Quantum geometric Langlands and TQFTs

Sam Raskin

Yale University

January 7, 2026

Contents

Introduction

In their work on geometric Langlands, Kapustin and Witten proposed (*i*) to consider the subject from the point of view of 4d TQFTs, and (*ii*) a 1-parameter quantum deformation of GLC (considered earlier in mathematics by Drinfeld, Feigin-Frenkel, and Stoyanovsky).

The goal of this talk is to explain how to take this proposal very literally in the purely quantum setting. The underlying mathematics connects to work of many people on real Chern-Simons, quantization of Teichmuller space, and related subjects. Unfortunately, all I really understand at this point is the abelian case, which is much more elementary.

TQFTs: review

Very briefly and informally, an n-dimensional QFT Z assigns a vector space $\mathsf{Z}(M)$ of *states* to any $(n-1)$-dimensional manifold with extra decorations (orientation, metric, etc.). For any n-dimensional manifold N with boundary $\overline{M}_1 \coprod M_2$, one obtains an evaluation map $\mathsf{Z}(M_1) \to \mathsf{Z}(M_2)$.

One requires $\mathsf{Z}(M_1 \coprod M_2) = \mathsf{Z}(M_1) \otimes \mathsf{Z}(M_2)$. In particular, $\mathsf{Z}(\varnothing) = \mathbf{C}$. Therefore, any *closed* n-manifold N is assigned a number $\mathsf{Z}(N) \in \mathbf{C}$ – this is the *partition function* of N.

In a TQFT, there is no dependence on geometry of the manifolds, only topology.

Of course, these maps should "compose well," etc. – I am not going to give all the axioms here. And there are various flavors of TQFTs – oriented, framed, spin, etc. – my priority is going to be on illustrating ideas rather than being maximally precise at every stage.

TQFTs: review

Of course, physical QFTs are typically non-topological.

But in the presence of *supersymmetry*, one can sometimes *deform* a physical theory to a topological theory by a procedure known as *twisting*. Experts say that very special, but still interesting quantities, may even survive the deformation.

Regardless, mathematicians like TQFTs because they relate to a great deal of interesting mathematics.

TQFTs: review

Experts in TQFTs teach us that it is convenient to consider *extended* TQFTs. These assign *categories* enriched over vector spaces to $(n-2)$-dimensional manifolds. These allows us to calculate $\mathsf{Z}(M^{n-1})$ by cutting and pasting.

More generally, one assigns ($\mathbf{C}$-linear) a 2-category to an $n-3$-dimensional manifold, ..., and an $(n-1)$-category to a point.

Lurie formalized this and proved that framed TQFTs of dimension n are the same as $n-1$-categories with various finiteness conditions. E.g., categories are (essentially) the same as framed $2d$ TQFTs. Moreover, if the finiteness conditions are "too weak," the theory may be defined only on manifolds of dimensions $[0, d]$ for $d < n$.

To orient you, in this talk, we mostly care about 3 and 4-dimensional TQFTs and I only want to think about their values up to dimension 3.

Kapustin-Witten summary

In their work on geometric Langlands, Kapustin and Witten said that for any $4d$ QFT Z with $\mathcal{N} = 4$ supersymmetry, there is a family of topological twists Z_c indexed by a single parameter $c = \frac{1}{\Psi} \in \mathbf{P}^1_{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. These are now called *Kapustin-Witten* twists.

For any reductive group G, there is such a theory YM_G, the maximally supersymmetric Yang-Mills theory with gauge group G^c (the compact form of G).

Kapustin and Witten argue that S-duality $\mathsf{YM}_G \simeq \mathsf{YM}_{\check{G}}$ exchanges the twists for c and $-\frac{1}{c}$ (up to linear factors I'll ignore henceforth), i.e., give rise to S-dualities $\mathsf{YM}_{G,c} \simeq \mathsf{YM}_{\check{G},-\frac{1}{c}}$.

Kapustin-Witten summary

Let Σ be a topological surface.

