{\mathfrak{g}}_\kappa$–mod$^{\mathring{I},\psi,\heartsuit}$, which was shown in Theorem 7.2.1 of [Ras16b].

We next show that the image lies in $\mathcal{O}^{loc}$. We saw in the proof of Lemma 5.1.6 that we may characterize

$$\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I},\psi,Z_f,\heartsuit} \subset \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I},\psi,\heartsuit}$$

as the full subcategory consisting of objects N expressible as a quotient of a direct sum of finite successive extensions of the Verma modules Δ_χ. By Lemma 5.1.7, it follows the essential image of $\Psi[\langle -2\check\rho, \rho\rangle]$ lies in $\mathcal{O}^{loc}$.

It remains to argue that the obtained map

$$\Psi : \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I},\psi,Z_f,\heartsuit} \to \mathcal{O}^{loc}$$

is an equivalence. As $\widehat{\mathfrak{g}}_\kappa$–mod$^{\mathring{I},\psi,Z_f,\heartsuit} \to \widehat{\mathfrak{g}}_\kappa$–mod$^{\mathring{I},\psi,\heartsuit}$ is a Serre subcategory preserved by taking filtered colimits, it follows from Theorem 4.0.1 that Ψ is fully faithful, and that its essential image is closed under taking filtered colimits, quotients, and extensions. As every object of $\mathcal{O}$ admits a filtration, indexed by the nonnegative integers, whose associated graded is a sum of quotients of Verma modules, cf. Proposition 6.8 of [Dhi], the essential surjectivity follows. $\square$

Remark 5.1.8. Having obtained an explicit realization of $\mathcal{O}^{loc}$ within the baby Whittaker category, one may also further ask for a description of the preimage of $\mathcal{O}$. Relatedly, one may ask for a notion of characters in the baby Whittaker category which determine the q-characters of the associated $\mathcal{W}_\kappa$-modules. We provide answers to both questions in the appendix.

5.2. **Arakawa's theorem.** While we will see a geometric proof of Arakawa's results on the 'minus' Drinfeld–Sokolov reduction for negative κ in the next section, let us obtain from Theorem 5.1.4 an algebraic proof here, which applies to any noncritical κ.

Recall the maximal torus T and Borel B of G. Write B^- for the Borel opposite to B with respect to T, I_0^- for the corresponding Iwahori subgroup of G_O, and I^- for its conjugate $\mathrm{Ad}_{t^{-\check\rho}} I_0^-$. Consider the reduction functor

$$(5.3) \qquad \Psi_s := \Psi[-2\langle\check\rho, \rho\rangle + \dim_N] : \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}^-} \to \mathcal{W}_\kappa\text{–mod}.$$

Arakawa showed, following a conjecture of Frenkel–Kac–Wakimoto [FKW92], that this functor is t-exact, sends Verma modules to Verma modules, and sends simple modules to simple modules or zero [Ara07]. As observed in [Ras16b], we may factor this through the baby Whittaker category, i.e.,

$$(5.4) \qquad \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I}^-} \xrightarrow{\mathrm{Av}_{\mathrm{Ad}_{t^{-\check\rho}} N, \psi, *}[\dim_N]} \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\mathring{I},\psi} \xrightarrow{\Psi[-2\langle\check\rho,\rho\rangle]} \mathcal{W}_\kappa\text{–mod}.$$

As explained in Corollary 7.3.1 of *loc. cit.*, the t-exactness is now a general feature of averaging from N^- to (N, ψ) invariants. We now reprove the remaining assertions listed above via our adolescent Whittaker construction of $\mathcal{O}$.

Let us introduce relevant notation. For any $\Lambda \in \mathfrak{t}^*$, where we view $\mathfrak{t}^*$ as the dual of the *abstract* Cartan,[8] consider the associated Verma module

$$\mathbb{M}_\Lambda := \mathrm{ind}_{\mathrm{Ad}_{t^{-\check\rho}}\mathfrak{b}^-}^{\mathrm{Ad}_{t^{-\check\rho}}\mathfrak{g}} \mathbb{C}_\Lambda,$$

and write $\mathbb{L}_\Lambda$ for its simple quotient. Similarly, write $\mathbb{M}_{\Lambda,\kappa}$ for the corresponding Verma module for $\widehat{\mathfrak{g}}_\kappa$, i.e.,

$$\mathbb{M}_{\Lambda,\kappa} := \mathrm{pind}_{\mathrm{Ad}_{t^{-\check\rho}}\mathfrak{g}}^{\widehat{\mathfrak{g}}_\kappa} \mathbb{M}_\Lambda,$$

and write $\mathbb{L}_{\lambda,\kappa}$ for its simple quotient. Finally, write λ for the central character of $\mathbb{M}_\Lambda$. Passing to $\mathcal{W}_\kappa$–mod$^\heartsuit$, for λ as before recall that L_λ denotes the simple quotient of M_λ.

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[8]The reader may prefer to work with the non-abstract Cartan, and should accordingly twist our results by $w_\circ$, the long element of the finite Weyl group, due to our working with I^-.

Theorem 5.2.1. *The functor Ψ_s, cf. Equation (5.3), sends Vermas to Vermas, i.e., for any $\Lambda \in \mathfrak{t}^*$, we have*

$$\Psi_s \, \mathbb{M}_{\Lambda, \kappa} \simeq M_\lambda.$$

Further, Ψ_s sends simples to simples or zero. Namely, recalling that $\mathfrak{I}$ indexes the finite simple coroots, for any $\Lambda \in \mathfrak{t}^*$ we have

$$(5.5) \qquad \Psi_s \, \mathbb{L}_{\Lambda, \kappa} \simeq \begin{cases} L_\lambda, & \text{if } \langle \Lambda, \check{\alpha}_i \rangle \notin \mathbb{Z}^{\geq 0}, \forall i \in \mathfrak{I}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Consider the factorization of Ψ_s through the baby Whittaker category as in Equation 5.4. We saw in Lemma 5.1.7 that the insertion of the baby Whittaker category sends Δ_λ to M_λ, and it follows from Theorem 5.1.4 that it exchanges their simple quotients. Therefore, we are reduced to showing the analogous assertions for the averaging functor

$$\mathrm{Av} : \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^{\check{I}^-} \xrightarrow{\mathrm{Av}_{\mathrm{Ad}_{t-\check{\rho}} \, N, \psi, *}[\dim_N]} \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^{\check{I}, \psi}.$$

Let us recall the analog of Theorem 5.2.1 in finite type. Namely, that

$$\mathrm{av} : \mathfrak{g}\text{–}\mathsf{mod}^{N^-} \xrightarrow{\mathrm{Av}_{N, \psi, *}[\dim_N]} \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^{N, \psi}.$$

sends $\mathbb{M}_\Lambda$ to the simple object V_λ, and $\mathbb{L}_\Lambda$ to V_λ or zero, for the exact same conditions on Λ as in Equation (5.5). While this is standard, we sketch a proof for the reader's convenience. Indeed, the calculation for Verma modules follows from using t-exactness and the central character of $\mathbb{M}_\Lambda$ to calculate

$$H^0 \, \mathrm{Hom}_{\mathfrak{g}\text{–}\mathsf{mod}}(V_\lambda, \mathrm{av} \, \mathbb{M}_\Lambda) \simeq H^0 \, \mathrm{Hom}_{\mathfrak{g}\text{–}\mathsf{mod}}(\mathrm{ind}_{\mathfrak{n}}^{\mathfrak{g}} \mathbb{C}_\psi, \mathrm{av} \, \mathbb{M}_\Lambda)$$
$$\simeq H^{\dim_N} \, \mathrm{Hom}_{\mathfrak{n}\text{–}\mathsf{mod}}(\mathbb{C}_\psi, \mathrm{Res} \, \mathbb{M}_\Lambda)$$
$$\simeq \det(\mathfrak{n}^*),$$

where in the last step one uses that $\mathbb{M}_\Lambda \simeq U(\mathfrak{n})$ as $\mathfrak{n}$-modules. The assertion about simples is now immediate from t-exactness and that the fact that every Verma module within a block was sent to the same object.

