ACYCLIC COMPLEXES AND 1-AFFINENESS

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ABSTRACT. This short note is an erratum to [Gai], correcting the proof of one of its main results. It includes some counterexamples regarding infinite-dimensional unipotent groups and affine spaces that may be of independent interest.

1.1. In [Gai] Theorem 2.4.5, one finds the claim that $\mathbb{A}^\infty := \operatorname{colim}_n \mathbb{A}^n$ is not 1-affine,[1] where we work relative to a field k of characteristic 0. The proof in *loc. cit.* is not correct, and the purpose of this note is to correct it.

Along the way, we also give some (possibly new) counterexamples on representations of $\mathbb{A}^\infty$ considered as additive group, and the formal completion of a pro-infinite dimensional affine space at the origin. These counterexamples will be a categorical level down, i.e., they will reveal pathological behavior for QCoh rather than ShvCat.

1.2. **What is the problem in** [Gai]**?** In §9.4.3 of *loc. cit.*, there is a claim that something is "easy to see," with no further explanation. In fact, the relevant claim is not true. This failure invalidates the argument about 1-affineness given in *loc. cit.*

1.3. **Structure of the argument.** There has only ever been one successful strategy to proving that a prestack is not 1-affine: showing that 1-affineness would imply some functor is co/monadic (usually by computing some tensor product of DG categories and applying the Beck-Chevalley formalism), and then showing that the relevant functor is not conservative.

We will exactly follow this strategy. Namely, in §1.4-1.7, we prove two results, Theorems 1 and 2, about the non-conservativeness of various functors. Then in §1.8, we deduce that $\mathbb{A}^\infty$ is not 1-affine.

The reader who is most invested in the non-1-affineness of $\mathbb{A}^\infty$ might prefer to read the statement of Theorem 1 and then skip ahead to 1.8, returning to read the proofs of the other results after seeing their application.

1.4. **Invariants and ind-unipotent groups.** Our first main result is the following.

Theorem 1. $\Gamma : \mathsf{QCoh}(\mathbb{B}\mathbb{A}^\infty) \to \mathsf{Vect}$ *is not conservative.*

In other words, the functor of invariants for ind-unipotent groups is not conservative, even though it is for unipotent groups.

Moreover, the construction will produce a non-zero representation in the heart of the[2] t-structure whose invariants vanish, and whose higher group cohomologies vanish as well. That is to say, this is an essential issue, not the sort resolved by some kind of renormalization procedure.

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[1]We refer to *loc. cit.* for the definition of this notion. We also follow the notational conventions of *loc. cit.*, so everything should be understood in the derived sense.

[2]Note that $\mathbb{B}\mathbb{A}^\infty$ is ind-smooth, in particular ind-flat, and therefore its QCoh indeed has an obvious t-structure.

1.5.   We will deduce Theorem 1 from the next result.

Let $A = k[t_1, t_2, \ldots]$, so $\mathrm{Spec}(A)$ is a pro-infinite dimensional affine space.[3]

Theorem 2. *There exists $0 \neq V \in A\text{–}\mathsf{mod}^\heartsuit$ with the properties that:*

- *Each t_i acts on V nilpotently (in fact, with square zero).*
- *$\mathrm{Tor}_j^A(V, k) = 0$ for all j, where k is equipped with the A-module structure where each t_i acts by zero.*

The proof can be found in §1.7.

Proof that Theorem 2 implies Theorem 1. Since each t_i acts on V nilpotently,[4] the induced action on $V^\vee \neq 0$ is also nilpotent. Moreover, all of the operators t_i acting on $V^\vee$ commute. Therefore, $V^\vee$ has a canonical structure as an object of $\mathsf{QCoh}(\mathbb{B}\mathbb{A}^\infty)^\heartsuit$, since this is the abelian category of a non-derived vector spaces equipped with $\mathbb{Z}^{>0}$-many locally nilpotent and pairwise commuting operators.

Finally, observe that:

$$H^j(\Gamma(\mathbb{B}\mathbb{A}^\infty, V^\vee)) = \mathrm{Ext}^j_{\mathsf{QCoh}(\mathbb{B}\mathbb{A}^\infty)}(k, V^\vee) = \mathrm{Tor}_j^A(k, V)^\vee = 0$$

since the complex computing these Exts is dual to the complex computing the Tors. This gives the claim.

$\square$

1.6. **Construction of V, and a heuristic.** We now construct V from Theorem 2, and explain why the relevant Tor_0 vanishes.

Namely, V has a basis v_S indexed by finite subsets $S \subseteq \mathbb{Z}^{>0}$, and we define $t_i \cdot v_S = v_{S \setminus \{i\}}$ if $i \in S$ and $t_i \cdot v_S = 0$ if $i \notin S$. Clearly t_i^2 acts by zero on V.

To see that $\mathrm{Tor}_0^A(V, k) = 0$, it suffices to see that there are no morphisms $f : V \to k$ in $A\text{–}\mathsf{mod}^\heartsuit$. For any such f, we should show that $f(v_S) = 0$ for all S as above. Since S is finite and $\mathbb{Z}^{>0}$ is infinite, we can find $i \notin S$. Then $t_i v_{S \cup \{i\}} = v_S$, so $f(v_S) = t_i f(v_{S \cup i}) = 0$ because each t_i acts on k by zero. (A similar argument also works for Tor_1.)