For $c = 0$, [KW] argue that $\mathsf{YM}_{G,0}$ compactified along Σ is the B-model of $\mathrm{LocSys}_G(\Sigma) = \{\sigma : \pi_1(\Sigma) \to G\}/G$. Concretely, this means $\mathsf{YM}_{G,0}(\Sigma) = \mathsf{QCoh}(\mathrm{LocSys}_G(\Sigma))$.

For $c = \infty$, [KW] argue that $\mathsf{YM}_{G,\infty}$ compactified along Σ is the A-model of $T^* \mathrm{Bun}_G(\Sigma)$, at least once you choose a holomorphic structure on Σ. Concretely, this might mean $\mathsf{YM}_{G,\infty}$ is something like $\mathrm{D}\text{–}\mathsf{mod}(\mathrm{Bun}_G)$.

One arrives at a statement

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}(\Sigma)) \simeq \mathrm{D}\text{–}\mathsf{mod}(\mathrm{Bun}_G),$$

which, up to various corrections, is now a theorem (to be discussed tomorrow). In this sense, Kapustin-Witten find connections between GLC and a 1-parameter deformation of it.

Quantum GLC in the de Rham setting

Let me write the conjecture briefly for later reference.

Conjecture

$$\mathrm{D\text{–}}\mathsf{mod}_{\psi}(\mathrm{Bun}_G) \simeq \mathrm{D\text{–}}\mathsf{mod}_{-\frac{1}{\psi}}(\mathrm{Bun}_{\check{G}})$$

As $\psi \to \infty$, one can see that $\mathrm{D\text{–}}\mathsf{mod}_{\psi}(\mathrm{Bun}_{\check{G}})$ tends to $\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}})$ in a precise sense.

In other words, quantum GLC is more symmetric – both sides look like D-modules on $\mathrm{Bun}_{G/\check{G}}$ – but in the limit, we recover the usual asymmetric statement.

Constructing TQFTs

Our goal now is to construct TQFTs. Sincerely: apologies for introducing brain twisters.

I already said that a 2d TQFT is essentially just a category $\mathcal{C}$.

One way to get a category is to start with an associative (or E_1) algebra A and take $\mathcal{C} = A\text{–}\mathsf{mod}$.

One can combine the two paragraphs above. One can take a *monoidal* category $\mathcal{A}$ and take the *2-category* $\mathcal{A}\text{–}\mathsf{mod}$. This gives a *3d* TQFT.

One way to get monoidal categories is to take $A\text{–}\mathsf{mod}$ where A is a commutative algebra. In fact, homotopy theorists teach us we only need A to be E_2 (if we work in a derived context).

So E_1-algebras give $2d$ theories, and E_2-algebras give $3d$ theories. (By the way, E_2-algebras are topological counterparts to VOAs, which relatedly have many ties to $3d$ TQFTs.)

Constructing TQFTs

By the same logic, an E_2-category $\mathcal{A}$ gives a $4d$ theory $Z_{\mathcal{A}}$. From here on out, we can more or less think of $4d$ TQFTs as braided monoidal categories with some funny notion of equivalence.

E_2-categories are not exotic for representation theorists/quantum topologists. They are also known as *braided monoidal categories* and were first considered by Drinfeld. In concrete situations, a braiding is the same as an R-matrix.

There is a favorite braided monoidal category attached to G and $q \in \mathbf{C}^\times$: $\mathsf{Rep}_q(G)$, the category of representations of the quantum group.

Definition

$\mathsf{YM}^{\mathrm{alg}}_{G,q} = \mathsf{YM}^{\mathrm{alg}}_{G,c}$ is the $4d$ theory associated with $\mathsf{Rep}_q(G)$ for $q = e^{2\pi i c}$ via the above procedure.

Example: $G = \mathbf{G}_m$

$\mathsf{Rep}_q(\mathbf{G}_m)$ consists of $\mathbf{Z}$-graded vector spaces $V = \oplus_{n \in \mathbf{Z}} V_n$. The monoidal structure is the usual one:

$$(V \otimes W)_n = \bigoplus_r V_r \otimes W_{n-r}.$$

(So tensoring a vector space in graded degree r and one in degree s gives a vector space in degree $r+s$.)