The assertion about affine Vermas now follows from the equivariance of parahoric induction with respect to the categorical action of the Levi, cf. Section A.2 of [CD19]. Namely, we have

$$\mathrm{Av} \, \mathbb{M}_{\Lambda, \kappa} = \mathrm{Av} \, \mathrm{pind}_{\mathrm{Ad}_{t-\check{\rho}} \, \mathfrak{g}}^{\widehat{\mathfrak{g}}_\kappa} \mathbb{M}_\Lambda \simeq \mathrm{pind}_{\mathrm{Ad}_{t-\check{\rho}} \, \mathfrak{g}}^{\widehat{\mathfrak{g}}_\kappa} \mathrm{av} \, \mathbb{M}_\Lambda \simeq \mathrm{pind}_{\mathrm{Ad}_{t-\check{\rho}} \, \mathfrak{g}}^{\widehat{\mathfrak{g}}_\kappa} V_\lambda = M_\lambda.$$

For the simples, let us write $\mathbb{A}_{\Lambda, \kappa}$ for the dual Verma module of co-highest weight Λ, i.e., the contragredient dual of $\mathbb{M}_{\Lambda, \kappa}$. Similarly, write $\mathbb{A}_\Lambda$ for the dual Verma module in $\mathfrak{g}\text{–}\mathsf{mod}^{B^-}$. For any central character μ, we have by adjunction

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}}(\Delta_\mu, \mathrm{Av} \, \mathbb{A}_{\Lambda, \kappa}) \simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}}(\Delta_\mu, \mathbb{A}_{\Lambda, \kappa})[\dim_N]$$

Write $\mathfrak{k}_1$ for the $\check{\rho}$-conjugated first congruence algebra, and write the superscript $h\mathfrak{k}_1$ for Lie algebra cohomology with respect to $\mathfrak{k}_1$. Using that $\mathbb{A}_{\Lambda, \kappa}$ is cofree as a $\mathfrak{k}_1$ module, we may continue

$$\simeq \mathrm{Hom}_{\mathfrak{g}\text{–}\mathsf{mod}}(V_\mu, \mathbb{A}_{\Lambda, \kappa}^{h\mathfrak{k}_1})[\dim_N] \simeq \mathrm{Hom}_{\mathfrak{g}\text{–}\mathsf{mod}}(V_\mu, \mathbb{A}_\Lambda)[\dim_N].$$

From the finite dimensional story, we see this is zero or an exterior algebra. In particular, on Homs in the abelian category we have

$$(5.6) \qquad \mathrm{Hom}_{\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^\heartsuit}(\Delta_\mu, \mathrm{Av} \, \mathbb{A}_{\Lambda, \kappa}) \simeq \begin{cases} \mathbb{C}, & \text{if } \mu = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the factorization of a nonzero map

$$\Delta_\lambda \twoheadrightarrow M \hookrightarrow \mathrm{Av} \, \mathbb{A}_{\Lambda, \kappa}$$

through its image. As M is a nonzero quotient of M_λ, we have a short exact sequence

$$0 \to K \to M \to L_\lambda \to 0,$$

where by mild abuse of notation we write L_λ for the simple quotient of Δ_λ. We claim K is zero. Indeed, if K is nonzero, then it admits a nonzero map $\Delta_\mu \to K$, for some $\mu \neq \lambda$. Composing with the injection $K \to \mathrm{Av}\, \mathbb{A}_{\Lambda,\kappa}$, we get a contradiction with (5.6), as desired. We therefore have an image factorization

$$(5.7) \qquad\qquad \Delta_\lambda \twoheadrightarrow L_\lambda \hookrightarrow \mathrm{Av}\, \mathbb{A}_{\Lambda,\kappa}.$$

If we take the sequence

$$\mathbb{M}_{\Lambda,\kappa} \twoheadrightarrow L_{\Lambda,\kappa} \hookrightarrow \mathbb{A}_{\Lambda,\kappa},$$

we may apply Av to obtain a sequence

$$(5.8) \qquad\qquad M_\lambda \twoheadrightarrow \mathrm{Av}\, L_{\Lambda,\kappa} \hookrightarrow \mathrm{Av}\, \mathbb{A}_{\Lambda,\kappa}.$$

As we show in the appendix, every object of $\widehat{\mathfrak{g}}_\kappa\text{--mod}^{\mathring{I},\psi,Z_f,\heartsuit}$ carries an energy grading, i.e. a locally finite action of (a slight modification of) the Segal–Sugawara energy operator, cf. Section A.2, and particularly Propositions A.2.3 and A.2.4. Looking at the lowest energy states in (5.8), we get the analogous sequence for $\mathfrak{g}$, i.e., the map is nonzero if and only if Λ satisfies the conditions of (5.5). Comparing with (5.6) and (5.7) gives the result. $\qquad\square$

Remark 5.2.2. Arakawa also shows that Ψ_s sends dual Verma modules to dual Verma modules. However, as these may be characterized up to isomorphism by (i) having the same Jordan-Hölder content as the corresponding Verma module, and (ii) admitting no maps from other Verma modules, this follows from (5.6).

6. The localization theorem

Having developed some properties of the adolescent Whittaker filtrations on the geometric and representation-theoretic sides of the localization theorem, we are ready to turn to its proof.

6.1. **The fully faithful embedding.** As discussed in the introduction, the existence of a fully faithful embedding follows combining (i) a sufficiently robust form of Kashiwara–Tanisaki localization, which incorporates renormalized DG categories and categorical loop group actions with (ii) the relation between $\mathcal{W}_\kappa$ representations and the Whittaker model of Kac–Moody representations established by the second named author in [Ras16b].

6.1.1. *Kashiwara–Tanisaki localization.* Let us state the form of (i) which we will need, and along the way fix convenient normalizations for discussion of monodromy.