1.7.   We now show that the higher Tors vanish as well.

Proof of Theorem 2.

Step 1. First, we rewrite the representation V in a more conceptual way.

For each $n > 0$, let $I_n \subseteq A$ be the ideal generated by $t_1^2, \ldots, t_n^2$ and all t_i for $i > n$.

Then we claim $V = \mathrm{colim}_n A/I_n$, where the structure morphisms:

$$A/I_n \to A/I_{n+1}$$

are given by multiplication by t_{n+1}. Indeed, the relevant structure morphism sends A/I_n to V by mapping the "vacuum vector" $1 \in A/I_n$ to $v_{\{1,\ldots,n\}}$ in the right hand side, which is an injection whose image identifies with the subspace of V spanned by v_S for $S \subseteq \{1, \ldots n\} \subseteq \mathbb{Z}^{>0}$.

Step 2. Next, let $\mathfrak{m} = (t_1, t_2, \ldots) \subseteq A$, and observe that the morphism:[5]

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[3]We avoid geometric notation here so that $\mathbb{A}^\infty$ has a unique meaning in this text.

[4]And not merely *locally* nilpotently.

[5]Here and always we use $\underset{A}{\otimes}$ for the derived tensor product, i.e., for $\underset{A}{\overset{L}{\otimes}}$.

$$A/I_n \underset{A}{\otimes} A/\mathfrak{m} \xrightarrow{t_{n+1}\cdot} A/I_{n+1} \underset{A}{\otimes} A/\mathfrak{m}$$

is zero, i.e., is canonically nullhomotopic; indeed, this follows from the fact that multiplication by t_{n+1} is zero on $A/\mathfrak{m}$.

Passing to the colimit over n and using our earlier expression for V, we see that $V \otimes_A A/\mathfrak{m} = 0$ as desired.

$\square$

1.8. **Application to non-1-affineness of $\mathbb{A}^\infty$.** We now show that $\mathbb{A}^\infty$ is not 1-affine.

Proof of Theorem 2.4.5 from [Gai]. We follow the beginning of the proof of the theorem from [Gai] 9.3.2.

Namely, it suffices to show that the canonical functor:

$$\mathsf{Vect} \underset{\mathsf{QCoh}(\mathbb{A}^\infty)}{\otimes} \mathsf{Vect} \to \mathsf{QCoh}(\mathrm{Spec}(k) \underset{\mathbb{A}^\infty}{\times} \mathrm{Spec}(k))$$

is not an equivalence.

Let $\mathbb{G}_m$ act on $\mathbb{A}^\infty$ by scaling. The above functor is a morphism of $\mathsf{QCoh}(\mathbb{G}_m)$-module[6] categories.

By 1-affineness of $\mathbb{B}\mathbb{G}_m$, it is equivalent to show that the above functor is not an equivalence after passing to $\mathbb{G}_m$-equivariant categories. Then using the "shift of grading trick" (c.f. [AG] A.2) and 1-affineness of $\mathbb{B}\mathbb{G}_m$ again, we see that it is equivalent to show that the functor:

$$\mathsf{Vect} \underset{\underset{n}{\lim}\left(\mathrm{Sym}(k^{\oplus n}[-2])\text{--}\mathsf{mod}\right)}{\otimes} \mathsf{Vect} \to \underset{n}{\lim}(k \underset{\mathrm{Sym}(k^{\oplus n}[-2])}{\otimes} k)\text{--}\mathsf{mod} = \underset{n}{\lim} \mathrm{Sym}(k^{\oplus n}[-1])\text{--}\mathsf{mod} = \mathsf{QCoh}(\mathbb{B}\mathbb{A}^\infty)$$

is not an equivalence, where all structure maps in the limits are induced by the projections $k^{\oplus n+1} \to k^{\oplus n}$ (so are given by tensor product functors everywhere).

By the Beck-Chevalley formalism, the (discontinuous) right adjoint to the canonical functor:

$$\mathsf{Vect} = \mathsf{Vect} \underset{\mathsf{Vect}}{\otimes} \mathsf{Vect} \to \mathsf{Vect} \underset{\lim_n \mathrm{Sym}(k^{\oplus n}[-2])\text{--}\mathsf{mod}}{\otimes} \mathsf{Vect}$$

(which is induced by the canonical functors $\mathsf{Vect} \to \mathrm{Sym}(k^{\oplus n}[-2])\text{--}\mathsf{mod}$ sending k to to the free module $\mathrm{Sym}(k^{\oplus n}[-2])$, i.e., the trivial representation in $\mathsf{QCoh}(\mathbb{B}\mathbb{A}^n)$) is monadic (c.f. [Gai] Lemma 9.3.3).

Therefore, it suffices to show that the right adjoint to the corresponding functor:

$$\mathsf{Vect} \to \underset{n}{\lim}(k \underset{\mathrm{Sym}(k^{\oplus n}[-2])}{\otimes} k)\text{--}\mathsf{mod} = \mathsf{QCoh}(\mathbb{B}\mathbb{A}^\infty)$$

is not conservative (in particular, not monadic). But this functor corresponds to group cohomology, and Theorem 1 says that it is not conservative.

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[6]Here $\mathsf{QCoh}(\mathbb{G}_m)$ is equipped with the convolution monoidal structure.

References

[AG]   Dima Arinkin and Dennis Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Mathematica*, 21(1):1–199, 2014.

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