The braiding $\beta_{V,W} : V \otimes W \simeq W \otimes V$ is given by

$$v_r \otimes w_s \mapsto q^{rs}\, w_s \otimes v_r, \quad v_r \in V_r, w_s \in W_s.$$

More on $\mathsf{YM}^{\mathrm{alg}}_{G,q}$

For $q = 1$, $\mathsf{YM}^{\mathrm{alg}}_{G,1}$ is easy to understand. For a 3-manifold M^3, it assigns $\mathrm{Fun}(\mathrm{LocSys}_G(M^3))$. For a 2-manifold, $M^2 = \Sigma$, it assigns $\mathsf{QCoh}(\mathrm{LocSys}_G(M^2))$. For a 1-manifold, it assigns sheaves of *categories* on $\mathrm{LocSys}_G(M^1)$, sometimes written as $2\mathsf{QCoh}(\mathrm{LocSys}_G(M^1))$. For a point, it assigns $3\mathsf{QCoh}(\mathbf{B}G) =$ sheaves of 2-categories on $\mathbf{B}G = \mathrm{LocSys}_G(pt)$.

In general, I picture $\mathsf{YM}_{G,q}$ as assigning q-deformed versions of the above objects. Sometimes I write $\mathsf{QCoh}_q(\mathrm{LocSys}_G(\Sigma))$, etc.

Note the contrast with the de Rham story: here everything is deformed from $\mathsf{QCoh}(\mathrm{LocSys}_G)$!

Example: $G = \mathbf{G}_m$

Say Σ is a genus g surface and choose a Darboux basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ for $H^1(\Sigma, \mathbf{Z})$ with $a_i \cap b_j = \delta_{ij}$.

Then up to mild corrections, $\mathrm{LocSys}_{\mathbf{G}_m}(\Sigma) = \mathbf{G}_m^{2g}$, so QCoh of it is $\mathbf{C}[x_1^{\pm 1}, \ldots, x_g^{\pm 1}, y_1^{\pm 1}, \ldots, y_g^{\pm 1}]$–mod. The q-deformed version is modules over the *quantum torus algebra* $A_{g,q} = \mathbf{C}\langle x_1^{\pm 1}, \ldots, x_g^{\pm 1}, y_1^{\pm 1}, \ldots, y_g^{\pm 1}\rangle / (x_i y_j = q y_j x_i)$.

In sum:

$$\mathsf{YM}^{\mathrm{alg}}_{\mathbf{G}_m, q}(\Sigma) \approx A_{g,q}\text{–}\mathsf{mod}.$$

Unhappy conjecture

Conjecture (Too naive)

$$\mathsf{YM}^{\mathrm{alg}}_{G,c} \simeq \mathsf{YM}^{\mathrm{alg}}_{\check{G},-\frac{1}{c}}.$$

There are many things to object to here. Most obviously, both sides depending algebraically on $q = e^{2\pi i c}$ and $\check{q} = e^{\frac{2\pi i}{c}}$, but the relationship between q and $\check{q}$ is transcendental.

There also are a number of very salient works in the literature (Ponsot-Teschner, Frenkel-Ip, Faddeev, Goncharov-Shen, ...).

The Weil representation

In the simplest case of $\mathbf{G}_m$, we are looking for a bimodule between $A_{g,q}$ and $A_{g,\check{q}}$ that should realize the equivalence. The literature suggests that this bimodule should be realized as:

$$\text{“}\mathrm{Fun}(\mathbf{R}^g)\text{”}$$

with the following operators, say for $g = 1$ for notational simplicity:

$$(x \cdot \phi)(t) = \phi(t + \sqrt{c}),$$
$$(y \cdot \phi)(t) = e^{2\pi i \sqrt{c} \cdot t} \phi(t)$$

and similarly for dual generators $\check{x}$ and $\check{y}$ with $\frac{1}{\sqrt{c}}$ in place of $\sqrt{c}$. Note that x and $\check{y}$ commute because $e^{2\pi i} = 1$.