6.1.2.   Recall the Iwahori subgroup $I^- \subset \mathrm{Ad}_{t^{-\check\rho}} G_O$. Let us present the affine flag variety as

$$\mathrm{Fl} = G_F/I^-.$$

It is well known that the twisting κ admits a unique trivialization on $\mathrm{Ad}_{t^{-\check\rho}} G_O$, and we use this to trivialize its restriction to I^-. In particular, for an element $\lambda \in \mathfrak{t}^*$ in the dual abstract Cartan, we may form

$$D_{\kappa,\lambda-mon}(\mathrm{Fl}) := D_\kappa(G_F)^{I^-,-\lambda-mon}.$$

Explicitly, this is the full subcategory of $D_\kappa(G_F)$ generated under shifts and colimits by the essential image of the forgetful map from the equivariant category

$$\mathrm{Oblv} : D_\kappa(G_F)^{I^-,-\lambda} \to D_\kappa(G_F).$$

6.1.3.   As in Section 7.11 of [Ras20] (and following [BD]), we *define* a global sections functor

$$(6.1) \qquad\qquad \Gamma(\mathrm{Fl},-) : D_{\kappa,\lambda-mon}(\mathrm{Fl}) \to \widehat{\mathfrak{g}}_\kappa\text{--mod}$$

as convolution with the *monodromic Verma module*

$$\mathbb{M}_{\widehat{\lambda}} \in \widehat{\mathfrak{g}}_\kappa\text{--mod}^{I^-,\lambda-mon}.$$

This functor is manifestly $D_\kappa(G_F)$-equivariant.

6.1.4.   We may now state the relevant form of localization for Kac–Moody representations.

Theorem 6.1.5 (Kashiwara-Tanisaki localization). *If λ is regular of level κ, then*

$$\Gamma(\mathrm{Fl}, -) : D_{\kappa, \lambda-mon}(\mathrm{Fl}) \to \widehat{\mathfrak{g}}_\kappa\text{–mod}$$

is fully faithful. Its essential image on $\mathring{I}$-equivariant objects is the subcategory of $\widehat{\mathfrak{g}}_\kappa$–mod compactly generated by Verma modules

$$\mathbb{M}_{\kappa, \theta} \qquad for \quad \theta \in W \cdot \lambda.$$

Moreover, if λ is antidominant of level κ, the functor $\Gamma(\mathrm{Fl}, -)$ is t-exact.

Remark 6.1.6. Using Section A.10 of [Ras20], one can show that $\Gamma(\mathrm{Fl}, -)$ coincides with the usual global sections functor considered in [KT95] and [FG09b]. Therefore, for antidominant λ, one can deduce Theorem 6.1.5 from these previous works. Alternatively, a direct proof is given in the work of J. Campbell and the first author on affine Harish–Chandra bimodules [CD20].

6.1.7.   Having stated Theorem 6.1.5, we may obtain from it a geometric construction of $\mathcal{W}$-algebra representations as follows. For any twist λ, let

(6.2) $$\Gamma(\mathrm{Fl}, -) : \mathsf{Whit}_{\kappa, \lambda-mon}(\mathrm{Fl}) \to \mathcal{W}_\kappa\text{–mod}$$

denote the functor obtained from (6.1) by passing to Whittaker invariants and using the affine Skryabin equivalence

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) \simeq \mathcal{W}_\kappa\text{–mod}$$

of [Ras16b].

Theorem 6.1.8. *If λ is regular of level κ, this functor is fully faithful. If λ is antidominant of level κ, then this functor is t-exact.*

Proof. Fully faithfulness follows formally from Theorem 6.1.5 using prounipotence of N_F.

For t-exactness, note that by construction the insertions

$$\mathsf{Whit}^{\leqslant 1}(D_{\kappa, \lambda-mon})(\mathrm{Fl}) \to \mathsf{Whit}(D_{\kappa, \lambda-mon}) \qquad \text{and} \qquad \mathsf{Whit}^{\leqslant 1}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) \to \mathsf{Whit}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$$

are t-exact. We saw in Corollary 3.0.2 that the left-hand map is an equivalence. Therefore, it is enough to check the t-exactness on baby Whittaker models. Here the assertion clear, as their forgetful functors to the ambient categories are t-exact. $\square$

Remark 6.1.9. One can directly show that (6.2) preserves compact objects for all λ. Indeed, this follows for (6.1) using [Ras20] Lemma 7.12.1. The similar result follows immediately for invariants for any prounipotent compact open subgroup of $G(K)$. We now obtain the claim for (6.2) using the baby Whittaker construction and the argument above.

6.2. **Identification of compact generators.** Our next task is to produce compact generators for the essential image.

6.2.1. *Geometric preliminaries.* Consider the Bruhat decomposition, i.e. the stratification of G_F with strata the Schubert cells

$$C_w \subset G_F, \qquad \text{for} \quad w \in W.$$

A convenient indexing for Schubert cells in our situation is

$$C_w := \mathrm{Ad}_{t^{-\check{\rho}} w_\circ} IwI, \qquad \text{for} \quad w \in W.$$

In particular, the closure relation between strata is the usual Bruhat order on W.

6.2.2.   It is standard that a cell C_w supports baby Whittaker sheaves, i.e. that $D_\kappa(C_w)^{\mathring{I},\psi \times I^-,-\lambda}$ is nontrivial if and only if w is of minimal length in $W_{\mathrm{f}} w$. In the latter case, it canonically identifies with Vect, and we write

$$j_{w,!}^\psi, \quad j_{w,!*}^\psi, \quad \text{and} \quad j_{w,*}^\psi, \qquad \text{for} \quad w \in W_{\mathrm{f}} \backslash W,$$

for the corresponding standard, simple and costandard objects of $D_{\kappa,\lambda-mon}(\mathrm{Fl})^\heartsuit$.

For any $w \in W$, similarly the category associated with a single stratum

$$D_\kappa(C_w)^{\mathring{I}^- \times I^-,-\lambda}$$

identifies with Vect, and we write

$$j_{w,!}, \quad j_{w,!*}, \quad \text{and} \quad j_{w,*}, \qquad \text{for} \quad w \in W,$$

for the corresponding standard, simple and costandard objects. If we vary the twist λ, we will use the same notations for the similar objects of the baby Whittaker and Iwahori-constructible categories.

6.2.3.   We will use the following standard assertion about averaging between these two categories.

Proposition 6.2.4. *Consider the averaging functor*

$$\mathrm{Av} := \mathrm{Av}_{I_1,\psi,*}[\dim_N] : D_\kappa(I_1^-\backslash G_F/I^-,-\lambda) \to D_\kappa(I_1,\psi\backslash G_F/I^-,-\lambda).$$

Then for any $w \in W$, if we again denote by w its image in $W_f\backslash W$, one has isomorphisms:

$$(6.3) \qquad\qquad \mathrm{Av}\, j_{w,!} \simeq j_{w,!}^\psi, \qquad \mathrm{Av}\, j_{w,*} \simeq j_{w,*}^\psi.$$

In the case when λ and κ are both trivial, this is well known, cf. Section 2 of Arkhipov–Bezrukavnikov [AB09].

Proof. Consider for $w \in W$ the convolution map

$$- \overset{I^-}{\star} - : D_\kappa(G_F/I^-,-w\lambda) \otimes D_\kappa(I^-,w\lambda\backslash G_F/I^-,-\lambda) \to D_\kappa(G_F/I^-,-\lambda).$$

It was shown in the proof of Theorem 3.7 of [CDR], which in turn adapted an argument of [LY19], that this satisfies

$$j_{e,*}^\psi \overset{I^-}{\star} j_{w,*} \simeq j_{w,*}^\psi \quad \text{and} \quad j_{e,*}^\psi \overset{I^-}{\star} j_{w,!} \simeq j_{w,!}^\psi.$$

Similarly one may show that

$$j_{e,*} \overset{I^-}{\star} j_{w,*} \simeq j_{w,*} \quad \text{and} \quad j_{e,*} \overset{I^-}{\star} j_{w,!} \simeq j_{w,!}.$$

Using these, by the commutation of Av and right convolution, we may reduce to the case of $w = e$, which in turn is a standard calculation on the finite flag variety. $\qquad\square$

6.2.5.   We are ready to describe a set of compact generators of the image. Recall the Harish–Chandra homomorphism

$$\pi : \mathfrak{t}^* \to W_{\mathrm{f}}\backslash\mathfrak{t}^*.$$

Theorem 6.2.6. *For a twist λ which is regular of level κ, the essential image of the fully faithful embedding*

$$\Gamma(\mathrm{Fl},-) : \mathsf{Whit}_{\kappa,\lambda-mon}(\mathrm{Fl}) \to \mathcal{W}_\kappa\text{--mod}$$

is compactly generated by the Verma modules

$$M_\chi, \qquad \text{for} \quad \chi \in \pi(W \cdot \lambda).$$

Proof. By construction, we have a commutative diagram

$$
\begin{array}{ccc}
D_{\kappa,\lambda-mon}(\mathrm{Fl})^{\mathring{I}^-} & \xrightarrow{\ \Gamma(\mathrm{Fl},-)\ } & \widehat{\mathfrak{g}}_\kappa\text{--mod}^{\mathring{I}^-} \\
{\scriptstyle\mathrm{Av}}\downarrow & & \downarrow{\scriptstyle\mathrm{Av}} \\
\mathsf{Whit}_{\kappa,\lambda-mon}^{\leqslant 1}(\mathrm{Fl}) & \xrightarrow{\ \Gamma(\mathrm{Fl},-)\ } & \mathsf{Whit}^{\leqslant 1}(\widehat{\mathfrak{g}}_\kappa\text{--mod}).
\end{array}
$$

By Proposition 6.2.4, the left functor generates under colimits. Therefore, it suffices to compute the category generated under colimits by the essential image of the induced functor:

$$D_{\kappa,\lambda-mon}(\mathrm{Fl})^{\mathring{I}^-} \to \mathsf{Whit}^{\leqslant 1}(\widehat{\mathfrak{g}}_\kappa-\mathsf{mod}).$$

By the diagram and Theorem 6.1.5, this is the subcategory generated by objects

$$\mathrm{Av}(\mathbb{M}_{\kappa,\theta}), \qquad \text{for} \quad \theta \in W \cdot \lambda.$$

By the proof of Theorem 5.2.1, we have $\mathrm{Av}(\mathbb{M}_{\kappa,\theta}) = M_{\pi(\theta)}$, giving the claim. $\square$

6.3. **The abelian equivalence at negative level.** In the remainder of this section, we specialize to the case of λ regular and antidominant of level κ. Recall we have already seen in this case that $\Gamma(\mathrm{Fl}, -)$ is t-exact. As we will show, it induces an equivalence between the abelian category of Whittaker sheaves and a direct sum of blocks of Category $\mathcal{O}$.

To simplify notation, let us denote the relevant Verma, simple, and dual Verma modules for $\mathcal{W}_\kappa$ by

$$M_w := M_{\pi(w\cdot\lambda)}, \quad L_w := L_{\pi(w\cdot\lambda)}, \quad \text{and} \quad A_w := A_{\pi(w\cdot\lambda)}, \qquad \text{for} \quad w \in W_{\mathrm{f}}\backslash W.$$

We will also use the integral Weyl group of λ, which we denote by W_λ; cf. Section 2.1.5 for more discussion. With these preparations, we may state the remaining theorem of this section.

Theorem 6.3.1. *For λ antidominant of level κ, the global sections functor induces an equivalence*

$$\Gamma(\mathrm{Fl}, -) : \mathsf{Whit}_{\kappa,\lambda-mon}(\mathrm{Fl})^{\heartsuit} \simeq \bigoplus_{w \in W_{\mathrm{f}}\backslash W/W_\lambda} \mathcal{O}_w^{loc}$$

where $\mathcal{O}_w^{loc}$ is an indecomposable summand of $\mathcal{O}^{loc}$ generated under extensions and colimits by the simple objects

$$L_y, \qquad \text{for} \quad y \in W_f\backslash W_f w W_\lambda.$$

Moreover, this equivalence exchanges standard, simple, and costandard objects, i.e.

(6.4) $\qquad \Gamma(\mathrm{Fl}, j_{w,!}^\psi) \simeq M_w, \quad \Gamma(\mathrm{Fl}, j_{w,!*}^\psi) \simeq L_w, \quad \text{and} \quad \Gamma(\mathrm{Fl}, j_{w,*}^\psi) \simeq A_w, \qquad \text{for} \quad w \in W_{\mathrm{f}}\backslash W.$

We now make two remarks before beginning the proof.

Remark 6.3.2. It is straightforward to deduce Theorem 6.3.1 using Arakawa's Theorem 5.2.1, by an argument similar to that of Theorem 6.2.6. However, we would like to provide an independent proof of the character formulas at negative level. For this reason, below we will only use the easy fact that the Drinfeld–Sokolov reduction Ψ_s sends Verma modules to Verma modules.

Remark 6.3.3. Passing to 'small' categories, an immediate consequence of Theorem 6.3.1 is that global sections induces an equivalence

$$\Gamma(\mathrm{Fl}, -) : \mathsf{Whit}_{\kappa,\lambda-mon}(\mathrm{Fl})^{\heartsuit, f.l.} \simeq \bigoplus_{w \in W_{\mathrm{f}}\backslash W/W_\lambda} \mathcal{O}_w,$$

where on the left-hand side we mean the finite length (equivalently, compact) objects of the heart and on the right-hand side we mean the corresponding blocks of $\mathcal{O}$. We remind the reader that while a general block of $\mathcal{O}$ contains objects of infinite length, this will not occur for the blocks obtained in our localization theorem at negative level (and indeed, this is a consequence of the theorem).

Proof of Theorem 6.3.1. For ease of notation, we below set $\Gamma(-) := \Gamma(\mathrm{Fl}, -)$. A basic property of $\Gamma(-)$ under our assumption on λ is that one has

$$\Gamma(j_{w,!}) \simeq \mathbb{M}_{\kappa,w\cdot\lambda} \in \widehat{\mathfrak{g}}_\kappa-\mathsf{mod}^{\mathring{I}^-}, \qquad \text{for} \quad w \in W.$$

Therefore, as in the proof of Theorem 6.2.6, we have

$$\Gamma(j_{w,!}^\psi) \simeq \Gamma(\mathrm{Av}\, j_{w,!}) \simeq \mathrm{Av}\,\Gamma(j_{w,!}) \simeq \mathrm{Av}\,\mathbb{M}_{\kappa,w\cdot\lambda} \simeq M_w,$$

which establishes the claim of (6.4) for standard objects. As a consequence, we deduce that the essential image of $\Gamma(-)$ lies in $\mathcal{O}^{loc}$.

We now turn to the claim for simple objects. We claim it is enough to show that

$$(6.5) \qquad \operatorname{Hom}_{\mathcal{W}_\kappa-\mathsf{mod}^\heartsuit}(M_\chi, \Gamma(j_{w,!*}^\psi)) \simeq 0 \qquad \text{for} \quad M_\chi \neq M_w.$$

Indeed, the tautological map $j_{w,!} \twoheadrightarrow j_{w,!*}$ yields a map $M_w \twoheadrightarrow \Gamma(j_{w,!*}^\psi)$. In particular, we obtain an exact sequence

$$0 \to K \to \Gamma(j_{w,!*}^\psi) \to L_w \to 0.$$

Applying (6.5), we deduce that K vanishes, as desired.