A comment: $\mathrm{Fun}(\mathbf{R}^g)$ should be thought of as the Weil representation. Note that this is not acted on by the mapping class group but a central extension of it; this will appear as an *anomaly* later in the talk.

Some comments on formalism

The previous slide suggests that in reality, we need some kind of functional analysis. But also I want to do homological algebra/derived categories (this was implicit all along).

A suitable formalism was introduced recently by Clausen-Scholze. They defined a symmetric monoidal derived category $\mathsf{LiqVect}$ containing Vect. Many usual types of topological vector spaces (Banach, nuclear Fréchet, etc.) embed here, and the tensor product corresponds to classical tensor products on these objects. It is a convenient, flexible formalism; you should black box it and just take liquid to mean "TVS+homological algebra."

We can tensor over Vect with $\mathsf{LiqVect}$. This is innocuous, it just means thinking of a vector space as a topological vector space. You might also picture tensoring over $\mathbf{C}$ with a larger ring $\mathbf{C}^{\mathrm{liq}}$.

Less naive expectation

Now define $\mathsf{YM}_{G,c}$ as $\mathsf{YM}^{\mathrm{alg}}_{G,c} \otimes_{\mathbf{C}} \mathbf{C}^{\mathrm{liq}}$.

We might less naively expect $\mathsf{YM}_{G,c} \simeq \mathsf{YM}_{\check{G},-\frac{1}{c}}$. Our main theorem essentially asserts this for $G = \mathbf{G}_m$, up to the *anomaly* indicated before.

Concretely, this means we have statements like $A_{g,q}$–$\mathsf{mod}(\mathsf{LiqVect}) \simeq A_{g,\check{q}^{-1}}$–$\mathsf{mod}(\mathsf{LiqVect})$, giving by tensoring by a bimodule in $\mathsf{LiqVect}$ (essentially the Weil representation).

Anomalies

Definition

An *invertible* n-dimensional TQFT τ, say defined up to dimension $(n-1)$, is one that assigns a line to every $(n-1)$-dimensional manifold, a *gerbe* (category non-canonically equivalent to Vect) to every $(n-2)$-dimensional manifold, etc.

For later use:

Definition

Given τ, a τ-anomalous TQFT Z of dimension $n-1$ is one that assigns $Z(M^{n-1})$ a vector in the line $\tau(M^{n-1})$, etc.

Statement of the main theorem

Theorem (Scholze-R.)

1. *There is a canonical invertible theory* τ *(valued in* LiqVect*) and an equivalence* $\mathsf{YM}_{\mathbf{G}_m,c} \simeq \mathsf{YM}_{\mathbf{G}_m,-\frac{1}{c}} \otimes \tau$.
2. *For* Σ *a compact oriented surface,* $\tau(\Sigma)$ *is the metaplectic/Stone-von Neumann gerbe of the real symplectic vector space* $H^1(\Sigma, \mathbf{R})$.
3. $\tau^{\otimes 2} = \mathsf{triv}$ *canonically.*
4. *As a spin TQFT,* τ *can be trivialized.*

Informally, this is all modeled on the example of the Weil representation I wrote before – the whole story is an appropriate retelling of representations of the Heisenberg group and its properties.

$\widetilde{\mathbf{G}}_m$

There is one major ingredient I left out: what do we mean by "Fun($\mathbf{R}$)"?

The role of $\mathbf{R}$ is played by an *analytic space* introduced by Scholze and denoted $\widetilde{\mathbf{G}}_m$ (but maybe think of it as an incarnation of $\mathbf{R}$). It is open in $\mathbf{A}^1_{\mathbf{C}}$ and a subgroup; its functions are entire functions with at most exponential growth along imaginary axes. There is a map $\exp(2\pi i \cdot -) : \widetilde{\mathbf{G}}_m \to \mathbf{G}_m$ that fits into a SES:

$$0 \to \mathbf{Z} \to \widetilde{\mathbf{G}}_m \to \mathbf{G}_m \to 1.$$

The quadratic Gaussian $\exp(\pi i(-)^2) : \widetilde{\mathbf{G}}_m \to \mathbf{G}_m$ induces a Cartier self-duality of $\widetilde{\mathbf{G}}_m$; the above SES is self-dual with respect to this self-duality. In particular, we have a Fourier transform $\mathsf{Rep}(\widetilde{\mathbf{G}}_m) \simeq \mathsf{QCoh}(\widetilde{\mathbf{G}}_m)$.