We now prove (6.5). From what we have already shown, it is enough to argue that, for any $\chi \notin \pi(W \cdot \lambda)$, one has

$$\operatorname{Hom}_{\mathcal{W}_\kappa-\mathsf{mod}^\heartsuit}(M_\chi, \Gamma(j_{w,!*}^\psi)) \simeq 0.$$

In fact we claim that for such a χ and any object N of $\mathsf{Whit}_{\kappa,\lambda-mon}(\mathrm{Fl})$ we have the vanishing

$$(6.6) \qquad \operatorname{Hom}_{\mathcal{W}_\kappa-\mathsf{mod}}(M_\chi, \Gamma(N)) \simeq 0.$$

By the compactness of M_χ, it is enough to show the vanishing for $N = M_y$, for any $y \in W_\mathrm{f} \backslash W$. To see this, pick a lift $\tilde\chi$ of χ, and use

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{W}_\kappa-\mathsf{mod}}(M_\chi, M_y) &\simeq \operatorname{Hom}_{\widehat{\mathfrak{g}}_\kappa-\mathsf{mod}^{\check{I},\psi}}(\Delta_{\tilde\chi}, \Delta_y) \\
&\simeq \operatorname{Hom}_{\widehat{\mathfrak{g}}_\kappa-\mathsf{mod}^{\check{I},\psi}}(\operatorname{Av} \mathbb{M}_{\kappa,\tilde\chi}, \operatorname{Av} \mathbb{M}_{\kappa,y\cdot\lambda}) \\
&\simeq \operatorname{Hom}_{\widehat{\mathfrak{g}}_\kappa-\mathsf{mod}^{I^-,\tilde\chi}}(\mathbb{M}_{\kappa,\tilde\chi}, \operatorname{Av}^R \operatorname{Av} \mathbb{M}_{\kappa,y\cdot\lambda})
\end{aligned}$$

The vanishing is now clear, as the functor $\operatorname{Av}^R \operatorname{Av}$ is given by convolution with an object of

$$D_\kappa(I^-, \tilde\chi \backslash G_F / I^-, -y \cdot \lambda) \simeq 0,$$

where the vanishing of the appearing category follows from our assumption on χ. Having proven (6.4) for standard and simple objects, the costandard objects follow by the argument of Remark 5.2.2.

Finally, it remains to show the essential image is a sum of blocks. However, we claim this follows from (6.6). Indeed, $\mathcal{O}^{loc}$ lies in the subcategory of $\mathcal{W}_\kappa-\mathsf{mod}$ compactly generated by the Verma modules M_χ, cf. Proposition A.2.11, and (6.6) implies this decomposes as the sum of the subcategory generated by M_w, for $w \in W$, and the subcategory generated by the remaining Verma modules

$$M_\chi, \qquad \text{for} \quad \chi \notin \pi(W \cdot \lambda).$$

By their indecomposability, the simple modules of L_χ, for $\chi \notin \pi(W \cdot \lambda)$, must lie in the latter category, as they admit a nontrivial map from M_χ. In particular, for any $w \in W$, we deduce the desired vanishing

$$\operatorname{Ext}^1_{\mathcal{W}_\kappa-\mathsf{mod}}(L_w, L_\chi) \simeq \operatorname{Ext}^1_{\mathcal{W}_\kappa-\mathsf{mod}}(L_\chi, L_w) \simeq 0.$$

Having shown the essential image is a sum of blocks, it remains to describe its own decomposition into blocks. However, this was determined in Theorem 3.7 of [CDR], completing the proof. $\square$

Remark 6.3.4. One can show that $\mathsf{Whit}_{\kappa,\lambda-mon}(\mathrm{Fl})$ is the renormalized derived category of its heart. Namely, its bounded below part canonically identifies with the bounded below derived category of its heart, and in general the canonical map

$$D^b(\mathsf{Whit}_{\kappa,\lambda-mon}(\mathrm{Fl})^{\heartsuit,f.l.}) \to \mathsf{Whit}_{\kappa,\lambda-mon}$$

exhibits the latter as the ind-completion of the former. It follows from this, combined with Theorems 6.1.8 and 6.3.1, that the subcategory of $\mathcal{W}_\kappa-\mathsf{mod}$ generated by the Verma modules $M_w, w \in W$, identifies with the renormalized derived category of its heart. This partially adresses a question raised in [Dhi].

Appendix A. Energy gradings in the baby Whittaker model

It follows from Theorem 5.1.4 that $\mathcal{O}$ is equivalent to subcategory of $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{\check{I},\psi,Z_f,\heartsuit}$. In this appendix we will explicitly characterize this subcategory by a positive energy condition. To account for the conjugation by $t^{-\check{\rho}}$, we as usual replace the Segal–Sugawara energy operator with the energy operator of a different conformal vector, as we explain next.

A.1. **Spectral flow and the Sugawara construction.** Let $\check{\lambda}$ be an integral coweight, i.e., a cocharacter of the adjoint torus. One has a corresponding spectral flow automorphism $\mathrm{Ad}_{t^{\check{\lambda}}}$ of $\widehat{\mathfrak{g}}_\kappa$. Explicitly, for $X \in \mathfrak{g}$, write X_n for the operator $X \otimes t^n$ of $\widehat{\mathfrak{g}}_\kappa$ and consider the corresponding field

$$X(z) = \sum_n X_n z^{-n-1}.$$

Using the root decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, the fields transform as

$$\mathrm{Ad}_{t^{\check{\lambda}}} h(z) = h(z) + \frac{\kappa(h, \check{\lambda})}{z}, \qquad \text{for } h \in \mathfrak{t},$$

$$\mathrm{Ad}_{t^{\check{\lambda}}} X_\alpha(z) = z^{\langle \alpha, \check{\lambda} \rangle} X_\alpha(z), \qquad \text{for } X_\alpha \in \mathfrak{g}_\alpha.$$

This induces an automorphism $\mathrm{Ad}_{t^{\check{\lambda}}}$ of the completed enveloping algebra $U_\kappa(\widehat{\mathfrak{g}}_\kappa)$, which we denote by the same symbol.

Remark A.1.1. Note that our $\mathrm{Ad}_{t^{\check{\lambda}}}$, which is compatible with the conventions for the adolescent Whittaker filtration, differs from that of Arakawa and Frenkel–Kac–Wakimoto by a sign, i.e., matches their $\mathrm{Ad}_{z^{-\check{\lambda}}}$.

A.1.2. Consider the usual Segal–Sugawara field $S(z) = \sum S_n z^{-n-2}$ of the Kac–Moody vacuum algebra $\mathbb{V}_\kappa$. I.e., writing J_a for a basis of $\mathfrak{g}$, and J^a for its dual basis with respect to κ, we have

$$S(z) = \frac{\kappa}{2(\kappa - \kappa_c)} \sum_a :J_a(z) J^a(z): \,.$$

We presently determine how it transforms under $\mathrm{Ad}_{t^{\check{\lambda}}}$.

Proposition A.1.3. *We have the equality of fields*

$$\mathrm{Ad}_{t^{\check{\lambda}}} S(z) = S(z) + \frac{\check{\lambda}(z)}{z} + \frac{\frac{1}{2}\kappa(\check{\lambda}, \check{\lambda})}{z^2}.$$

Proof. We must show, for any $n \in \mathbb{Z}$, the equality

$$\mathrm{Ad}_{t^{\check{\lambda}}} S_n = S_n + \check{\lambda}_n + \delta_{n,0} \frac{1}{2} \kappa(\check{\lambda}, \check{\lambda}).$$

We will explain the case of $n = 0$, and the others follow by a simpler variant of the same argument. Let h_i, for $i \in I$, be an orthonormal basis for $\mathfrak{t}$ with respect to κ, and for each positive root $\alpha > 0$ fix vectors e_α in $\mathfrak{g}_\alpha$ and f_α in $\mathfrak{g}_{-\alpha}$ with $\kappa(e_\alpha, f_\alpha) = 1$. Denoting again by α its image under $\kappa : \mathfrak{t}^* \simeq \mathfrak{t}$, one has $[e_\alpha, f_\alpha] = \alpha$. We have

$$\mathrm{Ad}_{t^{\check{\lambda}}} S_0 = \frac{\kappa}{2(\kappa - \kappa_c)} \Big(\sum_{\alpha > 0} \sum_{n \in \mathbb{Z}} \mathrm{Ad}_{t^{\check{\lambda}}} :e_{\alpha,n} f_{\alpha,-n}: + \mathrm{Ad}_{t^{\check{\lambda}}} :f_{\alpha,n} e_{\alpha,-n}: \Big)$$

$$+ \frac{\kappa}{2(\kappa - \kappa_c)} \sum_{i \in I} \sum_{m \in \mathbb{Z}} \mathrm{Ad}_{t^{\check{\lambda}}} :h_{i,m} h_{i,-m}: \,.$$

Introducing commutators to return each sum to its normal order, and recalling that

$$\mathrm{Ad}_{t^{\check{\lambda}}} h_{i,n} = h_{i,n} + \delta_{n,0} \kappa(h_i, \check{\lambda}),$$

we may continue as

$$\frac{\kappa}{2(\kappa - \kappa_c)}\left(\sum_{\alpha > 0}(2\langle\alpha, \check{\lambda}\rangle\alpha_0 + \langle\alpha, \check{\lambda}\rangle^2 + \sum_{n\in\mathbb{Z}}:e_{\alpha,n}f_{\alpha,-n}: + :f_{\alpha,n}e_{\alpha,-n}:)\right)$$

$$+ \frac{\kappa}{2(\kappa - \kappa_c)}\left(\sum_{i\in I}(2\kappa(h_i, \check{\lambda})h_{i,0} + \kappa(h_i, \check{\lambda})^2 + \sum_{m\in\mathbb{Z}}:h_{i,m}h_{i,-m}:)\right).$$

$$= S_0 + \frac{\kappa}{2(\kappa - \kappa_c)}\left(\sum_{\alpha>0}2\langle\alpha, \check{\lambda}\rangle\alpha_0 + \langle\alpha, \check{\lambda}\rangle^2 + \sum_{i\in I}2\kappa(h_i, \check{\lambda})h_{i,0} + \kappa(h_i, \check{\lambda})^2\right).$$

Recalling that κ_c is minus one half of the Killing form, we may sum over α and i to obtain that the above equals

$$S_0 + \frac{\kappa}{2(\kappa - \kappa_c)}\left(-\frac{2\kappa_c}{\kappa}\check{\lambda}_0 - \kappa_c(\check{\lambda}, \check{\lambda}) + 2\check{\lambda}_0 + \kappa(\check{\lambda}, \check{\lambda})\right)$$

$$= S_0 + \check{\lambda}_0 + \frac{1}{2}\kappa(\check{\lambda}, \check{\lambda}).$$

For the case of $n \neq 0$, one similarly obtains

$$\mathrm{Ad}_{t^{\check{\lambda}}} S_n = S_n + \frac{\kappa}{2(\kappa - \kappa_c)}\left(\sum_{\alpha>0}2\langle\alpha, \check{\lambda}\rangle\alpha_n + \sum_{i\in I}2\kappa(h_i, \check{\lambda})h_{i,n}\right) = S_n + \check{\lambda}_n$$

as desired. $\square$

A.1.4.   Let us use this to determine how $\mathrm{Ad}_{t^{\check{\lambda}}}$ interacts with positive energy conditions on representations. If V is a representation of $\mathrm{Ad}_{t^{\check{\lambda}}} \mathfrak{g}$, consider the parabolic induction $\mathrm{pind}_{\mathrm{Ad}_{t^{\check{\lambda}}} \mathfrak{g}}^{\widehat{\mathfrak{g}}_\kappa} V$. It is straightforward to see that the action of $\mathrm{Ad}_{t^{\check{\lambda}}} S_0$ preserves the image of the unit $V \to \mathrm{pind}_{\mathrm{Ad}_{t^{\check{\lambda}}} \mathfrak{g}}^{\widehat{\mathfrak{g}}_\kappa} V$, and acts on it via the element

$$\frac{\kappa}{2(\kappa - \kappa_c)}\Omega_\kappa,$$

where Ω_κ is the Casimir operator of $\mathrm{Ad}_{t^{\check{\lambda}}} \mathfrak{g}$ defined with respect to κ. On the other hand, if we consider the deformed conformal field $S^{\check{\lambda}}(z) := S(z) - \partial_z\check{\lambda}(z)$, this again generates a Virasoro algebra with energy operator

$$S_0^{\check{\lambda}} = S_0 + \check{\lambda}_0.$$

Using Proposition A.1.3 to compare the actions of $S_0^{\check{\lambda}}$ and $\mathrm{Ad}_{t^{\check{\lambda}}} S_0$, we obtain the following:

Corollary A.1.5. *If V is a representation of $\mathrm{Ad}_{t^{\check{\lambda}}} \mathfrak{g}$, then $S_0^{\check{\lambda}}$ acts the image of the unit $V \to \mathrm{pind}_{\mathrm{Ad}_{t^{\check{\lambda}}} \mathfrak{g}}^{\widehat{\mathfrak{g}}_\kappa} V$ via the operator of $U(\mathrm{Ad}_{t^{\check{\lambda}}} \mathfrak{g})$ given by*

$$\frac{\kappa}{2(\kappa - \kappa_c)}\Omega_\kappa - \frac{1}{2}\kappa(\check{\lambda}, \check{\lambda}).$$

In particular, if Ω_κ acts locally finitely on V, then $S_0^{\check{\lambda}}$ acts locally finitely on $\mathrm{pind}_{\mathrm{Ad}_{t^{\check{\lambda}}} \mathfrak{g}}^{\widehat{\mathfrak{g}}_\kappa} V$.

A.2. **Identifying Category $\mathcal{O}$.** Specializing the discussion of the previous subsection to $\check{\lambda} = -\check{\rho}$, consider the Kazhdan–Kostant field $S(z) + \partial_z\check{\rho}(z)$ and write $L_0 := S_0^{-\check{\rho}}$ for its energy operator.