Hopes

I (and others) hope there is a braided monoidal category $\mathsf{Rep}^{\mathrm{cts}}_{c,-\frac{1}{c}}(\mathbf{G})$ symmetric in G and $\check{G}$ receiving a braided monoidal functor

$$\mathsf{Rep}_q(G) \otimes \mathsf{Rep}_{\check{q}}(\check{G}) \to \mathsf{Rep}^{\mathrm{cts}}_{c,-\frac{1}{c}}(\mathbf{G})$$

with some desired properties to be spelled out below. In short, it should be the main player in the story.

Example

In the case of $\mathbf{G}_m$, this category consists of "$\mathbf{R}$-graded vector spaces" with braiding $V_r \otimes W_s \simeq W_s \otimes V_r$ given by $e^{2\pi irs}$, as before. $\mathsf{Rep}_q(\mathbf{G}_m)$ lives here as $\sqrt{c}\mathbf{Z}$-graded vector spaces, and similarly for $\check{q}$ and $\frac{1}{\sqrt{c}}\mathbf{Z}$. These objects commute with each other as $e^{2\pi i} = 1$. (Precisely, "$\mathbf{R}$"-graded vector spaces are quasi-coherent sheaves on $\widetilde{\mathbf{G}}_m$, of course.)

Hopes

In the non-abelian case, $\mathsf{Rep}^{\mathrm{cts}}_{c,\frac{1}{c}}(\mathbf{G})$ should contain the *self-dual principal series* representations considered by Schmüdgen, Ponsot-Teschner, Bytzko-Teschner, Frenkel-Ip, Ip, ... and these should play the role of Verma modules. Notably, these representations contain commuting actions of $U_q(\mathfrak{g})$ and $U_{\check{q}}(\check{\mathfrak{g}})$ that are are transcendentally related to each other.

In the case of $\mathbf{G}_m$, there is no dependence on c, but this should not expected in the non-abelian case.

From the QFT perspective, $\mathsf{Rep}^{\mathrm{cts}}_{c,\frac{1}{c}}(\mathbf{G})$ should be the category of line operators in Chern-Simons for the split group $G(\mathbf{R})$ with central charge c (or its "positive sector?"). (Traditional Chern-Simons is about the compact form of G.)

The category $\mathsf{Rep}^{\mathrm{cts}}_{c,\frac{1}{c}}(\mathbf{G})$ should be a Betti version of the representations of the affine $\mathcal{W}$-algebra, much as $\mathsf{Rep}_q(G)$ is a Betti version of $\mathsf{KL}_{G,\frac{1}{c}} = \widehat{\mathfrak{g}}_{\frac{1}{c}}\text{--}\mathsf{mod}^{\mathfrak{L}^+G}$.

Hopes

I then expect the following behavior:

- The $4d$ theory $\tau_{\mathbf{G},c,\frac{1}{c}}$ defined by $\mathsf{Rep}_{c,\frac{1}{c}}(\mathbf{G})$ is invertible.
- The "tautological boundary condition" for this theory, considered as an anomalous $3d$ theory, is Chern-Simons for $G(\mathbf{R})$.
- The canonical map $\tau_{\mathbf{G},c,\frac{1}{c}} \to \mathsf{YM}_{G,c} \otimes \mathsf{YM}_{\check{G},\frac{1}{c}}$ induces an equivalence:

$$\tau_{\mathbf{G},c,\frac{1}{c}} \otimes \mathsf{YM}_{\check{G},-\frac{1}{c}} \xrightarrow{\simeq} \mathsf{YM}_{G,c}.$$

At least it works for a torus!

Thanks!