Definition A.2.1. The category $\mathcal{O}'$ is the full subcategory of $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}^{\check{I},\psi,\heartsuit}$ consisting of objects M satisfying

   (1) Under the action of L_0, M decomposes into a sum of generalized eigenspaces

$$M = \bigoplus_{d\in\mathbb{C}} M_d, \quad M_d := \{m \in M : (L_0 - d)^N m = 0, N \gg 0\}.$$

   (2) For each $d \in \mathbb{C}$, M_d is finite length as an $\mathrm{Ad}_{t^{-\check{\rho}}} \mathfrak{g}$ module and M_{d-n} is nonzero for only finitely many $n \in \mathbb{Z}^{\geqslant 0}$.

For an object M of $\mathcal{O}'$, if we write $\ell(M_d)$ for the length of a generalized eigenspace as an $\mathrm{Ad}_{z-\check\rho}\,\mathfrak{g}$ module, then define its character to be

$$\mathrm{ch}\,M := \sum_{d\in\mathbb{C}} \ell(M_d) q^d.$$

In the remainder of this appendix, we first establish some basic properties of $\mathcal{O}'$. We then apply them to prove Theorem A.2.9, which shows that Drinfeld–Sokolov reduction identifies $\mathcal{O}$ with $\mathcal{O}'$, and moreover that the notions of character on either side are intertwined, up to multiplication by a universal q-series.

A.2.2.   We first show that Verma modules belong to our category.

Proposition A.2.3. *For any character χ of Z, the Verma module Δ_χ belongs to $\mathcal{O}'$ and we have*

$$\mathrm{ch}\,\Delta_\chi = q^{\frac{\kappa}{2(\kappa-\kappa_c)}\Omega_\kappa(\chi) - \frac12\kappa(\check\rho,\check\rho)} \prod_{i=1}^{\infty} \frac{1}{(1-q^i)^{\dim\mathfrak{g}}}.$$

Proof. As in the proof of Lemma 5.1.5, consider the associated graded of the PBW filtration

$$\mathrm{Sym}(\widehat{\mathfrak{g}}_\kappa / \mathrm{Ad}_{t-\check\rho}\,\mathfrak{g}_O) \otimes V_\chi$$

The energy of the zeroth associated graded is $\frac{\kappa}{2(\kappa-\kappa_c)}\Omega_\kappa(\chi) - \frac12\kappa(\check\rho,\check\rho)$ by Corollary A.1.5. Recalling that V_χ is simple, and that for a finite dimensional representation L of $\mathrm{Ad}_{z-\check\rho}\,\mathfrak{g}$, the tensor product $L \otimes V_\chi$ has length $\dim L$, the conclusion follows. $\square$

Let us next record the usual relationship between Verma modules and simple modules, and the usual variant of Jordan-Hölder series in the setting of infinite dimensional Lie algebras.

Proposition A.2.4. *For any character χ of Z, the Verma module Δ_χ has a unique simple quotient L_χ. Moreover, the assignment of L_χ to χ yields a bijection between central characters and isomorphism classes of simple objects in $\mathcal{O}'$.*

Proof. For the first assertion, by the simplicity of V_χ, it follows that a submodule of M_χ is proper if and only if its intersection with the lowest energy eigenspace is trivial. In particular, as every submodule is again L_0 locally finite, there exists a maximal submodule, and hence a unique simple quotient.

Given an arbitrary simple object L, it contains a lowest energy subspace, i.e., choose a $d \in \mathbb{C}$ with L_d nonzero and L_{d-n} zero for every $n \in \mathbb{Z}^{>0}$. As L_d is finite length, we may choose an embedding $V_\chi \to L_d$ of $\mathrm{Ad}_{t-\check\rho}\,\mathfrak{g}$ modules, which induces an equivalence $L_\chi \simeq L$. To see that the L_χ, as χ ranges over central characters, are mutually nonisomorphic, note that they are distinguished by the action of $\mathrm{Ad}_{z-\check\rho}\,\mathfrak{g}$ on their unique lowest energy subspaces. $\square$

A.2.5.   Let us record two further propositions.

Proposition A.2.6. *Given any scalar $d \in \mathbb{C}$, and any object M of $\mathcal{O}'$, it admits a finite filtration*

$$0 = M^0 \subset M^1 \subset M^2 \subset \cdots \subset M^n = M,$$

where each successive quotient M^i/M^{i-1} is either simple, or has no nontrivial subspaces with energy in $d + \mathbb{Z}^{\leqslant 0}$.

Proposition A.2.7. *Any object M of $\mathcal{O}'$ admits an exhaustive filtration*

$$0 = M^0 \subset M^1 \subset M^2 \subset \cdots, \qquad \underset{n}{\mathrm{colim}}\,M^n \simeq M,$$

where for each $n > 0$, the successive quotient M^n/M^{n-1} is a direct sum of highest weight modules, i.e., quotients of Verma modules.

As in Proposition A.2.4, one may prove these by adapting the usual arguments, cf. Lemma 9.6 of [Kac90] and Proposition 6.8 of [Dhi], to our setting. I.e., one considers lowest energy eigenspaces in lieu of highest weight spaces, and uses their finite length in lieu of their finite dimensionality.

A.2.8.   With these preparations, we are ready to prove the main result of this appendix. Recall our notation Ψ_s for the cohomologically shifted Drinfeld–Sokolov reduction Ψ, namely

$$\Psi_s = \Psi[-2\langle \check{\rho}, \rho \rangle].$$

Theorem A.2.9. *The composition* $\mathcal{O}' \to \widehat{\mathfrak{g}}_\kappa\text{–mod}^\heartsuit \xrightarrow{\ \Psi_s\ } \mathcal{W}_\kappa\text{–mod}$ *induces an equivalence*

$$\mathcal{O}' \simeq \mathcal{O}$$

Moreover, for object N of $\mathcal{O}'$, its character and that of its Drinfeld–Sokolov reduction are related by

(A.1)
$$\operatorname{ch} \Psi_s N = q^{\frac{1}{2}\kappa_c(\check{\rho},\check{\rho}) + \langle \rho, \check{\rho}\rangle} \prod_{i=1}^{\infty} (1 - q^i)^{\dim \mathfrak{g} - \operatorname{rk} \mathfrak{g}} \operatorname{ch} N.$$

Proof. We begin by proving (A.1), and start with the Verma modules. Recall that the character of the Verma module M_χ for $\mathcal{W}_\kappa$ is given by

(A.2)
$$\operatorname{ch} M_\chi = q^{\frac{\kappa}{2(\kappa - \kappa_c)} \Omega_\kappa(\chi) - \frac{1}{2}(\kappa - \kappa_c)(\check{\rho},\check{\rho}) + \langle \rho, \check{\rho}\rangle} \prod_{i=1}^{\infty} \frac{1}{(1 - q^i)^{\operatorname{rk} \mathfrak{g}}}.$$

While this is standard, let us quickly explain how to see the energy of its lowest energy eigenspace. Let us calculate $\Psi(\Delta_\chi)$ using the standard BRST complex. Recall the underlying vector space of the latter is $\Delta_\chi \otimes \mathcal{F}$, where $\mathcal{F}$ is the fermionic ghost vertex algebra associated to $\mathfrak{n}$. If we write $|\chi\rangle$ for the canonical generating line of Δ_χ, and $|0\rangle$ for the line spanned by the vacuum state of $\mathcal{F}$, the lowest energy line of M_χ is the image in cohomology of the line

$$|\chi\rangle \otimes \det(\operatorname{Ad}_{t^{-\check{\rho}}} \mathfrak{n}_O/\mathfrak{n}_O)|0\rangle.$$

Using Corollary A.1.5 to calculate the energy of the first tensor factor and considering the Kazhdan–Kostant grading of the determinant, namely $\frac{1}{2}\kappa_c(\check{\rho}, \check{\rho}) + \langle \rho, \check{\rho} \rangle$, yields the formula. Comparing (A.2) and Proposition A.2.3 yields (A.1) for Verma modules.

We now deduce the general case of (A.1) as follows. Fix $d \in \mathbb{C}/\mathbb{Z}$ and consider $\mathcal{O}'(d)$, the full subcategory of $\mathcal{O}'$ consisting of objects whose energy eigenvalues lie in the coset $d + \mathbb{Z}$. Similarly, consider $\mathcal{O}(d)$, the full subcategory of $\mathcal{O}$ consisting of objects whose energy eigenvalues lie in

$$d + \frac{1}{2}\kappa_c(\check{\rho}, \check{\rho}) + \langle \rho, \check{\rho} \rangle + \mathbb{Z}.$$

By the validity of Equation (A.1) for Verma modules and Proposition A.2.7, it follows that Ψ_s sends $\mathcal{O}'(d)$ into $\mathcal{O}(d)$. Moreover, if we pick a coset representative $D \in \mathbb{C}$ of d, and consider $\mathcal{O}'(D)$, the full subcategory of $\mathcal{O}'$ consisting of objects whose energy eigenvalues lie in $D + \mathbb{Z}^{\geqslant 0}$, and similarly define $\mathcal{O}(D)$, it follows that Ψ_s sends $\mathcal{O}'(D)$ into $\mathcal{O}(D)$. For any nonnegative integer $n \in \mathbb{Z}^{\geqslant 0}$, we obtain maps between the Serre quotients

$$\mathcal{O}'(D)/\mathcal{O}'(D + n) \to \mathcal{O}(D)/\mathcal{O}(D + n).$$

By Proposition A.2.6, the Grothendieck group of the left-hand category is freely generated as an abelian group by the classes of the Verma modules Δ_χ with lowest energy lying in

$$D, D + 1, \cdots, D + n - 1.$$

Similarly, by Proposition 6.4 of [Dhi] the Grothendieck group of the right-hand category is freely generated by the Verma modules M_χ with lowest energy in

$$D_s, D_s + 1 \cdots, D_s + n - 1,$$

where $D_s := D + \frac{1}{2}\kappa_c(\check{\rho}, \check{\rho}) + \langle \rho, \check{\rho} \rangle$. Note that the coefficients of

$$q^D, q^{D+1}, \cdots, q^{D+n-1}$$

in the character of any object of $\mathcal{O}'(D)$ depend only on its image in the left-hand Serre quotient, and similarly for the coefficients of the first n terms in the q-character of an object of $\mathcal{O}(D)$. It therefore follows from the Verma case that (A.1) holds for the first n terms of the characters of objects of $\mathcal{O}'(D)$. Since n and D

were arbitrary, it follows that (A.1) holds for objects of $\mathcal{O}'(d)$. As every object of $\mathcal{O}'$ is the direct sum of its maximal submodules lying in $\mathcal{O}'(d)$, for $d \in \mathbb{C}/\mathbb{Z}$, the general case of (A.1) follows.

Having shown Ψ_s sends $\mathcal{O}' \to \mathcal{O}$, we deduce from Theorem 5.1.4 that it is fully faithful with essential image closed under extensions and containing the highest weight modules. To see essential surjectivity, we will use an analog of Proposition A.2.7 for $\mathcal{O}$. Namely, for any object M of $\mathcal{O}$ we may choose an exhaustive filtration

$$M^0 \subset M^1 \subset \cdots,$$

whose successive quotients are sums of highest weight modules, cf. Proposition 6.8 of [Dhi]. As we may assume each summand lies in a fixed term of the associated graded lies in a distinct $\mathcal{O}(d)$, for $d \in \mathbb{C}/\mathbb{Z}$, it follows that each M^n lies in the essential image of $\mathcal{O}'$. By (A.1), so does the direct limit, i.e., M, completing the proof. $\square$

A.2.10. We finish by proving a statement used in showing the image of the localization functor is a sum of blocks at negative level, cf. the proof of Theorem 6.3.1.

Proposition A.2.11. *The abelian category $\mathcal{O}^{loc}$ belongs to the subcategory of $\mathcal{W}_\kappa$–mod compactly generated by the Verma modules.*

Proof. It suffices to show that any object of $\mathcal{O}'$ belongs to the subcategory of $\widehat{\mathfrak{g}}_\kappa$–mod$^{\mathring{I},\psi}$ generated by the Verma modules

$$\Delta_\chi, \qquad \text{for} \quad \chi \in W_\mathrm{f} \backslash \mathfrak{t}^*.$$

In the notation of the proof of Theorem A.2.9, we may further reduce to the case of an object N_0 of $\mathcal{O}'(D)$. By the proof of Lemma 5.1.6, we may find a short exact sequence

$$0 \to N_1 \to M_0 \to N_0 \to 0$$

where M_0 is a direct sum of finite successive extensions of Verma modules which lies in $\mathcal{O}'(D)$ and N_1 is an object of $\mathcal{O}'(D+1)$.

Iterating this, we obtain a complex

$$\cdots \to M_2 \to M_1 \to M_0$$

with each M_i in $\mathcal{O}'(D+i)$ and of the form above.

For any integer n, let us write $\sigma^{>n}M$ for its stupid truncation $M_{n-1} \to \cdots \to M_1 \to M_0$. Forming the colimit in $\widehat{\mathfrak{g}}_\kappa$–mod of the stupid truncations under the natural transition maps, it suffices to show that the obtained map $\mathrm{colim}_n \sigma^{>n}M \to N$ is an equivalence. To check the latter claim, it is enough to see that the obtained map

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_\kappa-\mathsf{mod}}(\mathrm{ind}_{\mathring{\mathfrak{i}}}^{\widehat{\mathfrak{g}}_\kappa}\mathbb{C}_\psi, \mathrm{colim}_n \sigma^{>n}M) \to \mathrm{Hom}_{\widehat{\mathfrak{g}}_\kappa-\mathsf{mod}}(\mathrm{ind}_{\mathring{\mathfrak{i}}}^{\widehat{\mathfrak{g}}_\kappa}\mathbb{C}_\psi, N)$$

is an equivalence.

To prove this last assertion, let us compute both sides using continuous Chevalley–Eilenberg cochain complexes. The left-hand side produces the direct sum totalization of the bicomplex with underlying vector space

$$\bigoplus_i M_i \otimes \mathrm{Sym}(\mathring{\mathfrak{i}}^*[-1]),$$

where the appearing dual is as a topological vector space. We therefore must show that that the natural augmentation to the continuous Chevalley–Eilenberg complex for N_0 is an equivalence. To see this note that by construction the energy operator L_0 acts locally finitely on the direct sum totalization, and acts on $M_i \otimes \mathrm{Sym}(\mathring{\mathfrak{i}}^*[-1])$ with energy at least $D+i$. Using this, we may safely conclude by applying the standard spectral sequence of a bicomplex, as its convergence is made clear by considering each generalized L_0 eigenspace separately. $\square$

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